Chapter 5

The Discrete-Time Fourier Transform

P5.1 Compute the DFS coefficients of the following periodic sequences using the DFS definition and then verify your answers using MATLAB .

1. $\tilde{x}_1(n) = \{4, 1, -1, 1\}$, $N = 4$

```
xtilde1 = [4,1,-1,1]; N = 4; Xtilde1 = dfs(xtilde1,N)
Xtilde1 =
   5.0000              5.0000 + 0.0000i   1.0000 - 0.0000i   5.0000 + 0.0000i
```

2. $\tilde{x}_2(n) = \{2, 0, 0, 0, -1, 0, 0, 0\}$, $N = 8$

```
xtilde2 = [2,0,0,0,-1,0,0,0]; N = 8; Xtilde2 = dfs(xtilde2,N)
Xtilde2 =
  Columns 1 through 4
   1.0000              3.0000 + 0.0000i   1.0000 - 0.0000i   3.0000 + 0.0000i
  Columns 5 through 8
   1.0000 - 0.0000i   3.0000 + 0.0000i   1.0000 - 0.0000i   3.0000 + 0.0000i
```

3. $\tilde{x}_3(n) = \{1, 0, -1, -1, 0\}$, $N = 5$

```
xtilde3 = [1,0,-1,-1,0]; N = 5; Xtilde3 = dfs(xtilde3,N)
Xtilde3 =
  Columns 1 through 4
  -1.0000              2.6180 + 0.0000i   0.3820 - 0.0000i   0.3820
  Column 5
   2.6180 + 0.0000i
```

4. $\tilde{x}_4(n) = \{0, 0, 2j, 0, 2j, 0\}$, $N = 6$

```
xtilde4 = [0,0,2j,0,2j,0]; N = 6; Xtilde4 = dfs(xtilde4,N)
Xtilde4 =
  Columns 1 through 4
        0 + 4.0000i   0.0000 - 2.0000i   0.0000 - 2.0000i  -0.0000 + 4.0000i
  Columns 5 through 6
   0.0000 - 2.0000i   0.0000 - 2.0000i
```

5. $\tilde{x}_5(n) = \{3, 2, 1\}$, $N = 3$

```
xtilde5 = [3,2,1]; N = 3; Xtilde5 = dfs(xtilde5,N)
Xtilde5 =
   6.0000             1.5000 - 0.8660i   1.5000 + 0.8660i
```

P5.2 Determine the periodic sequences given the following periodic DFS coefficients. First use the IDFS definition and then verify your answers using MATLAB .

1. $\tilde{X}_1(k) = \{4, 3j, -3j\}, N = 3$

```
Xtilde1 = [4,j*3,-j*3]; N = 3; xtilde1 = idfs(Xtilde1,N)

xtilde1 =
   1.3333            -0.3987 + 0.0000i   3.0654 - 0.0000i
```

2. $\tilde{X}_2(k) = \{j, 2j, 3j, 4j\}, N = 4$

```
Xtilde2 = [j,j*2,j*3,j*4]; N = 4; xtilde2 = idfs(Xtilde2,N)

xtilde2 =
        0 + 2.5000i   0.5000 - 0.5000i  -0.0000 - 0.5000i  -0.5000 - 0.5000i
```

3. $\tilde{X}_3(k) = \{1, 2+3j, 4, 2-3j\}, N = 4$

```
Xtilde3 = [1,2+j*3,4,2-j*3]; N = 4; xtilde3 = idfs(Xtilde3,N)

xtilde3 =
   2.2500            -2.2500 + 0.0000i   0.2500             0.7500 - 0.0000i
```

4. $\tilde{X}_4(k) = \{0, 0, 2, 0\}, N = 5$

```
Xtilde4 = [0,0,2,0,0]; N = 5; xtilde4 = idfs(Xtilde4,N)

xtilde4 =
  Columns 1 through 4
   0.4000            -0.3236 + 0.2351i   0.1236 - 0.3804i   0.1236 + 0.3804i
  Column 5
  -0.3236 - 0.2351i
```

5. $\tilde{X}_5(k) = \{3, 0, 0, 0, -3, 0, 0, 0\}, N = 8$

```
Xtilde5 = [3,0,0,0,-3,0,0,0]; N = 8; xtilde5 = idfs(Xtilde5,N)

xtilde5 =
  Columns 1 through 4
        0             0.7500 - 0.0000i        0 + 0.0000i   0.7500 - 0.0000i
  Columns 5 through 8
        0 + 0.0000i   0.7500 - 0.0000i        0 + 0.0000i   0.7500 - 0.0000i
```

P5.3 Let $\tilde{x}_1(n)$ be periodic with fundamental period $N = 40$ where one period is given by

$$\tilde{x}_1(n) = \begin{cases} 5\sin(0.1\pi n), & 0 \le n \le 19 \\ 0, & 20 \le n \le 39 \end{cases}$$

and let $\tilde{x}_2(n)$ be periodic with fundamental period $N = 80$, where one period is given by

$$\tilde{x}_2(n) = \begin{cases} 5\sin(0.1\pi n), & 0 \le n \le 19 \\ 0, & 20 \le n \le 79 \end{cases}$$

These two periodic sequences differ in their periodicity but otherwise have the same non-zero samples.

1. Computation of $\tilde{X}_1(k)$ using MATLAB :

```
n1 = [0:39]; xtilde1 = [5*sin(0.1*pi*[0:19]),zeros(1,20)]; N1 = length(n1);
[Xtilde1] = dft(xtilde1,N1); k1 = n1;
mag_Xtilde1 = abs(Xtilde1); pha_Xtilde1 = angle(Xtilde1)*180/pi;
zei = find(mag_Xtilde1 < 1000*eps);
pha_Xtilde1(zei) = zeros(1,length(zei));
Hf_1 = figure('Units','normalized','position',[0.1,0.1,0.8,0.8],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,5]);
set(Hf_1,'NumberTitle','off','Name','P5.3.1');
subplot(3,1,1); H_s1 = stem(n1,xtilde1,'filled'); set(H_s1,'markersize',3);
axis([-1,N1,-6,6]);
title('One period of the periodic sequence xtilde_1(n)','fontsize',10);
ntick = [n1(1):2:n1(N1),N1]'; ylabel('Amplitude');
set(gca,'XTickMode','manual','XTick',ntick,'FontSize',8);
subplot(3,1,2); H_s2 = stem(k1,mag_Xtilde1,'filled'); set(H_s2,'markersize',3);
axis([-1,N1,0,max(mag_Xtilde1)+10]);
title('Magnitude of Xtilde_1(k)','fontsize',10); ylabel('Magnitude');
ktick = [k1(1):2:k1(N1),N1]';
set(gca,'XTickMode','manual','XTick',ktick,'FontSize',8)
subplot(3,1,3); H_s3 = stem(k1,pha_Xtilde1,'filled'); set(H_s3,'markersize',3);
title('Phase of Xtilde_1(k)','fontsize',10); xlabel('k'); ylabel('Degrees');
ktick = [k1(1):2:k1(N1),N1]'; axis([-1,N1,-200,200]);
set(gca,'XTickMode','manual','XTick',ktick,'FontSize',8);
set(gca,'YTickMode','manual','YTick',[-180;-90;0;90;180]);
```

   Plots of $\tilde{x}_1(n)$ and $\tilde{X}_1(k)$ are shown in Figure 5.1.

2. Computation of $\tilde{X}_2(k)$ using MATLAB :

```
n2 = [0:79]; xtilde2 = [xtilde1, zeros(1,40)]; N2 = length(n2);
[Xtilde2] = dft(xtilde2,N2); k2 = n2;
mag_Xtilde2 = abs(Xtilde2); pha_Xtilde2 = angle(Xtilde2)*180/pi;
zei = find(mag_Xtilde2 < 1000*eps);
pha_Xtilde2(zei) = zeros(1,length(zei));
Hf_2 = figure('Units','normalized','position',[0.1,0.1,0.8,0.8],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,5]);
set(Hf_2,'NumberTitle','off','Name','P5.3.2');
subplot(3,1,1); H_s1 = stem(n2,xtilde2,'filled'); set(H_s1,'markersize',3);
title('One period of the periodic sequence xtilde2(n)','fontsize',10);
```

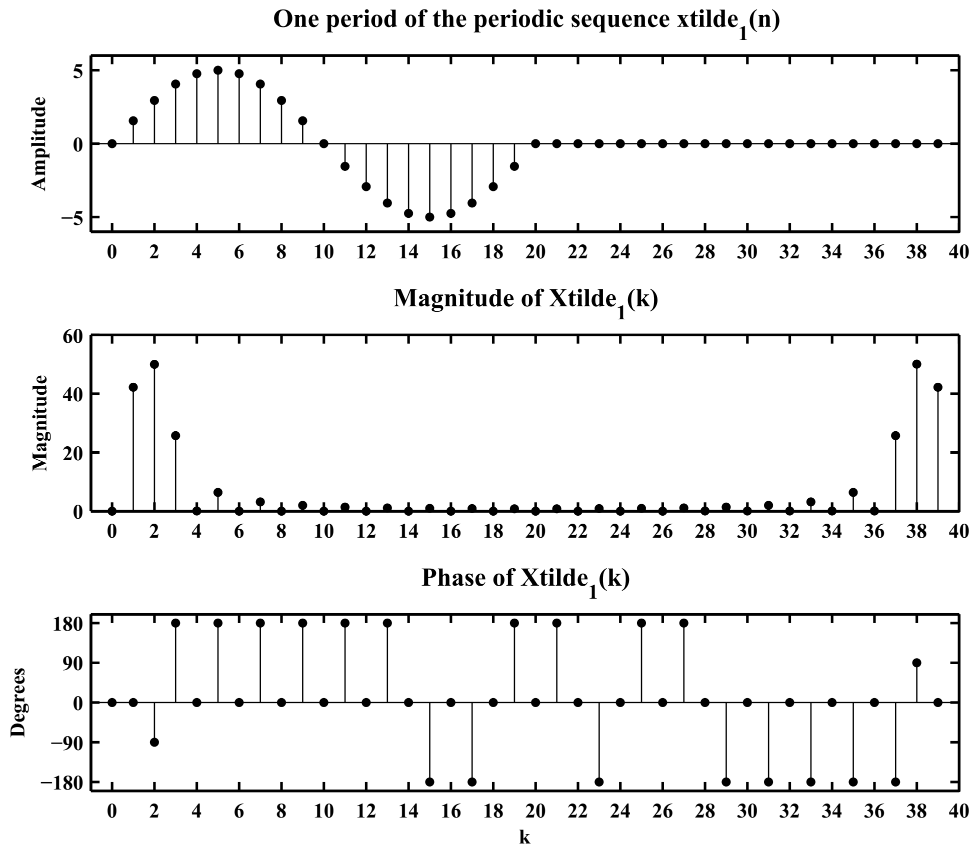


Figure 5.1: Plots of $\tilde{x}_1(n)$ and $\tilde{X}_1(k)$ in Problem 5.3a

```
ntick = [n2(1):5:n2(N2),N2]'; ylabel('xtilde2'); axis([-1,N2,-6,6]);
set(gca,'XTickMode','manual','XTick',ntick)
subplot(3,1,2); H_s2 = stem(k2,mag_Xtilde2,'filled'); set(H_s2,'markersize',3);
axis([-1,N2,0,60]);
title('Magnitude of Xtilde2(k)','fontsize',10); ylabel('|Xtilde2|')
ktick = [k2(1):5:k2(N2),N2]';
set(gca,'XTickMode','manual','XTick',ktick)
subplot(3,1,3); H_s3 = stem(k2,pha_Xtilde2,'filled'); set(H_s3,'markersize',3);
title('Phase of Xtilde2(k)','fontsize',10); xlabel('k'); ylabel('Degrees')
ktick = [k2(1):5:k2(N2),N2]'; axis([-1,N2,-200,200]);
set(gca,'XTickMode','manual','XTick',ktick)
set(gca,'YTickMode','manual','YTick',[-180;-90;0;90;180])
```

Plots of $\tilde{x}_2(n)$ and $\tilde{X}_2(k)$ are shown in Figure 5.2.

3. Changing the period from $N = 40$ to $N = 80$ resulted in a lower frequency sampling interval (higher frequency resolution) ω_1, i.e., in (5) $\omega_1 = \pi/20$ and in (5) $\omega_2 = \pi/40$. Hence there are more terms in the DFS expansion of $\tilde{x}_2(n)$. The shape of the DTFT begins to fill in with $N = 80$.

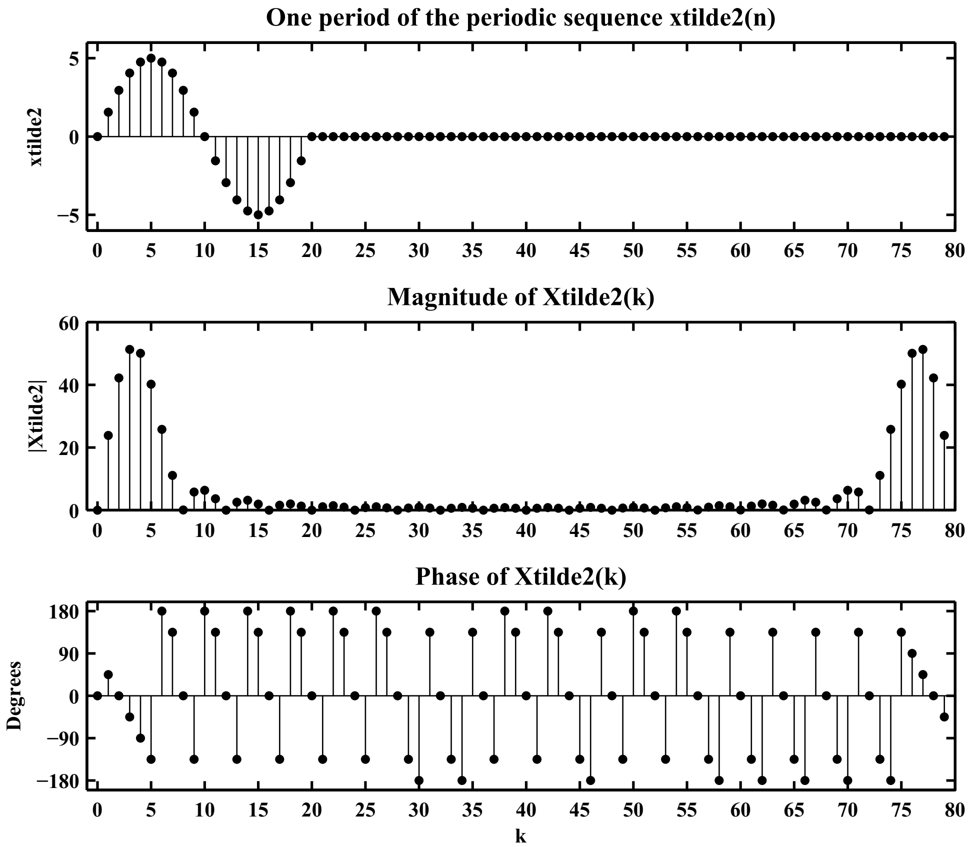


Figure 5.2: Plots of Magnitude and Phase of $\tilde{X}_2(k)$ in Problem 5.3b

P5.4 Consider the periodic sequence $\tilde{x}_1(n)$ given in Problem 5.3. Let $\tilde{x}_2(n)$ be periodic with fundamental period $N = 40$, where one period is given by

$$\tilde{x}_2(n) = \begin{cases} \tilde{x}_1(n), & 0 \le n \le 19 \\ -\tilde{x}_1(n-20), & 20 \le n \le 39 \end{cases}$$

1. Determine analytically the DFS $\tilde{X}_2(k)$ in terms of $\tilde{X}_1(k)$.
2. Computation of the DFS $\tilde{X}_2(k)$ using MATLAB :

```
n1 = [0:19]; xtilde1 = [5*sin(0.1*pi*n1)];
n2 = [0:39]; xtilde2 = [xtilde1, -xtilde1]; N2 = length(n2);
[Xtilde2] = dft(xtilde2,N2); k2 = n2;
mag_Xtilde2 = abs(Xtilde2); pha_Xtilde2 = angle(Xtilde2)*180/pi;
zei = find(mag_Xtilde2 < 1000*eps);
pha_Xtilde2(zei) = zeros(1,length(zei));
Hf_1 = figure('Units','normalized','position',[0.1,0.1,0.8,0.8],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_1,'NumberTitle','off','Name','P5.4.2');
subplot(3,1,1); H_s1 = stem(n2,xtilde2,'filled'); set(H_s1,'markersize',3);
axis([-1,N2,-6,6]);
title('One period of the periodic sequence xtilde_2(n)','fontsize',10);
ntick = [n2(1):5:n2(N2),N2]'; ylabel('Amplitude');
set(gca,'XTickMode','manual','XTick',ntick)
subplot(3,1,2); H_s2 = stem(k2,mag_Xtilde2,'filled'); set(H_s2,'markersize',3);
axis([-1,N2,0,100]);
title('Magnitude of Xtilde2(k)','fontsize',10); ylabel('Magnitude')
ktick = [k2(1):5:k2(N2),N2]';
set(gca,'XTickMode','manual','XTick',ktick)
subplot(3,1,3); H_s3 = stem(k2,pha_Xtilde2,'filled'); set(H_s3,'markersize',3);
title('Phase of Xtilde2(k)','fontsize',10); xlabel('k'); ylabel('Degrees')
ktick = [k2(1):5:k2(N2),N2]'; axis([-1,N2,-200,200]);
set(gca,'XTickMode','manual','XTick',ktick)
set(gca,'YTickMode','manual','YTick',[-180;-90;0;90;180])
```

   Plots of $\tilde{x}_2(n)$ and $\tilde{X}_2(k)$ are shown in Figure 5.3.
3. Verify your answer in part 1 above using the plots of $\tilde{X}_1(k)$ and $\tilde{X}_2(k)$?

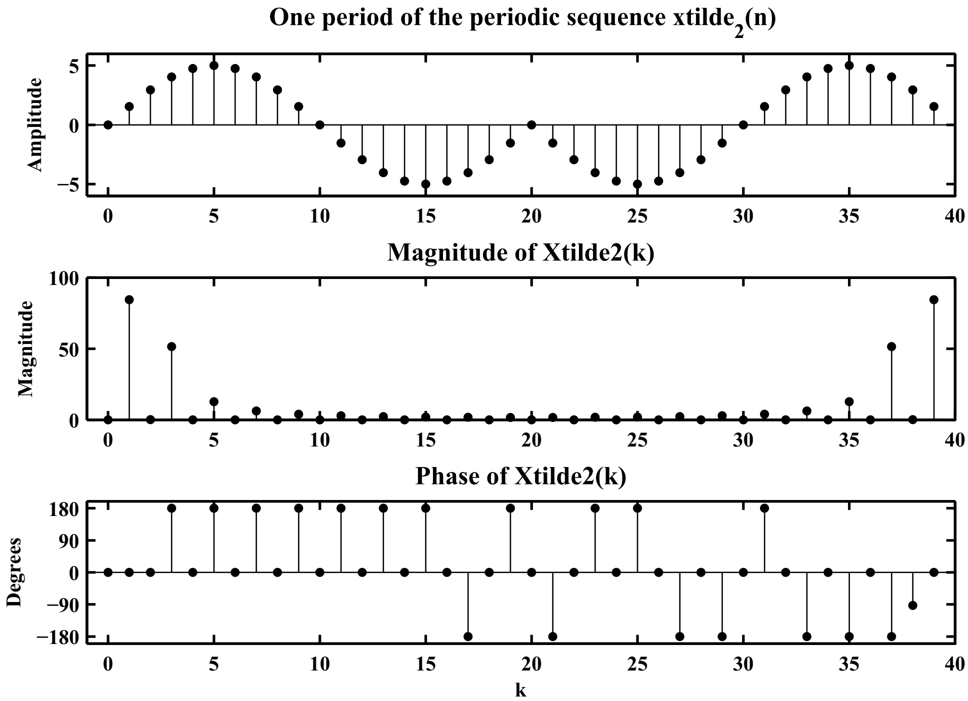


Figure 5.3: Plots of Magnitude and Phase of $\tilde{X}_2(k)$ in Problem 5.4b

P5.5 Consider the periodic sequence $\tilde{x}_1(n)$ given in Problem 5.3. Let $\tilde{x}_3(n)$ be periodic with period 80, obtained by concatenating two periods of $\tilde{x}_1(n)$, i.e.,

$$\tilde{x}_3(n) = \left[\tilde{x}_1(n), \tilde{x}_1(n)\right]_{\text{PERIODIC}}$$

Clearly, $\tilde{x}_3(n)$ is different from $\tilde{x}_2(n)$ of Problem 5.3 even though both of them are periodic with period 80.

1. Computation and plot of the DFS $\tilde{X}_3(k)$ using MATLAB :

```
n1 = [0:39]; xtilde1 = [5*sin(0.1*pi*[0:19]),zeros(1,20)];
n3 = [0:79]; xtilde3 = [xtilde1, xtilde1]; N3 = length(n3);
[Xtilde3] = dft(xtilde3,N3); k3 = n3;
mag_Xtilde3 = abs(Xtilde3); pha_Xtilde3 = angle(Xtilde3)*180/pi;
zei = find(mag_Xtilde3 < 0.00001);
pha_Xtilde3(zei) = zeros(1,length(zei));
Hf_1 = figure('Units','normalized','position',[0.1,0.1,0.8,0.8],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,5]);
set(Hf_1,'NumberTitle','off','Name','P5.5.1');
subplot(3,1,1); H_s1 = stem(n3,xtilde3,'filled'); set(H_s1,'markersize',3);
title('One period of the periodic sequence xtilde_3(n)','fontsize',10);
ylabel('Amplitude'); ntick = [n3(1):5:n3(N3),N3]';axis([-1,N3,-6,6]);
set(gca,'XTickMode','manual','XTick',ntick,'fontsize',8)
subplot(3,1,2); H_s2 = stem(k3,mag_Xtilde3,'filled'); set(H_s2,'markersize',3);
axis([-1,N3,min(mag_Xtilde3),max(mag_Xtilde3)]);
title('Magnitude of Xtilde_3(k)','fontsize',10); ylabel('Magnitude')
ktick = [k3(1):5:k3(N3),N3]'; set(gca,'XTickMode','manual','XTick',ktick)
subplot(3,1,3); H_s3 = stem(k3,pha_Xtilde3,'filled'); set(H_s3,'markersize',3);
title('Phase of Xtilde3(k)','fontsize',10); xlabel('k'); ylabel('Degrees');
ktick = [k3(1):5:k3(N3),N3]';axis([-1,N3,-180,180]);
set(gca,'XTickMode','manual','XTick',ktick)
set(gca,'YTickMode','manual','YTick',[-180;-90;0;90;180])
```

   Plots of $\tilde{x}_3(n)$ and $\tilde{X}_3(k)$ are shown in Figure 5.4.

2. Comparing the magnitude plot above with that of $\tilde{X}_1(k)$ in Problem (5), we observe that these plots are essentially similar. Plots of $\tilde{X}_3(k)$ have one zero between every sample of $\tilde{X}_1(k)$. (In general, for phase plots, we do get non-zero phase values when the magnitudes are zero. Clearly these phase values have no meaning and should be ignored. This happens because of a particular algorithm used by MATLAB. We avoided this problem by using the `find` function.) This makes sense because sequences $\tilde{x}_1(n)$ and $\tilde{x}_3(n)$, when viewed over $-\infty < n < \infty$ interval, look exactly same. The effect of periodicity doubling is in the doubling of magnitude of each sample.

