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ORIGINAL ARTICLE

Global Stability for Delay SIR and SEIR Epidemic Models with Nonlinear Incidence Rate

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Abstract In this paper, based on SIR and SEIR epidemic models with a general nonlinear incidence rate, we incorporate time delays into the ordinary differential equation models. In particular, we consider two delay differential equation models in which delays are caused (i) by the latency of the infection in a vector, and (ii) by the latent period in an infected host. By constructing suitable Lyapunov functionals and using the Lyapunov–LaSalle invariance principle, we prove the global stability of the endemic equilibrium and the disease-free equilibrium for time delays of any length in each model. Our results show that the global properties of equilibria also only depend on the basic reproductive number and that the latent period in a vector does not affect the stability, but the latent period in an infected host plays a positive role to control disease development.

Keywords Nonlinear incidence rate · Time delay · Lyapunov functional · Global stability

1. Introduction

It is well known that the spread of a communicable disease involves disease-related factors such as infectious agent, mode of transmission, incubation periods, infectious periods, susceptibility, and resistance. Communicable disease models describing a directly transmitted viral or bacterial agent in a closed population and consisting of susceptibles (S), infectives (I), and recovers (R) were considered by Kermack and McKendrick (1927). Based on the Kermack–McKendrick model, various epidemic models have been developed in recent decades, such as SIR models, SIS models, SEIR models with or without time delays (for example, Smith, 1983; Cooke et al., 1996, 1999; Beretta et al., 2001; Beretta and Takeuchi, 1995).

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Functional form of the incidence rate can have a crucial role for modeling of epidemic dynamics. It has been suggested by several authors that the disease transmission process may have a nonlinear incidence rate (see Derrick and van den Driessche, 2003; Korobeinikov and Maini, 2005; Korobeinikov, 2006, 2007; Kyrychko and Blyuss, 2005, and Liu et al., 1987 for survey of the possible reasons for nonlinearity of the incidence rate). To study the impact of the non-linearity, Korobeinikov and Maini (2005) considered a variety of models with the incidence rate of the form $g(I)h(S)$ and constructed Lyapunov functions that enable them to establish global properties for some of SIR and SEIR models. Thereafter, Korobeinikov (2006, 2007, 2009a, 2009b) also established global properties for a variety of epidemic models with incidence rate of a more general form $f(S, I)$.

Time delays are usually used to model the fact that an individual may not be infectious until some time after becoming infected. In the context of the epidemiology, delay can be caused by a variety of factors. The most notorious reasons for a delay are (i) the latency of the infection in a vector, and (ii) the latency of the infection in a infected host. In these cases, some time should elapse before the level of infection in the infected host or the vector will rise to a sufficiently high level to transmit the infection further. Delay is difficult to deal with mathematically, because straightforward incorporating it into a mathematical model generally leads to delay differential equations which are difficult to handle mathematically. A palliative that is commonly used is incorporating an extra compartment (or a few compartments), such as an exposed class of the SEIR model, that are accountable for the delay. While this approach is mathematically convenient as it leads to a higher order system of ordinary differential equations rather than to equation with a delay, it is also biologically questionable as it generally implies an assumption that the delay is exponentially distributed, whereas in epidemiological context it often appears that an assumption of a constant delay is more reasonable. The problem is, however, that the assumption of a constant delay leads to the delay differential equations. In Ma and Song (2004), McCluskey (2009a), and Rost and Wu (2008), SIR and SEIR models with discrete delays were analyzed, and delay epidemic models with nonlinear incidence further studied in Meng et al. (2009), Xu and Ma (2009), Zhao et al. (2008). Most of them only proved the global stability of disease-free equilibrium. Recently, McCluskey (2009a, 2009b) gave excellent proofs for the global stability of the endemic equilibrium for two classes of SIR and SEIR epidemic models with both finite time delay and infinite time delay by constructing suitable Lyapunov functionals. Motivated by these works, in this paper, we consider the global stability of two classes of SIR and SEIR epidemic models with more general incidence rate and with time delay, by constructing suitable Lyapunov functionals.

