C & A

Chap. II

Permutation and Combination

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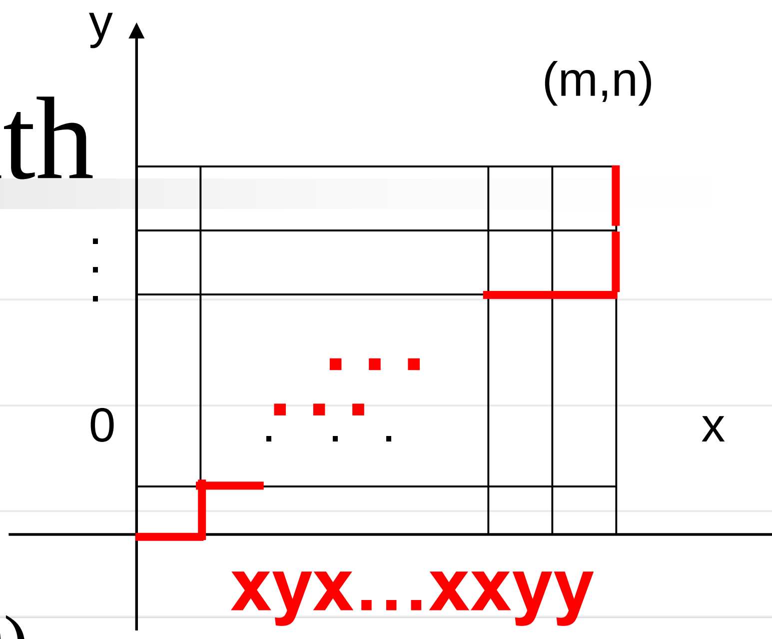
Review of the previous lesson

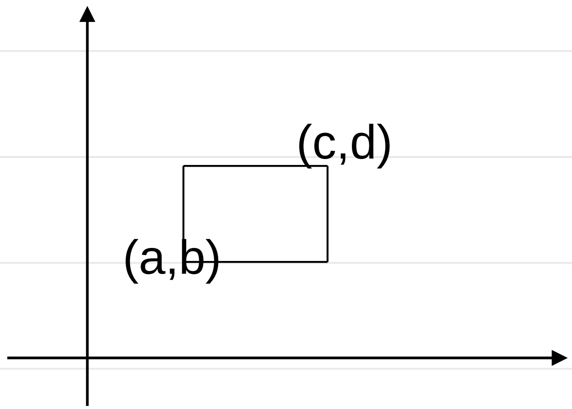
- Four basic counting principles
  - Addition
  - Multiplication
  - Substraction
  - Division
- Permutation and combination?
  - If the order **does** matter:
    - **Permutation: P(n,r)**= $n!/(n-r)!$
  - If the order **doesn't** matter:
    - **Combination: C(n,r)**= $n!/r!(n-r)!$.
- Ordered arrangement
  - Without repeating any objects, distinct: **P(n,r)**
  - **Circular permutation**

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}.$$

Lattice Path

- A path composed of connected horizontal and vertical line segments, each passing between adjacent lattice points。
- How many lattice paths from (0,0) to (m,n) ?
- **One-one correspondence**
  - Each path $(0,0)\rightarrow(m,n)$
  - Arrangement with m ‘*x*’*s and n* ‘*y*’*s*
  - $C(m+n,m)$

