

[REDACTED] Final Project (Entropic Lattice Boltzmann method)

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November 25, 2019

1 The Lattice Boltzmann Method

Macroscopic quantities in fluids are a result of microscopic interactions of all the molecules constituting that fluid. However, the number of molecules in a system is very large, about the order of Avogadro number (10^{23}) which makes it infeasible for simulations. In 1859, Maxwell proposed that a probabilistic understanding of microscopic interactions can give a sufficient idea of the behaviour of fluids on observable scales. This is the main idea of the Lattice Boltzmann Method (hereafter, LBM). In LBM, dynamics of fluids is explained by probability density functions, $f(\vec{x}, \vec{v}, t)$, called population vectors which provide the fraction of particles that have velocity in the range $(\vec{v}, \vec{v} + d\vec{v})$ at a location \vec{x} at time t .

$$\int f(\vec{x}, \vec{v}, t) d\vec{x} d\vec{v} = \text{Total number of particles in a system at time } t \quad (1)$$

Hereafter, we focus on the evolution of the population vectors over the phase space (over \vec{x}, \vec{v} and t). The discretised evolution of the population vectors over time dt can be approximated which leads to the Lattice Boltzmann Equation (hereafter LBE),

$$f(\vec{x} + d\vec{x}, \vec{v} + d\vec{v}, t + dt) - f(\vec{x}, \vec{v}, t) = \frac{df}{dt} + \vec{v} \cdot \frac{df}{d\vec{x}} + \frac{\vec{F}}{m} \cdot \frac{df}{d\vec{v}} = \Delta(f) \quad (2)$$

where \vec{F} is the external force and $\Delta(f)$ is the collision operator, which represents the change in the population vector due to collisions between the molecules.

However, in the derivation of LBE is done by considering only two particle collisions. Moreover, the molecules in this model act as point particles and interact over a short range in the phase space. In LBE, the left hand side represents the evolution of particles due to the external force and the right hand side is termed as the collision operator.

The next step is to discretise the LBE for simulations. However, before we look into the discretised form of the equation, we look into the equilibrium distribution function/ equilibrium population (f^{eq}) when all there are no more collisions in the system, i.e, the collision operator is zero. An 'acceptable' form of f must satisfy the conservation laws,

$$\int f d\vec{v} = \rho \quad \text{Mass conservation} \quad (3)$$

$$\int f \vec{v} d\vec{v} = \rho \vec{u} \quad \text{Momentum conservation} \quad (4)$$

$$\int f \frac{v^2}{2} d\vec{v} = \rho E \quad \text{Energy conservation} \quad (5)$$

In other words, the population vectors provide the necessary macroscopic quantities at each location in time.

In 1972, Boltzmann provided an expression for f^{eq} , i.e., the equilibrium thermodynamic state with energy level E ,

$$f^{eq}(E) = A e^{-\frac{E}{k_B T}} \quad (6)$$

where the constant A is found by normalising the f^{eq} over the velocity space. In three dimensions,

$$f^{eq}(E = \frac{mv^2}{2}) = \rho \left(\frac{m}{2\pi k_B T} \right)^{3/2} e^{-\frac{mv^2}{2k_B T}} \quad (7)$$

2 Discretised LBE and BGK collision operator

To discretise the LBE, a corresponding lattice model, indicating the discrete velocity directions must be specified. A $DmQn$ model is an m -dimensional lattice model with n discrete velocity directions.

The discretised LBE is,

$$f_i(\vec{x}_i + \vec{c}_i \Delta t, t + \Delta t) = f_i(\vec{x}_i, t) + \Delta(f_i) \quad (8)$$

f_i, x_i, c_i , ($i = 0, 1, \dots, n - 1$) are specified along the discrete velocity directions based on the lattice model.

