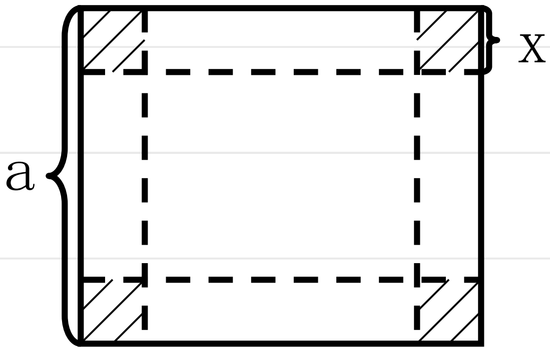
Linear Programming

Yuchun Ma

myc@tsinghua.edu.cn

Mathematical programming

- Mathematical programming is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.
- Given a paper in square shape, how to construct a box without the top lid such that the volume can be maximized?



$$\boldsymbol{v} = (\boldsymbol{a} - 2\boldsymbol{x})^2 \cdot \boldsymbol{x}$$

$$\frac{d\boldsymbol{v}}{d\boldsymbol{x}} = 0$$

$$2(a-2x)\cdot x \cdot (-2) + (a-2x)^2 = 0$$

$$\boldsymbol{x} = \frac{\boldsymbol{a}}{6}$$

Programming

- Convex programming
  - Linear programming
  - Second order cone programming
  - Semidefinite programming
  - ……
- Quadratic programming
- Integer programming
- Nonlinear programming
- Stochastic programming

Introduction

- A Diet Problem

eg: Polly wonders how much money she must spend on food in order to get all the energy (2,000 kcal), protein (50 g), and calcium (800 mg) that she needs every day. She choose six foods that seem to be cheap sources of the nutrients:

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (c)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole Milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

Introduction

- Servings-per-day limits on all six foods:

Oatmeal	at most 4 servings per day
Chicken	at most 3 servings per day
Eggs	at most 2 servings per day
Milk	at most 8 servings per day
Cherry pie	at most 2 servings per day
Pork with beans	at most 2 servings per day

- Now there are so many combinations seem promising that one could go on and on, looking for the best one. Trial and error is not particularly helpful here.

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (c)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole Milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

- A new way to express this—using inequalities:

minimize $$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

subject to
$$\begin{aligned}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\geq 2{,}000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\geq 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\geq 800
\end{aligned}$$

the energy (2,000 kcal), protein (50 g), and calcium (800 mg) that she needs every day

$$\begin{aligned}
0 \leq x_1 &\leq 4 \\
0 \leq x_2 &\leq 3 \\
0 \leq x_3 &\leq 2 \\
0 \leq x_4 &\leq 8 \\
0 \leq x_5 &\leq 2 \\
0 \leq x_6 &\leq 2
\end{aligned}$$

Oatmeal	at most 4 servings per day
Chicken	at most 3 servings per day
Eggs	at most 2 servings per day
Milk	at most 8 servings per day
Cherry pie	at most 2 servings per day
Pork with beans	at most 2 servings per day

Introduction

- Problems of this kind are called "linear programming problems" or "LP problems" for short; linear programming is the branch of applied mathematics concerned with these problems.
- A *linear programming problem* is the problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints.
- Standard form:

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i = 1, 2, \ldots, m)$$

$$x_j \ge 0 \quad (j = 1, 2, \ldots, n)$$

Linear Program

Linear programming (**LP**) is a mathematical method for determining a way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model for some list of requirements represented as linear equations.

Objective: $$\max(\min) Z = c_1 x_1 + c_2 x_2 + \cdots\cdots + c_n x_n \quad ①$$

Constraints:
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n \le (= \cdot \ge)\ b_1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad ② \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n \le (= \cdot \ge) b_m \\ x_1 \ge 0 \cdots\cdots x_n \ge 0 \qquad ③ \end{cases}$$

Different Forms

Objective: $\max(\min)\ Z = \sum_{j=1}^{n} c_j x_j$

Constraints: $\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le (= \cdot \ge)\, b_i & (i = 1 \cdot 2 \cdots m) \\ x_j \ge 0 & (j = 1 \cdot 2 \cdots n) \end{cases}$

$$C = (\, c_1 \quad c_2 \quad \cdots \quad c_n \,)$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad p_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad \Longrightarrow \quad \begin{aligned} &\max(\min)\, Z = CX \\ &\begin{cases} \sum p_j x_j \le (= \cdot \ge)\, b \\ X \ge 0 \end{cases} \end{aligned}$$

Different Forms

Coefficients matrix:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\max(\min)\ Z = CX$$

$$\begin{cases} AX \le (= \cdot \ge)\, b \\ X \ge 0 \end{cases}$$

History

The problem of solving a system of linear inequalities dates back at least as far as Fourier, after whom the method of Fourier-Motzkin elimination is named.