In this paper, we consider the impact of a constant delay on the dynamics of epidemics models with nonlinear incidence rate. We consider the incidence rate of the general form $F(S)G(I)$, and two type of the delays that correspond to (i) the latency in a vector, and (ii) the latent period in an infected hosts.

Respectively, the paper is organized as follows: In Section 2, we consider a class of SIR epidemic models with the nonlinear incidence rate and with a constant delay that is caused by latency in a vector. In Section 3, we consider a class of models with nonlinear incidence and with a constant delay that is caused by latency in a host. The global properties of these models are established by employing the direct Lyapunov method with suitable Lyapunov functional which we construct.

2. SIR epidemic model with delay caused by the latency in a vector

A number of the most dangerous infections are spread by vectors, that is agents (usually insects) by which infections are transmitted from one host to another. The most notorious examples of such vector-born infections and the vectors are: human malaria transmitted by mosquitoes of the *Anopheles* genus; *Aedes aegypti* mosquitoes that are vectors of avian malaria, Dengue fever and yellow fever; tsetse flies are vectors of human African trypanosomiasis ("African sleeping sickness"); Chagas' disease is spread by triatomine bugs; ticks of the genus *Ixodes* are vectors of Lyme disease and babesiosis; phlebotomine sand flies transmit leishmaniasis, bartonellosis and pappataci fever; fleas, such as *Pulex* and *Xenopsylla* transmit bubonic plague.

One of the major reasons for delay in epidemic models for the vector-born diseases is that some time (τ) needs before the infective organism develops in the vector to the level that is sufficient to pass the infection further.

If we assume that the force of infection is proportional to the infected human population time τ before, then the spread of a vector-born infection can be described by delay differential equation with the infection rate $F(s(t), i(t-\tau))$. (Note that under this assumption the incidence rate depends on the number of the susceptible host at this given moment t, and on the number of the infected host at the moment $t-\tau$.) Let $s(t)$ represent the number of individuals who are susceptible to the disease, that is, who are not yet infected at time t; $i(t)$ represent the number of infected individuals who are infectious and able to spread the disease contacting the susceptible individuals, and $r(t)$ represent the number of individuals who have been recovered from infection by time t. It is assumed that all recruitment goes into the susceptible class at a constant rate μ. The death rates for all classes are also μ, and the daily recovery rate is σ. Of course, $\mu, \sigma > 0$. We introduce time delay τ, which describes the infectious agents developing in a vector, into the model given by Korobeinikov and Maini (2005) and obtain the following time-delay SIR model:

$$\begin{aligned} \dot{s}(t) &= \mu - F(s(t))G(i(t-\tau)) - \mu s(t), \\ \dot{i}(t) &= F(s(t))G(i(t-\tau)) - (\sigma+\mu)i(t), \end{aligned} \tag{1}$$

and

$$\dot{r}(t) = \sigma i(t) - \mu r(t).$$

In this paper, we assume that $F(s)$ and $G(i)$ are always positive, continuous, and monotonically increasing for all $s > 0$ and $i > 0$, and $F(0) = G(0) = 0$. Since the variable $r(t)$ does not appear in the equations for $s(t)$ and $i(t)$, it is sufficient to consider the behavior of (1).

The initial condition of the system (1) is given as

$$s(\theta) = \varphi_1(\theta), \qquad i(\theta) = \varphi_2(\theta), \quad \theta \in [-\tau, 0], \tag{2}$$

where $\varphi = (\varphi_1, \varphi_2) \in C^+ \times C^+$, such that $\varphi_i(\theta) \geq 0$ $(-\tau \leq \theta \leq 0,\ i = 1, 2)$. Here, C denotes the Banach space $C([-\tau, 0], R)$ of continuous functions mapping the interval $[-\tau, 0]$ into R equipped with the sup-norm. The nonnegative cone of C is defined as $C^+ = C([-\tau, 0], R_+)$. System (1) always has a disease-free equilibrium $E_0(s_0, i_0)$, where