1. We can now generalize this argument. If

$$\tilde{x}_M(n) = \left\{ \underbrace{\tilde{x}_1(n), \tilde{x}_1(n), \dots, \tilde{x}_1(n)}_{M \text{ times}} \right\}_{\text{PERIODIC}}$$

   then there will be $(M-1)$ zeros between samples of $\tilde{X}_M(k)$. The magnitudes of non-zero samples of $\tilde{X}_M(k)$ will be M times the magnitudes of the samples of $\tilde{X}_1(k)$, i.e.,

$$\begin{aligned} \tilde{X}_M(Mk) &= M\tilde{X}_1(k) &&, k = 0, 1, \dots, N-1 \\ \tilde{X}_M(k) &= 0 &&, k \neq 0, 1, \dots, MN \end{aligned}$$

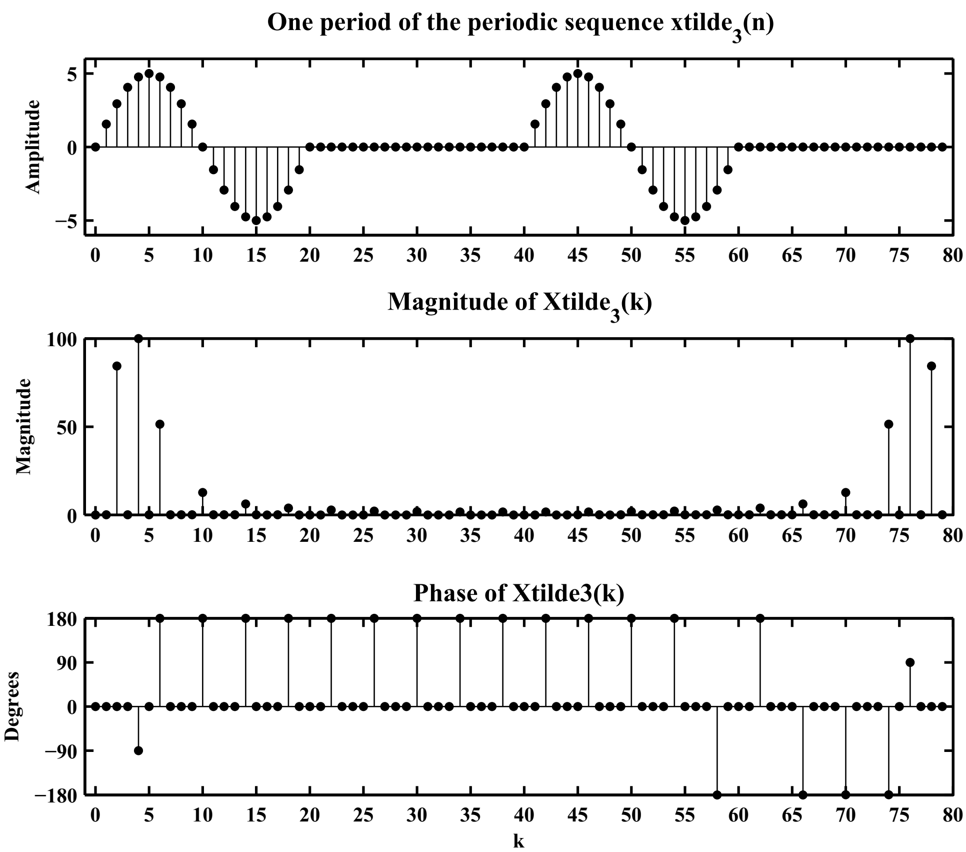


Figure 5.4: Plots of $\tilde{x}_3(n)$ and $\tilde{X}_3(k)$ in Problem 5.5a

P5.6 Let $X(e^{j\omega})$ be the DTFT of a finite-length sequence

$$x(n) = \begin{cases} n+1, & 0 \le n \le 49; \\ 100-n, & 50 \le n \le 99; \\ 0, & \text{otherwise.} \end{cases}$$

1. Let
$$y_1(n) = \overset{10\text{-point}}{\text{IDFS}} \left[X(e^{j0}), X(e^{j2\pi/10}), X(e^{j4\pi/10}), \dots, X(e^{j18\pi/10})\right]$$
which is a 10-point IDFS of ten samples of $X\left(e^{j\omega}\right)$ on the unit circle. Thus
$$\begin{aligned} y_1(n) = \sum_{r=-\infty}^{\infty} x(n-10r) &= \{1+11+\cdots+41+50+40+\cdots+10, \\ &\qquad 2+12+\cdots+42+49+\cdots+9, \cdots\}_{\text{periodic}} \\ &= \{255, 255, \dots, 255\}_{\text{periodic}} \end{aligned}$$
MATLAB verification:
```
n = 0:99; x = [n(1:50)+1,100-n(51:100)];
N1 = 10; k1 = 0:N1-1; w1 = 2*pi*k1/N1;
Y1 = dtft(x,n,w1); y1 = real(idfs(Y1,N1));
```
See the `stem` plot of $y_1(n)$ in Figure 5.5.
2. Let
$$y_2(n) = \overset{200\text{-point}}{\text{IDFS}} \left[X(e^{j0}), X(e^{j2\pi/200}), X(e^{j4\pi/200}), \dots, X(e^{j398\pi/200})\right]$$
which is a 200-point IDFS of 200 samples of $X\left(e^{j\omega}\right)$ on the unit circle. Thus
$$y_2(n) = \left\{ \begin{array}{ll} x(n), & 0 \le n \le 49; \\ 0, & 50 \le n \le 100. \end{array} \right\}_{\text{periodic}}$$
MATLAB verification:
```
n = 0:99; x = [n(1:50)+1,100-n(51:100)];
N2 = 200; k2 = 0:N2-1; w2 = 2*pi*k2/N2;
Y2 = dtft(x,n,w2); y2 = real(idfs(Y2,N2));
```
See the `stem` plot of $y_1(n)$ in Figure 5.5.
.
3. The sequence $y_1(n)$ is a 10-point aliasing version on $x(n)$ while $y_2(n)$ is a zero-padded version of $x(n)$.

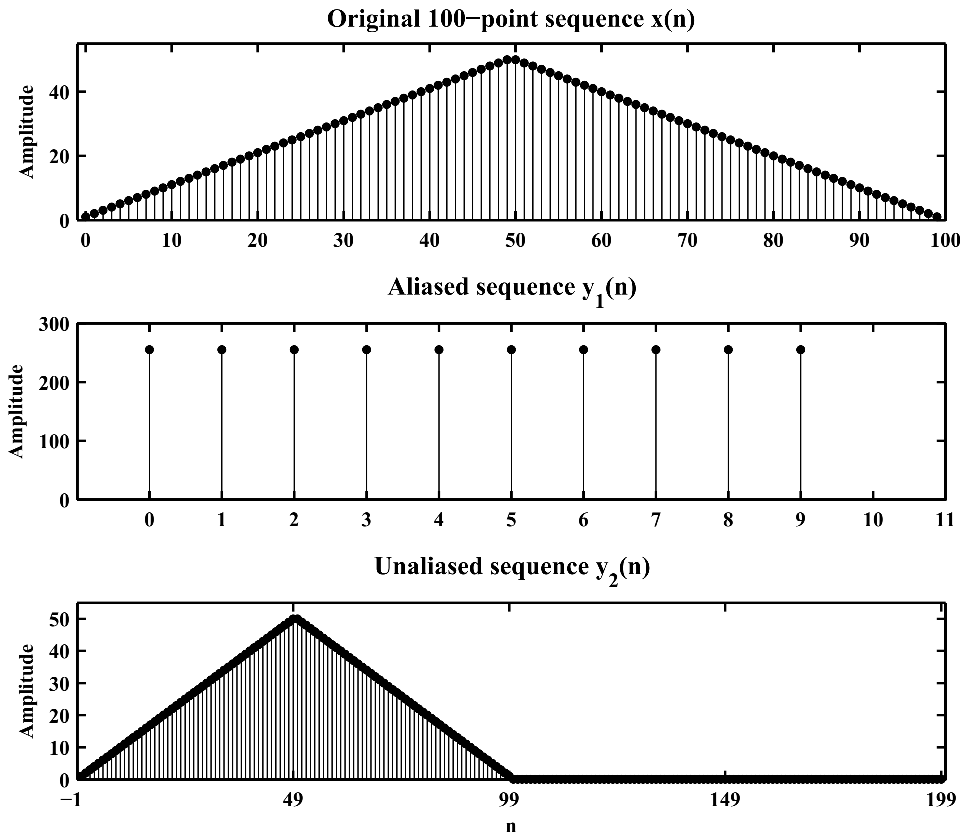


Figure 5.5: Plots of $y_1(n)$ and $y_2(k)$ in Problem 5.6

P5.7 Let $\tilde{x}(n)$ be a periodic sequence with period N and let

$$\tilde{y}(n) \triangleq \tilde{x}(-n) = \tilde{x}(N-n)$$

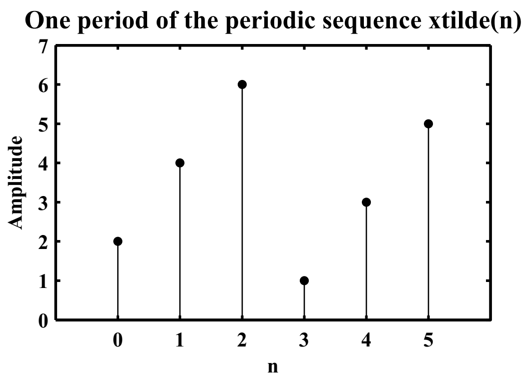
that is, $\tilde{y}(n)$ is a periodically folded version of $\tilde{x}(n)$. Let $\tilde{X}(k)$ and $\tilde{Y}(k)$ be the DFS sequences.

1. Consider

$$\begin{aligned}\tilde{Y}(k) &= \text{DFS}\left[\tilde{y}(n)\right] = \sum_{n=0}^{N-1} \tilde{y}(n) W_N^{nk} = \sum_{n=0}^{N-1} \tilde{x}(-n) W_N^{nk} = \sum_{n=0}^{N-1} \tilde{x}(N-n) W_N^{nk} \\ &= \sum_{\ell=1}^{N} \tilde{x}(\ell) W_N^{(N-\ell)k} = \sum_{\ell=0}^{N-1} \tilde{x}(\ell) W_N^{Nk} W_N^{-\ell k} = \sum_{\ell=0}^{N-1} \tilde{x}(\ell) W_N^{-\ell k} \quad (\because \text{ periodic}) \\ &= \tilde{X}(-k) = \tilde{X}(N-k)\end{aligned}$$

2. Let $\tilde{x}(n) = \{\underset{\uparrow}{2}, 4, 6, 1, 3, 5\}_{\text{PERIODIC}}$ with $N = 6$.

   (a) Sketch of $\tilde{y}(n)$ for $0 \leq n \leq 5$:



   (b) Computation of $\tilde{X}(k)$ for $0 \leq k \leq 5$:

```
X_tilde = dft(x_tilde,N)
X_tilde =
  Columns 1 through 4
  21.0000             1.0000 - 1.7321i  -6.0000 + 3.4641i   1.0000 - 0.0000i
  Columns 5 through 6
  -6.0000 - 3.4641i   1.0000 + 1.7321i
```

   (c) Computation of $\tilde{Y}(k)$ for $0 \leq k \leq 5$:

```
y_tilde = [x_tilde(1),fliplr(x_tilde(2:end))];
Y_tilde = dft(y_tilde,N)
Y_tilde =
  Columns 1 through 4
  21.0000             1.0000 + 1.7321i  -6.0000 - 3.4641i   1.0000 + 0.0000i
  Columns 5 through 6
  -6.0000 + 3.4641i   1.0000 - 1.7321i
```

   (d) MATLAB verification:

```
W_tilde = [X_tilde(1),fliplr(X_tilde(2:end))];
error = max(abs(Y_tilde - W_tilde))
error =
  2.5434e-014
```

P5.8 Consider the finite-length sequence given below.

$$x(n) = \begin{cases} \text{sinc}^2\{(n-50)/2\}, & 0 \le n \le 100; \\ 0, & \text{else.} \end{cases}$$

1. DFT $X(k)$:

```
n = 0:100; xn = sinc((n-50)/2).^2; N = length(xn); % given signal x(n)
Xk = dft(xn,N); k = 0:N-1;                         % DFT of x(n)
mag_Xk = abs(Xk); pha_Xk = angle(Xk)*180/pi;       % Mag and Phase of X(k)
zei = find(mag_Xk < 0.00001);                      % Set phase values to
pha_Xk(zei) = zeros(1,length(zei));                % zero when mag is zero

Hf_1 = figure('Units','normalized','position',[0.1,0.1,0.8,0.8],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,5]);
set(Hf_1,'NumberTitle','off','Name','P5.8');
subplot(2,1,1); H_s1 = stem(k,mag_Xk,'filled'); set(H_s1,'markersize',3);
set(gca,'XTick',[0:20:N],'fontsize',8); axis([0,N,0,2.5])
set(gca,'YTick',[0:0.5:2.5],'fontsize',8); ylabel('Magnitude');
title('Magnitude plots of DFT and DTFT','fontsize',10); hold on
subplot(2,1,2); H_s2 = stem(k,pha_Xk,'filled'); set(H_s2,'markersize',3);
set(gca,'XTick',[0:20:N],'fontsize',8); axis([0,N,-200,200])
set(gca,'YTick',[-180;-90;0;90;180],'fontsize',8);
xlabel('k'); ylabel('Degrees');
title('Phase plots of DFT and DTFT','fontsize',10); hold on
```

   The `stem` plot of $X(k)$ is shown in 5.6.

2. DTFT $X\left(e^{j\omega}\right)$:

```
[X,w] = freqz(xn,1,1000,'whole');             % DTFT of xn
mag_X = abs(X); pha_X = angle(X)*180/pi;      % mag and phase of DTFT
Dw = (2*pi)/N;                                % frequency resolution
subplot(2,1,1); plot(w/Dw,mag_X); grid
hold off
subplot(2,1,2); plot(w/Dw,pha_X); grid
hold off
```

   The continuous plot of $X\left(e^{j\omega}\right)$ is also shown in Figure 5.6.

3. Clearly, the DFT in part 1. is the sampled version of $X\left(e^{j\omega}\right)$.

4. It is possible to reconstruct the DTFT from the DFT if length of the DFT is larger than or equal to the length of sequence $x(n)$. We can reconstruct using the complex interpolation formula

$$X\left(e^{j\omega}\right) = \sum_{k=0}^{N-1} X(k)\,\phi\left(\omega - \frac{2\pi k}{N}\right), \quad \text{where} \quad \phi(\omega) = e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{N\sin(\omega/2)}$$

   For $N = 101$, we have

$$X\left(e^{j\omega}\right) = \sum_{k=0}^{100} X(k)\, e^{-j(50)\omega}\frac{\sin(50.5\omega)}{101\sin(\omega/2)}$$

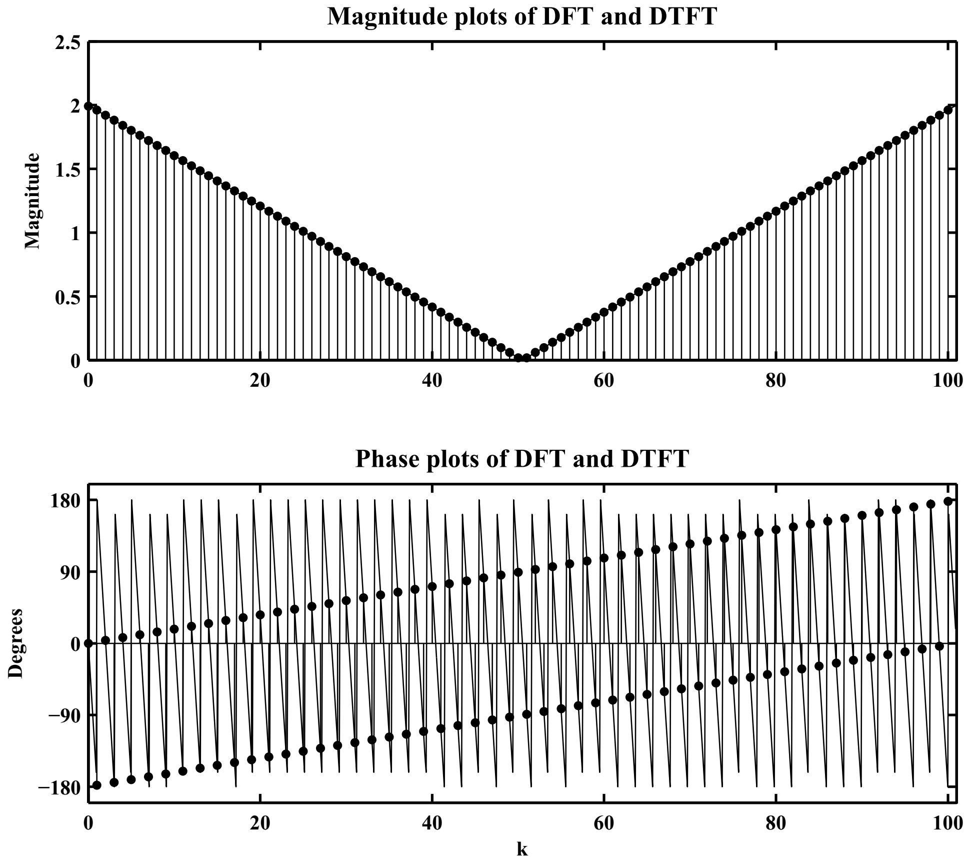


Figure 5.6: Plots of DTFT and DFT of signal in Problem 5.8

P5.9 The DTFT $X(e^{j\omega})$ of the following finite-length sequence using DFT as a computation tool

$$x(n) = \begin{cases} 2e^{-0.9|n|}, & -5 \leq n \leq 5; \\ 0, & \text{otherwise.} \end{cases}$$

MATLAB script:

```
clc; close all;
n = -5:5; xn = 2*exp(-0.9*abs(n)); N1 = length(xn);
N = 201; xn = [xn,zeros(1,N-N1)]; Xk = dft(xn,N); Xk = real(Xk);
w = linspace(-pi,pi,N); Xk = fftshift(Xk);
Hf_1 = figure('Units','normalized','position',[0.1,0.1,0.8,0.8],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,3]);
set(Hf_1,'NumberTitle','off','Name','P5.9');
plot(w/pi,Xk,'g','linewidth',1.5); axis([-1,1,-4,5]); hold on;
plot([-1,1],[0,0],'w',[0,0],[-4,5],'w','linewidth',0.5);
title('DTFT of x(n) = 2e^{-0.9|n|}, -5\leq n\leq 5','fontsize',10);
xlabel('\omega/\pi','fontsize',10); ylabel('Amplitude','fontsize',10);
```

The plot of the DTFT $X(e^{j\omega})$ is shown in 5.7.

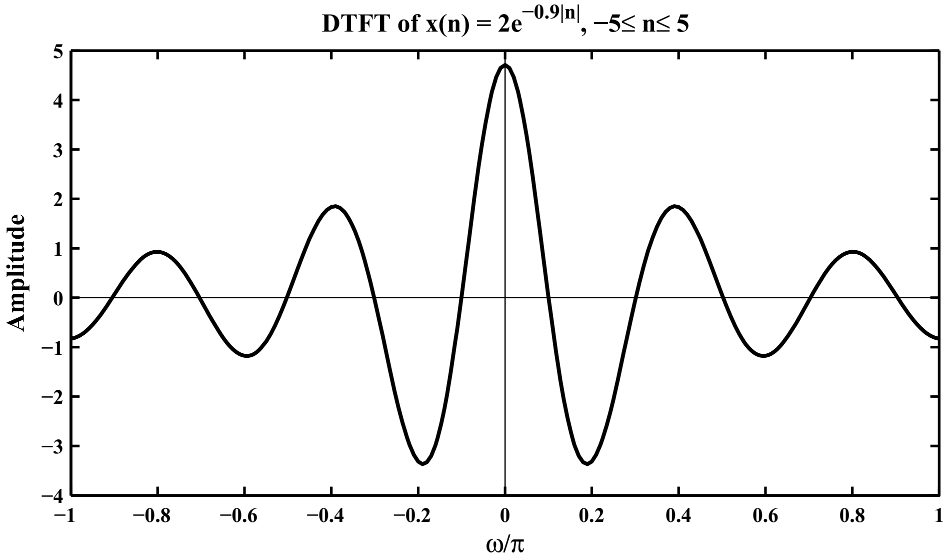


Figure 5.7: Plots of DTFT and DFT of signal in Problem 5.9

P5.10 Plot of the DTFT magnitude and angle of each of the following sequences using the DFT as a computation tool.

1. $x_1(n) = (0.6)^{|n|}[u(n+10) - u(n-10)]$. MATLAB script:

```
n1 = [-10:10]; x1 = (0.6).^abs(n1); N1 = length(n1); N = 200;   % Length of DFT
x1 = [x1(11:end), zeros(1,N-N1), x1(1:10)]; % Assemble x1
[X1] = fft(x1,N); w = (0:N/2)*2*pi/N;
mag_X1 = abs(X1(1:N/2+1)); pha_X1 = angle(X1(1:N/2+1))*180/pi;
Hf_1 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_1,'NumberTitle','off','Name','P5.10.1');
subplot(2,1,1); plot(w/pi,mag_X1,'g','linewidth',1); axis([0,1,0,11]);
title('Magnitude of DTFT X_1(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X1,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_1(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510a
```

The plot of the DTFT $X_1(e^{j\omega})$ is shown in 5.8.

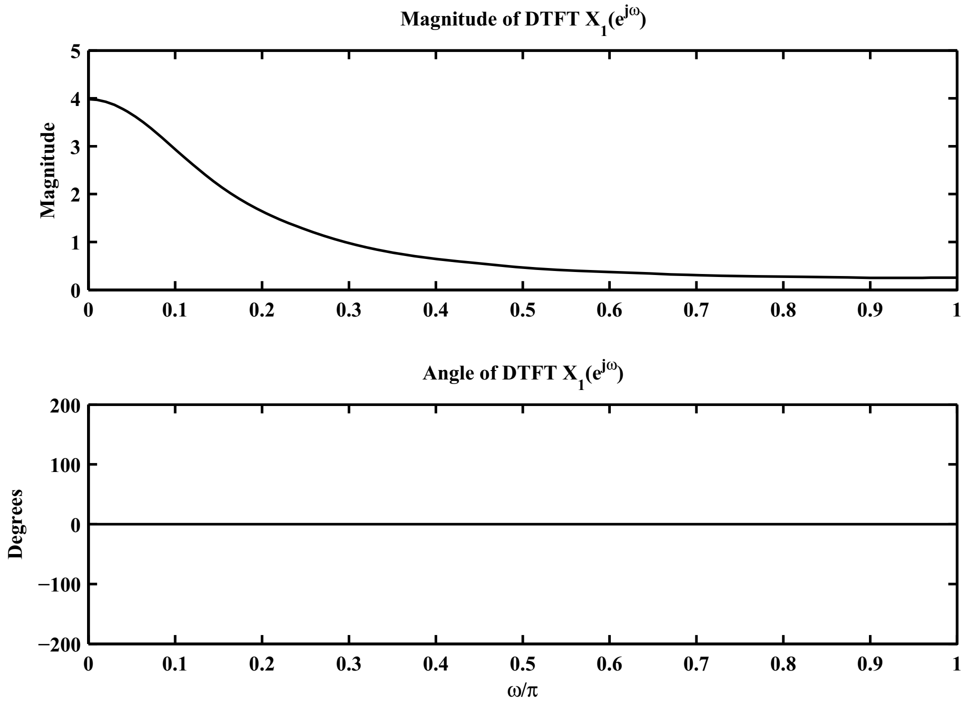


Figure 5.8: Plots of DTFT magnitude and phase in Problem 5.10.1

2. $x_2(n) = n(0.9)^n$, $0 \le n \le 20$. MATLAB script:

```
n2 = [0:20]; x2 = n2.*(0.9).^n2; N2 = length(n2); N = 400;   % Length of DFT
x2 = [x2,zeros(1,N-N2)]; % Assemble x2
[X2] = fft(x2,N); w = (0:N/2)*2*pi/N;
mag_X2 = abs(X2(1:N/2+1)); pha_X2 = angle(X2(1:N/2+1))*180/pi;
Hf_2 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_2,'NumberTitle','off','Name','P5.10.2');
subplot(2,1,1); plot(w/pi,mag_X2,'g','linewidth',1); %axis([0,1,0,5]);
title('Magnitude of DTFT X_2(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X2,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_2(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510b
```

The plot of the DTFT $X_2(e^{j\omega})$ is shown in 5.9.

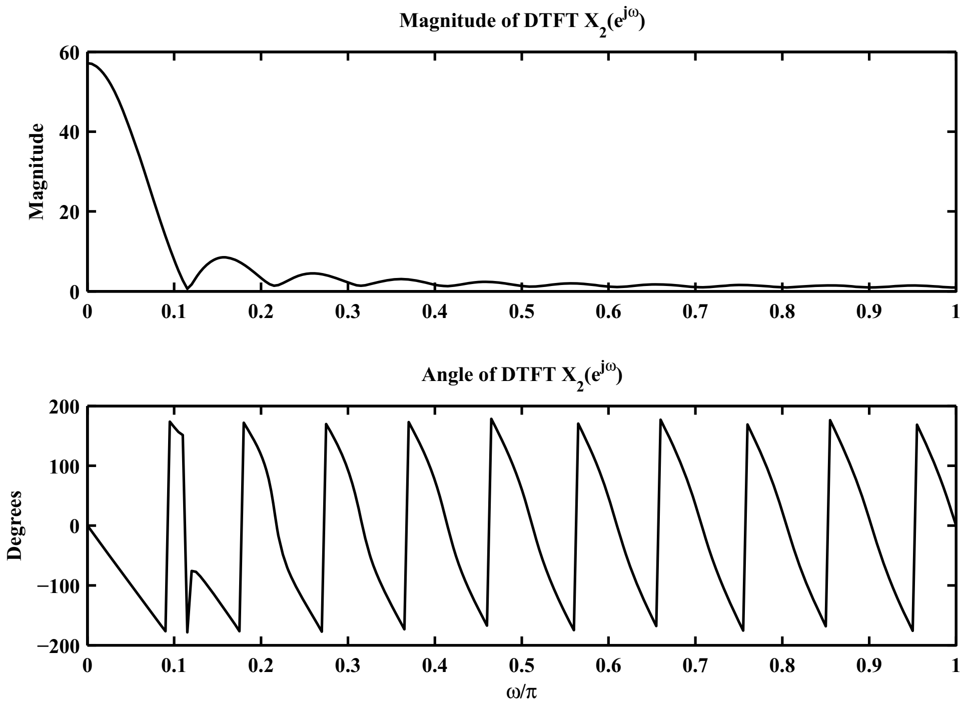


Figure 5.9: Plots of DTFT magnitude and phase in Problem 5.10.2

3. $x_3(n) = \cos(0.5\pi n) + j\sin(0.5\pi n)$, $0 \le n \le 50$. MATLAB script:

```
n3 = [0:50]; x3 = cos(0.5*pi*n3)+j*sin(0.5*pi*n3); N = 500;% Length of DFT
N3 = length(n3); x3 = [x3,zeros(1,N-N3)]; % Assemble x3
[X3] = dft(x3,N); w = (0:N/2)*2*pi/N;
mag_X3 = abs(X3(1:N/2+1)); pha_X3 = angle(X3(1:N/2+1))*180/pi;
Hf_3 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_3,'NumberTitle','off','Name','P5.10.3');
subplot(2,1,1); plot(w/pi,mag_X3,'g','linewidth',1); %axis([0,1,0,7000]);
title('Magnitude of DTFT X_3(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X3,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_3(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510c
```

The plot of the DTFT $X_3(e^{j\omega})$ is shown in 5.10.

