1 Definitions and reminders

1.1 Species and Operads

In all the following $\mathbb{k}$ is a field, $\mathbb{S}\text{et}$ is the category of finite sets and bijections, $\underline{\mathbb{S}\text{et}}$ the category of finite set and maps and $\mathbb{V}\text{ect}_{\mathbb{k}}$ the category of $\mathbb{k}$-vector spaces and linear maps.

Definition 1. A *vector* (resp *set*) *species* is a functor from $\mathbb{S}\text{et}$ to $\mathbb{V}\text{ect}_{\mathbb{k}}$ (resp $\underline{\mathbb{S}\text{et}}$). More informally, a vector (resp set) species S consists of the following data.

- For each finite set V, a vector space (resp set) $S[V]$.
- For each bijection of finite sets $\sigma : V \to V'$, a linear map (resp map) $S[\sigma] : S[V] \to S[V']$. These maps should be such that $S[\sigma \circ \tau] = S[\sigma] \circ S[\tau]$ and $S[\text{id}] = \text{id}$.

A *morphism* between vector (resp set) species is a natural transformation. More informally, a *morphism* $f : R \to S$ between vector (resp set) species is a collection of linear maps (resp maps) $f_V : R[V] \to S[V]$ satisfying the naturality axiom: for each bijection $\sigma : V \to V'$, $f_{V'} \circ R[\sigma] = S[\sigma] \circ f_V$.

We note $\mathcal{L}$ the functor from set species to vector species defined by $L(S)[V] = \mathbb{k}S[V]$, where $\mathbb{k}S[V]$ is the free $\mathbb{k}$ vector space on $S[V]$, and $L(f)_V$ the linear extension of f. We will also note $\mathbb{k}S$ for $\mathcal{L}(S)$.

We will use the term species to refer to vector species. For S a species, V a set and f a morphism from S we will note f instead of f_V when no confusion is possible.

We note 1 the set species defined by $1[V] = \{\emptyset\}$ if $V = \emptyset$ and $1[V] = \emptyset$ else; as well as X the set species defined by $X[V] = \{v\}$ if $V = \{v\}$ and $X[V] = \emptyset$ else.

A partition of V is a subset of $\{\pi_1, \dots, \pi_n\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ such that $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$ and $\sqcup_i \pi_i = V$. We note Π the set species of partitions.

Definition 2. Let R, S, R' and S' be four species and $f : R \to R'$ and $g : S \to S'$ two morphisms.

- The *sum* of R and S is the species $R + S$ defined by $(R + S)[V] = R[V] \oplus S[V]$ for all finite set V and $(R + S)[\sigma]|_{R[V]} = R[\sigma]$ and $(R + S)[\sigma]|_{S[V]} = S[\sigma]$ for all finite sets V, V' and bijections $\sigma : V \to V'$.

  The *sum* of f and g is the morphism $f + g : R + S \to R' + S'$ defined by $(f + g)_V = f_V \oplus g_V$.

- The *product* of R and S is the species $R \cdot S$ defined by $R \cdot S[V] = \bigoplus_{V_1 \sqcup V_2 = V} R[V_1] \otimes S[V_2]$ and $R \cdot S[\sigma] = \bigoplus_{V_1 \sqcup V_2 = V} R[\sigma|_{V_1}] \otimes S[\sigma|_{V_2}]$.

  The *product* of f and g is the morphism $f \cdot g : R \otimes S \to R' \otimes S'$ defined by $(f \cdot g)_V = \bigoplus_{V_1 \sqcup V_2 = V} f_{V_1} \otimes g_{V_2}$.

- The *Hadamard product* of R and S is the species $R \times S$ defined by $(R \times S)[V] = R[V] \otimes S[V]$ (resp $R[V] \times S[V]$) and $(R \times S)[\sigma] = R[\sigma] \otimes S[\sigma]$ (resp $R[\sigma] \times S[\sigma]$).

  The Hadamard product of f and g is the morphism $f \times g$ defined by $(f \times g)_V = f_V \otimes g_V$ (resp $(f \times g)_V = f_V \times g_V$).

- The *derivative* of R is the species $DR = R'$ defined by $R'[V] = R[V + \{*\}]$ where the $*$ is a "ghost element" not already in V and $R'[\sigma] = R[\sigma']$ where $\sigma' = \sigma$ on V and $\sigma(*) = *$. The *n-th derivative* of R is the species $D^n R = R^{(n)}$ recursively defined by $D^n R = D(D^{n-1} R)$.