$s_0 = 1$ and $i_0 = 0$. System (1) may also admit a unique endemic equilibrium $E^*(s^*, i^*)$, where s^*, i^* satisfy

$$\mu = F(s^*)G(i^*) + \mu s^*, \quad F(s^*)G(i^*) = (\sigma + \mu)i^*. \tag{3}$$

Since $(s(t) + i(t))' \le \mu - \mu(s(t) + i(t))$, we have that $\limsup(s(t) + i(t)) \le 1$. Hence we discuss system (1) in the closed set

$$\Omega = \{(s(t), i(t)) \in C^+ \times C^+ : \|s(t) + i(t)\| \le 1\}.$$

It is easy to show that Ω is positively invariant with respect to system (1).

The basic reproductive number R_0 represents the average number of secondary infections produced when one infected individual is introduced into a host virgin population. From the following van den Driessche and Watmough (2002) and Korobeinikov and Maini (2005), the basic reproduction number for system (1) is given as

$$R_0 = \frac{F(s_0)}{\sigma + \mu} \frac{\partial G(i_0)}{\partial i}. \tag{4}$$

Here, $1/(\sigma + \mu)$ is the average infection period, and $F(s_0)$ is the number of new cases infected per unit time by one infective individual which is introduced into the susceptible compartment in the case that all the members of the population are susceptible. Usually, $R_0 < 1$ implies that the number of infective tends to zero, and $R_0 > 1$ implies that the number of infective increases.

To prove our results, we need the following lemmas.

Lemma 1 (Korobeinikov and Maini, 2005). *If the following conditions hold,*

$$F(0) = G(0) = 0; \qquad F'(s) > 0, \qquad G'(i) > 0; \quad \frac{\partial^2 G(i)}{\partial i^2} \le 0, \quad \text{for all } s, i > 0, \tag{5}$$

and if $R_0 > 1$, then in addition to the disease-free equilibrium state, system (1) has a unique positive endemic equilibrium $E^*(s^*, i^*)$. If $R_0 \le 1$, then the disease-free equilibrium E_0 is the only nonnegative equilibrium of system (1).

By analyzing the characteristic equations and characteristic roots of system (1), it is not difficult to show the following lemma.

Lemma 2. *Suppose that the functions $F(s)$ and $G(i)$ satisfy the conditions (5).*

(i) *If $R_0 < 1$, then equilibrium E_0 is locally asymptotically stable.*
(ii) *If $R_0 > 1$, the equilibrium E^* is locally asymptotically stable.*

The proof of Lemma 2 is given in the Appendix.

In the following, we discuss the main result, that is the global stability of the endemic equilibrium E^* and the disease-free equilibrium E_0 of the system (1), respectively.

Theorem 1. *Suppose that conditions (5) are satisfied.*

(i) *If $R_0 > 1$, the positive equilibrium $E^*(s^*, i^*)$ is globally asymptotically stable for any $\tau \geq 0$;*
(ii) *If $R_0 \leq 1$, the disease-free equilibrium $E_0(s_0, 0)$ is globally asymptotically stable for any $\tau \geq 0$.*

Proof: (i) Let us consider any solution $(s(t), i(t))$ of the model (1) with the initial condition (2). For any $t \geq 0$, we define a function $U_1(t)$ as follows:

$$U_1(t) = s(t) - \int_{s^*}^{s(t)} \frac{F(s^*)}{F(\eta)} d\eta + i(t) - \int_{i^*}^{i(t)} \frac{G(i^*)}{G(\eta)} d\eta.$$

Korobeinikov (2007) showed that the point E^* is the only internal stationary point and minimum point of $U_1(t)$, and $U_1(t) \to \infty$ at the boundary of the positive quadrant. Consequently, E^* is the global minimum point, and the function is bounded from below.