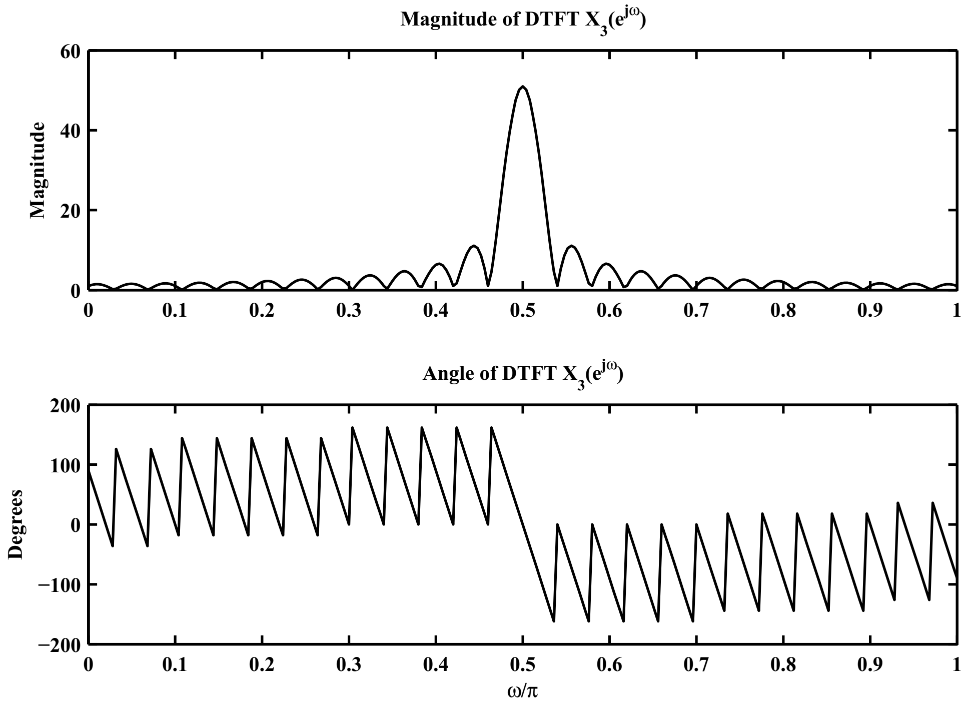


Figure 5.10: Plots of DTFT magnitude and phase in Problem 5.10.3

4. $x(n) = \{1, 2, 3, \underset{\uparrow}{4}, 3, 2, 1\}$. MATLAB script:

```
n4 = [-3:3]; x4 = [1,2,3,4,3,2,1]; N4 = length(n4); N = 100;   % Length of DFT
[X4] = dft([x4, zeros(1,N-N4)],N); w = (0:N/2)*2*pi/N;
mag_X4 = abs(X4(1:N/2+1)); pha_X4 = angle(X4(1:N/2+1))*180/pi;
Hf_4 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_4,'NumberTitle','off','Name','P5.10.4');
subplot(2,1,1); plot(w/pi,mag_X4,'g','linewidth',1); axis([0,1,0,20]);
title('Magnitude of DTFT X_4(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X4,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_4(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510d
```

The plot of the DTFT $X_4(e^{j\omega})$ is shown in 5.11.

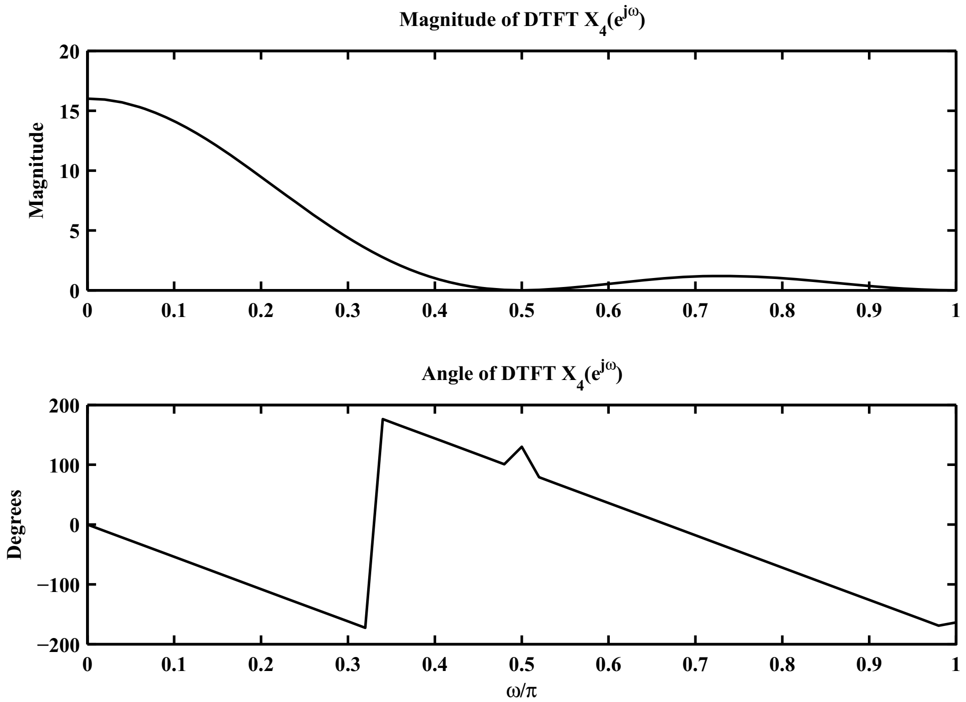


Figure 5.11: Plots of DTFT magnitude and phase in Problem 5.10.4

5. $x(n) = \{-1, -2, -3, \underset{\uparrow}{0}, 3, 2, 1\}$. MATLAB script:

```
n5 = [-3:3]; x5 = [-1,-2,-3,0,3,2,1]; N5 = length(n5); N = 100;   % Length of DFT
[X5] = dft([x5, zeros(1,N-N5)],N); w = (0:N/2)*2*pi/N;
mag_X5 = abs(X5(1:N/2+1)); pha_X5 = angle(X5(1:N/2+1))*180/pi;
Hf_5 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_5,'NumberTitle','off','Name','P5.10.5');
subplot(2,1,1); plot(w/pi,mag_X5,'g','linewidth',1); axis([0,1,0,20]);
title('Magnitude of DTFT X_5(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X5,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_5(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510e
```

The plot of the DTFT $X_5(e^{j\omega})$ is shown in 5.12.

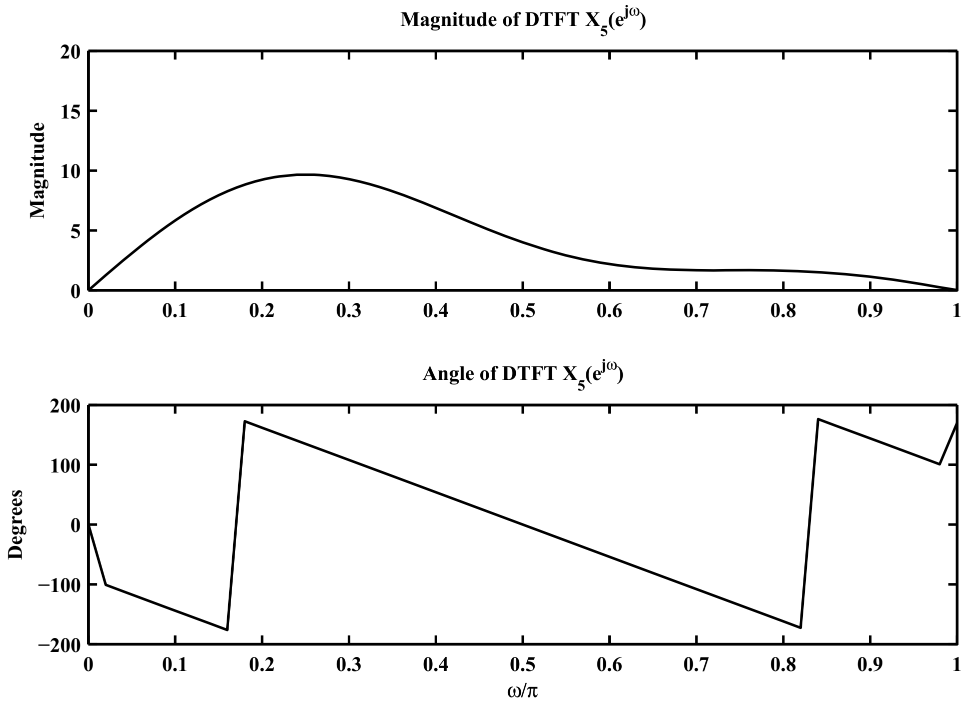


Figure 5.12: Plots of DTFT magnitude and phase in Problem 5.10.5

P5.11 Let $H(e^{j\omega})$ be the frequency response of a real, causal impulse response $h(n)$.

1. It is known that $\text{Re}\left\{H\left(e^{j\omega}\right)\right\} = \sum_{k=0}^{5} (0.9)^k \cos \omega k$. Consider

$$\text{Re}\left\{H\left(e^{j\omega}\right)\right\} = \text{Re}\left\{\sum_{k=0}^{\infty} h(k)e^{-j\omega k}\right\} = \sum_{k=0}^{\infty} h(k)\,\text{Re}\left\{e^{-j\omega k}\right\} = \sum_{k=0}^{\infty} h(k)\cos \omega k$$

Comparing with the given expression, we obtain

$$h(n) = \begin{cases} (0.9)^n, & 0 \le n \le 5 \\ 0, & \text{else} \end{cases}$$

MATLAB verification:

```
n = 0:5; h = (0.9).^n; N1 = length(h); N = 100; h = [h,zeros(1,N-N1)];
H = dft(h,N); Hr = real(H);
k = [0:5]; w = linspace(0,2*pi,N+1);
Hr_check = (0.9.^k)*cos(k'*w(1:end-1));
error = max(abs(Hr-Hr_check))
error =
  4.0856e-014
```

2. It is known that $\text{Im}\left\{H\left(e^{j\omega}\right)\right\} = \sum_{\ell=0}^{5} 2\ell \sin \omega\ell$ and $\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) d\omega = 0$. From the second condition

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) d\omega = h\left(0\right) = 0$$

Consider

$$\text{Im}\left\{H\left(e^{j\omega}\right)\right\} = \text{Im}\left\{\sum_{\ell=0}^{\infty} h(\ell)e^{-j\omega\ell}\right\} = \sum_{\ell=0}^{\infty} h\left(\ell\right)\text{Im}\left\{e^{-j\omega\ell}\right\} = -\sum_{\ell=0}^{\infty} h\left(\ell\right)\sin \omega\ell$$

Comparing with the given expression, we obtain

$$h(n) = \begin{cases} -2n, & 0 \le n \le 5 \\ 0, & \text{else} \end{cases}$$

MATLAB verification:

```
n = 0:5; h = -2*n; N1 = length(h); N = 100; h = [h,zeros(1,N-N1)];
H = dft(h,N); Hi = imag(H);
l = [0:5]; w = linspace(0,2*pi,N+1);
Hi_check = 2*l*sin(l'*w(1:end-1));
error = max(abs(Hi-Hi_check))
error =
  3.8014e-013
```

P5.12 Let $X(k)$ denote the N-point DFT of an N-point sequence $x(n)$. The DFT $X(k)$ itself is an N-point sequence.

1. The N-point DFT of $x(n)$: $X(k) = \sum_{m=0}^{N-1} x(m) W_N^{mk}$. The N-point DFT of $X(k)$: $y(n) = \sum_{k=0}^{N-1} X(k) W_N^{kn}$. Hence,

$$
\begin{aligned}
y(n) &= \sum_{k=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x(m) W_N^{mk} \right\} W_N^{kn} = \sum_{m=0}^{N-1} x(m) \sum_{k=0}^{N-1} W_N^{mk} W_N^{kn}, \ 0 \le n \le N-1 \\
&= \sum_{m=0}^{N-1} x(m) \sum_{k=0}^{N-1} W_N^{(m+n)k} = \sum_{m=0}^{N-1} x(m) \sum_{r=-\infty}^{\infty} N\delta(m+n-rN), 0 \le n \le N-1 \\
&= N \sum_{r=-\infty}^{\infty} x(-n+rN) = Nx((-n))_N, 0 \le n \le N-1
\end{aligned}
$$

This means that $y(n)$ is a "circularly folded and amplified (by N)" version of $x(n)$. Continuing further, if we take two more DFTs of $x(n)$ then

$$
x(n) \longrightarrow \boxed{N\text{-point DFT}} \longrightarrow \boxed{N\text{-point DFT}} \longrightarrow \boxed{N\text{-point DFT}} \longrightarrow \boxed{N\text{-point DFT}} \longrightarrow N^2 x(n)
$$

Therefore, if a given DFT function is working correctly then four successive applications of this function on any arbitrary signal will produce the same signal (multiplied by N^2). This approach can be used to verify a DFT function.

2. MATLAB function for circular folding:

```
function x2 = circfold(x1,N)
% Circular folding with respect to N
% ----------------------------------
% function x2 = circfold(x1,N)
%         x2(n) = x1((-n) mod N)
%
x2 = real(dft(dft(x1,N),N))/N;
```

3. MATLAB verification:

```
x = [1,3,5,7,9,-7,-5,-3,-1], N = length(x);

x =
     1     3     5     7     9    -7    -5    -3    -1


Y = circfold(x,N)

Y =
    1.000   -1.000   -3.000   -5.000   -7.000    9.000    7.000    5.000    3.000
```

P5.13 Let $X(k)$ be an N-point DFT of an N-point sequence $x(n)$. Let N be an even integer.

1. Given that $x(n) = x(n + N/2)$ for all n, consider

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk} \\
&= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x(n + N/2) W_N^{(n+N/2)k} \quad [\because n \to n + N/2] \\
&= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x(n) W_N^{nk} W_N^{Nk/2} \quad [\because x(n) = x(n + N/2)] \\
&= \sum_{n=0}^{N/2-1} x(n) \left\{1 + (-1)^k\right\} W_N^{nk} \quad [\because W_N^{N/2} = -1] = \begin{cases} 0, & k \text{ odd;} \\ \text{Non-zero}, & k \text{ even.} \end{cases}
\end{aligned}$$

Verification using $x(n) = \{1, 2, -3, 4, 5, 1, 2, -3, 4, 5\}$:

```
x = [1,2,-3,4,5,1,2,-3,4,5]; N = length(x); X = dft(x,N)
X =
  18.0000             -0.0000 + 0.0000i   4.7082 +13.9353i  -0.0000 + 0.0000i
  -8.7082 - 9.7881i    0.0000 - 0.0000i  -8.7082 + 9.7881i  -0.0000 - 0.0000i
   4.7082 -13.9353i    0.0000 - 0.0000i
```

2. Given that $x(n) = -x(n + N/2)$ for all n, consider

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk} \\
&= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x(n + N/2) W_N^{(n+N/2)k} \quad [\because n \to n + N/2] \\
&= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} - \sum_{n=0}^{N/2-1} x(n) W_N^{nk} W_N^{Nk/2} \quad [\because x(n) = -x(n + N/2)] \\
&= \sum_{n=0}^{N/2-1} x(n) \left\{1 - (-1)^k\right\} W_N^{nk} \quad [\because W_N^{N/2} = -1] \\
&= \begin{cases} 0, & k \text{ even;} \\ \text{Non-zero}, & k \text{ odd.} \end{cases}
\end{aligned}$$

Verification using $x(n) = \{1, 2, -3, 4, 5, -1, -2, 3, -4, -5\}$.

```
x = [1,2,-3,4,5,-1,-2,3,-4,-5]; N = length(x); X = dft(x,N)
X =
        0             -7.1803 -10.1311i  -0.0000 - 0.0000i  15.1803 -12.1392i
   0.0000 + 0.0000i  -6.0000 - 0.0000i   0.0000 - 0.0000i  15.1803 +12.1392i
  -0.0000 + 0.0000i  -7.1803 +10.1311i
```

P5.14 Let $X(k)$ be an N-point DFT of an N-point sequence $x(n)$. Let $N = 4\nu$ where ν is an integer.

1. It is given that $x(n) = x(n+\nu)$ for all n. Let $n = m + p\nu$; $0 \le m \le \nu - 1$, $0 \le p \le 3$, then

$$x(n) = x(n+\nu) \Rightarrow x(m+p\nu) = x(m), \quad 0 \le m \le \nu - 1 \tag{5.1}$$

Now the DFT $X(k)$ can be written as

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{p=0}^{3}\sum_{m=0}^{\nu-1} x(m+p\nu) W_N^{(m+p\nu)k} = \sum_{p=0}^{3}\sum_{m=0}^{\nu-1} x(m)\, W_N^{mk}\, W_N^{p\nu k} \qquad [\because (5.1)] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{3} W_N^{p\nu k} = \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{3} \left(W_N^{\nu k}\right)^p = \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \left[\frac{1 - W_N^{Nk}}{1 - W_N^{\nu k}}\right] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \left[\frac{1 - W_N^{Nk}}{1 - W_N^{N(k/4\ell)}}\right] = \begin{cases} \text{Non-zero}, & k = 4\ell \text{ for } 0 \le \ell \le \nu - 1; \\ 0, & k \ne 4\ell \text{ for } 0 \le \ell \le \nu - 1. \end{cases}
\end{aligned}$$

Verification for $x(n) = \{1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3\}$.

```
x = [1,2,3,1,2,3,1,2,3,1,2,3]; N = length(x); X = dft(x,N)
X =
  24.0000            -0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i
  -6.0000 + 3.4641i   0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i
  -6.0000 - 3.4641i   0.0000 - 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i
```

2. It is given that $x(n) = -x(n+\nu)$ for all n. Let $n = m + p\nu$; $0 \le m \le \nu - 1$, $0 \le p \le 3$, then

$$x(n) = -x(n+\nu) \Rightarrow x(m+p\nu) = (-1)^p x(m), \quad 0 \le m \le \nu - 1,\ 0 \le p \le 3 \tag{5.2}$$

Now the DFT $X(k)$ can be written as

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{p=0}^{3}\sum_{m=0}^{\nu-1} x(m+p\nu) W_N^{(m+p\nu)k} = \sum_{p=0}^{3}\sum_{m=0}^{\nu-1} (-1)^p x(m)\, W_N^{mk}\, W_N^{p\nu k} \qquad [\because (5.2)] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{3} (-1)^p W_N^{p\nu k} = \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{3} \left(W_N^{-N/2} W_N^{\nu k}\right)^p \qquad [\because -1 = W_N^{-N/2}] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \left[\frac{1 - W_N^{N(k-2)}}{1 - W_N^{\nu(k-2)}}\right] = \begin{cases} \text{Non-zero}, & k = 4\ell + 2 \text{ for } 0 \le \ell \le \nu - 1; \\ 0, & k \ne 4\ell + 2 \text{ for } 0 \le \ell \le \nu - 1. \end{cases}
\end{aligned}$$

Verification for $x(n) = \{1, 2, 3, -1, -2, -3, 1, 2, 3, -1, -2, -3\}$.

```
x = [1,2,3,-1,-2,-3,1,2,3,-1,-2,-3]; N = length(x); X = dft(x,N)
X =
        0            0            2.0000 -17.3205i   0.0000 + 0.0000i
        0            0            8.0000 + 0.0000i  -0.0000 + 0.0000i
        0            0            2.0000 +17.3205i  -0.0000 - 0.0000i
```

P5.15 Let $X(k)$ be an N-point DFT of an N-point sequence $x(n)$. Let $N = 2\mu\nu$ where μ and ν are integers.

1. It is given that $x(n) = x(n+\nu)$ for all n. Let $n = m + p\nu$; $0 \le m \le \nu - 1$, $0 \le p \le (2\mu - 1)$, then

$$x(n) = x(n+\nu) \Rightarrow x(m+p\nu) = x(m), \quad 0 \le m \le \nu - 1,\ 0 \le p \le (2\mu - 1) \tag{5.3}$$

Now the DFT $X(k)$ can be written as

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{p=0}^{2\mu-1}\sum_{m=0}^{\nu-1} x(m+p\nu) W_N^{(m+p\nu)k} = \sum_{p=0}^{2\mu-1}\sum_{m=0}^{\nu-1} x(m)\, W_N^{mk}\, W_N^{p\nu k} \qquad [\because (5.3)] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{2\mu-1} W_N^{p\nu k} = \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{2\mu-1} \left(W_N^{\nu k}\right)^p = \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \left[\frac{1 - W_N^{Nk}}{1 - W_N^{\nu k}}\right] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \left[\frac{1 - W_N^{Nk}}{1 - W_N^{N(k/2\mu)}}\right] = \begin{cases} \text{Non-zero}, & k = (2\mu)\ell \text{ for } 0 \le \ell \le \nu - 1; \\ 0, & k \ne (2\mu)\ell \text{ for } 0 \le \ell \le \nu - 1. \end{cases}
\end{aligned}$$

Verification for $x(n) = \{1, -2, 3, 1, -2, 3, 1, -2, 3, 1, -2, 3, 1, -2, 3, 1, -2, 3\}$.

```
x = [1,-2,3,1,-2,3,1,-2,3,1,-2,3,1,-2,3,1,-2,3]; N = length(x); X = dft(x,N)
X =
  12.0000             0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000             0.0000 + 0.0000i   3.0000 +25.9808i  -0.0000 - 0.0000i
  -0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i
   3.0000 -25.9808i   0.0000 + 0.0000i   0.0000             0.0000 - 0.0000i
   0.0000 - 0.0000i   0.0000 - 0.0000i
```

2. It is given that $x(n) = -x(n+\nu)$ for all n. Let $n = m + p\nu$; $0 \le m \le \nu - 1$, $0 \le p \le (2\mu - 1)$, then

$$x(n) = -x(n+\nu) \Rightarrow x(m+p\nu) = (-1)^p x(m), \quad 0 \le m \le \nu - 1,\ 0 \le p \le (2\mu - 1) \tag{5.4}$$

Now the DFT $X(k)$ can be written as

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{p=0}^{2\mu-1}\sum_{m=0}^{\nu-1} x(m+p\nu) W_N^{(m+p\nu)k} = \sum_{p=0}^{2\mu-1}\sum_{m=0}^{\nu-1} (-1)^p x(m)\, W_N^{mk}\, W_N^{p\nu k} \qquad [\because (5.4)] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{2\mu-1} (-1)^p W_N^{p\nu k} = \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \sum_{p=0}^{2\mu-1} \left(W_N^{-N/2} W_N^{\nu k}\right)^p \qquad [\because -1 = W_N^{-N/2}] \\
&= \sum_{m=0}^{\nu-1} x(m)\, W_N^{mk} \left[\frac{1 - W_N^{N(k-2)}}{1 - W_N^{N(k-2)/(2\mu)}}\right] = \begin{cases} \ne 0, & k = 2\mu\ell + 2 = 2(\mu\ell + 1), 0 \le \ell \le \nu - 1; \\ 0, & k \ne 2\mu\ell + 2 = 2(\mu\ell + 1), 0 \le \ell \le \nu - 1. \end{cases}
\end{aligned}$$

Verification for $x(n) = \{1, -2, 3, -1, 2, -3, 1, -2, 3, -1, 2, -3, 1, -2, 3, -1, 2, -3\}$.

```
x = [1,-2,3,-1,2,-3,1,-2,3,-1,2,-3,1,-2,3,-1,2,-3]; N = length(x); X = dft(x,N)
X =
         0           -0.0000            -0.0000 + 0.0000i  -9.0000 - 5.1962i
   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i
   0.0000 - 0.0000i  36.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i
  -0.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i  -9.0000 + 5.1962i
  -0.0000 - 0.0000i  -0.0000 - 0.0000i
```

P5.16 Let $X(k)$ and $Y(k)$ be 10-point DFTs of two 10-point sequences $x(n)$ and $y(n)$, respectively where

$$X(k) = \exp(j0.2\pi k), \quad 0 \leq k \leq 9$$

Computation of $Y(k)$ properties of the DFT.