In 1964, a simplified form of the collision operator was proposed by Bhatnagar, Gross and Krook (BGK) which used a single relaxation time (SRT) to dictate the evolution of the population vectors, while it satisfies the basic conservation laws given in Eq.(3-4). The BGK collision operator is given by,

$$\Delta(f_i) = \frac{f_i^{eq} - f_i}{\tau} \quad (9)$$

The BGK collision operator has a single relaxation time scale, τ which makes it simple for implementation while preserving the macroscopic dynamics. In conventional LBM, the expression for f^{eq} is derived based on a Taylor-series approximation in the regime of low Mach numbers, in terms of the speed of sound, $c_s = c/\sqrt{3}$ and c is the magnitude of a single discretised velocity component.

$$f_i^{eq} = \rho w_i \left(1 + \frac{\vec{c}_i \cdot \vec{u}_i}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u}_i)^2}{2c_s^4} - \frac{(\vec{u}_i \cdot \vec{u}_i)}{2c_s^2} \right) \quad (10)$$

3 Boltzmann H -theorem

Boltzmann provided the of the monotonously decreasing nature of the functional, H which he defined in the following way,

$$H(\vec{x}, t) = \int f \ln f d\vec{v} \quad (11)$$

Considering no external force on the system, the LBE equation simplifies to,

$$\frac{df}{dt} + \vec{v} \cdot \frac{df}{d\vec{x}} = \Delta(f) \quad (12)$$

$$\frac{d(f \ln f)}{dt} + \vec{v} \cdot \frac{d(f \ln f)}{d\vec{x}} = \Delta(f)(1 + \ln f) \quad (13)$$

Integrating over the velocity space,

$$\frac{d}{dt} \int f \ln f d\vec{v} + \frac{d}{d\vec{x}} \int \vec{v} f \ln f d\vec{v} = \int \Delta(f)(1 + \ln f) d\vec{v} \quad (14)$$

$$\frac{dH}{dt} + \frac{d\vec{H}_\beta}{d\vec{x}} = \int \Delta(f)(\ln f) d\vec{v} \quad (15)$$

Considering the BGK operator, $\int \Delta(f) d\vec{v} = 0$ from Eq. (3), the right hand side turns out to be a non positive quantity.

$$\frac{dH}{dt} + \frac{d\vec{H}_\beta}{d\vec{x}} \leq 0 \quad (16)$$

4 The Entropic Lattice Boltzmann Method

In the previous section, we derived the H -theorem, which defines a functional, H that acts like a negative entropy, ever decreasing with time. In the conventional LBM, the expression for f^{eq} is derived without respecting the H -theorem, which leads to instabilities in its implementation, for e.g., the system become unstable on coarser grids and in medium Mach number regimes. Therefore, deriving an expression for f^{eq} based on explicit minimisation of the H -functional was proposed by Ansumali and Karlin, given to the convex nature of the functional. It is possible to analytically obtain an expression for f^{eq} by minimising Eq.(11) subject to constraints given by Eq.(3-4). Before doing that, we look at the minimisation scheme.

We define a new population vector, called the auxiliary population vector, $f^* = f + \alpha \Delta(f)$, which has the same entropy as f , i.e, $H(f) = H(f^*)$.

$$f_i^* = f_i + \alpha \Delta(f_i) \quad (17)$$

Here, f_i is the pivot population vector around which we wish to travel in the direction of minimising the H -function. α acts as the full step in the direction of reaching the same entropy level. Around the equilibrium population vector, the auxiliary population vector can be expressed as,

$$f_i^* = f_i + \Delta t \frac{df_i}{dt} = f_i + \frac{f_i^{eq} - f_i}{\tau} \quad (18)$$

which shows that α is indeed a time step along the collision direction and α^{-1} is a full relaxation time scale along this direction. Now, due to the convexity of H , we know that, $H(f_{int}) \leq H(f) = H(f^*)$, $f_{int} \in [f, f^*]$ and the equality is true when equilibrium is reached ($f = f^* = f_{eq}$). As a consequence, we choose a population vector in the range (f, f^*) , which is bound to minimise the H -function, which is the basis of the Entropic lattice Boltzmann scheme (ELBM). Note that we have chosen to retain the single relaxation time model for simplicity and instead of referring to this scheme as SRT-ELBM, we simply call it ELBM.

We define the interior point in terms of a new variable β , $f_{int}(\beta) = (1 - \beta)f + \beta f^*$. β acts like a diffusion parameter in the entropy phase space walk, which respects the H -theorem. $(\beta\alpha)^{-1}$ is the effective relaxation time scale in the evolution of population vectors.

