投票 最多可选1项

设置

How much do you think you understand the preclass material?
Any comments are welcome by tweet or chatbox.

A More than 80%

B 50%-80%

C 20%-50%

D Less than 20%

提交

Generating Functions

组合数学 Combinatorics

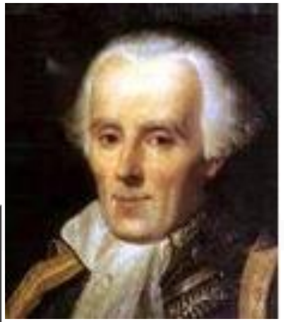
Tsinghua University
Associate Professor Yuchun Ma

§3 Lesson Summary

Definition 2-1 For sequence $a_0, a_1, a_2\ldots$, form a function

$$G(x)= a_0+a_1x+a_2x^2+\ldots,$$

Name $G(x)$ as the generating function for sequence $a_0, a_1, a_2\ldots$。



Laplace
Year 1812

$\square + \square = n \longrightarrow (x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$

Like A Function But Not A Function, It's Mapping

Bernoulli
Year 1705

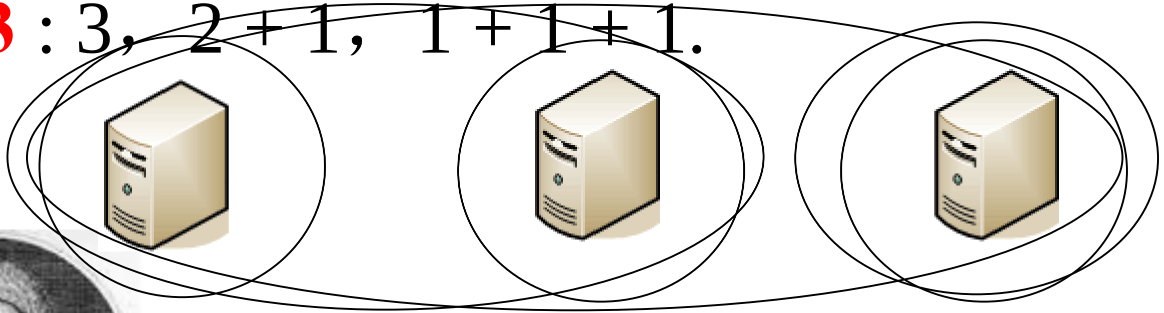
$c_0, c_1, \ldots, c_n \longleftarrow = x^2+2x^3+3x^4+4x^5+5x^6+\ldots\ldots$

Euler
Year 1764

Found the mapping relationship is a "Mathematic Discovery".
Finding mapping is an important mathematic thinking.

The Application of Generating Function: Integer Partition Number

- **Unordered Partition of Positive Integer**： Split a positive integer n into the summation of several integer, the order between numbers is ignored and allow repetition, its different partition number is $p(n)$。
  - Cryptography, Statistics, Biology……
- **$p(3)=3$** : 3， 2 + 1， 1 + 1 + 1.



Philippe Naudé

Year 1740, Naude asked Euler in the letter:

Solution number of partition of integer?

Euler

Exponent correspondence value

$$G(x) = \underbrace{(1+x+x^2+\dots)}_{\text{Generating Function of “1”}}\underbrace{(1+x^2+x^4+\dots)}_{\text{Generating Function of “2”}}\dots\underbrace{(1+x^m+x^{2m}+\dots)}_{\text{Generating Function of “m”}}\dots$$ coefficient of x^n

Weil A． Number Theory：An Approach Thought History [M]． Boston：Birkhouser，1984． 284

Generating Function And Recurrence Relation

组合数学 Combinatorics

清华大学 马昱春
Tsinghua University
Associate Professor Ma

Binomial Theorem

$$(1+x)^{-1} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + ...$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} x^k + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k \qquad \alpha \in R$$

$$(a)^n = a^n$$

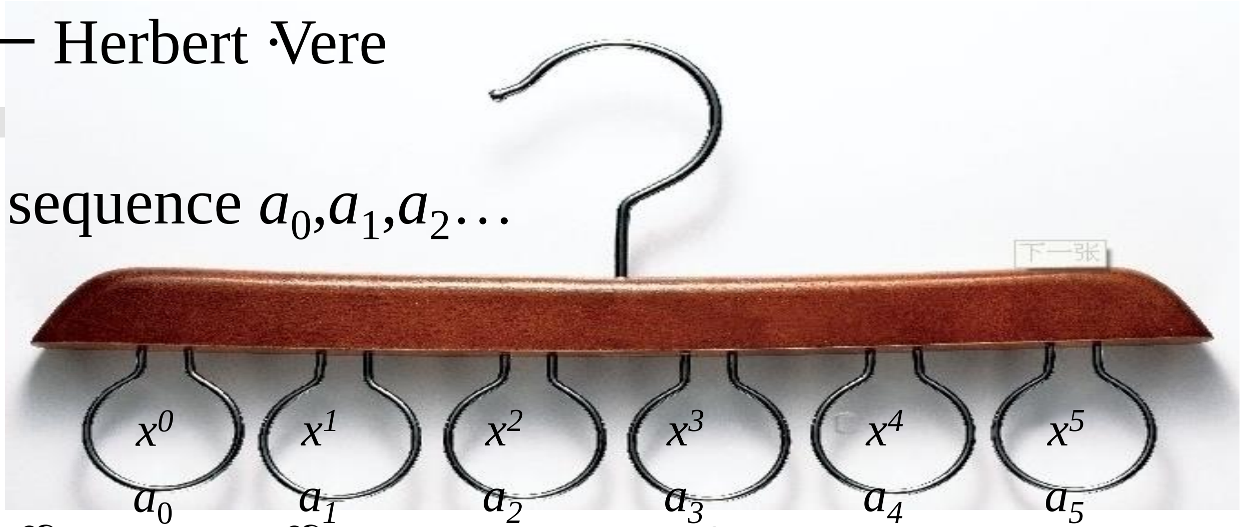
$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

$$(a+b+c)^n = \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{l!(k-l)!(n-k)!} a^{n-k} b^{k-l} c^l$$

$$(a+b+c+d)^n = \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{n!}{m!(l-m)!(k-l)!(n-k)!} a^{n-k} b^{k-l} c^{l-m} d^m$$

Generating function is a line of hangers which used to display a series of number sequences.