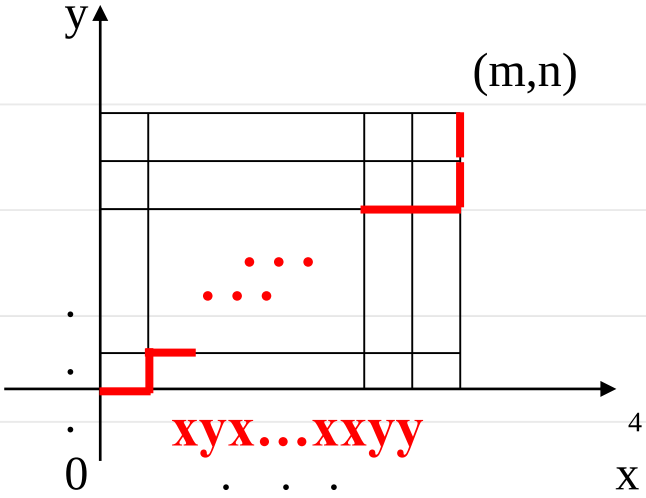


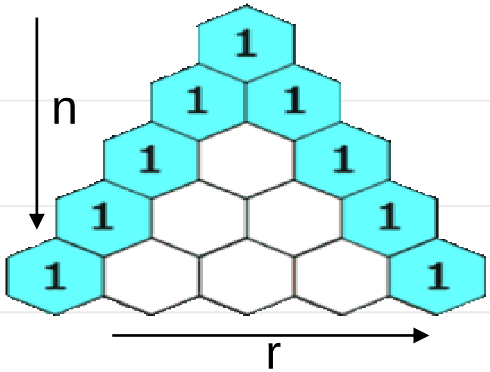


$$|(a,b)(c,d)| = \binom{(c-a)+(d-b)}{(c-a)}$$

Models of Combination

- Lattice - path：
- Walk along the positive directions of x-axis or y-axis from (0,0) to (m,n), 1 unit per step, there are $C(m+n,n)$ routes.





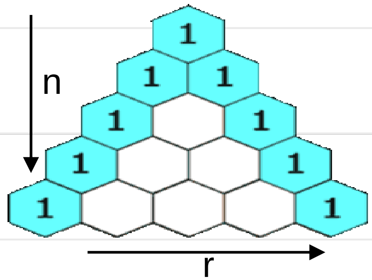
The n^{th} row, k^{th} column : $C(n,k)$

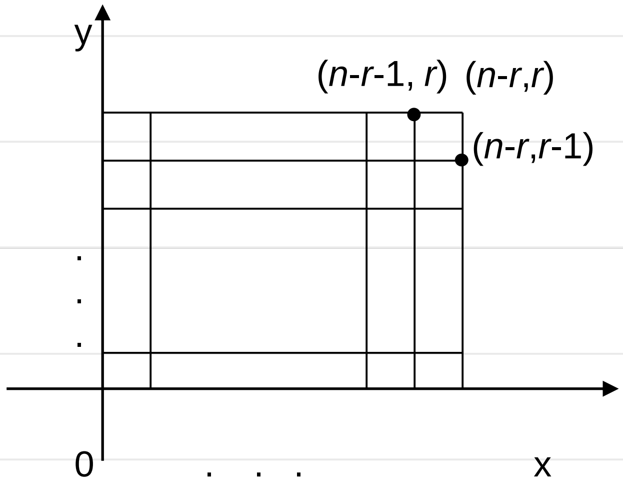
Coefficients of $(a+b)^n$

$$C(n,r) = C(n-1,r) + C(n-1,r-1)$$

Combinatorial Identities

- $C(n, r) = C(n-1, r) + C(n-1, r-1)$





- Left hand side: All lattice paths from (0,0) to $(n-r,r)$
- Right hand side:
  - (0,0) to $(n-r-1,r)$
  - (0,0) to $(n-r,r-1)$

Pascal'formula

- For all integers n and r with $1 \le r \le n-1$.

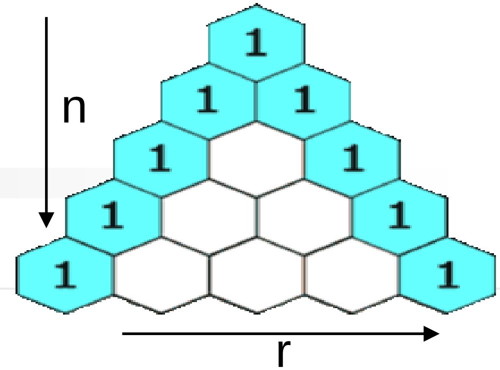
$$C(n,r) = C(n-1,r) + C(n-1,r-1)$$

- Proof: choose $a_1, a_2, \ldots, a_r$ from [1,n]
- Partition the combinations by with or without 1
  - With 1, other r-1 numbers from n-1 integers: $C(n-1,r-1)$
  - Without 1, r numbers from n-1 integers: $C(n-1,r)$
- Combing two parts: $C(n,r)=C(n-1,r)+C(n-1,r-1)$

