C & A

Chap. II
Revisit of the knowledge

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Review of the previous lesson

- Four basic counting principles
  - Addition
  - Multiplication
  - Substraction
  - Division
- Permutation and combination?
  - If the order **does** matter:
    - **Permutation: P(n,r)**= $n!/(n-r)!$
  - If the order **doesn't** matter:
    - **Combination: C(n,r)**= $n!/r!(n-r)!$.
- Ordered arrangement
  - Without repeating any objects, distinct: **P(n,r)**
  - **Circular permutation**

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}.$$

Grandma has six different candies and she want to give them to 3 kids. One kid gets 1 candy, one kid gets 2 and one gets 3. How many different arrangements could be?

作答

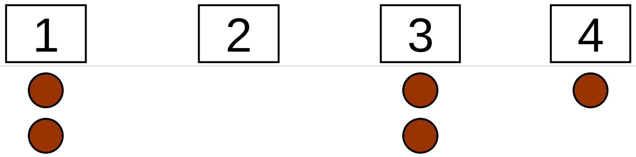
Permutations of multisets

- Permutations of multisets：
- How many permutations of "*pingpang*" are there？
  - 2 p's, 2 n's，2 g's，1 I's，1 a's，
  - The permutations are denoted by $\binom{8}{2\quad 2\quad 2\quad 1\quad 1}$
  - Differentiate it with subscripts $p_1p_2n_1n_2g_1g_2ia$
    - There are 2! arrangements for labes of p,n,g

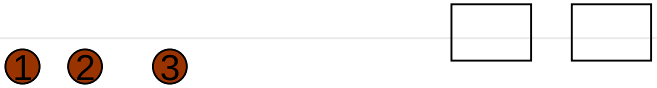
$$\binom{8}{2\quad 2\quad 2\quad 1\quad 1}2!2!2! = 8! \qquad \binom{8}{2\quad 2\quad 2\quad 1\quad 1} = \frac{8!}{2!2!2!}$$

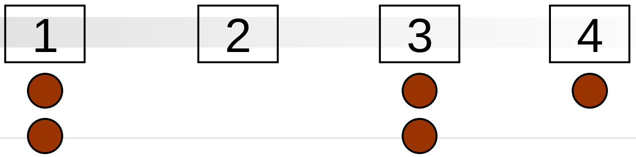
Combinations of Multisets

- Take 5 elements from A={1,2,3,4}, elements are repeatable.
- **Combinations of Multisets:**
- {1 1 3 3 4}

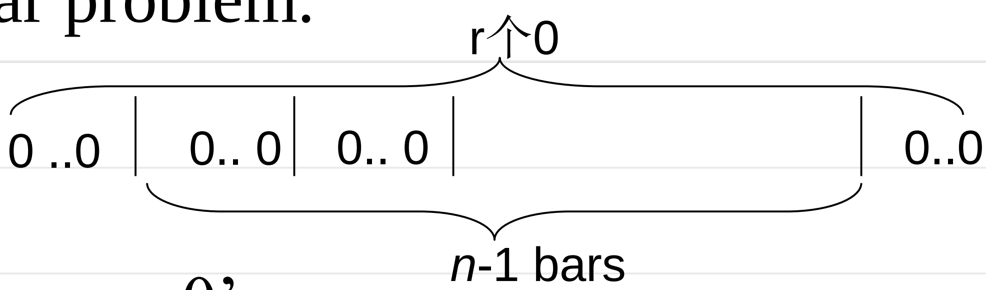


- Model for combinations of multisets: take r **non-labeled** balls, n **different** boxes. In each box there could be **0 or more than 1 balls.**
- Model for combinations without repetitions: n balls are **different**, r boxes are **the same**, put r balls into boxes, **1 ball** per box.





