Linear Programming

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$$\max(\min)\ Z = \sum c_j x_j$$
$$\begin{cases} \sum a_{ij} x_j \le (\ge =) b_i & (i = 1 \cdot 2 \cdots m) \\ x_j \ge 0 & (j = 1 \cdot 2 \cdots l\,) \end{cases}$$

$$\longrightarrow$$

$$\max\ \boldsymbol{Z} = \sum \boldsymbol{c_j x_j}$$
$$\begin{cases} \sum \boldsymbol{a_{ij} x_j} = \boldsymbol{b_i} & (i = 1 \cdot 2 \cdots m) \\ \boldsymbol{x_j} \ge 0 & (j = 1 \cdot 2 \cdots n\,) \end{cases}$$

Augmented Form

- Transformation:
  - Objective: min/max → max

$$\min \boldsymbol{Z} = \sum \boldsymbol{c_j x_j} \xrightarrow{\text{Multiply -1}} \max \boldsymbol{Z}' = -\boldsymbol{Z} = -\sum \boldsymbol{c_j x_j}$$

  - Variables: all the variables are non-negative

x_k has no constraint then let $x_k = x_k' - x_k''$, and x_k', x_k'' are non-negative

  - Constant term: non-negative

turn b_n to $-b_n$ by multiplying (-1) on both sides

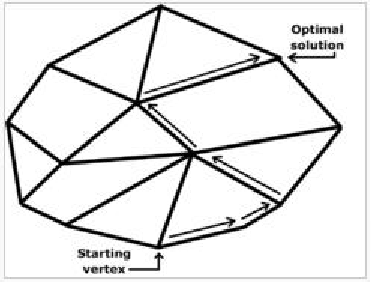
  - Constraints : replace non-equalities with equalities

non-negative slack variables

$$\sum \boldsymbol{a_{ij} x_j} \le \boldsymbol{b_i} \longrightarrow \sum a_{ij} x_j + x_{n+i} = b_i \qquad x_{n+i} \ge 0$$

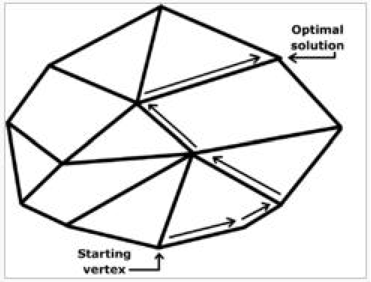
$$\sum \boldsymbol{a_{ij} x_j} \ge \boldsymbol{b_i} \longrightarrow \sum a_{ij} x_j - x_{n+i} = b_i \qquad x_{n+i} \ge 0$$

