Applications of IEP

- *Combinations with repetition*
- Derangements
- Permutations with forbidden positions

Combinations with repetition

- Example 1: Determine the number of 10-combinations of the multiset T = {3{a}, 4{b}, 5{c}}.
- Hint: Let $T^* = \{\infty\{a\}, \infty\{b\}, \infty\{c\}\}$, P_1 (resp., P_2, and P_3) be the property that a 10-combination of T^* has more than 3 a's (resp., 4 b's and 5 c's) and A_1 (resp., A_2 and A_3) be the 1—combinations of T^* which have property P_1 (resp., P_2 and P_3). We wish to determine the size of the set

$$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\right| = |S| - (|A_1| + |A_2| + |A_3|)$$
$$+ (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)$$
$$- (|A_1 \cap A_2 \cap A_3|)$$

10-combinations of the multiset T = {3{a}, 4{b}, 5{c}}.

- S: |S|=C(10+3-1,10) = 66
- A1:consists of all 10-conbinations of T^* in which *a* occurs at least 4 times.
  - The number of 10-combinations in A_1 equals the number of 6-combinations of T^*.
  - |A1|=C(6+3-1,6) = 28
- A2:consists of all 10-conbinations of T^* in which *b* occurs at least 5 times.
  - |A2|=C(5+3-1,5) = 21
- A3:consists of all 10-conbinations of T^* in which *c* occurs at least 6 times.
  - |A3|=C(4+3-1,4) = 15
- A1∩A2:consists of all 10-conbinations of T^* in which *a* occurs at least 4 times and *b* occurs at least 5 times.
  - |A1∩A2 |=C(1+3-1,1) = 3
- A1∩A3:consists of all 10-conbinations of T^* in which *a* occurs at least 4 times and c occurs at least 6 times.
  - |A1∩A3 |=1
- A2∩A3:
  - |A2∩A3 |=0
- A1 ∩ A2∩A3:
  - | A1 ∩ A2∩A3 |=0

$$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\right| = 66 - (28 + 21 + 15) + (3 + 1 + 0) - (0) = 6$$

Derangements

- A *derangement* of {1, 2, …, n} is a permutation $i_1 i_2 \ldots i_n$ of {1, 2, …, n} such that $i_1 \neq 1$, $i_2 \neq 2$, …, $i_n \neq n$ (i.e., no integer is in its natural position).
- We denote by D_n the number of derangements of {1, 2, …, n}.
- For n =1, there are no derangements. $D_1=0$
- For n =2, the only derangement is 2 1. $D_2=1$
- For n =3, there are two derangements: $D_3=2$

  2 3 1 and 3 1 2.
- For n = 4, there are 9 derangements: $D_4=9$

2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.

Examples

- At a party there are *n* men and *n* women. In how many ways can the *n* women choose male partners for the first dance? How many ways are there for the second dance if everyone has to change partners?
- Answer: for the first dance there are *n!* possibilities.
- For the second dance, the number of possibilities is D_n.

Formulas for Counting D_n

- For $n \geq 1$

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})$$

- Proof: A *derangement* of {1, 2, …, n} is a permutation $i_1 i_2 \ldots i_n$ of {1, 2, …, n} such that $i_1 \neq 1$, $i_2 \neq 2$, …, $i_n \neq n$ (i.e., no integer is in its natural position).
- Let S be the set of all n! permutations
- Let $P_j (j=1,2,\ldots n)$ be the property that, in a permutation, *j* is in its natural position.
- Let A_j denote the set of permutations with property $P_j (j=1,2,\ldots n)$

$$D_n = |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}|$$

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})$$

- S is the set of all n! permutations
  - $|S|=n!$
- A_j is the set of permutations with property P_j(j=1,2,…n) that j is in its natural position
  - $|A_j|=(n-1)!$
- $A_i \cap A_j$ is the set of permutations that i and j is in their natural positions
  - $|A_i \cap A_j|=(n-2)!$
- $a_k = |A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}|$.
  - $a_k=(n-k)!$
  - {i1,i2,…,ik} is a k-combination of {1,2…,n}
    - C(n,k)

