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Radiation stresses in water waves; a physical discussion, with applications

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Abstract—The radiation stresses in water waves play an important role in a variety of oceanographic phenomena, for example in the change in mean sea level due to storm waves (wave " set-up "); the generation of " surf-beats "; the interaction of waves with steady currents; and the steepening of short gravity waves on the crests of longer waves. In previous papers these effects have been discussed rigorously by detailed perturbation analysis. In the present paper a simplified exposition is given of the radiation stresses and some of their consequencies. Physical reasoning, though less rigorous, is used wherever possible. The influence of capillarity on the radiation stresses is fully described for the first time.

INTRODUCTION

In a series of recent papers (1960, 1961, 1962, see also TAYLOR 1962, WHITHAM 1962) we have attempted to elucidate some of the non-linear properties of surface gravity waves in terms of what we have called the " radiation stress." Some of these non-linear properties have turned out to be unexpected (or at least to differ from properties widely assumed previously in the literature). For this reason a major part of the above mentioned papers has been used for a careful check of the results obtained by using the radiation stress concept, by means of detailed perturbation analysis to the required order of approximation.

One effect of this approach (which we believe to have been necessary) has been that the papers are somewhat analytical, and the straightforward simplicity of the concept may have been partly obscured for some readers. It is the purpose of the present paper to attempt a simple exposition, setting forth the nature and uses of the radiation stress. In many cases results will be stated without proof; readers dissatisfied with any of these are referred to the previous papers. (We shall refer to LONGUET-HIGGINS and STEWART, 1960, 1961, and 1962 as I, II and III). At the same time we shall extend some of our previous results for pure gravity waves so as to include effects of capillarity.

In the first sections of the paper we describe a simple derivation of the radiation stress, both for gravity waves and for capillary waves. In the second part we shall describe the application of these results to a number of interesting phenomena observed in the oceans; in particular to wave " set-up," " surf beats," the steepening of short waves on adverse currents or tidal streams, and the generation of capillary waves by steep gravity waves.

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PART I—THE RADIATION STRESSES; A PHYSICAL DISCUSSION

It is well known that electromagnetic radiation impinging on a surface, or originating on a surface, produces a force which is referred to as the " radiation pressure." It is perhaps less well known that a similar phenomenon occurs in the case of acoustic waves and of waves on the surface of fluids (or indeed of internal waves in a stratified fluid). In each case the force is principally in the direction of wave propagation. It is therefore not an isotropic one unless the waves themselves are isotropically distributed (as is the case for electromagnetic waves in an isothermal enclosure). In fluid mechanics it has become customary to use the term " pressure " for the isotropic stress which figures in the equation of state. We have therefore considered it wise to coin the term radiation *stress* as a more general one which does not carry any implication of isotropy‡.

Qualitatively§, the following argument may serve to introduce the concept :

It is well known (LAMB, 1932, Section 250) that surface waves possess momentum which is directed parallel to the direction of propagation and is proportional to

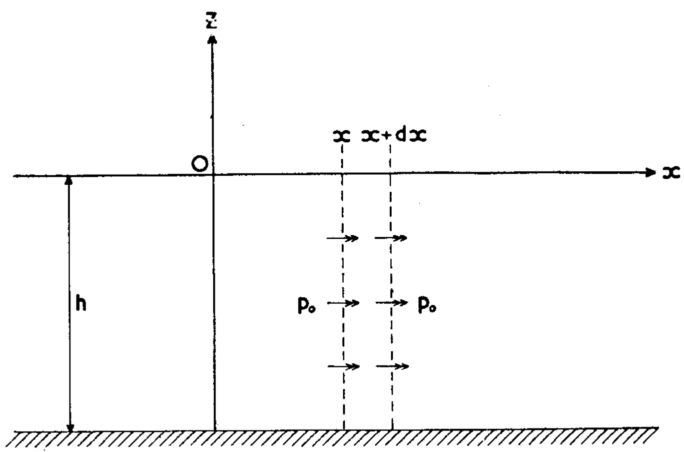

Fig. 1. The momentum flux in a stationary fluid.

the square of the wave amplitude. Now if a wave train is reflected from an obstacle, its momentum must be reversed. Conservation of momentum then requires that there be a force exerted on the obstacle, equal to the rate of change of a wave momentum. This force is a manifestation of the radiation stress.

A stress is by definition equivalent to a flow of momentum. The radiation stress may thus be defined as *the excess flow of momentum due to the presence of the waves.* Let us consider in detail how this stress arises in gravity waves.

1. *Progressive waves in water of uniform depth*

Consider first an undisturbed body of water of uniform depth h (Fig. 1). The pressure p at any point is equal to the hydrostatic pressure :

‡It might be argued that " radiative stress " would be grammatically more correct, but we wish to retain the implied analogy to the well established term " radiation pressure "—and in any case the use of nouns as adjectives is widespread in English.

§Quantitatively, in some special cases, it leads to difficulties and to errors, because some phenomena are incompletely described by the discussion in this paragraph.

$$p = -\rho g z, \tag{1}$$

here ρ, g and z denote density, gravity, and distance measured upwards from the ean surface. If we denote the above expression by p_0 then the flux of horizontal omentum across any vertical plane $x =$ constant is simply p_0 per unit vertical stance. Thus the total flux of horizontal momentum between surface and bottom $= -h$) is

$$\int_{-h}^{0} p_0 \, dz \tag{2}$$

nce this quantity is independent of x, the flux of momentum across an adjacent ane $(x + dx)$ just balances the flux across the first plane, and there is no net change momentum between the two planes (Fig. 1).

Now consider the momentum flux in the presence of a simple progressive wave otion (Fig. 2). The surface elevation $z = \zeta$ is given approximately by

$$\zeta = a \cos(kx - \sigma t) \tag{3}$$

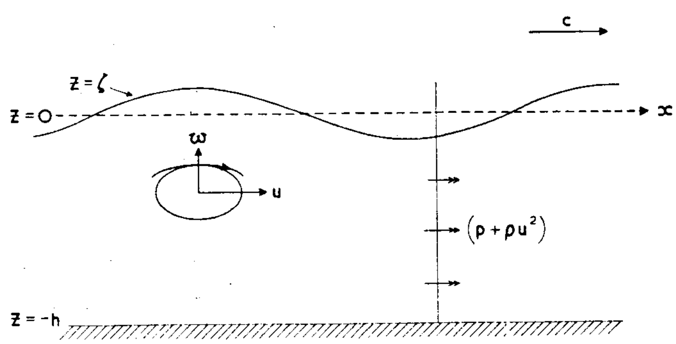

Fig. 2. The momentum flux in a progressive wave.

where a is the amplitude, $k = 2\pi/\text{wavelength}$ and $\sigma = 2\pi/\text{wave period}$. The particle orbits are roughly ellipses, with the major axes horizontal in general. (In deep water the orbits are circular). The corresponding components of velocity are given by

$$\left.\begin{aligned} u &= \frac{a\sigma}{\sinh kh} \cosh k(z + h) \cos(kx - \sigma t) \\ w &= \frac{a\sigma}{\sinh kh} \sinh k(z + h) \sin(kx - \sigma t). \end{aligned}\right\} \tag{4}$$

A quite general expression for the instantaneous flux of horizontal momentum across unit area of a vertical plane in the fluid is

$$p + \rho u^2. \tag{5}$$

In this expression the second term ρu^2 represents a bodily transfer of momentum ρu (per unit volume) at a rate u per unit time (Fig. 3). The term ρu^2 is evidently analogous to a pressure. Even if the mean value of u itself is zero, the mean value u^2 is of course generally positive.

(Similarly, fluid crossing the plane $x = \text{constant}$ possesses z-momentum associated with the velocity component w. A mean product such as $\rho \overline{uw}$, which represents the mean transport of z-momentum across a plane $x = \text{constant}$—or vice versa—is equivalent to a shear stress. In turbulence theory, such mean values are known collectively as Reynolds stresses, and it will be appreciated that the above

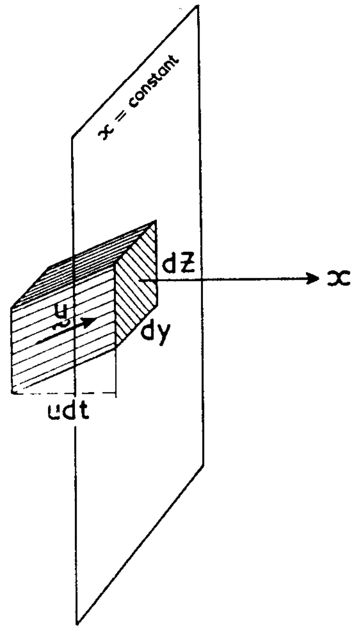

Fig. 3. Bodily transport of momentum across a plane x = constant. In time t a volume udt per unit area has been transported across the plane. The momentum transported is thus $\rho \underline{u}\, udt$.

concepts are also similar physically to those used to explain the origin of pressure and viscosity in the kinetic theory of gasses. If due to turbulent fluctuations, the Reynolds shear stress is frequently parameterised by the concept of "eddy viscosity." Reynolds stresses also occur in waves, but in this case we must seek a different kind of description. For a discussion of the Reynolds stresses particularly appropriate in the present context, see STEWART, (1956)).

To find the total flux of horizontal momentum across a plane $x = \text{constant}$ we have to integrate (5) between the bottom $z = -h$ and the free surface $z = \zeta$:

$$\int_{-h}^{\zeta} (p + \rho u^2)\, dz. \tag{6}$$

We now define the principal component S_{xx} of the radiation stress as the mean value of the function (6) with respect to time, minus the mean flux in the absence of the waves, that is to say

$$S_{xx} = \overline{\int_{-h}^{\zeta} (p + \rho u^2)\, dz} - \int_{-h}^{0} p_0\, dz. \tag{7}$$

Special care must be taken to take the mean value *after* integration, since the fluctuation of the free surface itself contributes to the momentum flux. We can see this best by separating the right-hand side of (7) into three parts, that is by writing

$$S_{xx} = S_{xx}^{(1)} + S_{xx}^{(2)} + S_{xx}^{(3)} \tag{8}$$

where

$$\left.\begin{aligned} S_{xx}^{(1)} &= \overline{\int_{-h}^{\zeta} \rho u^2\, dz} \\ S_{xx}^{(2)} &= \overline{\int_{-h}^{0} (p - p_0)\, dz} \\ S_{xx}^{(3)} &= \overline{\int_{0}^{\zeta} p\, dz}. \end{aligned}\right\} \tag{9}$$

Consider the first contribution $S_{xx}^{(1)}$. Since the integrand is of second order, the upper limit $z = \zeta$ may be replaced effectively by the mean level $z = 0$, since the additional range $0 < z < \zeta$ contributes only a third-order term. Now, both the limits of integration 0, h, being constant, we can transfer the mean value to the integrand. Thus

$$S_{xx}^{(1)} = \overline{\int_{-h}^{0} \rho u^2\, dz} = \int_{-h}^{0} \rho \overline{u^2}\, dz. \tag{10}$$

The contribution $S_{xx}^{(1)}$, then, is effectively *the Reynolds stress* $\rho\overline{u^2}$ *integrated from the bottom up to the free surface*. It is obviously positive in general.