Simplex Method

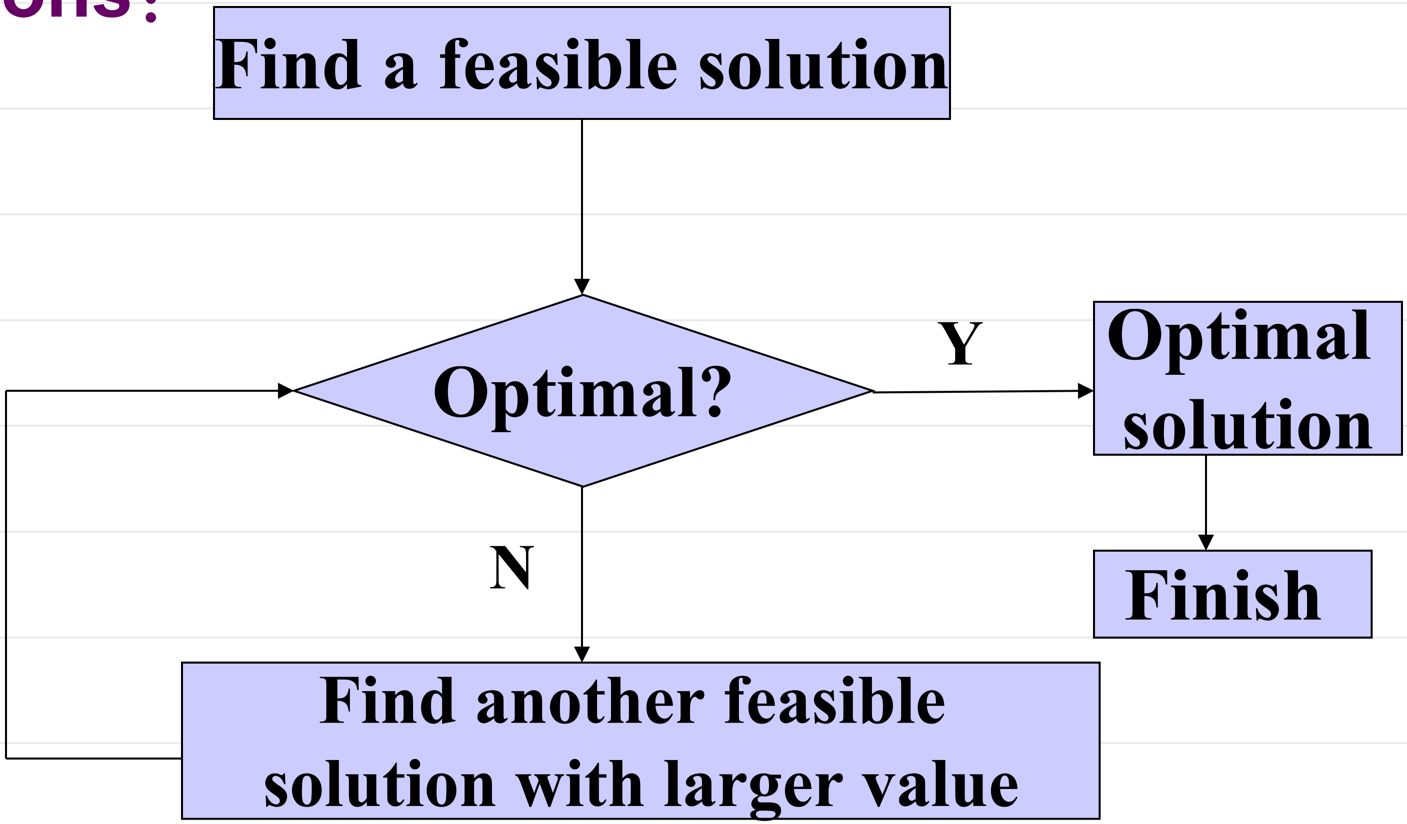


1. Begin the search at an extreme point (i.e., a basic feasible solution).
   - If $X = (x_1 \ldots x_n\ x_{n+1} \cdots x_{n+m})^T$ is an extreme point, then the coefficient vectors of non-zero variables x_i are linear independent.
2. Determine if the movement to an adjacent extreme point can improve on the objective function. If not, the current solution is optimal. If, however, improvement is possible, then proceed to the next step.
3. Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.
4. Continue steps 2 and 3 until the optimal solution is found or it can be shown that the problem is either unbounded or infeasible.

Simplex Method



Iterations：



$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} = [\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \boldsymbol{p}_3 \quad \boldsymbol{p}_4 \quad \boldsymbol{p}_5 \quad \boldsymbol{p}_6]$$

$$\boldsymbol{B} = [\boldsymbol{p}_3 \quad \boldsymbol{p}_4 \quad \boldsymbol{p}_5 \quad \boldsymbol{p}_6] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

Basic vectors

$\therefore\ x_3, x_4, x_5, x_6$ **are the basic variables**

x_1, x_2 **Are non-basic variables**

Accordingly, the feasible solution is（0 0 12 8 16 12）
while the objective is Z = 0

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{cases}$$

$$\begin{array}{cc} \mathbf{A} & \mathbf{b} \\ A = \begin{bmatrix} 2 & \boxed{2} & 1 & 0 & 0 & 0 \\ 1 & \boxed{2} & 0 & 1 & 0 & 0 \\ 4 & \boxed{0} & 0 & 0 & 1 & 0 \\ 0 & \boxed{4} & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} 12 \\ 8 \\ 16 \\ 12 \end{matrix} \end{array}$$

Let x_2 to be non-zero (to be basic variable)

x_1=0, some other variable should be 0 among x_3,x_4,x_5,x_6

$$x_3 = 12 - 2x_2 \ge 0$$

$$x_4 = 8 - 2x_2 \ge 0$$

$$x_5 = 16 \ge 0$$

$$\boxed{x_6 = 12 - 4x_2 \ge 0}$$

All variables non-negative

$$\beta_i \quad x_2 = \min\left(\frac{12}{2}, \frac{8}{2}, \frac{12}{4}\right) = 3$$

x_6=0, become non-basic variable to exchange with x_2

Then:

$$x_3 + 2x_2 = 12 - 2x_1 \quad (1)$$

$$x_4 + 2x_2 = 8 - x_1 \quad (2)$$

$$x_5 = 16 - 4x_1 \quad (3)$$

$$4x_2 = 12 - x_6 \quad (4)$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

Since (4)', we have $Z = 2x_1 + 9 + 0x_3 + 0x_4 + 0x_5 - \frac{3}{4}x_6$

$$P = \begin{array}{c} \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & -0.5 \\ 1 & 0 & 0 & 1 & 0 & -0.5 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.25 \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 2 \\ 16 \\ 3 \end{array}$$

Gaussian elimination to transform the coefficients for x_2 into I:

$$(4)' = \frac{(4)}{4}, (1)' = (1) - 2 \times (4)', (2)' = (2) - 2 \times (4)', (3)' = (3)$$

$$x_3 = 6 - 2x_1 + \frac{1}{2}x_6 \quad (1)'$$

$$x_4 = 2 - x_1 + \frac{1}{2}x_6 \quad (2)'$$

$$x_5 = 16 - 4x_1 \quad (3)'$$

$$x_2 = 3 - \frac{1}{4}x_6 \quad (4)'$$

$$\begin{aligned} x_3 \quad + 2x_2 &= 12 - 2x_1 & (1) \\ x_4 \quad + 2x_2 &= 8 - x_1 & (2) \\ x_5 \quad &= 16 - 4x_1 & (3) \\ 4x_2 &= 12 - x_6 & (4) \end{aligned}$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

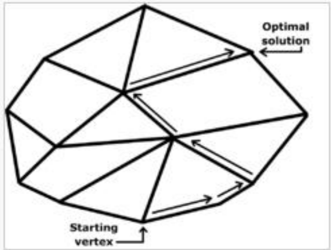
$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

$$Z = 2x_1 + 9 - \frac{3}{4}x_6 = 9 + 2x_1 - \frac{3}{4}x_6$$

$x_1 = x_6 = 0,\ Z = 9 \quad (0,\ 3,\ 6,\ 2,\ 16,\ 0)$

we can increase x_1 to enlarge Z.

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	6	2	0	1	0	0	-1/2	6/2
0	x_4	2	1	0	0	1	0	-1/2	2
0	x_5	16	4	0	0	0	1	0	16/4
3	x_2	3	0	1	0	0	0	1/4	—
-Z		-9	2 ↑	0	0	0	0	-3/4	



- Solves LP problems by constructing a feasible solution at a vertex of the polytope and then walking along a path on the edges of the polytope to vertices with non-decreasing values of the objective function until an optimum is reached.
  1. Locate an extreme point of the feasible region.
  2. Examine each boundary edge intersecting at this point to see whether movement along any edge increases the value of the objective function.
  3. If the value of the objective function increases along any edge, move along this edge to the adjacent extreme point. If several edges indicate improvement, the edge providing the greatest rate of increase is selected.
  4. Repeat steps 2 and 3 until movement along any edge no longer increases the value of the objective function.