Can you provide the proof for the following formula?

$$2^m = C(m,0) + C(m,1) + \ldots + C(m,m)$$

Pascal' formula

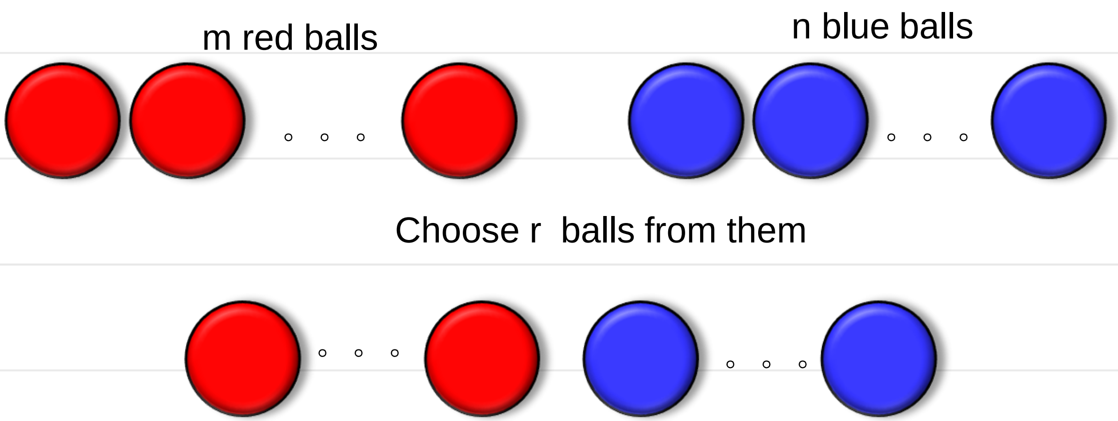


The n^{th} row, k^{th} column : $C(n,k)$
Coefficients of $(a+b)^{n}$

- JiaXian/Yanghui Triangle
- $(x+y)^m = C(m,0)x^m + C(m,1)x^{m-1}y + \ldots + C(m,m)y^m$
  - The coefficient of $x^a y^{m-a}$ means how many ways to choose a 'x' from m 'x' and the other m-a elements should be 'y: $C(m,a)$
- If x=y=1, then
- $2^m = C(m,0) + C(m,1) + \ldots + C(m,m)$

Chu-Vandermonde Identity

- **C**(m+n,r)=C(m,0)C(n,r)+C(m,1)C(n,r-1)+ …+C(m,r)C(n,0)
- C(m,0)C(n,r)+C(m,1)C(n,r-1)+…+C(m,r)C(n,0)



Examples

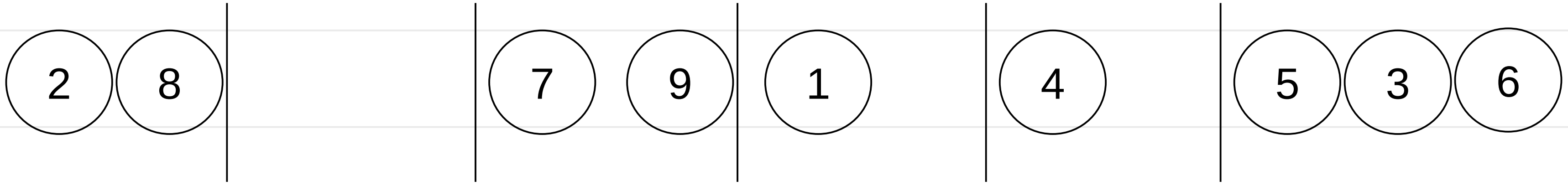
Q: There are 6 gates. How many ways to arrange 9 people to enter the gates in sequence?

The arrangement can be expressed as

XXffXXfXfXfXXX

Where X means somebody and f means the doorframe of gates.