Consider now the contribution $S_{xx}^{(2)}$. As in equation (10), we may take the mean value inside the limits of integration :

$$S_{xx}^{(2)} = \int_{-h}^{0} (\overline{p} - p_0)\, dz. \tag{11}$$

In other words $S_{xx}^{(2)}$ arises from *the change in mean pressure within the fluid*. Now the pressure $\overline{p}$ generally contains terms proportional to a^2, which can be found by a second order analysis. However, we do not need to calculate all the second-order terms explicitly since $\overline{p}$ may be found directly from a consideration of the *vertical* flux of vertical momentum as follows.

We know that the mean flux of vertical momentum across any horizontal plane, which is $\overline{p + \rho w^2}$, must be just sufficient to support the weight of the water above it. Since the mean level of the water is at $z = 0$, we have then

$$\overline{p + \rho w^2} = - \rho g z = p_0 \tag{12}$$

or

$$\overline{p} - p_0 = - \rho\overline{w^2} \tag{13}$$

Thus $\overline{p}$ is generally less than the hydrostatic pressure p_0. Substitution in equation (11) gives

$$S_{xx}^{(2)} = \int_{-h}^{0} (-\rho \overline{w^2})\, dz. \tag{14}$$

This contribution is then negative in general.

Combining equations (10) and (14) we have

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \int_{-h}^{0} \rho\, (\overline{u^2 - w^2})\, dz \geqslant 0. \tag{15}$$

For, since the major axes of the particle orbits are horizontal we have $\overline{u^2} \geqslant \overline{w^2}$. After substituting for the velocities from equations (4) and carrying out the integration we find in fact*

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \frac{1}{2} \frac{\rho a^2 \sigma^2 h}{\sinh^2 kh} = \frac{\rho g a^2 kh}{\sinh 2kh} \tag{16}$$

having used in the last step the frequency relation

$$\sigma^2 = gk \tanh ph \tag{17}$$

for waves of small amplitude.

In deep water the particle orbits are circles, and $\overline{u^2}$ equals $\overline{w^2}$; the Reynolds stresses are isotropic in x and z. The positive contribution $\rho\overline{u^2}$ from the horizontal Reynolds stress is then exactly cancelled by the pressure defect $-\rho\overline{w^2}$ arising from the vertical Reynolds stress. On the other hand in shallow water the particle orbits are elongated horizontally, and $\overline{w^2}$ becomes small compared with $\overline{u^2}$. Then $\rho\, (\overline{u^2 - w^2})$ becomes simply $\rho\overline{u^2}$. Since, for the same reason, the kinetic energy is then just $\frac{1}{2}\rho u^2$ per unit volume, we see that $S_{xx}^{(1)} + S_{xx}^{(2)}$ is then twice the kinetic energy density, that is, the total energy density of the waves.

There remains the important contribution $S_{xx}^{(3)}$. This is equal to the pressure p integrated† between 0 and ζ and then averaged with respect to time. It is easily evaluated, for near to the free surface p is nearly equal to the hydrostatic pressure below the free surface :

$$p = \rho g\, (\zeta - z). \tag{18}$$

Thus the pressure at any point near the surface fluctuates in phase with the surface elevation ζ. Substitution in the integral gives

$$S_{xx}^{(3)} = \tfrac{1}{2}\rho g \overline{\zeta^2}. \tag{19}$$

So $S_{xx}^{(3)}$ is generally positive and is in fact equal to the density of potential energy, that is to say half the total energy density E :

$$S_{xx}^{(3)} = \tfrac{1}{4}\, \rho g a^2 = \tfrac{1}{2} E \tag{20}$$

*It may be noted that $\overline{(u^2 - w^2)}$ is independent of z, for using the incompressible, irrotational and progressive character of the motion we have :

$$\frac{\partial}{\partial z}\overline{(u^2 - w^2)} = 2\,\overline{\left(u\frac{\partial u}{\partial z} - w\frac{\partial w}{\partial z}\right)} = 2\,\overline{\left(u\frac{\partial w}{\partial x} + w\frac{\partial u}{\partial x}\right)} = 2\frac{\partial}{\partial x}\overline{(uw)} = 0.$$

†When the free surface is below the mean level, the velocity field is assumed to be extended analytically up to the mean level. This device leads to the simplest algebra. If preferred, however, the upper limit of integration may be taken at any arbitrary fixed level in the fluid, within a distance of order a from the mean level; the final result is the same.

where

$$E = \tfrac{1}{2}\rho g a^2. \tag{21}$$

Altogether we have from equations (15) and (19)

$$S_{xx} = S_{xx}^{(1)} + S_{xx}^{(2)} + S_{xx}^{(3)} \geqslant 0, \tag{22}$$

or on inserting the values found in equations (16) and (20)

$$S_{xx} = E\left(\frac{2kh}{\sinh 2kh} + \tfrac{1}{2}\right). \tag{23}$$

The ratio $2kh/\sinh 2kh$ lies always between 0 and 1. In deep water ($kh \gg 1$) the ratio tends to 0 and so

$$S_{xx} = \tfrac{1}{2}E, \tag{24}$$

while the shallow water ($kh \ll 1$) it tends to 1 and so

$$S_{xx} = \tfrac{3}{2}E. \tag{25}$$

The transverse components of the radiation stress. Now let us consider in a similar way the flow of y-momentum (momentum parallel to the wave crests) across a plane $y =$ constant. Denoting this by S_{yy} we have corresponding to equation (7) the relation

$$S_{yy} = \overline{\int_{-h}^{\zeta} (p + \rho v^2)\, dz} - \int_{-h}^{0} p_0\, dz \tag{26}$$

where v is the transverse component of velocity. Just as for S_{xx} we can consider S_{yy} as the sum of three parts :

$$S_{yy} = S_{yy}^{(1)} + S_{yy}^{(2)} + S_{yy}^{(3)} \tag{27}$$

where

$$\left.\begin{aligned} S_{yy}^{(1)} &= \overline{\int_{-h}^{\zeta} \rho v^2\, dz} \\ S_{yy}^{(2)} &= \overline{\int_{-h}^{0} (p - p_0)\, dz} \\ S_{yy}^{(3)} &= \overline{\int_{0}^{\zeta} p dz} \end{aligned}\right\} \tag{28}$$

In gravity waves the transverse velocity vanishes everywhere and so

$$S_{yy}^{(1)} = 0, \tag{29}$$

while $S_{yy}^{(2)}$ and $S_{yy}^{(3)}$ are equal to $S_{xx}^{(2)}$ and $S_{xx}^{(3)}$ respectively.
Thus

$$S_{yy}^{(2)} = \int_{-h}^{0} (-\rho\overline{w^2})\, dz \leqslant 0, \tag{30}$$

$$S_{yy}^{(3)} = \tfrac{1}{2}\rho g\, \overline{\zeta^2} \geqslant 0. \tag{31}$$

Substitution for w from equations (4) and use of the frequency relation equation (17) leads to

$$S_{yy} = E \times \frac{kh}{\sinh 2kh}. \tag{32}$$

In deep water, $\overline{w^2} = \overline{u^2} = \frac{1}{2}\,\overline{(u^2 + v^2 + w^2)}$; then $S_{yy}^{(2)}$ is equal to minus the density of kinetic energy, which is $-\frac{1}{2}E$. Thus $S_{yy}^{(2)}$ just cancels $S_{yy}^{(3)}$ and S_{yy} vanishes :

$$S_{yy} = 0 \qquad (kh \gg 1) \tag{33}$$

In other words, the deficiency in the mean pressure $\bar{p}$ arising from the Reynolds stress $\rho\overline{w^2}$ is exactly cancelled, in deep water, by the surface deformation effect. In shallow water, on the other hand, the mean square vertical velocity $\overline{w^2}$ is small. Hence $S_{yy}^{(2)}$ is negligible and

$$S_{yy} = S_{yy}^{(3)} = \tfrac{1}{2}E. \tag{34}$$

Lastly the flow of x-momentum across the plane y = constant is given by

$$S_{xy} = \overline{\int_{-h}^{\zeta} \rho uv\, dz};$$

there is no contribution from the mean pressure. Since $\overline{uv}$ vanishes identically we have always

$$S_{xy} = 0 \tag{35}$$

provided, of course, that the x-direction is the direction of wave propagation. If for some reason we choose a co-ordinate system at an angle (other than a right angle), then there *will* be a non-zero shear stress S_{xy}. Its magnitude may be calculated by the ordinary tensor transformation rules from the two-dimensional tensor S, which in diagonal form is given by

$$S = E\begin{pmatrix} \dfrac{2kh}{\sinh 2kh} + \tfrac{1}{2} & 0 \\ 0 & \dfrac{kh}{\sinh 2kh} \end{pmatrix}. \tag{36}$$

2. *Standing gravity waves*

Let us combine two progressive waves of equal amplitude a and wavelength $2\pi/k$ so as to produce a standing wave. The free surface is then described by

$$\zeta = 2a \cos kx \cos \sigma t \tag{1}$$

and the components of velocity by

$$\left.\begin{aligned} u &= \frac{2a\,\sigma}{\sinh 2kh} \cosh k(z+h) \sin kx \sin \sigma t, \\ w &= \frac{2a\,\sigma}{\sinh 2kh} \sinh k(z+h) \cos kx \sin \sigma t. \end{aligned}\right\} \tag{2}$$

The surface elevation has antinodes at $kx = n\pi$ (where n is an integer) and nodes ta $kx = (n + \frac{1}{2})\pi$, as in Fig. 4. The two components of velocity fluctuate in-phase, proportionally to sin σt, so that the particle orbits are straight lines. Beneath the antinodes the orbits are vertical, beneath the nodes they are horizontal, and at intermediate positions the orbits are inclined generally to the horizontal. The mean product $\overline{uw}$ and also the shearing stress $\rho\overline{uw}$ do not vanish in general, and are functions of the horizontal co-ordinate x. The horizontal variation of $\rho\overline{uw}$ supports a difference in mean surface level between node and antinode. We can use the radiation stress to calculate this difference.

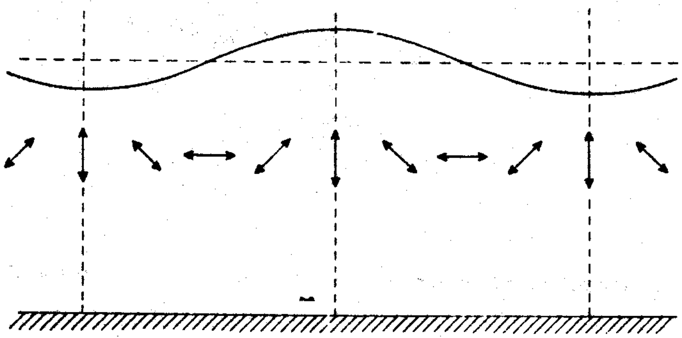

Fig. 4. Particle velocities in a standing wave. The components u and w fluctuate in-phase, and the mean product $\overline{uw}$ is non-zero in general.

