To what extent, can Fourier Analysis be used to solve Ordinary Differential Equations and Partial Differential Equations?

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Contents

1 Introduction

The allure of differential equations and Fourier analysis for me lies in their ability to unravel the intricate patterns of nature and unlock the hidden harmonies within complex systems. The ubiquitous presence of these fields across diverse scientific disciplines from physics and engineering to biology and economics help us understand the phenomena of celestial orbits and fluid flow to disease spread and stock market trends using one uniform language.

Recently, I utilized differential equations to come up with the following accurate and applicable epidemic model for policymakers that accounts for disease incubation, quarantine, immunity wear-off, and mortality rates.

$$\frac{dS}{dt} = -\frac{\beta SI}{N} + \mu R \tag{1}$$
$$\frac{dE}{dt} = \frac{\beta SI}{N} - \phi E \tag{2}$$
$$\frac{dI}{dt} = \phi E - \zeta I - \gamma I - \alpha I \tag{3}$$
$$\frac{dQ}{dt} = \zeta I - \kappa Q - \epsilon Q \tag{4}$$
$$\frac{dR}{dt} = \gamma I + \kappa Q - \mu R \tag{5}$$
$$\frac{dD}{dt} = \alpha I + \epsilon Q \tag{6}$$

where:

- N is the total population.
- β is the contact rate (1/days).
- ϕ is the incubation rate (1/days).
- ζ is the quarantine rate (1/days).
- γ is the recovery rate (non-quarantine) (1/days).

- κ is the recovery rate (quarantine) (1/days).
- μ is the immunity wearoff rate (1/days).
- α is the case fatality rate (non-quarantine) (1/days).
- ϵ is the case fatality rate (quarantine) (1/days).

I was astonished by how elegantly this model was described in the language of differential equations. Hence, through differential equations, I discovered a simple language that unveils the secrets of the behavior of a system and its change over time.

Fourier analysis, on the other hand, allowed me to deconstruct complex phenomena into fundamental sine and cosine components, revealing the underlying frequencies and patterns. In my research at my R&D internship, I utilized Fourier analysis to analyze the frequency content of a complex, noisy, non-stationary vibration signal from an IMU and perform order analysis to study and understand the fundamental orders/frequencies of the system. I was amazed how clear this coordinate transformation made the signal and how much information it revealed about the system's behavior. Thus, I became fascinated by their practical applications across scientific disciplines, enabling us to predict, simulate, and optimize the behavior of diverse systems. Embracing the power of differential equations and Fourier analysis, I embarked on a lifelong journey to explore and understand the world's mysteries, driven by an insatiable curiosity to uncover the harmony and order that underlie our complex reality.

In this paper, I will explore the extent to which Fourier analysis can be utilized to solve ordinary differential equations (ODEs) and partial differential equations (PDEs).

2 Background

2.1 Differential Equations

2.1.1 Why are Differential Equations important?

Calculus is the mathematics of change. Hence, differential equations, which relate the derivatives or integrals of a function to the function itself, can very elegantly summarize the behavior of otherwise complex, dynamic systems. As a result, differential equations are extremely powerful in their ability to model various systems in applied mathematics, physics, and engineering.

For example, the Lotka-Volterra equations describe the dynamics of populations of predators and prey. These equations are described below.

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{7}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{8}$$

where:

- x is the population density of the prey.
- y is the population density of the predator.
- α is the exponential growth rate of the prey.
- γ is the exponential decay rate of the predators.
- β is the effect of the predators on the prey growth rate.
- δ is the effect of the presence of prey on the predator growth rate.

Equations 7 and 8 are a set of first-order, nonlinear ODEs. This is further explained in Section 2.1.2 Properties of Differential Equations.

2.1.2 Properties of Differential Equations

Definition 2.1. The order of a system of differential equations is defined as the highest-order derivative the system contains. Since the highest order derivative in the Lotka-Volterra Equations 7 and 8 is $\frac{d}{dt}$ (no $\frac{d^2}{dt^2}$), the equations are first-order.