On a side note, there exists an optimal value $\beta_{optimal}$ where the entropy minimisation is maximum. It is easy to see that this point occurs where the direction of collision is orthogonal to the gradient of entropy. However, we do not explicitly look for this optimal solution in this project.

$$\langle \nabla H(f_{int}(\beta_{optimal})) | \Delta(f_{int}(\beta_{optimal})) \rangle = 0 \quad (19)$$

At this point, we understand the general frame work of the ELBM algorithm,

```

for all spatial locations do
    initialise macroscopic density and velocity
    initialise population vectors
end for
for all spatial locations do
    for all time steps do
        Calculate the equilibrium population vectors
        Get  $\alpha$ 
        Perform collision
        Perform streaming
        Update macroscopic variables
    end for
end for=0

```

4.1 Getting the equilibrium distribution

We derive the expression for f^{eq} by minimising the H -function subject to the constraints in Eq.(3-4) for a one dimensional case, by means of Lagrange multipliers. In this formulation, we consider the discrete form for H based on the Gauss-Hermite quadrature,

$$H = \sum_{i=0}^{i=n-1} f_i \ln \left(\frac{f_i}{w_i} \right) \quad (20)$$

In one dimension, the H -function is,

$$H = f_0 \ln \left(\frac{f_0}{4} \right) + f_1 \ln f_1 + f_2 \ln f_2 \quad (21)$$

where f_1, f_2 are the population vectors in the positive and negative directions, respectively with velocity magnitudes of $c_1 = -c_2 = c$ and $c_0 = 0$.

In terms of the Lagrange multipliers, λ and η , we get five equations in terms of five unknowns, $(f_0, f_1, f_2, \lambda, \eta)$

$$1 + \ln \left(\frac{f_0^{eq}}{4} \right) + \lambda = 0 \quad (22)$$

$$1 + \ln f_1^{eq} + \lambda + \eta = 0 \quad (23)$$

$$1 + \ln f_2^{eq} + \lambda - \eta = 0 \quad (24)$$

$$f_0^{eq} + f_1^{eq} + f_2^{eq} = \rho \quad (25)$$

$$f_1^{eq} - f_2^{eq} = \rho \frac{u}{c} \quad (26)$$

From Eq. (22-24), we get,

$$\ln(f_1^{eq} f_2^{eq}) = 2 \ln \left(\frac{f_0^{eq}}{4} \right) \quad (27)$$

In general, for an n -directional lattice system,

$$\ln \prod_{i=1}^{i=n-1} \frac{f_i^{eq}}{w_i} = (n-1) \ln \left(\frac{f_0^{eq}}{w_0} \right) \quad (28)$$

From Eq. (25-26) and Eq.(27), we get,

$$f_1^{eq} f_2^{eq} = \frac{(\rho - f_0^{eq})^2 - (\rho u/c)^2}{4} \implies \rho^2(1 - (u/c)^2) + (f_0^{eq})^2 - 2\rho f_0^{eq} = \frac{(f_0^{eq})^2}{4} \quad (29)$$

The acceptable solutions to the population vectors in (0,1) are

$$f_0^{eq} = \frac{2\rho}{3} \left(2 - \sqrt{1 + 3u^2/c^2} \right) \quad (30)$$

$$f_1^{eq} = \frac{\rho}{3} \left(\frac{3u - c}{2c} + \sqrt{1 + 3u^2/c^2} \right) \quad (31)$$

$$f_2^{eq} = \frac{\rho}{3} \left(-\frac{3u + c}{2c} + \sqrt{1 + 3u^2/c^2} \right) \quad (32)$$

It is possible to obtain the explicit equilibrium distributions for DmQn system, given by,

$$f_i^{eq} = \rho w_i \prod_{j=0}^{j=n-1} \left(2 - \sqrt{1 + 3u_j^2} \right) \left(\frac{2u_j + \sqrt{1 + 3u_j^2}}{1 - u_j} \right)^{c_{i,j}} \quad (33)$$