— Herbert ·Vere



- G(x) is the generating function for counting sequence $a_0,a_1,a_2\ldots$
  - **G(x)=a_0+a_1x+a_2x^2+……**

$$(1-ax)^{-1} = 1 + ax + a^2x^2 + \ldots..$$

$$\frac{2-3x}{(1-x)(1-2x)} = \frac{1}{1-x} + \frac{1}{1-2x} = \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} 2^k x^k = \sum_{k=0}^{\infty} (1+2^k)x^k$$

Generating Function | | **Number Sequence**

$$\frac{2-3x}{(1-x)(1-2x)}$$

Partial Fraction Decomposition

$$\Longleftrightarrow$$

$$(1-ax)^{-1} = 1 + ax + a^2x^2 + \ldots..$$

$$f(k) = 2^k + 1$$

G(x)

$$\Longleftrightarrow$$

$$h(k) = 2h(k-1) + 1$$

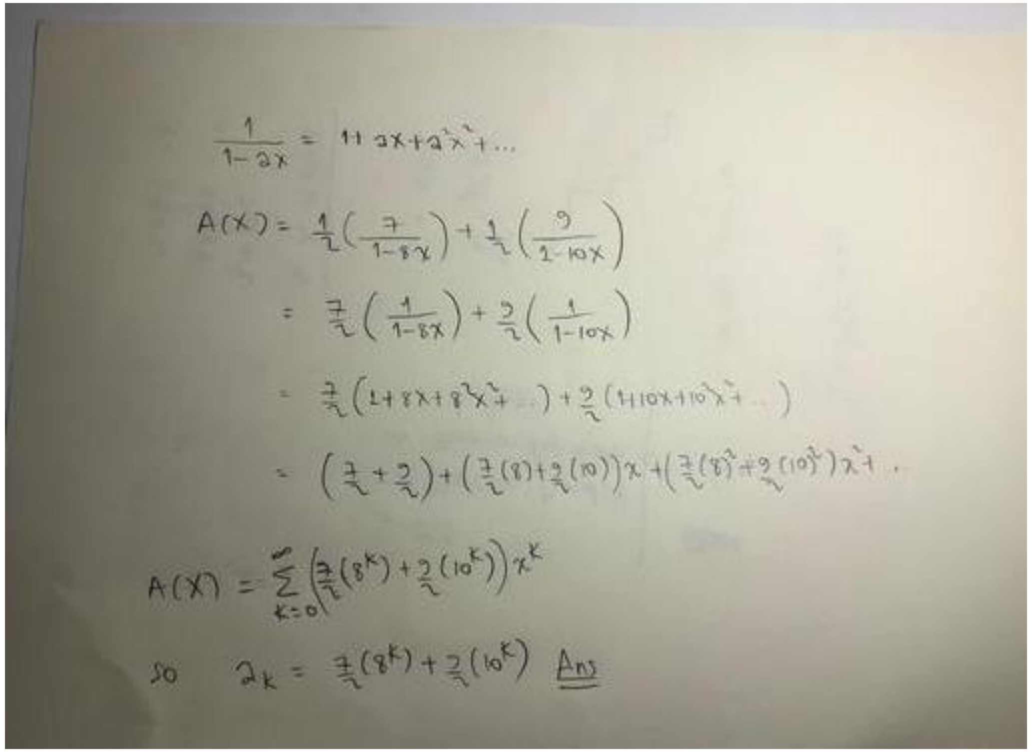
Recurrence Relation

?

If we have generating function
A(x)= $a_1x+a_2x^2+\ldots$ and we have

$$A(x)=\frac{1}{2}\left(\frac{7}{1-8x}+\frac{9}{1-10x}\right)$$

Please figure out the formula of a_k



$$\frac{1}{1-ax} = 1+ax+a^2x^2+\ldots$$

$$A(x) = \frac{1}{2}\left(\frac{7}{1-8x}\right)+\frac{1}{2}\left(\frac{9}{1-10x}\right)$$

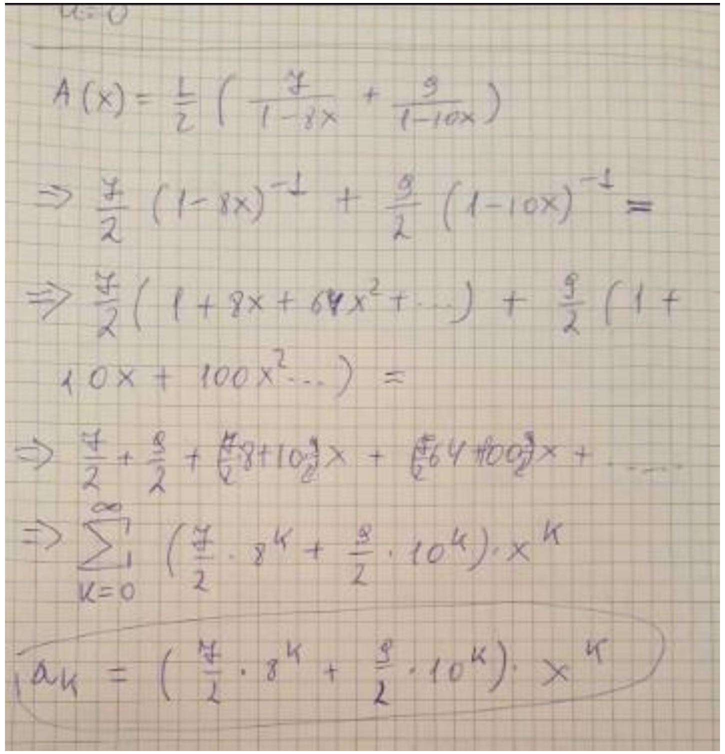
$$= \frac{7}{2}\left(\frac{1}{1-8x}\right)+\frac{9}{2}\left(\frac{1}{1-10x}\right)$$

