2. How many different weights can be weighed if we have the following weights? (All the weights with the same grams are the same. Please explain the answer using the corresponding generating function.)


(1+x)^4(1+x^2)^2(1+x^4)^2 = … all the powers of x between 0and 16 have a non-zero coefficient --> all the weights between 0 and 16g

$$G(x) = (1+x+x^2+x^3+x^4)(1+x^2+x^4)(1+x^4+x^8)$$

# *Chapter 7*

# *Recurrence Relations and Generating functions*

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# Generating Function & Counting Rules

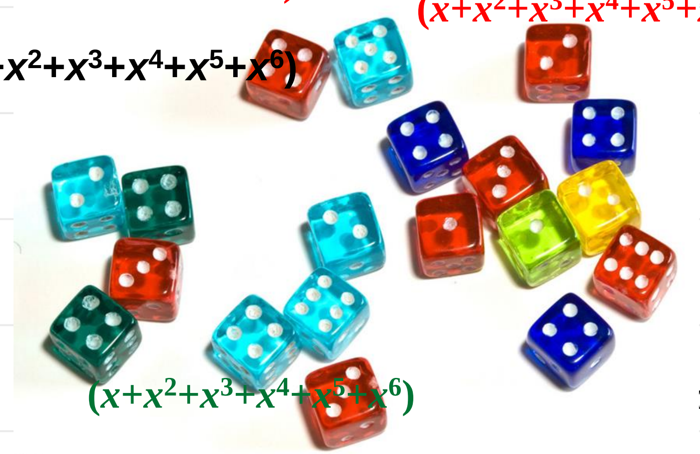
**Generating Function is mother，counting sequence is a child。**

- **For $m$ number of dice, what is the number of possibilities for the summation of points equals to $n$?**

$$G(x)=(x+x^2+x^3+x^4+x^5+x^6)^m$$

the coefficient of $x^n$ in the expansion equation

Jakob I. Bernoulli
Swiss Mathematician
Year 1654－1705

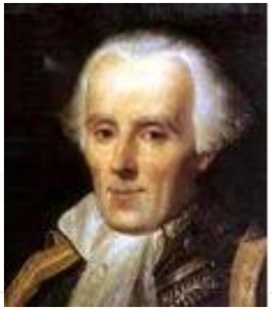


# Generating Function

- Given an infinite sequence of numbers: $h_0, h_1, h_2, \ldots, h_n, \ldots$
- The generating function is defined to be the infinite series

  $$G(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n + \ldots$$
- A **generating function** is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers $h_n$ that is indexed by the natural numbers.
- A finite sequence: $h_0, h_1, h_2, \ldots, h_m$
  - $h_0, h_1, h_2, \ldots, h_m, 0, 0, \ldots$
  - $G(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_m x^m$

## § 1.Generating Function & Counting Rule



Laplace

- **Definition 2-1** for sequence $c_0, c_1, c_2\ldots,$

$$G(x) = c_0 + c_1 x + c_2 x^2 + \ldots\ldots$$

Function $G(x)$ is the generating function for $c_0, c_1, c_2 \ldots$。

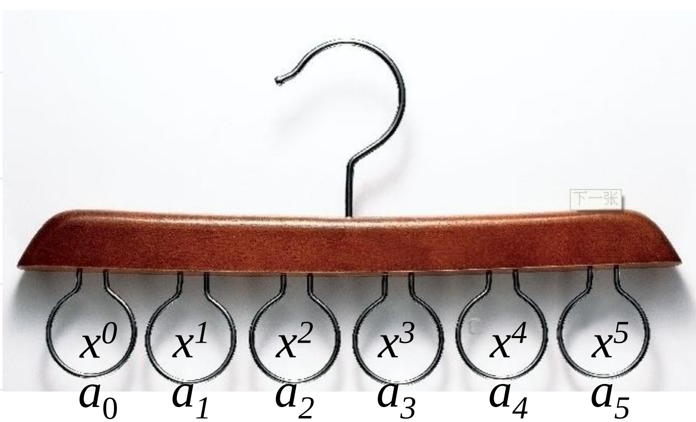
- In 1812， French mathematician Laplace was studying on generating function method and its theories while writing the 1st volume of "The Analysis Theory of Probability"
  - Counting Tool
  - Do not consider the convergence
  - Do not consider the actual value
  - Formal power series

Generating function is a line of hangers which used to display a series of number sequences 。

