Métodos Steklov para EDEs no lineales

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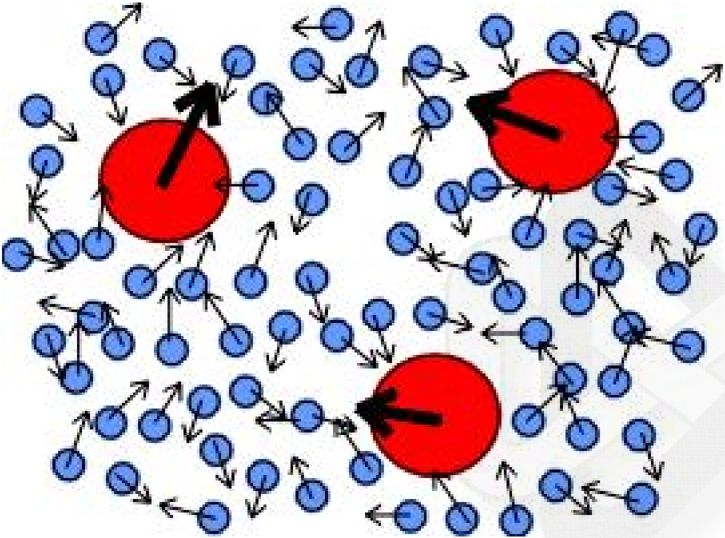
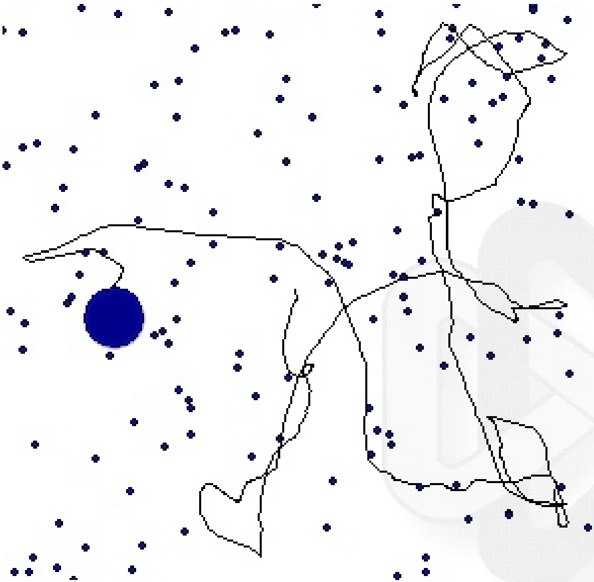
CIMAT A.C.

1 de diciembre de 2015





Ecuaciones de Movimiento

$$m\frac{d^2x}{dt^2} = \boxed{-\gamma\frac{dx}{dt}} + \boxed{\Gamma(t)}$$

- $x = x(t)$: posición a tiempo t.
- Fuerza de fricción, $\gamma = 6\pi\eta a$, η viscosidad laminar a radio coloide.
- $\Gamma(t)$: efecto estocástico debido a las colisiones.

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Al aplicar eliminación adiabática [?]

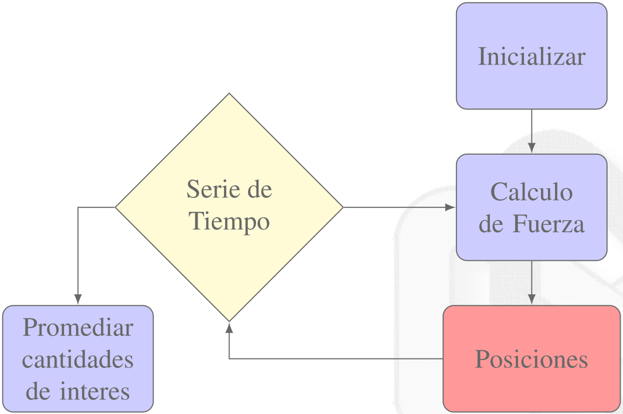
$$\frac{dx}{dt} = \frac{1}{k_B T} DF + D^{\frac{1}{2}} \xi.$$

- $x = x(t)$: posición a tiempo t.
- k_B, T: k_B constantes de Boltzmann, T temperatura,
- $F = -\dfrac{dU}{dx}$: fuerza de la partícula inmersa en un potencial U,
- $D = \dfrac{k_B T}{6\pi\eta a}$: coeficiente de difusión,
- ξ : ruido blanco,
  $\mathbb{E}(\xi(t)) = 0, \quad \mathbb{E}(\xi(t)\xi(t')) = 2\delta(t - t')$.

Gardiner, C. (1985).
Handbook of stochastic methods.
Springer Berlin.

Resolvemos $\dfrac{dx}{dt} = \dfrac{1}{k_B T} DF + D^{\frac{1}{2}}\xi.$

Para entender los mecanismos de difusión en una suspensión coloidal. *Sin embargo, en la práctica no se tiene solución analítica.*


Inicializar
Calculo de Fuerza
Serie de Tiempo
Posiciones
Promediar cantidades de interes

Euler-Mayurama (CBD)

$$
\begin{aligned}
Y_{j+1}^{(\alpha)}(h) =& Y_j^{(\alpha)} + \frac{D}{T} F_j^{(\alpha)} \Delta t + R_j^{(\alpha)} && (1)\\
\mathbb{E}\left[R_j^{(\alpha)}\right] =& 0 && (2)\\
\mathbb{E}\left[R_j^{(\alpha)} R_j^{(\beta)}\right] =& 2Dh\delta_{ij}\delta_{\alpha\beta} \qquad \alpha, \beta = x, y, z && (3)
\end{aligned}
$$

- $Y_j^{(\alpha)}$: posición.
- h : incremento temporal.
- $F_j^{(\alpha)}$: fuerza neta sobre la partícula i en la dirección α.
- $R_j^{(\alpha)}$: ruido blanco discreto, con media y covarianza como en (2) y (3).
- $D = \dfrac{k_B T}{\gamma}$: coeficiente de difusión de Stokes - Einstein

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- Es explicito, barato y fácil de implementar.
- Trabaja con un tamaño de paso restrictivo.

