Linear Homogeneous Recurrence Relation

组合数学 Combinatorics

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Generating Function

- Given an infinite sequence of numbers: $h_0, h_1, h_2, \ldots, h_n, \ldots$
- The generating function is defined to be the infinite series

  $$G(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n + \ldots$$

- A **generating function** is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers h_n that is indexed by the natural numbers.
- A finite sequence: $h_0, h_1, h_2, \ldots, h_m$
  - $h_0, h_1, h_2, \ldots, h_m, 0, 0, \ldots$
  - $G(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_m x^m$
- The generating function of the infinite sequence 1,1,1,…,1,… (h_i=1)
  - $g(x) = 1 + x + x^2 + \ldots + x^n + \ldots$

    $$= \frac{1}{1 - x}$$

Generating function and recurrence

- Given a linear homogeneous recurrence relation, find out the generating function in the form of P(x)/Q(x)

$$x^3 : F_3 = F_2 + F_1$$

$$x^4 : F_4 = F_3 + F_2$$

$$+)\quad \ldots\ldots\ldots\ldots\ldots$$

$$\therefore \quad (1-x-x^2)G(x) = x$$

- Turn the form of g(x) into

$$\therefore \quad G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

$$\therefore \quad \boxed{G(x) = \frac{A}{1-r_1 x} + \frac{B}{1-r_2 x}}$$

- Figure out A and B to be c_1 and c_2

$$A = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}$$

- $f_n = c_1 r_1^{\,n} + c_2 r_2^{\,n}$

$$\mathrm{F_n} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{\mathrm{n}} - \left(\frac{1-\sqrt{5}}{2}\right)^{\mathrm{n}}\right)$$

Fibonacci Recurrence

$F_n=F_{n-1}+F_{n-2}$ $F_0=0, F_1=1$

Assume $\boxed{G(x)=F_1x+F_2x^2+\cdots}$ $\therefore \quad (1-x-x^2)G(x)=x$

$$\therefore \quad G(x)=\frac{x}{1-x-x^2}=\frac{x}{\boxed{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)}}=\frac{A}{1-\frac{1+\sqrt{5}}{2}x}+\frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

Factoring?

$$(1-ax)^{-1}=1+ax+a^2x^2+\ldots..$$

$$(1-x-x^2)=(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)$$

Fibonacci sequence

- *Fibonacci recurrence relation*
- $\boxed{f_n - f_{n-1} - f_{n-2} = 0}$ ($n \geq 2$)
- Suppose that the solution of the form
  - **$f_n = q^n$ where q is non-zero**
  - $q^n - q^{n-1} - q^{n-2} = 0$
  - $(q^2 - q - 1)q^{n-2} = 0$
  - $\boxed{q^2 - q - 1 = 0}$
  - Find the roots for the quadratic equation $q_1 = \frac{1+\sqrt{5}}{2}, q_2 = \frac{1-\sqrt{5}}{2}$
- Suppose $f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$
- Use the initial conditions
- $n=0$, $f(0)=0$: $c_1 + c_2 = 0$
- $n=1$, $f(1)=1$: $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\Rightarrow c_1 = \frac{1}{\sqrt{5}}, c_2 = \frac{-1}{\sqrt{5}}$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Characteristic equation

- For a sequence $\{h_n\}$, it has the k-order linear homogeneous recurrence relation as
- Relations: $h_n + C_1 h_{n-1} + C_2 h_{n-2} + \cdots + C_k h_{n-k} = 0,$ $\quad f_n - f_{n-1} - f_{n-2} = 0$
- Initial values: $h_0 = d_0, h_1 = d_1, \cdots, h_{k-1} = d_{k-1},$ $\quad f(0)=0 \quad f(1)=1$
  $C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{k-1}$ are constants.
- The characteristic equation for $\{h_n\}$

$$C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k \qquad C(x)=x^2-x-1=0$$

- Suppose there are k *distinct* roots for C(x)

$$C(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_i) \qquad q_1 = \frac{1+\sqrt{5}}{2}, q_2 = \frac{1-\sqrt{5}}{2}$$

- Then the ***explicit formula of h_n***

$$h_n = l_1\alpha_1^n + l_2\alpha_2^n + \cdots + l_k\alpha_k^n \qquad f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

  – l_i : undetermined coefficient
- l_i can be determined using the initial values
  – n=0, $f(0)=0$: $c_1+c_2=0$
  – n=1, $f(1)=1$: $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$$\Rightarrow \quad c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = \frac{-1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Linear Homogeneous Recurrence Relation