Consider the stress S_{xx}, representing the flux of horizontal momentum parallel to the x-axis. This is given by the general relation in equation (7), Section 1, (in which an overbar denotes the average with respect to *time*). As before we may consider the right-hand side as the sum of three parts $S_{xx}^{(1)}$, $S_{xx}^{(2)}$, $S_{xx}^{(3)}$ given approximately by

$$\left.\begin{aligned} S_{xx}^{(1)} &= \int_{-h}^{0} \rho\overline{u^2}\,dz. \\ S_{xx}^{(2)} &= \int_{-h}^{0} (\overline{p} - p_0)\,dz \\ S_{xx}^{(3)} &= \overline{\int_{0}^{\zeta} p dz}. \end{aligned}\right\} \tag{3}$$

where $p_0 = -\rho g z$. The third component $S_{xx}^{(3)}$ is found to be

$$S_{xx}^{(3)} = \tfrac{1}{2}\rho g\overline{\zeta^2} \tag{4}$$

as before. In the second component $S_{xx}^{(2)}$, the time-mean pressure $\overline{p}$ cannot be deduced quite so simply as in the progressive wave, being no longer independent of x. However, it can be found from the more general relation for the vertical flux of vertical momentum :

$$\overline{p + \rho w^2} - \frac{\partial}{\partial x}\int_{z}^{0} \rho\overline{uw}\,dz = p_0 + \rho g\,\overline{\zeta}, \tag{5}$$

in which the terms on the right represent the total weight (per unit cross-section) of a vertical column of water from z to ζ (Fig. 5); the terms on the left show how this weight is supported : the first two terms represent the mean flux of vertical momentum through the base of the column, while the third term is the resultant of the fluxes of vertical momentum through the vertical sides of the column. Taking

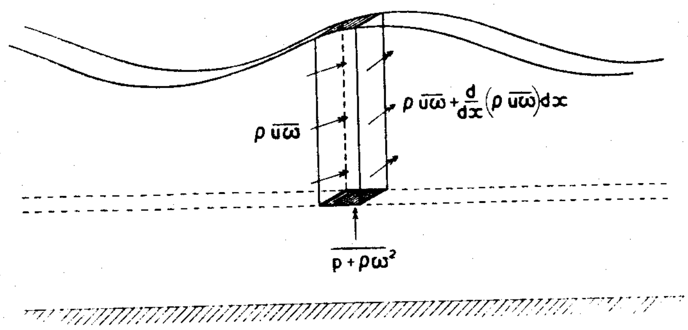

Fig. 5. The balance of momentum in a vertical column of unit cross-section.

$\bar{p} - p_0$ to the left-hand side and the other terms to the right, we have after integration with respect to z :

$$S_{xx}^{(2)} = \rho g h\,\zeta - \int_{-h}^{0} \rho\overline{w^2} + \frac{\partial}{\partial x}\int_{-h}^{0}\int_{z'}^{0} \rho\overline{uw}\,dz\,dz'. \tag{6}$$

Adding this to $S_{xx}^{(1)}$ and $S_{xx}^{(2)}$ we deduce

$$S_{xx} = \rho g h\,\zeta + \int_{-h}^{0} \rho\,\overline{(u^2 - w^2)}\,dz + \frac{\partial}{\partial x}\int_{-h}^{0}\int_{z'}^{0} \rho\overline{uw}\,dz\,dz' + \tfrac{1}{2}\rho g\,\overline{\zeta^2}. \tag{7}$$

Now clearly S_{xx} must be a constant, independent of x, for otherwise horizontal momentum would accumulate at some parts of the wave*. Therefore S_{xx} is equal to its horizontal average, that is to say its average with respect to x over a wavelength. Among the terms on the right of equation (7), the horizontal average of ζ is identically zero, while the horizontal average of the third term also vanishes by the periodicity (the momentum fluxes across two vertical walls a wavelenth apart just cancel). So we have simply

$$S_{xx} = \int_{-h}^{0} \rho\,\underline{\overline{(u^2 - w^2)}}\,dz + \tfrac{1}{2}\rho g\,\underline{\overline{\zeta^2}} \tag{8}$$

where an underbar denotes the horizontal mean value. Substituting from equations (1) and (2) we find

$$S_{xx} = \rho g a^2\left(\frac{2kh}{\sinh 2kh} + \tfrac{1}{2}\right). \tag{9}$$

Comparison with equation (23) Section 1 shows that the radiation stress in a

*This follows from the conservation equation for x- momentum: $\partial S_{xx}/\partial x + \partial S_{xy}/\partial y = 0$, and the fact that $S_{xy} = 0$ in these co-ordinates.

standing wave is exactly twice the value in each progressive wave; it represents the sum of the stresses in the incident and reflected waves, as we should expect.

The *local* mean level $\bar{\zeta}$ can now be found from equation (7) since all other terms in the equation are known to the required approximation. In this way we find†

$$\bar{\zeta} = a^2 k \coth 2kh \cos 2kx. \tag{10}$$

This shows that the mean surface level is slightly raised at the antinodes and correspondingly lowered at the nodes.

The various terms on the right of equation (7) do not all give contributions in the same sense; some tend to raise the level at the antinodes and others to lower it. A simpler way to estimate $\bar{\zeta}$ is to return to the momentum flux equation (5) and set $z = 0$. This gives us

$$\overline{(p + \rho w^2)}_{z=0} = \rho g\, \bar{\zeta} \tag{11}$$

(the flux of vertical momentum across the vertical sides of the column is of third order only). But on taking time averages in the Bernoulli equation

$$\left[p + \tfrac{1}{2}\rho\,(u^2 + w^2) + \frac{\partial \phi}{\partial t}\right]_{z=0} = f(t)$$

we have also

$$[\overline{p + \tfrac{1}{2}\rho\,(u^2 + w^2)}]_{z=0} = C, \tag{12}$$

where C is a constant, not necessarily zero. From equations (11) and (12) we deduce

$$g\,\bar{\zeta} = -\tfrac{1}{2}\,\overline{(u^2 - w^2)}_{z=0} - C. \tag{13}$$

The constant C is determined by the condition that $\underline{\xi} = 0$. Substitution for u and w now gives us equation (10) as before. We note in particular that in deep water ($kh \gg 1$) equation (10) becomes

$$\bar{\zeta} = a^2 k \cos 2kx \tag{14}$$

and in shallow water ($kh \ll 1$)

$$\bar{\zeta} = \frac{a^2}{2h} \cos 2kx. \tag{15}$$

As the depth h diminishes, a and k being fixed, the changes in mean level which are represented by $\bar{\zeta}$ become accentuated.

The evaluation of the transverse stress S_{yy} follows exactly similar lines; it is necessary only to replace $\overline{u^2}$ by $\overline{v^2}$, $= 0$ throughout. Hence

$$\begin{aligned} S_{yy} &= S_{xx} - \int_{-h}^{0} \rho \overline{u^2}\, dz \\ &= S_{xx} - \rho g a^2 \left(\frac{2kh}{\sinh 2kh} + 1\right) \sin^2 kx \\ &= \tfrac{1}{2}\rho g a^2 \left[\frac{2kh}{\sinh 2kh} + \left(\frac{2kh}{\sinh 2kh} + 1\right) \cos 2kx\right]. \end{aligned} \tag{16}$$

Hence S_{yy}, unlike S_{xx}, is a function of x in a standing wave. Perhaps surprisingly,

†This result is in agreement with TADJBAKHSH and KELLER (1960) provided that account is taken of a misprinted sign in their equation (30).

it will be noted that the maximum values of S_{yy} occur at the *nodes* of the surface elevation. The mean value of S_{yy} is given by

$$\underline{S_{yy}} = \rho g a^2 \frac{kh}{\sinh 2kh} \tag{17}$$

which is just twice the value for the progressive wave (equation (32) Section 1) as we should expect.

The radiation shear stress is given by

$$S_{xy} = S_{yx} = 0 \tag{18}$$

as in the progressive wave.

3. *Capillary-gravity waves*

The effect of capillarity is equivalent to the stretching of a thin membrane over the surface with constant tension T per unit length. This modifies the previous calculations in the following ways.

First, the tension produces a flux of x-momentum across the plane $x =$ constant given by $-T\cos\theta$, where θ is the inclination of the surface to the horizontal. The difference between this flux and the equivalent flux in the absence of waves is therefore

$$-T\cos\theta + T = T(1-\cos\theta) = \tfrac{1}{2}T\theta^2 \tag{1}$$

when θ is small. Hence the mean additional flux of momentum due to the presence of the wave is equal to $\tfrac{1}{2}T\overline{\theta^2}$, which must be added to equation (7) Section 1. Since $\theta \doteqdot \partial\zeta/\partial x$ we have

$$S_{xx} = \overline{\int_{-h}^{\zeta} (p+\rho u^2)\,dz} - \int_{-h}^{0} p_0\,dz + \tfrac{1}{2}T\overline{\left(\frac{\partial\zeta}{\partial x}\right)^2}. \tag{2}$$

For a progressive wave, the evaluation of $S_{xx}^{(1)}$ and $S_{xx}^{(2)}$ can be carried out as before, up to equation (15). However in calculating $S_{xx}^{(3)}$ the pressure p near the surface is to be decreased by an amount KT, where K is the curvature of the free surface, that is by an amount $T\partial^2\zeta/\partial x^2$. This adds to $S_{xx}^{(3)}$ the amount

$$-T\overline{\zeta\frac{\partial^2\zeta}{\partial x^2}} \tag{3}$$

which, because the wave is progressive, is equal to

$$T\overline{\left(\frac{\partial\zeta}{\partial x}\right)^2}. \tag{4}$$

For a progressive wave,

$$\overline{\left(\frac{\partial\zeta}{\partial x}\right)^2} = \tfrac{1}{2}a^2k^2. \tag{5}$$

Therefore altogether we have

$$S_{xx} = \tfrac{1}{2}\rho g a^2\left(\frac{\sigma^2 h}{g\sinh^2 kh} + \tfrac{1}{2} + \tfrac{3}{2}\frac{Tk^2}{\rho g}\right). \tag{6}$$

Secondly, the stretching of the surface by the waves stores additional energy T per unit extension of the surface, that is to say

$$T \sec \theta - T, \quad = \tfrac{1}{2} T\theta^2 \tag{7}$$

per unit horizontal area. The mean density of potential energy is therefore increased by an amount.

$$\tfrac{1}{2} T\overline{\theta^2} = \tfrac{1}{2} T \overline{\left(\frac{\partial \zeta}{\partial x}\right)^2} = \tfrac{1}{4}\, T\, a^2\, k^2. \tag{8}$$

Hence the total energy density E, being twice the potential energy density, becomes

$$E = \tfrac{1}{2} \rho g\, a^2 \left(1 + \frac{Tk^2}{\rho g}\right). \tag{9}$$

Thirdly, surface tension modifies the relation between σ and k, so that

$$\sigma^2 = gk \tanh kh \,.\left(1 + \frac{Tk^2}{\rho g}\right). \tag{10}$$

On combining the last two equations with equation (6) we find

$$S_{xx} = E\left(\frac{2kh}{\sinh 2kh} + \frac{1 + 3\epsilon}{2\,(1 + \epsilon)}\right) \tag{11}$$

where

$$\epsilon = \frac{Tk^2}{\rho g}. \tag{12}$$

This of course reduces to equation (23) Section 1 when $\epsilon = 0$. In the opposite limit when $\epsilon \gg 1$, that is to say for pure capillary waves, we have

$$S_{xx} = E\left(\frac{2kh}{\sinh 2kh} + \tfrac{3}{2}\right) \tag{13}$$

where

$$E = \tfrac{1}{2} Ta^2\, k^2, \tag{14}$$

and in particular in deep water ($kh \gg 1$)

$$S_{xx} = \tfrac{3}{2}\, E\,; \tag{15}$$

in the shallow-water case ($kh \ll 1$)

$$S_{xx} = \tfrac{5}{2}\, E. \tag{16}$$

To find the transverse stress S_{yy} we note that although the surface has no inclination in the y-direction nevertheless the corrugations of the wave system produce a greater surface area per unit distance in the y-direction and therefore more tensile stress. Hence equation (26) Section 1 is replaced by

$$S_{yy} = \overline{\int_{-h}^{\zeta} (p + \rho v^2)\, dz} - \int_{0}^{h} p_0\, dz - \tfrac{1}{2} T \overline{\left(\frac{\partial \zeta}{\partial x}\right)^2} \tag{17}$$