— Herbert · Vere

$$G(x)= x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$

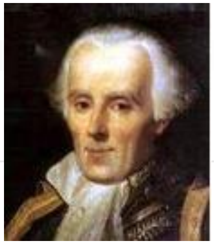
Function: $\boxed{f(x)}=\sum_{n=0}^{\infty} a_n \boxed{x^n}$ $\qquad G(x)=\sum_{n=0}^{\infty} \boxed{a_n} x^n$




**Definition 2-1** For sequence $a_0, a_1, a_2\ldots$, form a function

$$G(x) = a_0 + a_1x + a_2x^2 + \ldots,$$

Name $G(x)$ as the generating function for sequence $a_0, a_1, a_2\ldots$。



Laplace
Year 1812

Bernoulli
Year 1705

$$\begin{tikzcd} \square + \square = n \arrow[r] \arrow[d, "?"'] & (x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6) \arrow[d] \\ c_0, c_1, \ldots, c_n & = x^2+2x^3+3x^4+4x^5+\mathbf{5}x^6+\ldots\ldots \arrow[l] \end{tikzcd}$$

Euler
Year 1764

**Found the mapping relationship is a "Mathematic Discovery".**
**Finding mapping is an important mathematic thinking.**

*Like A Function But Not A Function,*
*It's Mapping*

2. How many different weights can be weighed if we have the following weights? (All the weights with the same grams are the same. Please explain the answer using the corresponding generating function.)


What if we constrain that at least one of the 1g weights should be used, how many different ways to weigh 8g? (Please explain the answer using the corresponding generating function.)

主观题 10分

设置

请您编辑题目

weights can be weighed if we have the following weights? (All the weights with the same grams are the same. Please explain the answer using the corresponding generating function.)


What if we constrain that at least one of the 1g weights should be used, how many different ways to weigh 8g? (Please explain the answer using the corresponding generating function.)

正常使用主观题需2.0以上版本雨课堂

作答

# Example Question

**Example 1:** If there is 1、2、4、8、16、32g of weights each, how different weight can be weighed? How many possible solutions?

$$G(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$$

$$\boxed{(1+x)(1-x) = (1-x^2)}$$

$$= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^8}{1-x^4}\frac{1-x^{16}}{1-x^8}\frac{1-x^{32}}{1-x^{16}}\frac{1-x^{64}}{1-x^{32}}$$

$$= \frac{1-x^{64}}{1-x} = (1+x+x^2+\cdots+x^{63}) = \sum_{k=0}^{63} x^k$$

$$(q^0+q^1+q^2+\ldots.+q^n) = \frac{1-q^{n+1}}{1-q}$$

$$\boxed{(1-x)^{-1} = 1+x+x^2+\ldots..}$$

The generating function of the infinite sequence 1,1,1,…,1,… ($h_i$=1)

$$g(x) = 1+x+x^2+\ldots+x^n+\ldots = \frac{1}{1-x}$$

# Example Question

**Example:** Integer $n$ is split into the summation of 1， 2， 3， …， $m$, and repetition is allowed, get its generating function.

If integer $n$ is split into the summation of 1， 2， 3， …， $m$, and repetition is allowed, its generating function is

$$\boxed{(1-x)^{-1}=1+x+x^2+.....}$$

$$G_1(x)=(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots$$

$$\cdots(1+x^m+x^{2m}+\cdots)$$

# Example Question

If $m$ appeared at least once， how is the generating function?

$$\boldsymbol{G_2(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\cdots(x^m + x^{2m} + \cdots)}$$

$$\boldsymbol{= \frac{x^m}{(1 - x)(1 - x^2)\cdots(1 - x^m)}}$$

$$G_2(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^m)} - \frac{1}{(1-x)(1-x^2)\cdots(1-x^{m-1})}$$

The above combination meaning: The partition number of integer $n$ which is split into the summation of 1 to $m$, minus the partition number of the split 1 to $m$-1, is the partition number of $m$ at least appeared once.

