An Introduction to the ϕRM and DCM methods

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This document is meant as a summary of the methods of characterising/fitting notch resonators and obtaining their Quality factors.
A major contribution to these methods was by [1, 2] and we shall be majorly discussing the results from these two documents[1].

1 Modelling a Notch Resonator

A resonator can be characterised by measuring its S_{21}. Such a notch resonator circuit is shown in 1(a) where the $\hat{C}$ and L form a frequency reject resonator which is weakly coupled to the circuit by capacitive and inductive modes, namely through C_c and L_1, M

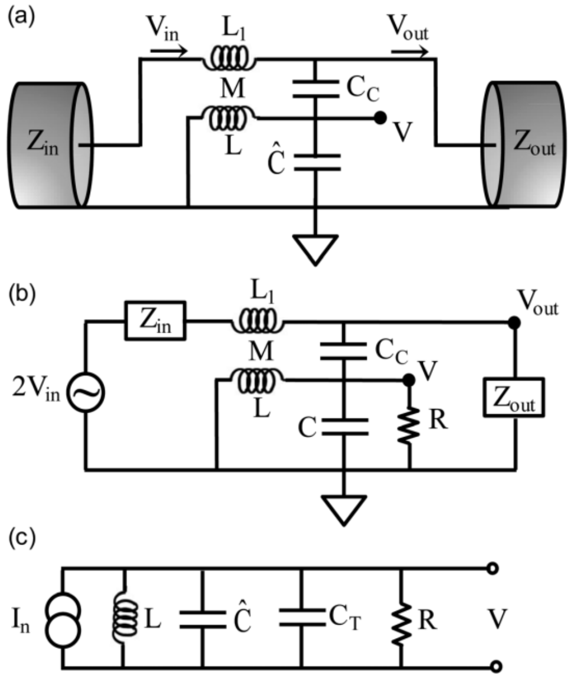

Figure 1: (a) Schematic of a notch resonator setup with both inductive and capacitive coupling and mismatched transmission lines. (b) Equivalent circuit to (a) assuming high internal Quality factors, where $\hat{C}$ has been separated into its capacitive part (C) and its resistive (dissipative) parts ($1/\omega R$) (c) Norton equivalent circuit for resonator, where V is the voltage across the capacitor $\hat{C}$ (Figure reproduced without permission from [1])

Solving Kirchoff's equations and expanding to the third order in small parameters ($(M/L), \omega C_c Z_{\text{out}}$),

[1]It is recommended to read [1] and Appendix E of [2] for greater clarity of material presented in this document

we get[2] (Equation 11 in [1] or equivalently E.1 in Appendix E of [2][3]),

$$S_{21} = ae^{-2\pi i f\tau}\left(1 - \frac{Q_r Q_c^{-1}}{1 + 2iQ_r \frac{f-f_r}{f_r}}\right) \tag{1}$$

$$\equiv S_{21} = ae^{-2\pi i f\tau}\left(1 - \frac{Q_r |Q_c|^{-1} e^{i\phi}}{1 + 2iQ_r \frac{f-f_r}{f_r}}\right) \tag{2}$$

where

- a: Complex constant correcting for the gain and phase shift (mostly the background)
- τ: Cable delay
- Q_r: Total Quality factor measured of the resonator
- Q_c: Coupling Quality Factor
- f_r: Resonant frequency (Remember that this might not be the $\mathrm{argmin}_f\, |S_{21}(f)|$ due to the asymmetric lineshape)
- ϕ: Phase of the coupling quality factor (This is the major source of asymmetry and the reason why the "Diameter Correction Method" was necessary)

As one might imagine, fitting 7 parameters to the obtained data is not a robust operation i.e. there are a lot of local minima that are hard to get out of. To solve this problem, the ϕRM method was invented. This method breaks the fitting into multiple fits with lower number of parameters to get an excellent initial guess for a final refining fit with all 7 parameters.