This may be split up as before. Since v vanishes, $S_{yy}^{(1)} = 0$. Further since $\overline{w^2}$ is related to the kinetic energy we have

$$S_{yy}^{(2)} = \int_{-h}^{0} (-\,\rho \overline{w^2})\, dz = \tfrac{1}{2} E \left(\frac{2kh}{\sinh 2kh} - 1\right) \tag{18}$$

as for pure gravity waves. The third component $S_{yy}^{(3)}$ is equal to $S_{xx}^{(3)}$:

$$S_{yy}^{(3)} = \tfrac{1}{4}\rho g a^2 \left(1 + \frac{2Tk^2}{\rho g}\right). \tag{19}$$

However

$$-\tfrac{1}{2}T\overline{\left(\frac{\partial \zeta}{\partial x}\right)^2} = -\tfrac{1}{4}\,Ta^2\,k^2 \tag{20}$$

so that the sum of the last two terms is

$$S_{yy}^{(3)} - \tfrac{1}{2}T\overline{\left(\frac{\partial \zeta}{\partial x}\right)^2} = \tfrac{1}{4}\,\rho g a^2\left(1 + \frac{Tk^2}{\rho g}\right) = \tfrac{1}{2}E. \tag{21}$$

Altogether then

$$S_{yy} = E\,\frac{kh}{\sinh 2kh}, \tag{22}$$

the form of which is independent of the surface tension. In deep water

$$S_{yy} = 0 \tag{23}$$

as for pure gravity waves, and in shallow water

$$S_{yy} = \tfrac{1}{2}E. \tag{24}$$

The radiation shear stress is unaffected by surface tension :

$$S_{xy} = S_{yx} = 0. \tag{25}$$

PART II APPLICATIONS

We propose now to describe some of the effects of the radiation stresses upon phenomena observable in the oceans.

First we shall consider instances where the radiation stresses either generate or modify motions on a scale larger than the waves themselves. As will be seen, such effects are liable to occur where there are horizontal gradients of the radiation stresses. Such gradients may arise in a variety of ways.

4. *Wave " set-up "*

One of the most important of these wave-driven effects occurs when deep water waves encounter a sloping beach. The waves shorten, steepen, and eventually break —but continue to advance with decreasing amplitude after breaking. The resulting changes in radiation stress lead to changes in the level of the mean surface.

In the steady state, the shoreward flux of momentum must be independent of x, which we take perpendicular to the shore. Let us now consider the momentum balance in a slice of water bounded by the (sloping) surface $z = \zeta$, the sloping bottom $z = -h$ and two vertical planes $x = x_0$ and $x = x_0 + dx$ (see Fig. 6). If the bottom slope is sufficiently small that $\overline{uw}$ and $\overline{w^2}$ at the bottom* are negligible, then the flux of momentum into the slice, crossing the plane $x = x_0$ is

*By bottom, of course, we refer to the bottom of the irrotational flow. The boundary layer between this and the true bottom we assume to be thin and inconsequential.

$$S_{xx} + \int_{-h}^{\bar{\zeta}} \rho g (\bar{\zeta} - z)\, dz = S_{xx} + \tfrac{1}{2}\rho g (\bar{\zeta} + h)^2. \tag{1}$$

Across the plane $x = x_0 + dz$ the flow out of the slice will be greater than this by

$$\frac{d}{dx}\left[S_{xx} + \tfrac{1}{2}\rho g (\bar{\zeta} + h)^2\right] dx. \tag{2}$$

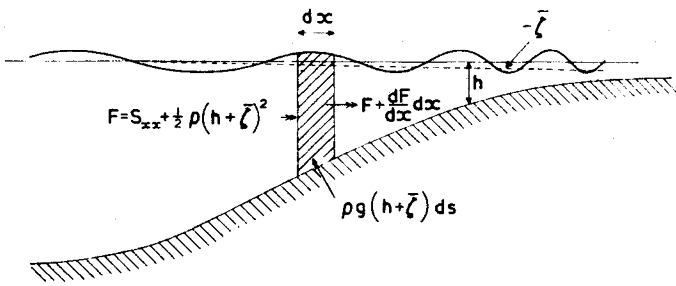

Fig. 6. The balance of horizontal momentum for waves entering shallow water.

There is an additional flux of horizontal momentum due to the bottom pressure, since the bottom is not horizontal, amounting to

$$\rho g (\bar{\zeta} + h) \frac{dh}{dx}\, dx. \tag{3}$$

(The validity of the approximations used here is discussed in III). Momentum balance then gives

$$\frac{dS_{xx}}{dx} + \rho g (\bar{\zeta} + h) \frac{d\bar{\zeta}}{dx} = 0 \tag{4}$$

and so, since $\bar{\zeta} \ll h$,

$$\frac{d\bar{\zeta}}{dx} = -\frac{1}{\rho g h} \frac{dS_{xx}}{dx}. \tag{5}$$

Wave energy approaching a shore may either be reflected or dissipated to heat. If the beach slope is very abrupt, for example like a sea wall, almost all of the energy will be reflected. Alternatively, the slope may be very gradual, so that almost no reflection takes place.

Here we shall consider in detail only the case of slopes sufficiently gentle that reflection is of negligible importance. Two distinct regions can be identified : seawards and shorewards of the line of breakers.

Seawards of the breaker line, in the absence of reflection, we can obtain an expression for the wave energy density as a function of water depth from the requirement that the shoreward flux of energy be independent of the distance from shore, e.g. if we take the simple but important case of wave crests normal to the direction of beach slope :

$$Ec_g = \text{constant}. \tag{6}$$

As the depth h changes, c_g changes and so E also changes. The radiation stress

S_{xx} thus varies because both kh and E vary. It is shown in III that with these conditions equation (5) may be integrated† to yield

$$\bar{\zeta} = -\tfrac{1}{2}\frac{a^2 k}{\sinh 2kh}. \tag{7}$$

In deep water, $\bar{\zeta}$ vanishes, while in shallow water ($kh \ll 1$) we have

$$\bar{\zeta} = -\frac{a^2}{4h}. \tag{8}$$

Formula (7) and (8) express the wave set-up in terms of the *local* wave amplitude, wavenumber and depth. However by using equation (6) it is also possible to express $\bar{\zeta}$ as a function of the (constant) wavenumber k_0 and amplitude a_0 in deep water, together with the local depth h, so that we gain an idea of the dependence of $\bar{\zeta}$ on the depth h for a given wave train. Thus if we substitute in equation (6) the known value of the group-velocity :

$$c_g = \frac{\sigma}{2k}\left(\frac{2kh}{\sinh 2kh} + 1\right) \tag{9}$$

we obtain

$$\frac{a^2\,\sigma}{2k}\left(\frac{2kh}{\sinh 2kh} + 1\right) = \text{constant} = \frac{a_0^{\,2}\,\sigma}{2k_0} \tag{10}$$

and so

$$a^2\,k = a_0^{\,2}\,k_0\left(\frac{k}{k_0}\right)^2\left(\frac{2kh}{\sinh 2kh} + 1\right)^{-1}. \tag{11}$$

But from the frequency relation equation (17) Section 1,

$$k/k_0 = \coth kh. \tag{12}$$

So we have from equation (7)

$$\bar{\zeta} = -\tfrac{1}{2}\,a_0^{\,2}\,k_0\,\frac{\coth^2 kh}{2kh + \sinh 2kh}. \tag{13}$$

Since from equation (17) Section 1

$$kh\tanh kh = \frac{\sigma^2 h}{g} = k_0\,h \tag{14}$$

it follows that we may write

$$\bar{\zeta} = -\,a_0^{\,2}\,k_0 f(k_0\,h)$$

where f is a function solely of the non-dimensional depth $k_0\,h$. The form of f is shown in Fig. 7. It will be seen that as the depth diminishes, the *depression* of the mean surface level increases very sharply. In shallow water, we have from equations (13) and (14) that

$$f \sim \tfrac{1}{8}\,(kh)^{-3} \sim \tfrac{1}{8}\,(k_0\,h)^{-3/2}$$

in agreement with equation (8), since in shallow water $a^2 \propto h^{-\frac{1}{2}}$ by energy conservation. It will be noted that as h decreases the mean water level is actually *lowered* by the presence of unbreaking waves, i.e. there is a " set-down." This is because, with no loss of energy, the radiation stress steadily increases.

†Alternatively equation (7) can be derived from equation (13) Section 2, by substituting for u and w from equations (4) Section 1.

On the other hand, inside the line of breakers the wave energy decreases shorewards leading to a decrease in radiation stress. No adequate theory exists for this situation, but we are nevertheless able to draw some approximate conclusions using a semi-empirical argument.

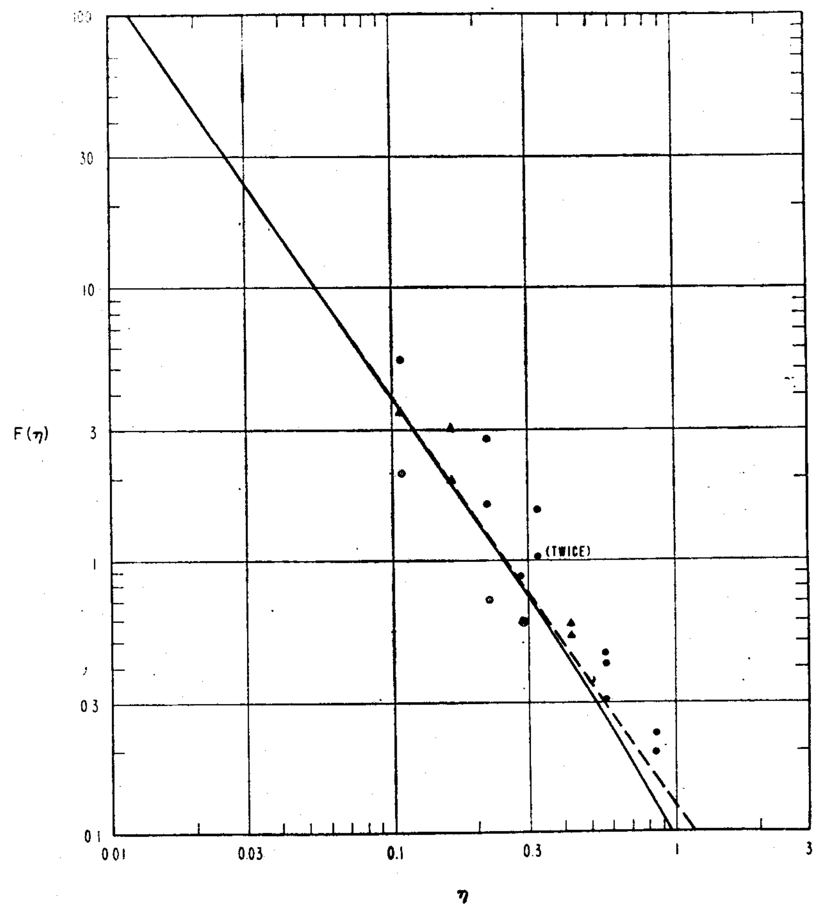

Fig. 7. (from LONGUET-HIGGINS and STEWART, 1963). The non-dimensional form of the wave set-up outside the breaker zone, compared with the observations of SAVILLE (1961).