Definition 2.2. If $x_1(t)$ and $x_2(t)$ are both solutions to a linear system of differential equations, then any linear superposition is also a solution to the system, namely anything in the form:

$$x(t) = c_1 x_1(t) + c_2 x_2(t) \tag{9}$$

More formally, a system of differential equations is said to be linear if and only if the equations follows the form:

$$a_0 y + a_1 y' + a_2 y'' \cdots + a_n y^{(n)} = b(x) \tag{10}$$

where $a_0, a_1, a_2, ..., a_n$ are any differentiable functions (do not need to be linear).

2.1.3 Solution to Differential Equations

Note that the solution to a differential equation is not one function, but rather a set of functions that all satisfy the differential equation. Some initial conditions must be given to reduce the solution to a single function. For example, for the Lotka-Volterra equations, the phase space shown in Figure 1 plots the various function solutions given various different initial conditions. The solution for a specific initial condition over time is plotted in Figure 2.

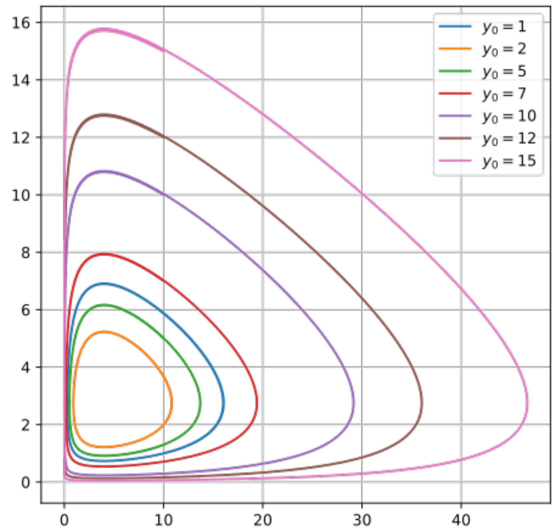

Figure 1: Solution to the Lotka-Volterra Equations given different predator initial conditions. The predator solution is shifted $\frac{\pi}{2}$ radians right of the prey solution.

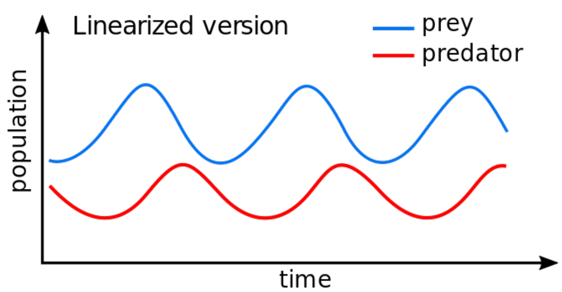

Figure 2: Solution to the Lotka-Volterra Equations given some initial conditions. The predator solution is shifted $\frac{\pi}{2}$ radians right of the prey solution.

2.1.4 ODE vs PDE

2.2 Fourier Analysis

2.2.1 What is Fourier Analysis?

Fourier Analysis is a field of mathematics that studies how complex function waveforms can be decomposed into a series of sinusoidal functions, whose frequencies form a harmonic series. In other words, the Fourier Transform (FT), the cornerstone of Fourier Analysis, turns a signal in time space into a signal in frequency space and a signal in real space into a signal in Fourier space.

2.2.2 Fourier and Inverse Fourier Transforms

Definition 2.3. For a continuous function $f(x)$, the continuous Fourier Transform $\mathcal{F}\{f(x)\}$ (CFT) is defined as below. The transform returns the frequency space function $F(\omega)$.

$$F(\omega) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \tag{11}$$

Definition 2.4. In order to reverse the Fourier Transform, the Inverse Fourier Transform $\mathcal{F}^{-1}\{F(\omega)\}$ (IFFT) can be applied as defined below.

$$f(x) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega \tag{12}$$

2.2.3 Fourier Series

Let $f(x)$ be defined as the following periodic, yet discontinuous square waveform signal (Heaviside Step Function):

$$f(x) = \begin{cases} 1 & \text{if } nT < x < \frac{T(2n+1)}{2}, n \in \mathbb{Z} \\ 0 & \text{if } x = \frac{nT}{2}, n \in \mathbb{Z} \\ -1 & \text{if } \frac{T(2n+1)}{2} < x < T(n+1), n \in \mathbb{Z} \end{cases} \tag{13}$$

In order to obtain the Fourier Series for $f(x)$, the Fourier Transform of $f(x)$ must be taken.