2 The ϕRM method aka the ϕ Rotation Method

This method takes advantage of the form of the expression for S_{21} and uses the fact that the $(\mathrm{Re}\{S_{21}\}, \mathrm{Im}\{S_{21}\})$ plot a circle on the complex plane when one removes the cable delay. So, the steps are,

1. Approximately remove the cable delay: This can be done by making sure that the phase of the data far from resonance does not change much in time after applying the phase correction i.e. multiplying the obtained S_{21} with $e^{2\pi i f\tau}$ and finding the τ by eye on the VNA
2. Approximately correct for a: One can correct for a by taking a point far away from resonance and dividing all the data by it making the assumption that a is a constant. A better correction can be obtained by linear detrending.
3. Fit a circle to the corrected data: Plotting the $(\mathrm{Re}\{S_{21}\}, \mathrm{Im}\{S_{21}\})$, fit it to a circle in the complex plane. This can be done very easily and quickly using multiple methods like the Kasa fit, or the Taubin fit. More methods and implementations of circle fits can be found at `https://people.cas.uab.edu/~mosya/cl/CPPcircle.html`. Using the fit, one can obtain the center z_c and the radius r of the circle which we shall make use of later.
4. Rotate and Translate the circle back to origin: First translate the circle to origin and then rotate it by the phase of z_c. This is the step which lends its name to the method.

[2] Those with some background in electronic systems are encouraged to try and derive these equations on their own. Some help in doing the same might be obtained by going through [1] which lays out the steps of performing such an exercise.

[3] The equation in this document intends to provide the best of both worlds from the two cited documents, that is, the clarity of notation of [2] and the technical correctness of [1]

5. Perform a fit in phase space: Let us call this corrected rotated and translated data as z. So, the $\theta = \arg z$ follows
$$\theta = -\theta_0 + 2\tan^{-1}\left(2Q_r\left(1 - \frac{f}{f_r}\right)\right) \tag{3}$$
Fitting our data to this, we get Q_r, f_r, θ_0 which can be used to obtain the other parameters as
$$|Q_c| = \frac{Q_r}{2r}$$
$$\phi_0 = \theta_0 - \arg z_c$$
6. Refine the fit by fitting the data to all the parameters using the obtained values as initial guesses. This can get us down to χ^2 per DOF very close to 1

Now, one might ask, "If the method fits the curve so well, why another method?". The answer to that would be in the details. So, using the $|Q_c|$ and Q_r obtained, if one goes about finding the internal quality factor of the resonator (Q_i), one would get one might erroneously use
$$\frac{1}{Q_r} = \frac{1}{|Q_c|} + \frac{1}{Q_i} \tag{4}$$
and as a matter of fact, this was the formula used for quite a while and no one could detect that there was something wrong with it. The "Diameter Correction Method" corrects for this and explains why.

3 The Diameter Correction Method aka DCM

The real formula for calculating Q_i is,
$$\frac{1}{Q_r} = \mathrm{Re}\left\{\frac{1}{Q_c}\right\} + \frac{1}{Q_i} \tag{5}$$
So, a simple substitution gives,
$$\frac{1}{Q_r} = \frac{\cos\phi}{|Q_c|} + \frac{1}{Q_i} \tag{6}$$
This makes it clear why the problem was not found earlier. There was not enough asymmetry to make ϕ large enough. As one knows, for small ϕ, $\cos\phi \approx 1 - \phi^2/2$, so the error is $\mathcal{O}(\phi^2)$ which is small enough for most cases. The reason why the method is called the "Diameter Correction Method" is that the method can be rewritten as ϕRM where after the rotation and translation step, another diameter correction step is added where the diameter is scaled by a factor of $\cos\phi$

References

[1] Khalil, M.S. et al. (2012) 'An analysis method for asymmetric resonator transmission applied to superconducting devices', Journal of Applied Physics, 111(5), p. 054510. doi:10.1063/1.3692073.

[2] Gao, J. 'The Physics of Superconducting Microwave Resonators', p. 197.