Linear programming arose as a mathematical model developed during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. It was kept secret until 1947.

Air Force initiated project SCOOP (Scientific Computing of Optimum Programs)

SCOOP began in June 1947 and at the end of the same summer, Dantzig and associates had developed:

1) An initial mathematical model of the <u>general</u> linear programming problem.
2) A general method of solution called the <u>simplex method.</u>

It soon became clear that a surprisingly wide range of apparently unrelated problems in production management could be stated in linear programming terms and solved by the simplex method.

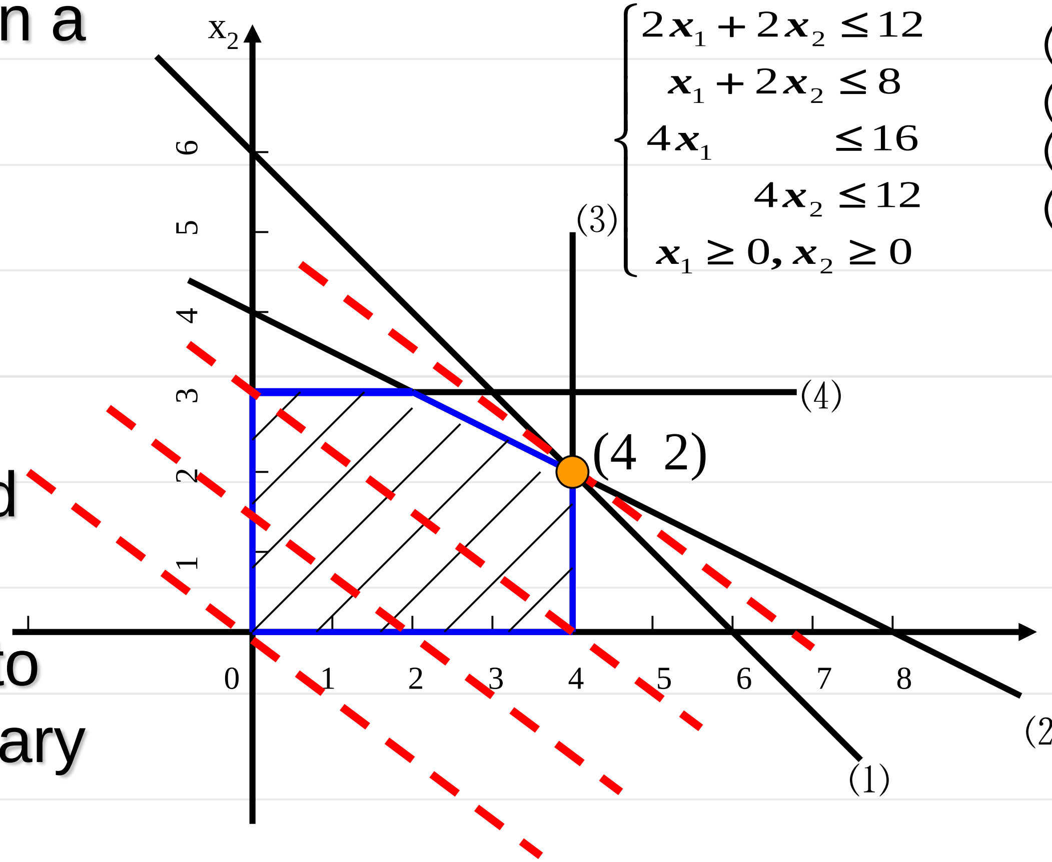
Later, it was used to solve problems of management. Its algorithm can also used to network flow problems.