$F_n - F_{n-1} - F_{n-2} = 0$ $\quad\quad$ $h(n) - 3h(n-1) + 2h(n-2) = 0$

$x^2 - x - 1 = 0$ $\quad\quad$ $x^2 - 3x + 2 = 0$

Def if sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1},$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{n-1}$ are constants, $C_k \neq 0$, then this expression is called a k^th-order linear homogeneous recurrence relation of $\{a_n\}$.

$$\boxed{C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k}$$

Characteristic Polynomial

$$G(x) = \frac{P(x)}{\left(1 + C_1 x + \cdots + C_k x^k\right)}$$

Linear Homogeneous Recurrence Relation

Def If sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0, \quad (2-5-1)$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1}, \quad (2-5-2)$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{k-1}$ are constants

Characteristic Polynomial $C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k$

1）Characteristic polynomial has k distinct real roots

$$C(x) = (x - a_1)(x - a_2)\cdots(x - a_k)$$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

In which $l_1, l_2, \cdots l_k$ are undetermined coefficients.

Given generating function of sequence {an}: $G(x) = \frac{3+78x}{1-3x-54x^2}$,

1) please find its recurrence relation(when n≥2);

2) please find the explicit expression for sequence {an}

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n \qquad \alpha \in R \qquad \sum_{k=0}^{\infty}(k+1)x^k = \frac{1}{(1-x)^2}$$

If $(1-ax)^{-m}$ then

$$= (1-ax)^{-m} = \sum_{k=0}^{\infty} C(m+k-1,k)2^k x^k$$

$$= \sum_{k=0}^{\infty} C(m+k-1,m-1)2^k x^k$$

(m-1)-order polynomial

In the form: $A_1+A_2n+A_3n^2+\ldots A_{m-1}*n^{m-1}$

- <u>Characteristic Polynomial has multiple roots</u>
- Eg $\quad \boldsymbol{a}_n - 4\boldsymbol{a}_{n-1} + 4\boldsymbol{a}_{n-2} = 0, \quad \boldsymbol{a}_0 = 1, \boldsymbol{a}_1 = 4$
- Generating Function Method

$$x^2 : a(2) = 4a(1) - 4a(0)$$
$$x^3 : a(3) = 4a(2) - 4a(1)$$
$$+) \qquad \cdots\ \cdots\ \cdots\ \cdots$$

$$A(x) = \frac{1}{1-4x+4x^2}$$

$$A(x) = \frac{1}{1-4x+4x^2} = \frac{1}{(1-2x)^2}$$

$$= (1-2x)^{-2} = \sum_{k=0}^{\infty} C(k+1,k)2^k x^k$$

$$= \sum_{k=0}^{\infty} (k+1)2^k x^k$$

$$\boxed{a_n = (n+1)2^n}$$

Characteristic Equation Method

Characteristic Equation: $x^2-4x+4=(x-2)^2$

Generating Function Form: $A(x) = \dfrac{ax+b}{(1-2x)^2}$

Partial Fractions: $A(x) = \dfrac{A}{(1-2x)} + \dfrac{B}{(1-2x)^2}$

$$a_n = A \times 2^n + B(n+1)2^n = (A' + Bn)2^n$$

$$a_0 = A' = 1, \quad a_1 = (1+B)2 = 4$$

$$A' = 1, \quad B = 1$$

$$\boxed{a_n = (n+1)2^n}$$

Characteristic equation

- If the root α has the multiplicity *m*

$$h_n = (A_0 + A_1 n + \cdots + A_{m-1} n^{m-1})\alpha^n$$

$$a_n - 4a_{n-1} + 4a_{n-2} = 0, \quad a_0 = 1, a_1 = 4$$

Characteristic equation $x^2 - 4x + 4 = (x-2)^2$ 2 has the multiplicity *2*

$$a_n = (A_1 + A_2 n)2^n$$

$$a_0 = A_1 = 1$$

$$a_1 = (1 + A_2)2 = 4, \qquad A_2 = 1$$

$$a_n = (1+n)2^n$$

Summary

The Characteristic equation C(x):

1) distinct roots $C(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$

$$a_n = l_1\alpha_1^n + l_2\alpha_2^n + \cdots + l_k\alpha_k^n$$

Where $l_1, l_2, \cdots, l_k$,are undetermined coefficients.

