Summary Class

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Final Exam

- 4-5 questions for you to answer with explanation
- An answer with no explanation will receive no credit.
- 90 min
- Calculators are NOT necessary

Contents

Magic Square

4	9	2
3	5	7
8	1	6

A magic square: a square array of numbers in which the sum of all rows, all columns and both diagonals is the same.

- a **magic square** of order *n* is an arrangement of n^2 numbers, usually distinct integers, in a square, such that the *n* numbers in all rows, all columns, and both diagonals sum to the same constant.
- A **normal** magic square contains the integers from 1 to n^2.
- The constant sum in every row, column and diagonal is called the magic constant or magic sum, *M*.

$$Sum = 1 + 2 + 3 \ldots + n^2 = \frac{n^2(n^2+1)}{2} \quad \xrightarrow[n \text{ rows}]{n*M = Sum} \quad M = \frac{n\ (n^2+1)}{2}$$

Construction

8	1	6
3	5	7
4	9	2

Given {1,2,3…9},
how to construct a magic square?

- A method for constructing magic squares of odd order was published by the French diplomat de la Loubère in his book. (Du Royaume de Siam, 1693),
  - Starting from the central column of the first row with the number 1,
  - the fundamental movement for filling the squares is diagonally up and right, one step at a time.
  - If a filled square is encountered, one moves vertically down one square instead, then continuing as before.
  - When a move would leave the square, it is wrapped around to the last row or first column, respectively.

Chap.2 Permutations and combinations

- Permutation and combination?
  - If the order **does** matter:
    - **Permutation: P(n,r)**= $n!/(n-r)!$
  - If the order **doesn't** matter:
    - **Combination: C(n,r)**= $n!/r!(n-r)!$.

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)}$$

- Ordered arrangement
  - Without repeating any objects, distinct: **P(n,r)**
  - With repetition of objects permitted: label the repeated items
- Unordered arrangements
  - Without repeating any object: **C(n,r)**
  - With repetition of objects permitted (but perhaps limited).
    - $x_1+x_2+\ldots+x_k = r$

$$\overbrace{\underset{x_1}{1..1} \mid \underset{x_2}{1..1} \mid \underset{x_3}{1..1} \mid \qquad \mid \underset{x_k}{1..1}}^{\text{The number of '1's is r}}$$