Graphical Solution

1. Plot model constraint on a set of coordinates in a plane
2. Identify the feasible solution space on the graph where all constraints are satisfied simultaneously
3. Plot objective function to find the point on boundary of this space that maximizes (or minimizes) value of objective function

$$\max Z = 2x_1 + 3x_2$$

$$\begin{cases} 2x_1 + 2x_2 \le 12 & (1) \\ x_1 + 2x_2 \le 8 & (2) \\ 4x_1 \le 16 & (3) \\ 4x_2 \le 12 & (4) \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

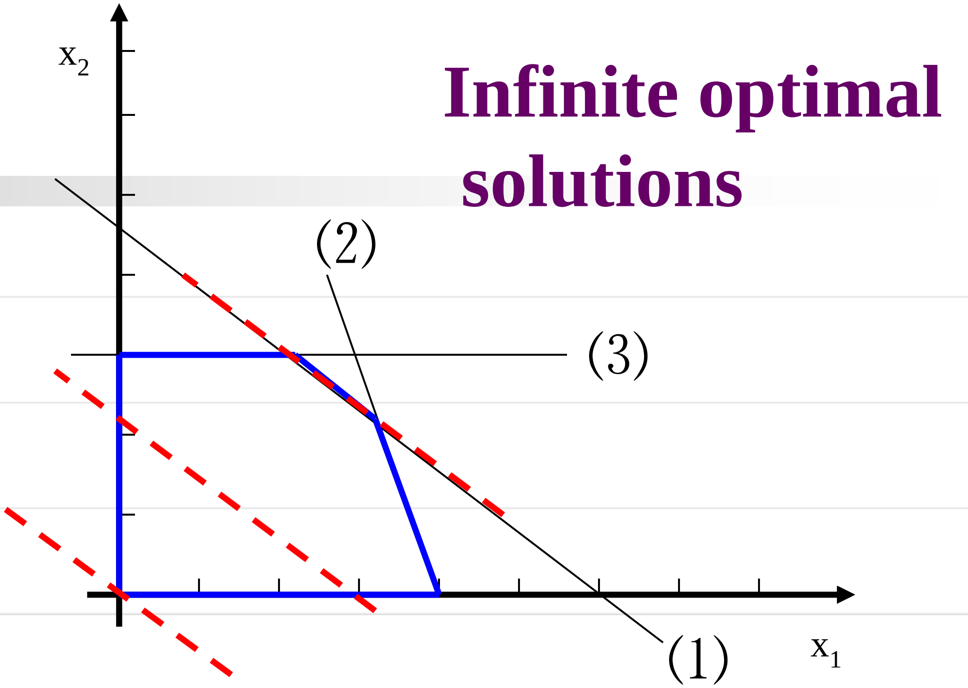


Maximal results Z=14, while $x_1 = 4$ $x_2 = 2$

Example 2

$$\max Z = x_1 + 2x_2$$

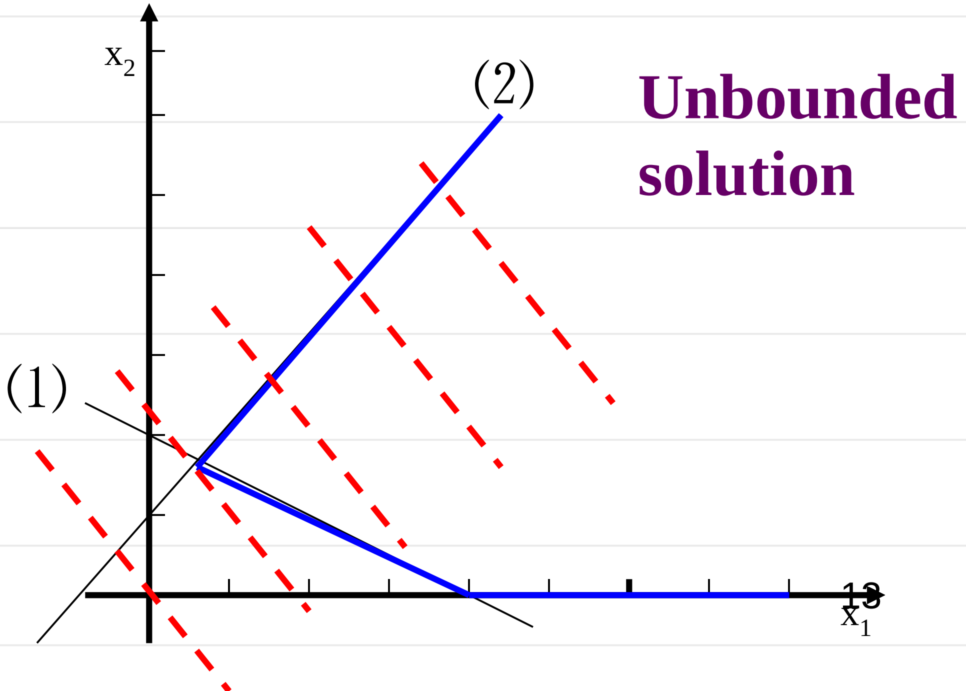
$$\begin{cases} x_1 + 2x_2 \le 6 \\ 3x_1 + 2x_2 \le 12 \\ x_2 \le 2 \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$