  The *n-derivative* of f is the morphism defined by $(D^n f)_V = f_{V + \{*_1, \dots, *_n\}}$

- The *substitution* by S in R is the species $R(S)$ defined by $R(S)[V] = \bigoplus_{P \in \Pi[V]} R[\pi] \bigotimes_{P \in \pi} S(P)$ and $R(S)[\sigma]$ is defined by following the definition of sum and product of species.

  The *substitution* by g in f is defined by following the definitions of sum and product of morphisms.

We have the same definition on set species by replacing direct sum of vector spaces by disjoint unions of sets and tensor product by Cartesian product.

Note that we stated these were species and morphisms without checking that these definition were functional and natural. Those verifications can be found in (ref...). These definitions are compatible with $\mathcal{L}$ i.e $\mathcal{L}(R+S)=\mathcal{L}(R)+\mathcal{L}(S)$, $\mathcal{L}(f+g)=\mathcal{L}(f)+\mathcal{L}(g)$, $\mathcal{L}(R\cdot S)=\mathcal{L}(R)\cdot\mathcal{L}(S)$ etc.

When no confusion is possible we will note S instead of id_S.

Definition 3. A *linear operad* is a triplet $(\mathcal{O},\eta,e)$ where:

- $\mathcal{O}$ is a vector species,
- η is a morphism of vector species $\mathcal{O}(\mathcal{O})\to\mathcal{O}$,
- e is a morphism of vector species $X\to\mathcal{O}$.

The two morphisms η and e should furthermore satisfy the associativity and naturality axioms, i.e the two following diagrams commute, where α, ρ and λ are the structural morphisms of the monoidal category of vector species (ref...) .

$$\begin{tikzcd}
\mathcal{O}(\mathcal{O}(\mathcal{O})) \arrow[r, "\mathcal{O}(\eta)"] \arrow[d, "\alpha"] & \mathcal{O}(\mathcal{O}) \arrow[r, "\eta"] & \mathcal{O} \\
(\mathcal{O}(\mathcal{O}))(\mathcal{O}) \arrow[r, "\eta(\mathcal{O})"] & \mathcal{O}(\mathcal{O}) \arrow[ur, "\eta"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\mathcal{O}(e)"] \arrow[dr, "\rho"'] & \mathcal{O}(\mathcal{O}) \arrow[d, "\eta"] & X(\mathcal{O}) \arrow[l, "e(\mathcal{O})"'] \arrow[dl, "\lambda"] \\
& \mathcal{O} &
\end{tikzcd}$$

We will use the term operad to refer to linear operad.

Let S be a species. We call *partial product* on S a morphism $\circ_*: S'\cdot S\to S$. A partial product naturally generalise to families of n morphisms $D^nS\otimes D^mS\to D^{n+m-1}S$ by the injections

$$\begin{aligned}
D^nS[V_1]\otimes D^mS[V_2] &\cong S'[V_1+\{*_1,\dots,*_{i-1},*_{i+1},\dots,*_n\}]\otimes S[V_2+\{*'_1,\cdots *'_m\}] \\
&\hookrightarrow (S'\cdot S)[V_1+V_2+\{*_1,\dots,*_{i-1},*_{i+1},\dots,*_n\}+\{*'_1,\cdots *'_m\}].
\end{aligned}$$

Proposition 4. Let $\mathcal{O}$ be a species equipped with a *partial product* $\circ_*:\mathcal{O}'\cdot\mathcal{O}\to\mathcal{O}$ and a morphism $e: X\to\mathcal{O}$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathcal{O}''\cdot\mathcal{O}^2 \arrow[r, "\circ_{*_1}"] \arrow[d, "\circ_{*_2}\circ\mathrm{id}\cdot\tau"'] & \mathcal{O}'\cdot\mathcal{O} \arrow[d, "\circ_{*_2}"] \\
\mathcal{O}'\cdot\mathcal{O} \arrow[r, "\circ_{*_1}"] & \mathcal{O}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}'\cdot\mathcal{O}'\cdot\mathcal{O} \arrow[r, "\circ_{*_1}\cdot\mathrm{id}"] \arrow[d, "\mathrm{id}\cdot\circ_{*_2}"'] & \mathcal{O}'\cdot\mathcal{O} \arrow[d, "\circ_{*_2}"] \\
\mathcal{O}'\cdot\mathcal{O} \arrow[r, "\circ_{*_1}"] & \mathcal{O}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}'\cdot X \arrow[r, "\mathcal{O}'\cdot e"] \arrow[dr, "p"'] & \mathcal{O}'\cdot\mathcal{O} \arrow[d, "\circ_*"] & X'\cdot\mathcal{O} \arrow[l, "e'\cdot\mathcal{O}"'] \arrow[dl, "\cong"] \\
& \mathcal{O} &
\end{tikzcd}$$