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

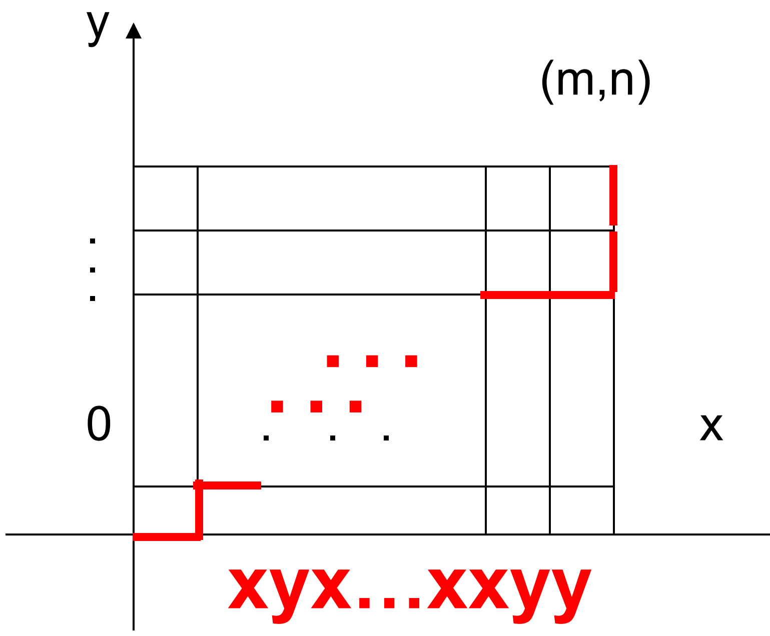
Combinations of Multisets

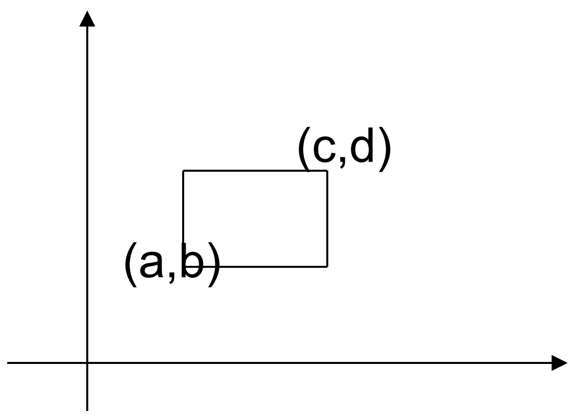
- If S is a multiset, then an *r-combination* of S is an unordered selection of r of the objects of S. Thus an r-combination is itself a multiset, a submultiset of S.
- Example. If $S = \{2\cdot a, 1\cdot b, 3\cdot c\}$
  - 3-combinations of S are $\{2\cdot a, 1\cdot b\}$, $\{2\cdot a, 1\cdot c\}$, $\{1\cdot a, 1\cdot b, 1\cdot c\}$, $\{1\cdot a, 2\cdot c\}$, $\{1\cdot b, 2\cdot c\}$, $\{3\cdot c\}$.
  - Let S be a multiset with objects of k different types where each has an infinite repetition number. Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

Lattice Path

- A path composed of connected horizontal and vertical line segments, each passing between adjacent lattice points。
- How many lattice paths from (0,0) to (m,n) ?
- **One-one correspondence**
  - Each path (0,0)→(m,n)
  - Arrangement with m *'x's and* n *'y's*
  - C(m+n,m)

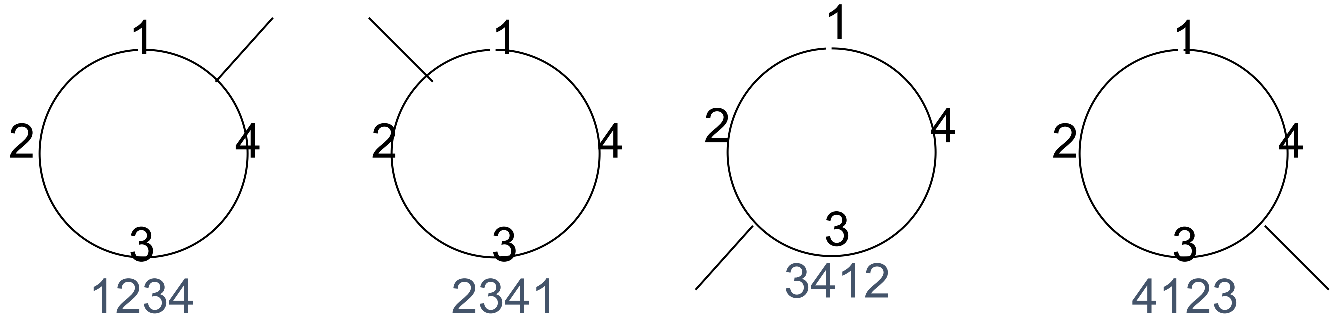




$$|(a,b)(c,d)| = \binom{(c-a)+(d-b)}{(c-a)}$$

Circular permutations

- The permutations that arrange objects in a line are called *linear permutations*. If the objects are arranged in a cycle, the permutations are called *circular permutations*.
- The number of circular r-permutations of a set of n elements is given by
$$\frac{p(n,r)}{r}=\frac{n!}{r(n-r)!}.$$
- In particular, the number of circular permutations of n elements is $(n-1)!$.