1. $y(n) = x((n-5)))_10$: Circular shift by 5. MATLAB script:

```
k = 0:9; X = exp(j*0.2*pi*k); N = length(X);
m = 5; WN = exp(-j*2*pi/N); Y = X.*WN.^(m*k)
% verification
x = real(idft(X,N)); y = cirshftt(x,m,N); Y1 = dft(y,N)
difference = abs(max(Y-Y1))
```

```
Y =
  Columns 1 through 4
   1.0000            -0.8090 - 0.5878i   0.3090 + 0.9511i   0.3090 - 0.9511i
  Columns 5 through 8
  -0.8090 + 0.5878i   1.0000 + 0.0000i  -0.8090 - 0.5878i   0.3090 + 0.9511i
  Columns 9 through 10
   0.3090 - 0.9511i  -0.8090 + 0.5878i
Y1 =
  Columns 1 through 4
   1.0000            -0.8090 - 0.5878i   0.3090 + 0.9511i   0.3090 - 0.9511i
  Columns 5 through 8
  -0.8090 + 0.5878i   1.0000 + 0.0000i  -0.8090 - 0.5878i   0.3090 + 0.9511i
  Columns 9 through 10
   0.3090 - 0.9511i  -0.8090 + 0.5878i
difference =
  2.7756e-015
```

2. $y(n) = x((n+4)))_10$: Circular shift by -4. MATLAB script:

```
k = 0:9; X = exp(j*0.2*pi*k); N = length(X);
m = -4; WN = exp(-j*2*pi/N); Y = X.*WN.^(m*k)
% verification
x = real(idft(X,N)); y = cirshftt(x,m,N); Y1 = dft(y,N)
difference = abs(max(Y-Y1))
```

```
Y =
  Columns 1 through 4
   1.0000            -1.0000 + 0.0000i   1.0000 - 0.0000i  -1.0000 + 0.0000i
  Columns 5 through 8
   1.0000 - 0.0000i  -1.0000 + 0.0000i   1.0000 - 0.0000i  -1.0000 + 0.0000i
  Columns 9 through 10
   1.0000 - 0.0000i  -1.0000 + 0.0000i
Y1 =
  Columns 1 through 4
   1.0000            -1.0000 - 0.0000i   1.0000 + 0.0000i  -1.0000 - 0.0000i
```

```
  Columns 5 through 8
   1.0000 - 0.0000i  -1.0000 - 0.0000i   1.0000 + 0.0000i  -1.0000 - 0.0000i
  Columns 9 through 10
   1.0000 - 0.0000i  -1.0000 + 0.0000i
difference =
  2.2249e-015
```

3. $y(n) = x((3-n)))_1 0$: Circular-fold and circular-shift by 3. MATLAB script:

```
k = 0:9; X = exp(j*0.2*pi*k); N = length(X);
Y = circfold(X,N); m = 3; WN = exp(-j*2*pi/N); Y = X.*WN.^(m*k)
% verification
x = real(idft(X,N)); y = circfold(x,N); y = cirshftt(x,m,N); Y1 = dft(y,N)
difference = abs(max(Y-Y1))
```

```
Y =
  Columns 1 through 4
   1.0000             0.3090 - 0.9511i  -0.8090 - 0.5878i  -0.8090 + 0.5878i
  Columns 5 through 8
   0.3090 + 0.9511i   1.0000 + 0.0000i   0.3090 - 0.9511i  -0.8090 - 0.5878i
  Columns 9 through 10
  -0.8090 + 0.5878i   0.3090 + 0.9511i
Y1 =
  Columns 1 through 4
   1.0000             0.3090 - 0.9511i  -0.8090 - 0.5878i  -0.8090 + 0.5878i
  Columns 5 through 8
   0.3090 + 0.9511i   1.0000 + 0.0000i   0.3090 - 0.9511i  -0.8090 - 0.5878i
  Columns 9 through 10
  -0.8090 + 0.5878i   0.3090 + 0.9511i
difference =
  2.6790e-015
```

4. $y(n) = x(n)e^{j3\pi n/5}$: Circular shift in the freq-domain by 3. MATLAB script:

```
k = 0:9; X = exp(j*0.2*pi*k); N = length(X); l = 3;
Y = cirshftt(X,l,N)
% verification
x = real(idft(X,N)); n = 0:9; WN = exp(-j*2*pi/N);
y = x.*WN.^(-l*n); Y1 = dft(y,N)
difference = abs(max(Y-Y1))
```

```
Y =
  Columns 1 through 4
  -0.3090 - 0.9511i   0.3090 - 0.9511i   0.8090 - 0.5878i   1.0000
  Columns 5 through 8
   0.8090 + 0.5878i   0.3090 + 0.9511i  -0.3090 + 0.9511i  -0.8090 + 0.5878i
  Columns 9 through 10
  -1.0000 + 0.0000i  -0.8090 - 0.5878i
```

```
Y1 =
  Columns 1 through 4
  -0.3090 - 0.9511i   0.3090 - 0.9511i   0.8090 - 0.5878i   1.0000 + 0.0000i
  Columns 5 through 8
   0.8090 + 0.5878i   0.3090 + 0.9511i  -0.3090 + 0.9511i  -0.8090 + 0.5878i
  Columns 9 through 10
  -1.0000            -0.8090 - 0.5878i
difference =
  3.3880e-015
```

5. $y(n) = x(n)(10)x((-n))_10$: Circular convolution with circularly-folded sequence. MATLAB script:

```
k = 0:9; X = exp(j*0.2*pi*k); N = length(X);
Y = circfold(X,N); Y = X.*Y
% verification
x = real(idft(X,N)); y = circfold(x,N); y = circonvt(x,y,N); Y1 = dft(y,N)
difference = abs(max(Y-Y1))
```

```
Y =
  Columns 1 through 4
   1.0000 - 0.0000i   1.0000 - 0.0000i   1.0000 + 0.0000i   1.0000 - 0.0000i
  Columns 5 through 8
   1.0000 - 0.0000i   1.0000 - 0.0000i   1.0000 - 0.0000i   1.0000 + 0.0000i
  Columns 9 through 10
   1.0000             1.0000 + 0.0000i
Y1 =
  Columns 1 through 4
   1.0000 + 0.0000i   1.0000 - 0.0000i   1.0000 - 0.0000i   1.0000 - 0.0000i
  Columns 5 through 8
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 - 0.0000i   1.0000 - 0.0000i
  Columns 9 through 10
   1.0000 - 0.0000i   1.0000 - 0.0000i
difference =
  4.7761e-015
```

P5.17 The first six values of the 10-point DFT of a real-valued sequence $x(n)$ are given by

$$\{10, -2 + j3, 3 + j4, 2 - j3, 4 + j5, 12\}$$

DFT computations using DFT properties:

1. $x_1(n) = x((2 - n))_1 0$: Circular-folding followed by circ-shifting by 2.
   MATLAB script:

```
N = 10; X = [10,-2+j*3,3+j*4,2-j*3,4+j*5,12];
X = [X,conj(X(5:-1:2))]; x = real(idft(X,N))
WN = exp(-j*2*pi/N); k = 0:N-1; m = 2;
X1 = circfold(X,N); X1 = (WN.^(m*k)).*X1
% Matlab Verification
x1 = circfold(x,N); x1 = cirshftt(x1,m,N); X12 = dft(x1,N)
difference = max(abs(X1-X12))
```

```
x =
  Columns 1 through 7
    3.6000   -2.2397    1.0721   -1.3951    3.7520    1.2000    0.6188
  Column 8 through 10
    0.0217    1.4132    1.9571
X1 =
  Columns 1 through 4
  10.0000 + 0.0000i  -3.4712 + 0.9751i  -4.7782 + 1.4727i  -3.3814 - 1.2515i
  Columns 5 through 8
   5.9914 + 2.2591i  12.0000 + 0.0000i   5.9914 - 2.2591i  -3.3814 + 1.2515i
  Columns 9 through 10
  -4.7782 - 1.4727i  -3.4712 - 0.9751i
X12 =
  Columns 1 through 4
  10.0000 + 0.0000i  -3.4712 + 0.9751i  -4.7782 + 1.4727i  -3.3814 - 1.2515i
  Columns 5 through 8
   5.9914 + 2.2591i  12.0000 + 0.0000i   5.9914 - 2.2591i  -3.3814 + 1.2515i
  Columns 9 through 10
  -4.7782 - 1.4727i  -3.4712 - 0.9751i
difference =
  1.2462e-014
```

2. $x_2(n) = x((n+5))_{10}$: 10-point circular shifting by -5.

   MATLAB script:

```
N = 10; X = [10,-2+j*3,3+j*4,2-j*3,4+j*5,12];
X = [X,conj(X(5:-1:2))]; x = real(idft(X,N))
WN = exp(-j*2*pi/N); k = 0:N-1; m = -5;
X2 = (WN.^(m*k)).*dft(x,N)
% Matlab verification
x2 = cirshftt(x,m,N); X22 = dft(x2,N)
difference = max(abs(X2-X22))

x =
  Columns 1 through 7
    3.6000   -2.2397    1.0721   -1.3951    3.7520    1.2000    0.6188
  Column 8 through 10
    0.0217    1.4132    1.9571
X2 =
  Columns 1 through 4
  10.0000             2.0000 - 3.0000i   3.0000 + 4.0000i  -2.0000 + 3.0000i
  Columns 5 through 8
   4.0000 + 5.0000i -12.0000 + 0.0000i   4.0000 - 5.0000i  -2.0000 - 3.0000i
  Columns 9 through 10
   3.0000 - 4.0000i   2.0000 + 3.0000i
X22 =
  Columns 1 through 4
  10.0000             2.0000 - 3.0000i   3.0000 + 4.0000i  -2.0000 + 3.0000i
  Columns 5 through 8
   4.0000 + 5.0000i -12.0000 - 0.0000i   4.0000 - 5.0000i  -2.0000 - 3.0000i
  Columns 9 through 10
   3.0000 - 4.0000i   2.0000 + 3.0000i
difference =
  3.2150e-014
```

3. $x_3(n) = x(n)x((-n))_{10}$: Multiplication by circularly-folded sequence.

   MATLAB script:

```
N = 10; X = [10,-2+j*3,3+j*4,2-j*3,4+j*5,12];
X = [X,conj(X(5:-1:2))]; x = real(idft(X,N))
X3 = circfold(X,N), X3 = circonvf(X,X3,N)/N
% Matlab verification
x3 = circfold(x,N); x3 = x.*x3; X32 = dft(x3,N)
difference = max(abs(X3-X32))
```

```
x =
  Columns 1 through 7
    3.6000   -2.2397    1.0721   -1.3951    3.7520    1.2000    0.6188
  Column 8 through 10
    0.0217    1.4132    1.9571
X3 =
  Columns 1 through 4
  10.0000 + 0.0000i  -2.0000 - 3.0000i   3.0000 - 4.0000i   2.0000 + 3.0000i
  Columns 5 through 8
   4.0000 - 5.0000i  12.0000 + 0.0000i   4.0000 + 5.0000i   2.0000 - 3.0000i
  Columns 9 through 10
   3.0000 + 4.0000i  -2.0000 + 3.0000i
X3 =
  Columns 1 through 4
  19.2000 + 0.0000i  10.2000 - 0.0000i  15.6000 + 0.0000i   6.4000 + 0.0000i
  Columns 5 through 8
  10.8000 - 0.0000i  24.4000 + 0.0000i  10.8000 + 0.0000i   6.4000 - 0.0000i
  Columns 9 through 10
  15.6000 + 0.0000i  10.2000 + 0.0000i
X32 =
  Columns 1 through 4
  19.2000 + 0.0000i  10.2000 - 0.0000i  15.6000 - 0.0000i   6.4000 + 0.0000i
  Columns 5 through 8
  10.8000 - 0.0000i  24.4000 - 0.0000i  10.8000 + 0.0000i   6.4000 - 0.0000i
  Columns 9 through 10
  15.6000 - 0.0000i  10.2000 - 0.0000i
difference =
  2.4416e-014
```

4. $x_4(n) = x(n) \circledast_{10} x((-n))_{10}$: 10-point circular convolution with a circularly-folded sequence.
MATLAB script:

```
N = 10; n = [0:N-1]; X = [10,-2+j*3,3+j*4,2-j*3,4+j*5,12];
X = [X,conj(X(5:-1:2))]; x = real(idft(X,N))
X0 = X(mod(-k,N)+1); X4 = X .* X0
% Verification
x4 = circonvt(x,x(mod(-n,N)+1),N); X42 = dft(x4,N)
difference = max(abs(X4-X42))

x =
  Columns 1 through 7
    3.6000   -2.2397    1.0721   -1.3951    3.7520    1.2000    0.6188
  Column 8 through 10
    0.0217    1.4132    1.9571
X4 =
   100    13    25    13    41   144    41    13    25    13
X42 =
  1.0e+002 *
  Columns 1 through 4
   1.0000             0.1300 + 0.0000i   0.2500 + 0.0000i   0.1300 - 0.0000i
  Columns 5 through 8
   0.4100 - 0.0000i   1.4400 + 0.0000i   0.4100 + 0.0000i   0.1300 + 0.0000i
  Columns 9 through 10
   0.2500 - 0.0000i   0.1300 - 0.0000i
difference =
  1.0378e-013
```

5. $x_5(n) = x(n)e^{-j4\pi n/5}$: Circular-shifting by -4 in the frequency-domain.

MATLAB script:

```
N = 10; n = [0:N-1]; X = [10,-2+j*3,3+j*4,2-j*3,4+j*5,12]; m = 4;
X = [X,conj(X(5:-1:2))]; x = real(idft(X,N))
X5 = [X(m+1:end),X(1:m)]
% Verification
WN = exp(-j*2*pi/N); x5 = x.*(WN.^(m*n)); X51 = dft(x5,N)
difference = max(abs(X5-X51))

x =
  Columns 1 through 7
    3.6000   -2.2397    1.0721   -1.3951    3.7520    1.2000    0.6188
  Column 8 through 10
    0.0217    1.4132    1.9571
X5 =
  Columns 1 through 4
   4.0000 + 5.0000i  12.0000             4.0000 - 5.0000i   2.0000 + 3.0000i
  Columns 5 through 8
   3.0000 - 4.0000i  -2.0000 - 3.0000i  10.0000            -2.0000 + 3.0000i
  Columns 9 through 10
   3.0000 + 4.0000i   2.0000 - 3.0000i
X51 =
  Columns 1 through 4
   4.0000 + 5.0000i  12.0000 + 0.0000i   4.0000 - 5.0000i   2.0000 + 3.0000i
  Columns 5 through 8
   3.0000 - 4.0000i  -2.0000 - 3.0000i  10.0000 + 0.0000i  -2.0000 + 3.0000i
  Columns 9 through 10
   3.0000 + 4.0000i   2.0000 - 3.0000i
difference =
  2.4895e-014
```

P5.18 Complex-valued N-point sequence $x(n)$ can be decomposed into N-point circular-conjugate-symmetric and circular-conjugate-antisymmetric sequences using the following relations

$$x_{\text{ccs}}(n) \triangleq \frac{1}{2}\left[x(n) + x^*((-n))_N\right]$$

$$x_{\text{cca}}(n) \triangleq \frac{1}{2}\left[x(n) - x^*((-n))_N\right]$$

If $X_{\text{R}}(k)$ and $X_{\text{I}}(k)$ are the real and imaginary parts of the N-point DFT of $x(n)$, then

$$\text{DFT}\left[x_{\text{ccs}}(n)\right] = X_{\text{R}}(k) \quad \text{and} \quad \text{DFT}\left[x_{\text{cca}}(n)\right] = jX_{\text{I}}(k)$$

1. Using the DFT properties of conjugation and circular folding, we obtain

$$\begin{aligned}\text{DFT}\left[x_{\text{ccs}}(n)\right] &= \frac{1}{2}\left\{\text{DFT}\left[x(n)\right] + \text{DFT}\left[x^*((-n))_N\right]\right\} \\ &= \frac{1}{2}\left\{X(k) + \hat{X}^*((-k))_N\right\}, \text{ where } \hat{X}(k) = \text{DFT}\left[x((-n))_N\right] \\ &= \frac{1}{2}\left\{X(k) + X^*(k)\right\} = \text{Re}\left[X(k)\right] = X_{\text{R}}(k)\end{aligned}$$

   similarly, we can show that

$$\text{DFT}\left[x_{\text{cca}}(n)\right] = j\,\text{Im}\left[X(k)\right] = jX_{\text{I}}(k)$$

2. The modified `circevod` function:

```
function [xccs, xcca] = circevod(x)
% Complex-valued signal decomposition into circular-even and circular-odd parts
% -----------------------------------------------------------------------------
% [xccs, xcca] = circecod(x)
%
N = length(x); n = 0:(N-1);
xccs = 0.5*(x + conj(x(mod(-n,N)+1)));
xcca = 0.5*(x - conj(x(mod(-n,N)+1)));
```

3. Let $X(k) = [3\cos(0.2\pi k) + j4\sin(0.1\pi k)][u(k) - u(k-20)]$ be a 20-point DFT. MATLAB verification:

```
N = 20; k = 0:N-1; X = 3*cos(0.2*pi*k) + j*sin(0.1*pi*k);
n = 0:N-1; x = idft(X,N); [xccs, xcca] = circevod(x);
Xccs = dft(xccs,N); Xcca = dft(xcca,N);
Hf_1 = figure('Units','inches','position',[1,1,6,4],...
    'paperunits','inches','paperposition',[0,0,6,4],'color',[0,0,0]);
set(Hf_1,'NumberTitle','off','Name','P5.18.3');
subplot(2,2,1); H_s1 = stem(n,real(X),'filled'); set(H_s1,'markersize',3);
title('X_R(k)'); ylabel('Amplitude'); axis([-0.5,20.5,-4,4]);
subplot(2,2,3); H_s2 = stem(n,real(Xccs),'filled'); set(H_s2,'markersize',3);
title('X_{ccs}(k)'); ylabel('Amplitude'); xlabel('k'); axis([-0.5,20.5,-4,4]);
subplot(2,2,2); H_s3 = stem(n,imag(X),'filled'); set(H_s3,'markersize',3);
title('X_I(k)'); ylabel('Amplitude'); axis([-0.5,20.5,-1.1,1.1]);
subplot(2,2,4); H_s4 = stem(n,imag(Xcca),'filled'); set(H_s4,'markersize',3);
title('X_{cca}(k)'); ylabel('Amplitude'); xlabel('k'); axis([-0.5,20.5,-1.1,1.1]);
```

   The plots are shown in Figure 5.13.

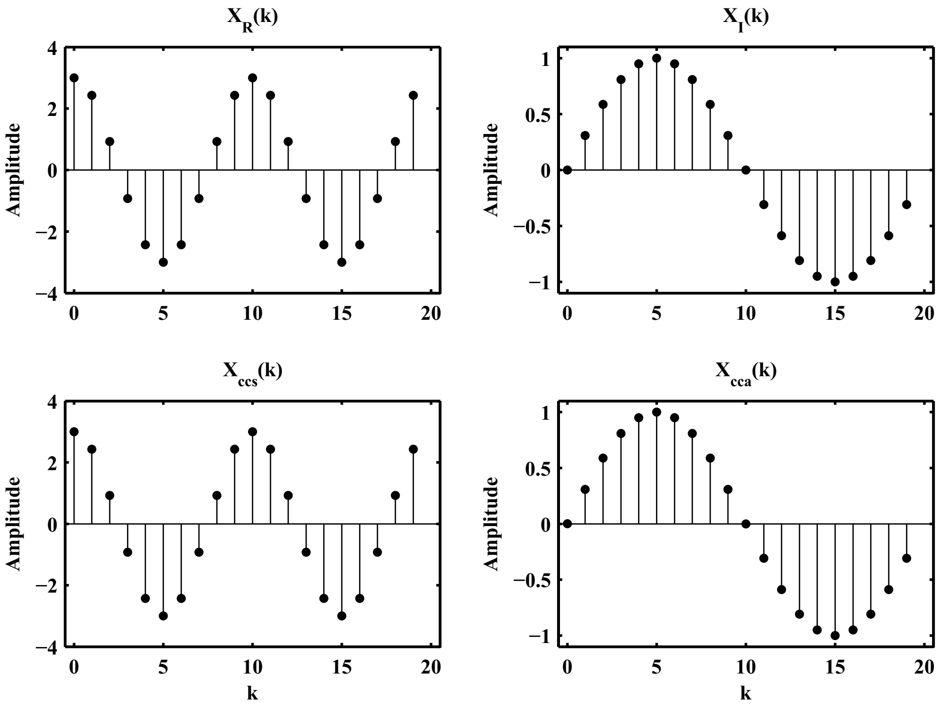


Figure 5.13: Plots in Problem P5.18.3

P5.19 (Two real DFTs using one complex DFT) If $X(k)$ is the N-point DFT of an N-point complex-valued sequence

$$x(n) = x_R(n) + jx_I(n)$$

where $x_R(n)$ and $x_I(n)$ are the real and imaginary parts of $x(n)$, then

$$\text{DFT}\,[x_R(n)] = X_{\text{ccs}}(k) \quad \text{and} \quad \text{DFT}\,[jx_I(n)] = X_{\text{cca}}(k)$$

where $X_{\text{ccs}}(k)$ and $X_{\text{cca}}(k)$ are the circular-even and circular-odd components of $X(k)$ as defined in Problem 5.18.

1. Analytical proof: Consider

$$\begin{aligned} X_R(k) \triangleq \text{DFT}\,[x_R(n)] &= \frac{1}{2}\left\{\text{DFT}\,[x(n)] + \text{DFT}\left[x^*(n)\right]\right\} \\ &= \frac{1}{2}\left\{X(k) + X^*((-k))_N\right\} \triangleq X_{\text{ccs}} \end{aligned}$$

   Similarly

$$\left.\begin{aligned} jX_I(k) &\triangleq \text{DFT}\,[jx_I(n)] = \frac{1}{2}\{\text{DFT}\,[x(n)] - \text{DFT}\,[x^*(n)]\} \\ &= \frac{1}{2}\{X(k) - X^*((-k))_N\} \triangleq X_{\text{cca}} \end{aligned}\right\} \Rightarrow X_I(k) = \frac{X_{\text{cca}}(k)}{j} = -jX_{\text{cca}}(k)$$

2. This property can be used to compute the DFTs of two real-valued N-point sequences using one N-point DFT operation. Specifically, let $x_1(n)$ and $x_2(n)$ be two N-point sequences. Then we can form a complex-valued sequence

$$x(n) = x_1(n) + jx_2(n)$$

   and use the above property. MATLAB function `real2dft`:

```
function [X1,X2] = real2dft(x1,x2,N)
% DFTs of two real sequences
% [X1,X2] = real2dft(x1,x2,N)
%  X1 = N-point DFT of x1
%  X2 = N-point DFT of x2
%  x1 = real-valued sequence of length <= N
%  x2 = real-valued sequence of length <= N
%   N = length of DFT
%
% Check for length of x1 and x2
if length(x1) > N
        error('*** N must be >= the length of x1 ***')
end
if length(x2) > N
        error('*** N must be >= the length of x2 ***')
end
N1 = length(x1); x1 = [x1 zeros(1,N-N1)];
N2 = length(x2); x2 = [x2 zeros(1,N-N2)];
x = x1 + j*x2;
X = dft(x,N);
[X1, X2] = circevod(X); X2 = X2/j;
```

We will also need the `circevod` function for complex sequences (see Problem P5.18). This can be obtained from the one given in the text by two simple changes.

```
function [xccs, xcca] = circevod(x)
% Complex signal decomposition into circular-even and circular-odd parts
% ----------------------------------------------------------------------
% [xccs, xcca] = circecod(x)
%
N = length(x); n = 0:(N-1);
xccs = 0.5*(x + conj(x(mod(-n,N)+1)));
xcca = 0.5*(x - conj(x(mod(-n,N)+1)));
```

3. Compute and plot the DFTs of the following two sequences using the above function

$$x_1(n) = \cos(0.1\pi n),\ x_2(n) = \sin(0.2\pi n); \quad 0 \le n \le 39$$

MATLAB verification:

```
N = 40; n = 0:N-1; x1 = cos(0.1*pi*n); x2 = sin(0.2*pi*n);
[X1,X2] = real2dft(x1,x2,N);
X11 = dft(x1,N); X21 = dft(x2,N);
difference = max(abs(X1-X11))
difference = max(abs(X2-X21))

difference =
  3.6876e-013
difference =
  3.6564e-013
```

P5.20 Circular shifting: The MATLAB routine `cirshftf.m` to implement circular shift is written using the frequency-domain property

$$y(n) \triangleq x((n-m))_N = \text{IDFT}\left[X(k)W_N^{mk}\right]$$

This routine will be used in the next problem to generate a circulant matrix and has the following features. If m is a scaler then $y(n)$ is circularly shifted sequence (or array). If m is a vector then $y(n)$ is a matrix, each row of which is a circular shift in $x(n)$ corresponding to entries in the vector m.

```
function y = cirshftf(x,m,N)
%  Circular shift of m samples wrt size N in sequence x: (freq domain)
%  -----------------------------------------------------------------
% function y=cirshift(x,m,N)
%       y : output sequence containing the circular shift
%       x : input sequence of length <= N
%       m : sample shift
%       N : size of circular buffer
%
%  Method: y(n) = idft(dft(x(n))*WN^(mk))
%
%   If m is a scalar then y is a sequence (row vector)
%   If m is a vector then y is a matrix, each row is a circular shift
%       in x corresponding to entries in vecor m
%   M and x should not be matrices
%
% Check whether m is scalar, vector, or matrix
[Rm,Cm] = size(m);
if Rm > Cm
    m = m'; % make sure that m is a row vector
end
[Rm,Cm] = size(m);
if Rm > 1
    error('*** m must be a vector ***') % stop if m is a matrix
end
% Check whether x is scalar, vector, or matrix
[Rx,Cx] = size(x);
if Rx > Cx
    x = x'; % make sure that x is a row vector
end
[Rx,Cx] = size(x);
if Rx > 1
    error('*** x must be a vector ***') % stop if x is a matrix
end
% Check for length of x
if length(x) > N
        error('N must be >= the length of x')
end
x=[x zeros(1,N-length(x))];
X=dft(x,N);
```

```
X=ones(Cm,1)*X;
WN=exp(-2*j*pi/N);
k=[0:1:N-1];
Y=(WN.^(m' * k)).*X;
y=real(conj(dfs(conj(Y),N)))/N;
```

MATLAB verification:

(a) $x(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1, 0, 0, 1, 2, 3, 4\}$, $0 \le n \le 10$; $m = -5$, $N = 11$

```
x = [5,4,3,2,1,0,0,1,2,3,4,5];
m = -5; N = 12;
y = cirshftf(x,m,N); y = real(y)

y =
     0     0     1     2     3     4     5     5     4     3     2     1
```

(b) $x(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1, 0, 0, 1, 2, 3, 4\}$, $0 \le n \le 10$; $m = 8$, $N = 15$

```
x = [5,4,3,2,1,0,0,1,2,3,4,5];
m = 8; N = 15;
y = cirshftf(x,m,N); y = real(y)

y =
     1     2     3     4     5     0     0     0     5     4     3     2
     1     0     0
```

P5.21 Parseval's relation for the DFT:

$$\begin{aligned}\sum_{n=0}^{N-1} |x(n)|^2 &= \sum_{n=0}^{N-1} x(n)\, x^*(n) = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \right\} x^*(n) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left\{ \sum_{n=0}^{N-1} x^*(n)\, W_N^{-nk} \right\} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left\{ \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \right\}^* \end{aligned}$$

Therefore,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, X^*(k) = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

MATLAB verification:

```
x = [5,4,3,2,1,0,0,1,2,3,4,5]; N = length(x);
% power of x(n) in the time-domain
power_x = sum(x.*conj(x))


power_x =
          110


% Power in the frequency-domain
X = dft(x,N); power_X = (1/N)*sum(X.*conj(X))


power_X =
          110
```

P5.22 A 512-point DFT $X(k)$ of a real-valued sequence $x(n)$ has the following DFT values:

$$X(0) = 20 + j\alpha; \quad X(5) = 20 + j30; \quad X(k_1) = -10 + j15; \quad X(152) = 17 + j23;$$
$$X(k_2) = 20 - j30; \quad X(k_3) = 17 - j23; \quad X(480) = -10 - j15; \quad X(256) = 30 + j\beta$$

and all other values are known to be zero.

