Experiment No.-5: Z-transform and inverse Z-transform

1. Overview

In this lab we will use MATLAB to find z-transform, inverse z-transform of LTI systems and analyze the stability of LTI systems in terms of the pole-zero location of their transfer function.

2. Z-transform

The bilateral z transform of a discrete-time signal $x[n]$,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

is a generalization of the discrete-time Fourier transform that is useful for studying discrete-time signals and systems. Note that if $z = e^{j\omega}$, the z-transform reduces to the discrete-time Fourier transform. However, the bilateral z-transform exists for a broader class of signals than the discrete-time Fourier transform does, and is useful for understanding the behavior of both stable and unstable systems. For a large class of signals, the z-transform can be represented as a ratio of polynomials in z, i.e.,

$$X(z) = \frac{N(z)}{D(z)}$$

These transforms are called rational transforms and arise as the system functions of LTI systems which satisfy linear constant-coefficient difference equations. The locations of the roots $N(z)$ and $D(z)$, known as the zeros and poles of the system, respectively, determine to within a constant multiplicative factor the behavior of LTI systems with rational transforms. Therefore, plots of the pole and zero locations can be used to analyze system properties.

A causal LTI system is one whose unit sample response $h[n]$ satisfies the condition

$$h[n] = 0, \quad n < 0$$

We also know that the ROC of the z-transform of a causal sequence is the exterior of a circle. Consequently, a linear time-invariant system is causal if and only if the ROC of the system function is the exterior of a circle or radius $r < \infty$ including the point $z = \infty$. The stability of a LTI system can also be expressed in terms the characteristics of the system function. The necessary and sufficient condition for a LTI system to be BIBO stable is

$$\sum_{n=-\infty}^{\infty} |h[n]| \leq \infty$$

In turn, this condition implies that $H(z)$ must contain the unit circle. Hence if the system is BIBO stable, the unit circle is contained in the ROC of $H(z)$. The converse is also true. Therefore, a linear time-invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle. Note that ROC cannot contain any poles of $H(z)$ and if we also consider the ROC of a causal system then it follows that a causal LTI system is BIBO stable if and only if all the poles are inside the unit circle.

3. Exercise

1. Write a program in MATLAB to find the Z – transform for the given input sequence :

i) $x(n) = \left(\frac{1}{4}\right)^n u(n)$ ii) $x(n) = \left(2(2)^n + 4\left(\frac{1}{2}\right)^n\right)u(n)$

2. Write a program in MATLAB to find the inverse Z - transform of :

   i) X (z) = $\frac{4z}{4z-1}$ ii) X (z) = $\frac{2z}{z-2}+\frac{4z}{z-\frac{1}{2}}$

3. Write a program in MATLAB to sketch pole – zero plot, impulse response and frequency response for the system described by the transfer function,

$$H(z) = \frac{z^{-1}}{1 - 0.25\, z^{-1} - 0.375z^{-2}}$$

   Also, comment on the stability of the system form its pole-zero plot.