where $\tau_V: x\otimes y\in\mathcal{O}^2[V]\mapsto y\otimes x\in\mathcal{O}^2[V]$ and $p_V: x\otimes\emptyset_{\{v\}}\mapsto\mathcal{O}[*\mapsto v](x)$ with $*\mapsto v: V\setminus\{v\}+\{*\}\to V$ the bijection that sends $*$ on v and is the identity on $V\setminus\{v\}$. Then $\mathcal{O}$ has an operad structure $(\mathcal{O},\eta,e)$ uniquely determined by $\circ_*$.

1.2 Multisets, graphs and co

Let V be a set. A *multiset* m over V is a set of couples $\{(v, m(v)) \mid v \in V\}$ in $V \times \mathbb{N}^*$. We call V the domain of m and note $D(m) = V$. We say that v is in m and note $v \in m$ if $v \in D(m)$. For any element v not in the domain of m, we note $m(v) = 0$.

We note $\mathcal{M}(V)$ the set of multisets with domain in $\mathcal{P}(V)$, $\mathcal{M}_k(V)$ the set of elements of $\mathcal{M}(V)$ of cardinality k (the cardinality of a multiset m over V being $\sum_{v\in V} m(v)$) and $\mathcal{M}(V)^*$ the set of multisets with domain in $\mathcal{P}(V)^* = \mathcal{P}(V) \setminus \{\emptyset\}$. We indentify sets with multisets constant equal to 1.

For m a multiset and V a set, we note $m \cap V = m \cap V \times \mathbb{N}^*$. If m' is an other multiset, we call the union of m and m' the multiset $\{(v, m(v) + m'(v)) \mid v \in D(m) \cup D(m')\}$ where $m(v) = 0$ if $v \not inD(m)$ and $m'(v) = 0$ if $v \notin D(m')$. A *decomposition* of a mutliset m is a tuple of multisets $(m_1, \ldots, m_k)$ such that $\bigcup D(m_i) = D(m)$ and $m(v) = m_i(v)$ for all $v \in D(m)$.

Definition 5. Let V de a set. A *multi-hypergraph* over V is a multiset with domain in $\mathcal{M}(V)^*$. In this context the elements of V are called *vertices*, the elements of a multi-hypergraphs are called *edges* and the elements of an edge are called its *ends*. A vertex contained in the domain of no edge is called *isolated vertex*. We note MHG the set species of multi-hypergraphs.

A *hypergraph* is a multi-hypergraph whose edges are sets. A *mutltigraph* is a multi-hypergraph whose edges have cardinality 2. A *graph* is a multi-hypergraph which is a hypergraph and a multigraph at the same time. Note MHG, HG, MG and G the set species corresponding to these structures.

We will also note F the species of forests, which is a subspecis of G.

In contrast with the standard definition, here the sole difference between graphs and multigraphs is that multigraphs can have loops while graphs no; but in both case we accept repetition of edges.

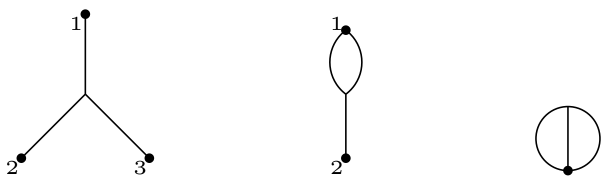

Figure 1: Three edges of cardinality 3.

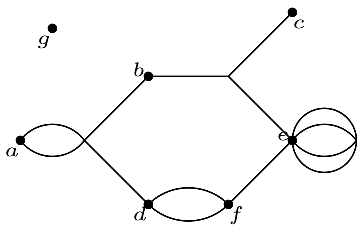

Figure 2: A mulit-hypergrah over $\{a, b, c, d, e, f, g\}$.

Example 6. We represented the three edges $\{(1,1),(2,1),(3,1)\}$, $\{(1,2),(2,1)\}$ and $\{(1,3)\}$ in Figure 1 and the multi-hypergraph

$$\{(\{(a,2),(b,1),(d,1)\},1),(\{(b,1),(c,1),(e,1)\},1),(\{e,4\},1),(\{(e,1),(f,1)\},1),(\{(d,1),(e,1)\},2)\}$$

over $\{a, b, c, d, e, f, g\}$ in Figure 2.