4.2 Getting the collision operator

The collision operator is chosen such that it satisfies the local hydrodynamic conservation laws and the local entropy production inequality,

$$\langle \{1, \vec{v}\} | \Delta \rangle = 0 \quad (34)$$

$$\langle \nabla H | \Delta \rangle \leq 0 \quad (35)$$

where ∇H is the gradient of H in the population vector space. The equality in latter case is satisfied at the equilibrium state. The BGK operator satisfies these constraints. We define an explicit construction for the new collision operator, Δ^* , a linearised version of the original collision operator around the equilibrium population,

$$\Delta^*(f) = \beta \alpha(f) \Delta(f) \quad (36)$$

$$\delta \Delta^*(f^{eq}) = \Delta^*(f^{eq} + \delta f) - \Delta^*(f^{eq}) = \Delta^*(f^{eq} + \delta f) \quad (37)$$

$$\Delta^*(f^{eq} + \delta f) = \beta \alpha(f^{eq} + \delta f) \Delta(f^{eq} + \delta f) = \beta \alpha(f^{eq}) \Delta'(f^{eq}) \delta f \quad (38)$$

$$\delta \Delta^* = \beta \alpha_{eq} L \delta f = \beta \alpha_{eq} (f - f^{eq}) \quad (39)$$

Here, L is the linearised collision integral around the equilibrium, which will be incorporated into the further development of this method.

4.3 Getting α

α is the non-trivial solution to the isentropic equation close to equilibrium,

$$H(f) = H(f + \alpha(f) \Delta(f)) \quad (40)$$

If we substitute f^{eq} directly into this equation, we do not get a solution since the relaxation parameter is defined based on the deviations from equilibrium. In the following equation, every term is zero due to the orthogonality property of the collision operator.

$$H(f^{eq}) = H(f^{eq} + \alpha_{eq} \Delta(f)) = H(f^{eq}) + \alpha_{eq} \Delta(f^{eq}) \nabla H(f^{eq}) + \frac{\alpha_{eq}^2 \Delta(f^{eq})^2}{2} \nabla \nabla H + O\left(\alpha_{eq}^3 \Delta(f^{eq})^3\right) \quad (41)$$

We expand the isentropic equation around a general population distribution,

$$H(f) = H(f + \alpha(f) \Delta(f)) = H(f) + \alpha(f) \Delta(f) \nabla H(f) + \frac{\alpha(f)^2 \Delta(f)^2}{2} \nabla \nabla H(f) + O\left(\alpha^3 \Delta(f)^3\right) \quad (42)$$

We omit the cubic and higher order terms,

$$-2\alpha(f)\Delta(f)\nabla H(f) = \alpha(f)^2\Delta(f)^2\nabla\nabla H(f) \quad (43)$$

Expanding around the equilibrium distribution, the left hand side is

$$-2\alpha(f^{eq}+\delta f)\Delta(f^{eq}+\delta f)\nabla H(f^{eq}+\delta f) = -2[\alpha_{eq}+\delta f\alpha'_{eq}][\Delta(f^{eq})+\delta f\Delta'(f^{eq})][\nabla H(f^{eq})+\delta f\nabla\nabla H(f^{eq})] \quad (44)$$

Zeroth and first order terms are zero upon simplification,

$$-2[\alpha_{eq}\nabla\nabla H(f^{eq})L]\delta f^2 + O(\delta f^3) \quad (45)$$

Expanding the right hand side around equilibrium,

$$\alpha(f)^2\Delta(f)^2\nabla\nabla H(f) = [\alpha_{eq}+\delta f\alpha'_{eq}]^2[\Delta(f^{eq})+\delta f\Delta'(f^{eq})]^2[\nabla\nabla H(f^{eq})+\delta f\nabla\nabla\nabla H(f^{eq})] \quad (46)$$

This simplifies to,

$$[\alpha_{eq}^2\nabla\nabla H(f_{eq})LL]\delta f^2 + O(\delta f^3) \quad (47)$$

Combining both sides and summing over the population vector space,

$$-2\delta f_i[\alpha_{eq}\nabla\nabla H(f_i^{eq})L]\delta f_i = \delta f_i[\alpha_{eq}^2\nabla\nabla H(f_{eq})LL]\delta f_i \quad (48)$$

$$\alpha_{eq} = -2\frac{\langle \delta f | \nabla\nabla H(f^{eq}) | L\delta f \rangle}{\langle L\delta f | \nabla\nabla H(f^{eq}) | L\delta f \rangle} \quad (49)$$