Example 3

$$\max Z = x_1 + x_2$$

$$\begin{cases} x_1 + 2x_2 \ge 2 \\ x_1 - x_2 \ge -1 \\ x_1, x_2 \ge 0 \end{cases}$$

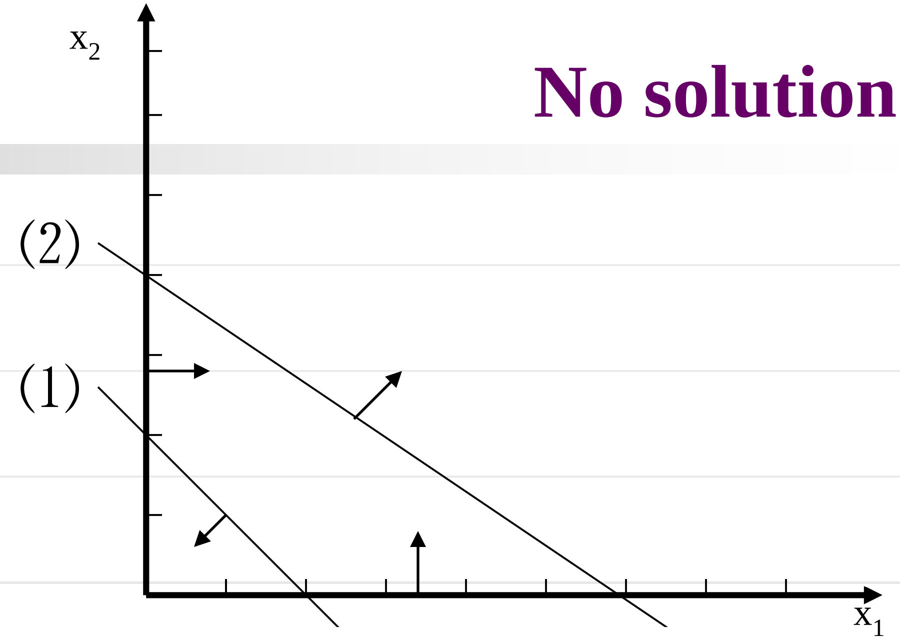


Example 4

No solution

$$\min Z = 3x_1 - 2x$$

$$\begin{cases} x_1 + \ x_2 \le \ 1 \\ 2x_1 + 3x_2 \ge 6 \\ x_1, x_2 \ge 0 \end{cases}$$

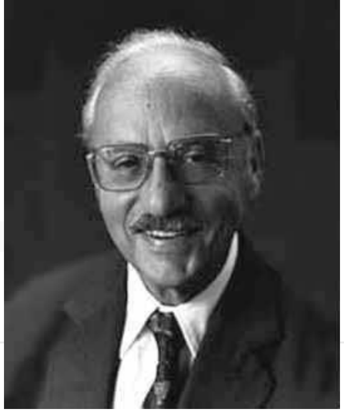


Solutions:

- **no feasible solution;**
- **Unbounded solution**
- **One optimal solution at the extreme point;**
- **Infinite optimal solutions**
- **More than one optimal solutions at the extreme points**

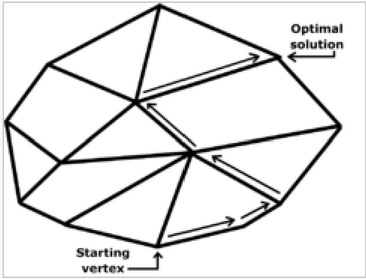
- The graphical method of solution may be extended to a case in which there are three variables. In this case, each constraint is represented by a plane in three dimensions, and the feasible region bounded by these planes is a polyhedron.
- A finite number of extreme points implies a finite number of solutions!
- Hence, search is reduced to a finite set of points
- However, a finite set can still be too large for practical purposes
- Simplex method provides an efficient systematic search guaranteed to converge in a finite number of steps.