The theory of similarity suggests that the amplitude of a breaking wave remains an approximately fixed proportion of the mean water depth, i.e.

$$a = \alpha h \tag{15}$$

where α is a constant of proportionality. Although the waves are now far too steep for our second-order treatment to remain valid, it is probably a not unreasonably inaccurate assumption to continue to assume that $S_{xx} = \frac{3}{2} E$. This gives

$$S_{xx} = \tfrac{3}{4}\, \rho g a^2 = \tfrac{3}{4}\, \rho g\, \alpha^2 h^2. \tag{16}$$

If we insert this expression in equation (5) we get

$$\frac{d\bar{\zeta}}{dx} = -\tfrac{3}{2}\alpha^2 \frac{dh}{dx}. \tag{17}$$

The observations of SAVILLE (1961) confirm that a rise in level starts to occur at about the point where the waves first break (Fig. 8). Moreover in the breaker zone it has been shown that $d\bar{\zeta}/dx$ was roughly proportional to dh/dx, with a constant of proportionality equal to about $-0{\cdot}15$: (LONGUET-HIGGINS and STEWART, (1963)). We therefore estimate α to be about 0·32.

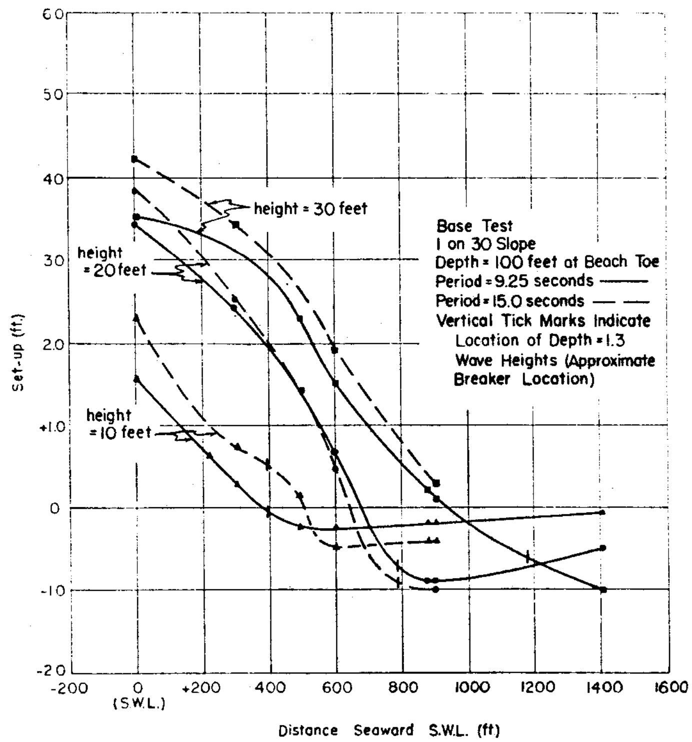

Fig. 8. (from SAVILLE, 1961). Observed wave set-up on beaches of different slope. The vertical tick marks the breaker point.

MUNK (1949) observed that swell tends to break when the depth is about 1·3 times the crest to trough height, i.e.

$$h = 1{\cdot}3 \times 2a. \tag{18}$$

This corresponds to $\alpha = 0{\cdot}39$, so the two sets of observations are entirely consistent.

The importance of waves in producing set-up seems only recently to have been realized. Contemporary with our work, contributions to the topic have been made by DORRESTEIN (1962), FORTAK (1962) and LUNDGREN (1963). HARRIS (1963) points out that since waves are subject to modification by refraction and diffraction, variations in wave set-up are to be expected even over short distances. He describes the observed variations in the height of storm surges to this effect. Since these variations may amount to " two to four feet in the peak water level within a distance of half a mile " it can be seen that wave set-up produces very far from negligible contributions to storm surges.

Another practically important effect of wave set-up is its influence on the apparent " tilt " due to wind stress on the surface of an enclosed body of water. Measurement of such tilts is one of the standard techniques for determining the magnitude of the wind stress. In his well-known critical article on " *Wave Generation by Wind*," URSELL (1956) speculated upon the possible importance of " radiation pressure " effects upon measured water levels.

As we have seen, such effects do occur, and are important. They may well account for much of the variability and unreliability which have beset efforts to determine the laws governing wind stress upon water.

5. *Groups of waves advancing in deep water*

Horizontal gradients of radiation stress can also arise when the waves have amplitudes which vary in time, and therefore in space. The simplest illustrative example is one where we have sinusoidal wave trains of nearly the same frequency and wavelength propagated in the same direction, resulting in the formation of " groups " of waves.

We shall assume that the groups are such that the energy density, rather than the envelope of the amplitude, varies sinusoidally. By this artifice we avoid some problems with non-linearities which are irrelevant to our present purpose. The energy density is then given by

$$E = E_0 \{1 + b \cos \Delta k (x - c_g t)\} \tag{1}$$

where Δk is a measure of the " band width " of wavenumbers making up the groups, which propagate with speed c_g.

Let us assume also that the depth h is large relative to the lengths of the individual waves, but not necessarily large relative to the length of the groups themselves, i.e. $kh \gg 1$, but not necessarily $\Delta kh \gg 1$. Accordingly, the radiation stress will be

$$S_{xx} = \tfrac{1}{2} E_0 \{1 + b \cos \Delta k (x - c_g t)\}. \tag{2}$$

We may now divide the depth into two regions : an upper one with thickness $D \sim k^{-1}$, in which virtually all of the radiation stress is concentrated, and a lower one which responds only to any variations in mean surface level produced by the radiation stresses. The problem is now analagous to that which arises in the study of flow induced by horizontal variations of surface tension.

Within the upper region, the horizontal momentum equation is :

$$\frac{\partial \bar{u}}{\partial t} = - \frac{1}{D\rho} \frac{\partial S_{xx}}{\partial x} - g \frac{\partial \bar{\zeta}}{\partial x}, \tag{3}$$

where $\bar{u}$ and $\bar{\zeta}$ are associated with the groups, i.e. the average is over one wavelength of the individual waves. Since D is small, it is not unreasonable to assume that the first term on the right of equation (3) is much larger than the second. We shall be able to check the validity of this assumption *post hoc*. We therefore put

$$D \frac{\partial \bar{u}}{\partial t} = - \frac{1}{\rho} \frac{\partial S_{xx}}{\partial x}. \tag{4}$$

Now if we integrate the equation of continuity over our upper region, we find

$$\overline{w}_D = -D\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{\zeta}}{\partial t} \tag{5}$$

where $\overline{w}_D$ is the mean vertical velocity at the depth D, the mean being taken over the individual waves as with $\overline{u}$ and $\overline{\zeta}$. These last two equations may be combined to give

$$\frac{\partial \overline{w}_D}{\partial t} - \frac{\partial^2 \overline{\zeta}}{\partial t^2} = \frac{1}{\rho}\frac{\partial^2 S_{xx}}{\partial x^2}. \tag{6}$$

Equation (6) may be interpreted as follows : variations in the radiation stress produce convergences in the upper layer. Continuity is preserved by pushing water up, thus producing variations in the surface elevation, and by pushing water down, resulting in an induced flow in the deeper region. Our equations are closed by the fact that these deep induced flows must be dynamically driven by pressure gradients produced by the variations in surface elevation.

The flow in the deep region is a periodic irrotational flow and so must be of the form of equation (4) Section 1, and derivable from a velocity potential :

$$\phi = \frac{Ac_g}{\sinh kh}\cosh \Delta k\,(z + h)\sin \Delta k\,(x - c_g\,t). \tag{7}$$

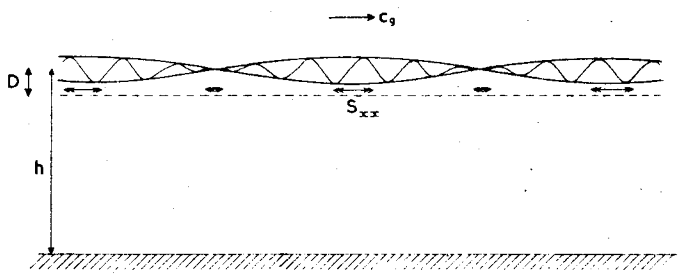

Fig. 9. Groups of waves in deep water. The radiation stress acts in a shallow layer near the surface.

For this flow we have, at $z = -D \doteqdot 0$, two requirements on the pressure $\overline{p}_D$: First, it must be given by the hydrostatic equation,

$$\frac{1}{\rho}\overline{p}_D = g\,(\overline{\zeta} + D). \tag{8}$$

Second it must satisfy the linearized Bernoulli equation.

$$\frac{\partial \overline{\phi}_D}{\partial t} + \frac{1}{\rho}\overline{p}_D - gD = 0. \tag{9}$$

Together, these conditions give us

$$\overline{\zeta} = -\frac{1}{g}\frac{\partial \overline{\phi}_D}{\partial t}. \tag{10}$$

We may now substitute for $\overline{w}_D$, $\overline{\zeta}$ and S_{xx} in equation (6), remembering that $D\Delta k \ll 1$:

$$-Ac_g^2 + \frac{A}{g}\Delta k\, c_g^4 \coth h\Delta k = -\frac{E_0 b}{2\rho}. \tag{11}$$

Since $kh \gg 1$, $c_g^2 = g/4k$, so we can write equation (11) as

$$A = \frac{E_0 bk}{2\rho g\{1 - (\Delta k/k)\coth h\Delta k\}} \tag{12}$$

We are now able to find $\bar{\zeta}$; from equation (10)

$$\bar{\zeta} = \frac{-E_0 b\,\Delta k \sin \Delta k(x - c_g t)}{2\rho g\{\tanh h\Delta k - \Delta k/k\}} = -\frac{(E - E_0)\,\Delta k}{2\rho g\{\tanh h\Delta k - \Delta k/k\}} \tag{13}$$

Since $E = \frac{1}{2}\rho g a^2$, where a is the individual wave amplitude,

$$\bar{\zeta} = -\frac{(a^2 - a_0^2)\,\Delta k}{4\{\tanh h\Delta k - \Delta k/k\}}. \tag{14}$$

This expression is in agreement with the result (3.19) of III, which was obtained by perturbation analysis. We note that $\bar{\zeta}$ is always out of phase with a^2, that is, the mean level is depressed under the largest waves.