2 Augmented multi-hypergraphs

Definition 7. Let A be a set and V a finite set. An *A-augmented end over V* is an element (v, a) in $V \times A$, we call v the vertex of the end and a the augmentation of the end. An *A-augmented edge over V* is multiset of A-augmented end over V and an *A-augmented multi-hypergraph over V* is a multiset of A-augmented edges. Note $A\text{-}MHG$ the set species of A-augmented multi-hypergraphs.

The forgetful functor $U : A\text{-}MHG \to MHG$ is defined by sending any A-augmented end (v, a) of a A-augmented edges of a A-augmented multi-hypergraph on the end v. Given an edge e over V, an A-augmentation of this edge is an A-augmented edge e' over V such that $U_V(e') = e$. Given a multi-hypergraph h, an A-augmentation of this multi-hypergraph is then a A-augmented multi-hypergraph g' over V such that $U_V(h') = h$.

Example 8. ...

Let A be a set and V a finite set. For $a \in A$ and $V \in v$, we will note:

- $v_a = (a, v)$ for $v \in V$ and $a \in A$,
- $\prod_{(v,a)\in e} v_a^{e((v,a))}$ for an A-augmented edge e over V,
- $\emptyset_V$ the empty A-augmented multi-hypergraph over V,
- $\bigoplus_{e_i\in h} h(e_i)e_i$ for an A-augmented multi-hypergraph h.

With these notations we can see A-augmented edges over V as monomials on the set of variables $\{v_a \,|\, v \in V, a \in A\}$ and non empty A-augmented multi-hypergraphs over V as polynomials with positive integer coefficients on the same set of variables. The binary operation $\oplus$ over A-augmented multi-hypergraphs can then be interpreted as the union of multisets of edges or the sum over polynomials. Note that we used $\oplus$ for the sum of polynomials / union of multisets such as to not confuse it with the $+$ of $\Bbbk A\text{-}MHG[V]$. In this context, the sum of polynomials $\oplus$ and the product of polynomials $\cdot$ are distributive over the sum of vectors $+$.

Example 9. ...

Note $V = \{v_1, \dots, v_n\}$ and $A = \{a_1, a_2, \dots\}$. Let be $h \in A\text{-}MHG[V]$, $1 \le i_1 < \dots < i_k \le n$ be $k \le n$ integers, V' be a finite set disjoint of V and for every $1 \le j \le k$, $a \in A$, $\emptyset \subsetneq V_{j,a} \subseteq V'$. We note $h|_{\{v_{i_j a} \leftarrow V_{j,a}\}_{1\le j\le k, a\in A}}$ the substitution of every $v_{i_j} \in V$ by $\sum_{v\in V_{j,a}} v_a$ in the polynomial h, i.e:

$$h(\dots, v_{i_1 a_1}, \dots, v_{i_j a_1}, \dots)|_{\{v_{i_j a} \leftarrow V_{j,a}\}_{1\le j\le k, a\in A}} = h(\dots, \sum_{v\in V_{1,a_1}} v_{a_1}, \dots, \sum_{v\in V_{j,a_1}} v_{a_1}, \dots).$$

In the case of the empty A-augmented multi-hypergraphs over V we note $\emptyset_V|_{\{v_{i_j a} \leftarrow V_{j,a}\}_{1\le j\le k, a\in A}} = \emptyset_{V\setminus\{v_{i_j}\}_{1\le j\le k}}$.

We note $\mathcal{F}_A[V]$ the set of maps from A to $\mathcal{P} \setminus \{\emptyset\}$. Let V_1 and V_2 be two finite disjoint sets. For $(h_1, f_1) \in (A\text{-}MHG \times \mathcal{F}_A)'[V_1]$ and $(h_2, f_2) \in (A\text{-}MHG \times \mathcal{F}_A)[V_2]$ we define the element of $\Bbbk((A\text{-}MHG \times \mathcal{F}_A)[V_1 + V_2]$:

$$(h_1, f_1) \circ_* (h_2, f_2) = \left(h_1|_{\{*_a \leftarrow f_2(a)\}_{a\in A}} \oplus h_2, f_1 \circ_* f_2 : a \to \begin{cases} f_1(a) & \text{if } * \notin f_1(a), \\ f_1(a) \setminus \{*\} + f_2(a) & \text{else.} \end{cases} \right)$$

where $(\sum_i h_i, f) = \sum_i (h_i, f)$. Let us also define $e : \Bbbk X \to \Bbbk(A\text{-}MHG \times \mathcal{F}_A)$ by $e_V(v) = (\emptyset_{\{v\}}, A \mapsto \{v\})$ if $V = \{v\}$ and $e_V = 0$ else.