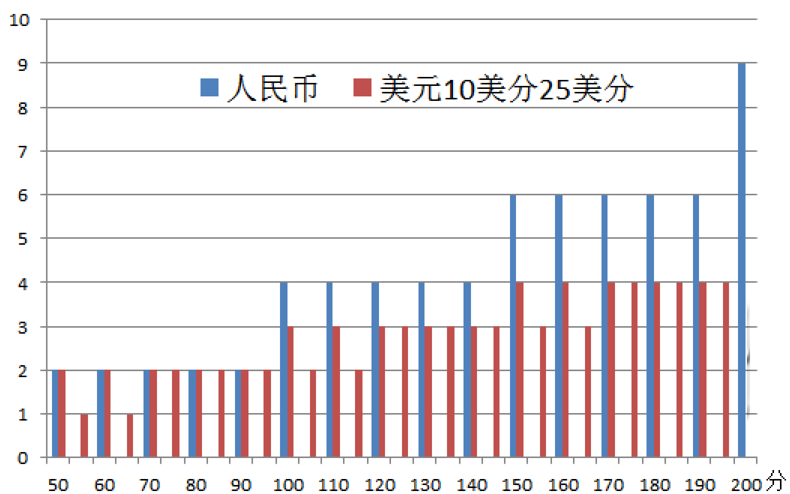
# Combinations of Coins

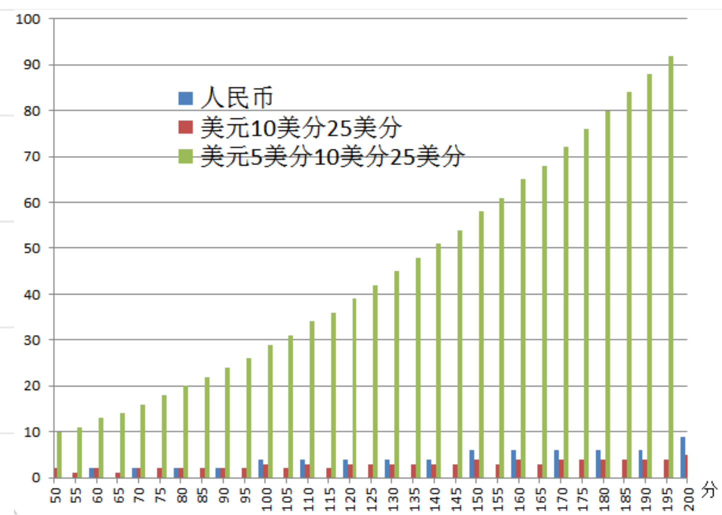
- China Yuan (RMB) common coins: 10 cents, 50 cents, 1 dollar
- The generating function for China Yuan coins

$$G(x) = (1 + x^{10} + x^{20} + \ldots)(1 + x^{50} + x^{100} + \ldots)(1 + x^{100} + x^{200} + \ldots)$$

- USD common coins: 10 cents, 25 cents

$$G(x) = (1 + x^{5} + x^{10} + \ldots)(1 + x^{10} + x^{20} + \ldots)(1 + x^{25} + x^{50} + \ldots)$$



# The Partition of Integer

Natural number (positive number) partition is to express a positive number as the summation of several positive number:

Order is considered within various parts is named as orderly partition (Composition) ;

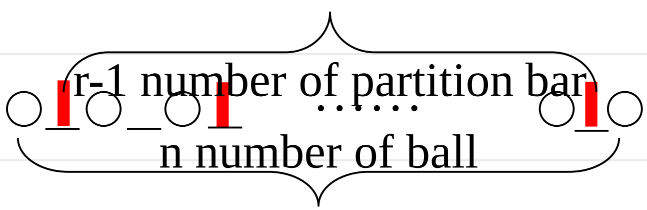
Otherwise, it is known as unordered partition (Partition).

3's orderly 2-splitting: 3=2+1=1+2

n's orderly r− splitting number is C(*n*-1,*r*-1)

n number of ball, split into r part，

Use r−1 of partition wall to put within n−1 gap between balls，solution number is C(n−1, r−1)




n-1 number of gap

Ball Placing Model: *n*'s single *r*-splitting is equaled to put *n* identical balls into *r* labelled boxes. Box cannot be left empty.

## Orderly partition of ball placing model: n’s single r-splitting is same as putting n identical balls into r **labelled** boxes, box cannot be left empty

- Unordered Partition
- 3’s unordered 2-splitting： 3=2+1
- 3’s all unordered splitting 3=3+0+0=2+1+0=1+1+1
- $x_1+x_2+\ldots+x_r=n$ number of solution of non-negative number? $C(n+r-1,n)$

$$\overbrace{\underset{x_1 \text{number of } 1}{1\,..1}\ \underbrace{|\ 1..\,1\ |\ 1..\,1\ |\qquad\qquad |}_{\text{r-1 number of door frame}}\ 1..1}^{\text{n number of 1}}$$

Equals to put n identical ball into r **labelled** box， box can be left empty

0+3+0

3+0+0

Unlike unordered partition

Integer Partition (**partition** of a positive integer *n* ) is to partition integer into the summation of several integers , same as putting n identical balls into n unlabeled boxes, box can be left empty, also allows to place more than 1 balls. Integer is partitioned into the summation of several integers with different ways, the total number of different splitting methods is known as partition number.