$$= \frac{7}{2}(1+8x+8^2x^2+\ldots)+\frac{9}{2}(1+10x+10^2x^2+\ldots)$$

$$= \left(\frac{7}{2}+\frac{9}{2}\right)+\left(\frac{7}{2}(8)+\frac{9}{2}(10)\right)x+\left(\frac{7}{2}(8^2)+\frac{9}{2}(10^2)\right)x^2+\ldots$$

$$A(x) = \sum_{k=0}^{\infty}\left(\frac{7}{2}(8^k)+\frac{9}{2}(10^k)\right)x^k$$

so $a_k = \frac{7}{2}(8^k)+\frac{9}{2}(10^k)$ Ans



$$A(x) = \frac{1}{2}\left(\frac{7}{1-8x}+\frac{9}{1-10x}\right)$$

$$\Rightarrow \frac{7}{2}(1-8x)^{-1}+\frac{9}{2}(1-10x)^{-1} =$$

$$\Rightarrow \frac{7}{2}(1+8x+64x^2+\ldots)+\frac{9}{2}(1+10x+100x^2\ldots) =$$

$$\Rightarrow \frac{7}{2}+\frac{9}{2}+\left(\frac{7}{2}8+10\frac{9}{2}\right)x+\left(\frac{7}{2}64+100\frac{9}{2}\right)x+\ldots$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(\frac{7}{2}\cdot 8^k+\frac{9}{2}\cdot 10^k\right)\cdot x^k$$

$$a_k = \left(\frac{7}{2}\cdot 8^k+\frac{9}{2}\cdot 10^k\right)\cdot x^k$$

Recurrence Relation

Recurrence Relation： Is difference equation, which is a recursively defined the formulae for a **sequence**： Each item of the sequence is defined as the function of “**Several Former Items**”。

- E.g.Hanoi Problem： Year 1883 France Mathematician Edouard Lucas
  - When the Great Brahma created the world, he made 3 diamond pillars, there are 64 golden discs from smallest to largest, from top to the bottom in each pillar.
  - The Great Brahma ordered Brahma to move all these discs to another pillar by following smallest to largest arrangement order, starting from the bottom.
  - No disc may be placed on top of its smaller disc, among the 3 pillars, only one disc may be moved at a time.
  - When the movement is completed, it will be the time when the world is destroyed
    - Algorithm design;
    - Estimation of complexity.

