Q1

(a) $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t-5)e^{-j\omega t}\,dt = e^{-j5\omega} = \cos 5\omega - j\sin 5\omega,$

by the sifting property of the unit impulse.

$$|X(\omega)| = |e^{j5\omega}| = 1 \qquad \text{for all } \omega,$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = \tan^{-1}\left(\frac{-\sin 5\omega}{\cos 5\omega}\right) = -5\omega$$

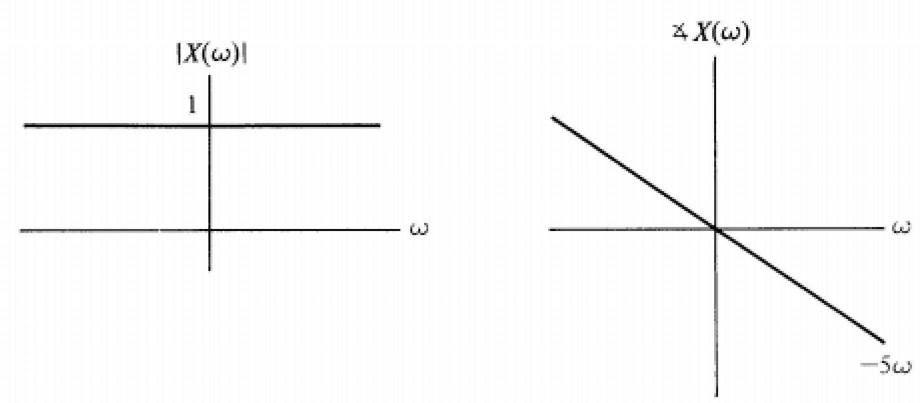


Figure S8.2-1

(b) $X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\,dt = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

Since $Re\{a\} > 0$, e^{-at} goes to zero as t goes to infinity. Therefore,

$$X(\omega) = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega},$$

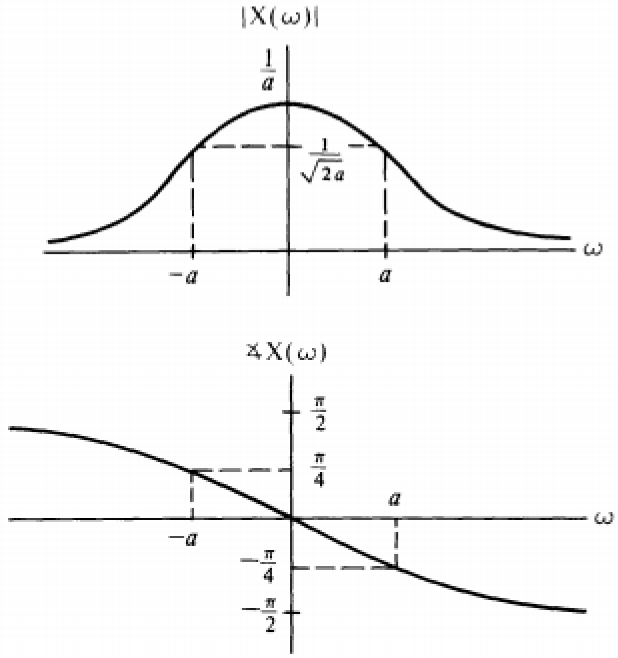
$$|X(\omega)| = [X(\omega)X^*(\omega)]^{1/2} = \left[\frac{1}{a+j\omega}\left(\frac{1}{a-j\omega}\right)\right]^{1/2} \frac{1}{\sqrt{a^2+\omega^2}},$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{a}{a^2+\omega^2},$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2} = \frac{-\omega}{a^2+\omega^2},$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}\frac{\omega}{a}$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-2.



$|X(\omega)|$
$\frac{1}{a}$
$\frac{1}{\sqrt{2}a}$
ω
$-a$
a
$\measuredangle X(\omega)$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
a
ω
$-a$
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$

Figure S8.2-2

Q2

Ex. 6

$$x(t) = \frac{b}{t^2+b}$$

If $X(j\omega)$ is the F.T. of $x(t)$, we have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega \cdot X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{j}{2\pi}\int \omega \cdot X(j\omega)\, e^{j\omega t}\, d\omega.$$

$$\therefore \left.\frac{d}{dt}x(t)\right|_{t=0} = \frac{j}{2\pi}\int_{-\infty}^{\infty} \omega \cdot X(j\omega)\, d\omega.$$

$$\therefore \int_{-\infty}^{\infty} \omega \cdot X(j\omega)\, d\omega = \left(\frac{2\pi}{j}\right)\cdot\left(\frac{d}{dt}x(t)\right)_{t=0}.$$

$$\frac{d}{dt}x(t) = \frac{-b}{(t^2+b)^2}\cdot 2t \quad \& \quad \left.\frac{d}{dt}x(t)\right|_{t=0} = 0.$$

$$\therefore \int_{-\infty}^{\infty} \omega\, X(j\omega)\, d\omega = 0.$$

Q3

S20.5

There are two ways to solve this problem.