单选题 1分

设置

Calculate the number of the orderly 3-partition (Composition) for integer 5:

A C(7, 2)

B C(5,2)

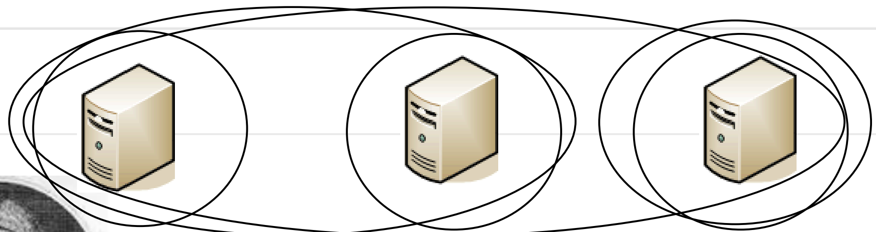
C C(8,3)

D C(4,2)

提交

# Application of Generating Function: Integer Partition Number

- **Unordered Partition of Positive Integer**： Split a positive integer *n* into the summation of several integer, the order between numbers is ignored and allow repetition, its different partition number is *p*(*n*)。
  - Cryptography, Statistics, Biology……
- ***p*(3)=3** : 3， 2 + 1， 1 + 1 + 1.



Philippe Naudé

Year 1740, Naude asked Euler in the letter:

*Solution number of partition of integer?*

**Exponent correspondence value**

Euler

$$G(x) = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots(1+x^m+x^{2m}+\ldots)\ldots$$ coefficient of $x^n$

Generating Function of "1" | Generating Function of "2" | Generating Function of "m"



*Weil A． Number Theory：An Approach Thought History [M]． Boston：Birkhouser，1984． 284*

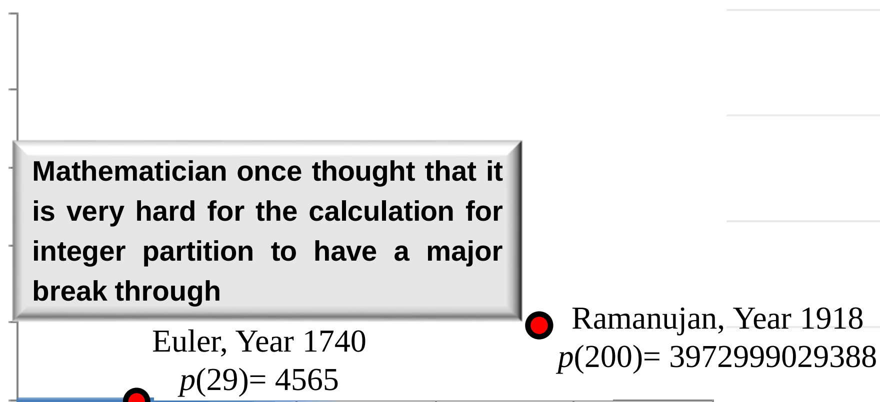
# Application of Generating Function: Integer Partition Number

- OEIS: On-line Encyclopedia of Integer Sequences
  - Number Theory Related Authoritative Database and Algorithm Library
  - $p(n)$: A000041 sequence
- Generating function of integer partition $p(n)$

# The Application of Generating Function: Integer Partition Number

- OEIS: On-line Encyclopedia of Integer Sequences
  - Number Theory Related Authoritative Database and Algorithm Library
  - $p(n)$: A000041 sequence
- Generating function of integer partition p(n)

$$G(x) = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ (1+x^3+x^6+\ldots)\ \ldots\ (1+x^m+x^{2m})\ldots$$

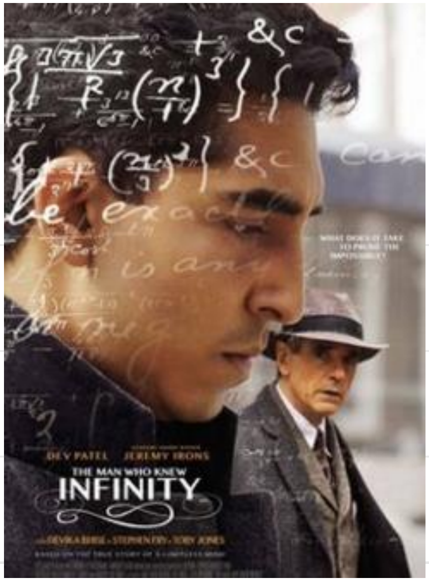



**Mathematician once thought that it is very hard for the calculation for integer partition to have a major break through**