Objective: $Z=C_BP0=C_BB^{-1}b$

$$A=\begin{array}{c} \mathbf{A} \\ \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{4} & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \mathbf{b} \\ \mathbf{12} \\ \mathbf{8} \\ \mathbf{16} \\ \mathbf{12} \end{array}$$

Transformation with B^{-1}

$$P=\begin{array}{c} \mathbf{P} \\ \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & -0.5 \\ 1 & 0 & 0 & 1 & 0 & -0.5 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.25 \end{bmatrix} \end{array} \quad \begin{array}{c} \mathbf{P0} \\ \mathbf{6} \\ \mathbf{2} \\ \mathbf{16} \\ \mathbf{3} \end{array}$$

$B^{-1}(\ b\ A_1\ A_2\ \ldots\ A_{n+m}\ )=(\ P0\ P1\ P2\ \ldots\ Pn+m\ ).$

$$\max\ Z = \sum c_j x_j$$

$$\begin{cases} \sum a_{ij}x_j = b_i & (\mathrm{i}=1\cdot 2\cdots \mathrm{m}) \\ x_j \geq 0 & (\mathrm{j}=1\cdot 2\cdots \mathrm{n}) \end{cases}$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

$$\begin{array}{cc} \mathbf{A} & \mathbf{b} \\ A=\begin{bmatrix} 2 & \boxed{2} & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} \mathbf{12} \\ \mathbf{8} \\ \mathbf{16} \\ \mathbf{12} \end{matrix} \end{array}$$

Let x_2 to be non-zero (to be basic variable)

x_1=0, some other variable should be 0 among x_3, x_4, x_5, x_6

$$x_3 = 12 - 2x_2 \geq 0$$

$$x_4 = 8 - 2x_2 \geq 0$$

$$x_5 = 16 \geq 0$$

$$\boxed{x_6 = 12 - 4x_2 \geq 0}$$

All variables non-negative

$$\beta_i \quad x_2 = \min\left(\frac{12}{2}, \frac{8}{2}, \frac{12}{4}\right) = 3$$

x_6=0, become non-basic variable to exchange with x_2

Then: $X_2 = (\alpha_1 - \beta\alpha_1 \ \ \alpha_2 - \beta\alpha_{2j} \ \ \ldots \ 0 \ \ldots \ \alpha_m - \beta\alpha_{mj} \ \ 0 \ \ldots \ \beta \ \ldots \ 0)^T$

$$z(X_2) = \sum_{i=1}^{m} c_i(\alpha_i - \beta\alpha_{ij}) + c_j\beta$$

$$= \sum_{i=1}^{m} c_i\alpha_i + \beta\left(c_j - \sum_{i=1}^{m} c_i\alpha_{ij}\right)$$

Z(X_1)

$$\Delta Z = \beta\left(c_j - \sum_{i=1}^{m} c_i\alpha_{ij}\right)$$

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	12	2	2	1	0	0	0	12/2
0	x_4	8	1	2	0	1	0	0	8/2
0	x_5	16	4	0	0	0	1	0	—
0	x_6	12	0	4	0	0	0	1	12/4

$$\Delta Z = \beta(c_j - \sum_{i=1}^{m} c_i \alpha_{ij})$$

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	6	2	0	1	0	0	-1/2	
0	x_4	2	1	0	0	1	0	-1/2	
0	x_5	16	4	0	0	0	1	0	
3	x_2	3	0	1	0	0	0	1/4	
-Z		-9	2	0	0	0	0	-3/4	

$$\lambda = (c_j - \sum_{i=1}^{m} c_i \alpha_{ij}) = c_j - C_B P_j$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