Summary

Sample	Order counts?	Repetition allowed?	Name	Number of ways
Choose 3 balls and put them in a box	No	No	r-combination	C(m,r)
People in a line	Yes	No	r-permutation	P(m,r)
Arrangement of fruits	No	Yes	r-combination of multi-sets	C(m+r-1,r)
4-letter word	Yes	Yes	r-permutation of multi-sets	m^r

Lexicographic Order

- **[Eg]**Calculate the next permutation of 839647521
- 1 Find the first decrease point from right to left: 4

839647521

- 2. Exchange: Find the smallest number larger than 4 in the suffix

839647521

- 3. Overturn the suffix 83965 1247

- The next permutation is: 839651247

PigeonHole

- Pigeonhole Principle: Simple Form
- If n+1 objects are put into n boxes, then at least one box contains two or more of the objects.
- Pigeonhole Principle: Strong Form
- Let $q1$, $q2$, …, qn be positive integers. If $q1$+ $q2$+ …+ qn- n +1 objects are put into n boxes, then either the first box contains at least $q1$ objects, or the second box contains at least $q2$ object, ….., or the nth box contains at least qn objects.
- Let $q_1 = q_2 = \ldots = q_n = r$. The principle reads as follows: If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more the objects.

Examples

- **A.5.** Given 101 integers from 1*, 2,…, 200, there are at least two integers such that one of them is divisible* by the other.
- *Proof. By factoring out as many 2's as possible, we see that any integer can be written in the form* $2^k \times a$*,*
  - $k \geq 0$ *and a is odd.*
  - *a can be one of the 100 numbers 1,3, 5,…,199.*
  - *Thus among the 101 integers* chosen, two of them must have the same *a's when they are written in this form*
    - $2^r \times a$ *and* $2^s \times a$ *with* $r \neq s$.
  - *If* $r < s$*, then the first one divides the second.*
  - *If* $r > s$*, then the second one divides the first.*

IEP

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |\mathrm{A} \cap \mathrm{C}| - |B \cap C| + |A \cap B \cap C|$$

Find A_i
Figure out the intersections
Apply IEP to get the final results

$$\left|A_1 \cup A_2 \cup \cdots \cup A_m\right| = \sum \left|A_i\right| - \sum \left|A_i \cap A_j\right| + \sum \left|A_i \cap A_j \cap A_k\right| - \cdots + (-1)^{m+1} \sum \left|A_1 \cap A_2 \cap \cdots \cap A_m\right|,$$

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}\right| = |S| - \sum \left|A_i\right| + \sum \left|A_i \cap A_j\right| - \sum \left|A_i \cap A_j \cap A_k\right| + \cdots + (-1)^m \sum \left|A_1 \cap A_2 \cap \cdots \cap A_m\right|$$

IEP: Inclusion-exclusion principle

Examples

Eg Calculate the number of primes which are ≤ 120.

As $11^2=121$, non-primes ≤ 120 must be multiples of 2, 3, 5 or 7, and their factors couldn't be larger than 11.

Suppose A_i is the set of i-multiples ≤ 120, i=2，3，5，7。

$$|A_2| = \left\lfloor \frac{120}{2} \right\rfloor = 60, |A_3| = \left\lfloor \frac{120}{3} \right\rfloor = 40,$$

$$|A_5| = \left\lfloor \frac{120}{5} \right\rfloor = 24, |A_7| = \left\lfloor \frac{120}{7} \right\rfloor = 17,$$

$$|A_2 \cap A_3| = \left\lfloor \frac{120}{2\times 3} \right\rfloor = 20, |A_2 \cap A_5| = \left\lfloor \frac{120}{10} \right\rfloor = 12,$$

$$|A_2 \cap A_7| = \left\lfloor \frac{120}{14} \right\rfloor = 8, |A_3 \cap A_5| = \left\lfloor \frac{120}{15} \right\rfloor = 8,$$

$$|A_3 \cap A_7| = \left\lfloor \frac{120}{21} \right\rfloor = 5, |A_5 \cap A_7| = \left\lfloor \frac{120}{35} \right\rfloor = 3,$$

$$|A_2 \cap A_3 \cap A_5| = \left\lfloor \frac{120}{2 \times 3 \times 5} \right\rfloor = 4,$$

$$|A_2 \cap A_3 \cap A_7| = \left\lfloor \frac{120}{2 \times 3 \times 7} \right\rfloor = 2,$$