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Existen varios esquemas para discretizar la ecuación ya mencionada [Branka and Heyes, 1999]. Sin embargo, no representan una mejora significativa a la precisión respecto al coste computacional.

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Branka, A. and Heyes, D. (1999).
Algorithms for brownian dynamics computer simulations: Multivariable case.
Physical Review E, 60(2):2381.

Si el coeficiente de deriva o difusión de una EDE, *crece más rápido que algo lineal*, entonces el EM diverge.

M. Hutzenthaler, A. Jentzen, and P. E. Kloeden.
Strong and weak divergence in finite time of euler's method for stochastic differential equations with non-globally lipschitz continuous coefficients.
Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 467(2130):1563–1576, December 2010.

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Ejemplo:

$$
\begin{aligned}
dy(t) &= -10\text{sign}(y(t))|y(t)|^{1.1}dt + 4dW_t, \\
y_0 &= 0, \quad t \in [0, 10] \\
&\approx \mathbb{E}\left[|y(10)|\right], \quad 10^4 \text{ trayectorias }, \\
h &= 10/N, \quad N = \{1, 2, \ldots, 50\}
\end{aligned}
$$

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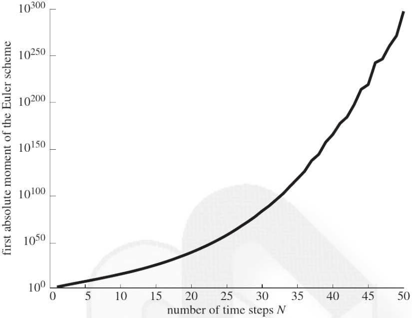

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- **Biología**
- Finanzas
- Física
- Química

Lotka Volterra

$$dX_t = (\lambda X_t - kX_tY_t)dt + \sigma X_t dW_t$$
$$dY_t = (kX_tY_t - mY_t)dt$$

M. Hutzenthaler and A. Jentzen, “Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients,” *Memoirs of the American Mathematical Society*, vol. 236, no. 1112, Jul. 2015. [Online]. Available: http://www.ams.org/

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Henston

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t \left(\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW^{(2)} \right)$$

$$dV_t = \kappa(\lambda - V_t)dt + \theta \sqrt{V_t} dW_t^{(2)}$$

M. Hutzenthaler and A. Jentzen, "Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients," *Memoirs of the American Mathematical Society*, vol. 236, no. 1112, Jul. 2015. [Online]. Available: http://www.ams.org/

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Langevin

$$dX_t = -(\nabla U)(X_t)dt + \sqrt{2\epsilon}dW_t$$

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Brusselator

$$dX_t = \left[\delta - (\alpha + 1)X_t + Y_t X_t^2\right] dt + g_1(X_t)dW_t^{(1)}$$
$$dY_t = \left[\alpha X_t + Y_t X_t^2\right] dt + g_2(X_t)dW_t^{(2)}$$

M. Hutzenthaler and A. Jentzen, "Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients," *Memoirs of the American Mathematical Society*, vol. 236, no. 1112, Jul. 2015. [Online]. Available: http://www.ams.org/

- **Implícitos**:
  - θ-BEM
  - FBEM
- **Explícitos**:
  - Tamed EM
  - Truncated
  - Sabanis

θ-Euler Maruyama

$$Y_{k+1} = Y_k + h(1-\theta)f(Y_k) + \theta f(Y_{k+1}) + g(Y_k)\Delta W_k,$$
$$\theta \in [0, 1].$$

Xuerong Mao and Lukasz Szpruch.
Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.
Journal of Computational and Applied Mathematics, 238:14–28, January 2013.

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Forward-Backward Euler Maruyama

$$
\begin{aligned}
Y_k &= Y_{k-1} + h(1-\theta)f(Y_{k-1}) + \theta f(Y_k) + g(Y_{k-1})\Delta W_{k-1} \\
\widehat{Y}_{k+1} &= \widehat{Y}_k + hf(Y_k) + g(Y_k)\Delta W_k, \qquad \theta \in [0,1].
\end{aligned}
$$

Xuerong Mao and Lukasz Szpruch.
Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.
Journal of Computational and Applied Mathematics, 238:14–28, January 2013.

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Tamed Euler Maruyama

$$Y_{k+1} = Y_k + \frac{hf(Y_k)}{1 + h\|f(Y_k)\|} + g(Y_k)\Delta W_k$$

Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden.
Strong convergence of an explicit numerical method for sdes with nonglobally lipschitz continuous coefficients.
The Annals of Applied Probability, 22(4):1611–1641, August 2012.

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Truncated Euler Maruyama

$$
\begin{aligned}
Y_{k+1} &= Y_k + f_\Delta(Y_k)h + g_\Delta(Y_k)\Delta_k, \\
f_\Delta(x) &:= \left(|x| \wedge \mu^{-1}(h(\Delta))\frac{x}{|x|}\right), \\
g_\Delta(x) &:= \left(|x| \wedge \mu^{-1}(h(\Delta))\frac{x}{|x|}\right)
\end{aligned}
$$

Xuerong Mao.
The truncated euler-maruyama method for stochastic differential equations.
Journal of Computational and Applied Mathematics, 290:370 – 384, 2015.