Method 1

This method is based on recognizing that the system input is a superposition of eigenfunctions. Specifically, the eigenfunction property follows from the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)\, d\tau$$

Now suppose $x(t) = e^{at}$. Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{a(t-\tau)}\, d\tau = e^{at} \int_{-\infty}^{\infty} h(\tau)e^{-a\tau}\, d\tau$$

Now we recognize that

$$\int_{-\infty}^{\infty} h(\tau)e^{-a\tau}\,d\tau = H(s)\bigg|_{s=a},$$

so that if $x(t) = e^{at}$, then

$$y(t) = \left[H(s)\bigg|_{s=a} \right] e^{at},$$

i.e., e^{at} is an eigenfunction of the system.

Using linearity and superposition, we recognize that if

$$x(t) = e^{-t/2} + 2e^{-t/3},$$

then

$$y(t) = e^{-t/2}H(s)\bigg|_{s=-1/2} + 2e^{-t/3}H(s)\bigg|_{s=-1/3}$$

so that

$$y(t) = 2e^{-t/2} + 3e^{-t/3} \qquad \text{for all } t.$$

Method 2

We consider the solution of this problem as the superposition of the response to two signals $x_1(t)$, $x_2(t)$, where $x_1(t)$ is the noncausal part of $x(t)$ and $x_2(t)$ is the causal part of $x(t)$. That is,

$$x_1(t) = e^{-t/2}u(-t) + 2e^{-t/3}u(-t),$$
$$x_2(t) = e^{-t/2}u(t) + 2e^{-t/3}u(t)$$

This allows us to use Laplace transforms, but we must be careful about the ROCs.

Now consider $\mathcal{L}\{x_1(t)\}$, where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform:

$$\mathcal{L}\{x_1(t)\} = X_1(s) = -\frac{1}{s+\frac{1}{2}} - \frac{2}{s+\frac{1}{3}}, \qquad Re\{s\} < -\frac{1}{2}$$

Now since the response to $x_1(t)$ is

$$y_1(t) = \mathcal{L}^{-1}\{X_1(s)H(s)\},$$

then

$$\begin{aligned} Y_1(s) &= -\frac{1}{(s+1)(s+\frac{1}{2})} - \frac{2}{(s+\frac{1}{3})(s+1)}, \qquad -1 < Re\{s\} < -\frac{1}{2}, \\ &= \frac{2}{s+1} + \frac{-2}{s+\frac{1}{2}} + \frac{-3}{s+\frac{1}{3}} + \frac{3}{s+1}, \\ &= \frac{5}{s+1} - \frac{2}{s+\frac{1}{2}} - \frac{3}{s+\frac{1}{3}}, \end{aligned}$$

so

$$y_1(t) = 5e^{-t}u(t) + 2e^{-t/2}u(-t) + 3e^{-t/3}u(-t)$$

The pole-zero plot and associated ROC for $Y_1(s)$ is shown in Figure S20.5-1.

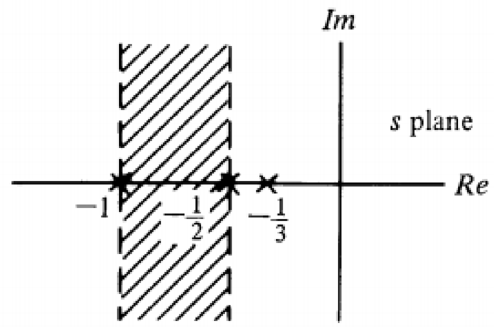


Figure S20.5-1

Next consider the response $y_2(t)$ to $x_2(t)$:

$$x_2(t) = e^{-t/2}u(t) + 2e^{-t/3}u(t),$$

$$X_2(s) = \frac{1}{s+\frac{1}{2}} + \frac{2}{s+\frac{1}{3}}, \qquad Re\{s\} > -\frac{1}{3},$$

$$Y_2(s) = X_2(s)H(s) = \frac{1}{(s+\frac{1}{2})(s+1)} + \frac{2}{(s+\frac{1}{3})(s+1)},$$

$$Y_2(s) = \frac{2}{s+\frac{1}{2}} + \frac{-2}{s+1} + \frac{3}{s+\frac{1}{3}} + \frac{-3}{s+1},$$

so

$$y_2(t) = -5e^{-t}u(t) + 2e^{-t/2}u(t) + 3e^{-t/3}u(t)$$

The pole-zero plot and associated ROC for $Y_2(s)$ is shown in Figure S20.5-2.