2) k-multiplicity root which is α 。

$$a_n = (A_0 + A_1 n + \cdots + A_{k-1} n^{k-1})\alpha^n$$

- Where $A_0, A_1, \cdots, A_{k-1}$ are undetermined coefficients.

Distinct real roots **Multiple real roots** **Conjugate complex roots?**

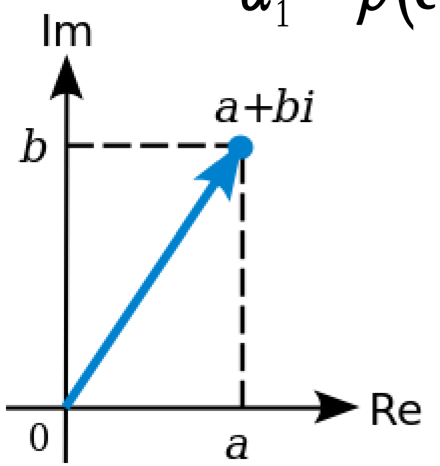
$$x^2 - x + 1 = 0$$

Conjugate Complex Roots

- Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- When b^2-4ac<0, there's no real root, two complex roots.

$$x_{1,2} = \frac{-b \pm i \times \sqrt{4ac - b^2}}{2a}$$

$$\alpha_1 = \rho(\cos\theta + i\sin\theta), \alpha_2 = \overline{\alpha_1} = \rho(\cos\theta - i\sin\theta)$$



Trigonometrical form of complex number $z = a + bi$:
$z = \rho(cos\theta + i\ sin\theta)$

$$\rho = \sqrt{a^2 + b^2}$$

(3) Characteristic Polynomial $C(x)$ has conjugate complex roots

Assume that a_1, a_2 are a pair of conjugate complex roots of $C(x)$.

$$\alpha_1 = \rho(\cos\theta + i\sin\theta), \alpha_2 = \overline{\alpha_1} = \rho(\cos\theta - i\sin\theta)$$

In $\frac{A_1}{1-\alpha_1 x} + \frac{A_2}{1-\alpha_2 x}$ the coefficient of x^n is:

$$A_1\alpha_1^n + A_2\alpha_2^n$$

$$A_1\alpha_1^n + A_2\alpha_2^n$$
$$= (A_1 + A_2)\rho^n \cos n\theta + i(A_1 - A_2)\rho^n \sin n\theta$$
$$= A\rho^n \cos n\theta + B\rho^n \sin n\theta$$

In which $A = A_1 + A_2,\quad B = i(A_1 - A_2)$

When calculating in reality, we could solve the conjugate complex roots at first, then calculate undetermined coefficients A, B to avoid the intermediate complex number calculations.

$A_1\alpha_1^n + A_2\alpha_2^n = A\rho^n \cos n\theta + B\rho^n \sin n\theta$

- Eg $a_n = a_{n-1} - a_{n-2}, a_1 = 1, a_2 = 0$

Characteristic equation： $x^2 - x + 1 = 0$

$$x = \frac{1 \pm \sqrt{-3}}{2} = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = e^{\pm\frac{\pi}{3}i}$$

$$a_n = A_1\cos\frac{n\pi}{3} + A_2\sin\frac{n\pi}{3}$$

$$a_1 = \frac{1}{2}A_1 + \frac{\sqrt{3}}{2}A_2 = 1$$

$$a_2 = -\frac{1}{2}A_1 + \frac{\sqrt{3}}{2}A_2 = 0$$

$$\Rightarrow \quad A_1 = 1; \qquad A_2 = \frac{\sqrt{3}}{3}$$

$$a_n = \cos\frac{n\pi}{3} + \frac{\sqrt{3}}{3}\sin\frac{n\pi}{3}$$

Summary of Linear Recurrence Relation

According to the non-zero roots of C(x)

1) k distinct non-0 real roots $C(x)=(x-a_1)(x-a_2)\cdots(x-a_k)$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

In which $l_1, l_2, \cdots, l_k$, are undetermined coefficients.

2)A pair of conjugate complex root $\alpha_1 = \rho e^{i\theta}$ and $\alpha_2 = \rho e^{-i\theta}$:

$$\boldsymbol{a_n = A\rho^n \cos n\theta + B\rho^n \sin n\theta}$$

In which A，B are undetermined coefficients.