- **Implícitos**:
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Euler Maruyama with varying coefficients

$$Y_{k+1} = Y_k + \frac{hf(Y_k) + g(Y_k)\Delta W_k}{1 + k^{-\alpha}\left(\|f(Y_k)\| + \|g(Y_k)\|\right)}, \quad \alpha \in (0, 1/2]$$

Sotirios Sabanis.
Euler approximations with varying coefficients : the case of superlinearly growing diffusion coefficients.
To appear in Annals of Applied Probability, 2015.

Objetivo

Método **explicito**, **barato**, con condiciones **local Lipschitz** y **crecimiento super lineal**.

Plan de Charla

1. Esquemas Steklov (EDEs escalares)
   - Constucción
   - Consistencia y estabilidad
   - Estabilidad lineal
   - Resultados numéricos
2. Linear Steklov (EDEs vectoriales)
   - Construcción
   - Convergencia
   - Resultados Numéricos
3. Comentarios Finales
   - Conclusiones
   - Perspectivas

Plan de Charla

Plan de Charla

Nuestra idea

En 2005, Matus et. al., usan una versión del promedio de Steklov, para logra un esquema en diferencias exacto que resolve EDOs no lineales de la forma

$$\frac{dx}{dt} = f_1(x) f_2(t)$$

Matus, P., Irkhin, U., and Lapinska, M. (2005).
Exact difference schemes for time-dependent problems.
Computational Methods In Applied Mathematics, 5(4):422.

Método Steklov para EDEs escalares

Queremos aproximar:

$$dy(t) = f(t, y(t))dt + g(t, y(t))dW_t, \quad y_0 = cte \quad t \in [0, T],$$
$$f, g : [0, T] \times \mathbb{R} \to \mathbb{R}.$$

Considerando su forma integral:

$$y(t) = y_0 + \int_0^t f(s, y(s))ds + \int_0^t g(s, y(s))dW_s$$

Existencia y unicidad de soluciones

Sean $f, g : \mathbb{R} \to \mathbb{R}$.

Hipótesis:

- $f(t, x) = f_1(t) f_2(x)$.
- Lipschitz globales. $\exists\, L_1 > 0$ t.q. $\forall x, y \in \mathbb{R}$, $t \in [0, T]$

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq L|x - y|^2.$$

- Crecimiento Lineal. $\exists\, L_2 > 0$ t.q. $\forall x, y \in \mathbb{R}$, $t \in [0, T]$

$$|f(x,t)|^2 \vee |g(x,t)|^2 \leq L_2(1 + |x|^2).$$

Existencia y unicidad de soluciones

Bajo estos supuestos $\exists!\ y(t)$ t.q.

$$\mathbb{E}\left(\int_0^T |y(t)|^2 dt\right) < \infty$$

Mao, X. (2007).
Stochastic Differential Equations and Application.
Horwood Pub.

Construcción de métodos Tipo Euler

Tipo base:
Euler-Maruyama
(EM).

Discretizamos $[0, T]$ con un paso uniforme h:

- $t_n = nh \; n = 0, 1, 2, \ldots, N$.
- $Y_n \approx y(t_n)$

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Discretizamos $[0, T]$ con un paso uniforme h:

- $t_n = nh \; n = 0, 1, 2, \ldots, N.$
- $Y_n \approx y(t_n)$

Para cada nodo

$$y(t_{n+1}) = y_{t_n} + \underbrace{\int_{t_n}^{t_{n+1}} f(y(s))ds}_{\approx \text{Con algún método}} + \underbrace{\int_{t_n}^{t_{n+1}} g(y(s))dW_s}_{\approx g(y_{t_n})\Delta W_n}$$

$$\Delta W_n := \left(W_{t_{n+1}} - W_{t_n}\right) \sim \sqrt{h}\mathcal{N}(0, 1).$$

Construcción de métodos Tipo Euler

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Euler-Maruyama
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$$\boxed{y(t_{n+1}) = y_{t_n} + \int_{t_n}^{t_{n+1}} f(y(s))ds + \int_{t_n}^{t_{n+1}} g(y(s))dW_s} \quad (*)$$

Para el Euler-Mayurama se considera

$$\int_{t_n}^{t_{n+1}} f(y(s))ds \approx f(Y_n)h,$$

EM para $(*)$:
$Y_{n+1} = Y_n + f(Y_n)h + g(Y_n)\Delta W_n, \quad n = 0, 1 \ldots, N-1, \; Y_0 = y_0.$

Promedio especial de Steklov

Estimamos la deriva con el promedio especial de Steklov

$$f(y(t)) \approx \varphi(Y_n, Y_{n+1}) := \left(\frac{1}{Y_{n+1} - Y_n} \int_{Y_n}^{Y_{n+1}} \frac{du}{f(u)} \right)^{-1},$$

$$t_n \leq t \leq t_{n+1},$$

$$Y_n = Y_{t_n}, \quad t_n = nh.$$

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$$\begin{aligned} t_n &\leq t \leq t_{n+1}, \\ Y_n &= Y_{t_n}, \quad t_n = nh. \end{aligned}$$

