C & A

Chap. II
Permutation and Combination

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Review of the pre-class video

- Four basic counting principles
  - Addition
  - Multiplication
  - Substraction
  - Division

Determine the number of positive integers which are factors of the number $3^4 \times 5^2 \times 11^7 \times 13^8$?

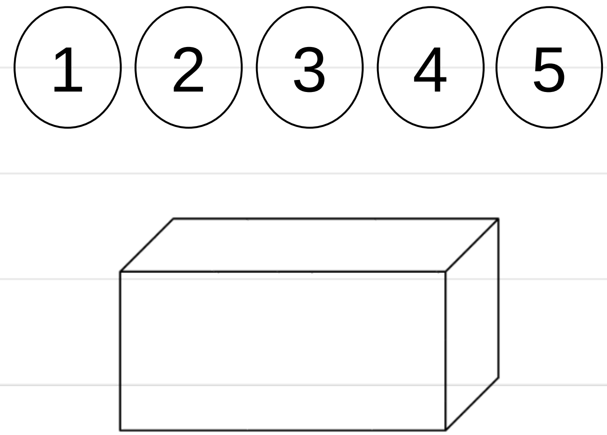
To give your aunt a basket of fruit. You have six oranges and nine apples. The only requirement is that an empty basket is not allowed. How many different baskets of fruit are possible?

Example

- Computer passwords are to consist a string of six symbols taken from the digits 0,1,2,…9 and the lower case letter a,b,…,z. How many computer passwords have a repeated symbol?
- Let U be the set of all computer passwords with and without repeated symbols.
  - 6 numbers, for each number, there are 10 + 26 choices
  - $|U|= (10+26)^6=36^6$
  - $\bar{A}$: the password without repeated symbol
  - $|\bar{A}|= 36\times35\times34\times33\times32\times31$
  - $|A| = |U| - |\bar{A}| = 774,372,906$

Combinations and Permutations

- Choose 3 balls and put them into a box
  – Not considering the order of picking, the balls in the box is a combination
  – Considering the order of picking, the sequence of picking is a permutation



If the order doesn't matter, it is a **Combination**.

If the order **does** matter it is a **Permutation**.

Put pingpang balls to boxes

- Permutation and Combination
- Indexed pingpang balls：
  - 4 pingpang balls：#1, #2, #3, #4
  - Pick 3 of them
  - If consider the order ⇒ Permutation P(4,3)
    - $P(4,3)=4\times3\times2 = 24$ Permutation without repetitions
  - If not **consider** the order ⇒ Combination C(4,3)
    - $C(4,3)=24/3! = 4$ Combination without repetitions

Permutations

- A *permutation* of a set S of objects is a sequence containing each object once.
- An ordered arrangement of r distinct elements of S is called an *r-permutation*.
- The number of r-permutations of a set with $n=|S|$ elements is
  - $P(n,r) = n(n-1)\ldots(n-r+1) = n!/(n-r)!$
- If $r > n$, then $P(n,r) = 0$.
- $P(n,1) = n$ for each positive integer n.
- $P(n,n) = n!$

http://www.mathsisfun.com/combinatorics/combinations-permutations.html

Permutation Example

- A terrorist has planted an armed nuclear bomb in your city, and it is your job to disable it by cutting wires to the trigger device. There are 10 wires to the device. If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode! If the wires all look the same, what are your chances of survival?

$P(10,3) = 10 \cdot 9 \cdot 8 = 720$,
so there is a 1 in 720 chance that you'll survive!
0.0013888888888

Example

- How many different five-digit numbers can be constructed out of the digits 1,1,1,3,8?
- Multiset with three objects(1,3,8), 1 repeated three times.
- If we identify three 1s as $1_a, 1_b, 1_c$
- Then the number of permutations is 5!
- The number of permutations of three 1s is 3!
- The answer should be 5!/3! = 20

Examples

- The number of 4-letter "words" that can be formed by using each of the letters *a, b, c, d, e* at most once equals to:
  - $P(5, 4) = 5!/(5-4)! = 120$.
- What is the number of ways to order the 26 letters of the alphabet so that no two of the vowels *a, e, i, o* and *u* occurs consecutively?
  - 21 Consonants (b,c,d….z) and 5 vowels (a,e,i,o,u)
  - Permutations of the consonants: 21!
  - Total 22 positions for vowels for 5 vowels
  - Permutations for the vowels between consonants: P(22,5)
  - The answer should be 21!*P(22,5)= 21!*22!/17!