Equation (14) simplifies to some extent at the two extreme cases $h\Delta k \ll 1$ and $h\Delta k \gg 1$. For $h\Delta k \ll 1$, when the group length is great compared with the depth, we find

$$\bar{\zeta} = -\frac{(a^2 - a_0^2)\,k}{4kh - 1} \tag{15}$$

or, since we have already assumed $kh \gg 1$,

$$\bar{\zeta} = -\frac{(a^2 - a_0^2)}{4h}. \tag{16}$$

For $h\Delta k \gg 1$, if we assume $\Delta k/k \ll 1$, equation (14) becomes

$$\bar{\zeta} = -\tfrac{1}{4}(a^2 - a_0^2)\,\Delta k. \tag{17}$$

To get a numerical order of magnitude, we might take $(a^2 - a_0^2)$ to be about $10\ \text{m}^2$. If the individual waves are about 100 m long, and the groups about 1 km long, we have $k \doteqdot 0{\cdot}06\ \text{m}^{-1}$, $\Delta k \doteqdot 0{\cdot}006\ \text{m}^{-1}$. In deep water ($h \geqslant 500$ m) this results in a surface depression of about 1·5 cm, while the water 100 m deep the depression would be 2·5 cm. These figures, of course, increase rapidly as the individual wave amplitude increases*.

*We are now in a position to make the *post hoc.* check of our assumption that

$$g\frac{\partial \bar{\zeta}}{\partial x} \ll \frac{1}{\rho D}\frac{\partial S_{xx}}{\partial x}.$$

Since $\Delta k \ll k$ and $kh \gg 1$,

$$\frac{\partial \bar{\zeta}}{\partial x} \doteqdot \frac{(a^2 - a_0^2)(\Delta k)^2}{4\tanh h\Delta k}.$$

Also

$$\frac{\partial S_{xx}}{\partial x} \sim \tfrac{1}{2}(E - E_0)\,\Delta k = \tfrac{1}{4}\rho g(a^2 - a_0^2)\,\Delta k.$$

Then

$$g\frac{\partial \bar{\zeta}}{\partial x}\Big/\frac{1}{\rho D}\frac{\partial S_{xx}}{\partial x} \doteqdot \frac{D\,\Delta k}{\tanh h\Delta k}.$$

Since

$$\tanh h\Delta k = \begin{Bmatrix} O(1), & h\Delta k > \frac{1}{2} \\ O(h\Delta k), & h\Delta k < \frac{1}{2} \end{Bmatrix}$$

our assumption is seen to be justified.

It is worth noting that the frequency of the induced motions is the group frequency —in practice periods are of the order of one minute. This may be important since often it is assumed that there is little motion in the ocean with such periods, and buoys are sometimes designed with a natural period of this order, in the hope that their free oscillations will not be excited.

6. *Wave groups in shallow water; surf beat*

Let us now consider the situation when waves enter water which is shallow enough so that kh is no longer large compared with unity. In this case we can no longer assume that the radiation stress acts in a thin layer near the surface. On the other hand since the length of the wave groups is certainly large compared to k^{-1} we may certainly assume that the groups are long compared to the depth, i.e. that $h\Delta k \ll 1$.

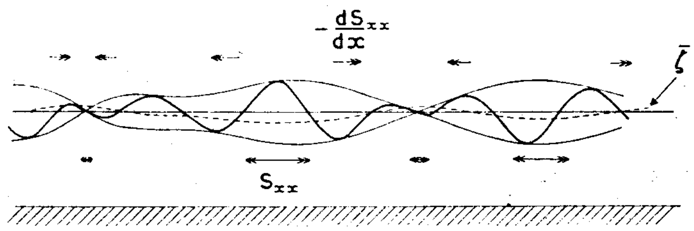

Fig. 10. Groups of waves entering shallow water, when the wavelength is no longer small compared to the depth.

Figure 10 illustrates the situation. Groups of waves (whose energy need not vary exactly sinusoidally) are propagated to the right with the group-velocity c_g. The depth h is at first assumed uniform. In regions of high energy the radiation stress S_{xx} is greater than in regions of low energy. Hence there is a tendency for fluid to be expelled from under regions of high energy density. The medium responds to the stress as to a horizontal force $-\partial S_{xx}/\partial x$ per unit distance, progressing with the group-velocity c_g.

The response of the system can be calculated as follows. The additional mean pressure due to a displacement $\bar{\zeta}$ of the free surface is $\rho g\bar{\zeta}$. Hence if M is the mean horizontal momentum we have for the rate of change of momentum

$$\frac{\partial M}{\partial t} = -\frac{\partial}{\partial x}(S_{xx} + \rho g h\bar{\zeta}). \tag{1}$$

On the other hand by continuity we have

$$\frac{\partial}{\partial t}(\rho\bar{\zeta}) = -\frac{\partial M}{\partial x}. \tag{2}$$

Since the pattern progresses with velocity c_g, we may replace $\partial/\partial t$ by $-c_g\,\partial/\partial x$ in each equation, giving

$$\left.\begin{aligned} c_g \frac{\partial M}{\partial x} + \rho g h \frac{\partial \bar{\zeta}}{\partial x} &= - \frac{\partial S_{xx}}{\partial x} \\ \frac{\partial M}{\partial x} + c_g \, \rho \frac{\partial \bar{\zeta}}{\partial x} &= 0. \end{aligned}\right\} \tag{3}$$

Solving these simultaneous equations for $\partial M/\partial x$ and $\partial\bar{\zeta}/\partial x$ and integrating with respect to x we find

$$\left.\begin{aligned} M &= - \frac{c_g \, S_{xx}}{gh - c_g^2} + \text{constant} \\ \rho\bar{\zeta} &= - \frac{S_{xx}}{gh - c_g^2} + \text{constant.} \end{aligned}\right\} \tag{4}$$

Now the group-velocity c_g cannot exceed the free-wave velocity $\sqrt{gh}$ of the long waves, so that $(gh - c_g^2) > 0$, i.e. the response of the free surface is in the same sense as if the group pattern were stationary ($c_g = 0$); below a group of high waves $\bar{\zeta}$ tends to be negative, and below a group of lower waves it is relatively positive. Since the groups are long, the mean pressure $\bar{p}$ on the bottom fluctuates in the same way as $\rho g\bar{\zeta}$, i.e. it tends to be negative under the higher waves. This is in agreement with some recent observations in swell off the Californian coast (see HASSELMAN, MUNK and MACDONALD 1962).

In very shallow water, c_g approaches $\sqrt{gh}$ and hence the denominator in equation (4) becomes small. Since in that case

$$c_g^2 \doteqdot gh\left[1 - (kh)^2\right] \tag{5}$$

we have

$$\bar{\zeta} \doteqdot - \frac{S_{xx}}{\rho\sigma^2 \, h^2} = - \frac{3ga^2}{2\sigma^2 \, h^2}. \tag{6}$$

If we now suppose that the depth is not quite uniform, but changes with x so slowly that dynamical equilibrium has time to be established, then, with no loss of energy, $a^2 \propto h^{-\frac{1}{2}}$ and so $\bar{\zeta} \propto h^{-5/2}$. Thus there will be a tendency for the displacement $\bar{\zeta}$ to be amplified as the waves enter shallower water.

It is possible that such an effect accounts for the occurrence of "surf-beats," as observed by MUNK (1949a) and TUCKER (1950). These are waves of long period associated with groups of high waves entering shallow water. TUCKER (1950) showed that in his observations there was a correlation between the surf beats at a point some way off-shore and the envelope of the incoming swell; but with a time-lag of a few minutes. The time-lag just corresponded with the time required for the swell to be propagated into the breaker zone and for the associated long wave to be reflected back as a free wave. If we suppose that, at some point shorewards of the wave recorder, the swell is destroyed by breaking but that the longer waves associated with the groups are reflected back as free waves (with relatively little attenuation in amplitude) then it seems possible to account for Tucker's observations. Tucker found that a group of high waves tended to be associated (after a time-lag) with a *negative* pressure pulse, which would accord with the present hypothesis.

7. *Interaction of waves and currents*

In the theory of elasticity and rheology, where stress is measured in force per unit area, it is well known that the product stress times rate of strain yields power per unit volume. Similarly, in our case of radiation stress (which is a force per unit length) stress times rate of strain is power per unit area. We expect that if a fluid, upon which are surface waves, is strained due to some other flow, the radiation stress due to the waves will interact with the rate of strain due to the other flow. In general, we argue that the straining flow must do work against the radiation stress. Energy must then be lost by the straining flow. In many cases we have been able to show that this energy is transferred to the waves. Indeed, if the sign of the interaction is changed, so that the stress does work against the rate of strain there seems to be no source for the energy added to the straining motion other than that residing in the waves. It thus seems legitimate to argue that the energy transfer will always be to or from the waves.

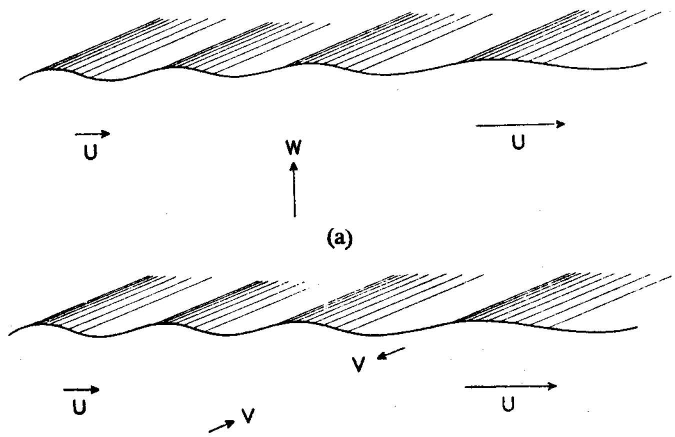

Fig. 11. Waves on a non-uniform current (a) with upwelling from below and (b) with horizontal inflow from the sides.

Interaction with irrotational plane strains. The simplest motions to deal with analytically are irrotational plane strains. They also serve as valuable examples of the nature of the interaction phenomenon.

Let us consider first a situation in which a contraction of the surface along the x-axis is combined with a vertical extension, i.e.

$$\frac{\partial U}{\partial x} = -\frac{\partial W}{\partial z} \tag{1}$$

where U and W are the mean straining velocities in the x and z directions (Fig. 11). For the moment we shall assume that the mean motion is not time-dependent. The situation we envisage is approximately that which occurs when a stream flows along a bed of fixed width but varying depth. The only component of radiation stress which is of consequence in this flow is S_{xx}. It interacts with the rate of strain $\partial U/\partial x$, which describes an extension of the surface, in such a way that work is done by the stress at the rate

$$S_{xx}\frac{\partial U}{\partial x} \tag{2}$$

per unit surface area. It seems that the only source of energy for this work is the energy residing in the waves. Since wave energy is propagated with celerity c_g and transported with velocity U, we postulate a "continuity" equation for wave energy in the form*

$$\frac{\partial}{\partial x}[E(U+c_g)] + S_{xx}\frac{\partial U}{\partial x} = 0. \tag{3}$$

We also have another general expression which might be called an expression for "conservation of phase." It states that in the steady state the rate at which wave crests enter a region must be equal to the rate at which they leave. Another way of stating it is that the apparent frequency observed is independent of the position of the observer. This general expression is

$$(U+c)\,k = \text{constant} = c_0\,k_0 \tag{4}$$

where the subscript 0 refers to some position, perhaps hypothetical, where $U = 0$.