$$|A_2 \cap A_5 \cap A_7| = \left\lfloor \frac{120}{2 \times 5 \times 7} \right\rfloor = 1,$$

$$\begin{aligned}
\left|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}\right| &= 120 - |A_2| - |A_3| - |A_5| \\
&\quad - |A_7| + |A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| \\
&\quad + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7| \\
&\quad - |A_2 \cap A_3 \cap A_5| - |A_2 \cap A_3 \cap A_7| \\
&\quad - |A_2 \cap A_5 \cap A_7| - |A_3 \cap A_5 \cap A_7| \\
&\quad + |A_2 \cap A_3 \cap A_5 \cap A_7| \\
&= 120 - (60 + 40 + 24 + 17) + (20 + 12 + 8 \\
&\quad + 8 + 5 + 3) - (4 + 2 + 1 + 1) \\
&= 27.
\end{aligned}$$

NOTE: as 2, 3, 5, and 7 are excluded in these 27 numbers while 1 is included, the number of primes ≤ 120 are:

27+4-1=30

Integers which are larger than 1 and has no other factors except itself and 1 is called primes

Eg Calculate the number of non-negative integer roots of $x_1+x_2+x_3=15$

Limitation is : $0\le x_1 \le 5$, $0\le x_2 \le 6$, $0\le x_3\le 7$。

Solution： The number of non-negative integer root of $x_1+x_2+\ldots+x_n=b$ is $C(n+b-1,b)$

The number of non-negative roots of $x_1+x_2+x_3=15$ without limitation is $C(15+3-1,15)= C(17,2)$

Assume A1 is the solution when $x_1\ge 6$, $y_1+6+x_2+x_3=15$

$|A1|= C(9+3-1,9)= C(11,2)$

Assume $A2$ is the solution when $x_2\ge 7$, $x_1+y_2+7+x_3=15$

$|A2|= C(8+3-1,8)= C(10,2)$

Assume $A3$ is the solution when $x_3\ge 8$, $x_1+x_2+ y_3+8=15$

$|A3|= C(7+3-1,7)= C(9,2)$

Eg Calculate the number of non-negative solutions of $x_1+x_2+x_3=15$

Limitation is: $0\le x_1 \le 5$, $0\le x_2 \le 6$; $0\le x_3\le 7$

Solution: Without limitation, $x_1+x_2+x_3=15$ has $C(15+3-1,15)= C(17,2)$ non-negative solutions.

$|A1|= C(9+3-1,9)= C(11,2)$

$|A2|= C(8+3-1,8)= C(10,2)$

$|A3|= C(7+3-1,7)= C(9,2)$

$A1\cap A2$: $y_1+6+y_2+7+x_3=15$ $|A1\cap A2| = C(2+3-1,2) = C(4,2)$

$A1\cap A3$: $y_1+6+x_2+y_3+8=15$ $|A1\cap A3| = C(1+3-1,1)= C(3,1)$

$A2\cap A3$: $x_1+y_2+7+y_3+8=15$ $|A2\cap A3| = 1$

$A1\cap A2 \cap A3$: $y_1+6+y_2+7+y_3+8=15$; $|A1\cap A2 \cap A3 |= 0$

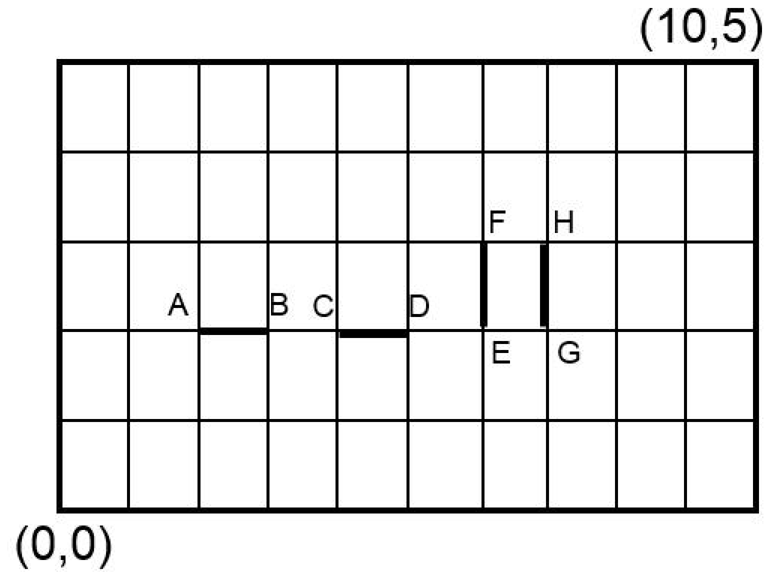
$|\overline{A1}\cap \overline{A2} \cap \overline{A3}|=C(17,2) - C(11,2) - C(10,2) - C(9,2)$
$+C(4,2) + C(3,1)+1=10$