Recurrence Relation

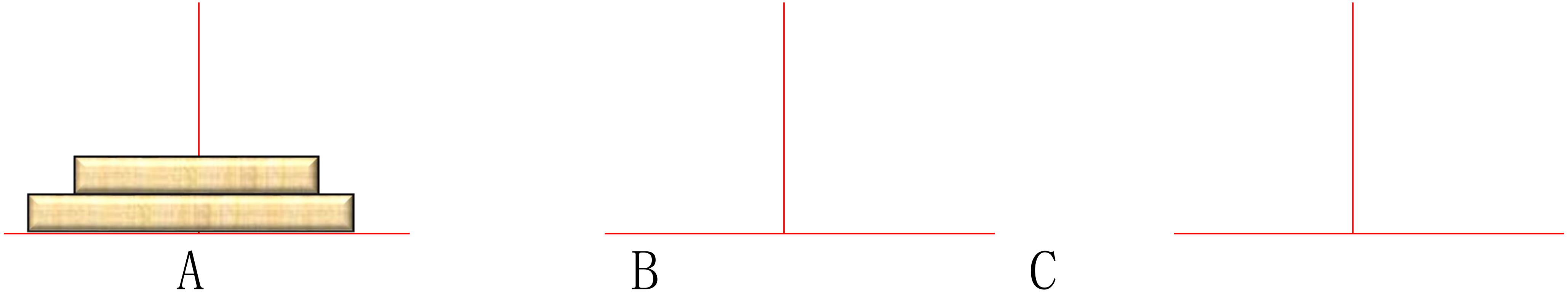
Algorithm: When N=2

1st Step: Move the top most disc to B

2nd Step: Move the bottom disc to C

Lastly, move the disc from B to C

The transmission is completed



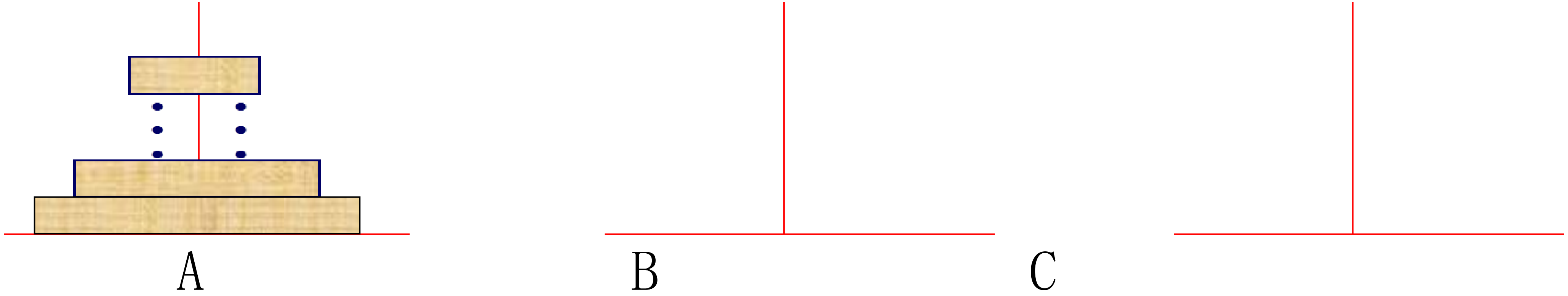
Recurrence Relation

Let $h(n)$ to represent the number of moves for n number of discs

- If the number of movements for n-1 discs is known to be with the complexity of $h(n\text{-}1)$
  - For typical problem like n number of discs, firstly, move the top n-1 of discs from C to B: $h(n\text{-}1)$
  - 2nd Step: Move the last disc from A to C: $h(1)$
  - Lastly, move n-1 number of disc from B to C through A: $h(n\text{-}1)$

Complexity of Algorithm： $h(n) = 2h(n-1)+1,\ \ h(1)=1$

Structure of Generating Function: $H(x) = h(1)x + h(2)x^2 + h(3)x^3 + \cdots,$



$$(1-x)^{-1}=1+x+x^2+\ldots..$$

$$h(n)=2h(n-1)+1,\ \ h(1)=1$$

Recurrence Relation

$h(0)=0$

If these exponent value is performing 4 arithmetic operations, it is same like the finite algebra expression.

$$H(x)=h(1)x+h(2)x^2+h(3)x^3+\cdots,$$

$$+)\quad -2xH(x)=\qquad -2h(1)x^2-2h(2)x^3+\cdots,$$

$$(1-2x)H(x)=h(1)x+[h(2)-2h(1)]x^2$$

$$+[h(3)-2h(2)]x^3+\cdots$$

$$\because\quad h(1)=1,h(2)-2h(1)=1,h(3)-2h(2)=1,\cdots$$

$$\therefore\quad (1-2x)H(x)=x+x^2+x^3+\cdots=x/(1-x)$$

$$\therefore\quad H(x)=\frac{x}{(1-2x)(1-x)}$$

$$h(n) = 2h(n-1)+1,\ \ h(1)=1$$

$h(0)=0$

Recurrence Relation

$$H(x) = h(1)x + h(2)x^2 + h(3)x^3 + \cdots,$$

Apply Recurrence Relation $x^2 : h(2) = 2h(1)+1$

$$x^3 : h(3) = 2h(2)+1$$

$$+)\qquad \cdots\cdots\cdots\cdots\cdots$$

Left side:

$$h(2)x^2 + h(3)x^3 + \cdots = H(x) - h(1)x = H(x) - x$$

1st term on the right side:

$$2h(1)x^2 + 2h(2)x^3 + \cdots = 2x[h(1)x + h(2)x^2 + \cdots]$$

$$= 2xH(x)$$

2nd term on the right side:

$$x^2 + x^3 + \cdots = x^2/(1-x)$$

$$\therefore \quad H(x) - x = 2xH(x) + x^2/(1-x)$$

$$H(x) = \frac{x}{(1-x)(1-2x)}$$

$$H(x)=\sum_{k=1}^{\infty}h(k)x^k=\frac{x}{(1-x)(1-2x)}$$

$$h(64)=18446744073709551615$$

How to find the number sequences based on generating function? $h(1),h(2),\cdots$

Transformed into partial fractional algorithm.

$$H(x)=\frac{A}{1-x}+\frac{B}{1-2x}=\frac{A(1-2x)+B(1-x)}{(1-x)(1-2x)}$$

$$=\frac{(A+B)-(2A+B)x}{(1-x)(1-2x)}$$

$$\therefore\quad (A+B)-(2A+B)x=x$$

From the above equation:

$$\begin{cases} A+B=0 \\ -2A-B=1 \end{cases}\Longrightarrow A=-1,\quad B=1.$$

Be: $$H(x)=\frac{1}{1-2x}-\frac{1}{1-x}$$

$$=(1+2x+2^2x^2+2^3x^3+\cdots)-(1+x+x^2+\cdots)$$

$$=(2-1)x+(2^2-1)x^2+(2^3-1)x^3+\cdots$$

$$=\sum_{k=1}^{\infty}(2^k-1)x^k$$

$$\therefore\quad h(k)=2^k-1$$

Conclusion

- $G(x)=a_0+a_1x+a_2x^2+\ldots\ldots$
- From **G(*x*)** obtains sequence **$\{a_n\}$.** The key is over the bridge between sequence to generating function, and between generating function to sequence.

$$\begin{aligned} x^2 &: h(2) = 2h(1)+1 \\ x^3 &: h(3) = 2h(2)+1 \\ +) & \quad \ldots\ldots\ldots\ldots \end{aligned}$$

$$H(x) = \sum_{k=1}^{\infty} h(k)x^k = \frac{x}{(1-x)(1-2x)}$$

$$= \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{k=1}^{\infty}(2^k-1)x^k$$

$$(1-ax)^{-1} = 1+ax+a^2x^2+\ldots..$$

Itemize representation of rational fraction

The denominator coefficients contain any special meaning?

The suitability of generating function method towards recurrence relation?

4 Linear Homogeneous Recurrence Relation

4-1 Fibonacci Rabbits

组合数学 Combinatorics

The delta of the *n^{th}* month and *n-1^{th}* month is given birth by the rabbits in *n-2* month. So

$$F_n = F_{n-1} + F_{n-2}$$

In the first month there's a pair of newly-born rabbits;
If a pair of rabbits could give birth to a new pair every month (one male, one female);
New rabbits could start giving birth since the third month;
The rabbits never die;
How many rabbits would there be in the 50^{th} month?

