Chapter 4

The z-Transform

P4.1 The z-transform computation using definition (4.1).

1. $x(n) = \{3, 2, \underset{\uparrow}{1}, -2, -3\}$: Then $X(z) = 3z^2 + 2z + 1 - 2z^{-1} + 3z^{-2}$, $0 < |z| < \infty$.

   MATLAB verification:
```
% P0401a.m
clc; close all;
b1 = [0 2 3]; a1 = [1]; [delta,n] = impseq(0,0,4);
xb1 = filter(b1,a1,delta); xb1 = fliplr(xb1); n1 = -fliplr(n);
b2 = [1 -2 -3]; a2 = [1]; xb2 = filter(b2,a2,delta); n2 = n;
[xa1,na1] = sigadd(xb1,n1,xb2,n2); xa2 = [0 0 3 2 1 -2 -3 0 0];
error = max(abs(xa1-xa2))
error =
     0
```

2. $x(n) = (0.8)^n u(n-2)$: Then

$$X(z) = \sum_{n=2}^{\infty}(0.8)^n z^{-n} = (0.8)^2 z^{-2}\sum_{n=0}^{\infty}(0.8)^n z^{-n} = \frac{0.64z^{-2}}{1-0.8z^{-1}};\ |z| > 0.8$$

   MATLAB verification:
```
% P0401b.m
clc; close all;
b = [0 0 0.64]; a = [1 -0.8]; [delta,n] = impseq(0,0,10);
xb1 = filter(b,a,delta);
[u,n] = stepseq(2,0,10); xb2 = ((0.8).^n).*u;
error =  max(abs(xb1-xb2))
error =
  1.1102e-016
```

3. $x(n) = [(0.5)^n + (-0.8)^n]u(n)$: Then

$$X(z) = \mathcal{Z}\left[(0.5)^n u(n)\right] + \mathcal{Z}\left[(-0.8)^n u(n)\right] = \frac{1}{1-0.5z^{-1}} + \frac{1}{1+0.8z^{-1}};\ \{|z| > 0.5\} \cap \{|z| > 0.8\}$$

$$= \frac{2+0.3z^{-1}}{1+0.3z^{-1}-0.4z^{-2}};\ |z| > 0.8$$

   MATLAB verification:

```
% P0401c.m
clc; close all;
b = [ 2 0.3]; a = [1 0.3 -0.4]; [delta,n] = impseq(0,0,7);
xb1 = filter(b,a,delta);
[u,n] = stepseq(0,0,7); xb2 = (((0.5).^n).*u)+(((-0.8).^n).*u);
error = max(abs(xb1-xb2))
error =
  1.1102e-016
```

4. $x(n) = 2^n \cos(0.4\pi n)u(-n)$: Consider

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} 2^n \cos(0.4\pi n)u(-n)z^{-n} = \sum_{n=-\infty}^{0} 2^n \cos(0.4\pi n)z^{-n} = \sum_{n=0}^{\infty} 2^{-n} \cos(0.4\pi n)z^{n} \\
&= \sum_{n=0}^{\infty} 2^{-n} \left(\frac{e^{j0.4\pi n} + e^{-j0.4\pi n}}{2}\right) z^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(2^{-1}e^{j0.4\pi}z\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(2^{-1}e^{-j0.4\pi}z\right)^n \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{j0.4\pi}z}\right) + \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{-j0.4\pi}z}\right);\ |z| < 2 \\
&= \frac{1-[0.5\cos(0.4\pi)]z}{1-[\cos(0.4\pi)]z + 0.25z^2};\ |z| < 2
\end{aligned}
$$

Hence

$$
X(z) = \frac{1-[0.5\cos(0.4\pi)]z}{1-2[0.5\cos(0.4\pi)]z + 0.25z^2} = \frac{1-[0.5\cos(0.4\pi)]z}{1-[\cos(0.4\pi)]z + 0.25z^2};\ |z| < 2
$$

MATLAB verification:

```
% P0401d.m
clc; close all;
b = 0.5*[2 -cos(0.4*pi)]; a = [1 -cos(0.4*pi) 0.25];
[delta,n1] = impseq(0,0,7); xb1 = filter(b,a,delta); xb1 = fliplr(xb1);
[u,n2] = stepseq(-7,-7,0); xb2 = ((2.^n2).*cos(0.4*pi*n2)).*u;
error = max(abs(xb1-xb2))
error =
  2.7756e-017
```

5. $x(n) = (n+1)(3)^n u(n)$: Consider

$$
x(n) = (n+1)(3)^n u(n) = n3^n u(n) + 3^n u(n)
$$

Hence

$$
\begin{aligned}
X(z) &= \mathcal{Z}\left[n3^n u(n)\right] + \mathcal{Z}\left[3^n u(n)\right] = -z\frac{d}{dz}\left(\frac{1}{1-3z^{-1}}\right) + \frac{1}{1-3z^{-1}};\ |z| > 3 \\
&= \frac{3z^{-1}}{1-6z^{-1}+9z^{-2}} + \frac{1}{1-3z^{-1}} = \frac{1-3z^{-1}}{1-9z^{-1}+27z^{-2}-27z^{-3}};\ |z| > 3
\end{aligned}
$$

MATLAB verification:

```
% P0401e.m
clc; close all;
b = [1 -3]; a = [1 -9 27 -27]; [delta,n1] = impseq(0,0,7);
xb1 = filter(b,a,delta);
[u,n2] = stepseq(0,0,7); xb2 = ((n2+1).*(3.^n2)).*u;
error = max(abs(xb1-xb2))
```

```
error =
  0
```

P4.2 Consider the sequence $x(n) = (0.9)^n \cos(\pi n/4)u(n)$. Let

$$y(n) = \begin{cases} x(n/2), & n = 0, \pm 2, \pm 4, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

1. The z-transform $Y(z)$ of $y(n)$ in terms of the z-transform $X(z)$ of $x(n)$: Consider

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n/2)z^{-n};\ n = 0, \pm 1, \pm 2, \ldots$$
$$= \sum_{m=-\infty}^{\infty} x(m)z^{-2m} = \sum_{m=-\infty}^{\infty} x(m)\left(z^2\right)^{-m} = X(z^2)$$

2. The z-transform of $x(n)$ is given by

$$X(z) = \mathcal{Z}\left[(0.9)^n \cos(\pi n/4)u(n)\right] = \frac{1 - [(0.9)\cos(\pi/4)]z^{-1}}{1 - 2[(0.9)\cos(\pi/4)]z^{-1} + (0.9)^2 z^{-2}}$$
$$= \frac{1 - 0.6364z^{-1}}{1 - 1.2728z^{-1} + 0.81z^{-2}};\ |z| > 0.9$$

Hence

$$Y(z) = \frac{1 - 0.6364z^{-2}}{1 - 1.2728z^{-2} + 0.81z^{-4}};\ |z| > \sqrt{0.9} = 0.9487$$

3. MATLAB verification:

```
% P0402c.m
clc; close all;
b = [1 0 -0.9*cos(pi/4)]; a = [1 0 2*-0.9*cos(pi/4) 0 0.81];
[delta,n1] = impseq(0,0,13); xb1 = filter(b,a,delta);
[u,n2] = stepseq(0,0,6); x1 = (((0.9).^n2).*cos(pi*n2/4)).*u;
xb2 = zeros(1,2*length(x1)); xb2(1:2:end) = x1;
error = max(abs(xb1-xb2))
error =
  1.2442e-016
```

P4.3 Computation of z-transform using properties and the z-transform table:

1. $x(n) = 2\delta(n-2) + 3u(n-3)$:

$$X(z) = 2z^{-2} + 3z^{-3}\frac{1}{1-z^{-1}} = \frac{2z^{-2}+z^{-3}}{1-z^{-1}};\ |z| > 1$$

MATLAB script:

```
clc; close all;
b = [0 -8 0 -1.5 0 -1/16]; a = [1 0 3/16 0 3/256 0 1/(256*16)];
[delta,n1] = impseq(0,0,9); xb1 = filter(b,a,delta);
[u,n2] = stepseq(0,0,9);xb2 = (((n2-3).*((1/4).^(n2-2))).*cos((pi/2)*(n2-1))).*u;
error = max(abs(xb1-xb2))
error =
  0
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0403e');
[Hz,Hp,Hl] = zplane(b,a); set(Hz,'linewidth',1); set(Hp,'linewidth',1);
title('Pole-Zero plot','FontSize',TFS); print -deps2 ../epsfiles/P0403e;
```

The pole-zero plot is shown in Figure 4.1.

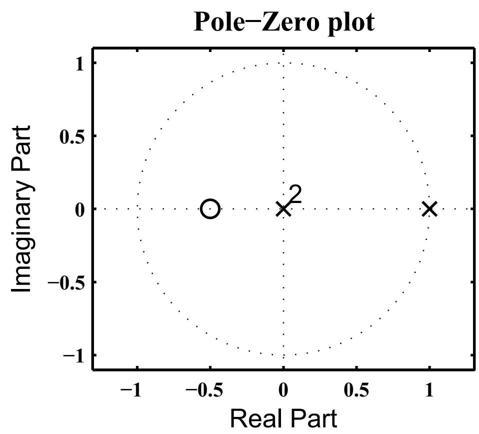


Figure 4.1: Problem P4.3.1 pole-zero plot

2. $x(n) = 3(0.75)^n \cos(0.3\pi n)u(n) + 4(0.75)^n \sin(0.3\pi n)u(n)$:

$$Z(z) = 3\frac{1 - [0.75\cos(0.3\pi)]z^{-1}}{1 - 2[0.75\cos(0.3\pi)]z^{-1} + (0.75)^2 z^{-2}} + 4\frac{[0.75\sin(0.3\pi)]z^{-1}}{1 - 2[0.75\cos(0.3\pi)]z^{-1} + (0.75)^2 z^{-2}}$$

$$= \frac{3 + [3\sin(0.3\pi) - 2.25\cos(0,3\pi)]z^{-1}}{1 - 1.5\cos(0,3\pi)z^{-1} + 0.5625z^{-2}} = \frac{3 + 1.1045z^{-1}}{1 - 0.8817z^{-1} + 0.5625z^{-2}}; \ |z| > 0.75$$

MATLAB script:

```
clc; close all;
b = [3 (3*sin(0.3*pi)-2.25*cos(0.3*pi))]; a = [1 -1.5*cos(0.3*pi) 0.5625];
[delta,n1] = impseq(0,0,7); xb1 = filter(b,a,delta); [u,n2] = stepseq(0,0, 7);
xb2 = 3*(((0.75).^n2).*cos(0.3*pi*n2)).*u+4*(((0.75).^n2).*sin(0.3*pi*n2)).*u ;
error = max(abs(xb1-xb2))
error =
  4.4409e-016
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0403b');
[Hz,Hp,Hl] = zplane(b,a); set(Hz,'linewidth',1); set(Hp,'linewidth',1);
title('Pole-Zero plot','FontSize',TFS); print -deps2 ../epsfiles/P0403b;
```

The pole-zero plot is shown in Figure 4.2.

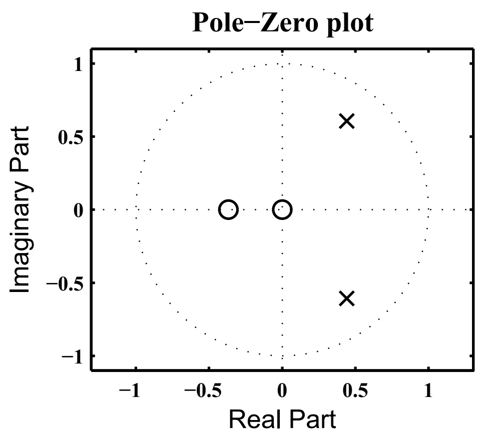


Figure 4.2: Problem P4.3.2 pole-zero plot

3. $x(n) = n\sin(\frac{\pi n}{3})u(n) + (0.9)^n u(n-2)$: Consider

$$\begin{aligned} x(n) &= n\sin\left(\frac{\pi n}{3}\right)u(n) + (0.9)^n u(n-2) = n\sin\left(\frac{\pi n}{3}\right)u(n) + 0.9^2(0.9)^{n-2}u(n-2) \\ &= n\sin\left(\frac{\pi n}{3}\right)u(n) + 0.81(0.9)^{n-2}u(n-2) \end{aligned}$$

Hence

$$\begin{aligned} X(z) &= \mathcal{Z}\left[n\sin\left(\frac{\pi n}{3}\right)u(n)\right] + \mathcal{Z}\left[0.81(0.9)^{n-2}u(n-2)\right] \\ &= -z\frac{d}{dz}\left(\mathcal{Z}\left[\sin\left(\frac{\pi n}{3}\right)u(n)\right]\right) + 0.81z^{-2}\mathcal{Z}\left[(0.9)^n u(n)\right] \\ &= -z\frac{d}{dz}\left(\frac{\sin(\pi/3)z^{-1}}{1-z^{-1}+z^{-2}}\right) + \frac{0.81z^{-2}}{1-0.9z^{-1}};\ |z|>1 \\ &= \frac{\sin(\pi/3)z^{-1} - \sin(\pi/3)z^{-3}}{1-2z^{-1}+3z^{-2}-2z^{-3}+z^{-4}} + \frac{0.81z^{-2}}{1-0.9z^{-1}};\ |z|>1 \\ &= \frac{0.866z^{-1}+0.0306z^{-2}-2.486z^{-3}+3.2094z^{-4}-1.62z^{-5}+0.81z^{-6}}{1-2.9z^{-1}+4.8z^{-2}-4.7z^{-3}+2.8z^{-4}-0.9z^{-5}};\ |z|>1 \end{aligned}$$

MATLAB script:

```
clc; close all;
b = [0 sin(pi/3) (0.81-0.9*sin(pi/3)) -(1.62+sin(pi/3)) ...
     (0.9*sin(pi/3)+2.43) -1.62 0.81];
a = [1 -2.9 4.8 -4.7 2.8 -0.9]; [delta,n1] = impseq(0,0,9);
xb1 = filter(b,a,delta);
[u2,n2] = stepseq(0,0,9); [u3,n3] = stepseq(2,0,9);
xb2 = (n2.*sin(pi/3*n2)).*u2+((0.9).^n3).*u3; error = max(abs(xb1-xb2))
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0403c');
[Hz,Hp,Hl] = zplane(b,a); set(Hz,'linewidth',1); set(Hp,'linewidth',1);
title('Pole-Zero plot','FontSize',TFS); print -deps2 ../epsfiles/P0403c;
error =
  2.1039e-014
```

The pole-zero plot is shown in Figure 4.3.

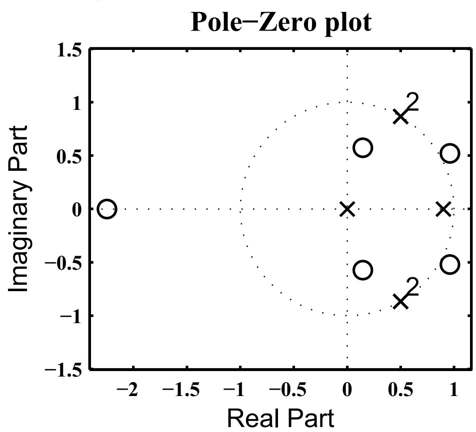


Figure 4.3: Problem P4.3.3 pole-zero plot

4. $x(n) = n^2(2/3)^{n-2}u(n-1)$: Consider

$$x(n) = n^2(2/3)^{n-2}u(n-1) = n^2(2/3)^{-1}(2/3)^{(n-1)}u(n-1) = \frac{3}{2}\left(n\left[n\left\{\left(\frac{2}{3}\right)^{(n-1)}u(n-1)\right\}\right]\right)$$

Let

$$x_1(n) = \left(\frac{2}{3}\right)^{(n-1)}u(n-1) \Rightarrow X_1(z) = \frac{z^{-1}}{1-\frac{2}{3}z^{-1}};\ |z| > \frac{2}{3}$$

Let

$$x_2(n) = n\,x_1(n) \Rightarrow X_2(z) = -z\frac{d}{dz}X_1(z) = \frac{z^{-1}}{1-\frac{4}{3}z^{-1}+\frac{4}{9}z^{-2}};\ |z| > \frac{2}{3}$$

Let

$$x_3(n) = n\,x_2(n) \Rightarrow X_3(z) = -z\frac{d}{dz}X_2(z) = \frac{z^{-1}-\frac{4}{9}z^{-3}}{1-\frac{8}{3}z^{-1}+\frac{8}{3}z^{-2}-\frac{32}{27}z^{-3}+\frac{16}{81}z^{-4}};\ |z| > \frac{2}{3}$$

Finally, $x(n) = \frac{3}{2}x_3(n)$. Hence

$$X(z) = \frac{3}{2}\left(\frac{z^{-1}-\frac{4}{9}z^{-3}}{1-\frac{8}{3}z^{-1}+\frac{8}{3}z^{-2}-\frac{32}{27}z^{-3}+\frac{16}{81}z^{-4}}\right) = \frac{\frac{3}{2}z^{-1}-\frac{2}{3}z^{-3}}{1-\frac{8}{3}z^{-1}+\frac{8}{3}z^{-2}-\frac{32}{27}z^{-3}+\frac{16}{81}z^{-4}};\ |z| > \frac{2}{3}$$

MATLAB script:

```
clc; close all; b = 3/2*[0 1 0 -4/9]; a = [1 -8/3 8/3 -32/27 16/81];
[delta,n1] = impseq(0,0,8); xb1 = filter(b,a,delta);
[u,n2] = stepseq(1,0,8); xb2 = ((n2.^2).*((2/3).^(n2-2))).*u;
error = max(abs(xb1-xb2))
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0403d');
[Hz,Hp,Hl] = zplane(b,a); set(Hz,'linewidth',1); set(Hp,'linewidth',1);
title('Pole-Zero plot','FontSize',TFS); print -deps2 ../epsfiles/P0403d;
error =
  9.7700e-015
```

The pole-zero plot is shown in Figure 4.4.