$X \in \{1,2,3,\ldots,9\}$ Five "f"s are same.



Examples

$XXf_a f_b XX\ f_c X f_d X f_e XXX$

A1: Each doorframe can be labeled.

- The number of permutation of labels is 5!
- The permutations for 14 elements are 14!.
- The total number of arrangements without labels is

$$14!/5!=726485760$$

	f		f		f			f					f

A2:There are 14 positions for all the elements.

Choose 5 positions for “f”s: C(14,5)

9 people can choose all the other positions: 9!

The total number of arrangements is

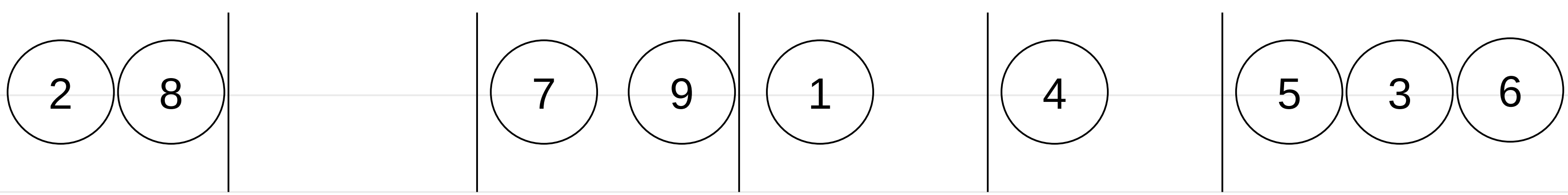
C(14,5)9! = 14!*9!/5!/9!=14!/5!=726485760

Examples

A3: We can arrange 9 people step by step

- The 1^{st} people has 6 choices;
- The 2^{nd} people has 7 choices
  - The same gate with 1^{st} people: before 1 or after 1
  - The other 5 gates
- The 3^{rd} people has 8 choices;
- ……
- The 9^{th} people has 14 choices.
- Totally, the number of arranges are

$$6*7*8*\ldots.*14 \;=14!/5!=726485760$$



Permutations of multi-sets

- If S is a multiset, an *r-permutation* of S is an ordered arrangement of r objects of S. If $|S| = n$, then an n-permutation of S will also be called a permutation of S.
  - $S=\{2\cdot a, 1\cdot b, 3\cdot c\}$
  - 4-permutation of S: *acbc*, *cacc*
- Let S be a multiset with objects of k different types where each has an infinite repetition number. Then the number of r-permutations of S is k^r.
  - $S=\{\infty\cdot 0, \infty\cdot 1, \infty\cdot 2\}$
  - The number of 4-permutation of S: 3^4

设置

The number of permutations of the letters in the word “MISSISSIPPI” is (   ).

作答

Finite Repetition Numbers

Let S be a multiset with objects of k different types with finite repetition numbers n_1, n_2, …, n_k, respectively. Let the size of S be $n = n_1 + n_2 + \ldots + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Specially, when k=2

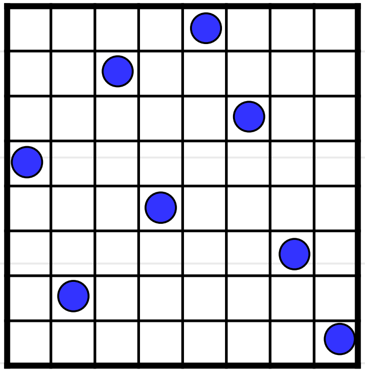
$$\frac{n!}{n_1!\,n_2!} = \frac{n!}{n_1!(n-n_1)!} = \binom{n}{n_1}$$

The number of permutations of the letters in the word “MISSISSIPPI” is $\dfrac{11!}{1!4!4!2!}$.