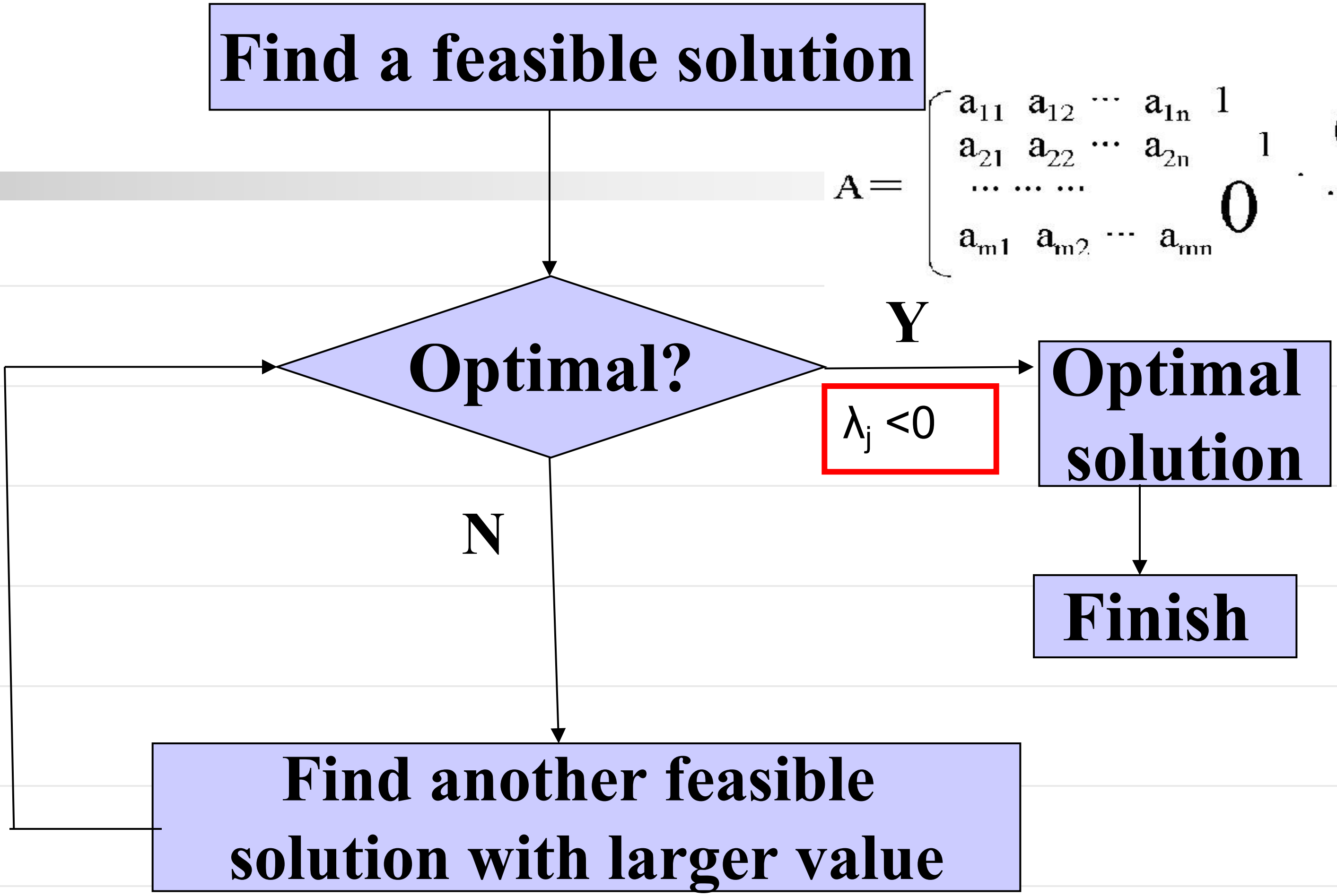
Entering Column

Leaving row

c_j			c_1	$\cdots$	c_m	c_{m+1}	$\cdots$	c_n	β_i
c_B	X_B	P_0	P_1	$\cdots$	P_m	P_{m+1}	$\cdots$	P_n	
c_1	x_1	b_1	1	$\cdots \quad \cdots$	0	$a_{1,m+1}$	$\cdots \quad \cdots$	a_{1n}	β_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_m	x_m	b_m	0	$\cdots \quad \cdots$	1	$a_{m,m+1}$	$\cdots \quad \cdots$	a_{mn}	β_m
$-Z$		$-\sum c_i b_i$	0	$\cdots \quad \cdots$	0	$\lambda_j = c_j - \sum c_i a_{ij}$			

P_0 remains non-negative

$$\beta = \min\left(\frac{b_i}{a_{kj}} \middle| a_{kj} > 0\right)$$



$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & & & \\ a_{21} & a_{22} & \cdots & a_{2n} & & 1 & & 0 \\ \cdots & \cdots & \cdots & & 0 & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & & & & 1 \end{pmatrix}$$

$$\begin{array}{ccc} & & c_1 c_2 \ldots\ldots c_{n+m} \\ & B^{-1}(\, b \; A_1 \; A_2 \; \ldots \; A_{n+m} \,) = & (\, P_0 \; P_1 \; P_2 \; \ldots \; P_{n+m} \,). \\ X_B & C_B & \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} \quad \beta \\ & -Z=-C_B P_0 & \lambda_j = c_j - CBP_j \end{array}$$

$$\beta = \min_i \left\{ \frac{\alpha_i}{\alpha_{ij}} \mid \alpha_{ij} > 0 \right\}$$

$$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$$

Example:

$$\begin{cases} 2x_1 + 2x_2 + x_3 = 12 \\ x_1 + 2x_2 + x_4 = 8 \\ 4x_1 + x_5 = 16 \\ 4x_2 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	12	2	2	1	0	0	0	
0	x_4	8	1	2	0	1	0	0	
0	x_5	16	4	0	0	0	1	0	
0	x_6	12	0	4	0	0	0	1	
		0	2	3	0	0	0	0	

c_j			2	3	0	0	0	0	
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	β_i
0	x_3	12	2	2	1	0	0	0	12/2
0	x_4	8	1	2	0	1	0	0	8/2
0	x_5	16	4	0	0	0	1	0	—
0	x_6	12	0	4	0	0	0	1	12/4
-Z		0	2	3 ↑	0	0	0	0	

c_j			2	3	0	0	0	0	
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	β_i
0	x_3	**6**	**2**	**0**	**1**	**0**	**0**	**-1/2**	
0	x_4	**2**	**1**	**0**	**0**	**1**	**0**	**-1/2**	
0	x_5	**16**	**4**	**0**	**0**	**0**	**1**	**0**	
3	x_2	**3**	**0**	**1**	**0**	**0**	**0**	**1/4**	
-Z			**2**	**0**	**0**	**0**	**0**	**-3/4**	

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	6	2	0	1	0	0	-1/2	6/2
0	x_4	2	1	0	0	1	0	-1/2	2
0	x_5	16	4	0	0	0	1	0	16/4
3	x_2	3	0	1	0	0	0	1/4	—
-Z		-9	2 ↑	0	0	0	0	-3/4	

c_j			2	3	0	0	0	0	β_i degradation
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	2	0	0	1	-2	0	1/2	4
2	x_1	2	1	0	0	1	0	-1/2	—
0	x_5	8	0	0	0	-4	1	2	4
3	x_2	3	0	1	0	0	0	1/4	12
-Z		-13	0	0	0	-2	0	1/4 ↑	