If the linearised collision operator has the property, $LL = -L$, then α_{eq} drastically simplifies to the value 2, meaning that the equilibrium population is reached at the same rate. This is the gist of the single relaxation time model (SRT). The BGK collision operator implicitly satisfies this projector property and we will leverage the same to simplify the simulations. Hereafter, we have our corrected collision operator to be the following around equilibrium,

$$\delta\Delta^* = 2\beta L\delta f = 2\beta(f - f^{eq}) \quad (50)$$

However, instead of imposing the projector property, we can also find an approximate value of α by solving the isentropic equation at each spatial location numerically via iterative methods such as Newton-Raphson method or bisection method. Since we can evaluate the derivative of the quadrature approximation of the H -function, we can find α numerically by Newton-Raphson method for faster convergence.

Since the equilibrium value of α is 2, we can save time in iterations by choosing the optimal starting point as 2 initially. In the subsequent time iterations, the previous value of α can be chosen as the initial guess.

However, in the trivial case when $\alpha = 0$, we employ a heuristic based guess for α given by,

$$\alpha = \min_{i=0,1,\dots,n-1} \left(\left| \frac{f_i}{\Delta(f_i)} \right| \right) \quad (51)$$

4.4 Understanding β

The viscosity as a function of parameter β is explicitly obtained based on Chapman-Enskog analysis, given by,

$$\beta = \frac{1}{6\nu + 1} \quad (52)$$

Therefore, full control over viscosity is retained in ELBM. In addition, β is a parameter that over-relaxes the forward time step, acting as a diffusion parameter.

The pseudo code for ELBM is given by,

```

for all spatial locations do
    initialise macroscopic density and velocity
    initialise population vectors
end for
for all spatial locations do
    for all time steps do
        Calculate the equilibrium population vectors via  $f_i^{eq} = \rho w_i \prod_{j=0}^{j=n-1} \left( 2 - \sqrt{1 + 3u_j^2} \right) \left( \frac{2u_j + \sqrt{1+3u_j^2}}{1-u_j} \right)^{c_{i,j}}$ 
        Get  $\alpha$  by solving  $H(f) = H(f + \alpha\Delta(f))$ 
        Perform collision  $f_i(\vec{x}_i, t + \Delta t) = f_i(\vec{x}_i, t) + \beta\alpha(f_i^{eq} - f_i)$ 
        Perform streaming
        Update macroscopic variables
    end for
end for
end for=0

```

5 Numerical experiments

We test our method against the well studied lid driven cavity flow problem LBM and ELBM for Re values of 100, 400 and 1000.

5.1 Problem description

The problem has a closed square cavity filled with a fluid of known viscosity, ν at rest and is driven by the top wall (lid) horizontally to the right during all positive times, while the rest of the walls remain stationary throughout the experiment. We have chosen the velocity of lid, u as 0.01 and the respective viscosity values, along with the β values are calculated accordingly. The grid size for simulation is [100,100] and the simulations are run for a total of 10^5 , 15×10^4 and 25×10^4 time iterations for Re values of 100, 400 and 1000 respectively in ELBM.

5.2 Boundary conditions

We employ the moving wall boundary condition for the lid. Moving wall BC is a generalisation of half-way bounce-back approach and hence is second-order convergent scheme, whose implementation is given by,

$$f_i(\vec{x}, t + \Delta t) = f_i(\vec{x} + \vec{c}_i \Delta t, t) + \frac{2w_i \rho}{c_s^2} \vec{c}_i \cdot \vec{u} \quad \text{for top wall} \quad (53)$$

On the rest of the walls, we impose a no-slip boundary condition,

$$f_i(\vec{x}, t + \Delta t) = f_i(\vec{x} + \vec{c}_i \Delta t, t) \quad \text{for left, right and bottom walls} \quad (54)$$