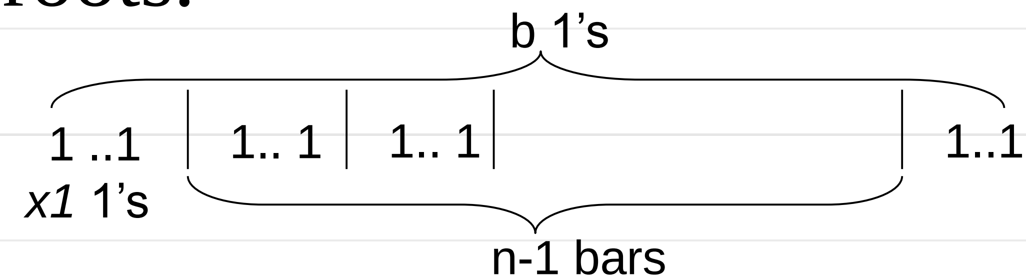
- **Picking *r* elements from *n* different ones, if repeatable, the number of combinations is C(*n*+*r*-1,*r*)**
- **Proof：** Convert it to the bar problem.



- Total: *n*+*r*-1 elements, *n*-1 bars, *r* 0's
- If both bars and 0's are indexed, it's a (n+r-1)-permutation.
- So the result is $\frac{(n+r-1)!}{r!(n-1)!} = C(n+r-1, r)$

Integer roots of linear equations

- $x_1+x_2\ldots+x_n=b$ has C($n+b$-1,b) non-negative integer roots.



- Number of roots: C($n+b$-1,b)

Example

- If S = {3 · a, 2 · b, 4 · c}
  - 8-permutations of S?
  - 8-combinations of S?
- 8-combinations
  - {2 · a, 2 · b, 4 · c}
  - {3 · a, 1 · b, 4 · c}
  - {3 · a, 2 · b, 3 · c}
  - Total: 3

$$x_1 + x_2 + x_3 + x_4 = 20$$

$$x_1 \geq 3, x_2 \geq 1, x_3 \geq 0 \; and \quad x_4 \geq 5$$

$$y_1 = x_1 - 3, y_2 = x_2 - 1, y_3 = x_3 \; and \quad y_4 = x_4 - 5$$

$$y_1 + y_2 + y_3 + y_4 = 20 - 3 - 1 - 5 = 11$$

$$x_1 + x_2 + x_3 + x_4 = 20$$

$$x_1 \leq 5, x_2 \leq 10, x_3 \leq 11, and x_4 \leq 8$$

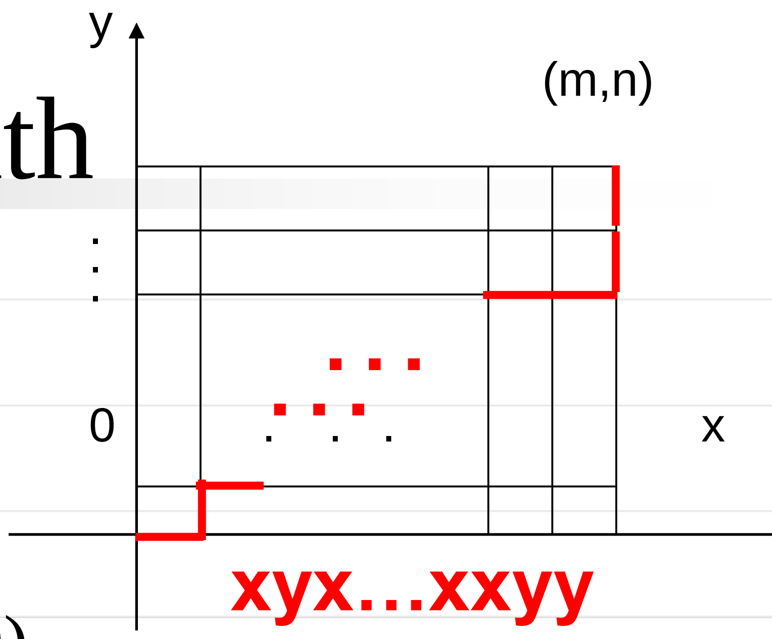
Complicated!