3)Has root α_1 with multiplicity of k.

$$(A_0 + A_1 n + \cdots + A_{k-1} n^{k-1})\alpha_1^n$$

In which $A_0, A_1, \cdots, A_{k-1}$ are k undetermined coefficients.

Linear Homogeneous Recurrence Relation

Eg： Solve $S_n = \sum_{k=0}^{n} k$

$$\boldsymbol{S_n = 1 + 2 + 3 + \cdots + (n-1) + n}$$

$$\boldsymbol{S_{n-1} = 1 + 2 + 3 + \cdots + (n-1)}$$

$$\therefore \quad S_n - S_{n-1} = n$$

Similarly $S_{n-1} - S_{n-2} = n-1$

Subtract, get $S_n - 2S_{n-1} + S_{n-2} = 1$

Similarly $S_{n-1} - 2S_{n-2} + S_{n-3} = 1$

$$\therefore \quad S_n - 3S_{n-1} + 3S_{n-2} - S_{n-3} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 3$$

$$\therefore \quad S_n - 3S_{n-1} + 3S_{n-2} - S_{n-3} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 3$$

Corresponding Characteristic Equation is

$$m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$$

$m = 1$ is a 3-multiple root

$$\therefore \quad S_n = (A + Bn + Cn^2)(1)^n = A + Bn + Cn^2$$

$$S_0 = 0, \quad \therefore \; A = 0$$

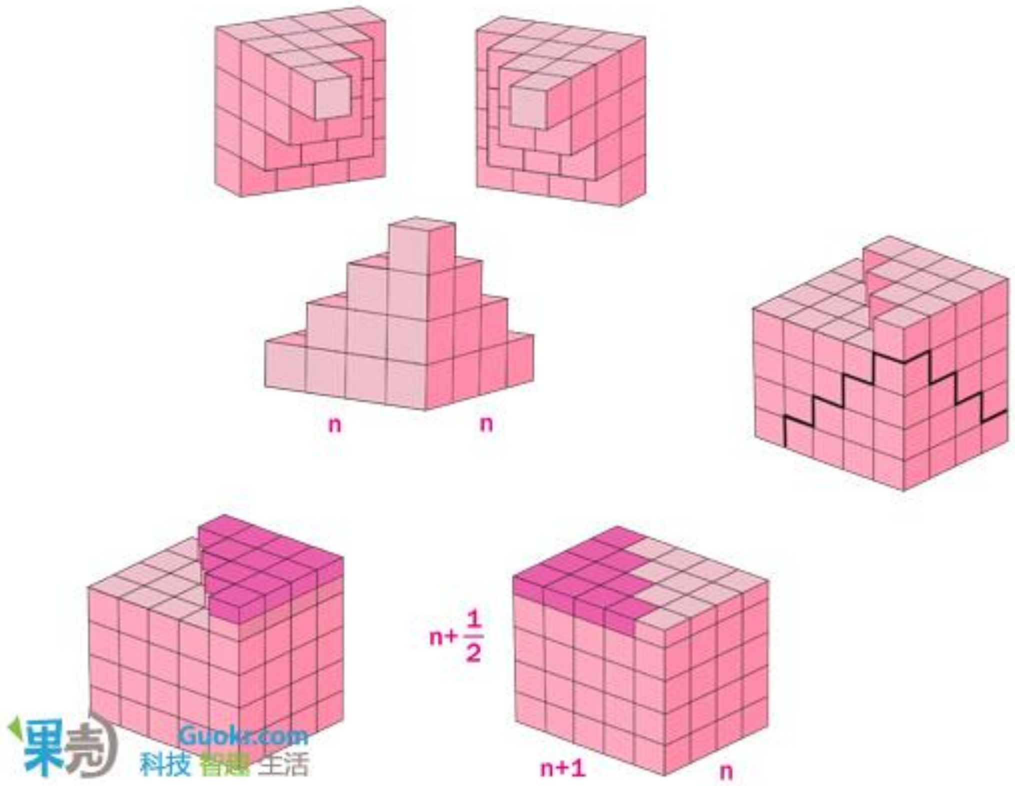
$$\boldsymbol{S_1 = 1, \quad B + C = 1}$$

$$\boldsymbol{S_2 = 3, \quad 2B + 4C = 3, \quad \therefore \; B = C = \frac{1}{2}}$$

So $\quad S_n = \frac{1}{2}n + \frac{1}{2}n^2 = \frac{1}{2}n(n+1)$

This proves $\quad 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

$$1^2+2^2+3^2+\cdots+n^2=\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)$$