Lemma 10. The partial product $\circ_*$ on $\mathcal{F}_A$ and the morphism $e : X \to \Bbbk\mathcal{F}_A$ defined by $e_V(v) = A \mapsto \{v\}$ if $V = \{v\}$ and $e_V = \emptyset$ else are such that the diagrams of Proposition 4 commute and hence define a set operad structure on $\mathcal{F}_A$

Proof. Since $\mathcal{F}_A[\sigma](f) = \sigma \circ f$ it is easy to see that $\circ_*$ is a morphism of species. Let now be V_1, V_2 and V_3 three disjoint sets.

- Let be $f_1 \in \mathcal{F}_A[V_1 + \{*_1, *_2\}]$, $f_2 \in \mathcal{F}_A[V_2]$ and $f_3 \in \mathcal{F}_A[V_3]$. Then:

$$\begin{aligned}
(f_1 \circ_{*_1} f_2) \circ_{*_2} f_3 &= a \mapsto \begin{cases} f_1(a) & \text{if } *_1 \notin f_1(a), \\ f_1(a) \setminus \{*_1\} + f_2(a) & \text{else.} \end{cases} \circ_{*_2} f_3 \\
&= a \mapsto \begin{cases} f_1(a) & \text{if } *_1 \notin f_1(a) \text{ and } *_2 \notin f_2(a), \\ f_1(a) \setminus \{*_2\} + f_3(a) & \text{if } *_1 \notin f_1(a) \text{ and } *_2 \in f_2(a), \\ f_1(a) \setminus \{*_1\} + f_2(a) & \text{if } *_1 \in f_1(a) \text{ and } *_2 \notin f_2(a), \\ f_1(a) \setminus \{*_1, *_2\} + f_2(a) + f_3(a) & \text{if } *_1 \in f_1(a) \text{ and } *_2 \in f_2(a). \end{cases} \\
&= a \mapsto \begin{cases} f_1(a) & \text{if } *_2 \notin f_1(a), \\ f_1(a) \setminus \{*_1\} + f_3(a) & \text{else.} \end{cases} \circ_{*_1} f_2 \\
&= (f_1 \circ_{*_2} f_3) \circ_{*_1} f_2
\end{aligned}$$

- Let be $f_1 \in \mathcal{F}_A[V_1 + \{*_1\}]$, $f_2 \in \mathcal{F}_A[V_2 + \{*_2\}]$ and $f_3 \in \mathcal{F}_A[V_3]$. Then:

$$\begin{aligned}
(f_1 \circ_{*_1} f_2) \circ_{*_2} f_3 &= a \mapsto \begin{cases} f_1(a) & \text{if } *_1 \notin f_1(a), \\ f_1(a) \setminus \{*_1\} + f_2(a) & \text{else.} \end{cases} \circ_{*_2} f_3 \\
&= a \mapsto \begin{cases} f_1(a) & \text{if } *_1 \notin f_1(a) \text{ and } *_2 \notin f_2(a), \\ f_1(a) & \text{if } *_1 \notin f_1(a) \text{ and } *_2 \in f_2(a), \\ f_1(a) \setminus \{*_1\} + f_2(a) & \text{if } *_1 \in f_1(a) \text{ and } *_2 \notin f_2(a), \\ f_1(a) \setminus \{*_1\} + f_2(a) \setminus \{*_2\} + f_3(a) & \text{if } *_1 \in f_1(a) \text{ and } *_2 \in f_2(a). \end{cases} \\
&= f_1 \circ_{*_1} \begin{cases} f_2(a) & \text{if } *_2 \notin f_2(a), \\ f_2(a) \setminus \{*_2\} + f_3(a) & \text{else.} \end{cases} \\
&= f_1 \circ_{*_1} (f_2 \circ_{*_2} f_3)
\end{aligned}$$

- Let be $f \in \mathcal{F}_1[V_1]$ and $f' \in \mathcal{F}_A[V_1 + \{*\}]$. Then $A \mapsto \{*\} \circ_* f = f$ and $f' \circ_* A \mapsto \{v\} = (* \mapsto v) \circ f'$.

□

Lemma 11. Let $A = \{a_1, \dots\}$ be a set, V, V_1 and V_2 pairwise disjoint finte sets.