Simplex method



- **George Bernard Dantzig**
  - November 8, 1914 – May 13, 2005
  - With the outbreak of World War II, George took a leave of absence from the doctoral program at Berkeley to join the U.S. Air Force Office of Statistical Control. In 1946, he returned to Berkeley to complete the requirements of his program and received his Ph.D. that year.
  - Dantzig is "generally regarded as one of the three founders of linear programming, along with John von Neumann and Leonid Kantorovich "
  - Movie *Good Will Hunting* :An event in Dantzig's life became the origin of a famous story in 1939 while he was a graduate student at UC Berkeley.
- The journal *Computing in Science and Engineering* listed it as one of the top 10 algorithms of the twentieth century

Simplex Method



1. Begin the search at an extreme point (i.e., a basic feasible solution).
2. Determine if the movement to an adjacent extreme can improve on the optimization of the objective function. If not, the current solution is optimal. If, however, improvement is possible, then proceed to the next step.
3. Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.
4. Continue steps 2 and 3 until the optimal solution is found or it can be shown that the problem is either unbounded or infeasible.

Augmented form

$$\max\ (\min)\ \ Z = \sum_{j=1}^{n} c_j x_j$$

General form

$$\sum_{j=1}^{n} a_{ij} x_j \le (= \ \ \ge) b_i \qquad (i = 1\ \ 2 \cdots m)$$

$$x_j \ge 0 \qquad (j = 1\ \ 2 \cdots l)$$

Transformation：

- Objective
- Variables
- Constraints
- Constant term

n **variables， n≥*l***

Augmented form

$$\max\ \ Z = \sum c_j x_j$$

***b_i* non-negative**

$$\begin{cases} \sum a_{ij} x_j = b_i & (i = 1 \cdot 2 \cdots m) \\ x_j \ge 0 & (j = 1 \cdot 2 \cdots n) \end{cases}$$

$$\max(\min)\ Z = \sum c_j x_j$$
$$\begin{cases} \sum a_{ij} x_j \le (\ge =) b_i & (i = 1 \cdot 2 \cdots m) \\ x_j \ge 0 & (j = 1 \cdot 2 \cdots l) \end{cases}$$

$$\max\ Z = \sum c_j x_j$$
$$\begin{cases} \sum a_{ij} x_j = b_i & (i = 1 \cdot 2 \cdots m) \\ x_j \ge 0 & (j = 1 \cdot 2 \cdots n) \end{cases}$$

- Transformation:
  - Objective: min/max $\rightarrow$ max

$$\min Z = \sum c_j x_j \xrightarrow{\text{Multiply -1}} \max Z' = -Z = -\sum c_j x_j$$

  - Variables: all the variables are non-negative

x_k has no constraint then let $x_k = x_k' - x_k''$, and x_k', x_k'' are non-negative

  - Constant term: non-negative

turn b_n to $-b_n$ by multiplying (-1) on both sides

  - Constraints : replace non-equalities with equalities

non-negative slack variables

$$\sum a_{ij} x_j \le b_i \Rightarrow \sum a_{ij} x_j + x_{n+i} = b_i \qquad x_{n+i} \ge 0$$

$$\sum a_{ij} x_j \ge b_i \Rightarrow \sum a_{ij} x_j - x_{n+i} = b_i \qquad x_{n+i} \ge 0$$