**Euler's generating function method is ingenious; but it is not suitable for calculating $p(n)$ 。**

Manual Calculation for Polynomial Calculation

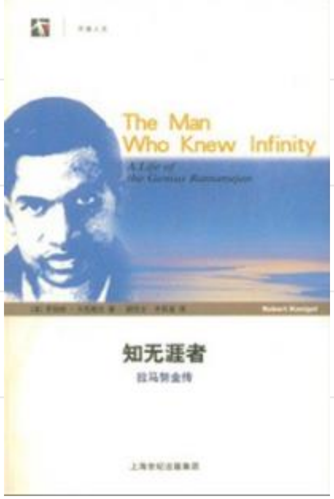
# Son of India, Ramanujan （1887-1920）




“The weirdest person in the history of mathematic and also science”

He had never exposed to any proper mathematic training but he owned very amazing sixth sense towards mathematic, he discovered almost 3900 mathematics formulas and propositions independently.

In year 2012, America mathematician Ken Ono and his colleagues had proved that as Ramanujan was laid dying, he left a miraculous function which can be used directly to explain the partial secret of our black holes.




He wrote down all his foreseen mathematic propositions into 3 notebooks; and many of them got proven later. For example, mathematician V. Deligne had proved in year 1973 on Ramanujan’s guess which was placed in the year 1916. And, he was awarded with Fields Metal in year 1973.

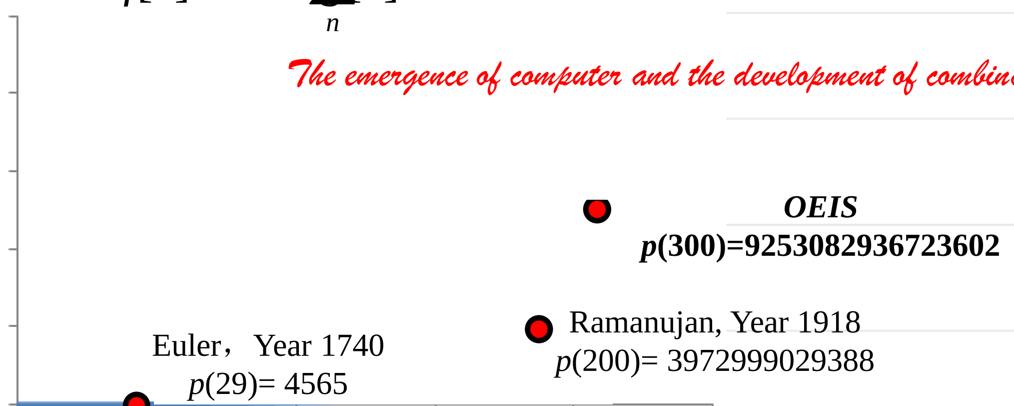
America University of Florida had founded 《 Ramanujan’s Periodic Magazine》 in year 1997, specifically to publish on the research papers which are related to “His Influenced Mathematic Field ”；

# Application of Generating Function: Integer Partition Number

- OEIS: On-line Encyclopedia of Integer Sequences
  - Number Theory Related Authoritative Database and Algorithm Library
  - p(n): A000041 sequence
- Generating function of integer partition p(n)

$$G(x) = (1+x+2x^2+2x^3+2x^4+2x^4 \ldots)\,(1+x^3+x^6+\ldots)\ \ldots\ (1+x^m+x^{2m})\ldots$$