$\square$b$\square$ c$\square$ d $\square$ f…… z$\square$

Counting Problems

- Ordered arrangement
  - Without repeating any objects, distinct
  - With repetition of objects permitted (but perhaps limited)
- Unordered arrangements
  - Fruits in basket
  - Without repeating any object,
  - With repetition of objects permitted (but perhaps limited).

Combinations

- An *r*-combination of elements of a set *S* is simply a subset $T \subseteq S$ with *r* members, $|T|=r$.
- The number of *r*-combinations of a set with $n=|S|$ elements is

$$C(n,r) = \binom{n}{r}$$

- The *r*-permutation from S with n elements can be constructed in 2 steps
  - Find the r-combinations of S: C(n,r)
  - Perturb r elements: r!

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

P(19,7) / C(19,7) = (   )

A 7!

B 19!

C 12!

D 26!

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Any question?

- $C(n,0)=?$
- $P(n,0)=?$
- $C(4,5)=?$
- ……

$$C(n,r) = C(n, n-r)$$
$$C(n,0) = C(n,n) = 1$$
$$C(n,r)=0 \text{ (if } r>n)$$

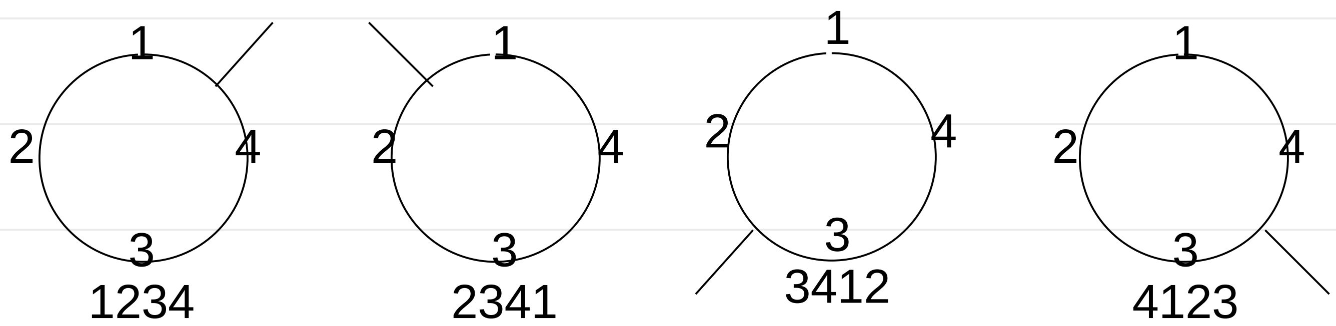
$$0! = 1$$

The origin of permutation and combination

- The problems of permutation and combinations firstly appeared in *I Ching*.
  - "Sixiang" is the permutation of two yao's. "Bagua" is the permutation of 3 yao's.
- Han Dynasty mathematician Xu Yue's *Shu Shu Ji Yi* (2nd century) mentioned a Bagua used for divination.
  - It's similar to the classic problem "8 people are sitting around a table, how many arrangements are there?"

Circular permutations

- The permutations that arrange objects in a line are called *linear permutations*. If the objects are arranged in a cycle, the permutations are called *circular permutations*.
- The number of circular *r*-permutations of a set of *n* elements is given by $\dfrac{p(n,r)}{r} = \dfrac{n!}{r(n-r)!}$.
- In particular, the number of circular permutations of *n* elements is $(n - 1)!$.



Examples

- Ten people, including two who do not wish to sit next to one another, are to be seated at a round table. How many circular seating arrangements are there?
  - Let 10 people to be P_1, P_2, P_3….,P_{10} , where P_1 and P_2 do not wish to be abutted.
  - Considering X, P_3….,P_{10} at the round table: 9!/9=8!
  - Replace X with $P_1 P_2$ or $P_2 P_1$, we can get the arrangement that P_1 and P_2 next to each other.
  - Arrangements without constraint: circular 10-permutations: 10!/10 = 9!
  - The answer should be 9!-2*8! =7*8!