Since when the depth is known, c and c_g are determined by k and S_{xx} by E, evidently equations (3) and (4) are sufficient for the determination of E as a function of U. In the general case this relation is analytically rather complex, but all the important features may be demonstrated by the example of the deep water case, which is simple. In deep water, we may assume

$$c = (g/k)^{\frac{1}{2}}, \quad c_g = \tfrac{1}{2}c, \quad S_{xx} = \tfrac{1}{2}E. \tag{5}$$

Thus equation (3) becomes

$$\frac{\partial}{\partial x}[E(U+\tfrac{1}{2}c)] + \tfrac{1}{2}E\frac{\partial U}{\partial x} = 0. \tag{6}$$

Equation (4), in view of (5), can be expressed as

$$\frac{\partial}{\partial x}\left[\frac{U+c}{c^2}\right] = 0 \tag{7}$$

or

$$\frac{\partial c}{\partial x} = \frac{\tfrac{1}{2}c}{U+\tfrac{1}{2}c}\,\frac{\partial U}{\partial x}. \tag{8}$$

Equation (6) has the exact integral

$$E(U+\tfrac{1}{2}c)\,c = \text{constant} = E_0\cdot\tfrac{1}{2}c_0^{\,2}, \tag{9}$$

as may be demonstrated by differentiating :

$$c\frac{\partial}{\partial x}[E(U+\tfrac{1}{2}c)] + E(U+\tfrac{1}{2}c)\frac{\partial c}{\partial x} = 0. \tag{10}$$

If equation (8) is substituted into (10), we obtain the differential equation (6). The corresponding variation in amplitude a, $\propto E^{\frac{1}{2}}$, is shown as a function of U by curve (1) in Fig. 12. The result (9) was also obtained by perturbation methods in II.

Laterally converging current. Another illustrative example of waves superimposed on a plane strain occurs when the mean motion is two-dimensional and horizontal (Fig. 11b). Such a situation may arise, for example, at a river mouth. For simplicity,

*For a further justification of this equation see Whitham (1962).

let us take the waves moving in one of the directions of principal rate of strain, so that if they are propagating in the x-direction

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{11}$$

V being the mean motion in the y-direction, which is parallel to the wave crests.

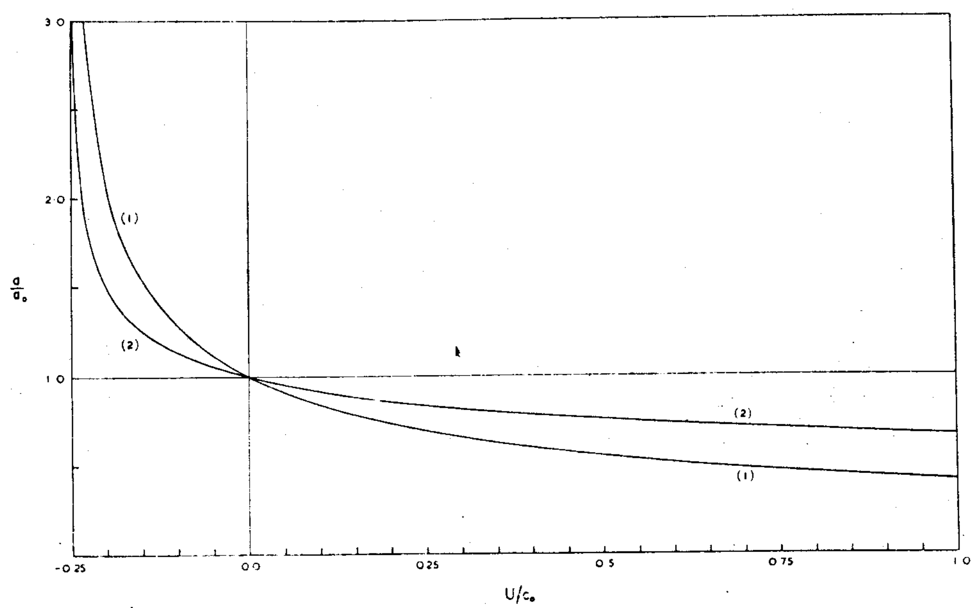

Fig. 12. (from LONGUET-HIGGINS and STEWART, 1961). The relative amplitude of waves propagated on a steady, non-uniform current U, (1) with upwelling from below (2) with horizontal inflow from the sides.

Equation (3) must now be modified to take account of the lateral divergence of wave energy and the work done by the lateral radiation stress S_{yy}, and so becomes

$$\frac{\partial}{\partial x}[E(U + c_g)] + \frac{\partial}{\partial y}(EV) + S_{xx}\frac{\partial U}{\partial x} + S_{yy}\frac{\partial V}{\partial y} = 0. \tag{12}$$

If we assume $\partial E/\partial y = 0$ and employ equation (11) this simplifies to

$$\frac{\partial E}{\partial x} U + E\frac{\partial c_g}{\partial x} + (S_{xx} - S_{yy})\frac{\partial U}{\partial x} = 0. \tag{13}$$

If we once more consider the simple deep water case, then $S_{yy} = 0$ and equation (8) is valid. Equation (13) can then be integrated, as demonstrated in II, to obtain

$$E(U + \tfrac{1}{2}c)/c = \text{constant} = E_0. \tag{14}$$

The corresponding change in amplitude ($\propto E^{\frac{1}{2}}$) is shown as a function of U by curve (2) in Fig. 12.

At the other extreme of very shallow water we have :

$$c_g = c = \text{constant}, \quad S_{xx} - S_{yy} = E. \tag{15}$$

It is then readily seen that (13) can be integrated to

$$E\,(U + c) = \text{constant} = E_0\, c_0. \tag{16}$$

It will be noted that in every case so far considered, E must diverge when $U = -c_g$. In practice, of course, the waves break. This result is to be expected since it is merely a statement of the fact that no energy can be propagated upstream against a current faster than c_g. Apart from this common property, it can be seen that the behaviour of wave energy differs from case to case.

TAYLOR (1962) has discussed a slightly different case, where a standing wave is compressed longitudinally, thus both reducing the wave length and increasing the frequency. There also work is done against the radiation stress. Taylor shows that in this situation, too, the energy used in the compression appears as increased wave energy.

Waves on a shear flow. We may use the same kind of arguments to discuss the interaction of a wave train with a shear flow. In this case, however, the waves will generally be refracted, so that it is not possible to use the direction of wave propagation as a fixed Cartesian co-ordinate direction. It is therefore necessary to put our radiation stress tensor into general, non-diagonalized form.

To keep the discussion as simple as possible we shall again confine ourselves to the case of waves on deep water. The diagonalized form of the radiation stress tensor is then

$$\boldsymbol{S} = \tfrac{1}{2}E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{17}$$

For a co-ordinate system orientated at an angle θ from the direction of propagation, the tensor transformation formula gives us

$$\boldsymbol{S} = \tfrac{1}{2}E\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}. \tag{18}$$

The rate-of-strain tensor for the mean flow is

$$\boldsymbol{\gamma} = \begin{pmatrix} \dfrac{\partial U}{\partial x} & \tfrac{1}{2}\left(\dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x}\right) \\ \tfrac{1}{2}\left(\dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x}\right) & \dfrac{\partial V}{\partial y} \end{pmatrix}. \tag{19}$$

Equations of the form (3) and (12) may thus be generalized to

$$\frac{\partial E}{\partial t} + \nabla\cdot[E\,(U + c_g)] + \boldsymbol{S} : \boldsymbol{\gamma} = 0. \tag{20}$$

As a simple illustrative example we choose the case of a mean flow with velocity in the y-direction only. If we assume steady state (except for the periodic motion) we have

$$U = 0, \quad V = V(x). \tag{21}$$

Putting in the appropriate value for the radiation stress, we get from (20) :

$$\frac{\partial}{\partial x}\left[E \cdot \tfrac{1}{2}c\cos\theta\right] + \tfrac{1}{2}E\frac{\partial V}{\partial x}\cos\theta\sin\theta = 0, \tag{22}$$

since the component of c_g in the x-direction is $\frac{1}{2}c\cos\theta$.

In order to solve (22) we require some relation between V and c. One is available from the "wave kinematics," because the wavenumber in the y-direction must be independent of x. Otherwise θ would be a function of y. Thus

$$k\sin\theta = \text{constant} = k_0\sin\theta_0. \tag{23}$$

As is shown in II, we are then able to integrate (22), getting

$$E\cos\theta\sin\theta = \text{constant} = E_0\cos\theta_0\sin\theta_0. \tag{24}$$

In less special cases, (20) can be integrated numerically. This was the procedure adopted by HUGHES and STEWART (1961), who studied the interaction of a wave train with a stable Couette shear flow. They found that their experimental observations were in quite good agreement with numerical calculations made from (20). However, HUGHES and STEWART were unaware of the full effect of capillarity on the radiation stress. (The influence of capillarity is given for the first time in the present paper). Since their waves were short enough to be influenced by surface tension, the actual effect of the radiation stress is greater than that which they assumed. It is noteworthy that the observations of HUGHES and STEWART indicated a somewhat greater influence of the radiation stress than was obtained from their calculations.

It should be emphasized that the changes in wave energy which are due to the non-linear interaction of waves with shear flow are of the same order of magnitude as those which occur due to the geometrical focussing effects produced by the currents. At first glance this may seem surprising, since the radiation stresses are a second order phenomenon, while the focussing effects appear to be first order. The fact is, however, that the focussing effects are first order in the *energy*, i.e. of second order in the amplitude and comparable with the radiation stresses.

8. *Non-linear interaction between waves*

In recent years there has been a considerable amount of interest shown in the problem of the non-linear interaction of surface waves. For some aspects of the problem, in particular in the study of the irreversible redistribution of energy over the wave spectrum (PHILLIPS, 1960; HASSELMANN, 1962, 1963), the interaction must be taken to the third or higher order. For such purposes the radiation stress concept is not particularly useful.

On the other hand there are cases where the radiation stress idea is valuable conceptually and, in some limiting situations, sufficient for calculations. These cases are ones in which one wave is much shorter than the other with which it interacts. Then it becomes reasonable to treat the long wave as a straining motion interacting with the radiation stress due to the short waves.

As a concrete example, we consider here the case of the long waves upon which are superimposed waves short enough that they are uninfluenced by the bottom. Most of the important features are illustrated by this example.

In any non-linear interaction between one Fourier component of wavenumber and frequency k_1, σ_1, and another specified by k_2, σ_2, the second-order terms describe the generation of components $k_1 \pm k_2$, $\sigma_1 \pm \sigma_2$. However, if one of the wave numbers, say k_1, is very much greater than the other, then the generated wave numbers will all be in the neighbourhood of k_1. The second-order interaction can thus be considered to describe the influence of the long waves on the shorter ones. For the reverse interaction of the shorter waves on the longer ones, higher order terms are needed.

If both short and long waves are progressing in the same direction, the problem is two-dimensional. Equation (20) Section 8 then becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[E(U + c_g)] + S_{xx}\frac{\partial U}{\partial x} = 0 \tag{1}$$

where we interpret E and c_g as pertaining to the short waves and U as the horizontal component of the orbital (particle) velocity of the long wave.