Fibonacci number

1 1 2 3 5 8 13 21 34 55……

OEIS: A000045
http://oeis.org/A000045

Leonardo of Pisa
Fibonacci，Bonacci's son
Bonacci: good, natural, simple

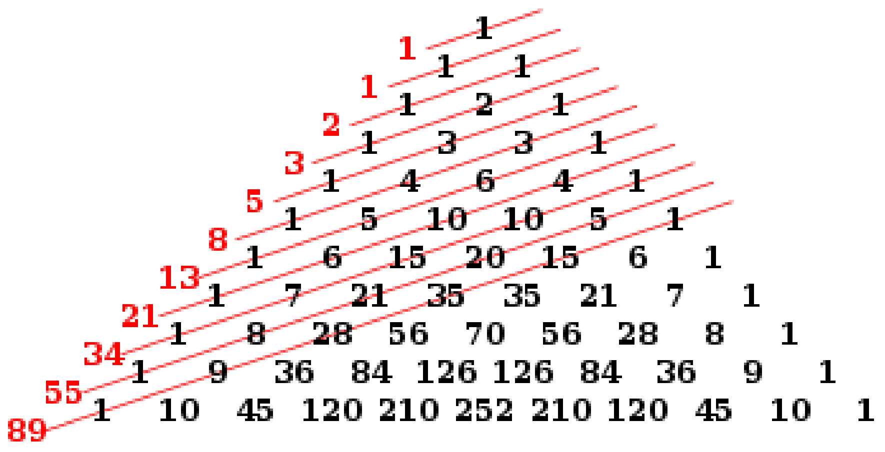
Recurrence Relation: $F(n)=F(n-1)+F(n-2)\quad n\geq 2$

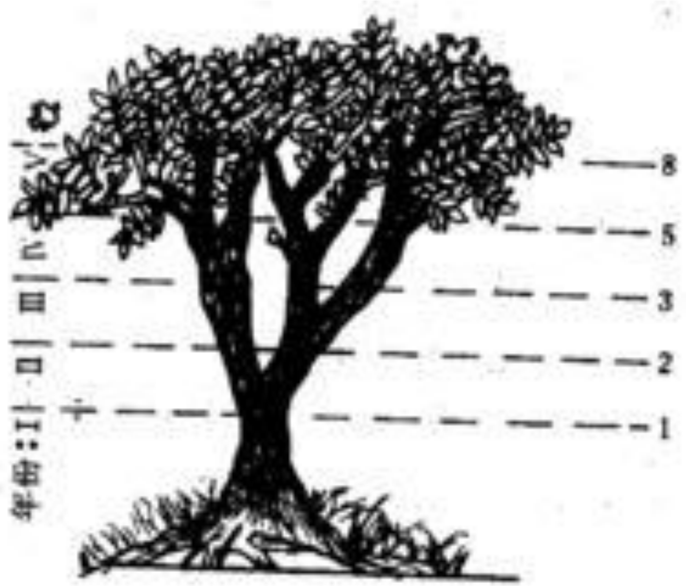
Initial values: $F(0)=0$, $F(1)=1$

- **In 1150, Indian mathematicians researched the number of arrangements to package items with length 1 and width 2 into boxes. And they described this sequence for the first time.**
- **In the western world, Fibonacci mentioned a problem about the reproduction of rabbits in Liber Abbaci in 1202.**
- Fibonacci，Leonardo 1175-1250
  - Member of the Bonacci family.
  - Travelled to Asia and Africa at 22 with his father and learned to calculate with Indian digits；
  - Played an important role in the recovery of Western Mathematics. And connected Western and Oriental mathematics.
  - G.Cardano: "We could assume that all mathematics we know except the Ancient Greek ones are gotten by Fibonacci.