Aproximamos

$$\int_{t_n}^{t_{n+1}} f(y(s))ds \approx \varphi(Y_n, Y_{n+1})h$$

Promedio especial de Steklov

Estimamos la deriva con el promedio especial de Steklov

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Métodos Steklov

Familia Steklov

$$Y_{n+1} = \boxed{Y_n + \varphi(Y_n, Y_{n+1})h} + \boxed{g(Y_n)\Delta W_n}$$

- $\approx \int_{Y_n}^{Y_{n+1}} \frac{du}{f(u)}$ (Cuadraturas)
- $\varphi(Y_n, Y^*_{n+1})$ (Pre-Corr)
- $Y^\star_n = Y_n + h\varphi(Y_n, Y^\star_n)$ (Split-Step)

Métodos Steklov

Familia Steklov

$$Y_{n+1} = Y_n + \varphi(Y_n, Y_{n+1})h + g(Y_n)\Delta W_n$$

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Métodos Steklov

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Métodos Steklov

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Steklov Explicito

$$dy(t) = f(t, y(t))dt + g(t, y(t))dW_t$$
$$f(t, y(t)) = f_1(t)f_2(y(t))$$

Steklov Explicito

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$\downarrow$

Steklov Implicito Determinista

$$y_{n+1} = y_n + h\varphi_1(t_n)\varphi_2(y_n, y_{n+1})$$

Define

$$H(x) := \int_0^x \frac{du}{f_2(u)}$$

Steklov Explicito

$$dy(t) = f(t, y(t))dt + g(t, y(t))dW_t$$
$$f(t, y(t)) = f_1(t) f_2(y(t))$$

$$y_{n+1} - y_n = \varphi_1(t_n) \frac{y_{n+1} - y_n}{H(y_{n+1}) - H(y_n)} h$$

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Resolviendo y_{n+1}

Steklov Explicito

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$$y_{n+1} = y_n + h\varphi_1(t_n)\varphi_2(y_n, y_{n+1})$$

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Resolviendo y_{n+1}

Steklov Explicito Determinista

$$y_{n+1} = \Psi_h(t_n, Y_n)$$
$$\Psi_h(t_n, Y_n) := H^{-1}\left[H(y_n) + h\varphi_1(t_n)\right]$$

Steklov Explicito

$$dy(t) = f(t, y(t))dt + g(t, y(t))dW_t$$
$$f(t, y(t)) = f_1(t) f_2(y(t))$$

Steklov Implicito Determinista

$$y_{n+1} = y_n + h\varphi_1(t_n)\varphi_2(y_n, y_{n+1})$$

Define

$$H(x) := \int_0^x \frac{du}{f_2(u)}$$

$$y_{n+1} - y_n = \varphi_1(t_n)\frac{y_{n+1} - y_n}{H(y_{n+1}) - H(y_n)} h$$

Resolviendo y_{n+1}

Steklov Explicito Determinista

$$y_{n+1} = \Psi_h(t_n, Y_n)$$
$$\Psi_h(t_n, Y_n) := H^{-1}\left[H(y_n) + h\varphi_1(t_n)\right]$$

Steklov Explicito Estocástico

$$Y_{n+1} = \Psi_h(t_n, Y_n) + g(t_n, Y_n)\Delta W_n$$

Definiciones y resultados previos

Kloeden, P. E. and Platen, E. (1991).
Numerical Solution of Stochastic Differential Equations.
Applications of Matematics. Springer-Verlag.

Consistencia convergencia y estabilidad en sentido fuerte

EDE

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0 \tag{EDE}$$

Y^h esquema con paso máx h.

$$\varepsilon(h) = \mathbb{E}\left(|y(T) - Y^h(T)|\right)$$

Definición (Consistencia)

Y^h a los tiempos $(\tau)_h = \{\tau_n : n = 0, 1, \cdots\}$ es consistente en sentido fuerte, si $\exists\, C = C(h) \geq 0, \quad h_0$ t.q. $\forall Y_n^h, n = 1, 2, \ldots, N, \quad h \in (0, h_0)$

- $\lim\limits_{h \downarrow 0} C(h) = 0$
- $\mathbb{E}\left(\left|\mathbb{E}\left(\frac{Y_{n+1}^h - Y_n^h}{h} \,|\mathcal{F}_{\tau_n}\right) - f\left(Y_n^h\right)\right|^2\right) \leq C(h).$
- $\mathbb{E}\left(\frac{1}{h}\left|Y_{n+1}^h - Y_n^h - \mathbb{E}\left(\frac{Y_{n+1}^h - Y_n^h}{h} \,|\mathcal{F}_{\tau_n}\right) - g\left(Y_n^h\right)\Delta W_n\right|^2\right) \leq C(h).$

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Definición (Convergencia fuerte)

Y^h converge en sentido fuerte a y a tiempo T si

$$\lim_{h \downarrow 0} \mathbb{E}\left(|y(T) - Y^h(T)|\right) = 0$$

Consistencia convergencia y estabilidad en sentido fuerte

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Definición (orden de convergencia)

Y^h converge en sentido fuerte con orden γ, si $\exists$ C independiente de h y h_0 t.q.

$$\epsilon(h) = \mathbb{E}\left(|y(T) - Y(T)|\right) \leq Ch^{\gamma} \qquad \forall h \in (0, h_0).$$

Consistencia convergencia y estabilidad en sentido fuerte

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$$dy(t) = f(y(t))dt + g(y(t))dW_t, \qquad y_0 = y(0)$$

Teorema

Bajo las condiciones del teorema de ***existencia y unicidad*** *(Lipschitz globales) para soluciones fuertes de (EDE). Si Y^h es consistente entonces Y^h converge en sentido fuerte a la solución $y(t)$.*

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Teorema

Bajo las mismas hipótesis, el esquema Steklov converge.