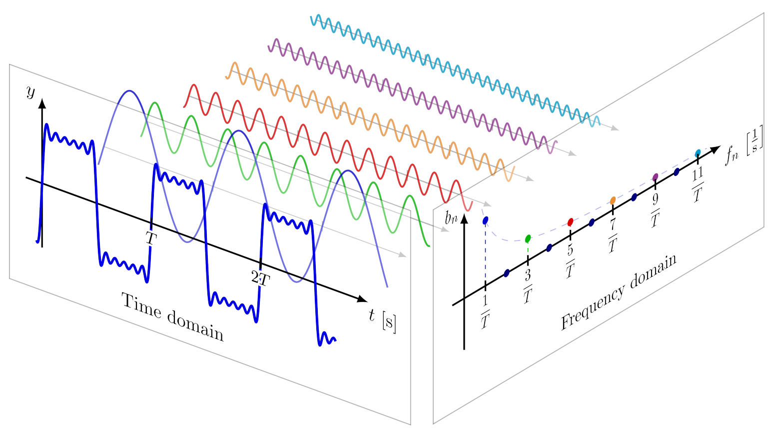

Figure 3: On the axis to the right, the Fourier Transform of $f(x)$ can be seen. If an infinite summation of all the frequencies with their corresponding amplitudes was performed as defined by the Fourier Transform, the solution would be exactly equal to $f(x)$.

Therefore, f(x) (Equation 12) can be defined by the infinite summation of sine functions through its Fourier Series as follows:

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{2\pi n x}{T}\right)$$

2.2.4 Properties of the Fourier Transform

The Fourier Transform, which is a coordinate transform, is important and useful because many mathematical operations in Fourier space are simplified compared to the original space. Below are some of the important properties of the Fourier Transform that will be useful in solving differential equations using Fourier Analysis.

Lemma 2.1. *If a function $f(x)$ is scaled by a constant a, the Fourier Transform will also be scaled by a.*

$$\mathcal{F}\{af(x)\} = a\mathcal{F}\{f(x)\} \tag{14}$$

Proof.

$$\mathcal{F}\{af(x)\} = \int_{-\infty}^{\infty} af(x)\, e^{-i\omega x}\, dx$$

$$= a\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = a\mathcal{F}\{f(x)\}$$

□

Theorem 2.1. *The Fourier Transform is a linear transformation. This means that the transform of a linear combination of functions is equal to the linear combination of the transforms of the individual functions.*

$$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\} \tag{15}$$

Proof.

$$\mathcal{F}\{af(x) + bg(x)\} = \int_{-\infty}^{\infty} (af(x) + bg(x))\, e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} af(x)\, e^{-i\omega x}\, dx + \int_{-\infty}^{\infty} bg(x)\, e^{-i\omega x}\, dx$$

Using Lemma 2.1,

$$= a\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx + b\int_{-\infty}^{\infty} g(x)\, e^{-i\omega x}\, dx$$

$$= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

□

Theorem 2.2. The Fourier Transform can be shifted easily along the axis of the transform. When a function $f(x)$ along the x-axis by x_0 units, the transform of the shifted function is equal to the transform of the original function multiplied by $e^{-i\omega x_0}$.

$$\mathcal{F}\{f(x - x_0)\} = e^{-i\omega x_0} F(\omega) \tag{16}$$

Proof.

$$\mathcal{F}\{f(x - x_0)\} = \int_{-\infty}^{\infty} f(x - x_0)\, e^{-i\omega x}\, dx$$

Let $u = x - x_0$. Thus, $x = u + x_0$ and $dx = du$.