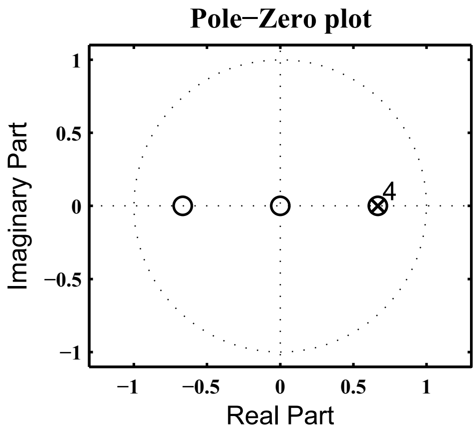


Figure 4.4: Problem P4.3.4 pole-zero plot

5. $x(n) = (n-3)\left(\frac{1}{4}\right)^{n-2} \cos\left\{\frac{\pi}{2}(n-1)\right\} u(n)$: Consider

$$\begin{aligned}
x(n) &= (n-3)\left(\frac{1}{4}\right)^{n-2} \cos\left\{\frac{\pi}{2}(n-1)\right\} u(n) = (n-3)\left(\frac{1}{4}\right)^{n-2} \sin\left\{\frac{\pi}{2}n\right\} u(n) \\
&= (n-3)\left(\frac{1}{4}\right)^{-2}\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n) = 16(n-3)\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n) \\
&= 16\,n\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n) - 48\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n)
\end{aligned}$$

Hence

$$\begin{aligned}
X(z) &= \mathcal{Z}\left[16\,n\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n)\right] - \mathcal{Z}\left[48\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n)\right] \\
&= -z\frac{d}{dz}\left(\mathcal{Z}\left[\left(\frac{1}{4}\right)^{n} \sin\left\{\frac{\pi}{2}n\right\} u(n)\right]\right) - 48\frac{[\left(\frac{1}{4}\right)\sin(\pi/2)]z^{-1}}{1-2[\frac{1}{4}\cos(\pi/2)]z^{-1}+\left(\frac{1}{4}\right)^2 z^{-2}};\ |z| > \frac{1}{4} \\
&= -z\frac{d}{dz}\left(\frac{\frac{1}{4}z^{-1}}{1+\frac{1}{16}z^{-2}}\right) - \frac{12\,z^{-1}}{1+\frac{1}{16}z^{-2}} = \frac{4z^{-1}-\frac{1}{4}z^{-3}}{1+\frac{1}{8}z^{-2}+\frac{1}{256}z^{-4}} - \frac{12\,z^{-1}}{1+\frac{1}{16}z^{-2}};\ |z| > \frac{1}{4} \\
&= \frac{-8z^{-1}-\frac{3}{2}z^{-3}-\frac{1}{16}z^{-5}}{1+\frac{3}{16}z^{-2}+\frac{3}{256}z^{-4}+\frac{1}{4096}z^{-6}};\ |z| > \frac{1}{4}
\end{aligned}$$

MATLAB script:

```
clc; close all; b = [0 -8 0 -1.5 0 -1/16]; a = [1 0 3/16 0 3/256 0 1/(256*16)];
[delta,n1] = impseq(0,0,9); xb1 = filter(b,a,delta);
[u,n2] = stepseq(0,0,9);xb2 = (((n2-3).*((1/4).^(n2-2))).*cos((pi/2)*(n2-1))).*u;
error = max(abs(xb1-xb2))
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0403e');
[Hz,Hp,Hl] = zplane(b,a); set(Hz,'linewidth',1); set(Hp,'linewidth',1);
title('Pole-Zero plot','FontSize',TFS); print -deps2 ../epsfiles/P0403e;
error =
  2.9392e-015
```

The pole-zero plot is shown in Figure 4.5.

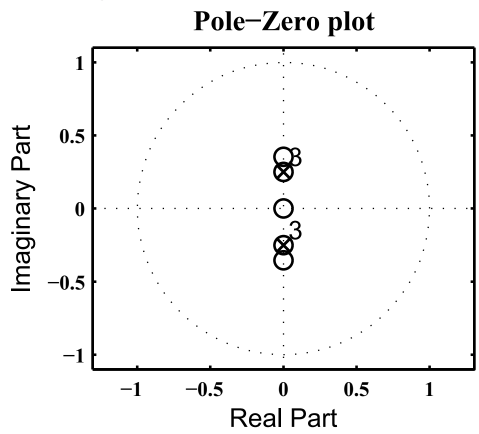


Figure 4.5: Problem P4.3.5 pole-zero plot

P4.4 Let $x(n)$ be a complex-valued sequence with the real part $x_R(n)$ and the imaginary part $x_I(n)$.

1. The z-transform relations for real and imaginary parts.: Consider

$$X_R(z) \triangleq \mathcal{Z}[x_R(n)] = \mathcal{Z}\left[\frac{x(n)+x^*(n)}{2}\right] = \frac{\mathcal{Z}[x(n)]+\mathcal{Z}[x^*(n)]}{2}$$
$$= \frac{X(z)+X^*(z^*)}{2} \tag{4.1}$$

and

$$X_I(z) \triangleq \mathcal{Z}[x_I(n)] = \mathcal{Z}\left[\frac{x(n)-x^*(n)}{2}\right] = \frac{\mathcal{Z}[x(n)]-\mathcal{Z}[x^*(n)]}{2}$$
$$= \frac{X(z)-X^*(z^*)}{2} \tag{4.2}$$

2. Verification using $x(n) = \exp\{(-1+j0.2\pi)n\}\,u(n)$: Consider

$$x(n) = \exp\{(-1+j0.2\pi)n\}\,u(n) = e^{-n}\,e^{j0.2\pi n}u(n)$$
$$= e^{-n}\{\cos(0.2\pi n)u(n) + j\sin(0.2\pi n)u(n)\}$$

Hence the real and imaginary parts of $x(n)$, respectively, are

$$x_R(n) = e^{-n}\cos(0.2\pi n)u(n) = (1/e)^n\cos(0.2\pi n)u(n) \tag{4.3}$$
$$x_I(n) = e^{-n}\sin(0.2\pi n)u(n) = (1/e)^n\sin(0.2\pi n)u(n) \tag{4.4}$$

with z-transforms , respectively,

$$X_R(z) = \frac{1-[(1/e)\cos(0.2\pi)]z^{-1}}{1-[(2/e)\cos(0.2\pi)]z^{-1}+(1/e^2)z^{-2}} = \frac{1-0.2976z^{-1}}{1-0.5952z^{-1}+0.1353z^{-2}},\ |z|>1/e \tag{4.5}$$
$$X_I(z) = \frac{[(1/e)\sin(0.2\pi)]z^{-1}}{1-[(2/e)\cos(0.2\pi)]z^{-1}+(1/e^2)z^{-2}} = \frac{0.2162z^{-1}}{1-0.5952z^{-1}+0.1353z^{-2}},\ |z|>1/e \tag{4.6}$$

The z-transform of $x(n)$ is

$$X(z) = \mathcal{Z}\left[\left(e^{-1+j0.2\pi}\right)^n u(n)\right] = \frac{1}{1-e^{-1+j0.2\pi}z^{-1}},\ |z|>1/e \tag{4.7}$$

Substituting (4.7) in (4.1),

$$X_R(z) = \frac{1}{2}\left[\frac{1}{1-e^{-1+j2\pi}z^{-1}} + \left(\frac{1}{1-e^{-1+j2\pi}z^{-*}}\right)^*\right]$$
$$= \frac{1}{2}\left[\frac{1}{1-e^{-1+j2\pi}z^{-1}} + \frac{1}{1-e^{-1-j2\pi}z^{-1}}\right] = \frac{1}{2}\left[\frac{2-0.5952z^{-1}}{1-0.5952z^{-1}+0.1353z^{-2}}\right]$$
$$= \frac{1-0.2976z^{-1}}{1-0.5952z^{-1}+0.1353z^{-2}},\ |z|>e^{-1} \tag{4.8}$$

as expected in (4.5). Similarly, Substituting (4.7) in (4.2),

$$X_I(z) = \frac{1}{2}\left[\frac{1}{1-e^{-1+j2\pi}z^{-1}} - \left(\frac{1}{1-e^{-1+j2\pi}z^{-*}}\right)^*\right]$$
$$= \frac{1}{2}\left[\frac{1}{1-e^{-1+j2\pi}z^{-1}} - \frac{1}{1-e^{-1-j2\pi}z^{-1}}\right] = \frac{1}{2}\left[\frac{0.2162z^{-1}}{1-0.5952z^{-1}+0.1353z^{-2}}\right]$$
$$= \frac{0.2162z^{-1}}{1-0.5952z^{-1}+0.1353z^{-2}},\ |z|>e^{-1} \tag{4.9}$$

as expected in (4.6).

P4.5 The z-transform of $x(n)$ is $X(z) = 1/(1 + 0.5z^{-1})$, $|z| \geq 0.5$.

1. The z-transforms of $x_1(n) = x(3-n) + x(n-3)$:

$$\begin{aligned} X_1(z) &= \mathcal{Z}[x_1(n)] = \mathcal{Z}[x(3-n) + x(n-3)] = \mathcal{Z}[x\{-(n-3)\}] + \mathcal{Z}[x(n-3)] \\ &= z^{-3}X(1/z) + z^{-3}X(z) = z^{-3}\left[\frac{1}{1+0.5z} + \frac{1}{1+0.5z^{-1}}\right],\ 0.5 < |z| < 2 \\ &= \frac{0.5z^{-3} + 2z^{-4} + 0.5z^{-5}}{0.5 + 1.25z^{-1} + 0.5z^{-2}},\ 0.5 < |z| < 2 \end{aligned}$$

2. The z-transforms of $x_2(n) = (1 + n + n^2)x(n)$:

$$\begin{aligned} X_2(z) &= \mathcal{Z}\left[\left(1+n+n^2\right)x(n)\right] = \mathcal{Z}\left[x(n) + n\,x(n) + n^2x(n)\right] = X(z) - z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z) \\ &= \frac{1}{1+0.5z^{-1}} - \frac{0.5z^{-1}}{\left(1+0.5z^{-1}\right)^2} - \frac{z^{-1}+0.5z^{-2}}{\left(1+0.5z^{-1}\right)^4} = \frac{1-0.25z^{-2}}{\left(1+0.5z^{-1}\right)^4},\ |z| > 0.5 \end{aligned}$$

3. The z-transforms of $x_3(n) = \left(\frac{1}{2}\right)^n x(n-2)$:

$$\begin{aligned} X_3(z) &= \mathcal{Z}\left[\left(\frac{1}{2}\right)^n x(n-2)\right] = \mathcal{Z}[x(n-2)]\Big|_{\left(\frac{1}{2}\right)^{-1}z} = \mathcal{Z}[x(n-2)]\Big|_{2z} \\ &= \left[z^{-2}X(z)\right]\Big|_{2z} = \left[\frac{z^{-2}}{1+0.5z^{-1}},\ |z| > 0.5\right]\Bigg|_{2z} = \frac{0.25z^{-2}}{1+0.25z^{-1}},\ |z| > 0.25 \end{aligned}$$

4. The z-transforms of $x_4(n) = x(n+2) * x(n-2)$:

$$\begin{aligned} X_4(z) &= \mathcal{Z}[x(n+2) * x(n-2)] = \left\{z^2X(z)\right\}\left\{z^{-2}X(z)\right\} = X^2(z) \\ &= \frac{1}{\left(1+0.5z^{-1}\right)^2},\ |z| > 0.5 \end{aligned}$$

5. The z-transforms of $x_5(n) = \cos(\pi n/2)x^*(n)$:

$$\begin{aligned} X_5(z) &= \mathcal{Z}\left[\cos(\pi n/2)x^*(n)\right] = \mathcal{Z}\left[\left(\frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2}\right)x^*(n)\right] \\ &= \frac{1}{2}\left(\mathcal{Z}\left[e^{j\pi n/2}x^*(n)\right] + \mathcal{Z}\left[e^{-j\pi n/2}x^*(n)\right]\right) \\ &= \frac{1}{2}\left(\mathcal{Z}\left[x^*(n)\right]\Big|_{e^{-j\pi/2}z} + \mathcal{Z}\left[x^*(n)\right]\Big|_{e^{j\pi/2}z}\right) \\ &= \frac{1}{2}\left[X^*\left(e^{j\pi/2}z^*\right) + X^*\left(e^{-j\pi/2}z^*\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{1+0.5e^{-j\pi/2}z^{-1}} + \frac{1}{1+0.5e^{j\pi/2}z^{-1}}\right] \\ &= \frac{1}{1+0.25z^{-2}},\ |z| > 0.5 \end{aligned}$$

P4.6 The z-transform of $x(n)$ is

$$X(z) = \frac{1 + z^{-1}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}; \ |z| > \frac{1}{2}$$

1. The z-transforms of $x_1(n) = x(3-n) + x(n-3)$:

$$\begin{aligned} X_1(z) &= \mathcal{Z}[x_1(n)] = \mathcal{Z}[x(3-n) + x(n-3)] = \mathcal{Z}[x\{-(n-3)\}] + \mathcal{Z}[x(n-3)] \\ &= z^{-3}X(1/z) + z^{-3}X(z) = z^{-3}\left[\frac{1+z}{1+\frac{5}{6}z+\frac{1}{6}z^2} + \frac{1+z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}\right], \ 0.5 < |z| < 2 \\ &= \frac{36z^{-1} + 72z^{-2} + 36z^{-3}}{6 + 35z^{-1} + 62z^{-2} + 35z^{-3} + 6z^{-4}}, \ 0.5 < |z| < 2 \end{aligned}$$

2. The z-transforms of $x_2(n) = (1 + n + n^2)x(n)$:

$$X_2(z) = \mathcal{Z}\left[\left(1 + n + n^2\right)x(n)\right] = \mathcal{Z}\left[x(n) + n\,x(n) + n^2x(n)\right] = X(z) - z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z)$$

   or $X_2(z) =$

$$\frac{1+\frac{17}{3}z^{-1}+\frac{1}{2}z^{-2}+\frac{1429}{108}z^{-3}+\frac{2399}{286}z^{-4}+\frac{215}{72}z^{-5}+\frac{829}{1944}z^{-6}-\frac{167}{1944}z^{-7}-\frac{7}{144}z^{-8}-\frac{2}{243}z^{-9}-\frac{1}{1944}z^{-10}}{1+5z^{-1}+\frac{137}{12}z^{-2}+\frac{425}{27}z^{-3}+\frac{6305}{432}z^{-4}+\frac{2694}{281}z^{-5}+\frac{1711}{374}z^{-6}+\frac{449}{281}z^{-7}+\frac{1258}{3103}z^{-8}+\frac{425}{5832}z^{-9}+\frac{137}{15552}z^{-10}+\frac{5}{7776}z^{-11}+\frac{1}{46656}z^{-12}}, \ |z| > 0.5$$

3. The z-transforms of $x_3(n) = \left(\frac{1}{2}\right)^n x(n-2)$:

$$\begin{aligned} X_3(z) &= \mathcal{Z}\left[\left(\frac{1}{2}\right)^n x(n-2)\right] = \mathcal{Z}[x(n-2)]|_{\left(\frac{1}{2}\right)^{-1}z} = \mathcal{Z}[x(n-2)]|_{2z} \\ &= \left[z^{-2}X(z)\right]\Big|_{2z} = [, \ |z| > 0.5]|_{2z} = \frac{\frac{1}{4}z^{-2} + \frac{1}{8}z^{-3}}{1 + \frac{5}{12}z^{-1} + \frac{1}{24}z^{-2}}, \ |z| > 0.25 \end{aligned}$$

4. The z-transforms of $x_4(n) = x(n+2) * x(n-2)$:

$$\begin{aligned} X_4(z) &= \mathcal{Z}[x(n+2) * x(n-2)] = \left\{z^2X(z)\right\}\left\{z^{-2}X(z)\right\} = X^2(z) \\ &= \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{5}{3}z^{-1} + \frac{37}{36}z^{-2} + \frac{5}{18}z^{-3} + \frac{1}{36}z^{-4}}, \ |z| > 0.5 \end{aligned}$$

5. The z-transforms of $x_5(n) = \cos(\pi n/2)x^*(n)$:

$$\begin{aligned} X_5(z) &= \mathcal{Z}\left[\cos(\pi n/2)x^*(n)\right] = \mathcal{Z}\left[\left(\frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2}\right)x^*(n)\right] \\ &= \frac{1}{2}\left(\mathcal{Z}\left[e^{j\pi n/2}x^*(n)\right] + \mathcal{Z}\left[e^{-j\pi n/2}x^*(n)\right]\right) \\ &= \frac{1}{2}\left(\mathcal{Z}\left[x^*(n)\right]\Big|_{e^{-j\pi/2}z} + \mathcal{Z}\left[x^*(n)\right]\Big|_{e^{j\pi/2}z}\right) \\ &= \frac{1}{2}\left[X^*\left(e^{j\pi/2}z^*\right) + X^*\left(e^{-j\pi/2}z^*\right)\right] \\ &= \frac{1}{2}\left[\frac{1 - jz^{-1}}{1 - j\frac{5}{6}z^{-1} - \frac{1}{6}z^{-2}}\frac{1 + jz^{-1}}{1 + j\frac{5}{6}z^{-1} - \frac{1}{6}z^{-2}}\right] \\ &= \frac{2 + \frac{4}{3}z^{-2}}{1 + \frac{13}{36}z^{-2} + \frac{1}{36}z^{-4}}, \ |z| > 0.5 \end{aligned}$$

P4.7 The inverse z-transform of $X(z)$ is $x(n) = (1/2)^n u(n)$. Sequence computation Using the z-transform properties:

1. $X_1(z) = \frac{z-1}{z} X(z)$: Consider

$$x_1(n) = \mathcal{Z}^{-1}[X_1(z)] = \mathcal{Z}^{-1}\left[\left(1 - \frac{1}{z}\right) X(z)\right] = \mathcal{Z}^{-1}\left[X(z) - z^{-1}X(z)\right]$$
$$= x(n) - x(n-1) = 0.5^n u(n) - 0.5^{n-1} u(n-1) = 1 - 0.5^n u(n-1)$$

2. $X_2(z) = zX(z^{-1})$: Consider

$$x_2(n) = \mathcal{Z}^{-1}[X_2(z)] = \mathcal{Z}^{-1}\left[zX(z^{-1})\right] = \mathcal{Z}^{-1}\left[X(z^{-1})\right]\Big|_{n\to(n+1)} = \mathcal{Z}^{-1}[X(z)]\Big|_{n\to-(n+1)}$$
$$= (0.5)^{-(n+1)} u(-n-1) = 2^{n+1} u(-n-1)$$

3. $X_3(z) = 2X(3z) + 3X(z/3)$: Consider

$$x_3(n) = \mathcal{Z}^{-1}[X_3(z)] = \mathcal{Z}^{-1}[2X(3z) + 3X(z/3)] = 2(3^{-n})x(n) + 3(3^n)x(n)$$
$$= 2(3^{-n})(2^{-n})u(n) + 3(3^n)(2^{-n})u(n) = \left[2\left(\frac{1}{6}\right)^n + 3\left(\frac{3}{2}\right)^n\right] u(n)$$

4. $X_4(z) = X(z)X(z^{-1})$: Consider

$$x_4(n) = \mathcal{Z}^{-1}[X_4(z)] = \mathcal{Z}^{-1}\left[X(z)X(z^{-1})\right] = x(n) * x(-n)$$
$$= \left[0.5^n u(n)\right] * \left[2^n u(-n)\right] = \sum_{k=-\infty}^{\infty} (0.5)^k u(k) 2^{n-k} u(-n+k)$$
$$= \begin{cases} 2^n \sum_{k=0}^{\infty} (0.5)^k 2^{-k}, & n < 0; \\ 2^n \sum_{k=n}^{\infty} (0.5)^k 2^{-k}, & n \ge 0. \end{cases} = \begin{cases} 2^n \sum_{k=-\infty}^{\infty} (0.25)^k, & n < 0; \\ 2^n 2^{-2n} \sum_{k=-\infty}^{\infty} (0.25)^k, & n \ge 0. \end{cases}$$
$$= \frac{4}{3} 2^{|n|}$$

5. $X_5(z) = z^2 \frac{dX(z)}{dz}$: Consider

$$x_5(n) = \mathcal{Z}^{-1}[X_5(z)] = \mathcal{Z}^{-1}\left[z^2 \frac{dX(z)}{dz}\right] = n^2 x(n)$$
$$= n^2 (1/2)^n u(n)$$

P4.8 If sequences $x_1(n)$, $x_2(n)$, and $x_3(n)$ are related by $x_3(n) = x_1(n) * x_2(n)$ then

$$\sum_{n=-\infty}^{\infty} x_3(n) = \left(\sum_{n=-\infty}^{\infty} x_1(n)\right)\left(\sum_{n=-\infty}^{\infty} x_2(n)\right)$$

1. Proof using the definition of convolution:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x_3(n) &= \sum_{n=-\infty}^{\infty} x_1(n) * x_2(n) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) = \left(\sum_{k=-\infty}^{\infty} x_1(k)\right)\sum_{n=-\infty}^{\infty} x_2(n-k) \\ &= \left(\sum_{n=-\infty}^{\infty} x_1(n)\right)\left(\sum_{n=-\infty}^{\infty} x_2(n)\right)\end{aligned}$$

   as expected.

2. Proof using the convolution property:

$$\begin{aligned}\mathcal{Z}\left[x_3(n)\right] &= \mathcal{Z}\left[x_1(n) * x_2(n)\right] = \mathcal{Z}\left[x_1(n)\right]\mathcal{Z}\left[x_2(n)\right] \\ \left.\left(\sum_{n=-\infty}^{\infty} x_3(n)z^{-n}\right)\right|_{z=1} &= \left.\left(\sum_{n=-\infty}^{\infty} x_1(n)z^{-n}\right)\right|_{z=1}\left.\left(\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}\right)\right|_{z=1} \\ \sum_{n=-\infty}^{\infty} x_3(n) &= \left(\sum_{n=-\infty}^{\infty} x_1(n)\right)\left(\sum_{n=-\infty}^{\infty} x_2(n)\right)\end{aligned}$$

   as expected.

3. MATLAB verification:

```
clc; close all;
N = 1000; n1 = [0:N]; x1 = rand(1,length(n1));
n2 = [0:N]; x2 = rand(1,length(n2)); [x3,n3] = conv_m(x1,n1,x2,n2);
sumx1 = sum(x1); sumx2 = sum(x2); sumx3 = sum(x3);
error = max(abs(sumx3-sumx1*sumx2))
error =
  2.9104e-011
```

P4.9 Polynomial operations using MATLAB:

1. $X_1(z) = (1 - 2z^{-1} + 3z^{-2} - 4z^{-3})(4 + 3z^{-1} - 2z^{-2} + z^{-3})$

```
% P0409a.m
clc; close all;
n1 = [0:3]; y1 = [1 -2 3 -4]; n2 = [0:3]; y2 = [4 3 -2 1];
[x1,n] = conv_m(y1,n1,y2,n2)
x1 =
     4    -5     4    -2   -20    11    -4
n =
     0     1     2     3     4     5     6
```

Hence
$$X_1(z) = 4 - 5z^{-1} + 4z^{-2} - 2z^{-3} - 20z^{-4}11 + z^{-5} - 4z^{-6}$$

2. $X_2(z) = (z^2 - 2z + 3 + 2z^{-1} + z^{-2})(z^3 - z^{-3})$

```
% P0409b.m
clc; close all;
n1 = [-2:2]; y1 = [1 -2 3 2 1]; n2 = [-3:3]; y2 = [1 0 0 0 0 0 1];
[x2,n] = conv_m(y1,n1,y2,n2)
x2 =
     1    -2     3     2     1     0     1    -2     3     2     1
n =
    -5    -4    -3    -2    -1     0     1     2     3     4     5
```

Hence
$$X_2(z) = z^5 - 2z^4 + 3z^3 + 2z^4 + z + z^{-1} - 2z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5}$$

3. $X_3(z) = (1 + z^{-1} + z^{-2})^3$

```
% P0409c.m
clc; close all;
n1 = [0 1 2]; y1 = [1 1 1]; [y2,n2] = conv_m(y1,n1,y1,n1);
[x3,n] = conv_m(y1,n1,y2,n2)
x3 =
     1     3     6     7     6     3     1
n =
     0     1     2     3     4     5     6
```

Hence
$$X_3(z) = 1 + 3z^{-1} + 6z^{-2} + 7z^{-3} + 6z^{-4} + 3z^{-5} + z^{-6}$$

4. $X_4(z) = X_1(z)X_2(z) + X_3(z)$

```
% P0409d.m
clc; close all;
n11 = [0:3]; y11 = [1 -2 3 -4]; n12 = [0:3]; y12 = [4 3 -2 1];
[y13,n13] = conv_m(y11,n11,y12,n12);
n21 = [-2:2]; y21 = [1 -2 3 2 1]; n22 = [-3:3]; y22 = [1 0 0 0 0 0 1];
[y23,n23] = conv_m(y21,n21,y22,n22);
n31 = [0 1 2]; y31 = [1 1 1];
[y32,n32] = conv_m(y31,n31,y31,n31); [y33,n33] = conv_m(y31,n31,y32,n32);
[y41,n41] = conv_m(y13,n13,y23,n23); [x4,n] = sigadd(y41,n41,y33,n33)
x4 =
  Columns 1 through 12
     4   -13    26   -17   -10    49   -79    -8    23    -8   -11    49
  Columns 13 through 17
   -86    -1   -10     3    -4
n =
  Columns 1 through 12
    -5    -4    -3    -2    -1     0     1     2     3     4     5     6
  Columns 13 through 17
     7     8     9    10    11
```

Hence

$$X_4(z) = 4z^5 - 13z^4 + 26z^3 - 17z^2 - 10z^1 + 49 - 79z^{-1} - 8z^{-2} + 23z^{-3} - 8z^{-4} - 11z^{-5} \\ + 49z^{-6} - 86z^{-7} - z^{-8} - 10z^{-9} + 3z^{-10} - 4z^{-11}$$

5. $X_5(z) = (z^{-1} - 3z^{-3} + 2z^{-5} + 5z^{-7} - z^{-9})(z + 3z^2 + 2z^3 + 4z^4)$

```
% P0409e.m
clc; close all;
n1 = [0:9]; y1 = [0 1 0 -3 0 2 0 5 0 -1]; n2 = [-4:0]; y2 = [4 2 3 1 0];
[x5,n] = conv_m(y1,n1,y2,n2)
x5 =
  Columns 1 through 12
     0     4     2    -9    -5    -1     1    26    12    11     3    -3
  Columns 13 through 14
    -1     0
n =
  Columns 1 through 12
    -4    -3    -2    -1     0     1     2     3     4     5     6     7
  Columns 13 through 14
     8     9
```

Hence

$$X_5(z) = 4z^3 + 2z^2 - 9z^1 - 5 - z^{-1} + z^{-2} + 26z^{-3} + 12z^{-4} + 11z^{-5} + 3z^{-6} - 3z^{-7} - z^{-8}$$

P4.10 The MATLAB function `deconv_m`:

```
function [p,np,r,nr] = deconv_m(b,nb,a,na)
% Modified deconvolution routine for noncausal sequences
% function [p,np,r,nr] = deconv_m(b,nb,a,na)
%
%  p = polynomial part of support np1 <= n <= np2
% np = [np1, np2]
%  r = remainder part of support nr1 <= n <= nr2
% nr = [nr1, nr2]
%  b = numerator polynomial of support nb1 <= n <= nb2
% nb = [nb1, nb2]
%  a = denominator polynomial of support na1 <= n <= na2
% na = [na1, na2]
%
[p,r] = deconv(b,a);
np1 = nb(1) - na(1); np2 = np1 + length(p)-1; np = [np1:np2];
nr1 = nb(1); nr2 = nr1 + length(r)-1; nr = [nr1:nr2];
```

MATLAB verification:

```
% P0410
clc; close all;
nb = [-2:3]; b = [1 1 1 1 1 1]; na = [-1:1]; a = [1 2 1];
[p,np,r,nr] = deconv_m(b,nb,a,na)
p =
      1    -1     2    -2
np =
     -1     0     1     2
r =
      0     0     0     0     3     3
nr =
     -2    -1     0     1     2     3
```

Hence

$$\frac{z^2+z+1+z^{-1}+z^{-2}+z^{-3}}{z+2+z^{-1}} = (z-1+2z^{-1}-2z^{-2}) + \frac{3z^{-2}+3z^{-3}}{z+2+z^{-1}}$$

P4.11 Inverse z-transforms using the partial fraction expansion method:

1. $X_1(z) = (1 - z^{-1} - 4z^{-2} + 4z^{-3})/(1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3})$. The sequence is right-sided.
   MATLAB script:

```
% P0611a: Inverse z-Transform of X1(z)
clc; close all;
b1 = [1,-1,-4,4]; a1 = [1,-11/4,13/8,-1/4];
[R,p,k] = residuez(b1,a1)
 R =
     0.0000
   -10.0000
    27.0000
 p =
     2.0000
     0.5000
     0.2500
 k =
    -16
```

   or

$$X_1(z) = \frac{1 - z^{-1} - 4z^{-2} + 4z^{-3}}{1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3}} = -16 + \frac{0}{1 - 2z^{-1}} - \frac{10}{1 - 0.5z^{-1}} + \frac{27}{1 - 0.25z^{-1}}, \ |z| > 0.5$$

   Note that from the second term on the right, there is a pole-zero cancellation. Hence

$$x_1(n) = -16\delta(n) - 10(0.5)^n u(n) + 27(0.25)^n u(n)$$

2. $X_2(z) = (1 + z^{-1} - 4z^{-2} + 4z^{-3})/(1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3})$. The sequence is absolutely summable.
   MATLAB script:

```
% P0611b: Inverse z-Transform of X2(z)
clc; close all;
b2 = [1,1,-4,4]; a2 = [1,-11/4,13/8,-1/4];
[R,p,k] = residuez(b2,a2)
 R =
     1.5238
   -12.6667
    28.1429
 p =
     2.0000
     0.5000
     0.2500
 k =
    -16
```

   or

$$X_2(z) = \frac{1 - z^{-1} - 4z^{-2} + 4z^{-3}}{1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3}} = -16 + \frac{1.5238}{1 - 2z^{-1}} - \frac{12.6667}{1 - 0.5z^{-1}} + \frac{28.1429}{1 - 0.25z^{-1}}, \ 0.5 < |z| > 2$$

Hence

$$x_2(n) = -16\delta(n) - 1.5238(2)^n u(-n-1) - 12.6667(0.5)^n u(n) + 28.1429(0.25)^n u(n)$$

3. $X_3(z) = (z^3 - 3z^2 + 4z + 1)/(z^3 - 4z^2 + z - 0.16)$. The sequence is left-sided. Consider

$$X_3(z) = \frac{z^3 - 3z^2 + 4z + 1}{z^3 - 4z^2 + z - 0.16} = \frac{1 - 3z^{-1} + 4z^{-2} + z^{-3}}{1 - 4z^{-1} + z^{-2} - 0.16z^{-3}}$$

MATLAB script for the PFE:

```
% P0611c: Inverse z-Transform of X3(z)
clc; close all;
b3 = [1,-3,4,1]; a3 = [1,-4,1,-0.16];
[R,p,k] = residuez(b3,a3)
% R =
%    0.5383
%    3.3559 + 5.7659i
%    3.3559 - 5.7659i
% p =
%    3.7443
%    0.1278 + 0.1625i
%    0.1278 - 0.1625i
% k =
%    -6.2500

r = abs(p(2))
% r =
%   0.2067

[b,a] = residuez(R(2:3),p(2:3),[])
% b =
%     6.7117   -2.7313
% a =
%     1.0000   -0.2557    0.0427
```

or

$$X_3(z) = -6.25 + \frac{0.5383}{1 - 3.7443z^{-1}} + \frac{3.3559 + j5.7659}{1 - (0.1278 + j0.1625)z^{-1}} + \frac{3.3559 - j5.7659}{1 - (0.1278 - j0.1625)z^{-1}}, \ |z| < 0.2067$$

Hence

$$\begin{aligned} x_3(n) = &-6.25\delta(n) - 0.5383(3.7443)^n u(-n-1) \\ &- (3.3559 + j5.7659)(0.1278 + j0.1625)^n u(-n-1) \\ &- (3.3559 - j5.7659)(0.1278 - j0.1625)^n u(-n-1) \end{aligned}$$

4. $X_4(z) = z/(z^3 + 2z^2 + 1.25z + 0.25)$, $|z| > 1$. Consider

$$X_4(z) = \frac{z}{z^3 + 2z^2 + 1.25z + 0.25} = \frac{z^{-2}}{1 + 2z^{-1} + 1.25z^{-2} + 0.25z^{-3}}$$

MATLAB script for the PFE:

```
% P0611d: Inverse z-Transform of X4(z)
clc; close all;
b4 = [0,0,1]; a4 = [1,2,1.25,0.25];
[R,p,k] = residuez(b4,a4)
 R =
    4.0000
   -0.0000 + 0.0000i
   -4.0000 - 0.0000i
 p =
   -1.0000
   -0.5000 + 0.0000i
   -0.5000 - 0.0000i
 k =
      []
```

or

$$X_4(z) = \frac{4}{1+z^{-1}} - \frac{4}{\left(1+0.5z^{-1}\right)^2} = \frac{4}{1+z^{-1}} - 8z\frac{0.5z^{-1}}{\left(1+0.5z^{-1}\right)^2},\ |z| > 1$$

Hence

$$x_4(n) = 4(-1)^n u(n) - 8(n+1)(0.5)^{n+1}u(n+1)$$

5. $X_5(z) = z/(z^2 - 0.25)^2$, $|z| < 0.5$. Consider

$$X_5(z) = \frac{z}{(z^2 - 0.25)^2} = \frac{z^{-3}}{(1 - 0.25z^{-2})^2},\ |z| < 0.5$$

MATLAB script for PFE:

```
% P0611e: Inverse z-Transform of X5(z)
clc; close all;
b5 = [0,0,0,1]; a5 = conv([1,0,-0.25],[1,0,-0.25]);
[R,p,k] = residuez(b5,a5)
 R =
    4.0000 + 0.0000i
   -2.0000
   -4.0000 - 0.0000i
    2.0000 + 0.0000i
 p =
   -0.5000
   -0.5000
    0.5000 + 0.0000i
    0.5000 - 0.0000i
 k =
      []
```

or

$$
\begin{aligned}
X_5(z) &= \frac{4}{1+0.5z^{-1}} - 2\frac{1}{(1+0.5z^{-1})^2} - 4\frac{1}{1-0.5z^{-1}} + 2\frac{1}{(1-0.5z^{-1})^2},\ |z| < 0.5 \\
&= \frac{4}{1-(-0.5)z^{-1}} + 4z\frac{(-0.5)z^{-1}}{[1-(-0.5)z^{-1}]^2} - 4\frac{1}{1-0.5z^{-1}} + 4z\frac{0.5z^{-1}}{(1-0.5z^{-1})^2},\ |z| < 0.5
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_5(n) &= -4(-0.5)^n u(-n-1) - 4(n+1)(-0.5)^n u[-(n+1)-1] + 4(0.5)^n u(-n-1) \\
&\quad - 4(n+1)(0.5)^n u[-(n+1)-1] \\
&= -4(-0.5)^n u(-n-1) - 4(n+1)(-0.5)^n u[-n-2] + 4(0.5)^n u(-n-1) \\
&\quad - 4(n+1)(0.5)^n u[-n-2] \\
&= 4(0.5)^n \left[1-(-1)^n\right] u(-n-1) - 4(n+1)(0.5)^n \left[1+(-1)^n\right] u(-n-2)
\end{aligned}
$$