- If $h \in A\text{-}MHG[V_1 + \{*_1, *_2\}]$ and $f_1 \in \mathcal{F}_A[V_1]$ and $f_2 \in \mathcal{F}_A[V_2]$ then $h|_{\{*_{1a} \leftarrow f_1(a)\}_{a \in A}}|_{\{*_{2a} \leftarrow f_2(a)\}_{a \in A}} = h|_{\{*_{2a} \leftarrow f_2(a)\}_{a \in A}}|_{\{*_{1a} \leftarrow f_1(a)\}_{a \in A}}$.
- If $h \in A\text{-}MHG[V_1 + \{*_1\}]$ and $f_1 \in \mathcal{F}_A[V_1 + \{*_2\}]$ and $f_2 \in \mathcal{F}_A[V_2]$ then $h|_{\{*_{1a} \leftarrow f_1(a)\}_{a \in A}}|_{\{*_{2a} \leftarrow f_2(a)\}_{a \in A}} = h|_{\{*_{1a} \leftarrow f_1 \circ_{*_2} f_2(a)\}_{a \in A}}$

Proof. This is just saying that the composition of multivariate polynomials is "symmetric" and associative.

-

$$\begin{aligned}
&h_1(\dots, *_{1a_1}, \dots, *_{2a_1}, \dots)|_{\{*_{1a} \leftarrow f_1(a)\}_{a \in A}}|_{\{*_{2a} \leftarrow f_2(a)\}_{a \in A}} \\
&\qquad = h_1(\dots, \sum_{v \in f_1(a_1)} v_{a_1}, \dots, *_{2a_1}, \dots)|_{\{*_{2a} \leftarrow f_2(a)\}_{a \in A}} \\
&\qquad = h_1(\dots, \sum_{v \in f_1(a_1)} v_{a_1}, \dots, \sum_{v \in f_2(a_1)} v_{a_1}, \dots) \\
&\qquad = h_1(\dots, *_{1a_1}, \dots, \sum_{v \in f_2(a_1)} v_{a_1}, \dots)|_{\{*_{1a} \leftarrow f_1(a)\}_{a \in A}} \\
&\qquad = h_1(\dots, *_{1a_1}, \dots, *_{2a_1}, \dots)|_{\{*_{2a} \leftarrow f_2(a)\}_{a \in A}}|_{\{*_{1a} \leftarrow f_1(a)\}_{a \in A}}
\end{aligned}$$

- $$\begin{aligned} h_1(\dots,*_{1a_1},\dots)|_{\{*_{1a}\leftarrow f_1(a)\}_{a\in A}}|_{\{*_{2a}\leftarrow f_2(a)\}_{a\in A}} &= h_1(\dots,\sum_{v\in f_1(a_1)} v_{a_1},\dots)|_{\{*_{2a}\leftarrow f_2(a)\}_{a\in A}} \\ &= h_1(\dots,\sum_{v\in f_1(a_1)\setminus\{*_2\}} v_{a_1}+\delta_{*_2\in f_1(a_1)}\sum_{v\in f_2(a_1)} v_{a_1},\dots) \\ &= h_1(\dots,\sum_{v\in f_1\circ_{*_2} f_2(a_1)} v_{a_1},\dots) \\ &= h_1(\dots,*_{1a_1},\dots)|_{\{*_{1a}\leftarrow f_1\circ_{*_2} f_2(a)\}_{a\in A}} \end{aligned}$$

  where $\delta_{*_2\in f_1(a_1)}=1$ if $*_2\in f_1(a_1)$ and $\delta_{*_2\in f_1(a_1)}=0$ else.

□

Theorem 12. The partial product $\circ_*$ and the morphism e are such that the diagrams of Proposition 4 commute and hence define an operad structure on $\Bbbk A\text{-}MHG\times\mathcal{F}_A$ as well as all its subspecies stable under partial product.

Proof. We must first show that $\circ_*$ is a morphism of species. Let V_1 and V_2 be two finite disjoint sets, $(h_1,f_1)\in(A\text{-}MHG\times\mathcal{F}_A)'[V_1]$ and $(h_2,f_2)\in(A\text{-}MHG\times\mathcal{F}_A)[V_2]$ and $\sigma: V_1\sqcup V_2\to V$ a bijection. Then,