Fibonacci number

1 1 2 3 5 8 13 21 34 55……





Trillium — 3 Petals

St. Johnswort — 5 Petals

Bloodroot — 8 Petals

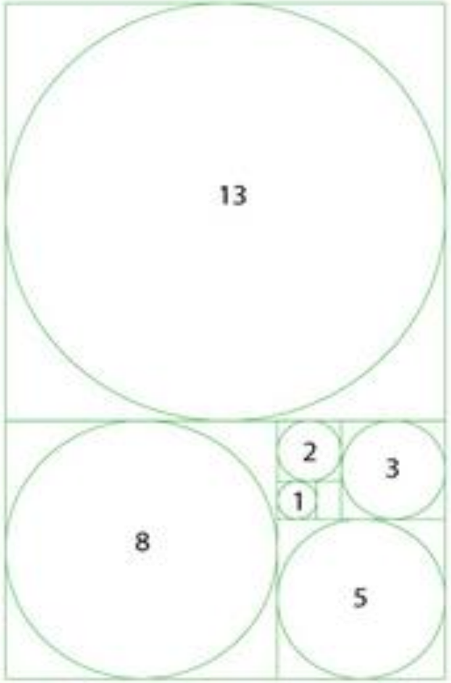
Black-eyed Susan — 13 Petals

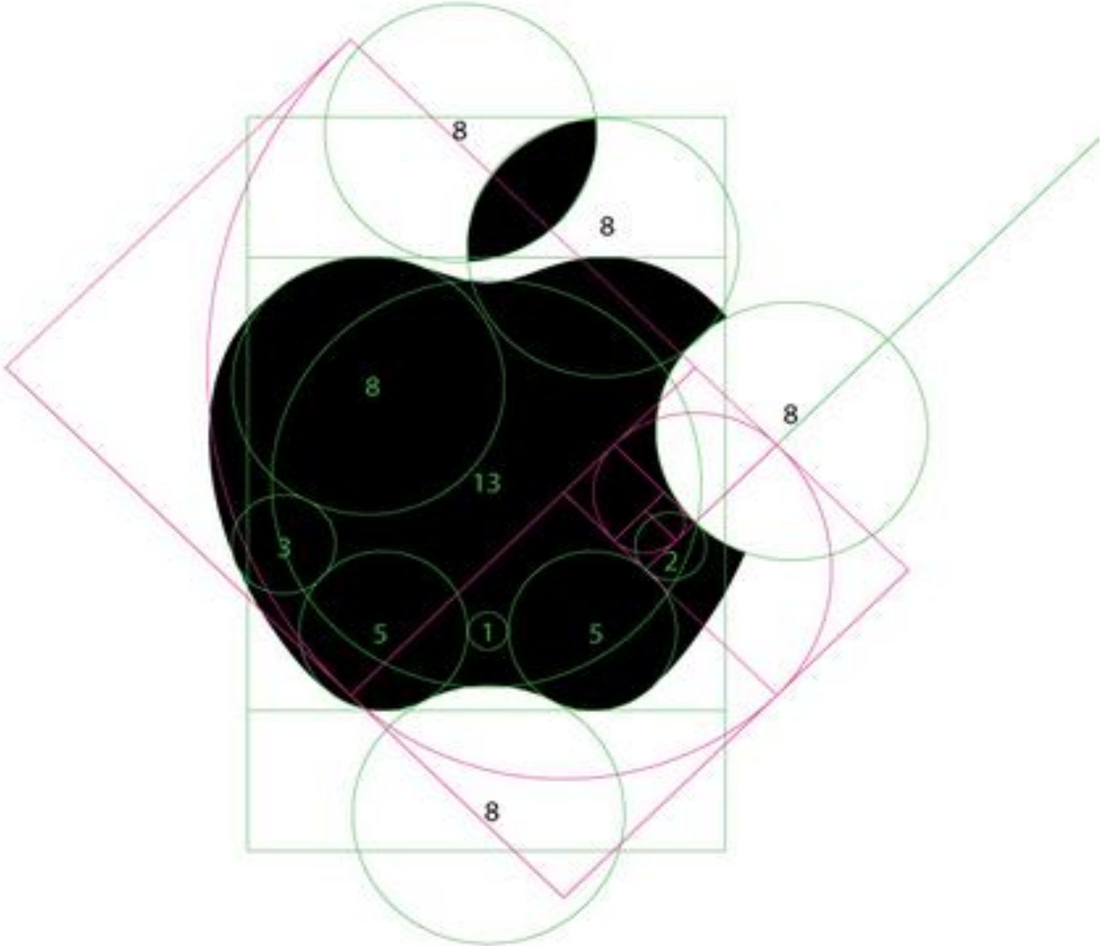
Devil's Paintbrush — 21 Petals

Ox-eyed Daisy — 34 Petals

Sunflower — 55 Petals

Daisy Fleabane — 89 Petals


13
2
3
1
8
5


8
8
8
13
8
3
2
5
1
5
8

$F_0 = 0, F_1 = 1, F_2 = 1 \ldots\ldots$

$F_n = F_{n-1} + F_{n-2}$

Recurrence Relation

Prove the identity: $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

Proof: $F_1^2 = F_2 F_1$

Magic

- There's a 80cm×80cm quadrate tablecloth. How to convert it to a 1.3m×50cm one?

0, 1, 1, 2, 3, 5, 8, 13, 21,……….

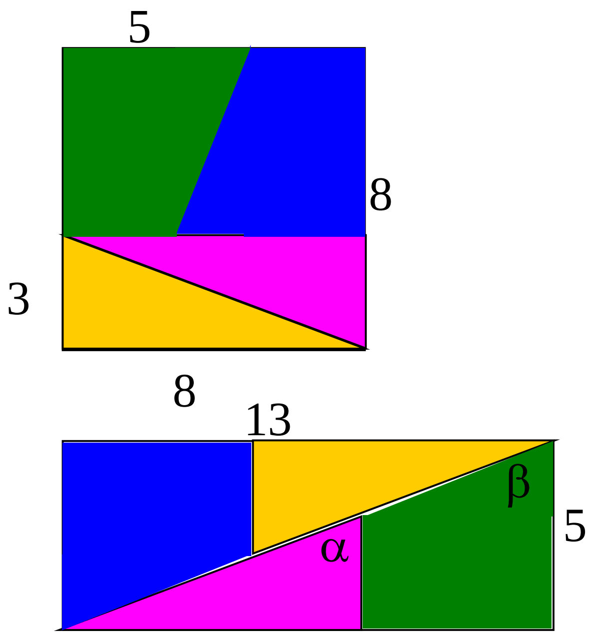
$$\boldsymbol{F(n)*F(n) - F(n-1)F(n+1) = (-1)^n}$$

$\boldsymbol{n=0,1,2}$

Larger tablecloths?

F(100)=?