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	2	0	0	1	-2	0	1/2	4
2	x_1	2	1	0	0	1	0	-1/2	—
0	x_5	8	0	0	0	-4	1	2	4
3	x_2	3	0	1	0	0	0	1/4	12
-Z		-13	0	0	0	-2	0	1/4 ↑	

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_6	4	0	0	2	-4	0	1	
2	x_1	4	1	0	1	-1	0	0	x1=4, x2=2
0	x_5	0	0	0	-4	4	1	0	Zmax=14
3	x_2	2	0	1	-1/2	1	0	0	
-Z		-14	0	0	-1/2	-1	0	0	

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	2	0	0	1	-2	0	1/2	4
2	x_1	2	1	0	0	1	0	-1/2	—
0	x_5	8	0	0	0	-4	1	$\boxed{2}$	4
3	x_2	3	0	1	0	0	0	1/4	12
-Z		-13	0	0	0	-2	0	1/4 ↑	

c_j			2	3	0	0	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_3	0	0	0	1	-1	-1/4	0	
2	x_1	4	1	0	0	0	1/4	0	X1=4,x2=2
0	x_6	4	0	0	0	-2	1/2	1	Zmax=14
3	x_2	2	0	1	0	1/2	-1/8	0	
-Z		-14	0	0	0	-3/2	-1/8	0	

$$\max Z = 2x_1 + 3x_2$$

$$\begin{cases} 2x_1 + 2x_2 \le 12 \\ x_1 + 2x_2 \le 8 \\ 4x_1 \le 16 \\ 4x_2 \le 12 \\ x_1, x_2 \ge 0 \end{cases}$$

***x*1=4, *x*2=2**

The optimal solution:

Z=2*4+3*2=14

2*4+2*2=12

4+2*2=8

4*4=16

4*2=8<12

$$\max Z = x_1 + 2x_5 - x_6$$

$$\begin{cases} -\frac{12}{7}x_1 + \frac{1}{14}x_2 + \frac{5}{14}x_3 + x_4 = \frac{45}{7} \\ \frac{1}{7}x_1 - \frac{3}{14}x_2 - \frac{1}{14}x_3 + x_5 = \frac{5}{7} \\ -\frac{3}{7}x_1 + \frac{1}{7}x_2 - \frac{2}{7}x_3 + x_6 = \frac{6}{7} \\ x_1, \ldots\ldots, x_6 \geq 0 \end{cases}$$

c_j			1	0	0	0	2	-1	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_4	45/7	-12/7	1/14	5/14	1	0	0	90
2	x_5	5/7	1/7	-3/14	-1/14	0	1	0	-10/3
-1	x_6	6/7	-3/7	1/7	-2/7	0	0	1	6
-Z		-4/7	2/7	4/7↑	-1/7	0	0	0	

c_j			1	0	0	0	2	-1	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
0	x_4	45/7	-12/7	1/14	5/14	1	0	0	90
2	x_5	5/7	1/7	-3/14	-1/14	0	1	0	-10/3
-1	x_6	6/7	-3/7	1/7	-2/7	0	0	1	6
-Z		-4/7	2/7	4/7 ↑	-1/7	0	0	0	

$$X_2 = (\alpha_1 - \beta\alpha_1 \quad \alpha_2 - \beta\alpha_{2j} \quad \dots \quad 0 \quad \dots \quad \alpha_m - \beta\alpha_{mj} \quad 0 \quad \dots \quad \beta \quad \dots \quad 0)^T$$

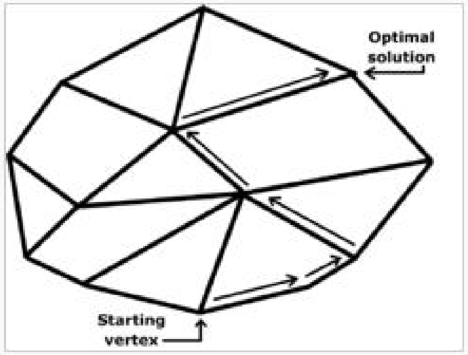
c_B	x_B	P_0	P_1	P_2	P_3	P_4	P_5	P_6	β_i
0	x_4	6	-3/2	0	1/2	1	0	-1/2	
2	x_5	2	-1/2	0	-1/2	0	1	0	
0	x_2	6	-3	1	-2	0	0	7	
-Z			2	0	1	0	0	0	

Since $c1 - z1 > 0$, But $P1 = (-3/2, -1/2, -3)^T$, all less than 0, unbounded solution.

Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.

$$\max Z = 3x_1 + 2x_2$$

$$\begin{cases} 2x_1 + x_2 \le 4 \\ 5x_1 + x_2 \le 5 \\ x_1, x_2 \ge 0 \end{cases}$$



c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
0	x_3	4	2	1	1	0	2
0	x_4	5	5	1	0	1	1
Z		0	3↑	2	0	0	

(1)

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
0	x_3	4	2	1	1	0	2
0	x_4	5	5	1	0	1	1
Z		0	3↑	2	0	0	

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
0	x_3	2	0	3/5	1	-2/5	10/3
3	x_1	1	1	1/5	0	1/5	5
Z		3	0	7/5↑	0	-3/5	

(2)

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
0	x_3	2	0	3/5	1	-2/5	10/3
3	x_1	1	1	1/5	0	1/5	5
Z		3	0	7/5 ↑	0	-3/5	