$$C(n,i)(n-i)! = \frac{n!}{(n-i)!\,i!}(n-i)! = \frac{n!}{i!}$$

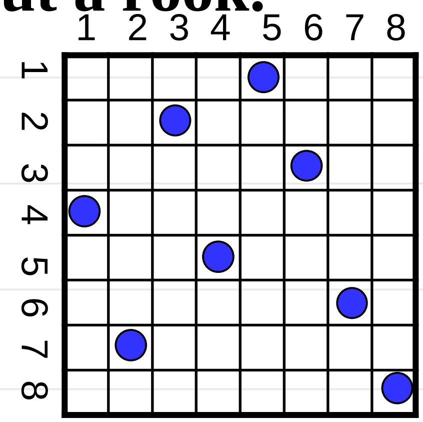
$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}| = n! - C(n,1)(n-1)!$$

$$+ C(n,2)(n-2)! - \cdots - \pm C(n,n)1!$$

$$= n!(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{n!})$$

Examples

- How many possibilities are there for 8 non-attacking rooks on an 8-by-8 chessboard?
- (1) The rooks are indistinguishable for one another;
- The coordinates of rooks: only 1 rook for each row/column
  - (1,5) (2,3), (3,6),(4,1),(5,4),(6,7),(7,2),(8,8)
  - 8-permutations of {1,2…8}: 8!
- **The permutations in $P(X_1, X_2,\ldots,X_n)$ correspond to placements of n non-attacking rooks on an n-by-n board in which there are certain squares in which it is forbidden to put a rook.**



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An example

- Determine the number of ways to place 5 non-attacking rooks on the following 5-by-5 board, with forbidden positions as shown.

	1	2	3	4	5
1	x				
2	x	x			
3			x	x	
4			x	x	
5					

Definition of $P(X_1, X_2,\ldots,X_n)$

- Let $X_1, X_2, \ldots, X_n$ be (possibly empty) subsets of $\{1, 2, \ldots, n\}$. We denote by ***$P(X_1, X_2,\ldots,X_n)$* the set of all permutations *$i_1 i_2 \ldots i_n$* of $\{1, 2, \ldots, n\}$ such that *i_1* is not in X_1, i_2 is not in X_2 … i_n is not in X_n.**
- **Let $p(X_1, X_2,\ldots,X_n) = |\, P(X_1, X_2,\ldots,X_n)\,|$**
- **Let $n = 4$ and let $X_1=\{1, 2\}$, $X_2 =\{2, 3\}$, $X_3 =\{3, 4\}$ and $X_4 =\{1, 4\}$. Then $P(X_1, X_2, X_3, X_4)$ consists of all permutations $i_1 i_2 i_3 i_4$ of $\{1,2 ,3 ,4\}$ such that $i_1 \neq 1,2$; $i_2 \neq 2,3$; $i_3 \neq 3,4$; $i_4 \neq 1,4$.**
- **Only two permutations**
  - **$P(X_1, X_2, X_3, X_4) = \{3412, 4123\}$**
  - **$p(X_1, X_2,\ldots,X_n) = 2$.**

Examples

- **Let $X_k = \{k\}$ ($k = 1, 2, \ldots, n$). Then the set $P(X_1, X_2, \ldots, X_n)$ equals the set of all permutations $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ for which $i_k \neq k$.**
- **We conclude that $P(X_1, X_2, \ldots, X_n)$ is the set of derangements of $\{1, 2, \ldots, n\}$ and we have $p(X_1, X_2, \ldots, X_n) = D_n$.**

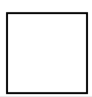
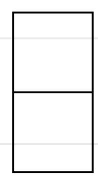
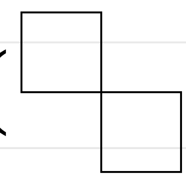
An example

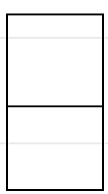
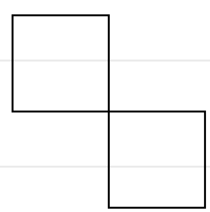
	1	2	3	4	5
1	x				
2	x	x			
3			x	x	
4			x	x	
5					