P4.12 Consider the sequence given below:

$$x(n) = A_c(r)^n \cos(\pi v_0 n)u(n) + A_s(r)^n \sin(\pi v_0 n)u(n) \tag{4.10}$$

1. The z-transform of $x(n)$ in (4.10): Taking z-transform of (4.10),

$$\begin{aligned} X(z) &= A_c \frac{1 - r\ \cos(\pi v_0)z^{-1}}{1 - 2r\ \cos(\pi v_0)z^{-1} + r^2 z^{-2}} + A_s \frac{r\ \sin(\pi v_0)z^{-1}}{1 - 2r\ \cos(\pi v_0)z^{-1} + r^2 z^{-2}} \\ &= \frac{A_c + r\left[A_s \sin(\pi v_0) - A_c \cos(\pi v_0)\right] z^{-1}}{1 - 2r\ \cos(\pi v_0)z^{-1} + r^2 z^{-2}} \triangleq \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}; \quad |z| > |r| \end{aligned}$$

   Comparing the last step above, we have

$$b_0 = A_c;\ b_1 = r[A_s \sin(\pi v_0) - A_c \cos(\pi v_0)];\ a_1 = -2r \cos(\pi v_0);\ a_2 = r^2 \tag{4.11}$$

2. The signal parameters A_c, A_s, r, and v_0 in terms of the rational function parameters b_0, b_1, a_1, and a_2: Using (4.11 and solving parameters in the following order: A_c, then r, then v_0, and finally A_s, we obtain

$$A_c = b_0; \quad r = \sqrt{a_2}; \quad v_0 = \frac{\arccos(-a_1/2r)}{\pi}; \quad A_s = \frac{2b_1 - a_1 b_0}{\sqrt{4a_2 - a_1^2}}$$

3. MATLAB function invCCPP:

```
function [Ac,As,r,v0] = inv_CC_PP(b0,b1,a1,a2)
% [Ac,As,r,v0] = inv_CC_PP(b0,b1,a1,a2)
Ac = b0;
r = sqrt(a2);
w0 = acos(-a1/(2*r));
As = (b1/r+Ac*cos(w0))/sin(w0);
v0 = w0/(pi);
```

P4.13 Suppose $X(z)$ is given as follows:

$$X(z) = \frac{2 + 3z^{-1}}{1 - z^{-1} + 0.81z^{-2}}, \ |z| > 0.9$$

1. The signal $x(n)$ in a form that contains no complex numbers: MATLAB script:

```
% P0413.m
clc; close all;
b0 = 2; b1 = 3; a1 = -1; a2 = 0.81;
% (a): Computation of sequence parameters
[Ac,As,r,v0] = invCCPP(b0,b1,a1,a2);
disp(sprintf('\nx(n) = %1i*(%3.1f)^n*cos(%5.4f*pi*n)u(n) ',Ac,r,v0));
disp(sprintf('      + %5.4f*(%3.1f)^n*sin(%5.4f*pi*n)u(n)\n',As,r,v0));

x(n) = 2*(0.9)^n*cos(0.3125*pi*n)u(n)
      + 5.3452*(0.9)^n*sin(0.3125*pi*n)u(n)
```

   Hence

$$x(n) = 2(0.9)^n \cos(0.3125\pi n)u(n) + 5.3452(0.9)^n \sin(0.3125\pi n)u(n)$$

2. MATLAB verification:

```
% (b) Matlab Verification
n = 0:20; x = Ac*(r.^n).*cos(v0*pi*n) + As*(r.^n).*sin(v0*pi*n);
y = filter([b0,b1],[1,a1,a2],impseq(0,0,20));
error = abs(max(x-y))
 error =
   1.7764e-015
```

P4.14 The z-transform of a causal sequence is given as:

$$X(z) = \frac{-2 + 5.65z^{-1} - 2.88z^{-2}}{1 - 0.1z^{-1} + 0.09z^{-2} + 0.648z^{-3}} \tag{4.12}$$

which contains a complex-conjugate pole pair as well as a real-valued pole.

1. Rearrangement of $X(z)$ into a first- and second-order sections: MATLAB Script:

```
% P0414
clc; close all;
b = [-2 5.65 -2.88]; a = [1 -0.1 .09 0.648]; [R,p,k] = residuez(b,a)
 R =
    1.0000 - 0.8660i
    1.0000 + 0.8660i
   -4.0000
 p =
    0.4500 + 0.7794i
    0.4500 - 0.7794i
   -0.8000
 k =
      []
[b1,a1] = residuez(R(1:2),p(1:2),k)
 b1 =
     2.0000    0.4500
 a1 =
     1.0000   -0.9000    0.8100
```

Hence

$$X(z) = \frac{(2) + (0.45)z^{-1}}{1 + (-0.9)z^{-1} + (0.81)z^{-2}} + \frac{(-4)}{1 - (-0.8)z^{-1}}$$

2. Computation of the causal sequence $x(n)$ from the $X(z)$ so that it contains no complex numbers: MATLAB Script:

```
[Ac,As,r,v0] = invCCPP(b1(1),b1(2),a1(2),a1(3));
disp(sprintf('\nx1(n) = %2.0f*(%3.1f)^n*cos(%5.4f*pi*n)u(n) ',Ac,r,v0));
disp(sprintf('       + %5.4f*(%3.1f)^n*sin(%5.4f*pi*n)u(n)\n',As,r,v0));
 x1(n) =  2*(0.9)^n*cos(0.3333*pi*n)u(n)
        + 1.7321*(0.9)^n*sin(0.3333*pi*n)u(n)
```

Hence the sequence $x(n)$ is:

$$x(n) = 2(0.9)^n \cos(\pi n/3)u(n) + \sqrt{3}(0.9)^n \sin(\pi n/3)u(n) - 4(-0.8)^n u(n)$$

P4.15 System representations and input/output calculations:

1. $h(n) = 5(1/4)^n u(n)$

   i. The system function: Taking the z-transform of $h(n)$,

   $$H(z) = \mathcal{Z}[h(n)] = \mathcal{Z}\left[5(1/4)^n u(n)\right] = \frac{5}{1 - 0.25z^{-1}}, \ |z| > 0.5$$

   ii. The difference equation representation: From $H(z)$ above,

   $$y(n) = 5x(n) + 0.25y(n-1)$$

   iii. The pole-zero plot is shown in Figure 4.6.

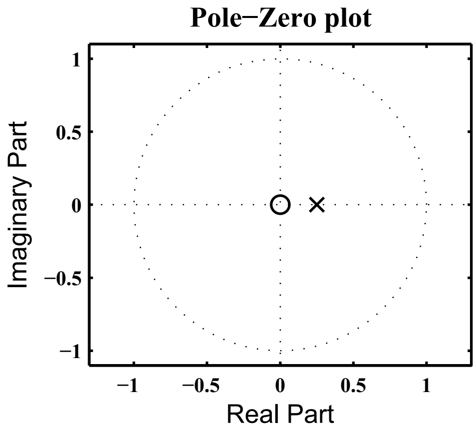


Figure 4.6: Problem P4.15.1 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = \left(\frac{1}{4}\right)^n u(n)$: Taking the z-transform of $x(n)$,

   $$X(z) = \mathcal{Z}\left[(1/4)^n u(n)\right] = \frac{1}{1 - 0.25z^{-1}}, \ |z| > 0.25$$

   Now the z-transform of $y(n)$ is

   $$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{5}{1 - 0.25z^{-1}}\right)\left(\frac{1}{1 - 0.25z^{-1}}\right) = \frac{5}{\left(1 - 0.25z^{-1}\right)^2}, \ |z| > 0.25 \\ &= 20z \frac{0.25z^{-1}}{\left(1 - 0.25z^{-1}\right)^2}, \ |z| > 0.25 \end{aligned}$$

   Hence

   $$y(n) = 20(n+1)(0.25)^{n+1} u(n+1)$$

2. $h(n) = n(1/3)^n u(n) + (-1/4)^n u(n)$

   i. The system function: Taking the z-transform of $h(n)$

$$\begin{aligned} H(z) = \mathcal{Z}[h(n)] &= \mathcal{Z}\left[n(1/3)^n u(n) + (-1/4)^n u(n)\right] \\ &= \frac{(1/3)z^{-1}}{\left[1-(1/3)z^{-1}\right]^2} + \frac{1}{1+(1/4)z^{-1}},\ |z| > (1/3) \\ &= \frac{1-\frac{1}{3}z^{-1}+\frac{7}{36}z^{-2}}{1-\frac{5}{12}z^{-1}-\frac{1}{18}z^{-2}+\frac{1}{36}z^{-3}},\ |z| > (1/3) \end{aligned}$$

   ii. The difference equation representation: From $H(z)$ above,

$$y(n) = x(n) - \frac{1}{3}x(n-1) + \frac{7}{36}x(n-2) + \frac{5}{12}y(n-1) + \frac{1}{18}y(n-2) - \frac{1}{36}y(n-3)$$

   iii. The pole-zero plot is shown in Figure 4.7.

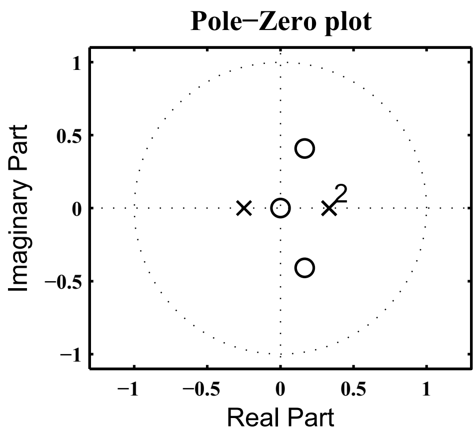


Figure 4.7: Problem P4.15.2 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = \left(\frac{1}{4}\right)^n u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[(1/4)^n u(n)\right] = \frac{1}{1-\frac{1}{4}z^{-1}},\ |z| > \frac{1}{4}$$

   Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1-\frac{1}{3}z^{-1}+\frac{7}{36}z^{-2}}{1-\frac{5}{12}z^{-1}-\frac{1}{18}z^{-2}+\frac{1}{36}z^{-3}}\right)\left(\frac{1}{1-0.25z^{-1}}\right),\ |z| > \frac{1}{3} \\ &= \frac{-16}{1-\frac{1}{3}z^{-1}} + \frac{4}{\left(1-\frac{1}{3}z^{-1}\right)^2} + \frac{\frac{1}{2}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{25}{2}}{1-\frac{1}{4}z^{-1}},\ |z| > \frac{1}{3} \\ &= \frac{-16}{1-\frac{1}{3}z^{-1}} + 12z\frac{\frac{1}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^2} + \frac{\frac{1}{2}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{25}{2}}{1-\frac{1}{4}z^{-1}},\ |z| > \frac{1}{3} \end{aligned}$$

   Hence

$$y(n) = -16(\tfrac{1}{3})^n u(n) + 12(n+1)(\tfrac{1}{3})^{n+1}u(n+1) + \frac{1}{2}(-\tfrac{1}{4})^n u(n) + \frac{25}{2}(\tfrac{1}{4})^n u(n)$$

3. $h(n) = 3(0.9)^n \cos(\pi n/4 + \pi/3)u(n+1)$: Consider

$$
\begin{aligned}
h(n) &= \frac{10}{3}\left[(0.9)^{n+1}\cos\left\{\frac{\pi(n+1)}{4}+\frac{\pi}{12}\right\}u(n+1)\right] \\
&= \left[\frac{10}{3}\cos\left(\frac{\pi}{12}\right)\right](0.9)^{n+1}\cos\left[\frac{\pi}{4}(n+1)\right]u(n+1) - \left[\frac{10}{3}\sin\left(\frac{\pi}{12}\right)\right](0.9)^{n+1}\sin\left[\frac{\pi}{4}(n+1)\right]u(n+1) \\
&= 3.2198(0.9)^{n+1}\cos\left[\frac{\pi}{4}(n+1)\right]u(n+1) - 0.8627(0.9)^{n+1}\sin\left[\frac{\pi}{4}(n+1)\right]u(n+1)
\end{aligned}
$$

i. The system function: Taking the z-transform of $h(n)$

$$
\begin{aligned}
H(z) = \mathcal{Z}[h(n)] &= z\left(3.2198\frac{1-0.6364z^{-1}}{1-1.2728z^{-1}+0.81z^{-2}} - 0.8627\frac{0.6364z^{-1}}{1-1.2728z^{-1}+0.81z^{-2}}\right) \\
&= \frac{3.2198z - 2.5981}{1-1.2728z^{-1}+0.81z^{-2}}, \ |z| > 0.9
\end{aligned}
$$

ii. The difference equation representation: From $H(z)$ above,

$$
y(n) = 3.2198x(n+1) - 2.5981x(n) + 1.2728y(n-1) - 0.81y(n-2)
$$

iii. The pole-zero plot is shown in Figure 4.8.

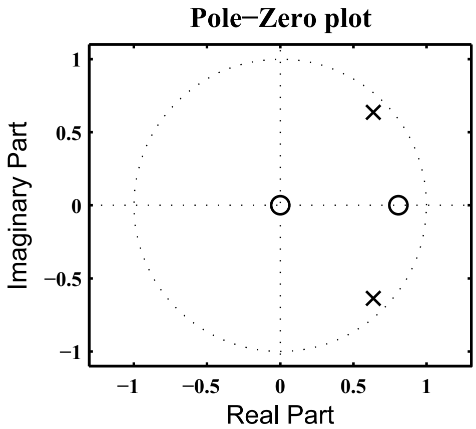


Figure 4.8: Problem P4.15.3 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = \left(\frac{1}{4}\right)^n u(n)$: The z-transform of $x(n)$ is $X(z) = \dfrac{1}{1-0.25z^{-1}}$, $|z| > 0.25$. Now the z-transform of $y(n)$ is

$$
\begin{aligned}
Y(z) = H(z)X(z) &= \left(\frac{3.2198z - 2.5981}{1-1.2728z^{-1}+0.81z^{-2}}\right)\left(\frac{1}{1-0.25z^{-1}}\right), \ |z| > 0.9 \\
&= z\left(\frac{4.0285 - 2.6203z^{-1}}{1.0000 - 1.2728z^{-1}+0.81z^{-2}} - \frac{0.8087}{1-\frac{1}{4}z^{-1}}\right), \ |z| > 0.9
\end{aligned}
$$

Hence

$$
y(n) = \left\{4.0285(0.9)^{n+1}\cos\left[\frac{\pi(n+1)}{4}\right] - 0.0889(0.9)^{n+1}\sin\left[\frac{\pi(n+1)}{4}\right] - 0.8087(\tfrac{1}{4})^{n+1}\right\}u(n+1)
$$

4. $h(n) = \dfrac{(0.5)^n \sin[(n+1)\pi/3]}{\sin(\pi/3)} u(n)$: Consider

$$\begin{aligned} h(n) &= \frac{1}{\sin(\frac{\pi}{3})}\left[(0.5)^n \sin\left\{\frac{\pi n}{3}+\frac{\pi}{3}\right\}u(n)\right] \\ &= \left[\frac{1}{\sin(\frac{\pi}{3})}\sin\left(\frac{\pi}{3}\right)\right](0.5)^n \sin\left[\frac{\pi}{3}n\right]u(n) + \left[\frac{1}{\sin(\frac{\pi}{3})}\cos\left(\frac{\pi}{3}\right)\right](0.5)^n \cos\left[\frac{\pi}{3}n\right]u(n) \\ &= (0.5)^n \sin\left[\frac{\pi}{3}n\right]u(n) + 0.5774(0.5)^n \cos\left[\frac{\pi}{3}n\right]u(n) \end{aligned}$$

i. The system function: Taking the z-transform of $h(n)$

$$\begin{aligned} H(z) = \mathcal{Z}[h(n)] &= \frac{0.4330z^{-1}}{1-0.5z^{-1}+0.25z^{-2}} + 0.5774\frac{1-0.25z^{-1}}{1-0.5z^{-1}+0.25z^{-2}} \\ &= \frac{0.5774+0.2887z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}, \; |z| > 0.5 \end{aligned}$$

ii. The difference equation representation: From $H(z)$ above,

$$y(n) = 0.5774x(n) + 0.2887x(n-1) + 0.5y(n-1) - 0.25y(n-2)$$

iii. The pole-zero plot is shown in Figure 4.9.

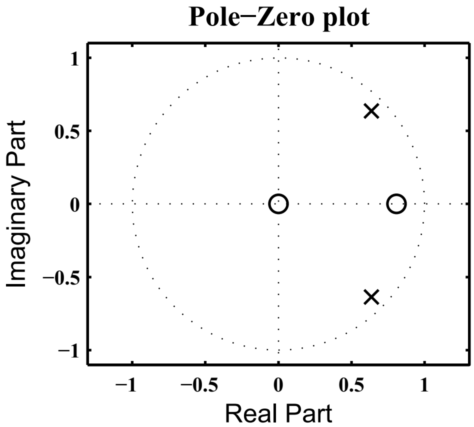


Figure 4.9: Problem P4.15.4 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = \left(\frac{1}{4}\right)^n u(n)$: The z-transform of $x(n)$ is $X(z) = \dfrac{1}{1-0.25z^{-1}}$, $|z| > 0.25$. Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{0.5774+0.2887z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}\right)\left(\frac{1}{1-0.25z^{-1}}\right), \; |z| > 0.5 \\ &= \frac{0.5774z^{-1}}{1.0000-0.5z^{-1}+0.25z^{-2}} - \frac{0.5774}{1-\frac{1}{4}z^{-1}}, \; |z| > 0.5 \end{aligned}$$

Hence

$$y(n) = \frac{4}{3}(0.5)^n \sin(\tfrac{\pi}{3}n)u(n) + 0.5774(\tfrac{1}{4})^n u(n)$$

5. $h(n) = [2 - \sin(\pi n)]u(n) = 2u(n)$

   i. The system function: $H(z) = \mathcal{Z}[h(n)] = \mathcal{Z}[2u(n)] = \dfrac{2}{1 - z^{-1}}$, $|z| > 1$.

   ii. The difference equation representation: $y(n) = 2x(n) + y(n-1)$

   iii. The pole-zero plot is shown in Figure 4.10.