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
2	x_2	10/3	0	1	5/3	-2/3	-5
3	x_1	1/3	1	0	-1/3	1/3	1
Z		23/3	0	0	-7/3	1/3 ↑	

(3)

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
2	x_2	10/3	0	1	5/3	-2/3	-5
3	x_1	1/3	1	0	-1/3	1/3	1
Z		23/3	0	0	-7/3	1/3	

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
2	x_2	4	2	1	1	0	
0	x_4	1	3	0	-1	1	
Z		8	-1	0	-2	0	

(4)

$x1=0, x2=4, \max Z = 8$

- How about to choose the less λ_j

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
0	x_3	4	2	1	1	0	4
0	x_4	5	5	1	0	1	5
Z		0	3	2 ↑	0	0	

c_j			3	2	0	0	β_i
c_B	x_B	P_0	P_1	P_2	P_3	P_4	
2	x_2	4	2	1	1	0	
0	x_4	1	3	0	-1	1	
Z		8	-1	0	-2	0	

(1)

$x1=0$, $x2=4$, maxZ = 8

- "The most improvement in the objective function" can not guarantee the least number of iterations.
- How to choose the entering column?

minimize $3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$

subject to $110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2{,}000$

$4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55$

$2x_1 + 12x_2 + 54x_3 + \ldots$

the energy (2,000 kcal), protein (50 g), and cal

$0 \leq x_1 \leq$

$0 \leq x_2 \leq$

$0 \leq x_3 \leq$

$0 \leq x_4 \leq$

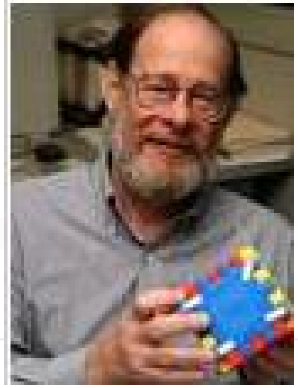
$0 \leq x_5 \leq$

$0 \leq x_6 \leq$

c_j			c_1	$\cdots$ $\cdots$	c_m	c_{m+1}	$\cdots$ $\cdots$	c_n	β_i
c_B	X_B	P_0	P_1	$\cdots$ $\cdots$	P_m	P_{m+1} Entering Column	$\cdots$ $\cdots$	P_n	
c_1	x_1	b_1	1	$\cdots$ $\cdots$	0	$a_{1,m+1}$	$\cdots$ $\cdots$	a_{1n}	β_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$ Leaving row		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_m	x_m	b_m	0	$\cdots$ $\cdots$	1	$a_{m,m+1}$	$\cdots$ $\cdots$	a_{mn}	β_m
$-Z$		$-\sum c_i b_i$	0	$\cdots$ $\cdots$	0	$\lambda_j = c_j - \sum c_i a_{ij}$			

First column with a positive cost.

Bland’s rule



- Bland’s rule
  - Robert G. Bland, now a professor of operations research at Cornell University.
  - An algorithmic refinement of the simplex method
  - Choose the **lowest-numbered** (i.e., leftmost) nonbasic column *t* with a positive cost.

Entering Column

Leaving row

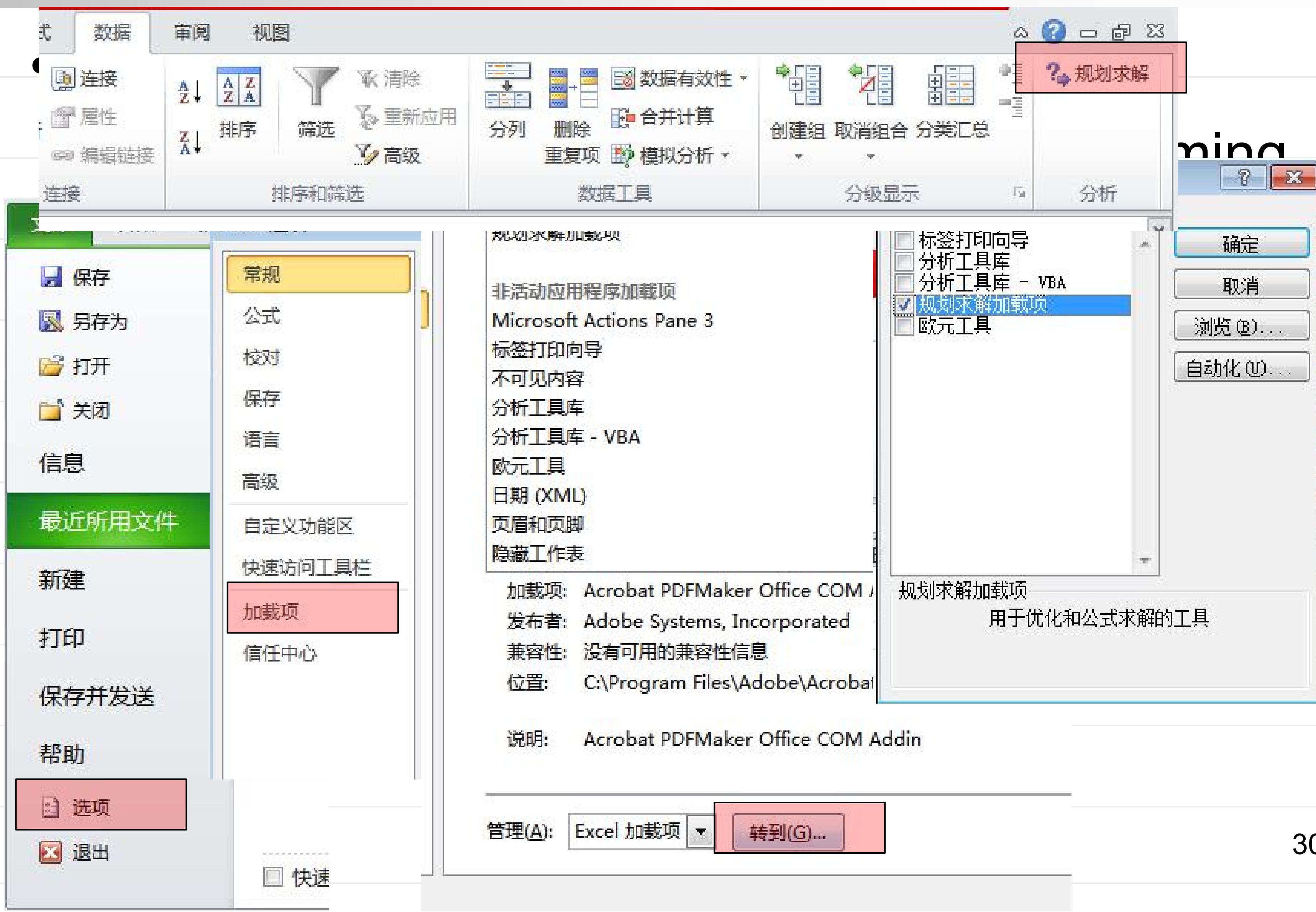
c_j			c_1	$\cdots$	$\cdots$	c_m	c_{m+1}	$\cdots$	$\cdots$	c_n	β_i
c_B	X_B	P_0	P_1	$\cdots$	$\cdots$	P_m	P_{m+1}	$\cdots$	$\cdots$	P_n	
c_1	x_1	b_1	1	$\cdots$	$\cdots$	0	$a_{1,m+1}$	$\cdots$	$\cdots$	a_{1n}	β_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$			$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$			$\vdots$	$\vdots$
c_m	x_m	b_m	0	$\cdots$	$\cdots$	1	$a_{m,m+1}$	$\cdots$	$\cdots$	a_{mn}	β_m
$-Z$		$-\sum c_i b_i$	0	$\cdots$	$\cdots$	0	$\lambda_j = c_j - \sum c_i a_{ij}$				