5.3 Results and discussion

5.3.1 Accuracy

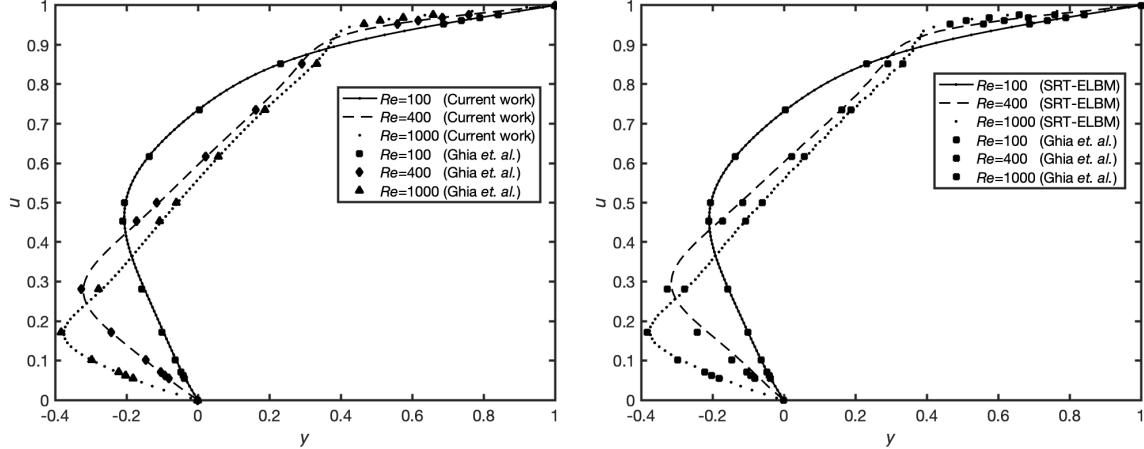


Figure 1: Profiles of horizontal velocity component along the line passing through the geometric center simulated using conventional LBM (left) and ELBM (right).

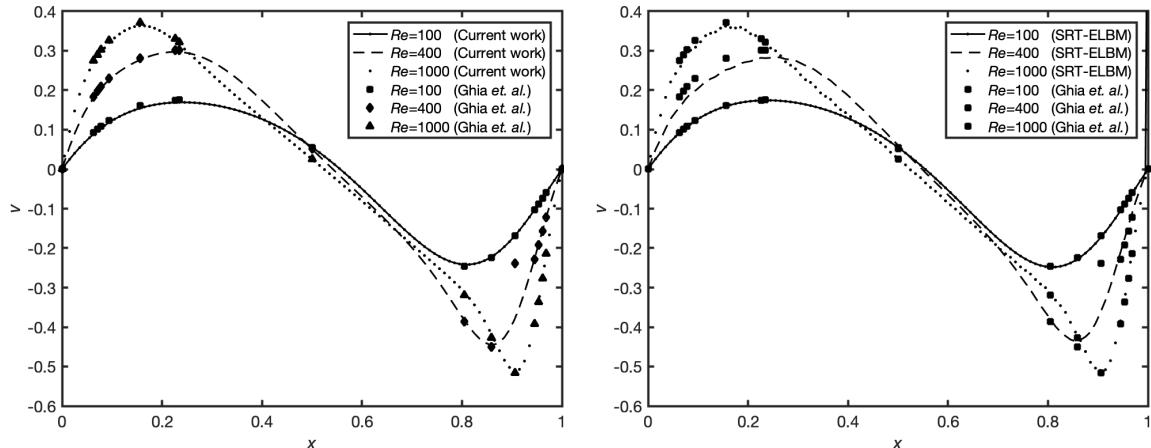


Figure 2: Profiles of vertical velocity component along the line passing through the geometric center simulated using conventional LBM (left) and ELBM (right).

The figures presented for center-line velocity components for both LBM and ELBM implementations indicate quantitative similarity between the both methods and both are accurate. Therefore, only the results for ELBM will be presented hereafter.