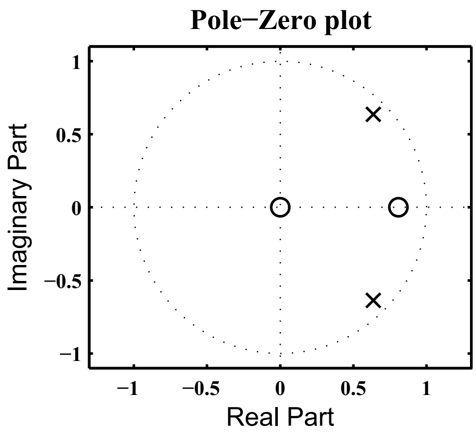


Figure 4.10: Problem P4.15.5 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = \left(\frac{1}{4}\right)^n u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[(1/4)^n u(n)\right] = \frac{1}{1 - 0.25z^{-1}},\ |z| > 0.25$$

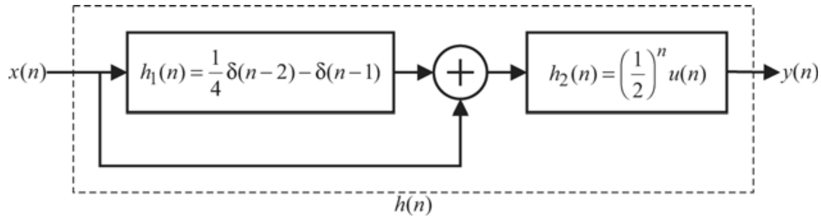
   Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{2}{1 - z^{-1}}\right)\left(\frac{1}{1 - 0.25z^{-1}}\right),\ |z| > 1 \\ &= \frac{8/3}{1 - z^{-1}} - \frac{2/3}{1 - \frac{1}{4}z^{-1}},\ |z| > 1 \end{aligned}$$

   Hence

$$y(n) = \frac{8}{3}u(n) - \frac{2}{3}\left(\frac{1}{4}\right)^n u(n)$$

P4.16 Consider the system shown below.



1. The overall system impulse response, $h(n)$, using the z-transform approach: The above system is given by

$$H(z) = H_2(z)\,[1 + H_1(z)] = \frac{1}{1 - 0.5z^{-1}}\left[1 + 0.25z^{-2} - z^{-1}\right]$$
$$= \frac{\left(1 - 0.5z^{-1}\right)^2}{1 - 0.5z^{-1}} = 1 - 0.5z^{-1},\ |z| \neq 0$$

Hence after taking inverse z-transform , we obtain

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

2. Difference equation representation of the overall system: From the overall system function $H(z)$,

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} \Rightarrow y(n) = x(n) - 0.5x(n-1)$$

3. Causality and stability: Since $h(n) = 0$ for $n < 0$, the system is causal. Since $h(n)$ is of finite duration (only two samples), $h(n)$ is absolutely summable. Hence BIBO stable.
4. Frequency response $H\left(e^{j\omega}\right)$ of the overall system.

$$H\left(e^{j\omega}\right) = \mathcal{F}[h(n)] = \mathcal{F}\left[\delta(n) - \frac{1}{2}\delta(n-1)\right] = 1 - \frac{1}{2}e^{-j\omega}$$

5. Frequency response over $0 \le \omega \le \pi$ is shown in Figure 4.11.

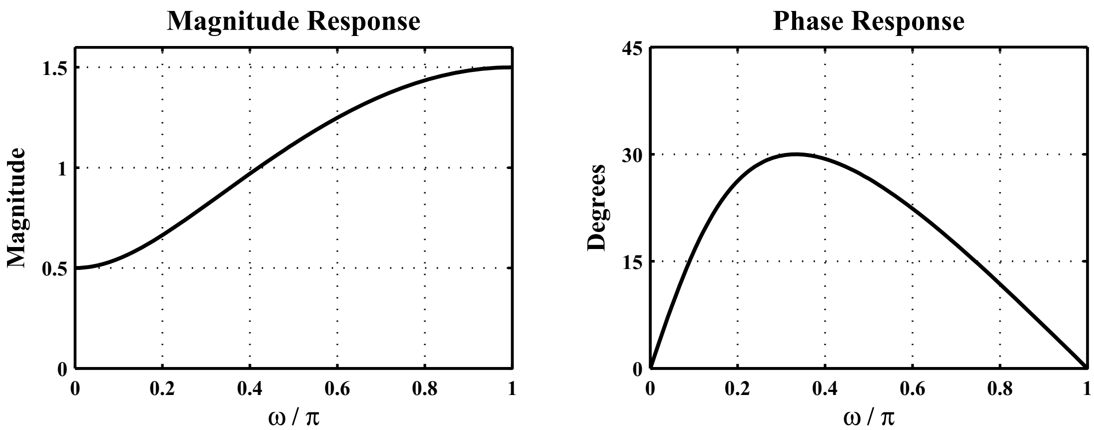


Figure 4.11: Problem P4.16 frequency-response plot

P4.17 System representations and input/output calculations:

1. $H(z) = (z+1)/(z-0.5)$, causal system. Consider

$$H(z) = \frac{z+1}{z-0.5} = \frac{1+z^{-1}}{1-0.5z^{-1}} = \frac{1}{1-0.5z^{-1}} + \frac{z^{-1}}{1-0.5z^{-1}}, \; |z| > 0.5$$

i. The impulse response: Taking the inverse z-transform of $H(z)$,

$$h(n) = \mathcal{Z}^{-1}[H(z)] = (0.5)^n u(n) + (0.5)^{n-1} u(n-1)$$

ii. The difference equation representation: From $H(z)$ above,

$$y(n) = x(n) + x(n-1) + 0.5y(n-1)$$

iii. The pole-zero plot is shown in Figure 4.12.

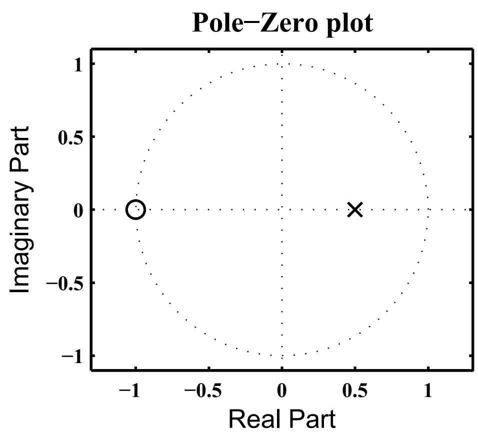


Figure 4.12: Problem P4.17.1 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = 3\cos(\pi n/3)u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}[3\cos(\pi n/3)u(n)] = 3\frac{1-[\cos(\pi/3)]z^{-1}}{1-[2\cos(\pi/3)]z^{-1}+z^{-2}} = 3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}, \; |z| > 1$$

Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1+z^{-1}}{1-0.5z^{-1}}\right)\left(3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right) = 3\frac{1+z^{-1}}{1-z^{-1}+z^{-2}}, \; |z| > 1 \\ &= 3\frac{1-0.5z^{-1}+1.5z^{-1}}{1-z^{-1}+z^{-2}} = 3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}} + 3\sqrt{3}\frac{\frac{\sqrt{3}}{2}z^{-1}}{1-z^{-1}+z^{-2}}, \; |z| > 1 \end{aligned}$$

Hence

$$y(n) = 3\cos(\pi n/3)u(n) + 3\sqrt{3}\sin(\pi n/3)u(n)$$

2. $H(z) = (1+z^{-1}+z^{-2})/(1+0.5z^{-1}-0.25z^{-2})$, stable system. Consider

$$H(z) = \frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}-0.25z^{-2}} = -4 + \frac{0.9348}{1+0.809z^{-1}} + \frac{4.0652}{1-0.309z^{-1}},\ |z| > 0.809$$

i. The impulse response: Taking the inverse z-transform of $H(z)$,

$$h(n) = \mathcal{Z}^{-1}[H(z)] = -4\delta(n) + 0.9348(-0.809)^n u(n) + 4.0652(0.309)^n u(n)$$

ii. The difference equation representation: From $H(z)$ above,

$$y(n) = x(n) + x(n-1) + x(n-2) - 0.5y(n-1) + 0.25y(n-2)$$

iii. The pole-zero plot is shown in Figure 4.13.

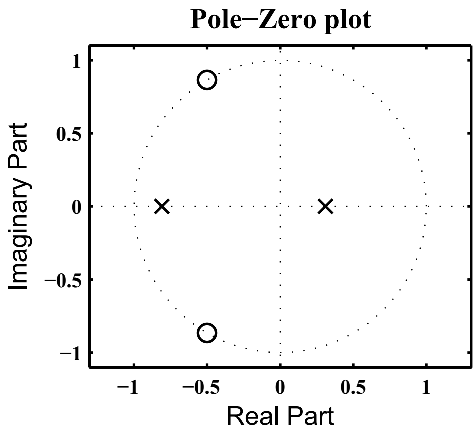


Figure 4.13: Problem P4.17.2 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = 3\cos(\pi n/3)u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}[3\cos(\pi n/3)u(n)] = 3\frac{1-[\cos(\pi/3)]z^{-1}}{1-[2\cos(\pi/3)]z^{-1}+z^{-2}} = 3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}},\ |z| > 1$$

Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}-0.25z^{-2}}\right)\left(3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right),\ |z| > 1 \\ &= \frac{1.2055}{1+0.809z^{-1}} - \frac{0.9152}{1-0.309z^{-1}} + \frac{2.7097\left(1-\frac{1}{2}z^{-1}\right)}{1-z^{-1}+z^{-2}} + \frac{3.3524\left(\frac{\sqrt{3}}{2}z^{-1}\right)}{1-z^{-1}+z^{-2}},\ |z| > 1 \end{aligned}$$

Hence

$$\begin{aligned} y(n) = {} & 1.2055(-0.809)^n u(n) - 0.9152(0.309)^n u(n) + 2.7097\cos(\pi n/3)u(n) \\ & + 3.3524\sin(\pi n/3)u(n) \end{aligned}$$

3. $H(z) = (z^2 - 1)/(z - 3)^2$, anti-causal system. Consider

$$H(z) = \frac{z^2-1}{(z-3)^2} = \frac{1-z^{-2}}{1-6z^{-1}+9z^{-2}} = -\frac{1}{9} + \frac{2/9}{1-3z^{-1}} + \frac{8z}{27}\frac{3z^{-1}}{\left(1-3z^{-1}\right)^2}, \; |z| < 3$$

i. The impulse response: Taking the inverse z-transform of $H(z)$,

$$h(n) = \mathcal{Z}^{-1}\left[H(z)\right] = -\frac{1}{9}\delta(n) - \frac{2}{9}3^n u(-n-1) - \frac{8}{27}(n+1)3^{n+1}u(-n-2)$$

ii. The difference equation representation: From $H(z)$ above,

$$y(n) = x(n) - x(n-2) + 6y(n-1) - 9y(n-2)$$

iii. The pole-zero plot is shown in Figure 4.14.

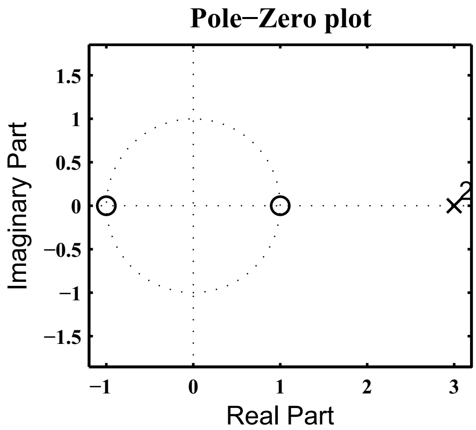


Figure 4.14: Problem P4.17.3 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = 3\cos(\pi n/3)u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[3\cos(\pi n/3)u(n)\right] = 3\frac{1-[\cos(\pi/3)]z^{-1}}{1-[2\cos(\pi/3)]z^{-1}+z^{-2}} = 3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}, \; |z| > 1$$

Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1-z^{-2}}{1-6z^{-1}+9z^{-2}}\right)\left(3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right), 1 < |z| < 3 \\ &= \frac{43/49}{1-3z^{-1}} + \frac{20z}{21}\frac{3z^{-1}}{\left(1-3z^{-1}\right)^2} + \frac{\frac{-36}{49}\left(1-\frac{1}{2}z^{-1}\right)}{1-z^{-1}+z^{-2}} + \frac{\frac{193}{1820}\left(\frac{\sqrt{3}}{2}z^{-1}\right)}{1-z^{-1}+z^{-2}}, \; |z| > 1 \end{aligned}$$

Hence

$$\begin{aligned} y(n) = &-\frac{43}{49}3^n u(-n-1) - \frac{20}{21}(n+1)3^{n+1}u(-n-2) - \frac{36}{49}\cos(\pi n/3)u(n) \\ &+ \frac{193}{1820}\sin(\pi n/3)u(n) \end{aligned}$$

4. $H(z) = \dfrac{z}{z-0.25} + \dfrac{1-0.5z^{-1}}{1+2z^{-1}}$, stable system. Consider

$$\begin{aligned} H(z) &= \frac{1}{1-0.25z^{-1}} + \frac{1-0.5z^{-1}}{1+2z^{-1}} = \frac{2+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}}{1+\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}} \\ &= -\frac{1}{4} + \frac{1}{1-0.25z^{-1}} + \frac{5/4}{1+2z^{-1}},\ 0.25 < |z| < 2 \end{aligned}$$

i. The impulse response: Taking the inverse z-transform of $H(z)$,

$$h(n) = \mathcal{Z}^{-1}[H(z)] = -\frac{1}{4}\delta(n) + \left(\frac{1}{4}\right)^n u(n) - \frac{5}{4}2^n u(-n-1)$$

ii. The difference equation representation: From $H(z)$ above,

$$y(n) = 2x(n) + \frac{5}{4}x(n-1) - \frac{1}{8}x(n-2) - \frac{7}{4}y(n-1) + \frac{1}{2}y(n-2)$$

iii. The pole-zero plot is shown in Figure 4.15.

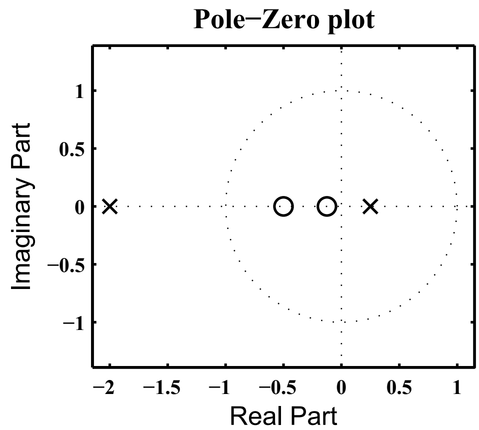


Figure 4.15: Problem P4.17.4 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = 3\cos(\pi n/3)u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}[3\cos(\pi n/3)u(n)] = 3\frac{1-[\cos(\pi/3)]z^{-1}}{1-[2\cos(\pi/3)]z^{-1}+z^{-2}} = 3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}},\ |z| > 1$$

Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{2+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}}{1+\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}\right)\left(3\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right),\ 1 < |z| < 3 \\ &= \frac{75/28}{1+2z^{-1}} - \frac{3/13}{1-\frac{1}{4}z^{-1}} + \frac{\frac{1293}{364}\left(1-\frac{1}{2}z^{-1}\right)}{1-z^{-1}+z^{-2}} - \frac{\frac{323}{2553}\left(\frac{\sqrt{3}}{2}z^{-1}\right)}{1-z^{-1}+z^{-2}},\ |z| > 1 \end{aligned}$$

Hence

$$\begin{aligned} y(n) = &-\frac{75}{28}2^n u(-n-1) - \frac{3}{13}\left(\frac{1}{4}\right)^n u(n) + \frac{1293}{364}\cos(\pi n/3)u(n) \\ &- \frac{323}{2553}\sin(\pi n/3)u(n) \end{aligned}$$

5. $H(z) = (1 + z^{-1} + z^{-2})^2$. Consider

$$H(z) = (1 + z^{-1} + z^{-2})^2 = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}, \ |z| > 0$$

i. The impulse response: Taking the inverse z-transform of $H(z)$,

$$h(n) = \mathcal{Z}^{-1}[H(z)] = \{\underset{\uparrow}{1}, 2, 3, 2, 1\}$$

ii. The difference equation representation: From $H(z)$ above,

$$y(n) = x(n) + 2x(n-1) + 3x(n-2) + 2x(n-3) + x(n-4)$$

iii. The pole-zero plot is shown in Figure 4.16.