Estabilidad lineal por trayectorias

$$dy(t) = \lambda y(t)dt + \beta dW_t, \qquad y_0 = cte., \lambda, \beta \in \mathbb{R}. \tag{E}$$

Pullback attractor

E. BUCKWAR, M. G. RIEDLER, and P. E. KLOEDEN.
The numerical stability of stochastic ordinary differential equations with additive noise.
Stochastics and Dynamics, 11(02n03):265–281, 2011.

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Teorema

Sea $\lambda < 0$, el método Steklov para (E) tiene el siguiente atractor

$$\widehat{O}_n^{(h)} := \xi \sum_{j=-\infty}^{n-1} \exp(\lambda h(n-1-j))\Delta B_j,$$

$$\widehat{O}_n^{(h)} \to \widehat{O}_t, \quad h \to 0, \quad \textit{pathwise.}$$

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Estabilidad en Media Cuadrática Ruido Multiplicativo

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MS-estabilidad Lineal

- diagonal (EM)
- vertical (Steklov)

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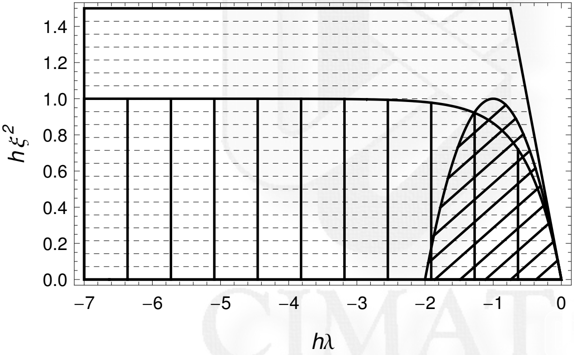

Ecuación Logística

$$dy(t) = \lambda y(t)(K - y(t))dt + \sigma y(t)^{\alpha}|K - y(t)|^{\beta} dW_t$$
$$X_0 = 50, K = 1000, \alpha = 1, \beta = 0.5, \lambda = 0{,}25, \rho = 0, \sigma = 0.05$$

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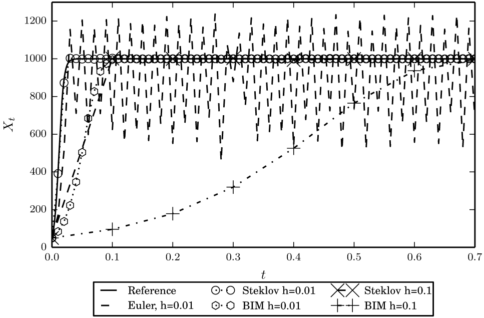

Schurz, H. (2007).
Modeling, analysis and discretization of stochastic logistic equations.
International Journal of Numerical Analysis and Modeling, 4(2):178–197.

Dinámica Browniana

$$\boxed{dX_t = -X_t^3 + \xi dB_t, \qquad Dta = 10^{-6}}$$

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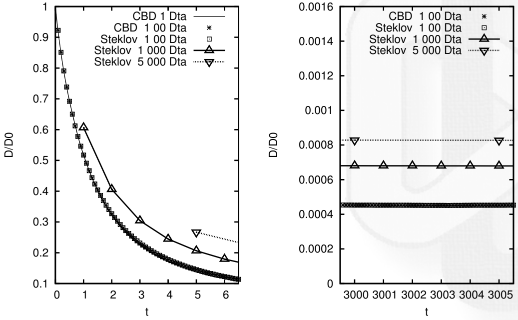

Branka, A. and Heyes, D. (1998).
Algorithms for brownian dynamics simulation.
Phys. Rev. E, 58:2611–2615.

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Difusión

$$g : \mathbb{R}^d \to \mathbb{R}^{d\times m},$$
$$g = \left(g^{(i,j)}\right)_{\substack{i\in\{1,\ldots,d\}\\ j\in\{1,\ldots,m\}}}$$
$$W = \left(W^{(1)}, \ldots, W^{(m)}\right)$$

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forma: $f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)}),\qquad a_j, b_j \in \mathcal{C}^1(\mathbb{R}^d)$

(EU-1) Lipschitz Local

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$\Rightarrow$ $\exists!\ y(t)$

Construcción

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)})$$

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)}))$$

$$\eta(t) := k,\ t \in [t_k, t_{k+1}), \quad k \geq 0,$$
$$\eta_+(t) := k+1,\ t \in [t_k, t_{k+1}), \quad k \geq 0$$

$$\varphi_f(y(t_{\eta_+(t)})) = \frac{y(t_{\eta_+(t)}) - y(t_{\eta(t)})}{\int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})} \frac{du}{a(y(t_{\eta(t)}))u+b}}$$

Construcción

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)})$$

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)}))$$

$$\varphi_f(y(t_{\eta_+(t)})) = \frac{y(t_{\eta_+(t)}) - y(t_{\eta(t)})}{\int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})} \frac{du}{a(y(t_{\eta(t)}))u+b}}$$

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$$\left(\varphi_{f^{(1)}}(Y_k^\star), \dots, \varphi_{f^{(d)}}(Y_k^\star)\right)$$

$$a_{j,k} = a_j\left(Y_k^{(1)}, \dots, Y_k^{(d)}\right),$$
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Construcción

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Hipótesis: $\forall x \in \mathbb{R}^d$