- Let n =5, X_1 ={1}, X_2 ={1,2}, X_3 = {3,4}, X_4 ={3,4}. Then P(X_1, X_2, X_3, X_4, X_5) are in one-to-one correspondence with the placement of 5 non-attacking rooks on the board with forbidden positions as shown.
- Let S be the set of all n! permutations without forbidden positions.
- Pj means the property that the rook in the jth row in in a column belonging to Xj $p(X_1, X_2, ..., X_n) = |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m|$
- Ai should be to Place n nonattacking rooks where the rook in row i is in one of the columns in Xi.
  - The ith element has |Xi| choices
  - |Ai|=|Xi|(n-1)! Σ |Ai|= Σ |Xi|*(n-1)! = $\mathbf{r_1}$(n - 1)!
- Ai∩Aj should be to Place n nonattacking rooks where the rook in rows i and j are in columns in Xi and Xj.
  - Suppose r_2 equal the number of ways to place two nonattacking rooks on the board in forbidden positions.
  - Σ | Ai∩Aj |= r_2(n - 2)!

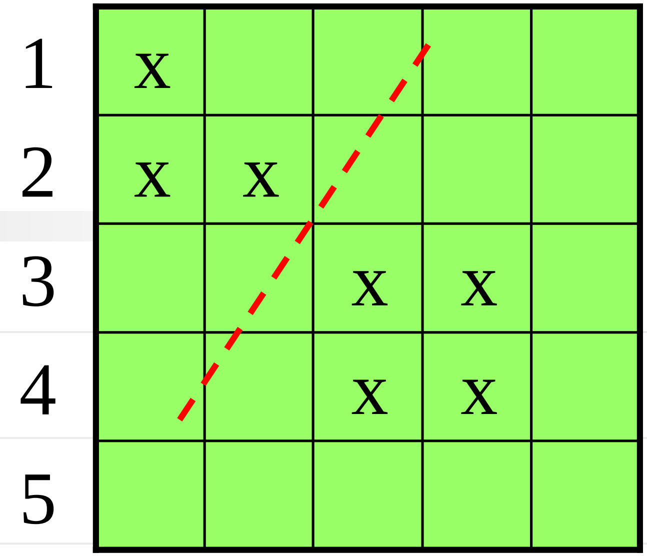
Placement of rooks in chess board

- r_k is the number of ways to place k non-attacking rooks on the n-by-n board where each of the k rooks is in a forbidden position (k=1, 2, …, n).
- The number of ways to place n non-attacking, indistinguishable rooks on an n-by-n board with forbidden positions equals
- $n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^k r_k(n-k)! + … + (-1)^r r_n$.

$r_1($  $)=1$， $r_1($  $)=2$， $r_1($  $)=2$，

$r_2($  $)=0$， $r_2($  $)=1$。

An example



- Determine the number of ways to place 5 non-attacking rooks on the following 5-by-5 board, with forbidden positions as shown.
- r1 = 7
- The set of forbidden positions can be partitioned into two "independent" parts
  - "Independent" means squares in different parts do not belong to a common row or column.
  - one part F_1 containing three positions and the other part F_2 containing four.
- r2: The rooks may be both in F_1, both in F_2 or one in F_1 and one in F_2.
  - r_2 = 1+2+3x4 =15.
- r3 = 1*4+3*2 = 10
- r4 = 1*2 = 2
- 5!-7*4!+15*3!-10*2!+2*1! = 226