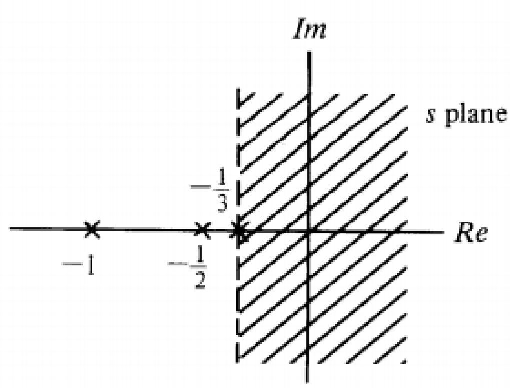


Figure S20.5-2

Since $y(t) = y_1(t) + y_2(t)$, then

$$y(t) = 2e^{-t/2} + 3e^{-t/3} \qquad \text{for all } t$$

Q4

(b) Calculating the Laplace transform of both sides, we obtain

$$\left[s^2Y(s) - s\underbrace{y(0^-)}_{=0} - \underbrace{\dot{y}(0^-)}_{=0}\right] + 4\left[sY(s) - \underbrace{y(0^-)}_{=0}\right] + 4Y(s) = \tfrac{1}{s}$$

or, $(s^2+4s+4)Y(s) = \frac{1}{s}$ or $Y(s) = \frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$,

where the partial fraction coefficients are calculated as

$$A = \left.\tfrac{1}{(s+2)^2}\right|_{s=0} = \frac{1}{4}, \text{ and } C = \left[\tfrac{1}{s}\right]_{s=-2} = -\frac{1}{2}$$

Expanding $Y(s)$, and comparing the numerator of both sides, we get

$$\begin{aligned}1 &= A(s+2)^2 + Bs(s+2) + Cs\\ &= (A+B)s^2 + (4A+2B+C) + 4A\end{aligned}$$

Comparing the coefficients of s^2 in both sides, we get $(A+B) = 0$ or $B = -1/4$.

In other words, $Y(s) = \frac{1}{s(s+2)^2} = \frac{1/4}{s} - \frac{1/4}{s+2} - \frac{1/2}{(s+2)^2}$

Calculating the inverse Laplace transform of Y(s) yields

$$y(t) = \tfrac{1}{4}\left[1 - e^{-2t} - 2te^{-2t}\right]u(t) = \tfrac{1}{4}\left[1 - (2t+1)e^{-2t}\right]u(t).$$

(c) Calculating the Laplace transform of both sides, we obtain

$$\left[s^2Y(s) - s\underbrace{y(0^-)}_{=1} - \underbrace{\dot{y}(0^-)}_{=1}\right] + 6\left[sY(s) - \underbrace{y(0^-)}_{=1}\right] + 8Y(s) = \tfrac{1}{(s+3)^2}$$

or, $(s^2+6s+8)Y(s) = \frac{1}{(s+3)^2} + (s+1+6)$ or $Y(s) = \frac{1}{(s+2)(s+3)^2(s+4)} + \frac{s+7}{(s+2)(s+4)}$.

Calculating the partial fraction expansion of the two terms separately, we obtain

$$\frac{1}{(s+2)(s+3)^2(s+4)} = \frac{1/2}{s+2} + \frac{0}{s+3} - \frac{1}{(s+3)^2} - \frac{1/2}{s+4}$$

$$\text{and } \frac{s+7}{(s+2)(s+4)} = \frac{5/2}{s+2} - \frac{3/2}{s+4}$$

Expanding $Y(s)$ as

$$Y(s) = \tfrac{1/2}{s+2} - \tfrac{1}{(s+3)^2} - \tfrac{1/2}{s+4} + \tfrac{5/2}{s+2} - \tfrac{3/2}{s+4} = \tfrac{3}{s+2} - \tfrac{1}{(s+3)^2} - \tfrac{2}{s+4}.$$

Calculating the inverse Laplace transform of Y(s) yields

$$y(t) = \left(3e^{-2t} - te^{-3t} - 2e^{-4t}\right)u(t).$$

Q5

Note that $$V_1(s) = \phi_1(s) - \theta(s)$$

and $$V_2(s) = K_1 G(s) V_1(s) = K_1 G(s)[\phi_1(s) - \theta(s)].$$

Substituting $\theta(s) = V_2(s)/s$ in the above equation, we get

$$V_2(s) = K_1 G(s)[\phi_1(s) - V_2(s)/s]$$

or, $$V_2(s) = \frac{sK_1 G(s)}{s + K_1 G(s)} \phi_1(s).$$

The output $$V(s) = K_2 V_2(s) = K_2 \frac{sK_1 G(s)}{s + K_1 G(s)} \phi_1(s),$$

which results in the transfer function $H(s) = \dfrac{V(s)}{\phi(s)} = K_1 K_2 \dfrac{sG(s)}{s + K_1 G(s)}$.

Differentiator: For the PLL to behave as an ideal differentiator, its transfer function $H(s) = Ks$, i.e.,

$$H(s) = Ks = K_1 K_2 \frac{sG(s)}{s + K_1 G(s)},$$

or, $$K(s + K_1 G(s)) = K_1 K_2 G(s).$$

Solving in terms of $G(s)$, we obtain $$G(s) = \frac{K}{K_1(K_2 - K)} s.$$

Another way of obtaining an ideal differentiator is to set $K_1 \to \infty$ in the transfer function as shown below:

$$\lim_{K_1 \to \infty} H(s) = \lim_{K_1 \to \infty} K_2 \frac{sG(s)}{s/K_1 + G(s)} = K_2 s.$$ ∎