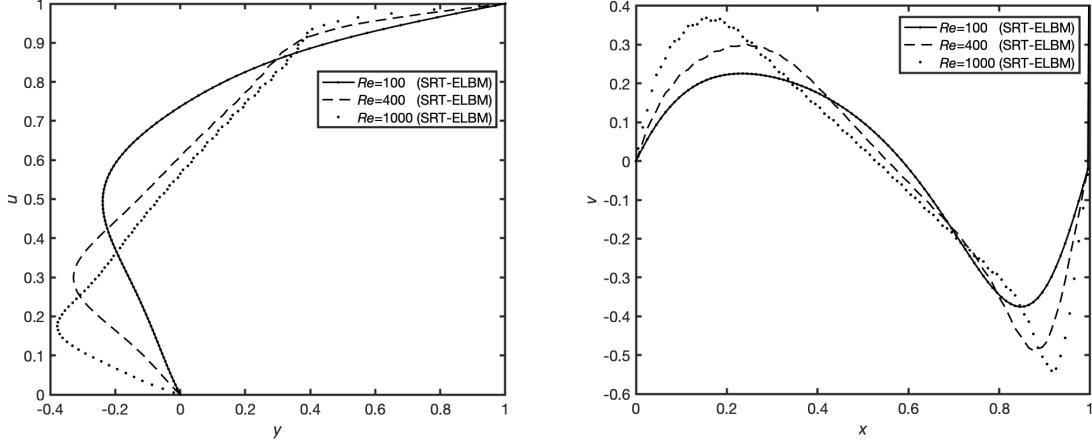


Figure 3: Profiles of velocity components along the lines passing through the primary vortex center.

Parameter	<i>Re = 100</i>			<i>Re = 400</i>		
	LBM	ELBM	Ghia et. al.	LBM	ELBM	Ghia et. al.
ψ_{min}	-0.1148	-0.1019	-0.1179	-0.1116	-0.1097	-0.1139
$x(\psi_{min})$	0.625	0.615	0.617	0.555	0.555	0.555
$y(\psi_{min})$	0.735	0.735	0.734	0.605	0.605	0.606
$\omega_{v.c.}$	3.1612	3.1598	3.1665	2.2425	2.2346	2.2947

Table 1: Location of vortex center and magnitudes of vorticity and minimum stream function for 100 and 400 Re .

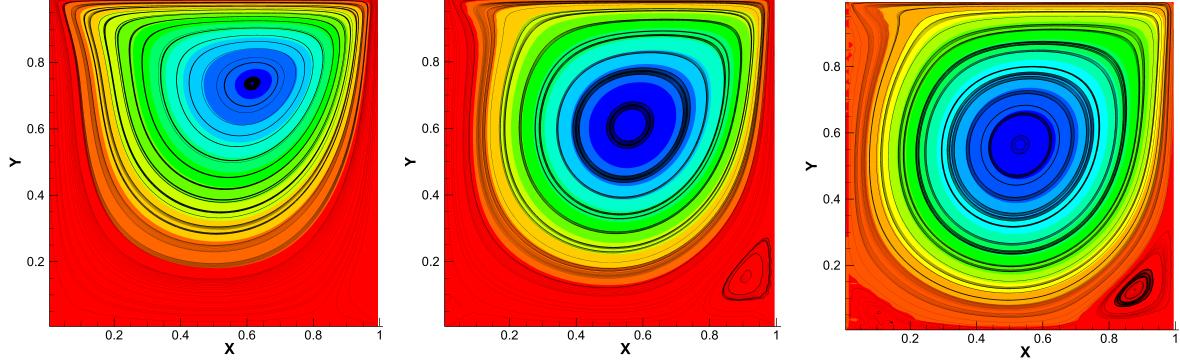


Figure 4: Streamlines for $Re = 100, 400$ and 1000 from left to right.

Parameter	<i>Re = 1000 [100 × 100]</i>		<i>Re = 1000 [60 × 60]</i>		<i>Ghia et. al.</i>
	LBM	ELBM	LBM	ELBM	
ψ_{min}	-0.1010	-0.1056	-	-0.1036	-0.1034
$x(\psi_{min})$	0.535	0.535	-	0.550	0.531
$y(\psi_{min})$	0.565	0.565	-	0.583	0.566
$\omega_{v.c.}$	2.0070	2.0532	-	3.4507	2.0497

Table 2: Location of vortex center and magnitudes of vorticity and minimum stream function for 1000 Re on coarse and fine grids.

5.3.2 Stability

Although both the schemes have similar accuracies at low Mach number regimes, the ELBM method has greater stability. Stability is compared between the two techniques using a metric for fluctuations in mean density field. The metric is defined as follows,

$$\Delta_\rho = \frac{1}{\bar{\rho}} \sqrt{\frac{\int (\rho - \bar{\rho})^2 d\vec{x}}{\int d\vec{x}}} \quad (55)$$

Parameter	$Re = 100$		$Re = 400$		$Re = 1000$	
	LBM	ELBM	LBM	ELBM	LBM	ELBM
Δ_ρ	4.70×10^{-5}	4.70×10^{-5}	-0.1179	4.10×10^{-5}	0.0014	5.60×10^{-5}
$\bar{\rho}$	1.0017	1.0017	1.0057	1.0018	1.0091	1.0018
$\sigma(\rho) \times 10^5$	4.71	4.71	29.27	4.11	10014	5.60

Table 3: Mean density fluctuations and variation in local density field.