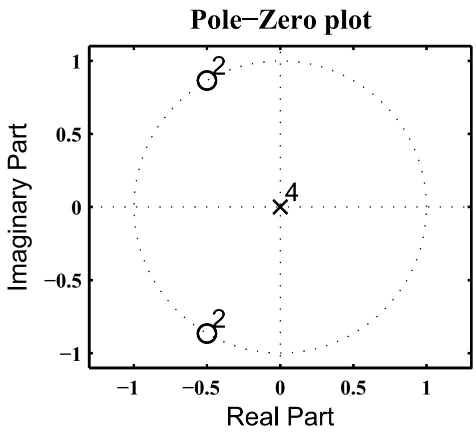


Figure 4.16: Problem P4.17.5 pole-zero plot

iv. The output $y(n)$ for the input $x(n) = 3\cos(\pi n/3)u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}[3\cos(\pi n/3)u(n)] = 3\frac{1 - [\cos(\pi/3)]z^{-1}}{1 - [2\cos(\pi/3)]z^{-1} + z^{-2}} = 3\frac{1 - 0.5z^{-1}}{1 - z^{-1} + z^{-2}}, \ |z| > 1$$

Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= 3\left[\frac{\left(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}\right)\left(1 - 0.5z^{-1}\right)}{1 - z^{-1} + z^{-2}}\right], |z| > 1 \\ &= 9 + \frac{3}{2}z^{-1} - \frac{3}{2}z^{-2} - \frac{3}{2}z^{-3} + \frac{\frac{1293}{364}\left(1 - \frac{1}{2}z^{-1}\right)}{1 - z^{-1} + z^{-2}} - \frac{\frac{328}{2553}\left(\frac{\sqrt{3}}{2}z^{-1}\right)}{1 - z^{-1} + z^{-2}}, |z| > 1 \end{aligned}$$

Hence

$$\begin{aligned} y(n) = 9\delta(n) + \frac{3}{2}\delta(n-1) - \frac{3}{2}\delta(n-2) - \frac{3}{2}\delta(n-3) + \frac{1293}{364}\cos(\pi n/3)u(n) \\ - \frac{328}{2553}\sin(\pi n/3)u(n) \end{aligned}$$

P4.18 System representations and input/output calculations:

1. $y(n) = [x(n) + 2x(n-1) + x(n-3)]/4$

   i. The system function representation: Taking the z-transform of the above difference equation,

   $$Y(z) = \frac{1}{4}[X(z) + 2z^{-1}X(z) + z^{-3}X(z)] \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-3}}{4}$$

   ii. The impulse response: Taking the inverse z-transform of $H(z)$,

   $$h(n) = \frac{1}{4}[\delta(n) + 2\delta(n-1) + \delta(n-3)]$$

   iii. The pole-zero plot is shown in Figure 4.17.

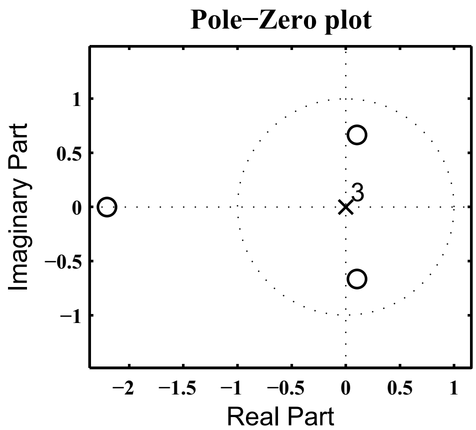


Figure 4.17: Problem P4.18.1 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = 2(0.9)^n u(n)$: Taking the z-transform of $x(n)$,

   $$X(z) = \mathcal{Z}\left[2(0.9)^n u(n)\right] = \frac{2}{1 - 0.9z^{-1}},\ |z| > 0.9$$

   Now the z-transform of $y(n)$ is

   $$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1 + 2z^{-1} + z^{-3}}{4}\right)\left(\frac{2}{1 - 0.9z^{-1}}\right),\ |z| > 0.9 \\ &= -\frac{1310}{729} - \frac{50}{81}z^{-1} - \frac{5}{9}z^{-2} + \frac{\frac{990}{431}}{1 - 0.9z^{-1}},\ |z| > 0.9 \end{aligned}$$

   Hence

   $$y(n) = -\frac{1310}{729}\delta(n) - \frac{50}{81}\delta(n-1) - \frac{5}{9}\delta(n-2) + \frac{990}{431}(0.9)^n u(n)$$

2. $y(n) = x(n) + 0.5x(n-1) - 0.5y(n-1) + 0.25y(n-2)$

   i. The system function representation: Taking the z-transform of the above difference equation,

$$Y(z) = X(z) + 0.5z^{-1}X(z) - 0.5z^{-1}Y(z) + 0.25z^{-2}Y(z)$$

   or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.5z^{-1}}{1+0.5z^{-1}-0.25z^{-2}} = \frac{1+0.5z^{-1}}{\left(1+0.809z^{-1}\right)\left(1-0.309z^{-1}\right)}, \; |z| > 0.809$$

   ii. The impulse response: Taking the inverse z-transform of $H(z)$,

$$\begin{aligned} h(n) &= \mathcal{Z}^{-1}\left[\frac{1+0.5z^{-1}}{\left(1+0.809z^{-1}\right)\left(1-0.309z^{-1}\right)}\right] = \mathcal{Z}^{-1}\left[\frac{0.2764}{1+0.809z^{-1}} + \frac{0.7236}{1-0.309z^{-1}}\right] \\ &= 0.2764(-0.809)^n u(n) + 0.7236(0.308)^n u(n) \end{aligned}$$

   iii. The pole-zero plot is shown in Figure 4.18.

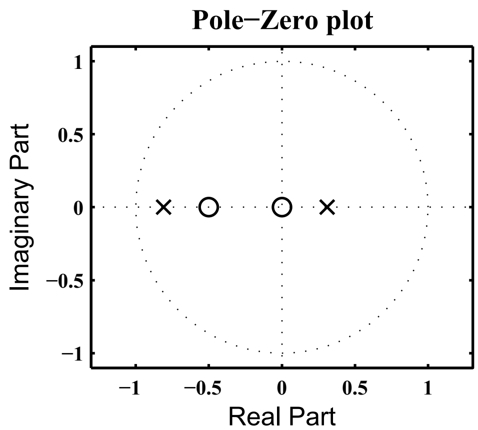


Figure 4.18: Problem P4.18.2 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = 2(0.9)^n u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[2(0.9)^n u(n)\right] = \frac{2}{1-0.9z^{-1}}, \; |z| > 0.9$$

   Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1+0.5z^{-1}}{1+0.5z^{-1}-0.25z^{-2}}\right)\left(\frac{2}{1-0.9z^{-1}}\right), \; |z| > 0.9 \\ &= \frac{0.2617}{1+0.809z^{-1}} - \frac{0.7567}{1-0.309z^{-1}} + \frac{2.495}{1-0.9z^{-1}}, \; |z| > 0.9 \end{aligned}$$

   Hence

$$y(n) = 0.2617(-0.809)^n u(n) - 0.7567(0.309)^n u(n) + 2.495(0.9)^n u(n)$$

3. $y(n) = 2x(n) + 0.9y(n-1)$

   i. The system function representation: Taking the z-transform of the above difference equation,

$$Y(z) = 2X(z) + 0.9z^{-1}Y(z)$$

   or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-0.9z^{-1}}, \ |z| > 0.9$$

   ii. The impulse response: Taking the inverse z-transform of $H(z)$,

$$h(n) = 2(0.9)^n u(n)$$

   iii. The pole-zero plot is shown in Figure 4.19.

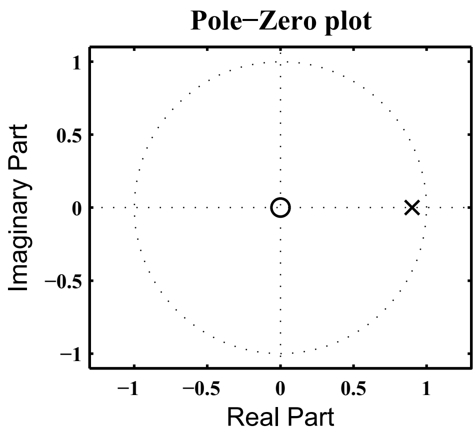


Figure 4.19: Problem P4.18.3 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = 2(0.9)^n u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[2(0.9)^n u(n)\right] = \frac{2}{1-0.9z^{-1}}, \ |z| > 0.9$$

   Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{2}{1-0.9z^{-1}}\right)\left(\frac{2}{1-0.9z^{-1}}\right), \ |z| > 0.9 \\ &= \frac{40}{9} z \frac{0.9z^{-1}}{\left(1-0.9z^{-1}\right)^2}, \ |z| > 0.9 \end{aligned}$$

   Hence

$$y(n) = \frac{40}{9}(n+1)(0.9)^{n+1}u(n+1)$$

4. $y(n) = -0.45x(n) - 0.4x(n-1) + x(n-2) + 0.4y(n-1) + 0.45y(n-2)$

   i. The system function representation: Taking the z-transform of the above difference equation,

$$Y(z) = -0.45X(z) - 0.4z^{-1}X(z) + z^{-2}X(z) + 0.4z^{-1}Y(z) + 0.45z^{-2}Y(z)$$

or
$$H(z) = \frac{Y(z)}{X(z)} = \frac{-0.45 - 0.4z^{-1} + z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}} = \frac{-0.45 - 0.4z^{-1} + z^{-2}}{\left(1+0.5z^{-1}\right)\left(1-0.9z^{-1}\right)}, \ |z| > 0.9$$

   ii. The impulse response: Taking the inverse z-transform of $H(z)$,

$$\begin{aligned} h(n) &= \mathcal{Z}^{-1}\left[\frac{-0.45 - 0.4z^{-1} + z^{-2}}{\left(1+0.5z^{-1}\right)\left(1-0.9z^{-1}\right)}\right] = \mathcal{Z}^{-1}\left[-\frac{20}{9} + \frac{1.5536}{1+0.5z^{-1}} + \frac{0.2187}{1-0.9z^{-1}}\right] \\ &= -\frac{20}{9}\delta(n) + 1.5536(-0.5)^n u(n) + 0.2187(0.9)^n u(n) \end{aligned}$$

   iii. The pole-zero plot is shown in Figure 4.20.

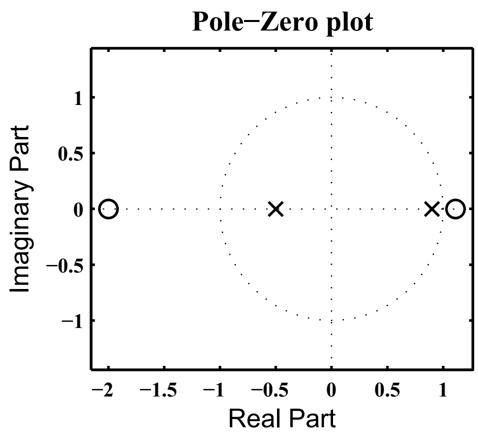


Figure 4.20: Problem P4.18.4 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = 2(0.9)^n u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[2(0.9)^n u(n)\right] = \frac{2}{1-0.9z^{-1}}, \ |z| > 0.9$$

Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{-0.45 - 0.4z^{-1} + z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}}\right)\left(\frac{2}{1-0.9z^{-1}}\right), |z| > 0.9 \\ &= \frac{1.1097}{1+0.5z^{-1}} - \frac{2.4470}{1-0.9z^{-1}} + 0.4859z\frac{0.9z^{-1}}{\left(1-0.9z^{-1}\right)^2}, \ |z| > 0.9 \end{aligned}$$

Hence

$$y(n) = 1.1097(-0.5)^n u(n) - 2.4470(0.9)^n u(n) + 0.4859(n+1)(0.9)^{n+1}u(n+1)$$

5. $y(n) = \sum_{m=0}^{4}(0.8)^m x(n-m) - \sum_{\ell=1}^{4}(0.9)^\ell y(n-\ell)$

   i. The system function representation: Taking the z-transform of the above difference equation,

$$Y(z) = \sum_{m=0}^{4}(0.8)^m z^{-m} X(z) - \sum_{\ell=1}^{4}(0.9)^\ell z^{-\ell} Y(z)$$

   or

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{m=0}^{4}(0.8)^m z^{-m}}{1+\sum_{\ell=1}^{4}(0.9)^\ell z^{-\ell}} \\ &= \frac{1+0.8z^{-1}+0.64+z^{-2}+0.512z^{-3}+0.4096z^{-4}}{\left(1-0.5562z^{-1}+0.81z^{-2}\right)\left(1+1.4562z^{-1}+0.81z^{-2}\right)}, \ |z| > 0.9 \end{aligned}$$

   ii. The impulse response: Taking the inverse z-transform of $H(z)$,

$$\begin{aligned} h(n) &= \mathcal{Z}^{-1}\left[0.6243 + \frac{0.1873+0.0651z^{-1}}{1-0.5562z^{-1}+0.81z^{-2}} + \frac{0.1884+0.1353z^{-1}}{1+1.4562z^{-1}+0.81z^{-2}}\right] \\ &= 0.1884\delta(n) + \left[0.1879(0.9)^n\cos(0.4\pi n + 4.63^\circ) + 0.1885(0.9)^n\cos(0.8\pi n + 1.1^\circ)\right]u(n) \end{aligned}$$

   iii. The pole-zero plot is shown in Figure 4.21.

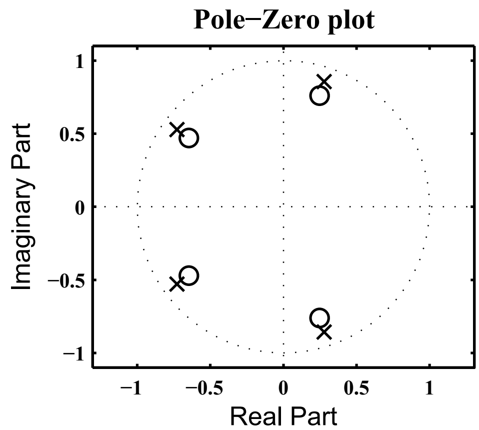


Figure 4.21: Problem P4.18.5 pole-zero plot

   iv. The output $y(n)$ for the input $x(n) = 2(0.9)^n u(n)$: Taking the z-transform of $x(n)$,

$$X(z) = \mathcal{Z}\left[2(0.9)^n u(n)\right] = \frac{2}{1-0.9z^{-1}}, \ |z| > 0.9$$

   Now the z-transform of $y(n)$ is

$$\begin{aligned} Y(z) = H(z)X(z) &= \left(\frac{1+0.8z^{-1}+0.64+z^{-2}+0.512z^{-3}+0.4096z^{-4}}{1+0.9z^{-1}+0.81+z^{-2}+0.279z^{-3}+0.6561z^{-4}}\right)\left(\frac{2}{1-0.9z^{-1}}\right) \\ &= \frac{0.2081+0.1498z^{-1}}{1-0.5562z^{-1}+0.81z^{-2}} + \frac{0.1896+0.1685z^{-1}}{1+1.4562z^{-1}+0.81z^{-2}} + \frac{1.6023}{1-0.9z^{-1}}, \ |z| > 0.9 \end{aligned}$$

   Hence

$$\begin{aligned} y(n) &= 0.3197(0.9)^n\cos(0.4\pi n - 49.37^\circ)u(n) + 0.1982(0.9)^n\cos(0.8\pi n - 16.9^\circ)u(n) \\ &\quad + 1.6023(0.9)^n u(n) \end{aligned}$$

P4.19 Separation of the total response $y(n)$ into (i) the homogeneous part, (ii) the particular part, (iii) the transient response, and (iv) the steady-state response for each of the systems given in Problem P4.18.

1. $y(n) = [x(n) + 2x(n-1) + x(n-3)]/4$: The total response is

$$y(n) = -\frac{1310}{729}\delta(n) - \frac{50}{81}\delta(n-1) - \frac{5}{9}\delta(n-2) + \frac{990}{431}(0.9)^n u(n)$$

   i. Homogeneous part: Since the system is an FIR filter, the homogeneous equation is $y(n) = 0$. Thus $y_{\text{h}}(n) = 0$.
   ii. Particular part: Hence the total response is the particular part, or

$$y_{\text{p}}(n) = -\frac{1310}{729}\delta(n) - \frac{50}{81}\delta(n-1) - \frac{5}{9}\delta(n-2) + \frac{990}{431}(0.9)^n u(n)$$

   iii. Transient response: Since the entire response decays to zero,

$$y_{\text{tr}}(n) = -\frac{1310}{729}\delta(n) - \frac{50}{81}\delta(n-1) - \frac{5}{9}\delta(n-2) + \frac{990}{431}(0.9)^n u(n)$$

   iv. Steady-state response: Clearly, $y_{\text{ss}}(n) = 0$.

2. $y(n) = x(n) + 0.5x(n-1) - 0.5y(n-1) + 0.25y(n-2)$: The total response is

$$y(n) = 0.2617(-0.809)^n u(n) - 0.7567(0.309)^n u(n) + 2.495(0.9)^n u(n)$$

   i. Homogeneous part: The first two terms in $y(n)$ are due to the system poles, hence

$$y_{\text{h}}(n) = 0.2617(-0.809)^n u(n) - 0.7567(0.309)^n u(n)$$

   ii. Particular part: The last term in $y(n)$ is due to the input pole, hence

$$y_{\text{p}}(n) = 2.495(0.9)^n u(n)$$

   iii. Transient response: Since all poles of $Y(z)$ are inside the unit circle,

$$y_{\text{tr}}(n) = 0.2617(-0.809)^n u(n) - 0.7567(0.309)^n u(n) + 2.495(0.9)^n u(n)$$

   iv. Steady-state response: Clearly, $y_{\text{ss}}(n) = 0$.

3. $y(n) = 2x(n) + 0.9y(n-1)$: The total response is

$$y(n) = \frac{40}{9}(n+1)(0.9)^{n+1}u(n+1)$$

   i. Homogeneous part: Since the system pole and the input pole are the same and hence are indistinguishable. Therefore, the total response can be equally divided into two parts or

$$y_{\text{h}}(n) = \frac{20}{9}(n+1)(0.9)^{n+1}u(n+1)$$

   ii. Particular part: Since the system pole and the input pole are the same and hence are indistinguishable. Therefore, the total response can be equally divided into two parts or

$$y_{\text{p}}(n) = \frac{20}{9}(n+1)(0.9)^{n+1}u(n+1)$$

iii. Transient response: Since all poles of $Y(z)$ are inside the unit circle,

$$y_{\text{tr}}(n) = \frac{40}{9}(n+1)(0.9)^{n+1}u(n+1)$$

iv. Steady-state response: Clearly, $y_{\text{ss}}(n) = 0$.