Direct expressions?



$$\tan\alpha = \frac{8}{3} \cong 2.67, \tan\beta = \frac{5}{2} = 2.5$$

Fibonacci Recurrence

$$F_0 = 0, \quad F_1 = 1, \quad \ldots\ldots$$
$$F_n = F_{n-1} + F_{n-2}$$

Assume $\boxed{G(x) = F_1 x + F_2 x^2 + \cdots}$

$$x^3 : F_3 = F_2 + F_1$$

$$x^4 : F_4 = F_3 + F_2$$

$$+) \quad \cdots\cdots\cdots\cdots\cdots$$

$$G(x) - x^2 - x = x(G(x) - x) + x^2 G(x)$$

$$\therefore \quad (1 - x - x^2)G(x) = x$$

$$\therefore \quad G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

Fibonacci Recurrence

$$\begin{cases} A+B=0 \\ \frac{\sqrt{5}}{2}(A-B)=1 \end{cases} \quad \begin{cases} A+B=0 \\ A-B=\frac{2}{\sqrt{5}} \end{cases} \quad A=\frac{1}{\sqrt{5}}, \quad B=-\frac{1}{\sqrt{5}}$$

$$\therefore \quad G(x)=\frac{1}{\sqrt{5}}\left[\frac{1}{1-\frac{1+\sqrt{5}}{2}x}-\frac{1}{1-\frac{1-\sqrt{5}}{2}x}\right] \quad =\frac{1}{\sqrt{5}}[(\alpha-\beta)x+(\alpha^2-\beta^2)x^2+\cdots]$$

$$\alpha=\frac{-2}{1-\sqrt{5}}=\frac{1+\sqrt{5}}{2}, \quad \beta=\frac{2}{1+\sqrt{5}}=\frac{1-\sqrt{5}}{2}$$

$$\boxed{F_n=\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

$$lim_{n\to\infty}\frac{F_n}{F_{n-1}}=\frac{1+\sqrt{5}}{2}\approx 1.618$$

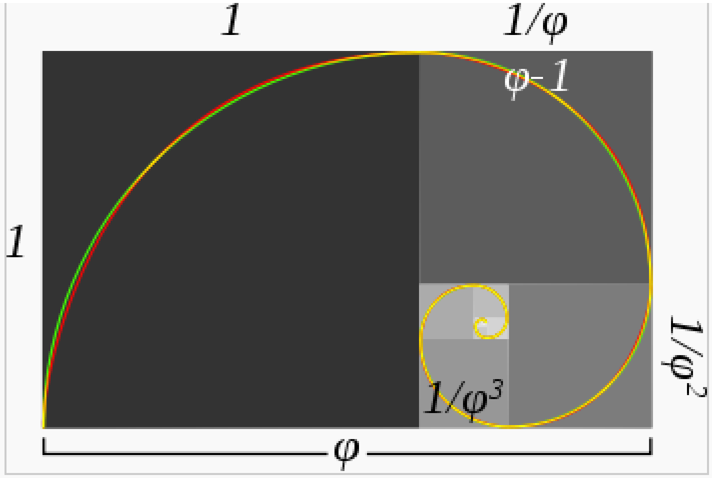
Fibonacci Sequence

$$F_n = F_{n-1} + F_{n-2}$$

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\varphi = [1; 1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$



$F_n - F_{n-1} - F_{n-2} = 0$ $h(n) - 3h(n-1) + 2h(n-2) = 0$

Linear ***Homogeneous*** Recurrence Relations

- Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be a sequence of numbers. This sequence is said to satisfy a linear recurrence relation of order k, provided that there exist quantities $a_1, a_2, \ldots, a_k$, with $a_k \neq 0$, and a quantity b_n (each of these quantities, $a_1, a_2, \ldots, a_k$, b_n may depend on n) such that $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \ldots + a_k h_{n-k} + b_n$
- If $b_n = 0$ and $a_1, a_2, \ldots, a_k$ are constants
- The recurrence relations of the form
- $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \ldots + a_k h_{n-k}$ is **linear homogeneous** recurrence relations

$$h(n) = 2h(n-1) + 1, \quad h(1) = 1$$

$$\boldsymbol{a_n = a_{n-1} + a_{n-2}\, a_{n-3} \quad a_{n-1} = a_{n-2} = a_{n-3} = 1}$$

Consider the following recurrence relations.

(i) $a_n = 3a_{n-1} + a_{n-2}$ √

(ii) $a_n = 3a_{n-1} + 5$

(iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$

(iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}a_{n-3}$ √

(v) $a_n = 3a_{n-1} + na_{n-2}$

Fibonacci Recurrence

$F_n = F_{n-1} + F_{n-2}$ $\quad F_0 = 0,\ F_1 = 1$

Assume $\boxed{G(x) = F_1 x + F_2 x^2 + \cdots}$ $\quad \therefore \quad (1 - x - x^2)G(x) = x$

$$\therefore \quad G(x) = \frac{x}{1 - x - x^2} = \frac{x}{\boxed{(1 - \frac{1-\sqrt{5}}{2}x)(1 - \frac{1+\sqrt{5}}{2}x)}} = \frac{A}{1 - \frac{1+\sqrt{5}}{2}x} + \frac{B}{1 - \frac{1-\sqrt{5}}{2}x}$$

Factoring?

$$(1 - ax)^{-1} = 1 + ax + a^2x^2 + \ldots..$$

$$(1 - x - x^2) = (1 - \frac{1-\sqrt{5}}{2}x)(1 - \frac{1+\sqrt{5}}{2}x)$$

Generating function and recurrence

- Given a linear homogeneous recurrence relation, find out the generating function in the form of P(x)/Q(x)

$$x^3 : F_3 = F_2 + F_1$$

$$x^4 : F_4 = F_3 + F_2$$

$$+) \quad \ldots\ldots\ldots\ldots\ldots$$

$$\therefore \quad (1-x-x^2)G(x) = x$$

- Turn the form of g(x) into

$$\therefore \quad G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

$$\therefore \quad \boxed{G(x) = \frac{A}{1-r_1x} + \frac{B}{1-r_2x}}$$

- Figure out A and B to be c_1 and c_2

$$A = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}$$

- $$f_n = c_1r_1^n + c_2r_2^n$$

$$\mathrm{F_n} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{\mathrm{n}} - \left(\frac{1-\sqrt{5}}{2}\right)^{\mathrm{n}}\right)$$