Inclusion-exclusion principle

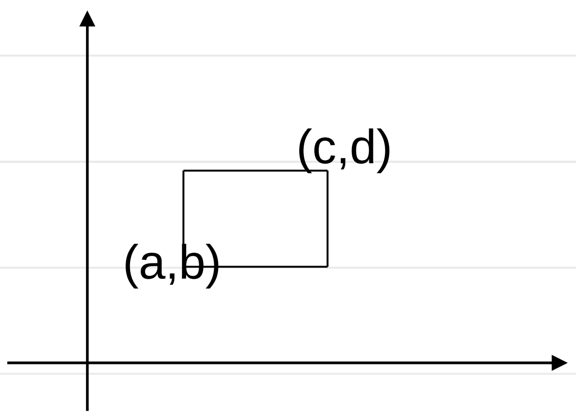
Summary

Sample	Order counts?	Repetition allowed?	Name	Number of ways
Choose 3 balls and put them in a box	No	No	r-combination	C(m,r)
People in a line	Yes	No	r-permutation	P(m,r)
Arrangement of fruits	No	Yes	r-combination of multi-sets	C(m+r-1,r)
4-letter word	Yes	Yes	r-permutation of multi-sets	m^r

Lattice Path

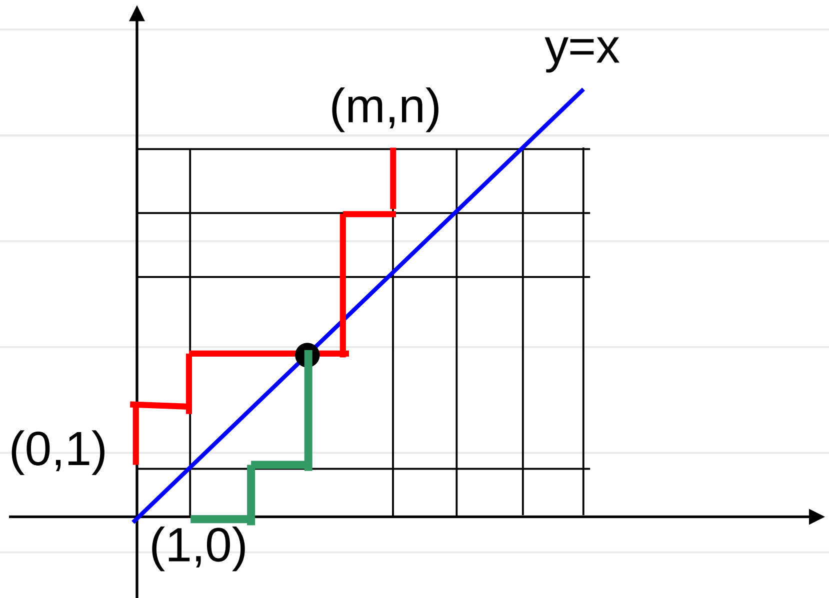


- A path composed of connected horizontal and vertical line segments, each passing between adjacent lattice points。
- How many lattice paths from (0,0) to (m,n)？
- **One-one correspondence**
  - Each path (0,0)→(m,n)
  - Arrangement with m *'x's and n 'y's*
  - C(m+n,m)



$$|(a,b)(c,d)| = \binom{(c-a)+(d-b)}{(c-a)}$$

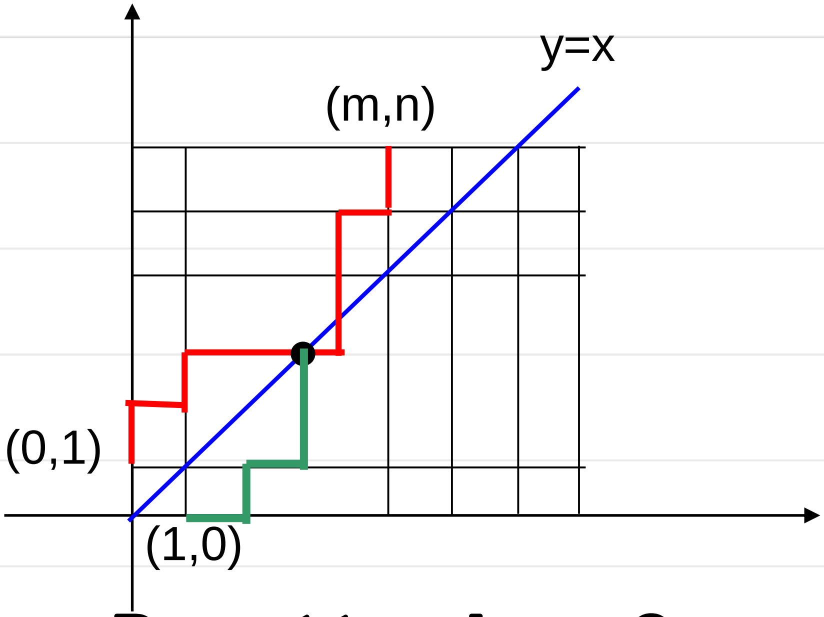
8 people is lining up to purchase tickets, 4 people are holding 10 Yuan, 4 people are holding 20 Yuan, the ticket price is 10 Yuan. The ticket booth does not have money at the first place, find out how many different possible ways of the arrangement of 8 people that they can successful purchase the tickets.