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega(u+x_0)}\, du$$

$$= e^{-i\omega x_0} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du$$

$$= e^{-i\omega x_0} F(\omega)$$

□

Theorem 2.3. The Fourier Transform of a function's derivative is equal to the transform of the original function multiplied by $i\omega$.

$$\mathcal{F}\{\frac{df(x)}{dx}\} = i\omega\mathcal{F}\{f(x)\} \tag{17}$$

Proof.

$$\mathcal{F}\{\frac{df(x)}{dx}\} = \int_{-\infty}^{\infty} \frac{df(x)}{dx}\, e^{-i\omega x}\, dx$$

Using integration by parts, let $u = f(x)$ and $dv = e^{-i\omega x} dx$.

$$= \left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \left(\frac{d}{dx} e^{-i\omega x}\right)\, dx$$

Since $f(x)$ must be continuous and integrable to be differentiable and take the Fourier Transform on $\mathbb{R}$, $\lim_{x\to-\infty} = \lim_{x\to\infty} = 0$ must be true. Thus, the first term is equal to zero.

$$= -\int_{-\infty}^{\infty} f(x)\left(-i\omega e^{-i\omega x}\right)\,dx$$

$$= i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$= i\omega \mathcal{F}\{f(x)\}$$

$\square$

3 ODE Simplification

3.1 Driven Simple Harmonic Oscillator with Damping

3.1.1 Brief Derivation

Let us define a Simple Harmonic Oscillator (SHO) of mass m and is attached to a spring with spring constant k. The SHO is subject to a damping force with damping coefficient γ and an additional driving force $f(t)$.

Define $\omega_0 = \sqrt{\frac{k}{m}}$ as the natural frequency of the SHO and the damping force as $F_d = -2m\gamma v$, where v is the velocity of the SHO.
Using Newton's Second Law,

$$\Sigma F = ma = -m\omega_0^2 x - 2m\gamma v + f(t) \tag{18}$$

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x - 2m\gamma\frac{dx}{dt} + f(t) \tag{19}$$

Dividing by m and rearranging terms, we get the following differential equation for a driven SHO with damping:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x(t) = \frac{f(t)}{m} \tag{20}$$

3.1.2 Application of the Fourier Transform

Let us take the Fourier Transform of Equation 20 with respect to t. Thus, let $X(\omega)$ be the Fourier Transform of $x(t)$ and $F(\omega)$ be the Fourier Transform of $f(t)$.

$$\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} + \mathcal{F}\left\{2\gamma\frac{dx}{dt}\right\} + \mathcal{F}\left\{\omega_0^2 x(t)\right\} = \mathcal{F}\left\{\frac{f(t)}{m}\right\} \tag{21}$$

Using Lemma 2.1,

$$\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} + 2\gamma\mathcal{F}\left\{\frac{dx}{dt}\right\} + \omega_0^2\mathcal{F}\left\{x(t)\right\} = \frac{\mathcal{F}\left\{f(t)\right\}}{m} \tag{22}$$

Using Theorem 2.3,

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = i\omega\mathcal{F}\left\{x(t)\right\} \tag{23}$$

$$= i\omega X(\omega) \tag{24}$$

$$\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = i\omega\mathcal{F}\left\{\frac{dx}{dt}\right\} \tag{25}$$

$$= -\omega^2 X(\omega) \tag{26}$$

Therefore,

$$-\omega^2 X(\omega) + 2\gamma i\omega X(\omega) + \omega_0^2 X(\omega) = \frac{F(\omega)}{m} \tag{27}$$

$$X(\omega)\left(\omega_0^2 - \omega^2 + 2\gamma i\omega\right) = \frac{F(\omega)}{m} \tag{28}$$

Thus,

$$X(\omega) = \frac{F(\omega)}{m\left(\omega_0^2 - \omega^2 + 2\gamma i\omega\right)} \tag{29}$$

By applying the Fourier Transform to the driven SHO equation, we have reduced the problem from solving a second order ODE (Equation 20) to solving a simple algebraic equation (Equation 29). This is a significant reduction in complexity.