Let

$$U_+ = \int_0^\tau \left\{ \frac{G(i(t-\theta))}{G(i^*)} - 1 - \ln \frac{G(i(t-\theta))}{G(i^*)} \right\} d\theta,$$

it is easy to see that $U_+ \geq 0$ and $U_+ = 0$ if and only if $i(t-\theta) = i^*$ for all $\theta \in [0, \tau]$. For any positive $i(t-\theta)$ for θ in $[0, \tau]$, U_+ will be finite and can be differentiated. Then the derivative of U_+ satisfies

$$\begin{aligned}
\frac{dU_+}{dt} &= \frac{d}{dt} \int_0^\tau \left\{ \frac{G(i(t-\theta))}{G(i^*)} - 1 - \ln \frac{G(i(t-\theta))}{G(i^*)} \right\} d\theta \\
&= \int_0^\tau \frac{d}{dt} \left\{ \frac{G(i(t-\theta))}{G(i^*)} - 1 - \ln \frac{G(i(t-\theta))}{G(i^*)} \right\} d\theta \\
&= -\int_0^\tau \frac{d}{d\theta} \left\{ \frac{G(i(t-\theta))}{G(i^*)} - 1 - \ln \frac{G(i(t-\theta))}{G(i^*)} \right\} d\theta \\
&= -\left\{ \frac{G(i(t-\theta))}{G(i^*)} - 1 - \ln \frac{G(i(t-\theta))}{G(i^*)} \right\} \Big|_{\theta=0}^{\tau} \\
&= -\frac{G(i(t-\tau))}{G(i^*)} + \frac{G(i(t))}{G(i^*)} + \ln \frac{G(i(t-\tau))}{G(i(t))}.
\end{aligned}$$

Now we study the behavior of the Lyapunov functional

$$V_1 = U_1(t) + (\sigma + \mu) i^* U_+.$$

The derivative of V_1 along solution of (1) is given by

$$\begin{aligned}
\frac{dV_1}{dt} &= \left(1 - \frac{F(s^*)}{F(s)}\right) \dot{s}(t) + \left(1 - \frac{G(i^*)}{G(i)}\right) \dot{i}(t) + (\sigma + \mu) i^* \frac{dU_+}{dt} \\
&= \left(1 - \frac{F(s^*)}{F(s)}\right) (\mu s^* - \mu s + F(s^*) G(i^*) - F(s) G(i(t-\tau)))
\end{aligned}$$

$$+\left(1-\frac{G(i^*)}{G(i)}\right)(F(s)G(i(t-\tau))-(\sigma+\mu)i)$$
$$-(\sigma+\mu)i^*\left(\frac{G(i(t-\tau))}{G(i^*)}-\ln\frac{G(i(t-\tau))}{G(i)}-\frac{G(i)}{G(i^*)}\right).$$

By noting that

$$\ln\frac{G(i(t-\tau))}{G(i)}=\ln\frac{F(s^*)}{F(s)}+\ln\frac{F(s)G(i(t-\tau))}{F(s^*)G(i)},$$

and

$$(\sigma+\mu)i^*=F(s^*)G(i^*),$$

it is easy to see that

$$\frac{dV_1}{dt}=\frac{\mu}{F(s)}(s-s^*)[F(s^*)-F(s)] \tag{6}$$
$$+(\sigma+\mu)i^*\left(\frac{G(i^*)}{G(i)}-1\right)\left(\frac{i}{i^*}-\frac{G(i)}{G(i^*)}\right) \tag{7}$$
$$-(\sigma+\mu)i^*\left(\frac{F(s^*)}{F(s)}-1-\ln\frac{F(s^*)}{F(s)}\right) \tag{8}$$
$$-(\sigma+\mu)i^*\left(\frac{F(s)G(i(t-\tau))}{F(s^*)G(i)}-1-\ln\frac{F(s)G(i(t-\tau))}{F(s^*)G(i)}\right). \tag{9}$$

Here,

$$\frac{F(s^*)}{F(s)}-1-\ln\frac{F(s^*)}{F(s)}\geq 0;\quad \text{for all } s>0,$$

and

$$\frac{F(s)G(i(t-\tau))}{F(s^*)G(i)}-1-\ln\frac{F(s)G(i(t-\tau))}{F(s^*)G(i)}\geq 0;\quad \text{for all } i(t-\tau)>0, s>0.$$