Augmented form

Turn the following linear program into augmented form

$$\min Z = -2x_1 + x_2 + 3x_3$$

$$\begin{cases} 5x_1 + x_2 + x_3 \leq 7 \\ x_1 - x_2 - 4x_3 \geq 2 \\ -3x_1 + x_2 + 2x_3 = -5 \\ x_1, x_2 \geq 0, x_3 \end{cases}$$

$$\sum a_{ij} x_j \leq b_i \Rightarrow \sum a_{ij} x_j + x_{n+i} = b_i$$

$$\sum a_{ij} x_j \geq b_i \Rightarrow \sum a_{ij} x_j - x_{n+i} = b_i$$

let $x_3=x_4-x_5$, and x_4, x_5 are non-negative

$$\max \quad Z = 2x_1 - x_2 - 3(x_4 - x_5) + 0x_6 + 0x_7$$

$$\begin{cases} 5x_1 + x_2 + (x_4 - x_5) + x_6 = 7 \\ x_1 - x_2 - 4(x_4 - x_5) - x_7 = 2 \\ 3x_1 - x_2 - 2(x_4 - x_5) = 5 \\ x_1, x_2, x_4, x_5, x_6, x_7 \geq 0 \end{cases}$$

- Objective: min/max $\rightarrow$ max
- Variables: all the variables are non-negative
- Constant term: non-negative
- Constraints : replace non-equalities with equalities

Augmented form

$$\max Z = -x_1 + 2x_2$$

$$\begin{cases} 3x_1 - 8x_2 \le 5 \\ x_1 - 3x_2 \ge 4 \\ x_1 \ge 0, x_2 \end{cases}$$

$$\max \ Z = -x_1 + 2(x_3 - x_4)$$

Augmented form

$$\begin{cases} 3x_1 - 8(x_3 - x_4) + x_5 = 5 \\ x_1 - 3(x_3 - x_4) - x_6 = 4 \\ x_1, x_3, x_4, x_5, x_6 \ge 0 \end{cases}$$

Augmented form

$$\max Z = \sum c_j x_j$$

$$\begin{cases} \sum a_{ij} x_j = b_i & (i = 1 \cdot 2 \cdots m) \\ x_j \geq 0 & (j = 1 \cdot 2 \cdots n) \end{cases}$$

Let $C = (c_1\ c_2\ \cdots c_n\ 0\ \cdots 0)$, (the number of 0s is *m*)

Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & & & \\ a_{21} & a_{22} & \cdots & a_{2n} & & 1 & & 0 \\ \cdots & \cdots & \cdots & & & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & & & 1 \end{pmatrix}$$

$$\max Z = CX$$

$X = (x_1 \ldots x_n\ x_{n+1}\ \cdots x_{n+m})^T$

$b = (b_1\ b_2\ \cdots b_m)^T$

$$\begin{cases} AX = b \\ X \geq 0 \end{cases}$$

A finite subset of n vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, from the vector space V, is *linearly dependent* if there exists a set of n scalars, $a_1, a_2, \ldots, a_n$, not all zero, such that $a_1v_1+a_2v_2+\ldots+a_nv_n=0$
linearly independent: $a_1, a_2, \ldots, a_n$, all zero, such that $a_1v_1+a_2v_2+\ldots+a_nv_n=0$

$$A=\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & & & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & & 1 & & \\ \cdots & \cdots & \cdots & & & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & & & 1 \end{array}\right)$$

- Principle theorems
- Max $Z=CX$, $AX=b$， $X\geq 0$
  - If the optimal solutions exist, then we can find the optimal solution on extreme points
  - If $X=(x_1 \ldots x_n\ x_{n+1}\ \cdots x_{n+m})^T$ is a extreme point, then the coefficient vectors of non-zero variables x_i are linear independent.
  - Total number of extreme points: C(m+n,n)

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

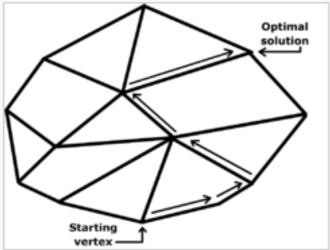
m=4 B

$$A=\begin{array}{c} \left[\begin{array}{cc|cccc} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cccccc} x1 & x2 & x3 & x4 & x5 & x6 \end{array} \end{array}$$

$Bx=b$
$x=B^{-1}b$

$x3=12;\quad x4=8;$
$x5=16;\quad x6=12$

$x1=x2=0$



- Solves LP problems by constructing a feasible solution at a vertex of the polytope and then walking along a path on the edges of the polytope to vertices with non-decreasing values of the objective function until an optimum is reached.
  1. Locate an extreme point of the feasible region.
  2. Examine each boundary edge intersecting at this point to see whether movement along any edge increases the value of the objective function.
  3. If the value of the objective function increases along any edge, move along this edge to the adjacent extreme point. If several edges indicate improvement, the edge providing the greatest rate of increase is selected.
  4. Repeat steps 2 and 3 until movement along any edge no longer increases the value of the objective function.