1. The real-valued coefficients α and β: Since the sequence $x(n)$ is real-valued, $X(k)$ is conjugate symmetric which means that $X(0)$ and $X(N/2)$ are also real-valued. Since $N = 512$, $X(0)$ and $X(256)$ are real-valued. Hence $\alpha = \beta = 0$.
2. The values of the integers k_1, k_2, and k_3: Again using the conjugate symmetry property of the DFT, we have $X(k) = X^*(N-k)$. Thus

$$X(5) = 20 + j30 = X^*(512-5) = X^*(507) \Rightarrow X(507) = 20 - j30 \Rightarrow k_2 = 507$$
$$X(480) = -10 - j15 = X^*(512-480) = X^*(32) \Rightarrow X(32) = -10 + j15 \Rightarrow k_1 = 32$$
$$X(152) = 17 + j23 = X^*(512-152) = X^*(360) \Rightarrow X(360) = 17 - j23 \Rightarrow k_3 = 360$$

3. The energy of the signal $x(n)$: Using Parseval's relation,

$$\begin{aligned}\mathcal{E}_x &= \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{N}\sum_{k=-\infty}^{\infty} |X(k)|^2 \\ &= \frac{1}{512}\left[|X(0)|^2 + 2|X(5)|^2 + 2|X(32)|^2 + 2|X(152)|^2 + |X(256)|^2\right] = 12.082\end{aligned}$$

4. Sequence $x(n)$ in a closed form: The time-domain sequence $x(n)$ is a linear combination of the harmonically related complex exponential. Hence

$$\begin{aligned}x(n) &= \frac{1}{512}\left[X(0) + X(5)e^{-2\pi 5n/512} + X^*(5)e^{2\pi 5n/512} + X(32)e^{-2\pi 32n/512} + X^*(32)e^{2\pi 32n/512}\right. \\ &\quad \left. + X(152)e^{-2\pi 152n/512} + X^*(152)e^{2\pi 152n/512} + X(256)e^{-2\pi 256n/512}\right] \\ &= \frac{1}{512}\left[X(0) + 2\mathrm{Re}\left\{X(5)e^{-2\pi 5n/512}\right\} + 2\mathrm{Re}\left\{X(32)e^{-2\pi 32n/512}\right\} + 2\mathrm{Re}\left\{X(152)e^{-2\pi 152n/512}\right\}\right. \\ &\quad \left. + X(256)(-1)^n\right] \\ &= \frac{1}{512}\left[20 + 72.111\cos(0.019531\pi n - 56.32^\circ) + 36.056\cos(0.125\pi n - 123.69^\circ)\right. \\ &\quad \left. + 57.201\cos(0.59375\pi n - 53.531^\circ) + 30(-1)^n\right]\end{aligned}$$

P5.23 Let $x(n)$ be a finite length sequence given by

$$x(n) = \left\{\ldots, 0, 0, 0, \underset{\uparrow}{1}, 2, -3, 4, -5, 0, \ldots\right\}$$

Then the sequence

$$\begin{aligned} x((-8-n))_7\mathcal{R}_7(n) &= x((-[n+8]))_7\mathcal{R}_7(n) = x((-[n+8-7]))_7\mathcal{R}_7(n) \\ &= x((-[n+1]))_7\mathcal{R}_7(n) \end{aligned}$$

where

$$\mathcal{R}_7(n) = \begin{cases} 1, & 0 \le n \le 6 \\ 0, & \text{else} \end{cases}$$

is a circularly folded and circularly shifted-by-(-1) version of the 7-point sequence $\{1, 2, -3, 4, -5, 0, 0\}$. Hence

$$x((-8-n))_7\mathcal{R}_7(n) = \{0, 0, -5, 4, -3, 2, 1\}$$

P5.24 Circular convolution using circulant matrix operation.

$$x_1(n) = \{1, 2, 2\}, \; x_2(n) = \{1, 2, 3, 4\}, \; x_3(n) \triangleq x_1(n) \, (4) \, x_2(n)$$

1. Using the results from Example 5.13, we can express the above signals as

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

   The matrix $\mathbf{X}_2$ has the property that its every row or column can be obtained from the previous row or column using circular shift. Such a matrix is called a *circulant* matrix. It is completely described by the first column or the row.

2. Circular convolution:

$$\mathbf{x}_3 = \mathbf{X}_2\mathbf{x}_1 = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \\ 9 \\ 14 \end{bmatrix}$$

P5.25 MATLAB function `circulnt`:

```
function C = circulnt(x,N)
%  Circulant matrix generation using vector data values
%  ---------------------------------------------------
%  function C = circulnt(h,N)
%
%   C : Circulant matrix
%   x : input sequence of length <= N
%   N : size of the circular buffer
%   Method: C = h((n-m) mod N);

  Mx = length(x);                             % length of x
   x = [x, zeros(N-Mx,1)];                    % zero-pad x
   C = zeros(N,N);                            % establish size of C
   m = 0:N-1;                                 % indices n and m
   x = circfold(x,N);                         % Circular folding
   C = cirshift(x,m,N);                       % Circular shifting
```

MATLAB verification on sequences in Problem 5.24:

```
N = 4; x1 = [1,2,2,0]; x2 = [1,2,3,4]; X2 = circulnt(x2,N)

X2 =
    1.0000    4.0000    3.0000    2.0000
    2.0000    1.0000    4.0000    3.0000
    3.0000    2.0000    1.0000    4.0000
    4.0000    3.0000    2.0000    1.0000

x3 = X2*x1'; x3 = x3'

x3 =
   15.0000   12.0000    9.0000   14.0000
```

P5.26 MATLAB function `circonvf`:

```
function y = circonvf(x1,x2,N)
%
%function y=circonvf(x1,x2,N)
%
%  N-point circular convolution between x1 and x2: (freq domain)
%  -------------------------------------------------------------
%       y  : output sequence containing the circular convolution
%       x1 : input sequence of length N1 <= N
%       x2 : input sequence of length N2 <= N
%       N  : size of circular buffer
%
%  Method: y(n) = idft(dft(x1)*dft(x2))

% Check for length of x1
if length(x1) > N
        error('N must be >= the length of x1')
end

% Check for length of x2
if length(x2) > N
        error('N must be >= the length of x2')
end

x1=[x1 zeros(1,N-length(x1))];
x2=[x2 zeros(1,N-length(x2))];

X1=fft(x1); X2=fft(x2);
y=real(ifft(X1.*X2));
```

Circular convolution $\{4, 3, 2, 1\} \, 4 \, \{1, 2, 3, 4\}$:

```
x1 = [4,3,2,1]; x2 = [1,2,3,4];
x3 = circonvf(x1,x2,4)

x3 =
    24    22    24    30
```

P5.27 The following four sequences are given:

$$x_1(n) = \{\underset{\uparrow}{1}, 3, 2, -1\};\;\; x_2(n) = \{\underset{\uparrow}{2}, 1, 0, -1\};\;\; x_3(n) = x_1(n) * x_2(n);\;\; x_4(n) = x_1(n) \, 5 \, x_2(n)$$

1. Linear convolution $x_3(n)$:

$$x_3(n) = x_1(n) * x_2(n) = \{\underset{\uparrow}{2}, 7, 7, -1, -4, -2, 1\}$$

2. Computation of $x_4(n)$ using $x_3(n)$ alone: The error in the two convolutions is given by

$$e(n) \triangleq x_4(n) - x_3(n) = x_3(n + N)$$

we have, for $N = 5$,

$$\begin{aligned}
e(0) &= x_4(0) - x_3(0) = x_3(5) \Rightarrow x_4(0) = x_3(0) + x_3(5) = 2 - 2 = 0 \\
e(1) &= x_4(1) - x_3(1) = x_3(6) \Rightarrow x_4(1) = x_3(1) + x_3(6) = 7 + 1 = 8 \\
e(2) &= x_4(2) - x_3(2) = x_3(7) \Rightarrow x_4(2) = x_3(2) + x_3(7) = 7 + 0 = 7 \\
e(3) &= x_4(3) - x_3(3) = x_3(8) \Rightarrow x_4(3) = x_3(3) + x_3(8) = -1 + 0 = -1 \\
e(4) &= x_4(4) - x_3(4) = x_3(9) \Rightarrow x_4(4) = x_3(4) + x_3(9) = -4 + 0 = -4
\end{aligned}$$

P5.28 Computation and plotting of the N-point circular convolutions between two finite-length sequences.

1. $x_1(n) = \sin(\pi n/3)\mathcal{R}_6(n)$, $\quad x_2(n) = \cos(\pi n/4)\mathcal{R}_8(n)$; $\quad N = 10$: MATLAB script:

```
N = 10; n = 0:N-1;n1 = 0:5; x1 = sin(pi*n1/3);
n2 = 0:7; x2 = cos(pi*n2/4); x3 = circonvt(x1,x2,N);
Hf_1 = figure('Units','inches','position',[1,1,5,2],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,5,2]);
set(Hf_1,'NumberTitle','off','Name','P5.28.1');
H_s1 = stem(n,x3,'filled'); set(H_s1,'markersize',3);
title('Circular Convolution {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); xlabel('{\itn}'); axis([-1,N,min(x3)-1,max(x3)+1]);
```

The sample plot is shown in Figure 5.14.

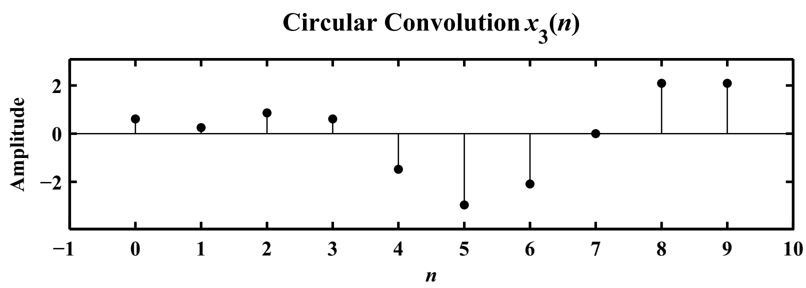


Figure 5.14: The sample plot in Problem P5.28.1

2. $x_1(n) = \cos(2\pi n/N)\,\mathcal{R}_N(n)$, $\quad x_2(n) = \sin(2\pi n/N)\,\mathcal{R}_N(n)$; $\quad N = 32$: MATLAB script:

```
N = 32; n = 0:N-1;
x1 = cos(2*pi*n/N); x2 = sin(2*pi*n/N); x3 = circonvt(x1,x2,N);
Hf_2 = figure('Units','inches','position',[1,1,5,1.5],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,5,1.5]);
set(Hf_2,'NumberTitle','off','Name','P5.28.2');
H_s2 = stem(n,x3,'filled'); set(H_s2,'markersize',3);
title('Circular Convolution {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); xlabel('{\itn}'); axis([-1,N,min(x3)-1,max(x3)+1]);
```

The sample plot is shown in Figure 5.15.

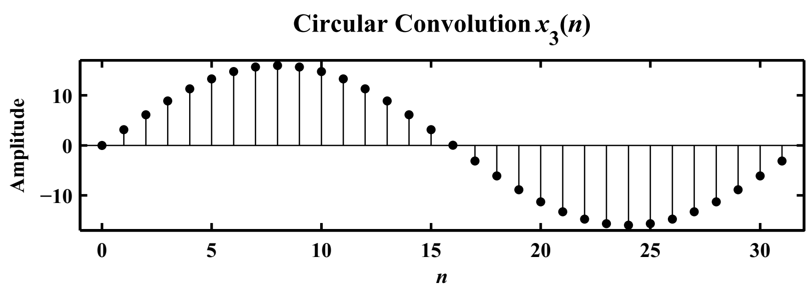


Figure 5.15: The sample plot in Problem P5.28.2

3. $x_1(n) = (0.8)^n\, \mathcal{R}_N(n), \quad x_2(n) = (-0.8)^n \mathcal{R}_N(n); \quad N = 20$: MATLAB script:

```
N = 20; n = 0:N-1;
x1 = (0.8).^n; x2 = (-0.8).^n; x3 = circonvt(x1,x2,N);
Hf_3 = figure('Units','inches','position',[1,1,5,1.5],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,5,1.5]);
set(Hf_3,'NumberTitle','off','Name','P5.28.3');
H_s3 = stem(n,x3,'filled'); set(H_s3,'markersize',3);
title('Circular Convolution {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); xlabel('{\itn}'); axis([-1,N,min(x3)-0.5,max(x3)+0.5]);
```

The sample plot is shown in Figure 5.16.

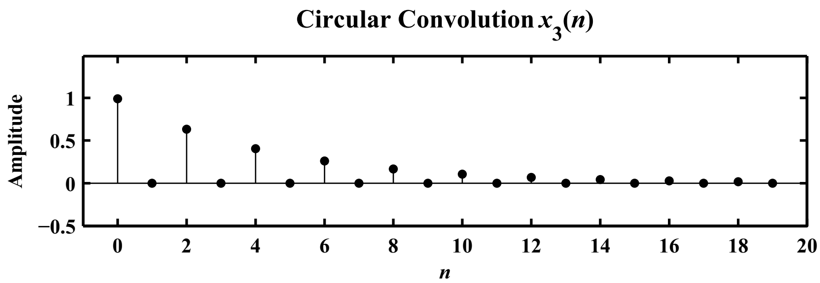


Figure 5.16: The sample plot in Problem P5.28.3

4. $x_1(n) = n\mathcal{R}_N(n), \quad x_2(n) = (N - n)\mathcal{R}_N(n); \quad N = 10$: MATLAB script:

```
N = 10; n = 0:N-1;
x1 = n; x2 = (N-n); x3 = circonvt(x1,x2,N);
Hf_4 = figure('Units','inches','position',[1,1,5,1.5],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,5,1.5]);
set(Hf_4,'NumberTitle','off','Name','P5.28.4');
H_s4 = stem(n,x3,'filled'); set(H_s4,'markersize',3);
title('Circular Convolution {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); xlabel('{\itn}'); axis([-1,N,min(x3)-10,max(x3)+10]);
```

The sample plot is shown in Figure 5.17.

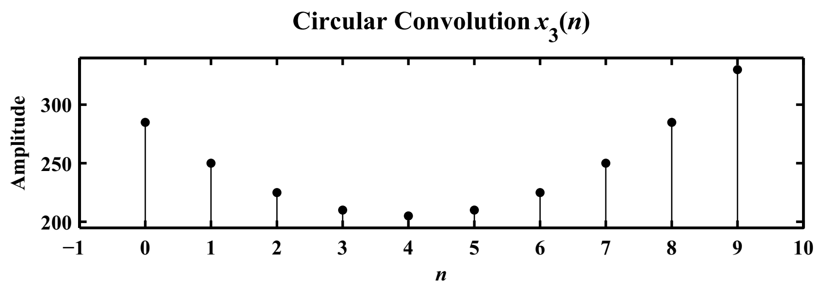


Figure 5.17: The sample plot in Problem P5.28.4

5. $x_1(n) = (0.8)^n R_{20}, x_2(n) = u(n) - u(n-40)$; $N = 50$: MATLAB script:

```
N = 50; n = 0:N-1; n1 = 0:19; x1 = (0.8).^n1;
n2 = 0:39; x2 = ones(1,40); x3 = circonvt(x1,x2,N);
Hf_5 = figure('Units','inches','position',[1,1,5,1.5],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,5,1.5]);
set(Hf_5,'NumberTitle','off','Name','P5.28.5');
H_s5 = stem(n,x3,'filled'); set(H_s5,'markersize',3);
title('Circular Convolution {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); xlabel('{\itn}'); axis([-1,N,min(x3)-1,max(x3)+1]);
```

The sample plot is shown in Figure 5.18.

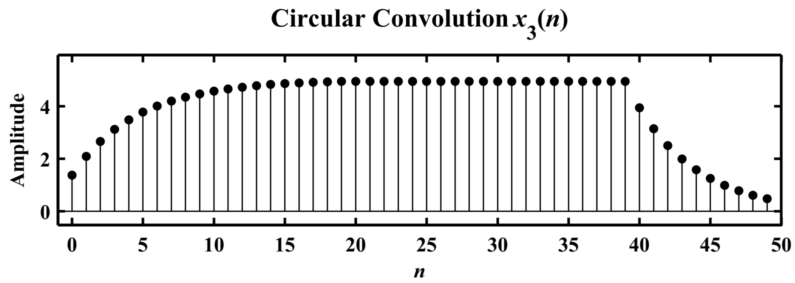


Figure 5.18: The sample plot in Problem P5.28.5

P5.29 Let $x_1(n)$ and $x_2(n)$ be two N-point sequences.

1. Since $y(n) = x_1(n) \circledcirc x_2(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2((n-k))_N$ we have

$$\begin{aligned}
\sum_{n=0}^{N-1} y(n) &= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} x_1(k)\, x_2((n-k))_N = \sum_{k=0}^{N-1} x_1(k) \sum_{n=0}^{N-1} x_2((n-k))_N \\
&= \sum_{n=0}^{N-1} x_1(n) \left[\sum_{n=0}^{k-1} x_2(n-k+N) + \sum_{n=k}^{N-1} x_2(n-k)\right] \\
&= \sum_{n=0}^{N-1} x_1(n) \left[\sum_{n=N-k}^{N-1} x_2(n) + \sum_{n=0}^{N-1-k} x_2(n)\right] = \sum_{n=0}^{N-1} x_1(n) \left[\sum_{n=0}^{N-k-1} x_2(n) + \sum_{n=N-k}^{N-1} x_2(n)\right] \\
&= \left(\sum_{n=0}^{N-1} x_1(n)\right)\left(\sum_{n=0}^{N-1} x_2(n)\right)
\end{aligned}$$

2. Verification using the following sequences:

$$x_1(n) = \{9, 4, -1, 4, -4, -1, 8, 3\}; \quad x_2(n) = \{-5, 6, 2, -7, -5, 2, 2, -2\}$$

Consider

$$x_1(n) = \{9, 4, -1, 4, -4, -1, 8, 3\} \Rightarrow \sum_{n=0}^{7} x_1(n) = 22$$

$$x_2(n) = \{-5, 6, 2, -7, -5, 2, 2, -2\} \Rightarrow \sum_{n=0}^{7} x_2(n) = -7$$

$$y(n) = x_1(n) \circledcirc x_2(n) = \{14, -9, -32, -74, -7, -16, -57, 27\} \Rightarrow \sum_{n=0}^{7} y(n) = -154$$

Hence

$$\sum_{n=0}^{7} y(n) = -154 = (22) \times (-7) = \left(\sum_{n=0}^{7} x_1(n)\right)\left(\sum_{n=0}^{7} x_2(n)\right)$$

P5.30 Let $X(k)$ be the 8-point DFT of a 3-point sequence $x(n) = \{\underset{\uparrow}{5}, -4, 3\}$. Let $Y(k)$ be the 8-point DFT of a sequence $y(n)$ where $Y(k) = W_8^{5k} X((-k))_8$. Then using the circular folding and the circular shifting properties of the DFT, we have

$$\begin{aligned}
y(n) &= \text{IDFT}\left[W_8^{5k} X((-k))_8\right] = \text{IDFT}\left[X((-k))_8\right]_{n\to(n-5)} \\
&= \left[x((-n))_8|_{n\to(n-5)}\right]\mathcal{R}_8(n) = x((5-n))_8\mathcal{R}_8(n) = \{\underset{\uparrow}{0}, 0, 0, 3, -4, 5, 0, 0\}
\end{aligned}$$

P5.31 Computation of (i) the N-point circular convolution $x_3(n) = x_1(n) \circledN x_2(n)$, (ii) the linear convolution $x_4(n) = x_1(n) * x_2(n)$, and (iii) the error sequence $e(n) = x_3(n) - x_4(n)$ for the following sequences:

1. $x_1(n) = \{1, 1, 1, 1\}$, $x_2(n) = \cos(\pi n/4)\mathcal{R}_6(n)$; $N = 8$:

```
x1 = [1,1,1,1]; x2 = cos(pi*[0:5]/4); N = 8; n = 0:N-1;
x3 = circonvt(x1,x2,N);
x4 = conv(x1,x2); n4 = 0:length(x4)-1;
e1 = x3 - x4(1:N); e2 = x4(N+1:end);
```

The plots of various signals are shown in Figure 5.19.

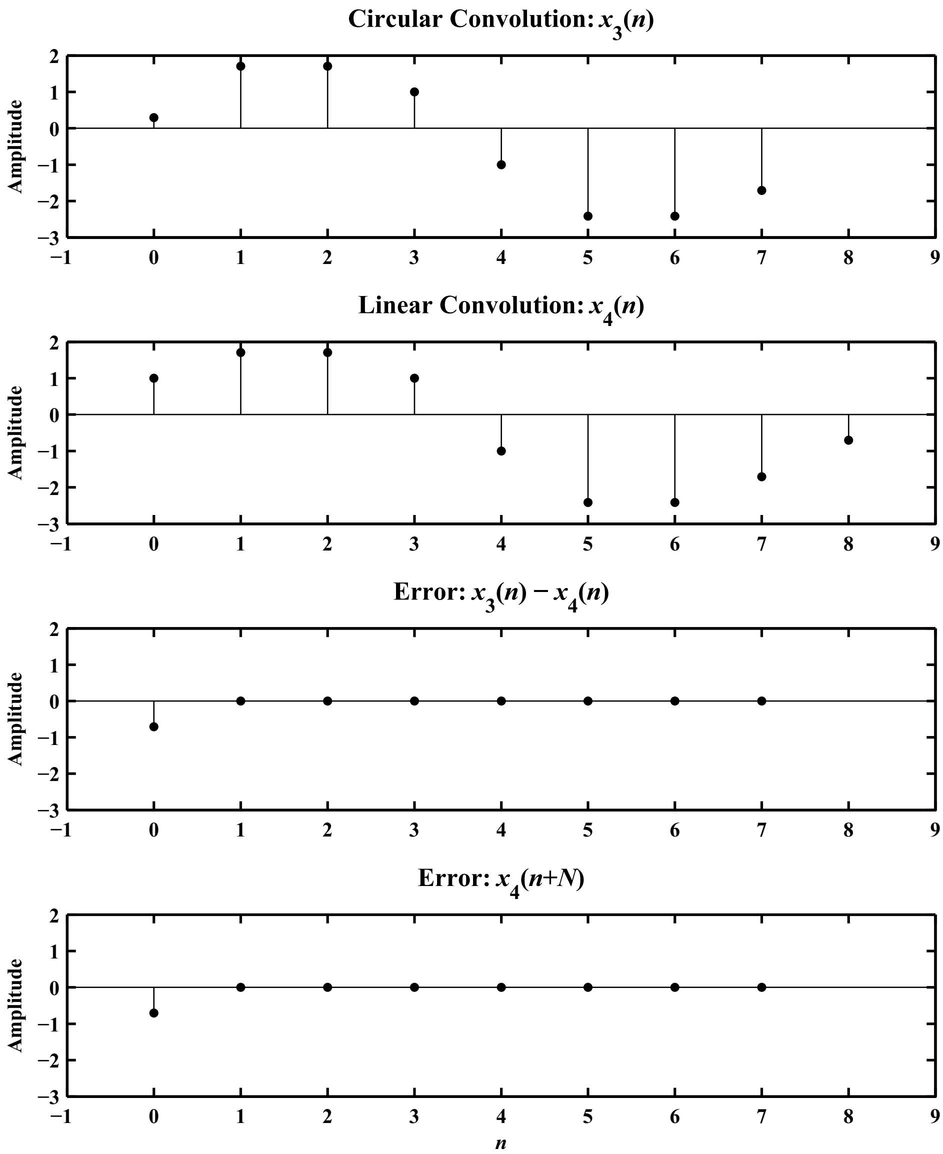


Figure 5.19: The sample plot of various signals in Problem P5.31.1

2. $x_1(n) = \cos(2\pi n/N)\mathcal{R}_{16}(n)$, $x_2(n) = \sin(2\pi n/N)\mathcal{R}_{16}(n)$; $N = 32$:

```
N = 32; x1 = cos(2*pi*[0:15]/N); x2 = sin(2*pi*[0:15]/N);
x3 = circonvt(x1,x2,N); n3 = 0:N-1;
x4 = conv(x1,x2); n4 = 0:length(x4)-1;
e1 = x3 - [x4,0];
e2 = x4(N+1:end);
```

The plots of various signals are shown in Figure 5.20.