Applications of Inclusion-Exclusion principals

Eg Lattice Path with barriers:

How many paths go from (0,0) to (10,5) without passing AB, CD, EF, GH?

The coordinates of the points are A(2,2),B(3,2),C(4,2),D(5,2),E(6,2),F(6,3),G(7,2),H(7,3)



Solution: Total number of paths:C(15,5);

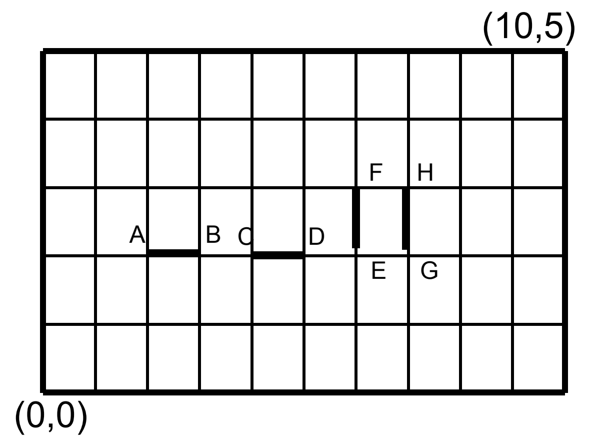
Paths that pass AB: $|A_1|$=C(2+2,2)C(7+3,3);

Paths that pass CD: $|A_2|$=C(4+2,2)C(5+3,3);

Paths that pass EF: $|A_3|$=C(8,2)C(6,2);

Paths that pass GH: $|A_4|$=C(9,2)C(5,2);

Applications of Inclusion-Exclusion Principle



Paths that pass AB, CD:

$|A_1 \cap A_2| = C(4,2)C(8,3)$;

Paths that pass AB, EF:

$|A_1 \cap A_3| = C(4,2)C(6,2)$;

Paths that pass AB, HG:

$|A_1 \cap A_4| = C(4,2)C(5,2)$;

Paths that pass CD, EF:

$|A_2 \cap A_3| = C(6,2)C(6,2)$;

Paths that pass CD, HG:

$|A_2 \cap A_4| = C(6,2)C(5,2)$;

Paths that pass EF, HG:

$|A_3 \cap A_4| = 0$;

Paths that pass AB, CD, EF:

$|A_1 \cap A_2 \cap A_3| = C(4,2)C(6,2)$;

Paths that pass AB, CD, HG:

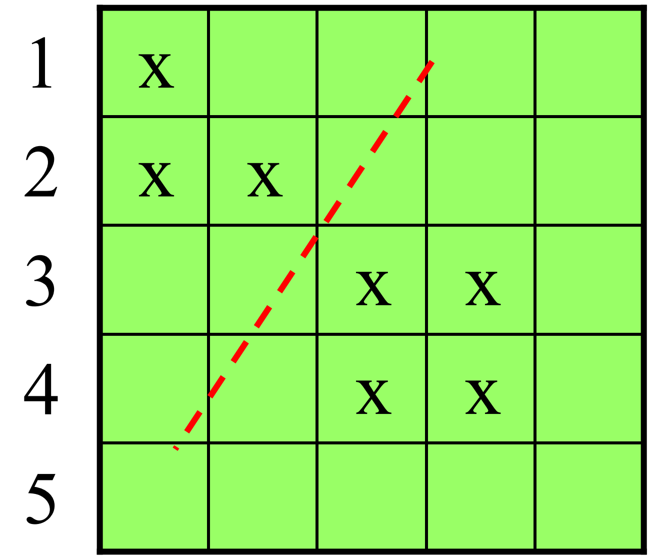
$|A_1 \cap A_2 \cap A_4| = C(4,2)C(5,2)$;