Once we have solved for $X(\omega)$, we can take the inverse Fourier Transform of Equation 29 to obtain $x(t)$.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega && (30) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(\omega)}{m\left(\omega_0^2-\omega^2+2\gamma i\omega\right)}e^{i\omega t}d\omega && (31) \end{aligned}$$

3.1.3 Boundary Conditions

3.1.4 Numerical Solution to Driven SHO with Damping Equation

3.2 Bessel's Equation

3.2.1 Brief Derivation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0 \tag{32}$$

To simplify Bessel's equation using the Fourier-Bessel transform, we can introduce a new variable, ρ, defined as:

$$x=\alpha\rho$$

where α is a constant.

Now, let's take the Fourier-Bessel transform of Bessel's equation.

The Fourier-Bessel transform of a function $f(x)$ is defined as:

$$F_\alpha(\rho)=\int_0^\infty f(x)J_n(\alpha\rho)xdx$$

where $J_n(\alpha\rho)$ is the Bessel function of the first kind of order n.

Applying the Fourier-Bessel transform to Bessel's equation, we get:

$$\frac{\alpha^2\rho^2 d^2 F_\alpha(\rho)}{d\rho^2} + \frac{\alpha\rho dF_\alpha(\rho)}{d\rho} + (\alpha^2\rho^2 - n^2)F_\alpha(\rho) = 0$$

This transformed equation simplifies to:

$$\frac{\rho^2 d^2 F_\alpha(rho)}{d\rho^2} + \frac{\rho dF_\alpha(\rho)}{d\rho} + (\frac{\rho^2 - n^2}{\alpha^2})F_\alpha(\rho) = 0$$

Notice that this transformed equation no longer contains the derivative with respect to x.

The solution to this transformed equation is given by the Bessel differential equation:

$$\frac{\rho^2 d^2 F_\alpha(\rho)}{d\rho^2} + \frac{\rho dF_\alpha(\rho)}{d\rho} + (\frac{\rho^2 - n^2}{\alpha^2})F_\alpha(\rho) = 0$$

The general solution to the Bessel differential equation is expressed in terms of Bessel functions:

$$F_\alpha(\rho) = c_1 J_n(\alpha\rho) + c_2 Y_n(\alpha\rho)$$

where $J_n(\alpha\rho)$ is the Bessel function of the first kind and $Y_n(\alpha\rho)$ is the Bessel function of the second kind. c1 and c2 are constants determined by the boundary conditions of the problem.

By taking the inverse Fourier-Bessel transform of $F_\alpha(\rho)$, we can obtain the solution $y(x)$ to Bessel's equation in terms of Bessel functions.

3.2.2 Application of the Fourier Transform

3.2.3 Boundary Conditions

3.2.4 Numerical Solution to Bessel's Equation

4 PDE to ODE Reduction

4.1 The Heat Equation

Fourier Analysis was first discovered when Joseph Fourier first developed the Heat Equation. The Heat Equation models the flow of heat along a certain heat profile over time. In this example, a 1D solution will be derived and solved.

4.1.1 Brief Derivation

Let us define a rod with the following assumptions:

- The rod is of length L and is composed of an homogenous material with a heat diffusion coefficient α^2.
- The rod is perfectly insulated along the Y and Z axes. Thus, heat can only flow along the X axis of the rod.
- The rod is thin enough such that the temperature of the rod at any cross-section is uniform.
- The rod is initially at a uniform temperature $u(x, 0) = f(x)$. Thus, the rod temperature at position x at time t is $u(x, t)$.