Linear Homogeneous Recurrence Relation

Eg2： Calculate $S_n = \sum_{k=0}^{n} k^2$

$S_n = 1 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 \qquad \therefore \quad S_n - S_{n-1} = n^2$

$S_{n-1} = 1 + 2^2 + 3^2 + \cdots + (n-1)^2$ Similarly $S_{n-1} - S_{n-2} = (n-1)^2$

Subtract, get $S_n - 2S_{n-1} + S_{n-2} = 2n - 1$

Similarly $S_{n-1} - 2S_{n-2} + S_{n-3} = 2(n-1) - 1$

Subtract, get $S_n - 3S_{n-1} + 3S_{n-2} - S_{n-3} = 2$

Similarly $S_{n-1} - 3S_{n-2} + 3S_{n-3} - S_{n-4} = 2$

$$\therefore \quad S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 5, \quad S_3 = 14$$

$$\therefore \quad S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 5, \quad S_3 = 14$$

Correspondent characteristic equation is:

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = (r-1)^4 = 0$$

$r = 1$ is a 4-multiple root

$$\therefore \quad S_n = (A + Bn + Cn^2 + Dn^3)(1)^n$$

As $S_0 = 0, S_1 = 1, S_2 = 5, S_3 = 14$ we have a equation group about A、B、C、D:

$$\begin{cases} A = 0 \\ B + C + D = 1 \\ 2B + 4C + 8D = 5 \\ 3B + 9C + 27D = 14 \end{cases}$$

$$1^2+2^2+3^2+\cdots+n^2=\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)$$

$$D_n=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}=\prod_{n\ge i>j\ge 1}(x_i-x_j).$$

$$\begin{cases} A=0 \\ B+C+D=1 \\ 2B+4C+8D=5 \\ 3B+9C+27D=14 \end{cases}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix}=12$$

=(4-3)(4-2)(4-1)(3-2)(3-1)(2-1)

=12

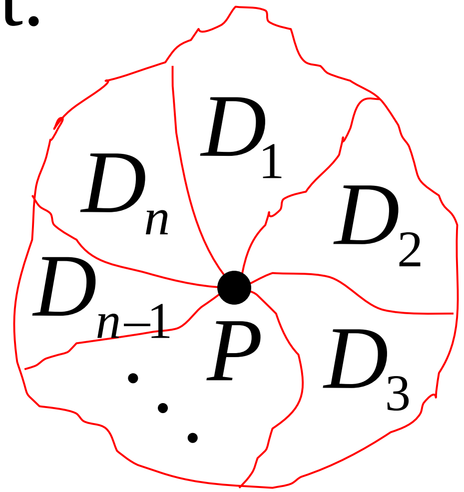
$$B=\frac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 8 \\ 14 & 9 & 27 \end{vmatrix}=\frac{1}{6}$$

$$C=\frac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 3 & 14 & 27 \end{vmatrix}=\frac{1}{2}$$

$$D=\frac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 9 & 14 \end{vmatrix}=\frac{1}{3}$$

Applications of generating function and recurrence relation

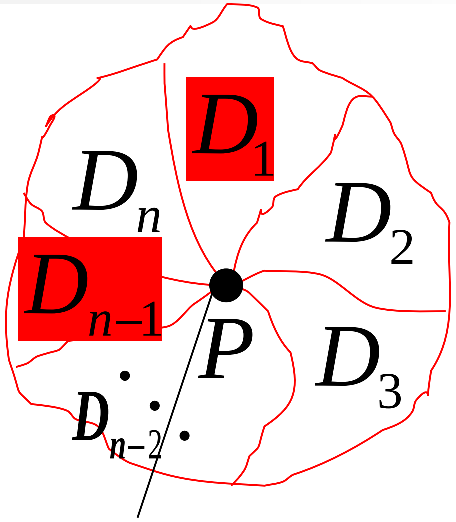
Eg: There's a point P on the plane. It's the cross of n fields $D_1, D_2, \cdots D_n$. Color these n fields with k colors. We require the color of two adjacent areas to be different. Calculate the number of arrangements.