1•M,4 • S,4 • I,2 • P

Examples

- How many possibilities are there for 8 non-attacking rooks on an 8-by-8 chessboard?
- (1) The rooks are indistinguishable for one another;
- The coordinates of rooks: only 1 rook for each row/column
  - (1,5) (2,3), (3,6),(4,1),(5,4),(6,7),(7,2),(8,8)
  - 8-permutations of {1,2…8}: 8!
- (2) We have 8 distinguished rooks;
  - 8! Color arrangement for each rook arrangement:
  - Total: 8!*8!
- (3) We have 1 red rook, 3 blue rooks and 4 yellow rooks
  - Divided with the repetitions: 8!*8!/(1!3!4!)



Combinations of Multisets

- If S is a multiset, then an *r-combination* of S is an unordered selection of r of the objects of S. Thus an r-combination is itself a multiset, a submultiset of S.
- Example. If $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$
  – 3-combinations of S are $\{2 \cdot a, 1 \cdot b\}$, $\{2 \cdot a, 1 \cdot c\}$, $\{1 \cdot a, 1 \cdot b, 1 \cdot c\}$, $\{1 \cdot a, 2 \cdot c\}$, $\{1 \cdot b, 2 \cdot c\}$, $\{3 \cdot c\}$.
- How to count the number of r-combinations of a multiset?

r-combinations

Theorem: Let S be a multiset with objects of k different types where each has an infinite repetition number. Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

Proof. Suppose that $S=\{\infty\cdot a_1, \infty\cdot a_2,\ldots, \infty\cdot a_k\}$

any r-combination of S is $\{x_1\cdot a_1, x_2\cdot a_2,\ldots, x_k\cdot a_k\}$, where $x_1, x_2,\ldots,x_k$ are nonnegative integers with $x_1+x_2+\ldots+x_k = r$.

The number of r-combinations of S equals the number of solutions of the equation

$x_1+x_2+\ldots+x_k = r$ where $x_1, x_2,\ldots,x_k$ are nonnegative integers

$$\overbrace{\underset{x_1}{1..1} \;\Big|\; \underbrace{\underset{x_2}{1..1} \;\Big|\; \underset{x_3}{1..1} \;\Big|\; \qquad \;\Big|}_{\text{k-1 frames}} \; \underset{x_k}{1..1}}^{\text{The number of '1's is r}}$$

$$\frac{(k-1+r)!}{r!(k-1)!} = \binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

Examples

- A bakery toasts 8 varieties of doughnuts. If a box of doughnuts contains 1 dozen how many different boxes can you buy?
- A: 12 combinations of a multiset with objects of 8 types, each having an infinite repetition number.
  - 12-combination of S={∞·*1*, ∞·2,…, ∞ ·8}

$$\binom{r+k-1}{r} = \binom{12+8-1}{12} = \binom{19}{12}.$$

$x_1+x_2+\ldots+x_k = r$ where $x_1, x_2,\ldots,x_k$ are nonnegative integers

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

- What is the number of integral solutions of the equation $x_1+x_2+x_3+x_4=20$, in which

$$x_1 \geq 3, x_2 \geq 1, x_3 \geq 0 \; and \quad x_4 \geq 5$$

- We introduce the new variables

$$y_1 = x_1 - 3, y_2 = x_2 - 1, y_3 = x_3 \; and \quad y_4 = x_4 - 5$$

$$y_1 + y_2 + y_3 + y_4 = 20 - 3 - 1 - 5 = 11$$

  - *y_i is nonnegative, and x_i satisfies the constraints.*
  - *The number of nonnegative solution of the equation is* $\binom{r+k-1}{r} = \binom{11+4-1}{11} = \binom{14}{11}.$

Putting balls into boxes

Putting 5 balls into 4 boxes：

1)5 different balls, 4 different boxes, the number of balls inside a box is not limited, allows empty box, and contains total _____ different solutions.【Please calculate the exact number】

2)5 identical balls, 4 different boxes, allows empty box, and contains total _____ different solutions.【Please calculate the exact number】

单选题 1分

设置

5 identical balls, 4 different boxes, allows empty box, and contains total _____ different solutions.

A 1024

B C(8,3)

C C(8,4)

D C(9,4)

提交

5 identical balls, 4 different boxes, allows empty box, and contains total ____ different solutions.