$$\begin{array}{cc} \mathbf{A} & \mathbf{b} \\ \boldsymbol{A}=\begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{4} & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} \mathbf{12} \\ \mathbf{8} \\ \mathbf{16} \\ \mathbf{12} \end{matrix} \end{array} \xrightarrow{\text{Transformation with } B^{-1}} \begin{array}{cc} \mathbf{P} & \mathbf{P0} \\ \boldsymbol{P}=\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & -0.5 \\ 1 & 0 & 0 & 1 & 0 & -0.5 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.25 \end{bmatrix} & \begin{matrix} \mathbf{6} \\ \mathbf{2} \\ \mathbf{16} \\ \mathbf{3} \end{matrix} \end{array}$$

$B^{-1}(\, b\ A_1\ A_2\ \ldots\ A_{n+m}\,) = (\, P0\ P1\ P2\ \ldots\ Pn+m\,)$.

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 & \boldsymbol{p}_5 & \boldsymbol{p}_6 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{p}_3 & \boldsymbol{p}_4 & \boldsymbol{p}_5 & \boldsymbol{p}_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

Basic vectors

$\therefore \; \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5, \boldsymbol{x}_6$ **are the basic variables**

$\boldsymbol{x}_1, \boldsymbol{x}_2$ **Are non-basic variables**

Accordingly, the feasible solution is（0 0 12 8 16 12）
while the objective is Z = 0

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

$$\begin{array}{cc} & \mathbf{A} \quad\quad\quad\quad \mathbf{b} \\ A = & \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \mathbf{12} \\ \mathbf{8} \\ \mathbf{16} \\ \mathbf{12} \end{array} \end{array}$$

Let x_2 to be non-zero (to be basic variable)

x_1=0, some other variable should be 0 among x_3,x_4,x_5,x_6

$$x_3 = 12 - 2x_2 \geq 0$$
$$x_4 = 8 - 2x_2 \geq 0$$
$$x_5 = 16 \geq 0$$
$$\boxed{x_6 = 12 - 4x_2 \geq 0}$$

All variables non-negative

$$\beta_i \quad x_2 = \min(\frac{12}{2}, \frac{8}{2}, \frac{12}{4}) = 3$$

x_6=0, become non-basic variable to exchange with x_2

Then:

$$x_3 + 2x_2 = 12 - 2x_1 \qquad (1)$$
$$x_4 + 2x_2 = 8 - x_1 \qquad (2)$$
$$x_5 = 16 - 4x_1 \qquad (3)$$
$$4x_2 = 12 - x_6 \qquad (4)$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

Since (4)', we have $Z = 2x_1 + 9 + 0x_3 + 0x_4 + 0x_5 - \frac{3}{4}x_6$

$$P = \begin{array}{c} \\ \end{array}\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & -0.5 \\ 1 & 0 & 0 & 1 & 0 & -0.5 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.25 \end{bmatrix}\begin{array}{c} \mathbf{6} \\ \mathbf{2} \\ \mathbf{16} \\ \mathbf{3} \end{array}$$

Gaussian elimination to transform the coefficients for x_2 into I:

$$(4)' = \frac{(4)}{4}, (1)' = (1) - 2\times(4)', (2)' = (2) - 2\times(4)', (3)' = (3)$$