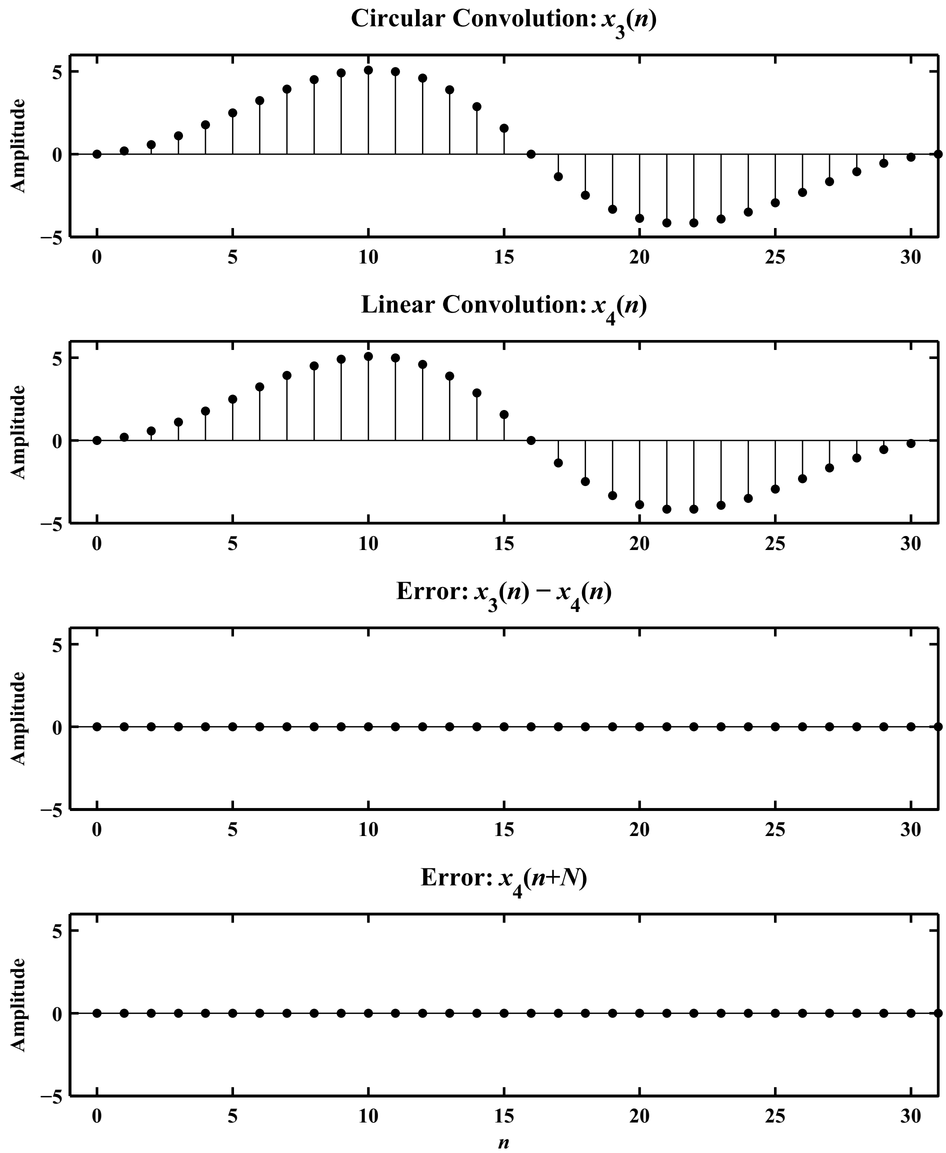


Figure 5.20: The sample plot of various signals in Problem P5.31.2

3. $x_1(n) = (0.8)^n \mathcal{R}_{10}(n), x_2(n) = (-0.8)^n \mathcal{R}_{10}(n)$; $N = 15$:

```
N = 15; x1 = (0.8).^[0:9]; x2 = (-0.8).^[0:9];
x3 = circonvt(x1,x2,N); n3 = 0:N-1;
x4 = conv(x1,x2); n4 = 0:length(x4)-1;
e1 = x3 - x4(1:N);
e2 = x4(N+1:end); Ne2 = length(e2);
```

The plots of various signals are shown in Figure 5.21.

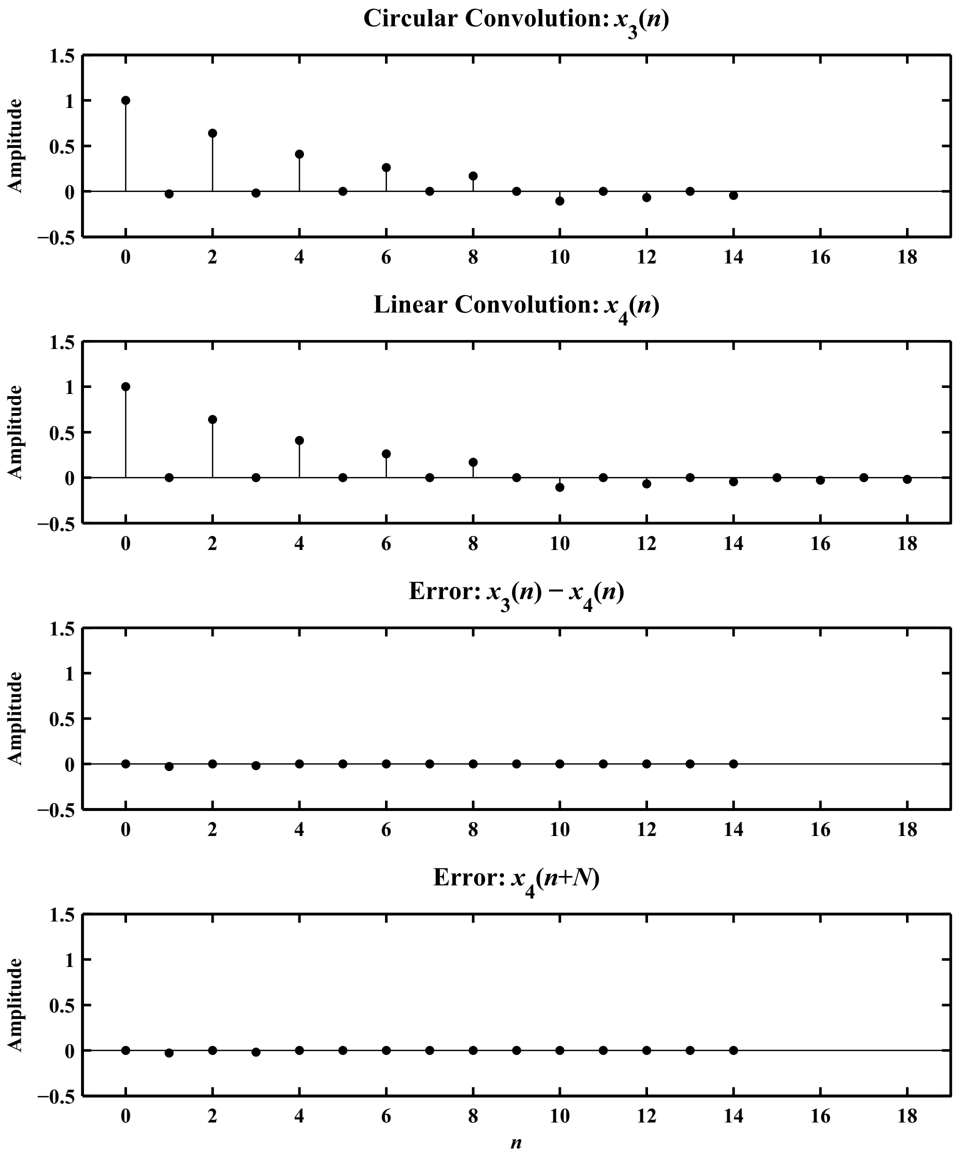


Figure 5.21: The sample plot of various signals in Problem P5.31.3

4. $x_1(n) = n\mathcal{R}_{10}(n), x_2(n) = (N - n)\mathcal{R}_{10}(n)$; $N = 10$:

```
N = 10; n = 0:N-1; x1 = n; x2 = N-n;
x3 = circonvt(x1,x2,N); n3 = 0:N-1;
x4 = conv(x1,x2); n4 = 0:length(x4)-1;
e1 = x3 - x4(1:N);
e2 = x4(N+1:end); Ne2 = length(e2);
```

The plots of various signals are shown in Figure 5.22.

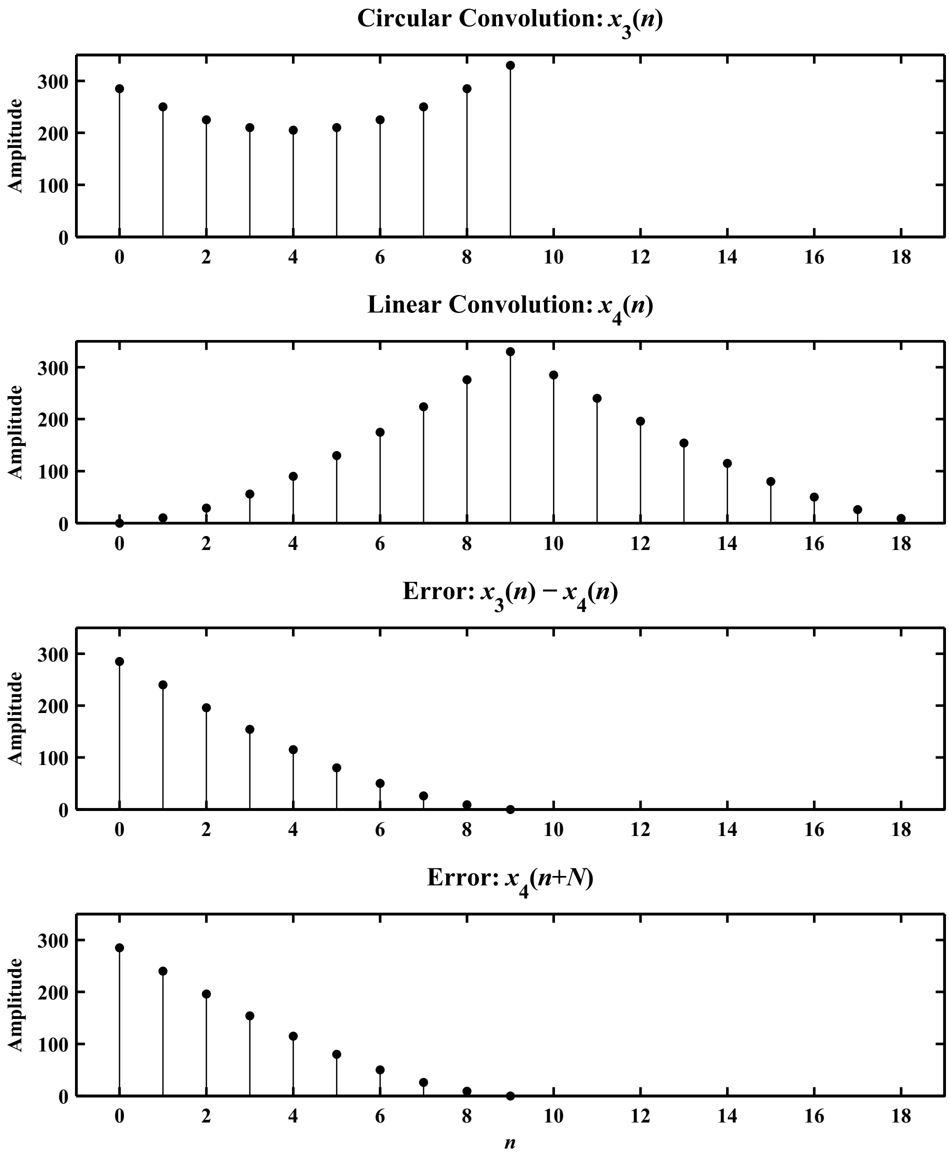


Figure 5.22: The sample plot of various signals in Problem P5.31.4

5. $x_1(n) = \{1, -1, 1, -1\}$, $x_2(n) = \{1, 0, -1, 0\}$; $N = 5$:

```
N = 5; n = 0:N-1; x1 = [1,-1,1,-1]; x2 = [1,0,-1,0];
x3 = circonvt(x1,x2,N); n3 = 0:N-1;
x4 = conv(x1,x2); n4 = 0:length(x4)-1;
e1 = x3 - x4(1:N);
e2 = x4(N+1:end); Ne2 = length(e2);
```

The plots of various signals are shown in Figure 5.23.

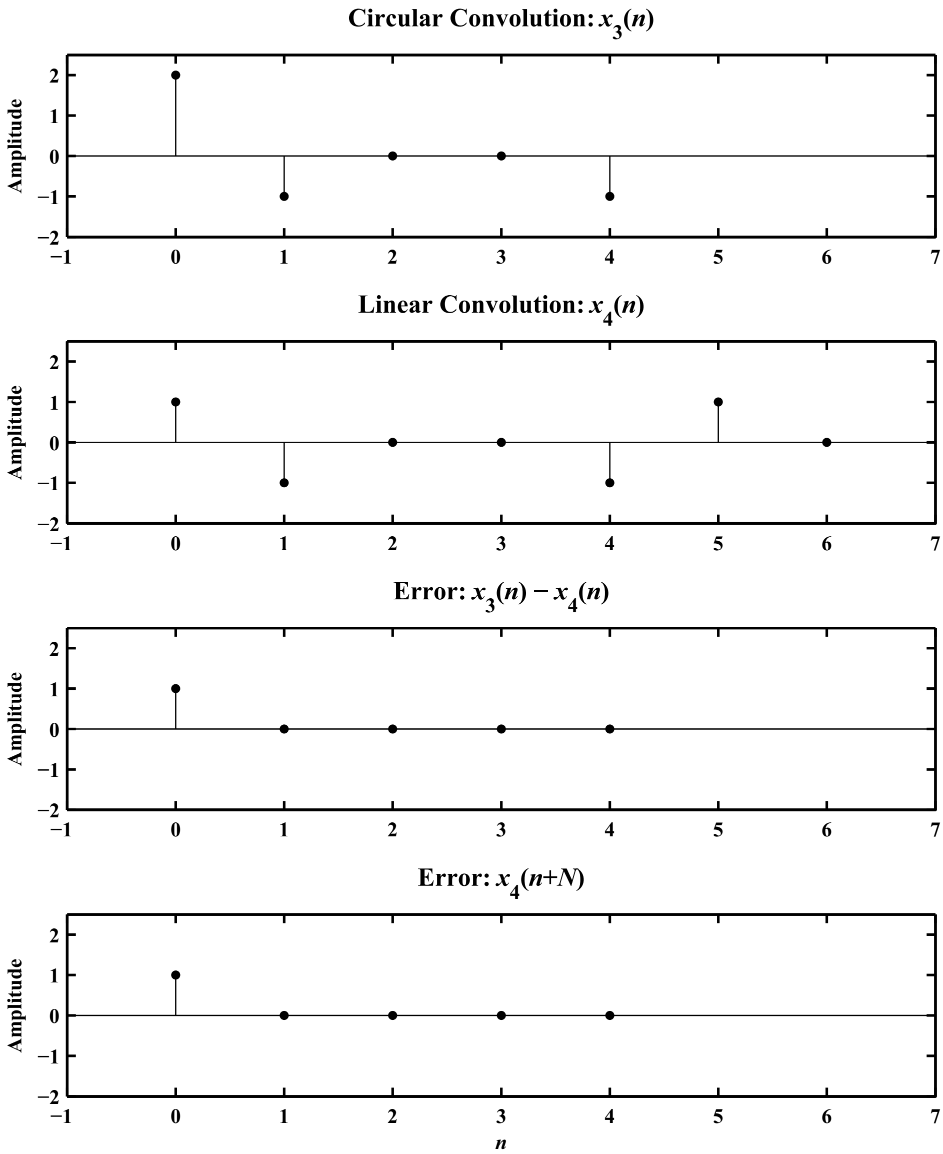


Figure 5.23: The sample plot of various signals in Problem P5.31.5

P5.32 The overlap-add method of block convolution is an alternative to the overlap-save method. Let $x(n)$ be a long sequence of length ML where $M, L \gg 1$. Divide $x(n)$ into M segments $\{x_m(n),\ m = 1, \dots, M\}$ each of length L

$$x_m(n) = \begin{cases} x(n), & mL \le n \le (m+1)L - 1 \\ 0, & \text{elsewhere} \end{cases} \quad \text{so that} \quad x(n) = \sum_{m=0}^{M-1} x_m(n)$$

Let $h(n)$ be an L-point impulse response, then

$$y(n) = x(n) * h(n) = \sum_{m=0}^{M-1} x_m(n) * h(n) = \sum_{m=0}^{M-1} y_m(n); \quad y_m(n) \triangleq x_m(n) * h(n)$$

Clearly, $y_m(n)$ is a $(2L-1)$-point sequence. In this method we have to save the intermediate convolution results and then properly overlap these before adding to form the final result $y(n)$. To use DFT for this operation we have to choose $N \ge (2L-1)$.

1. MATLAB function to implement the overlap-add method using the circular convolution operation:

```
function [y] = ovrlpadd(x,h,N)
% Overlap-Add method of block convolution
% ----------------------------------------
% [y] = ovrlpadd(x,h,N)
% y = output sequence
% x = input sequence
% h = impulse response
% N = DFT length >= 2*length(h)-1
%
Lx = length(x); L = length(h); L1 = L-1;
h = [h zeros(1,N-L)];
%
M = ceil(Lx/L);             % Number of blocks
x = [x, zeros(1,M*L-Lx)]; % append (M*N-Lx) zeros
Y = zeros(M,N);             % Initialize Y matrix
%
% convolution with succesive blocks
for m = 0:M-1
    xm = [x(m*L+1:(m+1)*L),zeros(1,N-L)];
    Y(m+1,:) = circonvt(xm,h,N);
end
%
% Overlap and Add
Y = [Y,zeros(M,1)]; Y = [Y;zeros(1,N+1)];
Y1 = Y(:,1:L); Y1 = Y1'; y1 = Y1(:);
Y2 = [zeros(1,L);Y(1:M,L+1:2*L)]; Y2 = Y2'; y2 = Y2(:);
y = y1+y2; y = y'; y = removetrailzeros(y);
```

2. The radix-2 FFT implementation for high-speed block convolution:

```
function [y] = hsolpadd(x,h)
% High-Speed Overlap-Add method of block convolution
```

```
% ---------------------------------------------------
% [y] = hsolpadd(x,h)
% y = output sequence (real-valued)
% x = input sequence (real-valued)
% h = impulse response (real-valued)
%

Lx = length(x); L = length(h); N = 2^ceil(log2(2*L-1));
H = fft(h,N);
%
M = ceil(Lx/L);           % Number of blocks
x = [x, zeros(1,M*L-Lx)]; % append (M*N-Lx) zeros
Y = zeros(M,N);           % Initialize Y matrix
%
% convolution with succesive blocks
for m = 0:M-1
    xm = [x(m*L+1:(m+1)*L),zeros(1,N-L)];
    Y(m+1,:) = real(ifft(fft(xm,N).*H,N));
end
%
% Overlap and Add
Y = [Y,zeros(M,1)]; Y = [Y;zeros(1,N+1)];
Y1 = Y(:,1:L); Y1 = Y1'; y1 = Y1(:);
Y2 = [zeros(1,L);Y(1:M,L+1:2*L)]; Y2 = Y2'; y2 = Y2(:);
y = y1+y2; y = y'; y = removetrailzeros(y);
```

3. Verification using the following two sequences

$$x(n) = \cos(\pi n/500)\,\mathcal{R}_{4000}(n), \quad h(n) = \{1, -1, 1, -1\}$$

```
n = 0:4000-1; x = cos(pi*n/500); h = [1,-1,1,-1];
y1 = ovrlpadd(x,h,7);
y2 = hsolpadd(x,h);
y3 = conv(x,h);
e1 = max(abs(y1-y3))
e2 = max(abs(y1-y2(1:end-1)))

e1 =
  2.7756e-017
e2 =
  3.6342e-016
```

P5.33 Given the sequences $x_1(n)$ and $x_2(n)$ shown below:

$$x_1(n) = \{2, 1, 1, 2\}, \quad x_2(n) = \{1, -1, -1, 1\}$$

1. Circular convolutions $x_1(n) \, N \, x_2(n)$:

   $N = 4 \ : x_1(n) \, 4 \, x_2(n) = \{0, 0, 0, 0\}$

   $N = 7 \ : x_1(n) \, 7 \, x_2(n) = \{2, -1, 2, 0, -2, 1, -2\}$

   $N = 8 \ : x_1(n) \, 8 \, x_2(n) = \{2, -1, 2, 0, -2, 1, -2, 0\}$
2. The linear convolution: $x_1(n) * x_2(n) = \{2, -1, 2, 0, -2, 1, -2\}$.
3. From the results of the above two parts, the minimum value of N to make the circular convolution equal to the linear convolution is 7.
4. If we make N equal to the length of the linear convolution which is equal to the length of $x_1(n)$ plus the length of $x_2(n)$ minus one, then the desired result can be achieved. In this case then $N = 4 + 4 - 1 = 7$, as expected.

P5.34 Let

$$x(n) = \begin{cases} A\cos(2\pi\ell n/N), & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases} = A\cos(2\pi\ell n/N)\mathcal{R}_N(n)$$

where ℓ is an integer. Notice that $x(n)$ contains *exactly* ℓ periods (or cycles) of the cosine waveform in N samples. This is a windowed cosine sequence containing *no leakage*.

1. Consider the DFT $X(k)$ of $x(n)$ which is given by

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} A\cos\left(\frac{2\pi\ell n}{N}\right) e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \\ &= \frac{A}{2}\sum_{n=0}^{N-1}\left\{e^{j\frac{2\pi}{N}\ell n} + e^{-j\frac{2\pi}{N}\ell n}\right\} e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \\ &= \frac{A}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-\ell)n} + \frac{A}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+\ell)n}, \quad 0 \le k \le N-1 \\ &= \frac{AN}{2}\delta(k-\ell) + \frac{AN}{2}\delta(k-N+\ell); \quad 0 \le k \le (N-1), \quad 0 < \ell < N \end{aligned}$$

   which is a real-valued sequence.

2. If $\ell = 0$, then the DFT $X(k)$ is given by

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} A\, e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \\ &= AN\delta(k); \quad 0 \le k \le (N-1) \end{aligned}$$

3. If $\ell < 0$ or $\ell > N$, then we must replace it by $((\ell))_N$ in the result of part 1., i.e.

$$X(k) = \frac{AN}{2}\delta\left[k - ((\ell))_N\right] + \frac{AN}{2}\delta\left[k - N + ((\ell))_N\right]; \quad 0 \le k \le (N-1)$$

4. Verification of the results of parts 1., 2., and 3. above using MATLAB and the following sequences:

   (a) $x_1(n) = 3\cos(0.04\pi n)\mathcal{R}_{200}(n)$:

```
N = 200; n = 0:N-1; x1 = 3*cos(0.04*pi*n); l = 4;
k = 0:N-1; X1 = real(fft(x1,N));
Hf_1 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_1,'NumberTitle','off','Name','P5.34.4(a)');
subplot(2,1,1); H_s1 = stem(n,x1,'g','filled'); set(H_s1,'markersize',1);
title('Sequence: {\itx}_1({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-4,4]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X1,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_1({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-10,310]); xlabel('\itk');
set(gca,'xtick',[0,l,N-l],'ytick',[0,300])
```

   The sequence $x_1(n)$ and its DFT $X_1(k)$ are shown in Figure 5.24.

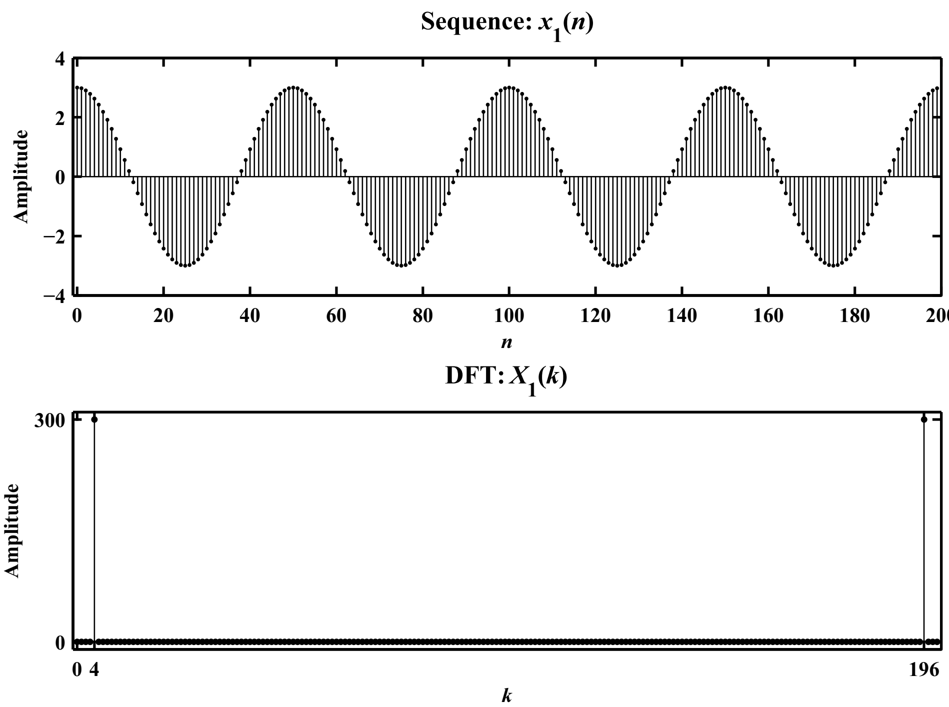


Figure 5.24: The signal $x_1(n)$ and its DFT $X_1(k)$ in Problem P5.34.4(a)

(b) $x_2(n) = 5\mathcal{R}_{50}(n)$:

```
N = 50; n = 0:N-1; x2 = 5*cos(0*pi*n); l = 0;
k = 0:N-1; X2 = real(fft(x2,N));
Hf_2 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_2,'NumberTitle','off','Name','P5.34.4(b)');
subplot(2,1,1); H_s1 = stem(n,x2,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_2({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1,6]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X2,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_2({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-10,260]); xlabel('\itk');
set(gca,'xtick',[0,N-1],'ytick',[0,250])
```

The sequence $x_2(n)$ and its DFT $X_2(k)$ are shown in Figure 5.25.