Thus, the heat equation in one dimension is defined as follows:

$$\frac{\partial u}{\partial t} = \alpha^2 \nabla^2 u = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{33}$$

Or in subscript notation,

$$u_t = \alpha^2 u_{xx} \tag{34}$$

4.1.2 Application of the Fourier Transform

Since $u(x,t)$ is defined as the temperature of the rod at position x and time t, we can apply the Fourier Transform to $u(x,t)$ with respect to position x to reduce the Equation 34 PDE into an ODE. Thus, let $U(\kappa,t)$ be the Fourier Transform of $u(x,t)$ with respect to x.

Taking the Fourier Transform of Equation 34 with respect to x,

$$\mathcal{F}\left\{u_t\right\} = \mathcal{F}\left\{\alpha^2 u_{xx}\right\} \tag{35}$$

Using Lemma 2.1,

$$\mathcal{F}\left\{u_t\right\} = \alpha^2 \mathcal{F}\left\{u_{xx}\right\} \tag{36}$$

Using Theorem 2.3,

$$\mathcal{F}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = i\kappa\mathcal{F}\left\{\frac{\partial u(x,t)}{\partial x}\right\} \tag{37}$$

$$= -\kappa^2 \mathcal{F}\{u(x,t)\} \tag{38}$$

$$= -\kappa^2 U \tag{39}$$

Therefore,

$$\mathcal{F}\left\{u_t\right\} = -\alpha^2\kappa^2 U \tag{40}$$

$$\frac{dU}{dt} = -\alpha^2\kappa^2 U \tag{41}$$

4.1.3 Boundary Conditions

4.1.4 Numerical Solution to Heat Equation

Equation 41 is a decoupled ODE that can be easily numerically integrated. A numerical solution to Equation 41 using a fifth-order Runge-Kutta approximation in Python is presented below:

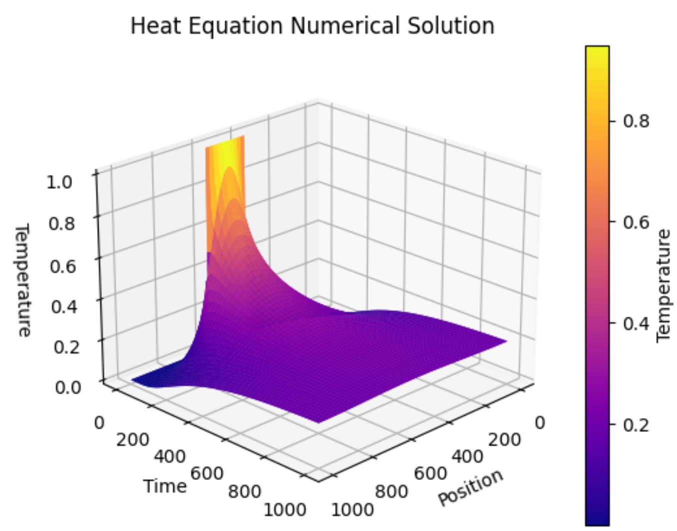

Figure 4: Using a simple square waveform as the initial temperature function of the rod, this plot shows the temperature change along x and t by performing a numerical integration of Equation 41.

4.2 The Wave Equation

4.2.1 Brief Derivation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \frac{\partial^2 u}{\partial x^2} \tag{42}$$

Or in subscript notation,

$$u_{tt} = c^2 u_{xx} \tag{43}$$

4.2.2 Application of the Fourier Transform

Since $u(x,t)$ is defined as the displacement of the wave at position x and time t, we can apply the Fourier Transform to $u(x,t)$ with respect to position x to reduce the Equation 43 PDE into an ODE. Thus, let $U(\kappa,t)$ be the Fourier Transform of $u(x,t)$ with respect to x.