(A-1) $\exists\, L_a,\ a_j(x) \leq L_a$

(A-2) $|b_j(x^{(-j)})|^2 \leq L_b(1 + |x|^2)$

(A-3) Condiciones ceros de $a_j(\cdot)$

Teorema

Sea $u \in \mathbb{R}^d$

$$\begin{aligned} v &= u + h\varphi_f(v), \\ v &= A^{(1)}(h,u)u + A^{(2)}(h,u)b(u). \end{aligned}$$

Def

Construcción

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)})$$

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Teorema

$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$
$F_h(u) = v, \ \ \varphi_{f_h}(u) = \varphi_f(F_h(u)),$
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$\Rightarrow$ *son local Lipschitz*

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(A-2) $|b_j(x^{(-j)})|^2 \leq L_b(1 + |x|^2)$

(A-3) Condiciones ceros de $a_j(\cdot)$

Método Explícito

$$\begin{aligned} Y_k^\star &= A^{(1)}(h, Y_k)Y_k + A^{(2)}(h, Y_k)b(Yk), \\ Y_{k+1} &= Y_k^\star + g(Y_k^\star)\Delta W_k, \end{aligned}$$

Resultados para el EM

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0. \tag{EDE}$$

Hipótesis:

(H1) $\forall R > 0, \exists\, C_R > 0$

$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C_R|x - y|^2$

$\forall x, y \in \mathbb{R}^d |x| \vee |y| \leq R.$

(H2) Para algún $p > 2$, $\exists\, A > 0$ t.q.

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq A.$$

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(H2) Para algún $p > 2, \exists\, A > 0$ t.q.

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0 \leq t \leq T} |y(t)|^p\right] \leq A.$$

Higham, D. J., Mao, X., and Stuart, A. M. (2002). Strong convergence of euler-type methods for nonlinear stochastic differential equations. *SIAM Journal on Numerical Analysis*, 40(3):1041–1063.

Resultados para el EM

$$\boxed{dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0.} \tag{EDE}$$

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$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq A.$$

$$\overline{Y}(t) := Y_{\eta(t)} + (t - t_{\eta(t)}) f(Y_{\eta(t)}) + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}),$$
$$\eta(t) := k, \text{ for } t \in [t_k, t_{k+1})$$

Higham, D. J., Mao, X., and Stuart, A. M. (2002). Strong convergence of euler-type methods for nonlinear stochastic differential equations. *SIAM Journal on Numerical Analysis*, 40(3):1041–1063.

Resultados para el EM

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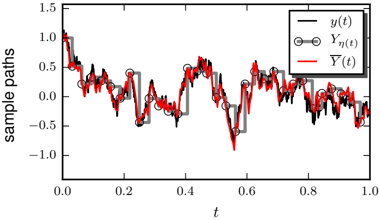

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$$\begin{aligned}\overline{Y}(t) := {} & Y_{\eta(t)} + (t - t_{\eta(t)})f(Y_{\eta(t)}) \\ & + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}),\end{aligned}$$
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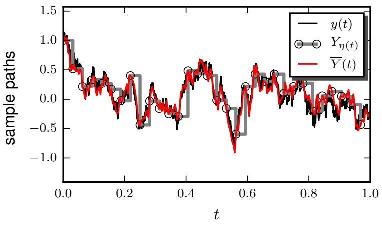

Teorema

EM converge

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] = 0.$$

Convergencia: Tecnica de Higham-Mao-Stuart

$$\boxed{dy(t) = f(y(t))dt + g(y(t))dW_t \text{ (EDE)}}$$

Paso 1:

Convergencia: Tecnica de Higham-Mao-Stuart

$$dy(t) = f(y(t))dt + g(y(t))dW_t \text{ (EDE)}$$

Paso 1: LS para (EDE) $\Leftrightarrow$ EM para (mEDE)

mEDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t)$$

Teorema

$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$
$F_h(u) = v,\ \varphi_{f_h}(u) = \varphi_f(F_h(u)),$
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$$dy(t) = f(y(t))dt + g(y(t))dW_t \text{ (EDE)}$$

Paso 1: LS para (EDE) $\Leftrightarrow$ EM para (mEDE)

Paso 2: $\mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t)|^p\right] \leq C\left(1 + \mathbb{E}\left[|y_0|^p\right]\right)$

mEDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t)$$

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$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$
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Paso 3: $\mathbb{E}\left[\sup_{kh\in[0,T]} |Y_k|^{2p}\right] \leq C,$

mEDE

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mEDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t)$$

Teorema

$$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$$
$$F_h(u) = v, \quad \varphi_{f_h}(u) = \varphi_f(F_h(u)),$$
$$g_h(u) = g(F_h(u)),$$
$$\Rightarrow |\varphi_{f_h}(u)| \leq L_\Phi |f(u)|.$$

Convergencia: Tecnica de Higham-Mao-Stuart

$$\boxed{dy(t) = f(y(t))dt + g(y(t))dW_t \text{ (EDE)}}$$

Paso 1: LS para (EDE) $\Leftrightarrow$ EM para (mEDE)

Paso 2: $\mathbb{E}\left[\sup_{0\le t\le T} |y_h(t)|^p\right] \le C\left(1+\mathbb{E}\left[|y_0|^p\right]\right)$

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y(t)-y_h(t)|^2\right] = 0,$$

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Paso 5: $\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right]$

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$$\lim_{h\to 0}\left\{\mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)-y(t)|^2\right]+\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y_h(t)|^2\right]\right\} = 0.$$

Teorema

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EDE con difusión superlineal

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EDE con difusión superlineal