4. $y(n) = -0.45x(n) - 0.4x(n-1) + x(n-2) + 0.4y(n-1) + 0.45y(n-2)$: The total response is

$$y(n) = 1.1097(-0.5)^n u(n) - 2.4470(0.9)^n u(n) + 0.4859(n+1)(0.9)^{n+1}u(n+1)$$

i. Homogeneous part: There are two system poles, $p_1 = -0.5$ and $p_2 = 0.9$. Clearly, p_2 is also an input pole. Hence the response due to p_2 has to be divided to include in both parts. Hence

$$y_{\text{h}}(n) = 1.1097(-0.5)^n u(n) - 1.1135(0.9)^n u(n) + 0.24295(n+1)(0.9)^{n+1}u(n+1)$$

ii. Particular part: from above,

$$y_{\text{p}}(n) = -1.1135(0.9)^n u(n) + 0.24295(n+1)(0.9)^{n+1}u(n+1)$$

iii. Transient response: Since all poles of $Y(z)$ are inside the unit circle,

$$y_{\text{tr}}(n) = 1.1097(-0.5)^n u(n) - 2.4470(0.9)^n u(n) + 0.4859(n+1)(0.9)^{n+1}u(n+1)$$

iv. Steady-state response: Clearly, $y_{\text{ss}}(n) = 0$.

5. $y(n) = \sum_{m=0}^{4}(0.8)^m x(n-m) - \sum_{\ell=1}^{4}(0.9)^\ell y(n-\ell)$: The total response is

$$\begin{aligned} y(n) = {} & 0.3197(0.9)^n \cos(0.4\pi n - 49.37^\circ)u(n) + 0.1982(0.9)^n \cos(0.8\pi n - 16.9^\circ)u(n) \\ & + 1.6023(0.9)^n u(n) \end{aligned}$$

i. Homogeneous part: The first two terms in $y(n)$ are due to the system poles, hence

$$y_{\text{h}}(n) = 0.3197(0.9)^n \cos(0.4\pi n - 49.37^\circ)u(n) + 0.1982(0.9)^n \cos(0.8\pi n - 16.9^\circ)u(n)$$

ii. Particular part: The last term in $y(n)$ is due to the input pole, hence

$$y_{\text{p}}(n) = 1.6023(0.9)^n u(n)$$

iii. Transient response: Since all poles of $Y(z)$ are inside the unit circle,

$$\begin{aligned} y_{\text{tr}}(n) = {} & 0.3197(0.9)^n \cos(0.4\pi n - 49.37^\circ)u(n) + 0.1982(0.9)^n \cos(0.8\pi n - 16.9^\circ)u(n) \\ & + 1.6023(0.9)^n u(n) \end{aligned}$$

iv. Steady-state response: Clearly, $y_{\text{ss}}(n) = 0$.

P4.20 A stable system has the following pole-zero locations:

$$\text{zeros: } \pm 1,\ \pm j1 \qquad \text{Poles: } \pm 0.9,\ \pm j0.9$$

It is also known that $H\left(e^{j\pi/4}\right) = 1$.

1. The system function $H(z)$ and its region of convergence: Consider

$$H(z) = K\frac{(z-1)(z+1)(z-j)(z+j)}{(z-0.9)(z+0.9)(z-j0.9)(z+j0/9)} = K\frac{1-z^{-4}}{1-0.6561z^{-4}},\ |z| > 0.9$$

Now at $z = e^{j\pi/4}$, we have $H\left(e^{j\pi/4}\right) = 1$. Hence

$$1 = H\left(e^{j\pi/4}\right) = K\frac{1-e^{j\pi}}{1-0.6561e^{j\pi}} = K \times 1.2077 \Rightarrow K = 0.8281$$

or

$$H(z) = \frac{0.8281\left(1-z^{-4}\right)}{1-0.6561z^{-4}},\ |z| > 0.9$$

2. The difference equation representation: From

$$H(z) = \frac{0.8281\left(1-z^{-4}\right)}{1-0.6561z^{-4}} = \frac{Y(z)}{X(z)}$$

we have

$$y(n) = 0.8281x(n) - 0.8281x(n-4) + 0.6561y(n-4)$$

3. The steady-state response $y_{ss}(n)$ for the input $x(n) = \cos(\pi n/4)u(n)$: From the z-transform table,

$$X(z) = \frac{1-[\cos(\pi/4)]z^{-1}}{1-[2\cos(\pi/4)]z^{-1}+z^{-2}} = \frac{1-\frac{1}{\sqrt{2}}z^{-1}}{1-\sqrt{2}z^{-1}+z^{-2}}$$

Hence

$$\begin{aligned} Y(z) = H(z)X(z) &= \left[\frac{0.8281\left(1-z^{-4}\right)}{1-0.6561z^{-4}}\right]\left(\frac{1-\frac{1}{\sqrt{2}}z^{-1}}{1-\sqrt{2}z^{-1}+z^{-2}}\right) \\ &= \frac{1-\frac{1}{\sqrt{2}}z^{-1}}{1-\sqrt{2}z^{-1}+z^{-2}} - \frac{0.0351}{1-0.9z^{-1}} - \frac{0.0509}{1+0.9z^{-1}} + \frac{-0.0860-0.1358z^{-1}}{1-0.81z^{-2}},\ |z| > 1 \end{aligned}$$

The first term above has poles on the unit circle and hence gives the steady-state response

$$y_{ss}(n) = \cos(\pi n/4)$$

4. The transient response $y_{tr}(n)$ for the input $x(n) = \cos(\pi n/4)u(n)$: The remaining terms in $y(n)$ are the transient response terms. Using the `inv_CC_PP` function we have

$$Y_{tr}(z) = -\frac{0.0351}{1-0.9z^{-1}} - \frac{0.0509}{1+0.9z^{-1}} - 0.0860\frac{1}{1-0.81z^{-2}} - 0.1509\frac{0.9z^{-1}}{1-0.81z^{-2}},\ |z| > 1$$

Hence

$$\begin{aligned} y_{tr}(n) = &-0.0351(0.9)^n u(n) - 0.0509(-0.9)^n u(n) - 0.086(0.9)^n\cos(\pi n/2)u(n) \\ &- 0.1509(0.9)^n\sin(\pi n/2)u(n) \end{aligned}$$

P4.21 A digital filter is described by the frequency response function

$$H(e^{j\omega}) = [1 + 2\cos(\omega) + 3\cos(2\omega)]\cos(\omega/2)e^{-j5\omega/2}$$

which can be written as

$$\begin{aligned} H(e^{j\omega}) &= \left[1 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 3\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)\right]\frac{e^{j\frac{1}{2}\omega} + e^{-j\frac{1}{2}\omega}}{2}e^{-j\frac{5}{2}\omega} \\ &= \frac{3}{4} + \frac{5}{4}e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + \frac{5}{4}e^{-j4\omega} + \frac{3}{4}e^{-j5\omega} \end{aligned}$$

or after substituting $e^{-j\omega} = z^{-1}$, we obtain

$$H(z) = \frac{3}{4} + \frac{5}{4}z^{-1} + z^{-2} + z^{-3} + \frac{5}{4}z^{-4} + \frac{3}{4}z^{-5}$$

1. The difference equation representation: From $H(z)$ above

$$y(n) = \frac{3}{4}x(n) + \frac{5}{4}x(n-1) + x(n-2) + x(n-3) + \frac{5}{4}x(n-4) + \frac{3}{4}x(n-5)$$

2. The magnitude and phase response plots are shown in Figure 4.22.

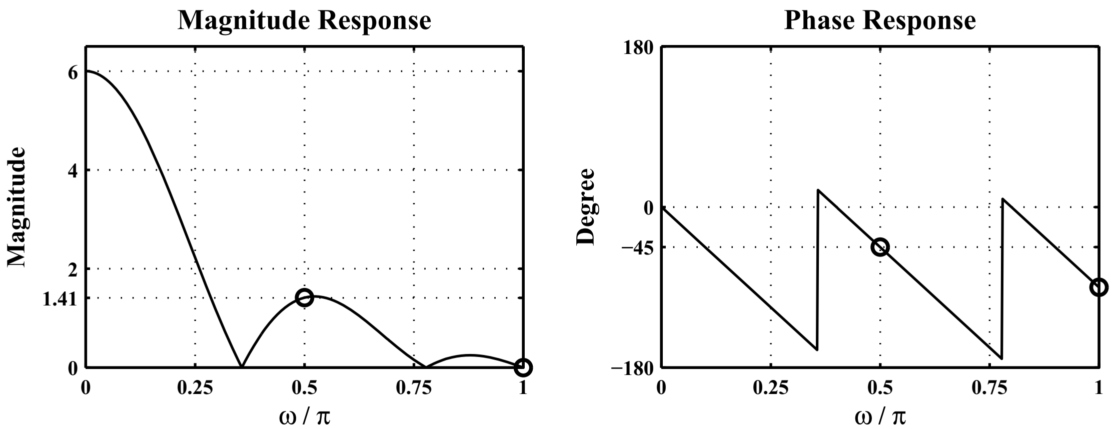


Figure 4.22: Problem P4.21.2 frequency-response plots

The magnitude and phase at $\omega = \pi/2$ are $\sqrt{2}$ and $-45°$, respectively. The magnitude at $\omega = \pi$ is zero.

3. The output sequence $y(n)$ for the input $x(n) = \sin(\pi n/2) + 5\cos(\pi n)$: MATLAB script:

```
clc; close all;  set(0,'defaultfigurepaperposition',[0,0,7,5]);
b = [3/4 5/4 1 1 5/4 3/4]; a = [1 0];
n = 0:200; x = sin(pi*n/2)+5*cos(pi*n); y = filter(b,a,x);
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0421c');
subplot(2,1,1); Hs = stem(n,x); set(Hs,'markersize',2); axis([-2 202 -7 6]);
xlabel('n','FontSize',LFS); ylabel('x(n)','FontSize',LFS);
title('x(n) = sin(\pi \times n / 2)+5 \times cos(\pi \times n)',...
      'FontSize',TFS);
subplot(2,1,2); Hs = stem(n,y); set(Hs,'markersize',2); axis([-2 202 -2 4]);
xlabel('n','FontSize',LFS); ylabel('y(n)','FontSize',LFS);
title('Output sequence after filtering','FontSize',TFS);
print -deps2 ../epsfiles/P0421c;
```

The input and output sequence plots are shown in Figure 4.23. It shows that the sinusoidal sequence with the input frequency $\omega = \pi$ is completely suppressed in the steady-state output. The steady-state response of $x(n) = sin(\pi n/2)$ should be (using the magnitude and phase at $\omega = \pi/2$ computed in part 2. above)

$$\begin{aligned} y_{ss}(n) &= \sqrt{2}\sin(\pi n/2 - 45^\circ) = \sqrt{2}\cos(45^\circ)\sin(\pi n/2) - \sqrt{2}\sin(45^\circ)\cos(\pi n/2) \\ &= sin(\pi n/2) - cos(\pi n/2) = \{\ldots, \underset{\uparrow}{-1}, 1, 1, -1, -1, \ldots\} \end{aligned}$$

as verified in the bottom plot of Figure 4.23.

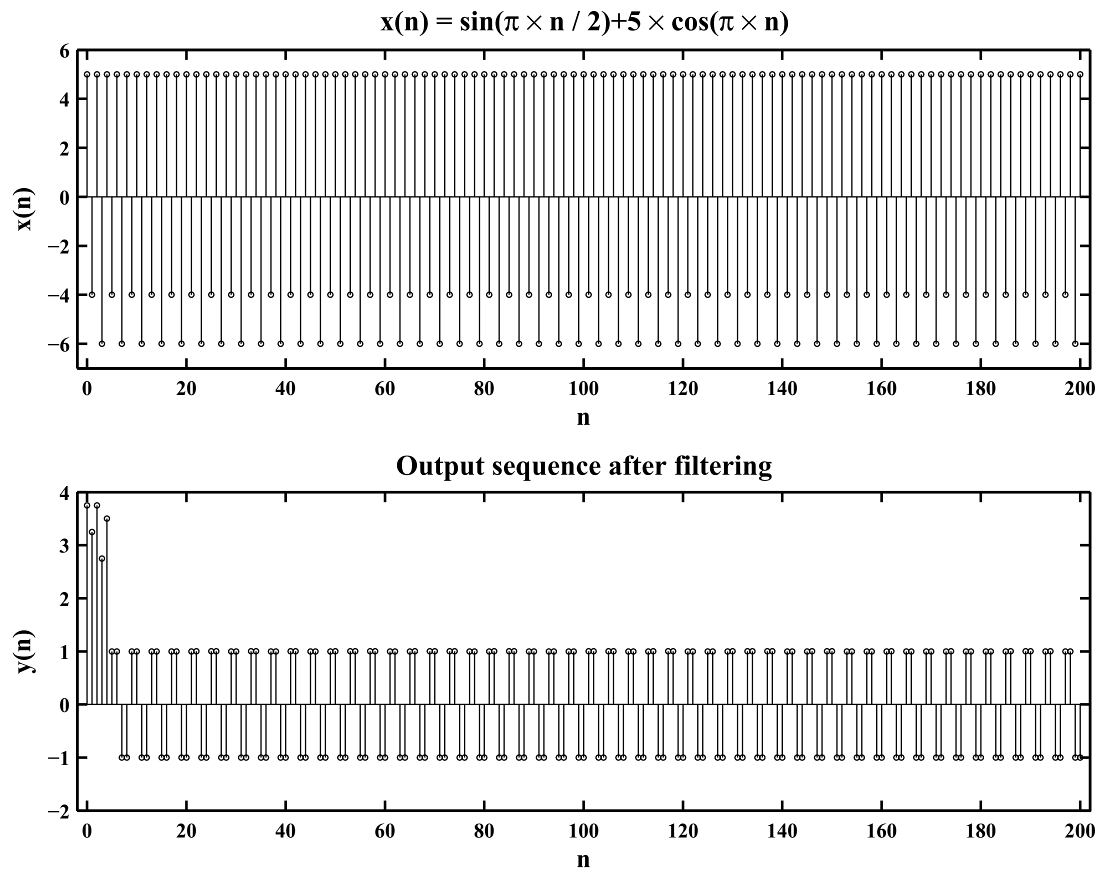


Figure 4.23: Problem P4.21.3 input and output sequence plots

P4.22 A digital filter is described by the frequency response function

$$H(e^{j\omega}) = \frac{1 + e^{-j4\omega}}{1 - 0.8145e^{-j4\omega}}$$

which after substituting $e^{-j\omega} = z^{-1}$ becomes

$$H(z) = \frac{1 + z^{-4}}{1 - 0.8145z^{-4}}$$

1. The difference equation representation: From $H(z)$ above

$$y(n) = x(n) + x(n-4) + 0.8145y(n-4)$$

2. The magnitude and phase response plots are shown in Figure 4.24.

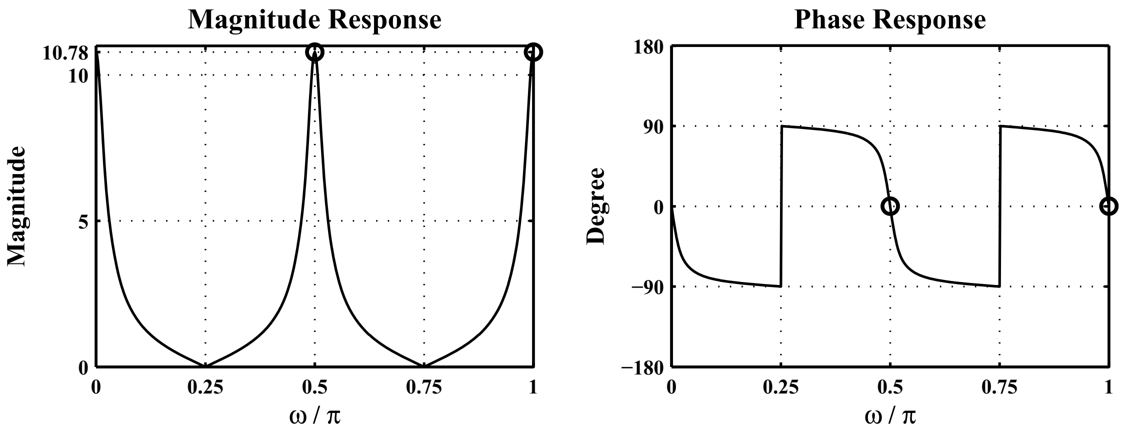


Figure 4.24: Problem P4.22.2 frequency-response plots

The magnitudes and phases at both $\omega = \pi/2$ and $\omega = \pi$ are 10.78 and 0°, respectively.

3. The output sequence $y(n)$ for the input $x(n) = \sin(\pi n/2) + 5\cos(\pi n)$: MATLAB script:

```
clc; close all;  set(0,'defaultfigurepaperposition',[0,0,7,5]);
b = [1 0 0 0 1]; a = [1 0 0 0 -0.8145];
n = 0:200; x = sin(pi*n/2)+5*cos(pi*n); y = filter(b,a,x);
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0422c');
subplot(2,1,1); Hs = stem(n,x); set(Hs,'markersize',2); axis([-2 202 -7 7]);
xlabel('n','FontSize',LFS); ylabel('x(n)','FontSize',LFS);
title('x(n) = sin(\pi \times n / 2)+5 \times cos(\pi \times n)',...
      'FontSize',TFS);
subplot(2,1,2); Hs = stem(n,y); set(Hs,'markersize',2); axis([-2 202 -70 70]);
xlabel('n','FontSize',LFS); ylabel('y(n)','FontSize',LFS);
title('Output sequence after filtering','FontSize',TFS);
print -deps2 ../epsfiles/P0422c;
```

The input and output sequence plots are shown in Figure 4.25.The steady-state response of $x(n)$ should be (using the magnitude and phase at $\omega = \pi/2$ computed in part 2. above)

$$\begin{aligned} y_{ss}(n) &= 10.78\sin(\pi n/2) + 10.78 \times 5\cos(\pi n) \\ &= 10.78 sin(\pi n/2) + 53.91 cos(\pi n/2) \end{aligned}$$

The bottom plot of Figure 4.25 shows that both sinusoidal sequences have the scaling of 10.78 and no delay distortion.