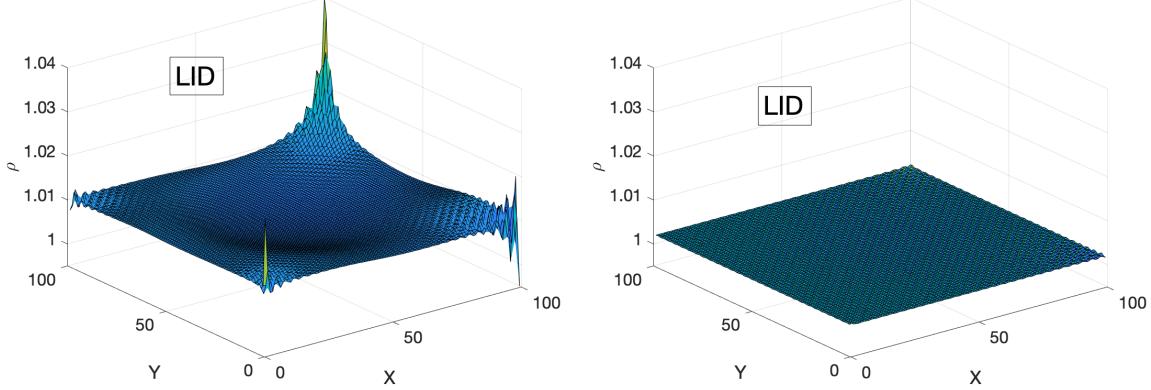


Figure 5: Fluctuations in local density field in LBM (top) and ELBM (bottom) at $Re = 1000$. Both the images are plotted on the same scale, indicating a large amplitude of fluctuations in the conventional approach. Large fluctuations are observed near the corners, and the amplitude of fluctuation is maximum at the right top corner due to maximum momentum transfer.

In addition, it was possible to simulate the flow for high Re , up to 5000 via ELBM and the simulation was stable even after 2×10^5 time iterations on a coarse grid of 60×60 lattices, although the results were inaccurate. Simulation was performed on 320×320 grid which had a very low rate of convergence. The simulation was not performed due to lack of necessary computational hardware.

5.3.3 Time performance

The time taken for ELBM for one iteration at different Re is about 1.5 times the time taken by conventional approach, given to the iterative scheme employed by the former method. However, this delay can be reduced by employing faster iterative algorithms for root finding. The range of values of α for the simulations were [1.9984, 2.0015], [1.9972, 2.0029], [1.9941, 2.0067] for Re values of 100, 400 and 1000 respectively. Therefore, an approximation of $\alpha = 2$, close to equilibrium can be imposed to increase convergence.

Parameter	$Re = 100$		$Re = 400$		$Re = 1000$	
	LBM	ELBM	LBM	ELBM	LBM	ELBM
Time per iteration (sec)	0.0346	0.0583	0.0336	0.0498	0.0268	0.0402
Ratio	1.6870		1.4793		1.5000	

Table 4: Time data for both simulation methods.

6 Conclusion

In this project, ELBM approach was developed from first principles and tested against a well known benchmark problem. The ELBM approach produces accurate results on fine grid sizes. However, the approach has similar accuracies as conventional formulation and is not beneficial at low Mach number regimes in terms of accuracy. However, in terms of stability, ELBM was shown to be more reliable. Even in coarse grid (60×60) simulations at Re of 5000, the approach was stable up to 2×10^5 iterations, although the results were inaccurate. A fine grid of 320×320 was employed to solve the problem for Re of 5000. In addition, the time taken per iteration was shown to be about 1.5 times the time taken per iteration using conventional approach. This is lower than the values reported in literature due to the optimised root finding scheme that we have employed for finding α . There is much scope in this method in terms of further reduction in time taken to find α .

7 References

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