$$\begin{aligned} &(A\text{-}MHG\times\mathcal{F}_A)[\sigma]((h_1,f_1)\circ_*(h_2,f_2)) \\ &\quad= (A\text{-}MHG\times\mathcal{F}_A)[\sigma]((h_1|_{\{*_a\leftarrow f_2(a)\}_{a\in A}}\oplus h_2, f_1\circ_* f_2) \\ &\quad= (A\text{-}MHG[\sigma](h_1|_{\{*_a\leftarrow f_2(a)\}_{a\in A}}\oplus h_2),\mathcal{F}_A[\sigma](f_1\circ_* f_2)) \\ &\quad= (A\text{-}MHG'[\sigma_{|V_1}](h_1)|_{\{*_a\leftarrow\sigma_{|V_2}(f_2(a))\}_{a\in A}}\oplus A\text{-}MHG[\sigma_{|V_2}](h_2),\mathcal{F}'_A[\sigma_{|V_1}](f_1)\circ_*\mathcal{F}_A[\sigma_{|V_2}](f_2)) \\ &\quad= (A\text{-}MHG\times\mathcal{F}_A)'[\sigma]((h_1,f_1))\circ_*(A\text{-}MHG\times\mathcal{F}_A)[\sigma]((h_2,f_2)) \end{aligned}$$

Let now be V_1, V_2 and V_3 three disjoint sets and note $A=\{a_1,\dots\}$.

- Let be $(h_1,f_1)\in(A\text{-}MHG\times\mathcal{F}_A)[V_1+\{*_1,*_2\}]$, $(h_2,f_2)\in(A\text{-}MHG\times\mathcal{F}_A)[V_2]$ and $(h_3,f_3)\in(A\text{-}MHG\times\mathcal{F}_A)[V_3]$. Then:

  $$\begin{aligned} &((h_1,f_1)\circ_{*_1}(h_2,f_2))\circ_{*_2}(h_3,f_3) \\ &\quad= (h_1|_{\{*_{1a}\leftarrow f_2(a)\}_{a\in A}}\oplus h_2, f_1\circ_{*_1} f_2)\circ_{*_2}(h_3,f_3) \\ &\quad= (h_1|_{\{*_{1a}\leftarrow f_2(a)\}_{a\in A}}|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}\oplus h_2|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}\oplus h_3,(f_1\circ_{*_1} f_2)\circ_{*_2} f_3) \\ &\quad= (h_1|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}|_{\{*_{1a}\leftarrow f_2(a)\}_{a\in A}}\oplus h_3|_{\{*_{1a}\leftarrow f_2(a)\}_{a\in A}}\oplus h_2,(f_1\circ_{*_2} f_3)\circ_{*_1} f_2) \\ &\quad= (h_1|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}\oplus h_3, f_1\circ_{*_2} f_3)\circ_{*_1}(h_2,f_2) \\ &\quad= ((h_1,f_1)\circ_{*_2}(h_3,f_3))\circ_{*_1}(h_2,f_2), \end{aligned}$$

  where the third equality comes from Lemma 10 and Lemma 11.

- Let be $(h_1,f_1)\in(A\text{-}MHG\times\mathcal{F}_A)[V_1+\{*_1\}]$, $(h_2,f_2)\in(A\text{-}MHG\times\mathcal{F}_A)[V_2+\{*_2\}]$ and $(h_3,f_3)\in(A\text{-}MHG\times\mathcal{F}_A)[V_3]$. Then:

  $$\begin{aligned} &((h_1,f_1)\circ_{*_1}(h_2,f_2))\circ_{*_2}(h_3,f_3) \\ &\quad= (h_1|_{\{*_{1a}\leftarrow f_2(a)\}_{a\in A}}\oplus h_2, f_1\circ_{*_1} f_2)\circ_{*_2}(h_3,f_3) \\ &\quad= (h_1|_{\{*_{1a}\leftarrow f_2(a)\}_{a\in A}}|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}\oplus h_2|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}\oplus h_3,(f_1\circ_{*_1} f_2)\circ_{*_2} f_3) \\ &\quad= (h_1|_{\{*_{1a}\leftarrow f_2\circ_{*_2} f_3(a)\}_{a\in A}}\oplus h_2|_{\{*_{2a}\leftarrow f_3(a)\}_{a\in A}}\oplus h_3, f_1\circ_{*_1}(f_2\circ_{*_2} f_3)) \\ &\quad= (h_1,f_1)\circ_{*_1}((h_2,f_2)\circ_{*_2}(h_3,f_3)), \end{aligned}$$

  where the third equality comes from Lemma 10 and Lemma 11.