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$$a(x) := -x^4 + x^2, \quad b := 1, \quad E = \{-1, 0, 1\}$$

$$Y_{k+1} = \exp(ha(Y_k))Y_k + \frac{\exp(ha(Y_k)) - 1}{a(Y_k)} \mathbf{1}_{\{E^c\}}$$
$$+ h\mathbf{1}_{\{E\}} + Y_k^2 \Delta W_k.$$

EDE con difusión superlineal

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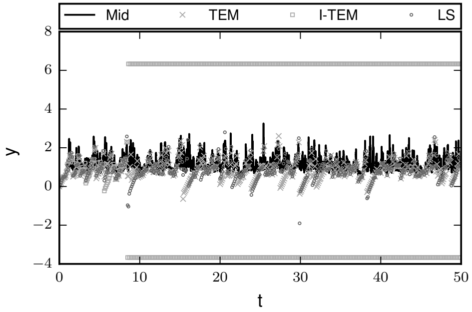

Tretyakov, M. V. and Zhang, Z. (2013).
A fundamental mean-square convergence theorem for sdes with locally lipschitz coefficients and its applications.
SIAM Journal on Numerical Analysis, 51:3135–3162.

Sistemas (Ecuación de Langevin)

$$U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$$

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	TEM		LS		BEM	
h	ms-error	ECO	ms-error	ECO	ms-error	ECO
2^{-2}	1.703 88	—	1.553 94	—	1.381 57	—
2^{-3}	1.169 77	0.54	1.107 75	0.48	1.053 09	0.39
2^{-7}	0.278 95	0.48	0.277 95	0.48	0.276 895	0.48
2^{-11}	0.070 10	0.50	0.070 09	0.50	0.070 07	0.50
2^{-15}	0.017 39	0.51	0.017 39	0.51	0.017 39	0.51

Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. (2012). Strong convergence of an explicit numerical method for sdes with nonglobally lipschitz continuous coefficients.

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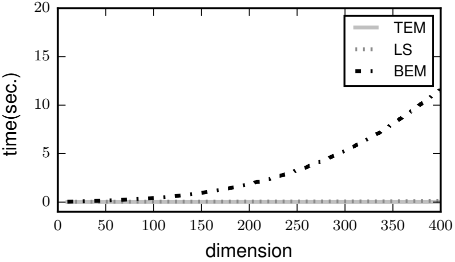

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Contra ejemplo para los tamed

$$
\begin{aligned}
dy_1(t) &= (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)}, \\
dy_2(t) &= ((1-\gamma)\beta y_1(t) y_3(t) - \alpha y_2(t))\, dt - \sigma_1 y_2(t) dW_t^{(1)}, \\
dy_3(t) &= ((1-\eta) N_0 \alpha y_2(t) - \mu y_3(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_2 y_3(t) dW_t^{(2)}
\end{aligned}
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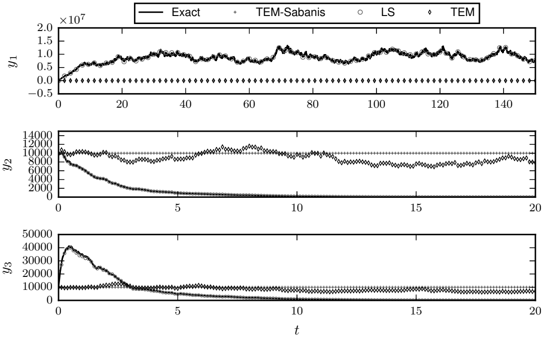

$\gamma = 0.5$, $\eta = 0.5$, $\lambda = 10^6$, $\delta = 0.1$, $\beta = 10^{-8}$, $\alpha = 0.5$, $N_0 = 100$, $\mu = 5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$,
$y_0 = (10\,000, 10\,000, 10\,000.)^T$,
$h = 0.125$.
Exacta: BEM $h = 10^{-5}$

Contra ejemplo para los tamed

$$dy_1(t) = (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)},$$
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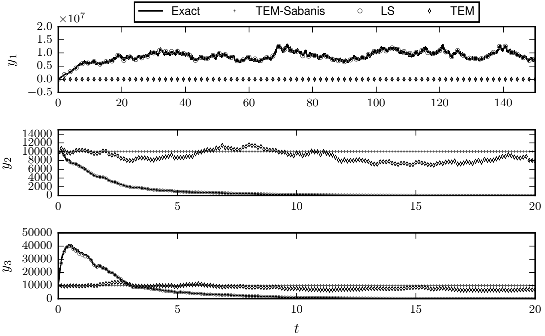

Dalal, N., Greenhalgh, D., and Mao, X. (2008). A stochastic model for internal hiv dynamics. *Journal of Mathematical Analysis and Applications*, 341(2):1084–1101.

Conclusiones

- En el caso escalar logramos un esquema con buenas propiedades de estabilidad.
- Obtuvimos una extensión para sistemas y coeficientes más generales.

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Trabajo Futuro

$$\boxed{dy(t) = f(y(t))dt + g(y(t))dW_t \text{ (EDE)}}$$

- Simulaciones sugieren: Steklov para difusión super-lineal.
- Combinar Multilevel-Montecarlo y Promedio de Steklov para Dinámica Browniana.
- Estabilidad usando teoría de Sistemas Random.
- Positividad.
- EDEs más generales:
  - Delay
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Buckwar, E., Riedler, M. G., and Kloeden, P. E. (2011).
The numerical stability of stochastic ordinary differential equations with additive noise.
Stochastics and Dynamics, 11(02n03):265–281.