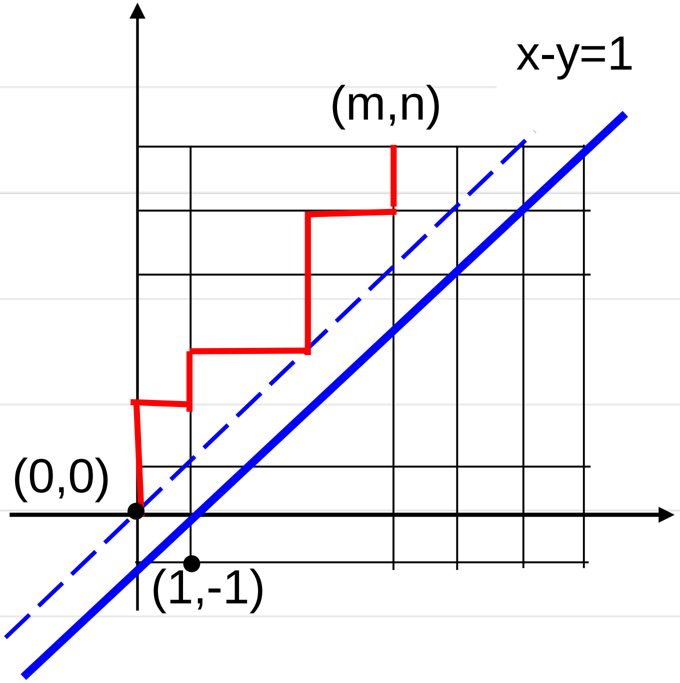
Let n=4, assume of using 2n dimensional 0,1 vector to represent a type of queueing condition, make this vector as , represents the ith of customer is holding 10 Yuan, represents the ith of customer is holding 20 Yuan.

If so, there are vector of n number of ‘0’ element, n number of ‘1’ element, total C(2n,n).

Each vector may have a one-to-one correspondence from (0,0) point to (n,n) point, which is departed from (0,0) point， means move an unit along x axis, means move an unit along y axis.To ensure the customer may purchase a ticket successfully, it may not appear the condition of unable to change 10 Yuan, the path must satisfy condition.



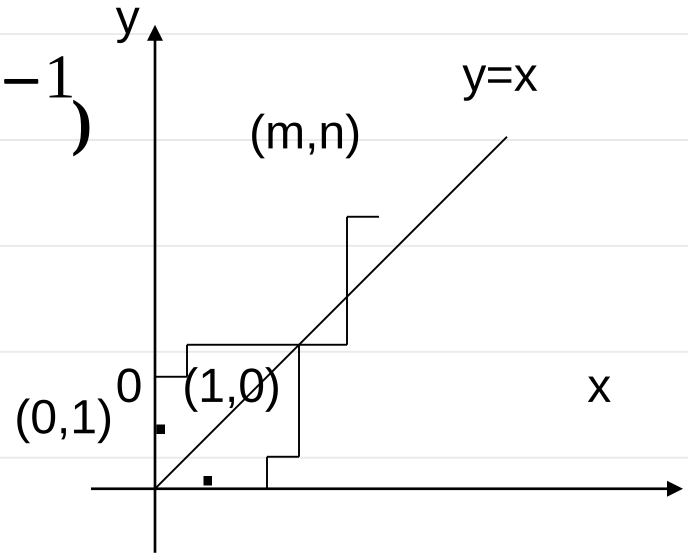
Do not touch x=y?



