Combinatorics HW Generating Function and Integer Partition

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1. **Integer composition: Integer 5 is partitioned into orderly partitions which are made up by numbers 1,2,3,4. Such as (1+1+3, or 1+3+1 or 2+3, 4+1,....) How many different ways are there?**

   Since order is considered, since no location can contain 0, then r-1 partitions could be used for n-1 available locations, giving c(n-1, r-1). Therefore, since we need at least 2 numbers to create 5, then r would be in range [2, 5]. Accordingly, we should count the number of each possible orderly r-partition for n, where n is 5, giving

   $$\sum_{r=2}^{5} C(5-1, r-1) = c(4,1) + c(4,2) + c(4,3) + c(4,4) = 4 + 6 + 4 + 1 = 15$$

   Therefore, there are 15 different ways to orderly partition Integer 5.

2. **Integer partition: How many ways to partition n into several numbers that the order between numbers is ignored. Please write the corresponding generating function.**

   Since order is not considered, various numbers of each digit could be used to construct integer n. For instance, integer 1 being chosen 0, 1, 2, ... times would have generating function $(1 + x + x^2 + \ldots)$ and so on, which gives the following generative function for n:

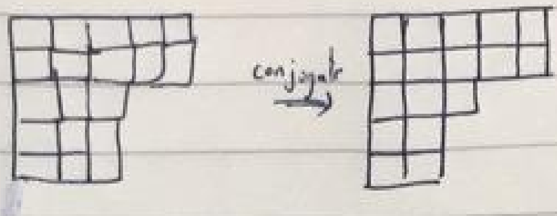
   $$G(x) = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)\ldots(1 + x^n + x^{2n} + \ldots)$$
   $$G(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\ldots\left(\frac{1}{1-x^n}\right) = \prod_{i=1}^{n}\left(\frac{1}{1-x^i}\right)$$

   Hence, the coefficient of x^n in the expanded equation would be the number of ways that n could be partitioned into several numbers regardless of their order.

3. **Provide proof that the partition number for integer n using different odd numbers (ordering is ignored), equals to the partition number of n being partitioned into the self-conjugated Ferrers Diagrams.**
   (1st row exchanged with 1st column, 2nd row exchanged with 2nd column, …, as image is rotated by the dotted line as axis shown in slices; is still Ferrers diagram. 2 Ferrers diagrams are known as a pair of conjugated Ferrers diagrams. The diagram is called self-conjugated if its conjugated diagram is the same with the original diagram.)

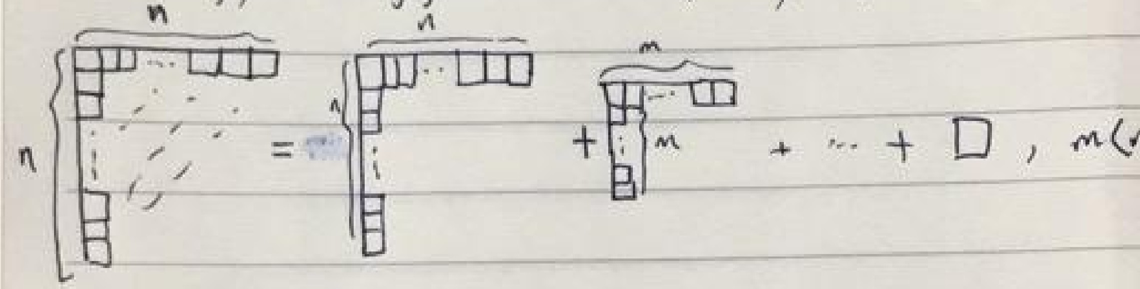
Self-conjugated diagram ⇒ conjugate looks similar to the original

For example



⇒ self-conjugated

Accordingly self-conjugates could be generally represented as



Hence, each self-conjugate can be divided into L-shaped parts that consist of odd numbers of □ (for example, $\overbrace{2n-1}^{odd}, \overbrace{2m-1}^{odd}, \dots, \overbrace{1}^{odd}$). Therefore, the partition number of n partitioned into a self-conjugated Ferrers diagram would be the same as partition number of n using different odd numbers ∴