$$x_3 = 6 - 2x_1 + \frac{1}{2}x_6 \quad (1)'$$

$$x_4 = 2 - x_1 + \frac{1}{2}x_6 \quad (2)'$$

$$x_5 = 16 - 4x_1 \quad (3)'$$

$$x_2 = 3 - \frac{1}{4}x_6 \quad (4)'$$

$$x_3 + 2x_2 = 12 - 2x_1 \quad (1)$$

$$x_4 + 2x_2 = 8 - x_1 \quad (2)$$

$$x_5 = 16 - 4x_1 \quad (3)$$

$$4x_2 = 12 - x_6 \quad (4)$$

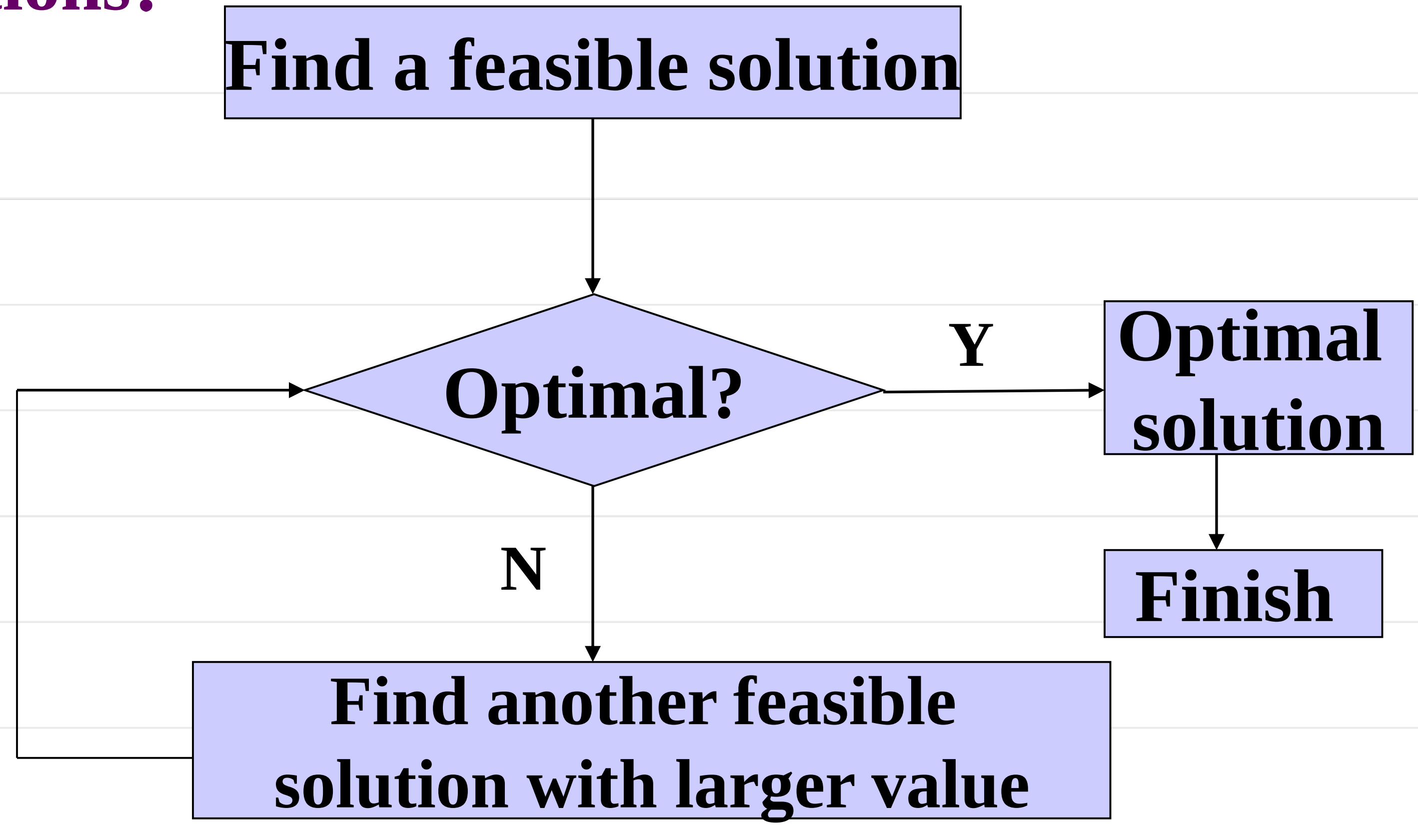
$$Z = 2x_1 + 9 - \frac{3}{4}x_6 = 9 + 2x_1 - \frac{3}{4}x_6$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 & = 12 \\ x_1 + 2x_2 \quad + x_4 & = 8 \\ 4x_1 \quad\quad + x_5 & = 16 \\ 4x_2 \quad\quad + x_6 & = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

$x_1 = x_6 = 0,\ Z = 9$ （0，3，6，2，16，0）

Iterations：



Thank you