http://en.wikipedia.org/wiki/Bland's_rule

LP solver

- Lindo/Lingo
- Matlab
- Glpk: GNU Linear Programming Kit.
- **Excel**
- ……

Excel 2010



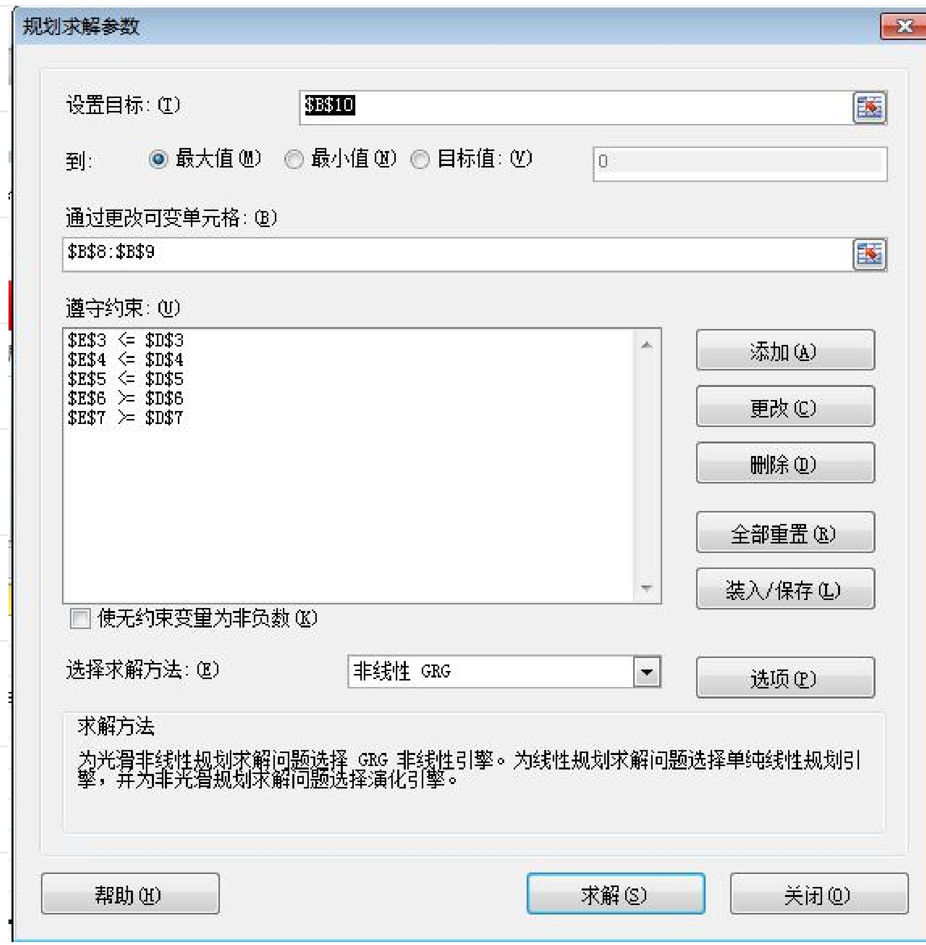
数据
审阅
视图
连接
属性
编辑链接
排序
筛选
清除
重新应用
高级
排序和筛选
分列
删除
重复项
数据有效性
合并计算
模拟分析
数据工具
创建组
取消组合
分类汇总
分级显示
规划求解
分析
保存
另存为
打开
关闭
信息
最近所用文件
新建
打印
保存并发送
帮助
选项
退出
常规
公式
校对
保存
语言
高级
自定义功能区
快速访问工具栏
加载项
信任中心
快速
非活动应用程序加载项
Microsoft Actions Pane 3
标签打印向导
不可见内容
分析工具库
分析工具库 - VBA
欧元工具
日期 (XML)
页眉和页脚
隐藏工作表
加载项: Acrobat PDFMaker Office COM
发布者: Adobe Systems, Incorporated
兼容性: 没有可用的兼容性信息
位置: C:\Program Files\Adobe\Acrobat
说明: Acrobat PDFMaker Office COM Addin
管理(A): Excel 加载项
转到(G)...
标签打印向导
分析工具库
分析工具库 - VBA
规划求解加载项
欧元工具
确定
取消
浏览(B)...
自动化(U)...
规划求解加载项
用于优化和公式求解的工具