Chap. III
Pigeonhole principle

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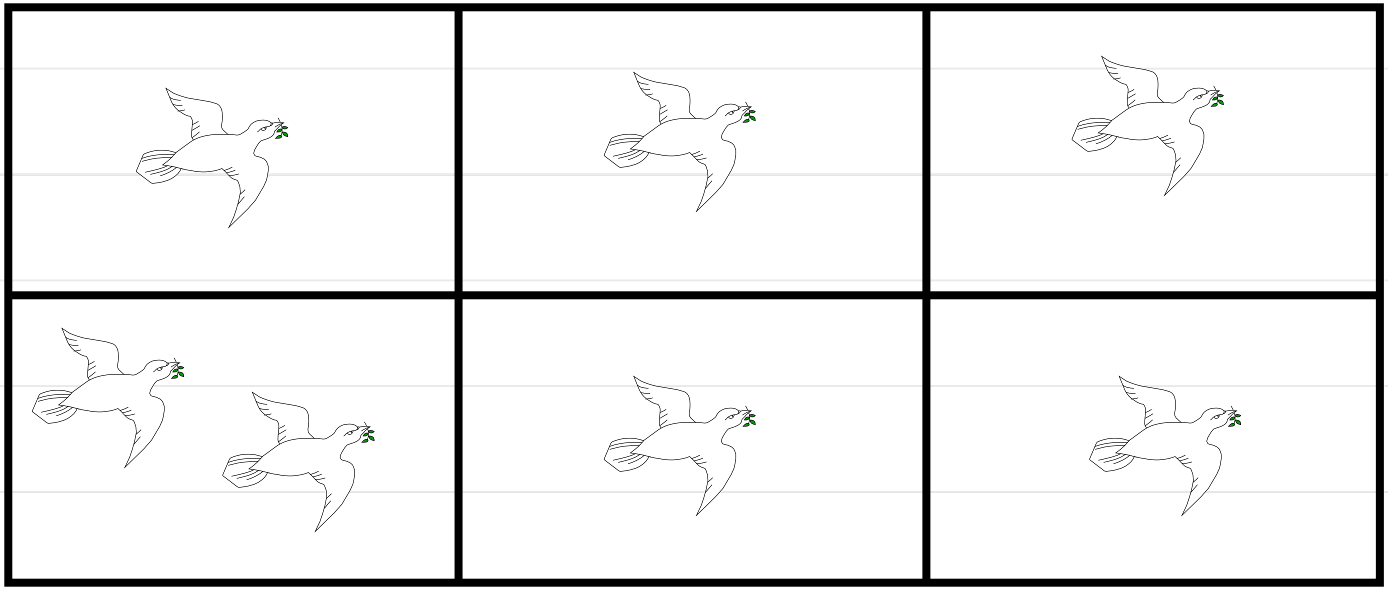
§ 3. Pigeonhole principle

If there is n+1 pigeons are flying to n holes,
then at least one hole contains two pigeons.

7 pigeons
6 holes



Variety of names: pigeonhole principle, Dirichlet drawer principle, shoebox principle……

Dirichlet

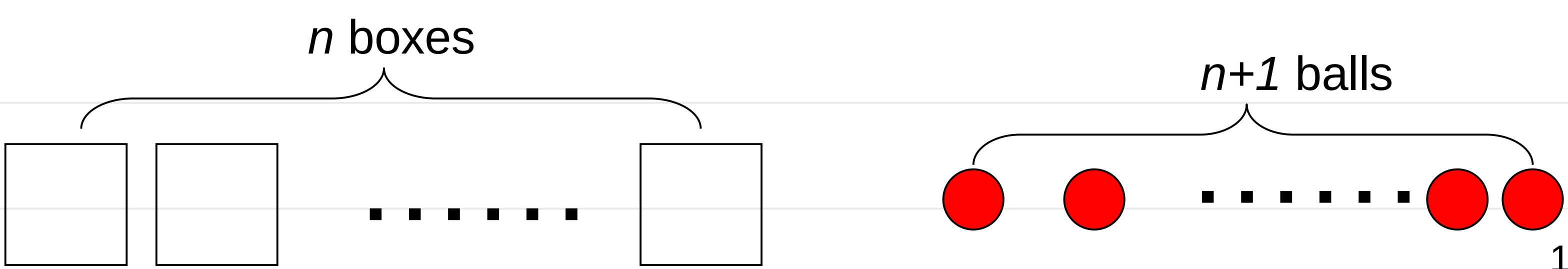
- **Johann Dirichlet**（1805～1859）
- German mathematician, credited with the modern formal definition of a function and the foundation of number theory.
  - Fermat's last theorem : no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2
    - In 1825, a partial proof for the case $n = 5$;
    - Later, a full proof for the case $n = 14$.
    - The first successful proof was released in 1994 by Andrew Wiles
  - Pigeonhole principle
    - In 1834, under the name *Schubfachprinzip* ("drawer principle" or "shelf principle")
    - Taken from: http://episte.math.ntu.edu.tw/people/p_dirichlet/

Simple Form

- **Theorem**. If $n+1$ objects are put into n boxes, then at least one box contains two or more of the objects.
- **Proof.**

Proof by contradiction

If each of the n boxes contains at most one of the objects, then the total number of objects is at most n.
Since we start with n+1 objects, some box contains at least two of the objects. □



Another Form

- **Pigeonhole principle** states that if n items are put into m holes with $n > m$, then at least one hole must contain more than one item.
- Example. Among 400 people there are two who have the same birthday.