The motion due to the long wave will be described by (4) Section 1. We are concerned only with motion near the surface, so that the horizontal and vertical velocities are given by

$$\left.\begin{aligned} U &= A\,\sigma_2 \coth k_2 h \cos (k_2 x - \sigma_2 t) \\ W &= A\,\sigma_2 \sin (k_2 x - \sigma_2 t) \end{aligned}\right\} \tag{2}$$

The horizontal variation of E and c_g arises only because of the interaction, and so is irrelevant to (1) if the equation is taken only to the lowest order. Since the short waves are uninfluenced by the bottom, $S_{xx} = \frac{1}{2}E$. Hence (1) becomes

$$\frac{\partial E}{\partial t} + \tfrac{3}{2}E\frac{\partial U}{\partial x} = 0 \tag{3}$$

or, since U is due to a wave motion and so $\partial/\partial x = -(1/c_2)\,\partial/\partial t$,

$$\frac{\partial E}{\partial t} - \tfrac{3}{2}\frac{E}{c_2}\frac{\partial U}{\partial t} = 0 \tag{4}$$

where c_2 is the phase speed of the long wave. This may be integrated to give

$$E\left(1 - \tfrac{3}{2}\frac{U}{c_2}\right) = \text{constant} = E_0 \tag{5}$$

or, considering that $U \ll c_2$,

$$E = E_0[1 + \tfrac{3}{2}A\,k_2 \coth k_2 h \cos (k_2 x - \sigma_2 t)]. \tag{6}$$

Although (6) describes the energy variation, the noticeable feature will be the amplitude variation. As shown in I, here the relation between amplitude and energy is not quite so straight-forward as it is in most cases. The water surface upon which the short waves are running is subject to vertical acceleration due to the presence of the long wave. This results in a distribution of E between potential energy and kinetic energy which differs from that which obtains in the absence of the vertical acceleration.

The question is discussed in some detail in I. In the present paper we shall be content with an outline. To an observer moving in an (accelerated) frame of reference tied to one point on the surface of the *long* wave, the apparent value of

g is $g' = g + \partial W/\partial t$. To this observer, the short wave energy is equally distributed between kinetic and potential, i.e.

$$K.E.' = P.E.' = \tfrac{1}{4}\rho g' a^2 \tag{7}$$

where a is the short wave amplitude. An observer in an inertial frame of reference finds himself in agreement with the accelerated observer as to the kinetic energy, but calculates a different potential energy. Thus

$$\left.\begin{aligned} K.E. &= \tfrac{1}{4}\rho g' a^2 = K.E.' \\ P.E. &= \tfrac{1}{4}\rho g\, a^2 \neq P.E.' \end{aligned}\right\} \tag{8}$$

so

$$E = K.E. + P.E. = \tfrac{1}{2}\rho g\, a^2 \left(1 + \frac{1}{2g}\frac{\partial W}{\partial t}\right). \tag{9}$$

Since $\partial W/\partial t \ll g$, we can write this as

$$a = \left(\frac{2E}{\rho g}\right)^{\frac{1}{2}} \left(1 - \frac{1}{4g}\frac{\partial W}{\partial t}\right). \tag{10}$$

After using (2) and (6) this becomes

$$a = a_0 \left[1 + A\left(\tfrac{3}{4}k_2 \coth k_2 h + \frac{\sigma_2^2}{4g}\right)\cos\left(k_2 x - \sigma_2 t\right)\right]. \tag{11}$$

if the long waves are also effectively in deep water, the expression simplifies to

$$a = a_0\left[1 + A\,k_2 \cos\left(k_2 x - \sigma_2 t\right)\right]. \tag{12}$$

It will be noted that the maximum small-wave amplitude occurs on the crests of the long waves. Such amplification of short waves on the crests of the long wave is a matter of common observation.

These effects may be of some consequence in the spectrum of wind-raised waves. It is generally considered (Phillips, 1957) that on a wind swept sea all waves shorter than a certain length are "saturated." That is, they possess as much energy as they are statistically able to. If they gain more energy, wave breaking becomes so widespread in both time and space that the energy rapidly reverts to the "saturated" level.

We see from the above discussion, however, that for waves riding on the backs of longer waves peak amplitudes occur at the crests of the longer waves. It is there that the shorter waves break, and there that the overall energy of these shorter waves is controlled. Since the average short wave energy will be less than that at the crests of the long waves, it seems entirely possible that the average energy of short waves may be somewhat less when they are superimposed upon longer waves than when the long waves are absent.

Long waves develop only after high winds have blown for long times over long fetches. If we envisage a situation where the wind speed increases to a high level and then remains constant for some time, it seems possible that the spectral energy density corresponding to the short waves will first rise to the saturation level and then actually decrease as the long waves grow to significant amplitude. Similarly it seems possible that the short wave energy may be less at longer fetches than at shorter fetches.

Another point is worthy of consideration: The excess energy in the short waves at the crests of the long ones must have been gained at the expense of the long waves. If these short waves then loose their energy due to breaking at the crest, it is no longer available to be fed back into the long waves during their next half cycle. The net result is a mechanism for the dissipation of *long wave* energy. This has been discussed in detail by PHILLIPS (1963). A similar, and equally important, mechanism involving capillary waves is described in the next section.

9. *Damping of gravity waves by capillary waves*

Capillary waves on the surface of the sea can be generated by at least two mechanisms. One is instability of the shearing flow of wind over water, as described by MILES (1962). Capillaries generated in this way can occur, in theory, on any part of the sea surface which is exposed to a sufficiently strong wind. A second cause of capillaries is the sharp *curvature* near the crests of steep gravity waves which produces a local accentuation of the surface tension forces. If the waves are progressive, these forces act like any other travelling disturbance to produce capillary waves ahead of the disturbance. Capillaries generated this second way are observed only on the forward face of steep gravity waves; they may occur in the absence of wind. Their steepness has been shown theoretically to be given approximately by

$$\frac{2\pi}{3} \exp\left(-\frac{\rho g}{6TK^2}\right) \tag{1}$$

where K is the maximum curvature at the crest of the gravity wave (LONGUET-HIGGINS, 1963*), a result in agreement with observations made by COX (1958).

Whatever their origin, however, capillary waves will subsequently undergo rapid modification from two causes: damping by viscosity and non-linear interaction with the surrounding velocity field. The interactions with gravity waves may be especially strong owing to the relatively short wavelength of the capillaries.

Consider pure capillary waves, of energy density E and wavelength $2\pi/k$, riding on the surface of a two-dimensional flow $U = (U, V, W)$ where $V = 0$ and U, W are independent of y. We suppose that x is measured along the surface of the free gravity flow, and z normal to it, and we assume that the curvature of the mean surface is always small compared to k. Now the rate of dissipation of energy by viscosity in a capillary wave is $4\nu k^2 E$ (LAMB 1932, Section 347). Hence, as in Section 7, we have the following equation for the capillary wave energy:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[E(U + c_g)] + S_{xx}\frac{\partial U}{\partial x} + 4\nu k^2 E = 0. \tag{2}$$

Since for capillary waves

$$c_g = \tfrac{3}{2}\, c, \quad S_{xx} = \tfrac{3}{2}\, E, \quad c^2 = Tk, \tag{3}$$

this can also be written

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[E(U + \tfrac{3}{2}\, c)] + \tfrac{3}{2}\, E\frac{\partial U}{\partial x} + 4\,(\nu/T^2)\, Ec^4 = 0. \tag{4}$$

To apply this equation in any particular case we need a further relation between U and c. As an example let us take the case of capillary waves propagated on the

*This paper will be referred to as (IV).

forward face of a gravity wave (as described above) and sufficiently far from the wave crest that the curvature of the gravity wave profile is small compared to k. The gravity wave being progressive, we may take axes moving with the wave and so reduce the motion to a steady state. The velocity U will have one component due to the orbital velocity in the gravity wave and another due to the negative velocity associated with the forwards motion of the frame of reference. Since the capillary waves originate at the (stationary) crest of the gravity wave, they will appear stationary in the new frame of reference. Hence their phase velocity c must be equal to $-U$. In the steady state $\partial E/\partial t$ vanishes and (4) becomes now

$$\frac{\partial}{\partial x}(-\tfrac{1}{2}EU) + \tfrac{3}{2}E\frac{\partial U}{\partial x} + 4(\nu/T^2)EU^4 = 0. \tag{5}$$

Ignoring for one moment the viscous term in (5) we have

$$\frac{1}{E}\frac{\partial E}{\partial x} = \frac{2}{U}\frac{\partial U}{\partial x} \tag{6}$$

and so

$$E \propto U^2. \tag{7}$$

In other words the energy of the capillaries increases proportionally to the square of the opposing current. This increase is due not only to the shortening of the wavelength by the contracting current but also to the work done by the current against the radiation stress. The same result (7) was derived also by a perturbation analysis in (IV).

If we take full account of the viscous damping in (5) we have now

$$\frac{1}{E}\frac{\partial E}{\partial x} = \frac{2}{U}\frac{\partial U}{\partial x} + \frac{8\nu}{T^2}U^3 \tag{8}$$

which has the integral

$$E \propto U^2 \exp\left[\frac{8\nu}{T^2}\int_0^x U^3\,dx\right], \quad (U < 0). \tag{9}$$

We see that E may at first increase, owing to the radiation stresses, but ultimately the waves are damped out by viscosity. From (8) it follows that the maximum amplitude is attained where

$$\frac{1}{U^4}\frac{\partial U}{\partial x} = -\frac{4\nu}{T^2}. \tag{10}$$

the law of energy variation (9) was shown in (IV) to be in good agreement with observation.

All the energy in the capillary waves is ultimately dissipated by viscosity, including any work done against the radiation stresses by contraction of the current U. Even without the radiation stresses, the energy lost in the capillary waves could be several times that in the basic gravity wave (see IV, Section 10), so that the capillaries must be important in damping the gravity waves when they are near to their maximum steepness. The effect is enhanced by the action of the radiation stresses.

Moreover, capillary waves of any origin, whether due to sharp crests or direct wind action, may dissipate energy derived from the gravity waves through the radiation stresses.

CONCLUSION

As has been shown in the series of examples outlined above, the radiation stress concept permits straightforward calculation of a range of important phenomena. In every case the same results could have been obtained by a detailed perturbation analysis, but comparison with the original papers (I, II, III & IV) in which such analyses were carried out will reveal the considerable reduction of effort required and gain in clarity achieved.

It is our belief that the radiation stress should be regarded not as a " virtual " effect but as real, at least in the same sense as the radiation pressure in electromagnetic theory and the Reynolds stress in turbulence theory are real. Viewed thus, such phenomena as wave set-up (and set-down), where the stress must be balanced by hydrostatic pressure, become entirely natural and expected. Also the non-linear energy exchanges between waves and currents and among waves can, with this concept, be regarded as strictly analogous to corresponding cases in the theory of elasticity and the theory of turbulence, where the rate of energy exchange is given by the product stress times rate of strain.

Radiation stresses will arise not only due to surface waves, but due to internal waves. In the oceans the interaction of internal waves and currents may be considerably more important than interaction involving surface waves, because of the much lower propagation speeds. Small propagation speeds tend to increase the strength of the interaction, as can be seen from two points of view : First, in any wave current interaction, the energy exchange can be written so as to be seen to be proportional to U/c or U/c_0, so small values of c lead to large interactions. From the other point of view, we note that for almost all species of wave the ratio of energy density : momentum density equals the phase speed. Surface and internal waves are no exception, so internal waves, with their low propagation speed, are particularly efficient at transferring momentum.

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