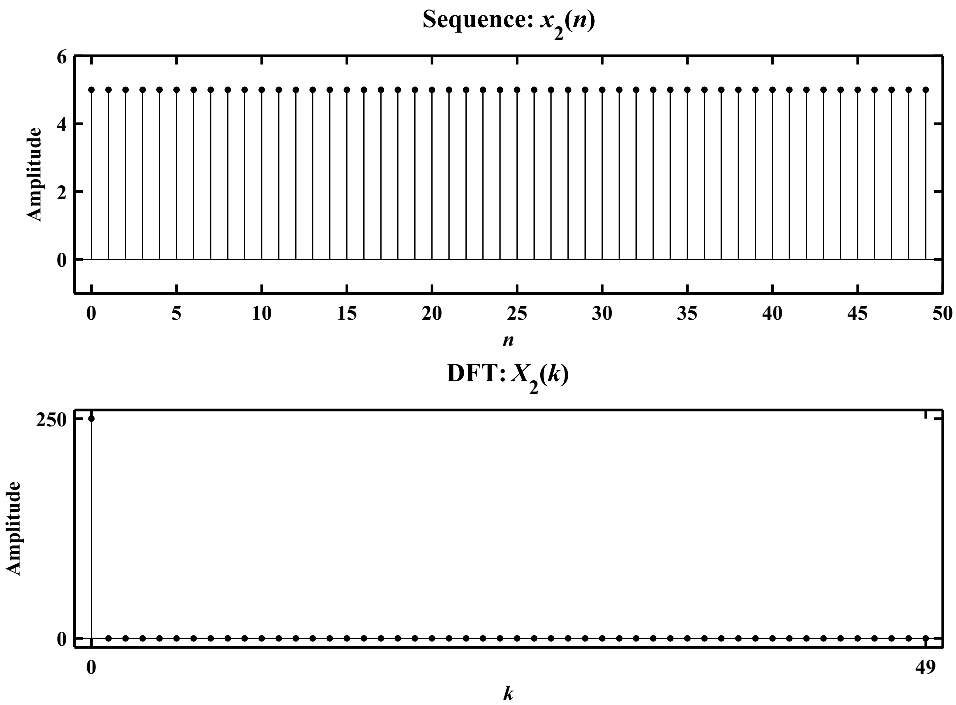


Figure 5.25: The signal $x_2(n)$ and its DFT $X_2(k)$ in Problem P5.34.4(b)

(c) $x_3(n) = [1 + 2\cos(0.5\pi n) + \cos(\pi n)]\mathcal{R}_{100}(n)$:

```
N = 100; n = 0:N-1; x3 = 1+2*cos(0.5*pi*n)+cos(pi*n); l1 = 0; l2 = 25; l3 = 50;
k = 0:N-1; X3 = real(fft(x3,N));
Hf_3 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_3,'NumberTitle','off','Name','P5.34.4(c)');
subplot(2,1,1); H_s1 = stem(n,x3,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1,5]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X3,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_3({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-10,110]); xlabel('\itk');
set(gca,'xtick',[l1,l2,l3,N-l2,N-1],'ytick',[0,100])
```

The sequence $x_3(n)$ and its DFT $X_3(k)$ are shown in Figure 5.26.

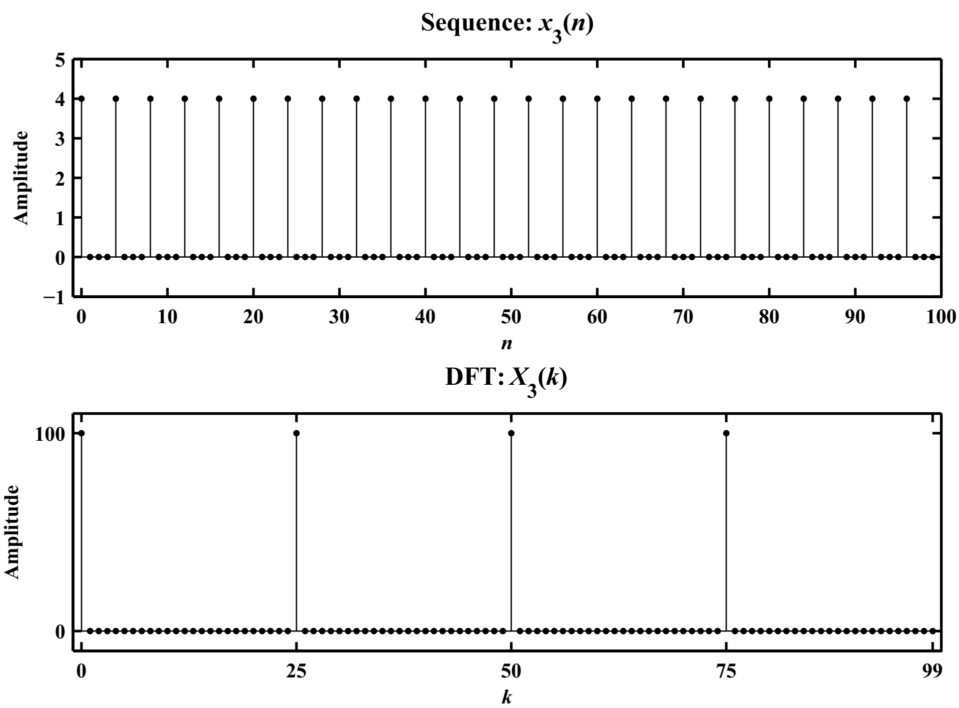


Figure 5.26: The signal $x_3(n)$ and its DFT $X_3(k)$ in Problem P5.34.4(c)

(d) $x_4(n) = \cos(25\pi n/16)\mathcal{R}_{64}(n)$:

```
N = 64; n = 0:N-1; x4 = cos(25*pi*n/16); l = 50;
k = 0:N-1; X4 = real(fft(x4,N));
Hf_4 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_4,'NumberTitle','off','Name','P5.34.4(d)');
subplot(2,1,1); H_s1 = stem(n,x4,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_4({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1.1,1.1]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X4,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_4({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-5,35]); xlabel('\itk');
set(gca,'xtick',[0,N-1,l,N-1],'ytick',[0,32])
```

The sequence $x_4(n)$ and its DFT $X_4(k)$ are shown in Figure 5.27.

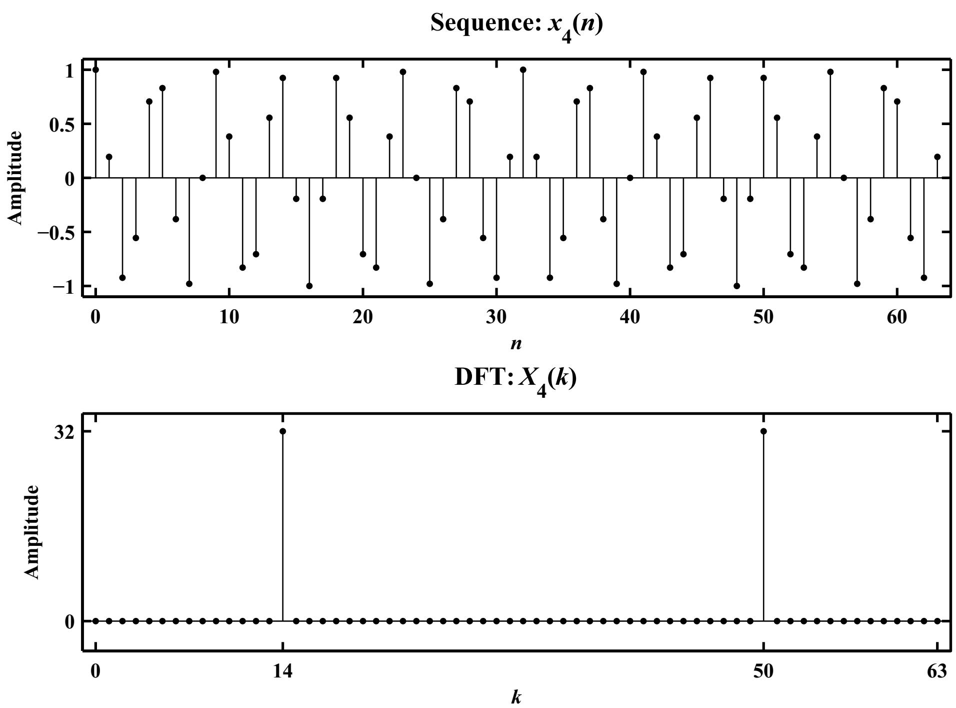


Figure 5.27: The signal $x_4(n)$ and its DFT $X_4(k)$ in Problem P5.34.4(d)

(e) $x_5(n) = [\cos(0.1\pi n) - 3\cos(1.9\pi n)]\mathcal{R}_{40}(n)$:

```
N = 40; n = 0:N-1; x5 = 4*cos(0.1*pi*n)-3*cos(1.9*pi*n); l1 = 2; l2 = 38;
k = 0:N-1; X5 = real(fft(x5,N));
Hf_5 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_5,'NumberTitle','off','Name','P5.34.4(e)');
subplot(2,1,1); H_s1 = stem(n,x5,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_5({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1.1,1.1]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X5,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_5({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-5,25]); xlabel('\itk');
set(gca,'xtick',[0,l1,l2,N],'ytick',[0,20])
```

The sequence $x_5(n)$ and its DFT $X_5(k)$ are shown in Figure 5.28.

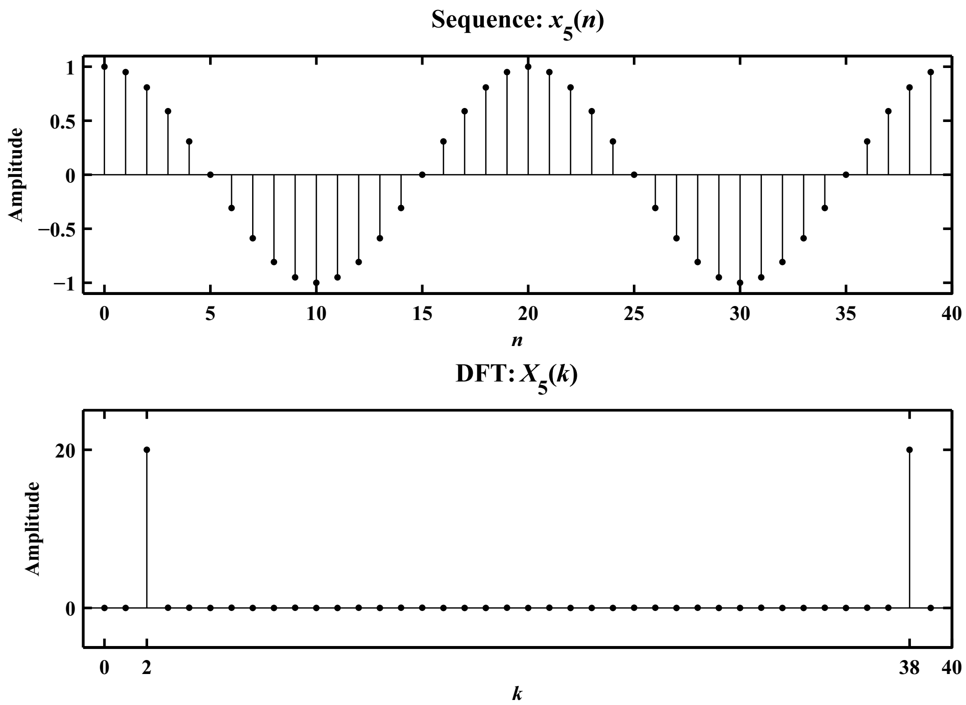


Figure 5.28: The sample plot of various signals in Problem P5.34.4(e)

P5.35 Let $x(n) = A\cos(\omega_0 n)\mathcal{R}_N(n)$, where ω_0 is a real number.

1. Consider

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}\right)nk} = A\sum_{n=0}^{N-1}\cos(\omega_0 n)\left\{\cos\left(\frac{2\pi}{N}nk\right) - j\sin\left(\frac{2\pi}{N}nk\right)\right\}$$

$$X_{\mathrm{R}}(k) + jX_{\mathrm{I}}(k) = A\sum_{n=0}^{N-1}\cos(\omega_0 n)\cos\left(\frac{2\pi}{N}nk\right) - jA\sum_{n=0}^{N-1}\cos(\omega_0 n)\sin\left(\frac{2\pi}{N}nk\right)$$

Hence

$$X_{\mathrm{R}}(k) = A\sum_{n=0}^{N-1}\cos(\omega_0 n)\cos\left(\frac{2\pi}{N}nk\right) \tag{5.5}$$

$$X_{\mathrm{I}}(k) = -A\sum_{n=0}^{N-1}\cos(\omega_0 n)\sin\left(\frac{2\pi}{N}nk\right) \tag{5.6}$$

Consider the real-part in (5.5),

$$\begin{aligned}
X_{\mathrm{R}}(k) &= A\sum_{n=0}^{N-1}\cos(\omega_0 n)\cos\left(\frac{2\pi}{N}nk\right) = \frac{A}{2}\sum_{n=0}^{N-1}\left\{\cos\left(\omega_0 n - \frac{2\pi}{N}nk\right) + \cos\left(\omega_0 n + \frac{2\pi}{N}nk\right)\right\} \\
&= \frac{A}{2}\sum_{n=0}^{N-1}\left\{\cos\left(2\pi f_0 n - \frac{2\pi}{N}nk\right) + \cos\left(2\pi f_0 n + \frac{2\pi}{N}nk\right)\right\} \qquad [\because \omega_0 = 2\pi f_0] \\
&= \frac{A}{2}\sum_{n=0}^{N-1}\left\{\cos\left[\frac{2\pi}{N}(f_0 N - k)n\right] + \cos\left[\frac{2\pi}{N}(f_0 N + k)n\right]\right\} \\
&= \frac{A}{2}\sum_{n=0}^{N-1}\left\{\cos\left[\frac{2\pi}{N}(k - f_0 N)n\right] + \cos\left[\frac{2\pi}{N}\{k - (N - f_0 N)\}n\right]\right\},\ 0 \le k < N
\end{aligned} \tag{5.7}$$

To reduce the sum-of-cosine terms in (5.7), consider

$$\begin{aligned}
\sum_{n=0}^{N-1}\cos\left(\frac{2\pi}{N}vn\right) &= \frac{1}{2}\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)vn} + \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)vn} = \frac{1}{2}\left(\frac{1-e^{j2\pi v}}{1-e^{j\frac{2\pi}{N}v}}\right) + \frac{1}{2}\left(\frac{1-e^{-j2\pi v}}{1-e^{-j\frac{2\pi}{N}v}}\right) \\
&= \frac{1}{2}e^{-j\pi v\left(\frac{N-1}{N}\right)}\frac{\sin(\pi v)}{\sin(\pi v/N)} + \frac{1}{2}e^{j\pi n\left(\frac{N-1}{N}\right)}\frac{\sin(\pi v)}{\sin(\pi v/N)} \\
&= \cos\left\{\frac{\pi v(N-1)}{N}\right\}\frac{\sin(\pi n)}{\sin(\pi v/N)}
\end{aligned} \tag{5.8}$$

Now substituting (5.8) in the first term of (5.7) with $v = (k - f_0 N)$ and in the second term of (5.7) with $v = (k - [N - f_0 N])$, we obtain the desired result

$$\begin{aligned}
X_{\mathrm{R}}(k) &= \frac{A}{2}\cos\left\{\frac{\pi(N-1)}{N}(k - f_0 N)\right\}\frac{\sin[\pi(f_0 N - k)]}{\sin[\frac{\pi}{N}(f_0 N - k)]} \\
&\quad + \frac{A}{2}\cos\left\{\frac{\pi(N-1)}{N}(k - [N - f_0 N])\right\}\frac{\sin\{\pi(k - [N - f_0 N])\}}{\sin[\frac{\pi}{N}(f_0 N - k)]}
\end{aligned} \tag{5.9}$$

Similarly, we can show that

$$\begin{aligned} X_{\mathrm{I}}(k) = &-\frac{A}{2}\sin\left\{\tfrac{\pi(N-1)}{N}(k-f_0N)\right\}\frac{\sin[\pi(f_0N-k)]}{\sin[\frac{\pi}{N}(f_0N-k)]} \\ &-\frac{A}{2}\sin\left\{\tfrac{\pi(N-1)}{N}(k-[N-f_0N])\right\}\frac{\sin\{\pi(k-[N-f_0N])\}}{\sin[\frac{\pi}{N}(f_0N-k)]} \end{aligned} \tag{5.10}$$

2. The above result implies that the original frequency ω_0 of the cosine waveform has *leaked* into other frequencies that form the harmonics of the time-limited sequence and hence it is called the leakage property of cosines. It is a natural result due to the fact that bandlimited periodic cosines are sampled over noninteger periods. Due to this fact, the periodic extension of $x(n)$ does not result in a continuation of the cosine waveform but has a jump at every N interval. This jump results in the leakage of one frequency into the abducent frequencies and hence the result of the Problem P5.34.1 do not apply.
3. Verification of the leakage property using $x(n) = \cos(5\pi n/99)\,\mathcal{R}_{200}(n)$: The sequence $x(n)$, the real-part of its DFT $X_{\mathrm{R}}(k)$, and the imaginary part of its DFT $X_{\mathrm{I}}(k)$ are shown in Figure 5.29.

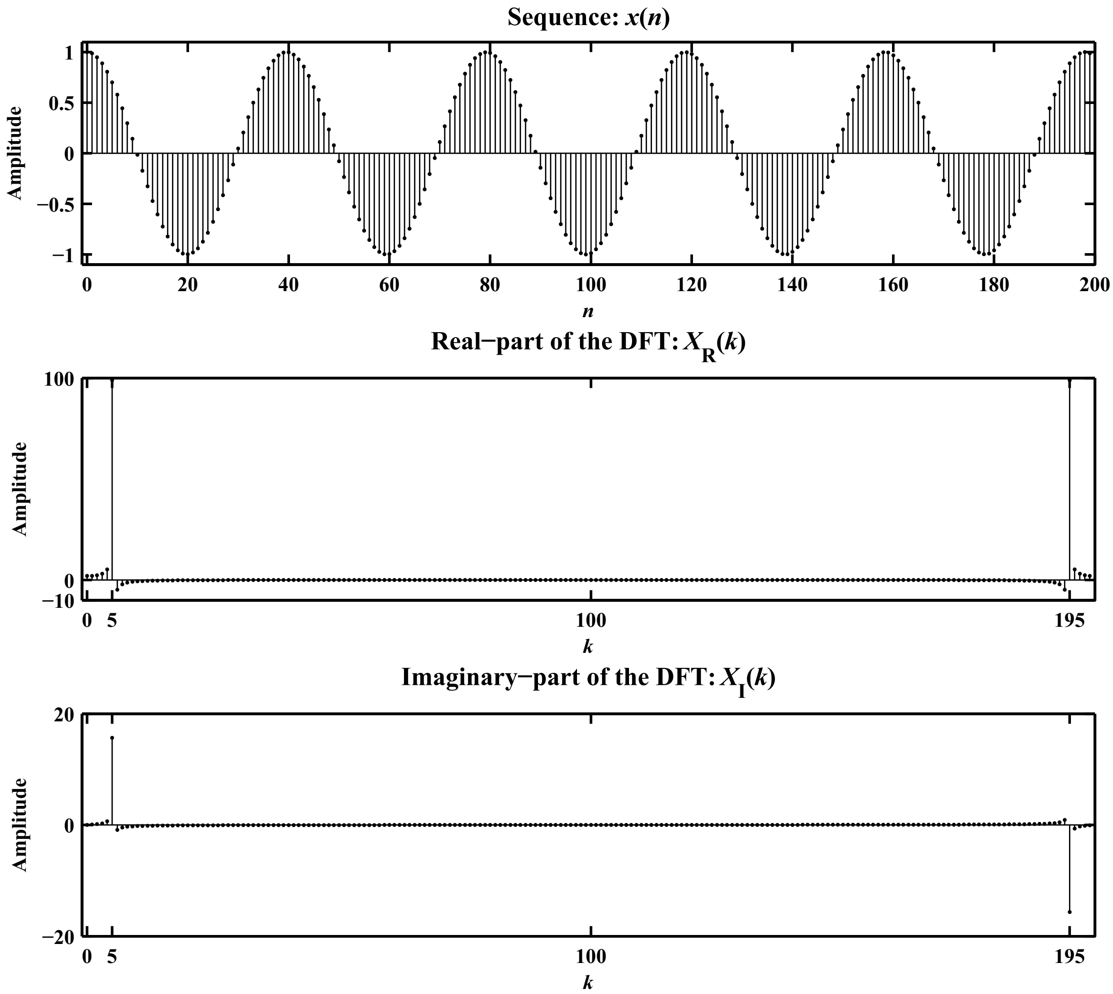


Figure 5.29: The leakage property of a cosine signal in Problem P5.35.3

P5.36 Let

$$x(n) = \begin{cases} A\sin(2\pi \ell n/N), & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases} = A\sin(2\pi \ell n/N)\mathcal{R}_N(n)$$

where ℓ is an integer. Notice that $x(n)$ contains *exactly* ℓ periods (or cycles) of the sine waveform in N samples. This is a windowed sine sequence containing *no leakage*.

1. Consider the DFT $X(k)$ of $x(n)$ which is given by

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} A\sin\left(\frac{2\pi \ell n}{N}\right) e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \\
&= \frac{A}{j2}\sum_{n=0}^{N-1}\left\{e^{j\frac{2\pi}{N}\ell n} - e^{-j\frac{2\pi}{N}\ell n}\right\} e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \\
&= \frac{A}{j2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-\ell)n} - \frac{A}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+\ell)n}, \quad 0 \le k \le N-1 \\
&= \frac{AN}{j2}\delta\,(k-\ell) - \frac{AN}{j2}\delta(k-N+\ell); \quad 0 \le k \le (N-1), \quad 0 < \ell < N
\end{aligned}$$

   which is a purely imaginary-valued sequence.
2. If $\ell = 0$, then the DFT $X(k)$ is given by

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} 0\, e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le (N-1) \\
&= 0; \quad 0 \le k \le (N-1)
\end{aligned}$$

3. If $\ell < 0$ or $\ell > N$, then we must replace it by $((\ell))_N$ in the result of part 1., i.e.

$$X(k) = \frac{AN}{j2}\delta\left[k - ((\ell))_N\right] - \frac{AN}{j2}\delta\left[k - N + ((\ell))_N\right]; \quad 0 \le k \le (N-1)$$

4. Verification of the results of parts 1., 2., and 3. above using MATLAB and the following sequences:

   (a) $x_1(n) = 3\sin(0.04\pi n)\mathcal{R}_{200}(n)$:

```
N = 200; n = 0:N-1; x1 = 3*sin(0.04*pi*n); l = 4;
k = 0:N-1; X1 = imag(fft(x1,N));
Hf_1 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_1,'NumberTitle','off','Name','P5.34.4(a)');
subplot(2,1,1); H_s1 = stem(n,x1,'g','filled'); set(H_s1,'markersize',1);
title('Sequence: {\itx}_1({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-4,4]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X1,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_1({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-350,350]); xlabel('\itk');
set(gca,'xtick',[0,l,N-l],'ytick',[-300,0,300])
```

   The sequence $x_1(n)$ and its DFT $X_1(k)$ are shown in Figure 5.30.

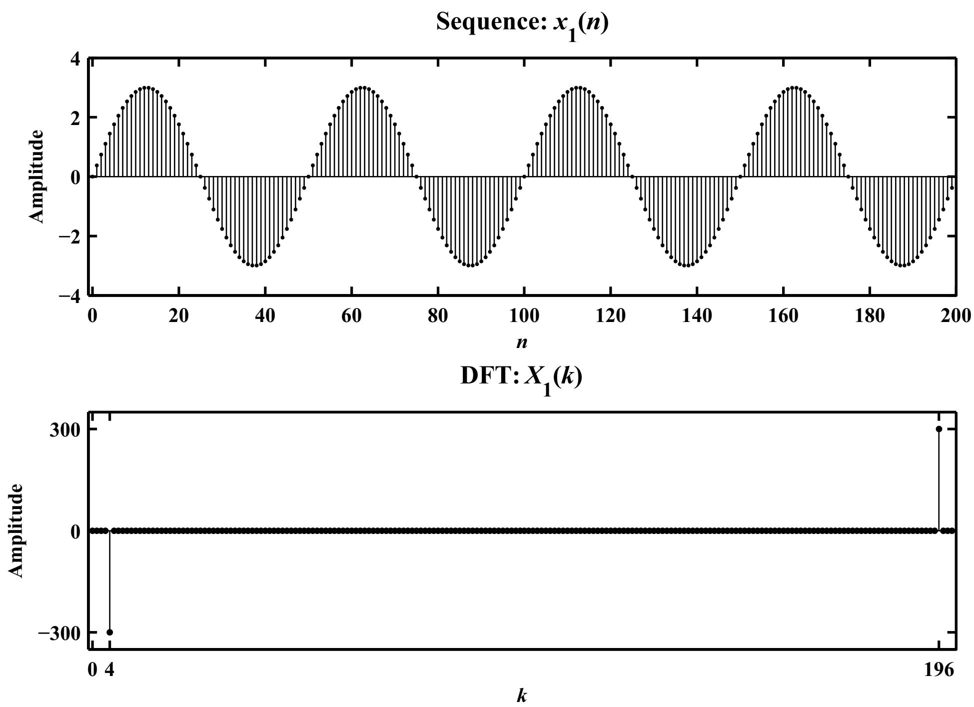


Figure 5.30: The signal $x_1(n)$ and its DFT $X_1(k)$ in Problem P5.36.6(a)

(b) $x_2(n) = 5\sin(10\pi n)\mathcal{R}_{50}(n)$:

```
N = 50; n = 0:N-1; x2 = 5*sin(10*pi*n); l = 0;
k = 0:N-1; X2 = imag(fft(x2,N));
Hf_2 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_2,'NumberTitle','off','Name','P5.34.4(b)');
subplot(2,1,1); H_s1 = stem(n,x2,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_2({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1,1]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X2,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_2({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1,1]); xlabel('\itk');
set(gca,'xtick',[0,N-1],'ytick',[-1,0,1])
```

The sequence $x_1(n)$ and its DFT $X_1(k)$ are shown in Figure 5.31.