单选题 1分

设置

Example There are 4 pairs of red socks, 5 pairs of pink socks in a box. We randomly pick one sock from them for each time. How many picks are needed to guarantee that **a pair of socks** is selected?

A 2

B 3

C 4

D 5

提交

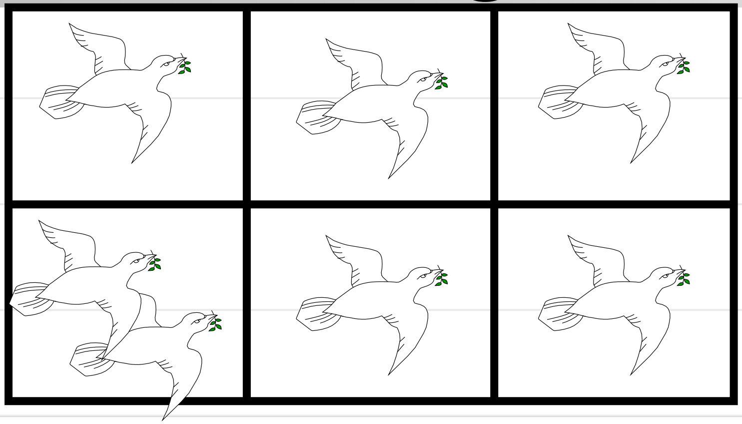
Simple Application

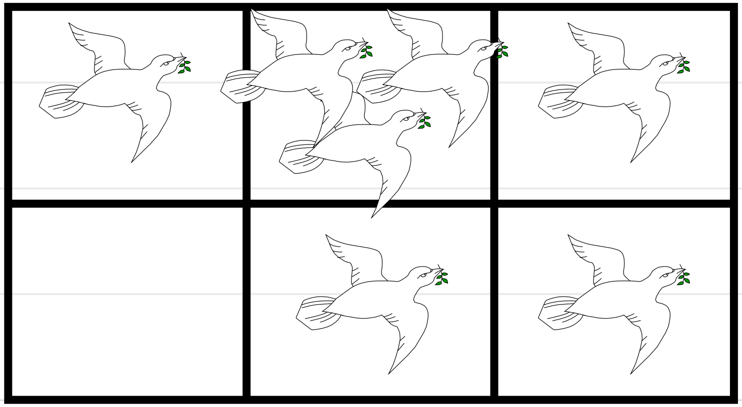
Example There are 4 pairs of red socks, 5 pairs of pink socks in a box. We randomly pick one sock from them for each time. How many picks are needed to guarantee that **a pair of socks** is selected?

m = 2 holes, using one pigeonhole per color

need only three socks (n = 3 items).

Pigeonhole Principle





- If n+1 objects are put into n boxes, then at least one box contains two or more of the objects.
  - Only guarantee the **existence**
  - No help in finding a box that contains two or more of the objects
  - Keys: what are pigeons and what are holes?

Generalized Pigeonhole Principle

GPP. If N objects are assigned to k boxs, then at least one box must be assigned at least $\lceil N/k \rceil$ objects.

Top integral function
Ceiling function

- *E.g.*, there are N=280 students in this class. There are k=52 weeks in the year.
  - Therefore, there must be at least 1 week during which at least $\lceil 280/52 \rceil = \lceil 5.38 \rceil = 6$ students in the class have his or her birthday in this week.

Proof of G.P.P.

G.P.P: If N objects are assigned to k boxs, then at least one box must be assigned at least $\lceil N/k \rceil$ objects.

- Proof By contradiction. Suppose every box has < $\lceil N/k \rceil$ objects, thus the number of objects in each box $\leq (\lceil N/k \rceil - 1)$.
- Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\left(\frac{N}{k}\right) = N$$

- There are less than N objects, which contradicts our assumption of N objects. The original statement is true.

Pigeonhole Principle

Mark Six (a lottery game)

49 labeled balls (1 to 49),
Draw 6 balls randomly and then
the 7th as a special number

Pigeonhole Principle

Every time there must be two numbers among the 6 such that the first digit is the same. (Assume 1=01, 2=02, 3=03, 4=04).