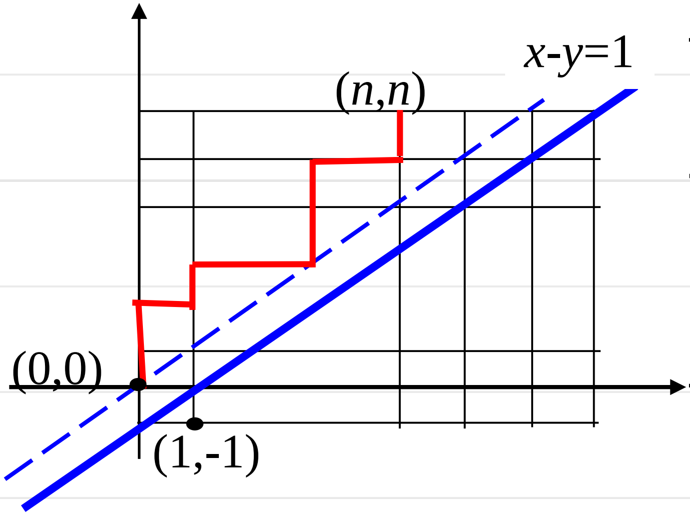
$$\binom{m+n-1}{m}-\binom{m+n-1}{m-1}=\frac{(m+n-1)!}{m!(n-1)!}-\frac{(m+n-1)!}{(m-1)!n!}$$

$$=\frac{(m+n-1)!}{(m-1)!(n-1)!}\left(\frac{1}{m}-\frac{1}{n}\right)=\frac{(m+n-1)!}{m(m-1)!(n-1)!}\left(1-\frac{m}{n}\right)$$

$$=\frac{n-m}{n}\binom{m+n-1}{m}=\left(1-\frac{m}{n}\right)\binom{m+n-1}{m}$$



Lattice Path



Limit the line to either move one grid down or one grid to the right. $x-y=1$;

The problem turns into a lattice path going from (0,0) to (n,n) without touching the x-y=1 line.

The symmetry point of (0,0) about $x-y=1$ is (1,-1).

Using the method used in the last question, the number of lattice paths equal to:

$$C_n = C(2n,n) - C(2n,n-1)$$

$$= \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k}$$

the path that we are looking for is the point which is reachable by y=x, but does not go through y=x or the point which is not touching y=x. According to the last question, the path number that we are looking is from（0,0） point until（n,n+1）is:

$$C(2n,n)-C(2n,n-1)=C(8,4)-C(8,3)=14$$

And, the person which is lining up to purchase ticket are of different person, so the queueing solution number is 14x4!x4!=8064

Inclusion-Exclusion principle

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j|$$

$$+ \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| - \ldots$$

$$+ (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$|\overline{A}| = N - |A|,$$

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}| = N - |A_1 \cup A_2 \cup \ldots \cup A_{n-1} \cup A_n|$$

$$= N - \sum_{i=1}^{n} |A_i| + \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j| \quad - \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| + \ldots$$

$$+ (-1)^{n} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Special case for IEP

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}\right| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k|$$
$$+ \cdots + (-1)^m \sum |A_1 \cap A_2 \cap \cdots \cap A_m|$$

Assume that the size of the set $A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}$ that occurs in the IEP depends only on k and not on which k sets are used in the intersection. Then

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}\right| = a_0 - \binom{m}{1} a_1 + \binom{m}{2} a_2 - \binom{m}{3} a_3$$
$$+ \cdots + (-1)^k \binom{m}{k} a_k + \cdots + -1)^m a_m$$

where $a_k = |A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}|$. This is because the *k*th summation that occurs in the IEP contains C(*m*, *k*) summands each equal to a_k.