$|A_2 \cap A_3 \cap A_4| = 0$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 2049$$

An example



- Determine the number of ways to place 5 non-attacking rooks on the following 5-by-5 board, with forbidden positions as shown.
- r1 = 7
- The set of forbidden positions can be partitioned into two “independent” parts
  - “Independent” means squares in different parts do not belong to a common row or column.
  - one part F_1 containing three positions and the other part F_2 containing four.
- r2: The rooks may be both in F_1, both in F_2 or one in F_1 and one in F_2.
  - $r_2 = 1+2+3x4 = 15$.
- r3 = 1*4+3*2 = 10
- r4 = 1*2 = 2
- 5!-7*4!+15*3!-10*2!+2*1! = 226

Generating Function

- Generating Function **Combinations**
  - $G(x) = h_0+h_1x+h_2x^2+\ldots+h_nx^n+\ldots$
  - A **generating function** is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers h_n that is indexed by the natural numbers.

$$(1-x)^{-1} = 1 + x + x^2 + \ldots.. \qquad (1-ax)^{-1} = 1 + ax + a^2x^2 + \ldots..$$

- Exponential Generating Function **Permutations**
  - The exponential generating function for the sequence $h_0,h_1,\ldots h_n,\ldots$ is defined to be

$$g^e(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = h_0 + h_1x + h_2\frac{x^2}{2!} + \ldots. + h_n\frac{x^n}{n!}$$

Linear Homogeneous Recurrence Relations

- Let $h_0,h_1,h_2,\ldots,h_n,\ldots$ be a sequence of numbers. This sequence is said to satisfy a linear recurrence relation of order k, provided that there exist quantities $a_1, a_2, \ldots,a_k$, with $a_k \neq 0$, and a quantity b_n (each of these quantities, $a_1, a_2, \ldots,a_k$, b_n may depend on n) such that $h_n=a_1h_{n-1}+a_2h_{n-2}+\ldots+a_kh_{n-k}+b_n$
- If $b_n=0$ and $a_1, a_2, \ldots,a_k$ are constants
- The recurrence relations of the form
- $h_n=a_1h_{n-1}+a_2h_{n-2}+\ldots+a_kh_{n-k}$ is **linear homogeneous** recurrence relations

- Hanoi problem

$$h(n)-2h(n-1)=1$$

$$h(n-1)-2h(n-2)=1 \text{ substract}$$

$$\boxed{h(n)-3h(n-1)+2h(n-2)=0}$$

$$H(x)=\frac{x}{(1-x)(1-2x)}=\frac{x}{1-3x+2x^2}$$

$$\boxed{C(x) = x^2-3x+2}$$

The root of C(x)=0 is 1 and 2

Characteristic equation

- For a sequence $\{h_n\}$, it has the k-order linear homogeneous recurrence relation as
- Relations: $$h_n + C_1 h_{n-1} + C_2 h_{n-2} + \cdots + C_k h_{n-k} = 0,$$ $f_n - f_{n-1} - f_{n-2} = 0$
- Initial values: $$h_0 = d_0, h_1 = d_1, \cdots, h_{k-1} = d_{k-1},$$ $f(0)=0 \quad f(1)=1$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{k-1}$ are constants.