For a monotonically increasing function $F(s)$, $F(s)\geq F(s^*)$ holds when $s\geq s^*$ and $F(s)\leq F(s^*)$ holds when $s\leq s^*$, and hence the following inequality holds:

$$(s-s^*)[F(s^*)-F(s)]\leq 0. \tag{10}$$

Furthermore, since the function $G(i)$ is concave, the inequalities

$$\begin{cases} G(i)/G(i^*)\geq i/i^*, & \text{for } 0<i\leq i^*,\\ G(i)/G(i^*)\leq i/i^*, & \text{for } i\geq i^*, \text{ hold,}\end{cases} \tag{11}$$

implying that

$$\left(\frac{G(i^*)}{G(i)}-1\right)\left(\frac{i}{i^*}-\frac{G(i)}{G(i^*)}\right)\leq 0.$$

Therefore, all the conditions of Corollary 5.2 of Kuang (1993, p. 30) are satisfied. This proves that E^* is globally asymptotically stable for any $\tau \geq 0$ when $R_0 > 1$.

(ii) We consider a Lyapunov functional

$$V_2 = s(t) - \int_{s_0}^{s(t)} \frac{F(s_0)}{F(\eta)} d\eta + i(t) + F(s_0) \int_0^\tau G(i(t-\theta)) d\theta. \tag{12}$$

Let

$$U_- = \int_0^\tau G(i(t-\theta)) d\theta.$$

The derivative of U_- is

$$\frac{dU_-}{dt} = \int_0^\tau \frac{d}{dt} G(i(t-\theta)) d\theta = -\int_0^\tau \frac{d}{d\theta} G(i(t-\theta)) d\theta = -G(i(t-\tau)) + G(i(t)).$$

Hence, we obtain

$$\begin{aligned}
\frac{dV_2}{dt} &= \left(1 - \frac{F(s_0)}{F(s)}\right) \dot{s}(t) + \dot{i}(t) + \frac{dU_-}{dt} F(s_0) \\
&= \left(1 - \frac{F(s_0)}{F(s)}\right) (\mu s_0 - \mu s - F(s) G(i(t-\tau))) \\
&\quad + F(s) G(i(t-\tau)) - (\sigma + \mu) i(t) - F(s_0) G(i(t-\tau)) + F(s_0) G(i) \\
&= -\frac{\mu}{F(s)} (s - s_0)(F(s) - F(s_0)) + (\sigma + \mu) \left(\frac{F(s_0)}{(\sigma + \mu)} G(i) - i \right).
\end{aligned}$$

Here,

$$(s - s_0)(F(s) - F(s_0)) \geq 0 \tag{13}$$

and the conditions (5) ensure that $G(i) \leq \frac{\partial G(0)}{\partial i} i$ for all $i > 0$. Hence,

$$\frac{F(s_0)}{\sigma + \mu} G(i) - i \leq \left(\frac{F(s_0)}{\sigma + \mu} \frac{\partial G(0)}{\partial i} - 1 \right) i = (R_0 - 1) i. \tag{14}$$

Therefore, $R_0 < 1$ ensures that $dV_2/dt \leq 0$ for all $s(t), i(t) \geq 0$. Hence, again from Corollary 5.2 of Kuang (1993, p. 30), we have that E_0 is stable. Furthermore, also for $R_0 = 1$, $dV_2/dt = 0$ implies that $s(t) = s_0$. Hence, it is easy to show that $E_0(s_0, 0)$ is the largest invariant set in $\{(s(t), i(t)) | \dot{V}_2 = 0\}$. By the classical Lyapunov–LaSalle invariance principle (Theorem 5.3 of Kuang 1993, p. 30), E_0 is globally stable.

This completes the proof of Theorem 1. □

3. SEIR epidemic model with delay caused by latent period

The major reason for time delay in epidemic modeling is that all infectious diseases have so-called latent period. The latent period is the time elapsed between exposure of a host

to a pathogenic organism, and the infectiousness of this host. It is assumed that this time is needed for the pathogenic organism to reproduce within the infected host in sufficient numbers to become infectious for the others.