- Let $v \notin V_1$ and $(h, f) \in (A\text{-}MHG \times \mathcal{F}_A)[V_1]$ and $(h', f') \in (A\text{-}MHG \times \mathcal{F}_A)[V_1 + \{*\}]$. Then $({}_{\{*\}}, A \mapsto \{*\}) \circ_* (h, f) = (h, f)$ and $(h', f') \circ_* (\emptyset_{\{v\}}, A \mapsto \{v\}) = A\text{-}MHG[* \mapsto v](h', f')$

□

3 Extending PreLie

Lemma 13. Let A be a set. The set sub-species $\mathcal{F}_A^{max}$ of $\mathcal{F}_A$ defined by $\mathcal{F}_A^{max}[V] = \{f : a \in A \mapsto V\}$ is stable under partial product and contains $e_{\{v\}}(v)$ for all singleton $\{v\}$ and is hence a set sub-operad of $\mathcal{F}_A$.

Proof. By definition of e and $\mathcal{F}_A^{max}$ we have that $\mathcal{F}_A^{max}[\{v\}] = \{e_{\{v\}}(v)\}$. Let V_1 and V_2 be two disjoint sets and $\mathcal{F}_A^{max}[V_1 + \{*\}] = \{f_1\}$ and $\mathcal{F}_A^{max}[V_2] = \{f_2\}$. Then for all $a \in A$, $* \in f_1(a)$ and hence $f_1 \circ_* f_2(a) = f_1(a) \setminus \{*\} + f_2(a) = V_1 + V_2$ which conclude this proof. □

Definition 14 (Operad of multi-hypergraphs and oriented multi-hypergraphs)**.** The species $\Bbbk MHG$ is isomorphic to $\Bbbk(\{0\}\text{-}MHG \times \mathcal{F}_{\{0\}}^{max})$. This second species being stable under $\circ_*$ and containing $e(X)$, this gives us an operad structure on $\Bbbk MHG$.

We call *oriented multi-hypergraph* the elements of $\{s, t\}\text{-}MHG$. The ends augmented with s are called *sources* and the one augmented with t *targets*. We note MHG_{or} this set species. $\Bbbk MHG_{or}$ as an operad structure given by its isomorphism with $\Bbbk(\{s, t\}\text{-}MHG \times \mathcal{F}_{\{0\}}^{max})$. An *orientation* of a multi-hypergraph is then a $\{s, t\}$-augmentation.

We have analogous definitions for HG, MG, G and F

Note that the notion of orientation for a graph differs from the usual one where each edge has exactly one source and one target.

Let S one of the set species HMG, HG, MG or G. We note S_c the set subspecies of S of connected elements, S_{orc} the set subspecies of S_{or} of connected elements. We note $T = F_c$ the set species of trees and T_{or} the set species of oriented trees.

Lemma 15. The monomorphism of vector species given by, for V a finite set:

$$\begin{aligned} \phi_V : \Bbbk T[V] &\to \mathrm{PreLie}[V] \cong \Bbbk T^\bullet[V] \\ t &\mapsto \sum_{v \in V} (t, v) \end{aligned}$$

is a morphism of operads.

Proof. Let V_1 and V_2 be two disjoint sets and $t_1 \in T[V_1 + \{*\}]$ and $t_2 \in T[V_2]$. Then we have by noting $p(v)$ the parent of v, $c(v)$ its children and $n(v) = p(v) + c(v)$ its neighbours:

$$
\begin{aligned}
\phi_{V_1}(t_1) \circ_* \phi_{V_2}(t_2) &= \sum_{v_1 \in V_1 + \{*\}} (t_1, v_1) \circ_* \sum_{v_2 \in V_2} (t_2, v_2) \\
&= \sum_{v_1 \in V_1 + \{*\}} \sum_{v_2 \in V_2} (t_1, v_1) \circ_* (t_2, v_2) \\
&= \sum_{v_1 \in V_1} \left(t_1 \cap V_1^2 \oplus \bigoplus_{v \in c(*)} v \left(\sum V_2 \right) \oplus p(*) \left(\sum V_2 \right) \oplus t_2, v_1 \right) \\
&\quad + \sum_{v_2 \in V_2} \left(t_1 \cap V_1^2 \oplus \bigoplus_{v \in c(*)} v \left(\sum V_2 \right) \oplus t_2, v_2 \right) \\
&= \sum_{v \in V_1 + V_2} \left(t_1 \cap V_1^2 \oplus \bigoplus_{v' \in n(*)} v' \left(\sum V_2 \right) \oplus t_2, v \right) \\
&= \sum_{v \in V_1 + V_2} (t_1|_{* \leftarrow \sum V_2} \oplus t_2, v) \\
&= \phi_{V_1 + V_2}(t_1 \circ_* t_2)
\end{aligned}
$$

□