Date	Draw Number	Draw Results
20/12/2002	02/110	13 18 23 24 26 33 + 15
17/12/2002	02/109	6 18 39 40 41 42 + 9
12/12/2002	02/108	7 15 16 23 31 35 + 8
10/12/2002	02/107	5 36 37 38 46 49 + 17
05/12/2002	02/106	11 21 27 31 37 44 + 1
03/12/2002	02/105	9 11 14 17 24 28 + 46
28/11/2002	02/104	17 19 26 31 37 43 + 38
26/11/2002	02/103	19 21 40 42 46 47 + 33
21/11/2002	02/102	4 16 18 25 29 41 + 21
19/11/2002	02/101	3 15 22 23 42 47 + 18

Pigeonhole Principle

- Pick 6 master numbers every time.
- For every number, there are {0,1,2,3,4} 5 choices for the first digit;
- By pigeonhole principle, 6 pigeons are flying to 5 pigeonholes. So there’s at least one pigeonhole with 2 pigeons. This means that at least 2 numbers share their first-digits.

Pigeonhole Principle (2)

- There are 20 shirts in a drawer, in which 4 are blue, 7 are grey, 9 are red. How many shirts do we need to pick to ensure that we have at least 4 shirts in the same color?
- Pigeonhole Principle (2): n pigeonholes, kn+1 pigeons, at least 1 pigeonhole has k+1 pigeons.
- Solution: 3 colors, 3 pigeonholes, so k+1=4.
- K=3, kn+1=10, we need to pick at least 10 shirts.

单选题 1分

设置

There are 20 shirts in a drawer, in which 4 are blue, 7 are grey, 9 are red. How many shirts do we need to pick to ensure that we have at least 6 shirts in the same color?

A 15

B 16

C 18

D 20

提交

Pigeonhole Principle (2)

- There are 20 shirts in a drawer, in which there are **4** blue ones,7 are grey, 9 are red. How many do we need to pick to ensure 6 same-colored shirts?
- Solution:（for 6 same-colored shirts） If we pick 4 blue ones at first, then choosing from red and grey ones: n=2,k+1=5
- So we need to take 4+5×2+1=15 shirts to have 6 with the same color
- ……

Pigeonhole Principle

- We know n+1 positive integers, all of them are $\leq 2n$, prove that at least 2 of them are relatively prime.
- Famous Hungarian mathematician Paul Erdos (1913-1996) asked 11-year-old Louis Pósa this problem. Pósa answered it in half minute.
- （Hint ….）
- Pósa thought: take *n* boxes, put 1 and 2 in the first one, 3 and 4 in the second one, 5 and 6 in the third one, so forth, 2n-1 and 2n in the n^{th} one.
- Now we take n+1 numbers from n boxes, so at least one box would be emptied. So there must be a pair of adjacent numbers among these n+1 ones, and they are relatively prime.

主观题 10分

设置

Take any n+1 integers from 1 to 2n, among them there's at least one pair such that one is the multiple of the other.

作答

Pigeonhole Principle

Eg Take any n+1 integers from 1 to 2n, among them there's at least one pair such that one is the multiple of the other.

Proof Assume the n+1 numbers are $a_1, a_2, \cdots, a_{n+1}$.
Dividing 2's until all of them becomes odd numbers.
Then it construct a sequence $r_1, r_2, \cdots, r_{n+1}$.
These n+1 numbers are still in [1,2n] and they are all odd.
While there are only n odd numbers in [1,2n].
So There must be $r_i = r_j = r$, then $a_i = 2^{ki} r$, $a_j = 2^{kj} r$
If $a_i > a_j$, a_i is a multiple of a_j .

Pigeonhole Principle

Eg Assume $a_1, a_2, \cdots, a_{100}$ is a sequence consists of 1 and 2. And any subsequence of 10 consecutive in it has a sum that is ≤ 16:

$$a_i + a_{i+1} + \ldots + a_{i+9} \leq 16, \quad 1 \leq i \leq 91$$

So $\exists$ h and k such that $k > h$ and

$$a_h + a_{h+1} + \ldots + a_k = 39$$

Proof Let $S_j = \sum_{i=1}^{j} a_i, \quad j = 1, 2, \ldots, 100$

$S_1 < S_2 < \ldots < S_{100}$,

And $S_{100} = (a_1 + \ldots + a_{10})$

$+ (a_{11} + \ldots + a_{20}) + \ldots + (a_{91} + \ldots + a_{100})$

§ 3 .7 Pigeonhole Principle

According to assumption $a_i + a_{i+1} + \ldots + a_{i+9} \leq 16$， $1 \leq i \leq 91$

We have $S_{100} \leq 10 \times 16 = 160$

Create sequence $S_1, S_2, \ldots, S_{100}, S_1 + 39, \ldots, S_{100} + 39$.