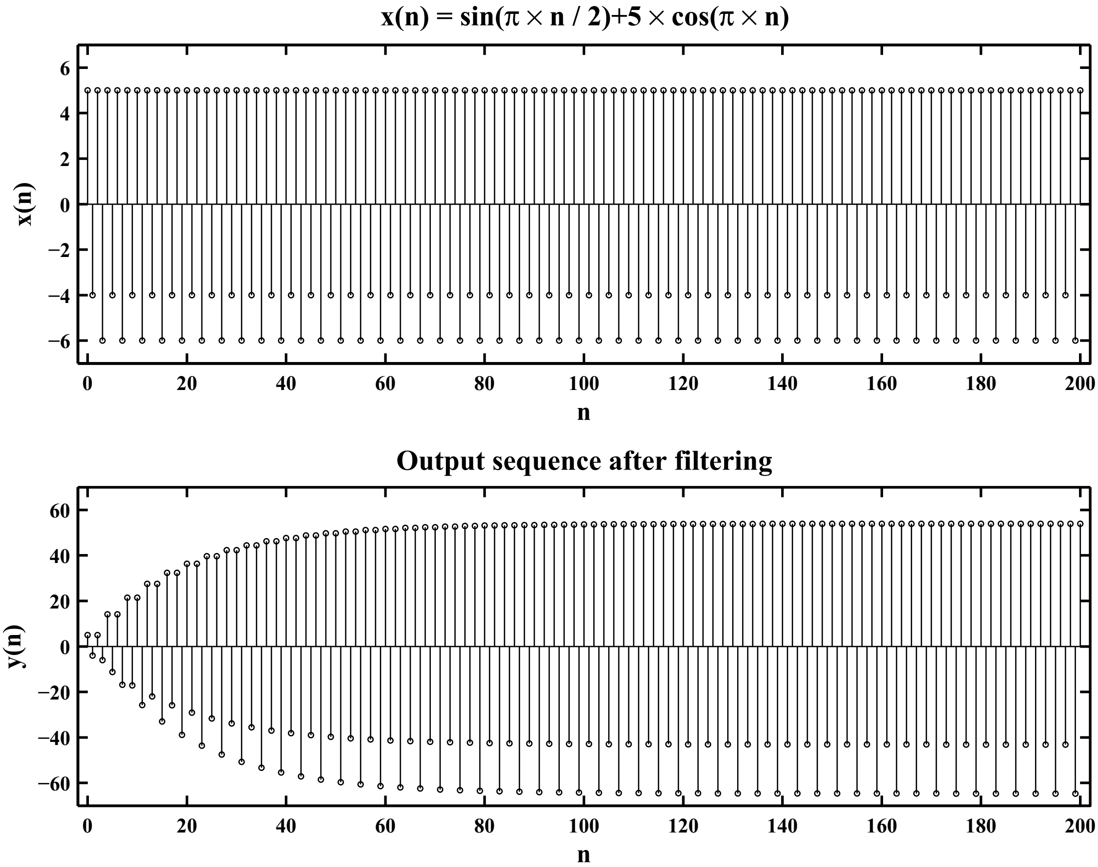


Figure 4.25: Problem P4.22.3 input and output sequence plots

P4.23 Difference equation solution using the one-sided z-transform approach.

$$\begin{aligned} y(n) &= 0.81y(n-2) + x(n) - x(n-1), \; n \ge 0; \quad y(-1) = 2, \; y(-2) = 2 \\ x(n) &= (0.7)^n u(n+1) \end{aligned}$$

Notice that

$$x(n) = (0.7)^n u(n+1) = \begin{cases} (0.7)^{-1}, & n = -1; \\ (0.7)^n u(n), & n \ge 0. \end{cases}$$

After taking the one-sided z-transform of the above difference equation, we obtain

$$\begin{aligned} Y^+(z) &= 0.81\left[y(-2) + y(-1)z^{-1} + z^{-2}Y^+(z)\right] + X^+(z) - \left[x(-1) + z^{-1}X^+(z)\right] \\ &= 0.81z^{-2}Y^+(z) + \left[1 - z^{-1}\right]X^+(z) + [0.81y(-2) - x(-1)] + 0.81y(-1)z^{-1} \end{aligned}$$

or

$$Y^+(z) = \frac{1 - z^{-1}}{1 - 0.81z^{-1}}X^+(z) + \frac{[0.81y(-2) - x(-1)] + 0.81y(-1)z^{-1}}{1 - 0.81z^{-1}}$$

After substituting the initial conditions and $X^+(z) = \mathcal{Z}\left[0.7^n u(n)\right] = \dfrac{1}{1 - 0.7z^{-1}}$, we obtain

$$\begin{aligned} Y^+(z) &= \left(\frac{1 - z^{-1}}{1 - 0.81z^{-1}}\right)\left(\frac{1}{1 - 0.7z^{-1}}\right) + \frac{0.1914 + 1.62z^{-1}}{1 - 0.81z^{-1}} \\ &= \frac{1.1914 + 0.4860z^{-1} - 1.1340z^{-2}}{\left(1 - 0.81z^{-1}\right)\left(1 - 0.7z^{-1}\right)} = 2 + \frac{0.4642}{1 - 0.81z^{-1}} + \frac{2.7273}{1 - 0.7z^{-1}} \end{aligned}$$

Hence upon inverse transformation

$$y(n) = 2\delta(n) + 0.4642(0.81)^n u(n) + 2.7273(0.7)^n u(n)$$

MATLAB verification:

```
clc; close all;
b1 = [1 -1]; nb1 = [0 1]; a11 = [1 0 -0.81]; na11 = [0 1 2]; a12 = [1 -0.7];
na12 = [0 1]; [a1,na1] = conv_m(a11,na11,a12,na12);
b2 = [0.1914 1.62]; nb2 = [0 1]; a2 = [1 0 -0.81]; na2 = [0 1 2];
[bnr1,nbnr1] = conv_m(b1,nb1,a2,na2); [bnr2,nbnr2] = conv_m(b2,nb2,a1,na1);
[b,nb] = sigadd(bnr1,nbnr1,bnr2,nbnr2); [a,na] = conv_m(a1,na1,a2,na2);
[R,p,k] = residuez(b,a);

 R =

   -0.2106-0.0000i
    0.0000
    0.7457+0.0000i
    0.0000-0.0000i
    0.6562

 p =
```

```
   -0.9000
   -0.9000
    0.9000+0.0000i
    0.9000-0.0000i
    0.7000

 k = []

n = [0:20]; x = 0.7.^n; xic = [0.1914 1.62];
yb1 = filter(b1,a11,x,xic);
yb2 = R(1)*((p(1)).^n)+R(3)*((p(3)).^n)+R(5)*((p(5)).^n);
error = max(abs(yb1-yb2))
error =
  6.2150e-008
```

P4.24 Difference equation solution for $y(n),\ n \geq 0$: Given

$$y(n) - 0.4y(n-1) - 0.45y(n-2) = 0.45x(n) + 0.4x(n-1) - x(n-2)$$

driven by the input $x(n) = 2 + \left(\frac{1}{2}\right)^n u(n)$ and subject to $y(-1) = 0,\ y(-2) = 3;\ x(-1) = x(-2) = 2$. Taking the one-sided z-transform of the difference equation, we obtain

$$\begin{aligned} &Y^+(z) - 0.4y(-1) - 0.4z^{-1}Y^+(z) - 0.45y(-2) - 0.45y(-1)z^{-1} - 0.45z^{-2}Y^+(z) \\ &= 0.45X^+(z) + 0.4x(-1) + 0.4z^{-1}X^+(z) - x(-2) - x(-1)z^{-1} - z^{-2}X^+(z) \end{aligned}$$

or

$$\begin{aligned} Y^+(z) = &\frac{0.45 + 0.4z^{-1} - z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}} X^+(z) \\ &+ \frac{[0.4y(-1) + 0.45y(-2) + 0.4x(-1) - x(-2)] + [0.45y(-1) - x(-1)]z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}} \end{aligned}$$

After substituting the initial conditions and $X^+(z) = \mathcal{Z}\left[[2 + \left(\frac{1}{2}\right)^n]u(n)\right] = \dfrac{2}{1 - z^{-1}} + \dfrac{1}{1 - 0.5z^{-1}}$, we obtain

$$\begin{aligned} Y^+(z) &= \left(\frac{0.45 + 0.4z^{-1} - z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}}\right)\left(\frac{2}{1 - z^{-1}} + \frac{1}{1 - 0.5z^{-1}}\right) + \frac{0.15 - 2z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}} \\ &= \frac{1.35 + 0.3z^{-1} - 3.8z^{-2} + 2z^{-3}}{\left(1 - 0.9z^{-1}\right)\left(1 + 0.5z^{-1}\right)\left(1 - z^{-1}\right)\left(1 - 0.5z^{-1}\right)} + \frac{0.15 - 2z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}} \qquad (4.13) \\ &= \frac{1.5 - 1.925z^{-1} - 0.725z^{-2} + z^{-3}}{1 - 1.9z^{-1} + 0.65z^{-2} + 0.475z^{-3} - 0.225z^{-4}} \\ &= \frac{-2}{1 - z^{-1}} + \frac{2.1116}{1 - 0.9z^{-1}} + \frac{1.7188}{1 - 0.5z^{-1}} - \frac{0.3304}{1 + 0.5z^{-1}} \end{aligned}$$

Hence after inverse transformation

$$y(n) = \left[-2 + 2.1116(0.9)^n + 1.7188(0.5)^n - 0.3303(-0.5)^n\right]u(n)$$

(a) Transient response: This response is due to the poles inside the unit circle or equivalently, the part of $y(n)$ that decays to zero as $n \nearrow \infty$. Thus

$$y_{\text{tr}}(n) = \left[2.1116(0.9)^n + 1.7188(0.5)^n - 0.3303(-0.5)^n\right]u(n)$$

(b) Steady-state response: This response is due to poles on the unit circle. Hence $y_{\text{ss}}(n) = -2$.

(c) Zero input response: In (4.13), the last term on the right corresponds to the initial condition or zero-input response. Hence

$$Y^+_{\text{ZI}}(z) = \frac{0.15 - 2z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}} = \frac{-1.3321}{1 - 0.9z^{-1}} + \frac{1.4821}{1 + 0.5z^{-1}}$$

or

$$y_{\text{ZI}}(n) = \left[-1.3321(0.9)^n + 1.4821(-0.5)^n\right]u(n)$$

(d) Zero-state response: In (4.13), the first term on the right corresponds to the input excitation or is the zero-state response. Hence

$$\begin{aligned} Y^+_{\text{ZS}}(z) &= \frac{1.35 + 0.3z^{-1} - 3.8z^{-2} + 2z^{-3}}{\left(1 - 0.9z^{-1}\right)\left(1 + 0.5z^{-1}\right)\left(1 - z^{-1}\right)\left(1 - 0.5z^{-1}\right)} \\ &= \frac{-2}{1 - z^{-1}} + \frac{3.4438}{1 - 0.9z^{-1}} + \frac{1.7187}{1 - 0.5z^{-1}} - \frac{1.8125}{1 + 0.5z^{-1}} \end{aligned}$$

or

$$y_{\text{ZS}}(n) = \left[-2 + 3.4438(0.9)^n + 1.7187(0.5)^n - 1.8125(-0.5)^n\right]u(n)$$

P4.25 A stable, linear and time-invariant system is given by the following system function

$$H(z) = \frac{4z^2 - 2\sqrt{2}z + 1}{z^2 - 2\sqrt{2}z + 4} = \frac{4 - 2\sqrt{2}z^{-1} + z^{-2}}{1 - 2\sqrt{2}z^{-1} + 4z^{-2}} \tag{4.14}$$

$$= \frac{\left(1 - 0.5e^{j45^\circ}z^{-1}\right)\left(1 - 0.5e^{-j45^\circ}z^{-1}\right)}{\left(1 - 2e^{j45^\circ}z^{-1}\right)\left(1 - 2e^{-j45^\circ}z^{-1}\right)}, \ |z| < 2 \text{ (for stability)} \tag{4.15}$$

$$= 0.25 + \frac{2.1866e^{-j30.96^\circ}}{1 - 2e^{j45^\circ}z^{-1}} + \frac{2.1866e^{j30.96^\circ}}{1 - 2e^{-j45^\circ}z^{-1}}, \ |z| < 2 \tag{4.16}$$

which is an anti-causal system.

1. The difference equation representation: From (4.14) above,

$$y(n) - 2\sqrt{2}y(n-1) + 4y(n-2) = 4x(n) - 2\sqrt{2}x(n-1) + x(n-2)$$

   Hence for anti-causal implementation

$$y(n) = \frac{1}{4}x(n) - \frac{1}{\sqrt{2}}x(n+1) + x(n+2) + \frac{1}{\sqrt{2}}y(n+1) - \frac{1}{4}y(n+2)$$

2. The pole-zero plot: From (4.15), the zeros are at $0.5e^{\pm 45^\circ}$ and poles are at $2e^{\pm 45^\circ}$. The pole-zero plot is shown in Figure 4.26.

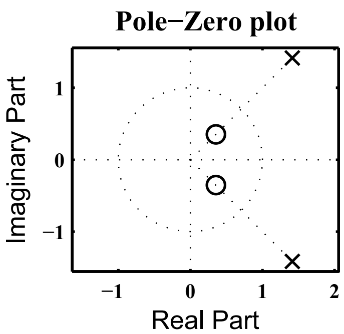


Figure 4.26: Problem P4.25 pole-zero plot

3. The unit sample response $h(n)$: From (4.16),

$$\begin{aligned}
h(n) &= 0.25\delta(n) - \left[2.1866e^{-j30.96^\circ}\left(2e^{j45^\circ}\right)^n + 2.1866e^{j30.96^\circ}\left(2e^{-j45^\circ}\right)^n\right]u(-n-1) \\
&= 0.25\delta(n) - 2.1866(2)^n\left[e^{-j30.96^\circ}e^{j\pi/4n} + e^{j30.96^\circ}e^{-j\pi/4n}\right]u(-n-1) \\
&= 0.25\delta(n) - 2.1866(2)^n\cos(\pi n/4 - 30.96^\circ)u(-n-1)
\end{aligned}$$

4. The system is anti-causal. The causal (but not stable) unit-sample response is given by the system function

$$H(z) = 0.25 + \frac{2.1866e^{-j30.96^\circ}}{1 - 2e^{j45^\circ}z^{-1}} + \frac{2.1866e^{j30.96^\circ}}{1 - 2e^{-j45^\circ}z^{-1}}, \ |z| > 2$$

   Hence

$$h(n) = 0.25\delta(n) + 2.1866(2)^n\cos(\pi n/4 - 30.96^\circ)u(n)$$

P4.26 The zero-input, zero-state, and steady-state responses of the system

$$y(n) = 0.9801y(n-2) + x(n) + 2x(n-1) + x(n-2), \ n \geq 0; \quad y(-2) = 1, \ y(-1) = 0$$

to the input $x(n) = 5(-1)^n u(n)$: After taking the one-sided z-transform of the above difference equation, we obtain

$$\begin{aligned} Y^+(z) = 0.9801\left[y(-2) + y(-1)z^{-1} + z^{-2}Y^+(z)\right] + X^+(z) + 2\left[x(-1) + z^{-1}X^+(z)\right] \\ + \left[x(-2) + x(-1)z^{-1} + z^{-2}X^+(z)\right] \end{aligned}$$

After substituting the initial conditions and $X(z) = \mathcal{Z}\left[5(-1)^n u(n)\right] = \dfrac{5}{1+z^{-1}}$, we obtain

$$\begin{aligned} Y^+(z) &= \frac{1+2z^{-1}+z^{-2}}{1-0.9801z^{-2}}X^+(z) + \frac{[2x(-1)+x(-2)+0.9801y(-2)] + [2+x(-1)]z^{-1}}{1-0.9801z^{-2}} \\ &= \left(\frac{1+2z^{-1}+z^{-2}}{1-0.9801z^{-2}}\right)\left(\frac{5}{1+z^{-1}}\right) + \frac{0.9801+2z^{-1}}{1-0.9801z^{-2}} \qquad (4.17) \\ &= \frac{5+10z^{-1}+5z^{-2}}{1+z^{-1}-0.9801z^{-2}-0.9801z^{-3}} + \frac{0.9801+2z^{-1}}{1-0.9801z^{-2}} \qquad (4.18) \\ &= \frac{5.9801+12.9801z^{-1}+7z^{-2}}{1+z^{-1}-0.9801z^{-2}-0.9801z^{-3}} \\ &= \frac{-0.5453}{1+0.99z^{-1}} + \frac{6.5254}{1-0.99z^{-1}} \qquad (4.19) \end{aligned}$$

Hence

$$y(n) = -0.5453(-0.99)^n u(n) + 6.5254(0.99)^n u(n)$$

(a) Zero-input response: From (4.17), we have

$$H_{\text{ZI}}(z) = \frac{0.9801+2z^{-1}}{1-0.9801z^{-2}} = \frac{-0.5201}{1+0.99z^{-1}} + \frac{1.5002}{1-0.99z^{-1}}$$

Hence

$$y_{\text{ZI}}(n) = -0.5201(-0.99)^n u(n) + 1.5002(0.99)^n u(n)$$

(b) Zero-state response: From (4.18), we have

$$H_{\text{ZS}}(z) = \frac{5+10z^{-1}+5z^{-2}}{1+z^{-1}-0.9801z^{-2}-0.9801z^{-3}} = \frac{-0.0253}{1+0.99z^{-1}} + \frac{5.0253}{1-0.99z^{-1}}$$

Hence

$$y_{\text{ZS}}(n) = -0.0253(-0.99)^n u(n) + 5.0253(0.99)^n u(n)$$

(c) Steady-state response: Since the total response $y(n)$ goes to zero as $n \nearrow \infty$, there is no steady-state response.