Higham, D. J., Mao, X., and Stuart, A. M. (2002).
Strong convergence of euler-type methods for nonlinear stochastic differential equations.
SIAM Journal on Numerical Analysis, 40(3):1041–1063.

Hutzenthaler, M. and Jentzen, A. (2015).
Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients.
Memoirs of the American Mathematical Society, 236(1112).

Mao, X. and Szpruch, L. (2013).
Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.
Journal of Computational and Applied Mathematics, 238:14–28.

Apendice A: Existencia y unicidad (condiciones locales)

[Mao and Szpruch, 2013]

Teorema

(EU-1)-(EU-3) $\Rightarrow \exists! \ \{y(t)\}_{t \geq 0}$,
$\forall y(0) = y_0 \in \mathbb{R}^d$.
Además $0 < T < \infty$,

- $\mathbb{E}\left[y(T)\right] < \left(|y_0|^2 + 2\alpha T\right) \exp(2\beta T)$,
- $\tau_n := \inf\{t \geq 0 : |y(t)| > n\}, \ n \in \mathbb{N}$,
- $\mathbb{E}\left[|y(t)|^p\right] \leq 2^{\frac{p-2}{2}} \left(1 + \mathbb{E}\left[|y_0|^p\right]\right) e^{Cpt}$.

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◂ Construccón

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$$\mathbb{P}[\tau_n \leq T] \leq \frac{\left(|y_0|^2 + 2\alpha T\right) \exp(2\beta T)}{n},$$

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Apendice A

$$A^{(1)}(h,u) := \begin{pmatrix} e^{ha_1(u)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(u)} \end{pmatrix},$$

$$A^{(2)}(h,u) := \begin{pmatrix} \left(\dfrac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\dfrac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix},$$

$$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}, \qquad b(u) := \left(b_1(u^{(-1)}), \ldots, b_d(u^{(-d)})\right)^T.$$

Teorema

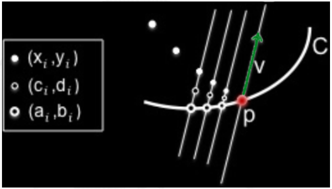
Apendice B: Resultato para ceros no aislados

Teorema (L'hôpital Multivariable)

- *$\mathcal{N}$ vecindad en $\mathbb{R}^2$ de $\mathbf{p}$ donde $f : \mathcal{N} \to \mathbb{R}$, $g : \mathcal{N} \to \mathbb{R}$ diferenciables son cero.*
- $C = \{x \in \mathcal{N} : f(x) = g(x) = 0\}$,
- *Supón C suave, que pasa por $\mathbf{p}$.*
- *$\exists$ $\mathbf{v}$ no tangente a C en $\mathbf{p}$ t.q $D_{\mathbf{v}}g$ en la dirección $\mathbf{v}$ es no nula en $\mathcal{N}$.*
- *$\mathbf{p}$ es punto limite de $\mathcal{N} \setminus C$.*

Entonces $$\lim_{(x,y)\to\mathbf{p}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to\mathbf{p}\\ (x,y)\in\mathcal{N}\setminus C}} \frac{D_{\mathbf{v}}f}{D_{\mathbf{v}}g},$$

siempre que exista el limite.



Hipótesis

Gary R Lawlor.
A l'hospital's rule for multivariable functions.
arXiv preprint arXiv:1209.0363, 2012.

Apendice B: Resultado para ceros aislados

Definición (DD respecto a p)

$u, \mathbf{p} \in \mathbb{R}^2$, α angulo positivo respecto a eje-x segmento $\overline{u\mathbf{p}}$.

$$f_\alpha(u) = \frac{\langle q - u, \nabla f(u) \rangle}{|u - q|}$$

derivada direccional respecto $\mathbf{p}$ *en* u.

Definición (Star-like set)

$S \subset \mathbb{R}^2$ es *star-like* respecto $\mathbf{p}$, $\forall\, s \in S$ el segmento abierto $\overline{s\mathbf{p}}$ esta en S.

Teorema

- $\mathbf{p} \in \mathbb{R}^2$, $S \subset \mathbb{R}^2$ *star-like respecto* $\mathbf{p}$ *en el dominio de* f, g.
- *En* S, f, g *diferenciables* , $g_\alpha(s) \neq 0$,
- $f(\mathbf{p}) = g(\mathbf{p}) = 0, \qquad \lim_{x \to \mathbf{p}} \frac{f_\alpha(x)}{g_\alpha(x)} = L,$

Entonces $\lim_{x \to \mathbf{p}} \frac{f(x)}{g(x)} = L.$

AI Fine and S Kass.
Indeterminate forms for multi-place functions.
Annales Polonici Mathematici, 18(1):59–64, 0 1966.

Apendice B: Condiciones para ceros de $a_j(\cdot)$

$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisface alguno de los puntos:

(I) $p \in E_j$ es un cero no aislado de $a_j(\cdot)$ y:

- $D := \{u : e^{ha_j(u)} - 1 = a_j(u) = 0\}$, es una curva suave que pasa por p.
- El vector canónico e_j es no tangente a D.
- Para cada $p \in E_j$, existe una bola $B_r(p)$ t.q.

$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in D \setminus B_r(p).$$

(II) $p \in E_j$ es un cero aislado de $a_j(\cdot)$ y:

- Para cada $q \in E_j$, p no es punto limite de $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- Para cada $p \in E_j$ existe $B_r(p)$, t.q. la derivada direccional respecto a p satiface

$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in B_r(p).$$