With 200 terms. The largest term $S_{100} + 39 \leq 160 + 39 = 199$

By pigeonhole principle, there must be two equal terms.

And it must be a term in the first part and a term in the second part. Assume

$S_k = S_h + 39$， $k > h$ $S_k - S_h = 39$ So

$a_h + a_{h+1} + \ldots + a_k = 39$

Examples

Example: Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Hint. Consider the m sums

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \ldots,\ a_1+a_2+a_3+\ldots+a_m.$$

If any of these sums is divisible by m, then the conclusion holds.

Thus suppose that each of the sums has a non-zero remainder when divided by m, and so a remainder equal to one of 1, 2, … m-1.

Since there are m sums and only m-1 remainders , two of the sums have the same remainder when divided by m.

$a_1+a_2+a_3+\ldots+a_k = bm+r$ $\qquad$ $a_1+a_2+a_3+\ldots+a_l = cm+r \ \ (k<l)$

Subtracting: $a_{k+1}+a_{k+2}+a_{k+3}+\ldots+a_l=(c-b)m;$

Thus, $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Examples

Example: Given m integers a_1, a_2, …, a_m, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Let m=7, and let our integers be 2, 4, 6, 3, 5, 5 and 6.

Compute the sums of

	remainders when divided by 7 are
a1=2,	2
a1+a2=6,	6
a1+a2+a3=12,	5
a1+a2+a3+a4=15,	1
a1+a2+a3+a4+a5=20,	6
a1+a2+a3+a4+a5+a6=25,	4
a1+a2+a3+a4+a5+a6+a7=31,	3

a1+a2=6,

a1+a2+a3+a4+a5=20, $\Rightarrow$ a3+a4+a5= 6+3+5=14

Divisible by 7!

Examples

- Example: **Hand shaking problem**: If there are n number of people who can shake hands with one another (where $n > 1$), the pigeonhole principle shows that there is always a pair of people who will shake hands with the same number of people.
- Hint: As the 'holes', or m, correspond to number of hands shaken, and each person can shake hands with anybody from 0 to $n - 1$ other people
- $n - 1$ possible holes.
  - either the '0' or the '$n - 1$' hole must be empty
  - if one person shakes hands with everybody, it's not possible to have another person who shakes hands with nobody;
  - if one person shakes hands with no one there cannot be a person who shakes hands with everybody.
- This leaves n people to be placed in at most $n - 1$ non-empty holes, guaranteeing duplication.

Pigeonhole Principle: Strong Form

Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 object, ….., or the nth box contains at least q_n objects.

- Suppose that we distribute $q_1 + q_2 + \ldots + q_n - n + 1$ objects among n boxes.
- If for each $i = 1, 2, \ldots, n$ the ith box contains fewer than q_i objects
  - The total number of objects in all boxes does not exceed $(q_1-1) + (q_2-1) + \ldots + (q_n-1) = q_1 + q_2 + \ldots + q_n - n$.
- Since this number is one less than the number of objects distributed, we conclude that for some $i = 1, 2, \ldots, n$, the ith box contains at least q_i objects.

Application Examples

- A bag contains 100 apples, 100 bananas, 100 oranges and 100 pears. How many fruits should be taken out such that we can be sure a dozen pieces of them are of the same kind?
  - Let $q_1 = q_2 = \ldots = q_n = r$. The principle reads as follows: If $n(r\text{-}1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more the objects.
  - 4 boxes, $q_1 = q_2 = \ldots = q_n = 12$
  - If 4*(12-1)+1 = 45 fruits are taken out, then at least one of the boxes contains 12 fruits.

To Do List

- OJ tasks
- HW sheet
- Pre-class videos and quizzes
  - Generating Function
  - RainclassRoom