Fibonacci sequence

- *Fibonacci recurrence relation*
- $\boxed{f_n - f_{n-1} - f_{n-2} = 0}$ $(n \geq 2)$
- Suppose that the solution of the form
  - **$f_n = q^n$ where q is non-zero**
  - $q^n - q^{n-1} - q^{n-2} = 0$
  - $(q^2 - q - 1)q^{n-2} = 0$
  - $\boxed{q^2 - q - 1 = 0}$
  - Find the roots for the quadratic equation $q_1 = \frac{1+\sqrt{5}}{2}, q_2 = \frac{1-\sqrt{5}}{2}$
- Suppose $f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$
- Use the initial conditions
- $n=0$, $f(0)=0$: $c_1 + c_2 = 0$
- $n=1$, $f(1)=1$: $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$$\Rightarrow c_1 = \frac{1}{\sqrt{5}}, c_2 = \frac{-1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Factor Theorem: a polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$

Turn the form of Q(x) into $(1-\alpha_1 x)(1-\alpha_2 x)\ldots$

Replace x with x^{-1} : a polynomial $f(x^{-1})$ has a factor $(x^{-1} - a)=(1-ax)/x$ if and only if $f(a) = 0$

- Fibonacci sequence

$F_0 = 0, \quad F_1 = 1,$

$\boxed{F_n = F_{n-1} + F_{n-2}}$

$$\therefore \quad G(x) = \frac{P(x)}{Q(x)} = \frac{x}{1-x-x^2}$$

Let $F(x) = 1-x-x^2=x^2((x^{-1})^2-x^{-1}-1)=x^2((m)^2-m-1) \quad m=x^{-1}$

$C(m) = \boxed{m^2-m-1}=(m-\alpha)(m-\beta) \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$

$m=x^{-1}$ into F(x)

$F(x) = x^2(x^{-1}- \alpha)(x^{-1}- \beta)=(1- \alpha x)(1- \beta x)$

$$G(x) = \frac{x}{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

- Hanoi Problem

$h(n)-2h(n-1)=1$

$h(n-1)-2h(n-2)=1$ substract

$\boxed{h(n)-3h(n-1)+2h(n-2)=0}$

$$H(x) = \frac{x}{(1-x)(1-2x)} = \frac{x}{1-3x+2x^2}$$

$\boxed{C(x) = x^2-3x+2}$

The root of C(x)=0 is 1 and 2

Characteristic equation

- For a sequence $\{h_n\}$, it has the k-order linear homogeneous recurrence relation as
- Relations: $h_n + C_1 h_{n-1} + C_2 h_{n-2} + \cdots + C_k h_{n-k} = 0,$ $f_n - f_{n-1} - f_{n-2} = 0$
- Initial values: $h_0 = d_0, h_1 = d_1, \cdots, h_{k-1} = d_{k-1},$ $f(0)=0 \quad f(1)=1$
  $C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{k-1}$ are constants.
- The characteristic equation for $\{h_n\}$
  $$C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k$$
  $C(x)=x^2-x-1=0$
- Suppose there are k *distinct* roots for C(x)
  $$C(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_i)$$
  $$q_1 = \frac{1+\sqrt{5}}{2}, q_2 = \frac{1-\sqrt{5}}{2}$$
- Then the ***explicit formula of h_n***
  $$h_n = l_1\alpha_1^n + l_2\alpha_2^n + \cdots + l_k\alpha_k^n$$
  $$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$
  - l_i : undetermined coefficient
- l_i can be determined using the initial values
  - $n=0$, $f(0)=0$: $c_1+c_2=0$
  - $n=1$, $f(1)=1$: $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$$\Rightarrow \quad c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = \frac{-1}{\sqrt{5}}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Linear Homogeneous Recurrence Relation

$F_n - F_{n-1} - F_{n-2} = 0$ $\qquad$ $h(n) - 3h(n-1) + 2h(n-2) = 0$

$x^2 - x - 1 = 0$ $\qquad$ $x^2 - 3x + 2 = 0$

Def if sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1},$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{n-1}$ are constants, $C_k \neq 0$, then this expression is called a k[th]-order linear homogeneous recurrence relation of $\{a_n\}$.

$$\boxed{C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k}$$

Characteristic Polynomial

单选题 1分

设置

Please choose the corresponding characteristic equation for the following recurrence relation

$$a_n = 10a_{n-1} + 40a_{n-2},$$

A $x^n - 10x^{n-1} - 40x^{n-2} = 0$

B $40x^2 + 10x - 1 = 0$

C $x^2 - 10x - 40 = 0$

D $x^2 + 10x + 40 = 0$

提交

Example

- Solve the recurrence relation

  $h_n=5h_{n-1}-6h_{n-2}$,n≥2, $h_0=1$, $h_1=-2$

$$h_n + C_1 h_{n-1} + C_2 h_{n-2} + \cdots + C_k h_{n-k} = 0,$$

- Characteristic equation

  $x^2-5x+6=0$

$$C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k$$

  The roots are 2 and 3

$$h_n = l_1 \alpha_1^n + l_2 \alpha_2^n + \cdots + l_k \alpha_k^n$$

  $h_n = A(2)^n + B(3)^n$

  $h_0=1$, $h_1=-2$

  $h_0=1$: $A+B=1$

  $h_1=2$: $2A+3B=-2$ $\Longrightarrow$ $A=5$, $B=-4$

  $h_n = 5(2)^n - 4(3)^n$

Linear Homogeneous Recurrence Relation

Def If sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0, \quad (2-5-1)$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1}, \quad (2-5-2)$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{k-1}$ are constants

Characteristic Polynomial $C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k$

1）Characteristic polynomial has k distinct real roots

$$C(x) = (x - a_1)(x - a_2)\cdots(x - a_k)$$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

In which $l_1, l_2, \cdots l_k$ are undetermined coefficients.

Todo List

- Homework sheet due on Monday
- No preClass material

Thanks