Taking the Fourier Transform of Equation 43 with respect to x,

$$\mathcal{F}\left\{u_{tt}\right\} = \mathcal{F}\left\{\alpha^2 u_{xx}\right\} \tag{44}$$

Using Lemma 2.1,

$$\mathcal{F}\left\{u_{tt}\right\} = \alpha^2 \mathcal{F}\left\{u_{xx}\right\} \tag{45}$$

Using Theorem 2.3,

$$\mathcal{F}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = i\kappa\mathcal{F}\left\{\frac{\partial u(x,t)}{\partial x}\right\} \tag{46}$$

$$= -\kappa^2 \mathcal{F}\{u(x,t)\} \tag{47}$$

$$= -\kappa^2 U \tag{48}$$

Therefore,

$$\mathcal{F}\left\{u_{tt}\right\} = -\alpha^2\kappa^2 U \tag{49}$$

$$\frac{d^2U}{dt^2} = -c^2\kappa^2 U \tag{50}$$

4.2.3 Boundary Conditions

4.2.4 Numerical Solution to Wave Equation

Equation 50 is a decoupled ODE that can be easily numerically integrated. A numerical solution to Equation 50 using a fifth-order Runge-Kutta approximation in Python is presented below:

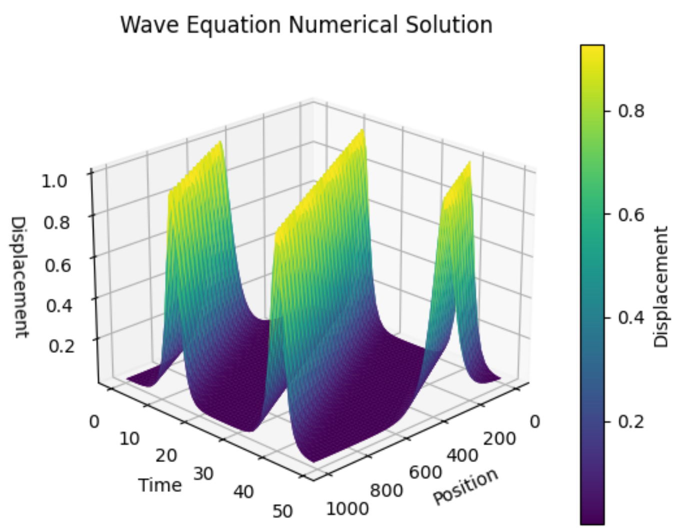

Figure 5: Defining the initial wave displacement $u(x, 0) = \text{sech}(x)$, this plot shows the temperature change along x and t by performing a numerical integration of Equation 50.

5 Limitations of the Fourier Transform

As shown by the various examples, Fourier transforms are a powerful tool for solving differential equations. However, there are certain situations where Fourier transforms may not be applicable or effective. Here are a few cases:

1. Nonlinear Equations: Fourier transforms are generally not applicable

to nonlinear differential equations. The transforms rely on linearity properties, such as superposition and scaling, which do not hold for nonlinear equations. In such cases, other techniques like numerical methods or perturbation methods may be more suitable.

2. Variable Coefficients: Fourier transforms are most commonly used for differential equations with constant coefficients. When the coefficients of the differential equation are functions of the independent variable or have a complicated dependence, the application of Fourier transforms becomes more challenging. In such cases, specialized techniques like Laplace transforms or numerical methods may be employed.

3. Boundary Conditions: Fourier transforms are well-suited for solving differential equations on unbounded domains or for periodic problems. However, when dealing with differential equations on finite intervals or with non-periodic boundary conditions, additional techniques like separation of variables, finite difference methods, or numerical techniques may be required.

4. Discontinuous Functions: Fourier transforms rely on the assumption that the functions involved are well-behaved and continuous. If the functions or their derivatives exhibit discontinuities, the Fourier transform may not be directly applicable. Techniques like generalized functions (e.g., distributions) or other specialized methods may be employed in these cases.

5. Stochastic or Random Processes: Fourier transforms are primarily used for deterministic differential equations. When dealing with stochastic or random processes, such as in the field of stochastic differential equations, other tools like stochastic calculus, probability theory, or numerical simulation methods may be more appropriate.

6 Conclusion