- The characteristic equation for $\{h_n\}$

$$C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k$$

$C(x)=x^2-x-1=0$

- Suppose there are k *distinct* roots for $C(x)$

$$C(x) = (x - a_1)(x - a_2) \cdots (x - a_i)$$

$$q_1 = \frac{1+\sqrt{5}}{2}, q_2 = \frac{1-\sqrt{5}}{2}$$

- Then the ***explicit formula of h_n***

$$h_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

  - l_i : undetermined coefficient
- l_i can be determined using the initial values
  - $n=0$, $f(0)=0$: $c_1+c_2=0$
  - $n=1$, $f(1)=1$: $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \Rightarrow c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = \frac{-1}{\sqrt{5}}$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Summary of Linear Recurrence Relation

According to the non-zero roots of C(x)

1) k distinct non-0 real roots $C(x)=(x-a_1)(x-a_2)\cdots(x-a_k)$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

In which $l_1, l_2, \cdots, l_k$, are undetermined coefficients.

2)A pair of conjugate complex root $\alpha_1 = \rho e^{i\theta}$ and $\alpha_2 = \rho e^{-i\theta}$:

$$\boldsymbol{a_n} = \boldsymbol{A}\rho^{\boldsymbol{n}} \cos \boldsymbol{n}\theta + \boldsymbol{B}\rho^{\boldsymbol{n}} \sin \boldsymbol{n}\theta$$

In which A， B are undetermined coefficients.

3)Has root α_1 with multiplicity of k.

$$(A_0 + A_1 n + \cdots + A_{k-1} n^{k-1})\alpha_1^n$$

In which $A_0, A_1, \cdots, A_{k-1}$ are k undetermined coefficients.

Recurrence Relation

How many n-digit positive decimal numbers in which the digit “5” appears even times.

Suppose a_n is the number of positive decimals with even 5s,

b_n is the number of decimals with odd 5s

$a_n + b_n = 9 \times 10^{n-1}$(the first digit can not be 0)，

Let $A(x) = a_0 + a_1x + a_2x^2 + \ldots$,

We can classify a_n by the last digit：

The last digit is not 5: $9a_{n-1}$

The last digit is 5, the other digits has odd 5s：

$$b_{n-1} = 9 \times 10^{n-2} - a_{n-1}$$

$$a_n = 9a_{n-1} + 9 \times 10^{n-2} - a_{n-1}$$

$$\therefore \quad a_n = 8a_{n-1} + 9 \times 10^{n-2}, \qquad a_1 = 8$$

$$a_n - 8a_{n-1} = 9\times 10^{n-2} \qquad a_{n-1} - 8a_{n-2} = 9\times 10^{n-3}$$

$$10a_{n-1} - 80a_{n-2} = 9\times 10^{n-2}$$

$$a_n \text{-} 18a_{n-1} + 80a_{n-2} = 0$$

- The Characteristic equation C(x):

$$x^2 \text{-} 18x + 80 = 0 \qquad x_1 = 8, x_2 = 10 \qquad a_n = l_1 8^n + l_2 10^n$$

a_1=8,a_2=73

$$a_1 = l_1 8 + l_2 10 = 8$$
$$a_2 = l_1 64 + l_2 100 = 73$$

$$l_1 = \frac{7}{16}, l_2 = \frac{9}{20}$$

$$\therefore \quad a_k = \frac{7}{16}\cdot 8^k + \frac{9}{20}\cdot 10^k$$

With digits 1, 3, 5, 7, 9, how many n-digit numbers are there, where 3 and 7 appear an even number of times, and 1, 5, 9 don't have any conditions.

Assume an r-digit number that satisfies the conditions is a_r. Then, the exponential generating function of the seq. of $a_1, a_2, \ldots, a_3$ is:

$$G_e(x) = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)^2 (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^3$$

$$G_e(x)=(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)^2(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)^3$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots,\ e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots,$$

$$\therefore\qquad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\frac{1}{2}(e^x+e^{-x}).$$

$$G_e(x)=\frac{1}{4}(e^x+e^{-x})^2e^{3x}=\frac{1}{4}(e^{2x}+2+e^{-2x})e^{3x}$$

$$=\frac{1}{4}(e^{5x}+2e^{3x}+e^x)=\frac{1}{4}(\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n+2\sum_{n=0}^{\infty}\frac{3^nx^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!})$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}(5^n+2\cdot 3^n+1)\frac{x^n}{n!}.$$

$$\therefore\qquad a_n=\frac{1}{4}(5^n+2\cdot 3^n+1).$$

Catalan Numbers

- Named after the Belgian mathematician Eugene Charles Catalan (1814–1894)
- OEIS A000108
- 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, ...