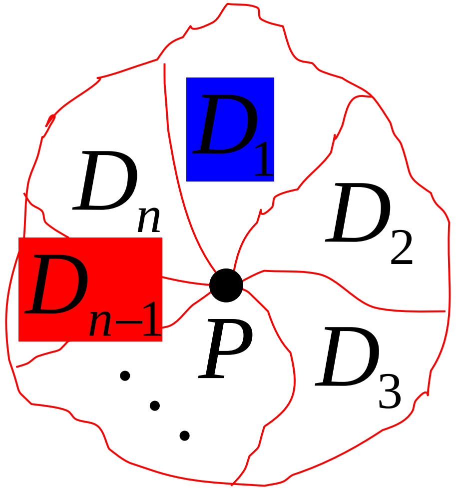
Let a_n be the number of arrangement to color these areas. There are 2 situations:

Applications of generating function and recurrence relation

(1) D_1 and D_{n-1} have the same color;
D_n has k-1 choices, which is all colors except the one used by D_1 and D_{n-1}; the arrangements for D_{n-2} to D_1的 are one-to-one correspondent to the arrangements for *n-2* areas. $(k-1)a_{n-2}$



(2) D_1 and D_{n-1} have different colors.
D_n has k-2 choices; the arrangements from D_1 to D_{n-1} are one-to-one correspondent to the arrangements for *n-1* areas.



$(k-2)a_{n-1}$

$$\therefore\ a_n = (k-2)a_{n-1} + (k-1)a_{n-2},$$

$$a_2 = k(k-1),\ a_3 = k(k-1)(k-2).$$

Applications of generating function and recurrence relation

$$\therefore\ a_n = (k-2)a_{n-1} + (k-1)a_{n-2},$$

$$a_2 = k(k-1),\ a_3 = k(k-1)(k-2).$$

$$a_1 = 0, a_0 = k.$$

$$x^2 - (k-2)x - (k-1) = 0,$$

$$x_1 = k-1,\quad x_2 = -1.$$

$$a_n = A(k-1)^n + B(-1)^n.$$

$$\begin{cases} A + B = k, \\ (k-1)A - B = 0. \end{cases} \qquad \begin{cases} A = 1, \\ B = k-1. \end{cases}$$

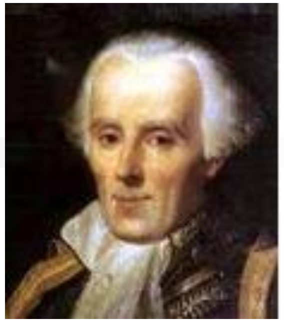
$$\therefore\quad a_n = (k-1)^n + (k-1)(-1)^n,\ n \ge 2.$$

$$a_1 = k.$$

Def 2-1 For sequence $a_0, a_1, a_2 \ldots$, construct a function

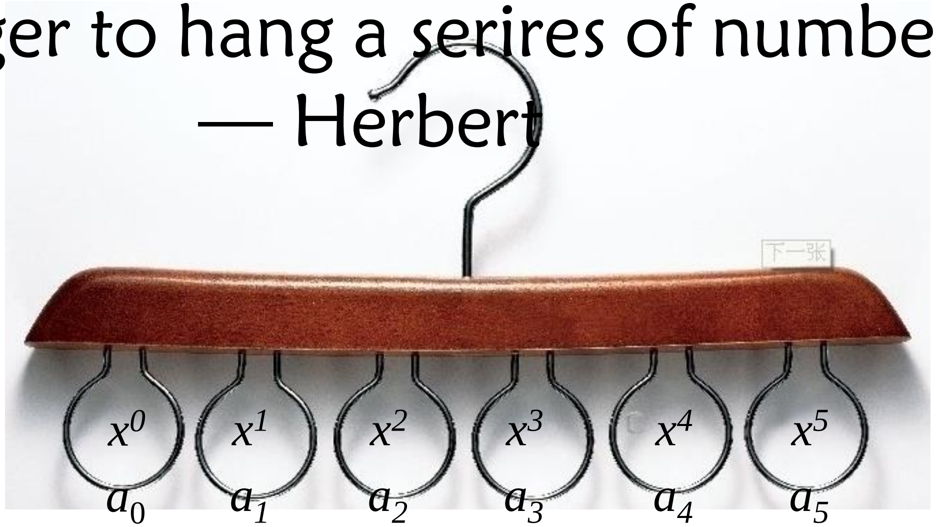
$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

Then $G(x)$ is called the generating function of $\boldsymbol{a_0, a_1, a_2 \ldots\ldots}$



Laplace
1812 AD

Generating functions are a hanger to hang a serires of numbers.
— Herbert



ToDo List

- Homework sheet due on Monday
- No preclass video

Thanks