Explanation:

It is equivalence to find the solution number of

$x_1+x_2+x_3+x_4=5$,

$x_1, x_2, x_3, x_4, x_5>=0$.

$$C_{n+r-1}^{r} = C_{4+5-1}^{5} = C_8^5 = C_8^3 = 56$$

Example

- If $S = \{3 \cdot a, 2 \cdot b, 4 \cdot c\}$
  - 8-permutations of *S*?
  - 8-combinations of *S*?
- 8-permutations
  - $\{2 \cdot a, 2 \cdot b, 4 \cdot c\}$: $8!/(2!2!4!)=420$
  - $\{3 \cdot a, 1 \cdot b, 4 \cdot c\}$:$8!/(3!1!4!)=280$
  - $\{3 \cdot a, 2 \cdot b, 3 \cdot c\}$:$8!/(3!2!3!)=560$
  - Total: 420+280+560 = 1260

Example

- If $S = \{3 \cdot a, 2 \cdot b, 4 \cdot c\}$
  - 8-permutations of S?
  - 8-combinations of S?
- 8-combinations
  - $\{2 \cdot a, 2 \cdot b, 4 \cdot c\}$
  - $\{3 \cdot a, 1 \cdot b, 4 \cdot c\}$
  - $\{3 \cdot a, 2 \cdot b, 3 \cdot c\}$
  - Total: 3

$$x_1 + x_2 + x_3 + x_4 = 20$$

$$x_1 \geq 3, x_2 \geq 1, x_3 \geq 0 \; and \quad x_4 \geq 5$$

$$y_1 = x_1 - 3, y_2 = x_2 - 1, y_3 = x_3 \; and \quad y_4 = x_4 - 5$$

$$y_1 + y_2 + y_3 + y_4 = 20 - 3 - 1 - 5 = 11$$

$$x_1 + x_2 + x_3 + x_4 = 20$$

$$x_1 \leq 5, x_2 \leq 10, x_3 \leq 11, and x_4 \leq 8$$

Complicated!

Inclusion-exclusion principle in Chapter 6

Summary

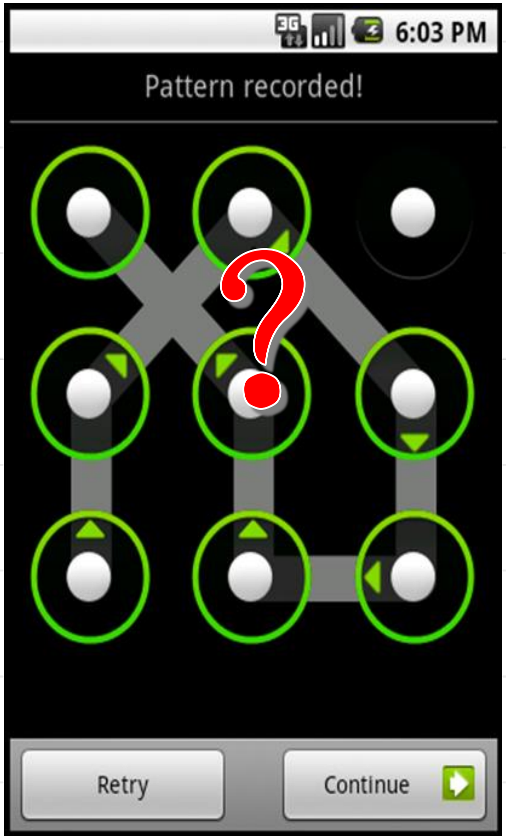
Sample	Order counts?	Repetition allowed?	Name	Number of ways
Choose 3 balls and put them in a box	No	No	r-combination	C(m,r)
People in a line	Yes	No	r-permutation	P(m,r)
Arrangement of fruits	No	Yes	r-combination of multi-sets	C(m+r-1,r)
4-letter word	Yes	Yes	r-permutation of multi-sets	m^r

To Do Homework

- Homework sheet
- Pre-class video on RainClass room
  - Generating Permutations
  - OJ task 1: Cellphone passwords (Due on Oct.5)

Is your cell phone password safe?

Which type is better?



OJ task 1: Cellphone passcode

Thanks