Necklace

- Necklace: Similar to real necklaces. Based on circular permutation. Here two sides of the necklace are considered a same permutation.
- **Eg.** The next two arrangements are actually the same permutation of 3 elements.
- The number of necklace permutation of picking *r* elements from *n* is :$P(n, r)/2r$, $3 \le r \le n$

单选题 1分

设置

1. How many 4-strings could be formed with the 26 English letters?

2. How many **non-repeated** 4-strings could be formed with 26 English letters?

A $26^4, 4^{26}$

B $4^{26}, \ 26^4$

C $26^4, P(26,4)$

D $4^{26}, \ C(26,4)$

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Permutation with repetitions

Eg How many 4-strings could be formed with the 26 English letters?

26^4

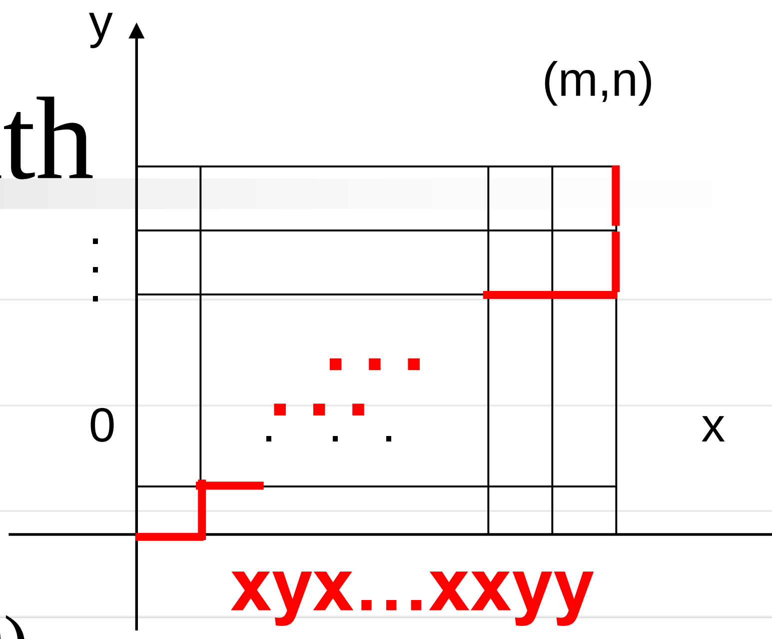
Eg How many non-repeated 4-strings could be formed with 26 English letters？

P(26,4)

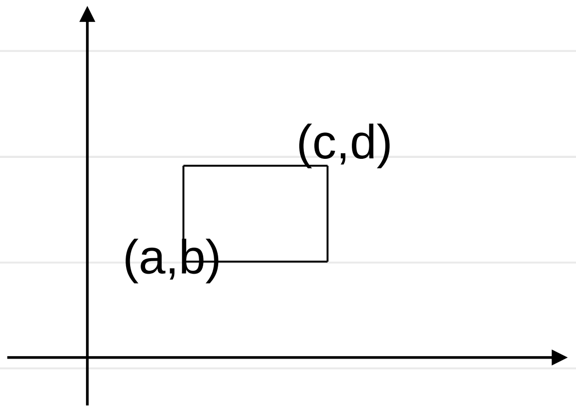
Eg How many non-repeating 4-strings such that *b* and *d* are not adjacent can be formed with 26 English letters?

P(26,4) –C(24,2)*3!*2

Lattice Path



- A path composed of connected horizontal and vertical line segments, each passing between adjacent lattice points。
- How many lattice paths from (0,0) to (m,n) ?
- **One-one correspondence**
  - Each path (0,0)→(m,n)
  - Arrangement with *m ‘x’s and n ‘y’s*
  - C(m+n,m)



$$|(a,b)(c,d)| = \binom{(c-a)+(d-b)}{(c-a)}$$

To do list

- HW
  - Two sheets
  - Due on Monday 9:00
  - Programming tasks on OJ
- Teaching Material
  - Slides
  - Videos

Thank you!