Combinatorics IEP HW

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1. how many integer numbers from 1 to 10000 are not squares of integers or cubes of integers?

Let A be the number of squares in the given range, while B is the number of cubes between 1 to 10000. Hence, in order to find the number of integers that are not squares nor cubes, we should find

$$|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = S - |A| - |B| + |A \cap B|$$

Where S is the total amount of numbers = 10000. Moreover, since 1^2, 2^2, 3^2, ..., 100^2 are all squares in the given range, $|A| = 100$. In addition, the largest square less than or equal to 10000 would be round($\sqrt[3]{10000}$) = 21. Hence, the squares in the given range are 1^3, 2^3, 3^3, ..., 21^3, giving $|B| = 21$. Lastly, the number of integers that are both cubes and squares (power of 6) can be calculated as round($\sqrt[4]{10000}$) = 4.

$$|\overline{A} \cap \overline{B}| = 10000 - 100 - 21 + 4 = 9883$$

Therefore, there are 9883 integer numbers from 1 to 10000 that are not squares nor cubes of integers.

2. How many permutations of 1, 2, 3，……， 9 have at least one odd number in its natural position?

Let A_1, A_2, A_3, A_4, and A_5 denote the set of permutations that numbers 1, 3, 5, 7, and 9 are in their natural position respectively. Then, since the total number of permutations is 9!, the number of permutations in which at least one of these five odd numbers is in its natural position would be $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$, where $|A_1| = |A_2| = |A_3| = |A_4| = |A_5| = 8!$ since one location is filled with the odd number. Similarly, the total cases could be presented by 1, 2, 3, 4, and all 5 odd numbers being in their natural positions, giving

$$\binom{5}{1} 8! + \binom{5}{2} 7! + \binom{5}{3} 6! + \binom{5}{4} 5! + \binom{5}{5} 4! = 157824$$

The number of permutations decreases by one each time an additional odd number is fixed to its natural position, resulting in the number of permutations to go from 8! to 4!.Hence, by calculating the above statement, we would have

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = 157824$$

Therefore, there are <u>157824</u> permutations of 1, 2, 3, ..., 9 in which at least one odd number is in its natural position.

3 $x_1 + x_2 + x_3 + x_4 = 20$, *where* $1 \le x_1 \le 6, 0 \le x_2 \le 7, 4 \le x_3 \le 8, 2 \le x_4 \le 6$. please calculate the number of integral solutions.

Firstly, in order to have the same lower boundary for all variables, x_1 to x_4 are replaced with y_1 to y_4 as follows

$$y_1 = x_1 - 1, y_2 = x_2, y_3 = x_3 - 4, \text{and } y_4 = x_4 - 2$$

giving

$$0 \le y_1 \le 5, 0 \le y_2 \le 7, 0 \le y_3 \le 4, 0 \le y_4 \le 4$$

$$y_1 + y_2 + y_3 + y_4 = 20 - 7 = 13$$

Now we can calculate all the non-negative solutions of this equation. Firstly, without any of the upper bounds $y_1 + y_2 + y_3 + y_4 = 13$ would have c(13+4-1, 13) =

$c(16, 13)$ non-negative solutions. Assuming that A_1, A_2, A_3, and A_4 are the solutions when $y_1 \geq 6, y_2 \geq 8, y_3 \geq 5$ *and* $y_4 \geq 5$ respectively, gives

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + y_2 + y_3 + y_4 = 13$$

$$z_1 + y_2 + y_3 + y_4 = 7$$

$$|A_1| = c(7 + 4 - 1, 3) = c(10, 3)$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + z_2 + 8 + y_3 + y_4 = 13$$

$$y_1 + z_2 + y_3 + y_4 = 5$$

$$|A_2| = c(5 + 4 - 1, 3) = c(8, 3)$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + y_2 + z_3 + 5 + y_4 = 13$$

$$y_1 + y_2 + z_3 + y_4 = 8$$

$$|A_3| = c(8 + 4 - 1, 3) = c(11, 3)$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + y_2 + y_3 + z_4 + 5 = 13$$

$$z_1 + y_2 + y_3 + y_4 = 8$$

$$|A_4| = c(8 + 4 - 1, 3) = c(11, 3)$$

Accordingly

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + z_2 + 8 + y_3 + y_4 = 13$$

$$z_1 + z_2 + y_3 + y_4 = -1$$

$$|A_1 \cap A_2| = 0$$

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + y_2 + z_3 + 5 + y_4 = 13$$

$$z_1 + y_2 + z_3 + y_4 = 2$$

$$|A_1 \cap A_3| = c(2 + 4 - 1, 3) = c(5, 3)$$

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + y_2 + y_3 + z_4 + 5 = 13$$

$$z_1 + y_2 + y_3 + z_4 = 2$$

$$|A_1 \cap A_4| = c(2 + 4 - 1, 3) = c(5, 3)$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + z_2 + 8 + z_3 + 5 + y_4 = 13$$

$$y_1 + z_2 + y_3 + z_4 = 0$$

$$|A_2 \cap A_3| = c(4 - 1, 3) = c(3, 3)$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + z_2 + 8 + y_3 + z_4 + 5 = 13$$

$$y_1 + z_2 + z_3 + y_4 = 0$$

$$|A_2 \cap A_4| = c(4 - 1, 3) = c(3, 3)$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + y_2 + z_3 + 5 + z_4 + 5 = 13$$

$$y_1 + z_2 + z_3 + y_4 = 3$$

$$|A_3 \cap A_4| = c(3 + 4 - 1, 3) = c(6, 3)$$

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + z_2 + 8 + z_3 + 5 + y_4 = 13$$

$$z_1 + z_2 + z_3 + y_4 = -6$$

$$|A_1 \cap A_2 \cap A_3| = 0$$

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + z_2 + 8 + y_3 + z_4 + 5 = 13$$

$$z_1 + z_2 + y_3 + z_4 = -6$$

$$|A_1 \cap A_2 \cap A_4| = 0$$

$$y_1 + y_2 + y_3 + y_4 = y_1 + z_2 + 8 + z_3 + 5 + z_4 + 5 = 13$$

$$y_1 + z_2 + z_3 + z_4 = -5$$

$$|A_2 \cap A_3 \cap A_4| = 0$$

$$y_1 + y_2 + y_3 + y_4 = z_1 + 6 + z_2 + 8 + z_3 + 5 + z_4 + 5 = 13$$

$$z_1 + z_2 + z_3 + z_4 = -11$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$$

Therefore,

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| =$$

$$c(16,13) - c(10,3) - c(8,3) - c(11,3) - c(11,3) +$$

$$c(5,3) + c(5,3) + c(3,3) + c(3,3) + c(6,3) =$$

$$560 - 120 - 56 - 165 - 165 + 10 + 10 + 1 + 1 + 20 = 96$$

Hence there are <u>96 </u>integral solutions to this equation.