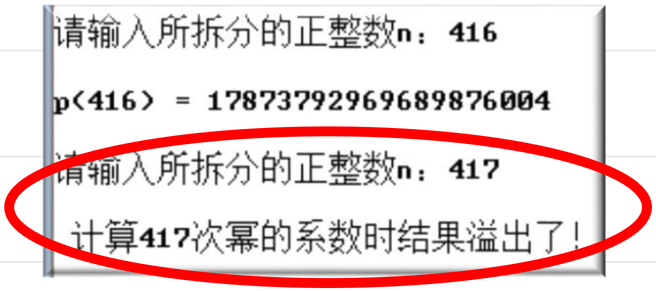
$$(f*g)[k] = \sum_n f[i]\,g[k-i]$$




**Euler's generating function method is ingenious; but it is not suitable for calculating $p(n)$ 。**

Manual Calculation for Polynomial Calculation

# Application of Generating Function: Integer Partition Number




64-bit of computer unsigned integer

unsigned_int64 – largest representation is $2^{64}$-1

**18,4**46,744,073,709,551,615

$p$(416)= **17,8**73,792,969,689,876,004

$p$(417)= **18,9**87,964,267,331,664,557

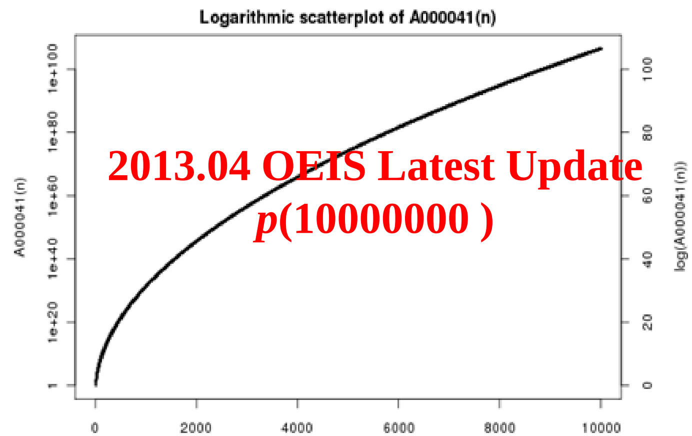
**The polynomial calculation which based on integer representation can only be calculated until *p*(416)**

- How large the integer partition number can be calculated to?

**Algorithm for Big Number Calculation?**

$$(f*g)[m]=\sum_{n} f[n]g[m-n]$$

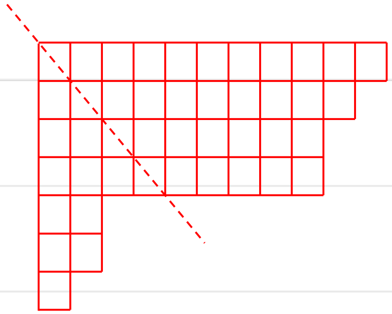
- Related Guess： BSD Guess
  - Birch and Swinnerton-Dyer’s Guess
  - 7 Big Problems of Mathematics
  - 1 million USD Awards

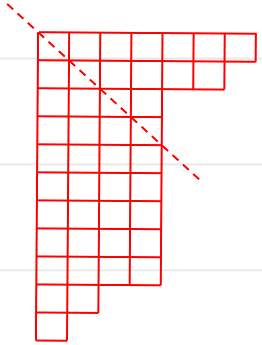



**Do you want to have a try to accurately calculate the integer partition number for n?**

# Ferrers Diagram

From top to bottom n level of grids, $m_i$ is the number of grids for level $i$, when $m_i \geq m_{i+1}$, where the total grids of level above is not less than the level below（weakly decreasing ）， known as Ferrers diagram





Ferrers Diagram owns the following characteristics:

1. Each level contains at least 1 grid.
2. 1st row exchanged with 1st column, 2nd row exchanged with 2nd column, …, as image is rotated by following the dotted line as axis; is still Ferrers diagram. 2 Ferrers diagrams are known as a pair of conjugated Ferrers diagram.

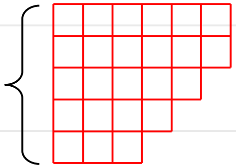
# Ferrers Diagram

Through Ferrers diagram, it managed to get a very interesting result for integer partition.

(a) the number of ways to partition n into k numbers would be the same to the number of ways to partition n with the largest number of k.

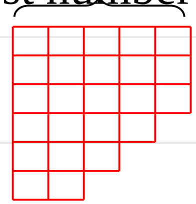
Because integer n is split into the summation of k numbers and its partition can use one k row of diagram to represent. The conjugated Ferrers diagram contains k grids on its top level. For example:

5 numbers



24=6+6+5+4+3  5 numbers，largest is 6

Largest number as 5



24=5+5+5+4+3+2  6 numbers, largest is 5

(b) The partition number of integer n is split into the summation of not more than m numbers, is equaled to n is split with the partition number that is not more than m.

Reason is similar to (a).

The generating function for the partition number of partition where the summation of not more than m numbers is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}$$

The generating function of the partition number of partitioning into the summation of not more than m−1 numbers is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^{m-1})}$$

The generating function of the partition number of the summation of exact partitioning into m numbers is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}-\frac{1}{(1-x)(1-x^2)\cdots(1-x^{m-1})}=\frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)}$$

# *ToDo List*

- Homework sheet
- PreClass Video
  - Recurrence Relations