$$C(n) = C(0)*C(n\text{-}1) + C(1)*C(n\text{-}2) + \ldots\ldots + C(n\text{-}2)*C(1) + C(n\text{-}1)*C(0)$$

$$C_n = C(2n,n) - C(2n,n-1)$$

$$= \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k}$$

Augmented form

Turn the following linear program into augmented form

$$\min Z = -2x_1 + x_2 + 3x_3$$

$$\begin{cases} 5x_1 + x_2 + x_3 \leq 7 \\ x_1 - x_2 - 4x_3 \geq 2 \\ -3x_1 + x_2 + 2x_3 = -5 \\ x_1, x_2 \geq 0, x_3 \end{cases}$$

$$\sum a_{ij}x_j \leq b_i \Rightarrow \sum a_{ij}x_j + x_{n+i} = b_i$$

$$\sum a_{ij}x_j \geq b_i \Rightarrow \sum a_{ij}x_j - x_{n+i} = b_i$$

let $x_3 = x_4 - x_5$, and x_4, x_5 are non-negative

$$\max \quad Z = 2x_1 - x_2 - 3(x_4 - x_5) + 0x_6 + 0x_7$$

$$\begin{cases} 5x_1 + x_2 + (x_4 - x_5) + x_6 = 7 \\ x_1 - x_2 - 4(x_4 - x_5) \quad -x_7 = 2 \\ 3x_1 - x_2 - 2(x_4 - x_5) = 5 \\ x_1, x_2, x_4, x_5, x_6, x_7 \geq 0 \end{cases}$$

- Objective: min/max $\rightarrow$ max
- Variables: all the variables are non-negative
- Constant term: non-negative
- Constraints : replace non-equalities with equalities

Assignment

- Solving the following linear program using simplex method

$$\max Z = 3x_1 + 6x_2 + 2x_3$$

$$\begin{cases} 3x_1 + 4x_2 + x_3 \le 2 \\ x_1 + 3x_2 + 2x_3 \le 1 \\ \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

$$\max Z = 3x_1 + 6x_2 + 2x_3$$

$$\begin{cases} 3x_1 + 4x_2 + x_3 \le 2 \\ x_1 + 3x_2 + 2x_3 \le 1 \\ \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

$$\max Z = 3x_1 + 6x_2 + 2x_3$$

$$\begin{cases} 3x_1 + 4x_2 + x_3 + x_4 = 2 \\ x_1 + 3x_2 + 2x_3 + x_5 = 1 \\ x_1, x_2, x_3, x_4, x_5 \ge 0 \end{cases}$$

c_j			3	6	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	
0	x_4	2	3	4	1	1	0	2/4
0	x_5	1	1	3	2	0	1	1/3
-Z		0	3	6 ↑	2	0	0	

c_j			3	6	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	
0	x_4	2	3	4	1	1	0	2/4
0	x_5	1	1	3	2	0	1	1/3
-Z		0	3	6 ↑	2	0	0	

c_j			3	6	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	
0	x_4	2/3	5/3	0	-5/3	1	-4/3	2/5
6	x_2	1/3	1/3	1	2/3	0	1/3	1
-Z		-2	1 ↑	0	-2	0	-2	

c_j			3	6	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	
0	x_4	2/3	5/3	0	-5/3	1	-4/3	2/5
6	x_2	1/3	1/3	1	2/3	0	1/3	1
-Z		-2	1 ↑	0	-2	0	-2	

c_j			3	6	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	
3	x_1	2/5	1	0	-1	3/5	-4/5	
6	x_2	1/5	0	1	1	-1/5	3/5	
-Z		-12/5	0	0	-1	-3/5	-6/5	

Optimal solution:

$x_1=2/5, x_2=1/5 \quad z=12/5$

Good Luck to you all