主观题 0.5分

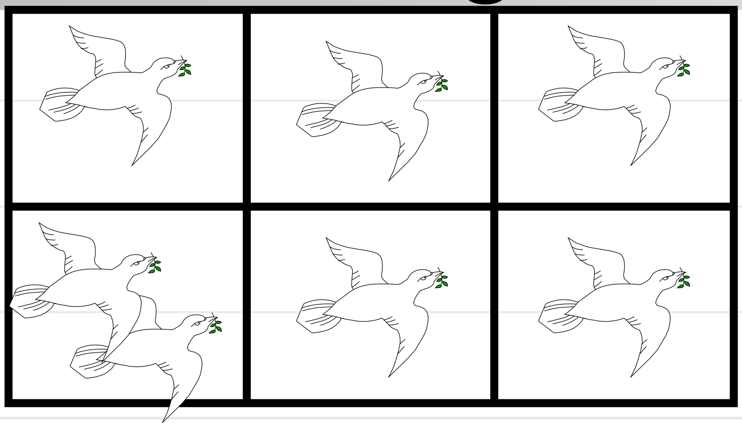
设置

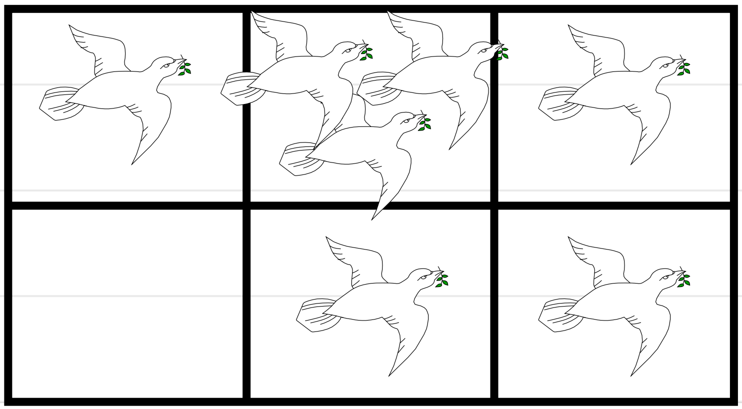
How many positive integers between [200,2000] are relative prime with 105?

正常使用主观题需2.0以上版本雨课堂

作答

Pigeonhole Principle





- If *n*+1 objects are put into *n* boxes, then at least one box contains two or more of the objects.
  - Only guarantee the **existence**
  - No help in finding a box that contains two or more of the objects
  - Keys: what are pigeons and what are holes?

Examples

Example: Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Hint. Consider the m sums

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \ldots,\ a_1+a_2+a_3+\ldots+a_m.$$

If any of these sums is divisible by m, then the conclusion holds.

Thus suppose that each of the sums has a non-zero remainder when divided by m, and so a remainder equal to one of 1, 2, … m-1.

Since there are m sums and only m-1 remainders , two of the sums have the same remainder when divided by m.

$a_1+a_2+a_3+\ldots+a_k = bm+r$ $\qquad$ $a_1+a_2+a_3+\ldots+a_l = cm+r \quad (k<l)$

Subtracting: $a_{k+1}+a_{k+2}+a_{k+3}+\ldots+a_l=(c-b)m;$

Thus, $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Examples

Example: Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Let m=7, and let our integers be 2, 4, 6, 3, 5, 5 and 6.

Compute the sums of

	remainders when divided by 7 are
a1=2,	2
a1+a2=6,	6
a1+a2+a3=12,	5
a1+a2+a3+a4=15,	1
a1+a2+a3+a4+a5=20,	6
a1+a2+a3+a4+a5+a6=25,	4
a1+a2+a3+a4+a5+a6+a7=31,	3

a1+a2=6,

a1+a2+a3+a4+a5=20, $\Rightarrow$ a3+a4+a5= 6+3+5=14

Divisible by 7!

Examples

- Example: **Hand shaking problem**: If there are n number of people who can shake hands with one another (where $n > 1$), the pigeonhole principle shows that there is always a pair of people who will shake hands with the same number of people.
- Hint: As the 'holes', or m, correspond to number of hands shaken, and each person can shake hands with anybody from 0 to $n - 1$ other people
- $n - 1$ possible holes.
  - either the '0' or the '$n - 1$' hole must be empty
  - if one person shakes hands with everybody, it's not possible to have another person who shakes hands with nobody;
  - if one person shakes hands with no one there cannot be a person who shakes hands with everybody.
- This leaves n people to be placed in at most $n - 1$ non-empty holes, guaranteeing duplication.

Examples

- Example Chinese Remainder Theorem
- *Hanxin Dianbing* (韩信点兵):
  - a military general who served Liu Bang
  - *Han Xin count his troops*
    - *3 soldiers in a line……2 left at the end*
    - *5 soldiers in a line……3 left at the end*
    - *7 soldiers in a line……2 left at the end*
    - *The officer told Han Xin there are total 2395 soldiers?*
    - *Han Xin said "No, you are wrong, there should be 2333 soldiers."*
- A third-century AD book *Sun Zi Suanjing* (孙子算经 The Mathematical Classic by Sun Zi)
  - 今有物，不知其数，三三数之，剩二，五五数之，剩三，七七数之，剩二，问物几何
  - We want to count the number of a pile of things, we only know that the remainder divided by 3 is 2, and the remainder divided by 5 is 3, the remainder divided by 7 is 2, what the number would be?