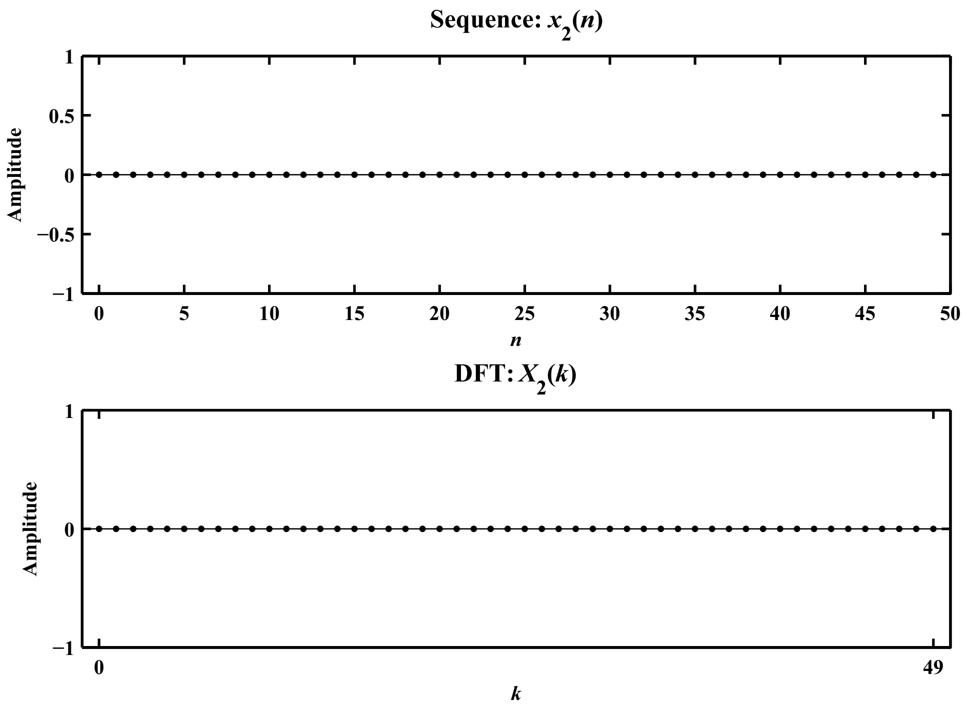


Figure 5.31: The signal $x_1(n)$ and its DFT $X_1(k)$ in Problem P5.34.6(b)

(c) $x_3(n) = [2\sin(0.5\pi n) + \sin(\pi n)]\mathcal{R}_{100}(n)$:

```
N = 100; n = 0:N-1; x3 = 2*sin(0.5*pi*n)+0*sin(pi*n); l1 = 0; l2 = 25; l3 = 50;
k = 0:N-1; X3 = imag(fft(x3,N));
Hf_3 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_3,'NumberTitle','off','Name','P5.34.4(c)');
subplot(2,1,1); H_s1 = stem(n,x3,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_3({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-3,3]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X3,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_3({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-120,120]); xlabel('\itk');
set(gca,'xtick',[l1,l2,l3,N-l2,N-1],'ytick',[-100,0,100])
```

The sequence $x_1(n)$ and its DFT $X_1(k)$ are shown in Figure 5.32.

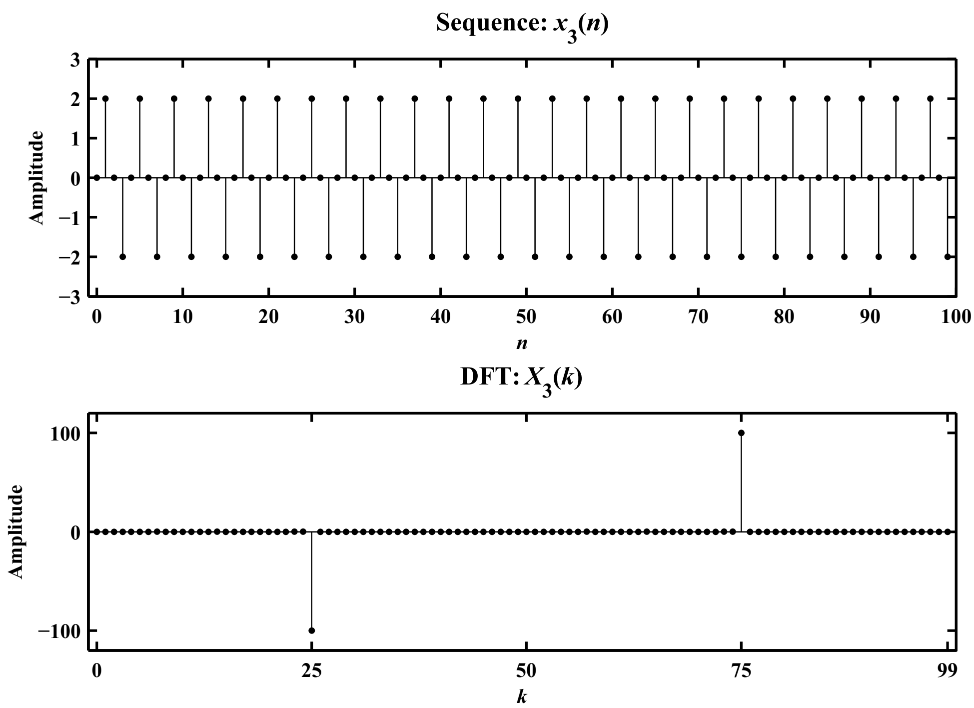


Figure 5.32: The signal $x_1(n)$ and its DFT $X_1(k)$ in Problem P5.34.6(c)

(d) $x_4(n) = \sin(25\pi n/16)\mathcal{R}_{64}(n)$:

```
N = 64; n = 0:N-1; x4 = sin(25*pi*n/16); l = 50;
k = 0:N-1; X4 = imag(fft(x4,N));
Hf_4 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_4,'NumberTitle','off','Name','P5.34.4(d)');
subplot(2,1,1); H_s1 = stem(n,x4,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_4({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-1.1,1.1]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X4,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_4({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-40,40]); xlabel('\itk');
set(gca,'xtick',[0,N-1,l,N-1],'ytick',[-32,0,32])
```

The sequence $x_1(n)$ and its DFT $X_1(k)$ are shown in Figure 5.33.

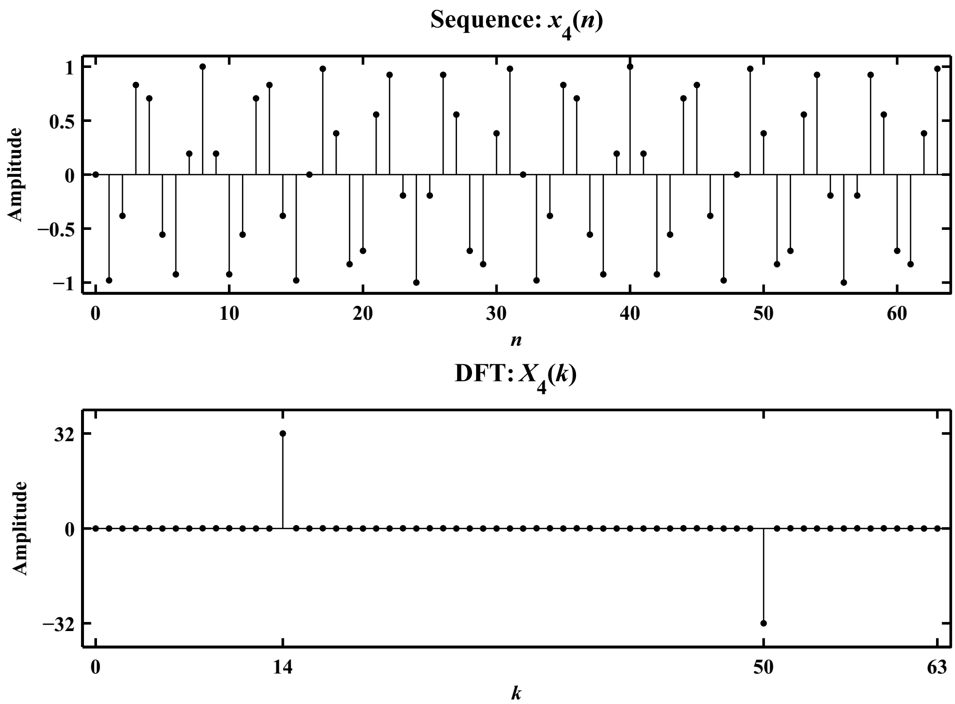


Figure 5.33: The signal $x_1(n)$ and its DFT $X_1(k)$ in Problem P5.34.6(d)

(e) $x_5(n) = [4\sin(0.1\pi n) - 3\sin(1.9\pi n)]\mathcal{R}_{20}(n)$:

```
N = 20; n = 0:N-1; x5 = 4*sin(0.1*pi*n)-3*sin(1.9*pi*n); l1 = 1; l2 = 19;
k = 0:N-1; X5 = imag(fft(x5,N));
Hf_5 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_5,'NumberTitle','off','Name','P5.34.4(e)');
subplot(2,1,1); H_s1 = stem(n,x5,'g','filled'); set(H_s1,'markersize',2);
title('Sequence: {\itx}_5({\itn})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-10,10]); xlabel('\itn');
subplot(2,1,2); H_s2 = stem(n,X5,'r','filled'); set(H_s2,'markersize',2);
title('DFT: {\itX}_5({\itk})','fontsize',10);
ylabel('Amplitude'); axis([-1,N,-80,80]); xlabel('\itk');
set(gca,'xtick',[0,l1,l2,N],'ytick',[-70,0,70])
```

The sequence $x_1(n)$ and its DFT $X_1(k)$ are shown in Figure 5.34.

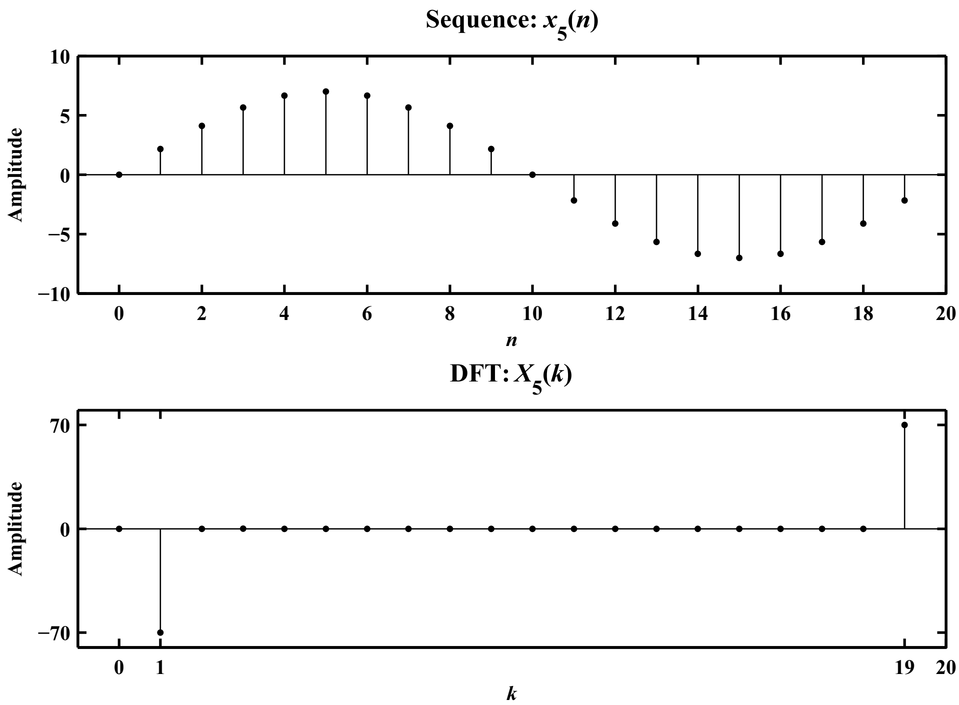


Figure 5.34: The signal $x_1(n)$ and its DFT $X_1(k)$ in Problem P5.34.6(e)

P5.37 Let $x(n) = A\sin(\omega_0 n)\mathcal{R}_N(n)$, where ω_0 is a real number.

1. Consider

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\left(\frac{2\pi}{N}\right)nk} = A\sum_{n=0}^{N-1}\sin(\omega_0 n)\left\{\cos\left(\frac{2\pi}{N}nk\right) - j\sin\left(\frac{2\pi}{N}nk\right)\right\}$$

$$X_R(k) + jX_I(k) = A\sum_{n=0}^{N-1}\sin(\omega_0 n)\cos\left(\frac{2\pi}{N}nk\right) - jA\sum_{n=0}^{N-1}\sin(\omega_0 n)\sin\left(\frac{2\pi}{N}nk\right)$$

Hence

$$X_R(k) = A\sum_{n=0}^{N-1}\sin(\omega_0 n)\cos\left(\frac{2\pi}{N}nk\right) \tag{5.11}$$

$$X_I(k) = -A\sum_{n=0}^{N-1}\sin(\omega_0 n)\sin\left(\frac{2\pi}{N}nk\right) \tag{5.12}$$

Consider the real-part in (5.11),

$$\begin{aligned}
X_R(k) &= A\sum_{n=0}^{N-1}\sin(\omega_0 n)\cos\left(\frac{2\pi}{N}nk\right) = \frac{A}{2}\sum_{n=0}^{N-1}\left\{\sin\left(\omega_0 n - \frac{2\pi}{N}nk\right) + \sin\left(\omega_0 n + \frac{2\pi}{N}nk\right)\right\}\\
&= \frac{A}{2}\sum_{n=0}^{N-1}\left\{\sin\left(2\pi f_0 n - \frac{2\pi}{N}nk\right) + \sin\left(2\pi f_0 n + \frac{2\pi}{N}nk\right)\right\} \qquad [\because \omega_0 = 2\pi f_0]\\
&= \frac{A}{2}\sum_{n=0}^{N-1}\left\{\sin\left[\frac{2\pi}{N}(f_0 N - k)n\right] + \sin\left[\frac{2\pi}{N}(f_0 N + k)n\right]\right\}\\
&= \frac{A}{2}\sum_{n=0}^{N-1}\left\{-\sin\left[\frac{2\pi}{N}(k - f_0 N)n\right] + \sin\left[\frac{2\pi}{N}\{k - (N - f_0 N)\}n\right]\right\},\ 0 \le k < N
\end{aligned} \tag{5.13}$$

To reduce the sum-of-sine terms in (5.13), consider

$$\begin{aligned}
\sum_{n=0}^{N-1}\sin\left(\frac{2\pi}{N}vn\right) &= \frac{1}{j2}\sum_{n=0}^{N-1}e^{j\left(\frac{2\pi}{N}\right)vn} - \frac{1}{j2}\sum_{n=0}^{N-1}e^{-j\left(\frac{2\pi}{N}\right)vn} = \frac{1}{j2}\left(\frac{1-e^{j2\pi v}}{1-e^{j\frac{2\pi}{N}v}}\right) - \frac{1}{j2}\left(\frac{1-e^{-j2\pi v}}{1-e^{-j\frac{2\pi}{N}v}}\right)\\
&= \frac{1}{j2}e^{j\pi v\left(\frac{N-1}{N}\right)}\frac{\sin(\pi v)}{\sin(\pi v/N)} - \frac{1}{j2}e^{-j\pi n\left(\frac{N-1}{N}\right)}\frac{\sin(\pi v)}{\sin(\pi v/N)}\\
&= \sin\left\{\frac{\pi v(N-1)}{N}\right\}\frac{\sin(\pi v)}{\sin(\pi v/N)}
\end{aligned} \tag{5.14}$$

Now substituting (5.14) in the first term of (5.13) with $v = (k - f_0 N)$ and in the second term of (5.13) with $v = (k - [N - f_0 N])$, we obtain the desired result

$$\begin{aligned}
X_R(k) = &-\frac{A}{2}\sin\left\{\frac{\pi(N-1)}{N}(k - f_0 N)\right\}\frac{\sin[\pi(f_0 N - k)]}{\sin[\frac{\pi}{N}(f_0 N - k)]}\\
&+\frac{A}{2}\sin\left\{\frac{\pi(N-1)}{N}(k - [N - f_0 N])\right\}\frac{\sin\{\pi(k - [N - f_0 N])\}}{\sin\{\frac{\pi}{N}(f_0 N - k)\}}
\end{aligned} \tag{5.15}$$

Similarly, we can show that

$$X_{\mathrm{I}}(k) = -\frac{A}{2}\sin\left\{\frac{\pi(N-1)}{N}(k - f_0N)\right\}\frac{\sin[\pi(f_0N-k)]}{\sin[\frac{\pi}{N}(f_0N-k)]} + \frac{A}{2}\sin\left\{\frac{\pi(N-1)}{N}(k-[N-f_0N])\right\}\frac{\sin\{\pi(k-[N-f_0N])\}}{\sin[\frac{\pi}{N}(f_0N-k)]} \tag{5.16}$$

2. The above result is the leakage property of sines. It implies that the original frequency ω_0 of the sine waveform has *leaked* into other frequencies that form the harmonics of the time-limited sequence. It is a natural result due to the fact that bandlimited periodic sines are sampled over noninteger periods. Due to this fact, the periodic extension of $x(n)$ does not result in a continuation of the sine waveform but has a jump at every N interval. This jump results in the leakage of one frequency into the abducent frequencies and hence the result of the Problem P5.36.1 do not apply.
3. Verification of the leakage property using $x(n) = \sin(5\pi n/99)\mathcal{R}_{100}(n)$: The sequence $x(n)$, the real-part of its DFT $X_{\mathrm{R}}(k)$, and the imaginary part of its DFT $X_{\mathrm{I}}(k)$ are shown in Figure 5.35.

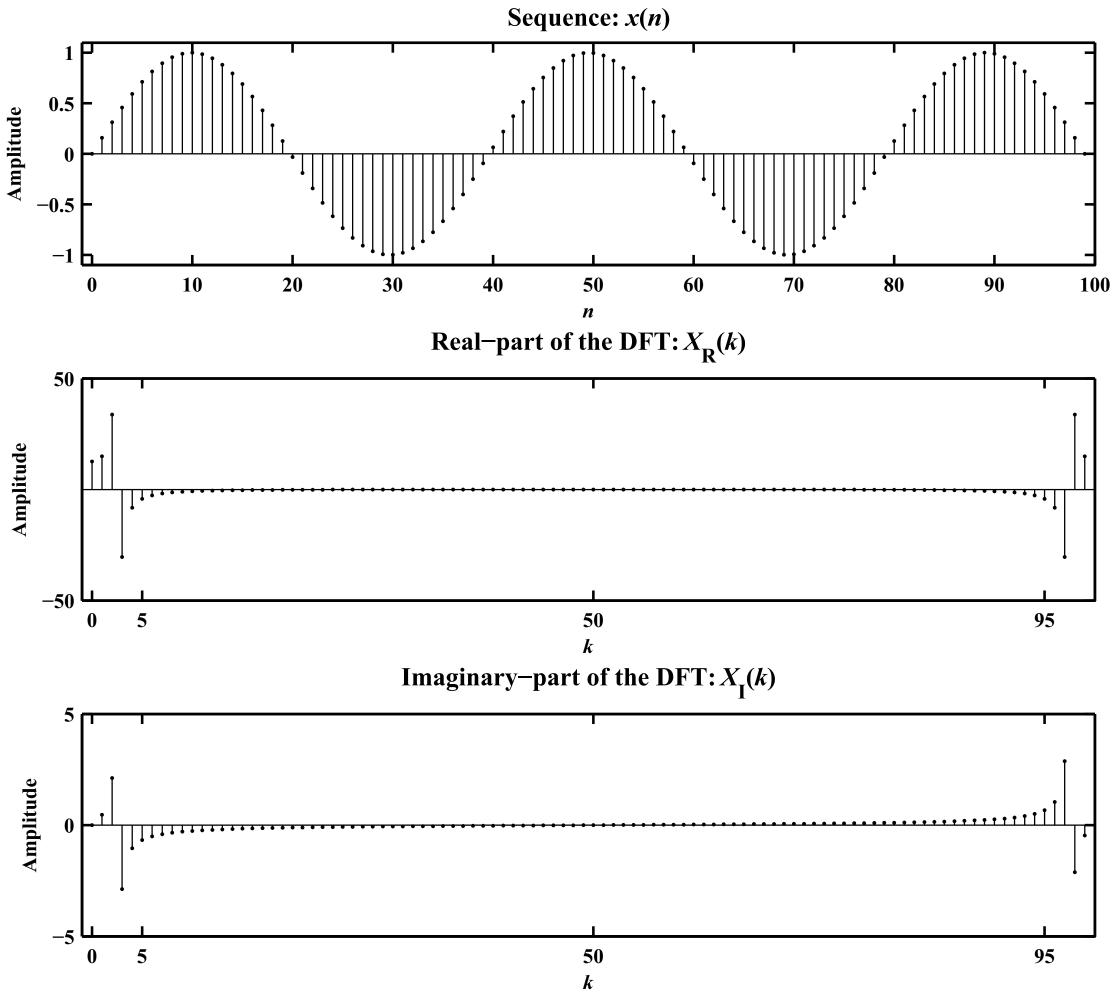


Figure 5.35: The leakage property of a sine signal in Problem P5.37.3

P5.38 An analog signal $x_a(t) = 2\sin(4\pi t) + 5\cos(8\pi t)$ is sampled at $t = 0.01n$ for $n = 0, 1, \ldots, N-1$ to obtain an N-point sequence $x(n)$. An N-point DFT is used to obtain an estimate of the magnitude spectrum of $x_a(t)$.

1. Out of the given three values, $N = 50$ provides complete cycles of both the sine and the cosine components. Thus $N = 50$ provides the most accurate estimate as shown in Figure 5.36.

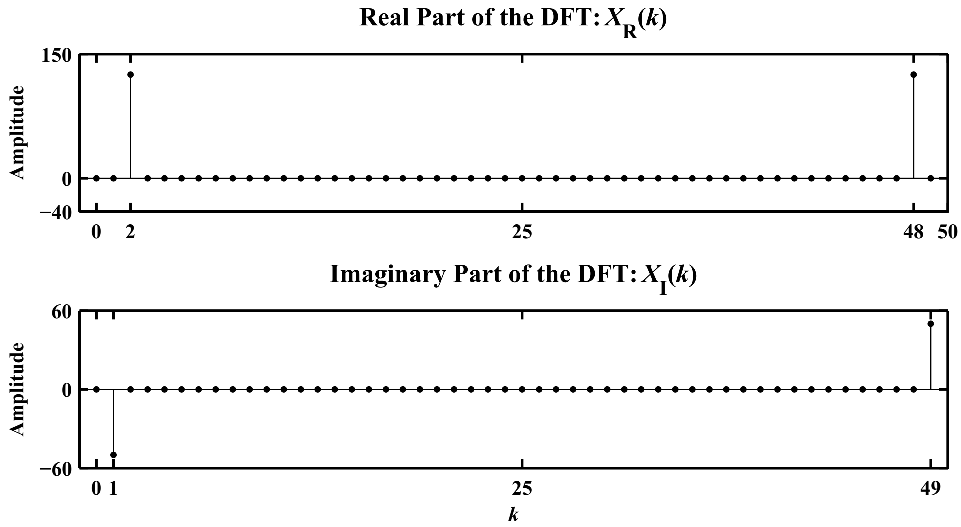


Figure 5.36: The accurate spectrum of the signal in Problem P5.38.1

2. Out of the given three values, $N = 99$ provides almost complete cycles of both the sine and the cosine components. Thus $N = 99$ provides the least amount of leakage as shown in Figure 5.37.

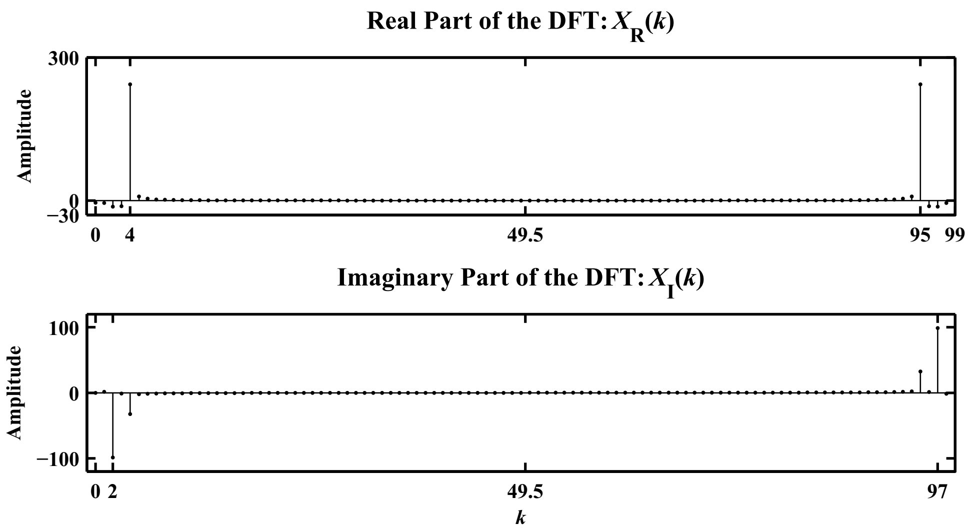


Figure 5.37: The least amount of leakage in the spectrum of the signal in Problem P5.38.2

P5.39 Using (5.49), determine and draw the signal flow graph for the $N = 8$ point, radix-2 decimation-in-frequency FFT algorithm. Using this flow graph, determine the DFT of the sequence

$$x(n) = \cos(\pi n/2), \quad 0 \le n \le 7.$$

P5.40 Using (5.49), determine and draw the signal flow graph for the $N = 16$ point, radix-4 decimation-in-time FFT algorithm. Using this flow graph determine the DFT of the sequence

$$x(n) = \cos(\pi n/2), \quad 0 \le n \le 15.$$

P5.41 Let $x(n)$ be a uniformly distributed random number between $[-1, 1]$ for $0 \le n \le 10^6$. Let

$$h(n) = \sin(0.4\pi n), \quad 0 \le n \le 100$$

1. Using the `conv` function, determine the output sequence $y(n) = x(n) * h(n)$.
2. Consider the overlap-and-save method of block convolution along with the FFT algorithm to implement high-speed block convolution. Using this approach, determine $y(n)$ with FFT sizes of 1024, 2048, and 4096.
3. Compare the above approaches in terms of the convolution results and their execution times.