Programming

- "Data"-> "Programming"

Max Z=600x+1000y

$$\begin{cases} 10x+4y \le 300 \\ 5x+4y \le 200 \\ 4x+9y \le 360 \\ x \ge 0 \\ y \ge 0 \end{cases}$$



Simplex Method

Max **A**

$$\lambda_j = c_j - z_j$$

all $\lambda_j \le 0$ — Y → *optimal solution*

N

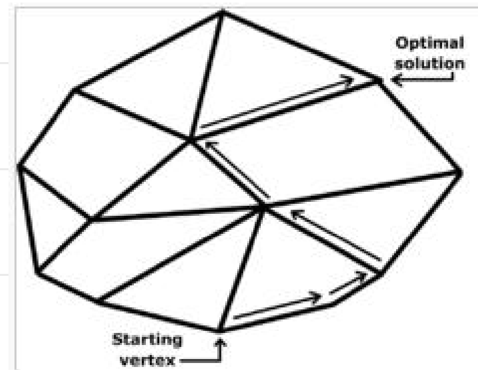
$$find(\lambda_j)_{\max}\ as\ \lambda_k$$

$a_{ik} \le 0$ $(exist\ \lambda_j > 0)$ — Y → *unbounded*

N

$$\beta_i = Min\left(\frac{b_i}{a_{lk}} \middle| a_{lk} > 0\right)$$

→ *entering column* x_k, *leaving row* x_l → *next tableau* → $\lambda_j = c_j - z_j$



		c_j	$c_1 \cdots \cdots c_m$	$c_{m+1} \cdots \cdots c_n$	
c_B	X_B	P_0	$P_1 \cdots \cdots P_m$	$P_{m+1} \cdots \cdots P_n$ (Entering Column)	β_i
c_1	x_1	b_1	$1 \cdots \cdots 0$	$a_{1,m+1} \cdots \cdots a_{1n}$	β_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$ Leaving row $\vdots$	$\vdots \quad \vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
c_m	x_m	b_m	$0 \cdots \cdots 1$	$a_{m,m+1} \cdots \cdots a_{mn}$	β_m
$-Z$		$-\sum c_i b_i$	$0 \cdots \cdots 0$	$\lambda_j = c_j - \sum c_i a_{ij}$	

If no identity matrix in the augmented form?

$$\max Z = 3x_1 - x_2 - x_3$$

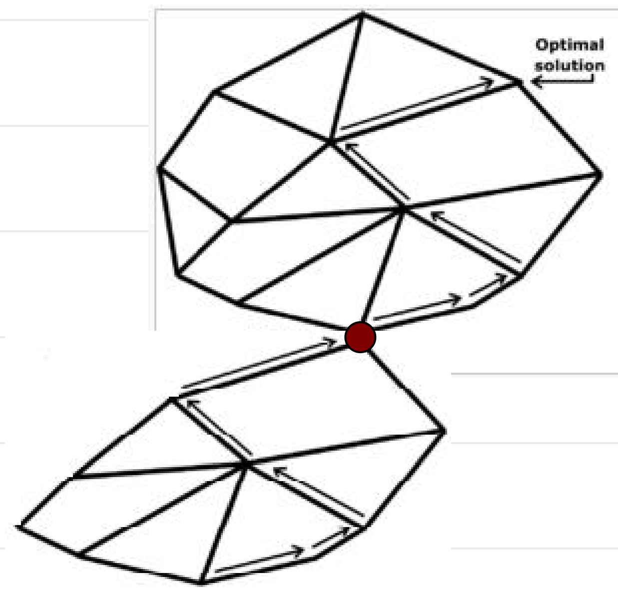
$$\begin{cases} x_1 - 2x_2 + x_3 \le 11 \\ -4x_1 + x_2 + 2x_3 \ge 3 \\ -2x_1 + x_3 = 1 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$$\max Z = 3x_1 - x_2 - x_3$$

$$\begin{cases} \mathbf{x_1 - 2x_2 + x_3 + x_4 = 11} \\ \mathbf{-4x_1 + x_2 + 2x_3 - x_5 = 3} \\ \mathbf{-2x_1 + x_3 = 1} \\ \mathbf{x_1, x_2, x_3 \ge 0} \end{cases}$$

$$\mathbf{min}\, Z = x_6 + x_7$$

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 11 \\ -4x_1 + x_2 + 2x_3 - x_5 + x_6 = 3 \\ -2x_1 + x_3 + x_7 = 1 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0 \end{cases}$$



Think over: What if the optimal solution for the slack variables are not “0”s?

Assignment

- Solving the following linear program using simplex method

$$\max Z = 3x_1 + 6x_2 + 2x_3$$

$$\begin{cases} 3x_1 + 4x_2 + x_3 \le 2 \\ x_1 + 3x_2 + 2x_3 \le 1 \\ \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

Thanks