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

- Find the smallest integer to satisfy this:
- List numbers such that $x \div 3 \equiv 2$：
  - 2， 5， 8， 11， 14， 17， 20， 23， 26…
- List numbers such that $x \div 5 \equiv 3$：
  - 3， 8， 13， 18， 23， 28 …
- The first common number is 8. The least common multiple of 3 and 5 is 15.
  - Combine those 2 requirements, we need $8 + 15 \times integer$:
  - 8， 23， 38， …，

Chinese Remainder Theorem

- Find out the smallest non-negative integers solutions for the following equations: $\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$
- Construct the numbers
  - *s* is the smallest number which can be divided by 5 and 7 but the remainder divided by 3 is 1. s = 70
  - s*2=140 will be divisible by both 5 and 7 but the remainder divided by 3 is 2.
  - t is the smallest number which can be divided by 3 and 7 but the remainder divided by 5 is 1. t = 21
  - t*3=63 will be divisible by both 3 and 7 but the remainder divided by 5 is 3.
  - h is the smallest number which can be divided by 3 and 5 but the remainder divided by 7 is 1. t = 15
  - h*2=30 will be divisible by both 3 and 5 but the remainder divided by 7 is 2.
  - 2s + 3t + 2h = 233 should satisfy the equation array
  - To find the smallest one, 105 is the least common multiple of 3,5 and 7.
    - 233-105=128>105, 128-105=23.
  - The answer is 23

Chinese Remainder Theorem

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

- Chinese Remainder Theorem: m and n are relatively prime, for any non-negative integer a and b($a<m$, $b<n$), there must be positive integer x which makes the equations solvable.

$$\begin{cases} x = pm + a \\ x = qn + b \end{cases}$$

p,q are non-negative integers

- Proof: Consider n integers：a, $m+a$, $2m+a$, …..$(n\text{-}1)m+a$

  There's no command remainders for the n numbers divided by n. [0,1,2…,n-1], a total of n ones.

So for b($b<n$), there must exist a number in the sequence which satisfies $x=qn+b$

Chinese Remainder Theorem

- （**Chinese Remainder Theorem, RT**）Assume $m_1, m_2, ..., m_k$ are relative prime, so $gcd(m_i, m_j) = 1$, $i \neq j$, $i, j = 1, 2, ..., k$, and the congruence equations：

  $x \equiv b_1 \bmod m_1$

  $x \equiv b_2 \bmod m_2$

  ...

  $x \equiv b_k \bmod m_k$

Mod $[m_1, m_2, ..., m_k]$ has solutions, this means with $[m_1, m_2, ..., m_k]$ there exists x which satisfies $x \equiv b_i \bmod [m_1, m_2, ..., m_k]$, $i = 1, 2, ..., k$

Pigeonhole Principle: Strong Form

Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 object, ….., or the nth box contains at least q_n objects.

- Suppose that we distribute $q_1 + q_2 + \ldots + q_n - n + 1$ objects among n boxes.
- If for each $i = 1, 2, \ldots, n$ the ith box contains fewer than q_i objects
  – The total number of objects in all boxes does not exceed $(q_1 - 1) + (q_2 - 1) + \ldots + (q_n - 1) = q_1 + q_2 + \ldots + q_n - n$.
- Since this number is one less than the number of objects distributed, we conclude that for some $i = 1, 2, \ldots, n$, the ith box contains at least q_i objects.

Application Examples

- A bag contains 100 apples, 100 bananas, 100 oranges and 100 pears. How many fruits should be taken out such that we can be sure a dozen pieces of them are of the same kind?
  - Let $q_1 = q_2 = \ldots = q_n = r$. The principle reads as follows: If $n(r\text{-}1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more the objects.
  - 4 boxes, $q_1 = q_2 = \ldots = q_n = 12$
  - If 4*(12-1)+1 = 45 fruits are taken out, then at least one of the boxes contains 12 fruits.

To Do List

- OJ tasks
- Pre-class videos and quizzes
  - Generating Function with recurrence relations

Thank you!