One approach to incorporate the latency into a model is introduction to an extra class (usually termed "exposed" and denoted by E) or a few classes for the hosts in latent state; the corresponding models that are based on this approach are the SEIR and SEIRS models. While such an approach is mathematically convenient as it leads to a system of ordinary differential equations, its relevance in context of epidemiology may cause some doubts, as this approach implies the assumption of the exponentially distributed time delay. Whereas in reality it often appears that the latent period can be better assumed to be constant and the same for all infected hosts. This later assumption leads to a delay model. A model that is based on this assumption is considered in this section.

In this section, we consider an SEIR model with nonlinear incidence rate and time delay caused by latency in a host. Here, $s(t)$, $e(t)$, $i(t)$, and $r(t)$ denote four classes, the susceptible, exposed, infectious, and recovered. Based on biological meaning, we consider the following delay SEIR epidemic model:

$$\begin{aligned}
\dot{s}(t) &= \mu - F(s(t))G(i(t)) - \mu s(t),\\
\dot{e}(t) &= F(s(t))G(i(t)) - e^{-\mu\omega}F(s(t-\omega))G(i(t-\omega)) - \mu e(t),\\
\dot{i}(t) &= e^{-\mu\omega}F(s(t-\omega))G(i(t-\omega)) - (\sigma+\mu)i(t),\\
\dot{r}(t) &= \sigma i(t) - \mu r(t).
\end{aligned}$$

Here, the time delay ω is the latent period of disease in a host. The term $e^{-\mu\omega}F(s(t-\omega), i(t-\omega))$ represents the individuals who were exposed at time $t-\omega$ and survive to time t (with the death rate μ), that is, represents the transformation of the exposed to the infectious group i.

Note that the variables $e(t)$ and $r(t)$ do not appear in the first and third equations of the above system. This allows us to consider the following two dimensional system:

$$\begin{aligned}
\dot{s}(t) &= \mu - F(s(t))G(i(t)) - \mu s(t),\\
\dot{i}(t) &= e^{-\mu\omega}F(s(t-\omega))G(i(t-\omega)) - (\sigma+\mu)i(t).
\end{aligned} \tag{15}$$

The system (15) has a disease-free equilibrium $\widetilde{E}_0(s_0, 0)$ where $s_0 = 1$ and endemic equilibrium $\widetilde{E}^*(\widetilde{s}^*, \widetilde{i}^*)$, where $\widetilde{s}^*$ and $\widetilde{i}^*$ satisfy

$$\begin{aligned}
\mu &= F(\widetilde{s}^*)G(\widetilde{i}^*) + \mu\widetilde{s}^*,\\
e^{-\mu\omega}F(\widetilde{s}^*)G(\widetilde{i}^*) &= (\sigma+\mu)\widetilde{i}^*.
\end{aligned}$$

The basic reproductive number for system (15) is

$$\widetilde{R}_0 = \frac{F(s_0)e^{-\mu\omega}}{\sigma+\mu}\frac{\partial G(0)}{\partial i}. \tag{16}$$

Remark 1. In particular, the basic reproductive number $\widetilde{R}_0$ of (15) is a decreasing function on the time delay ω. As $\omega \to \infty$, the basic reproductive number tends to zero. $\widetilde{R}_0$ is

different from the basic reproductive number R_0 for system (1). R_0 is independent of the delay τ, but $\widetilde{R}_0$ does depend on the delay ω since it includes a survival rate term $e^{-\mu\omega}$. When no delay in latent period is considered ($\omega = 0$), $\widetilde{R}_0$ reduces to R_0, which is the same to the basic reproduction number for ODE model in Korobeinikov and Maini (2005).

Firstly, for the local stability, it is easy to have the following.

Lemma 3. *Suppose that the functions $F(s)$ and $G(i)$ satisfy the conditions (5).*

(i) *If $\widetilde{R}_0 < 1$, then $\widetilde{E}_0$ is locally asymptotically stable;*
(ii) *If $\widetilde{R}_0 > 1$, then $\widetilde{E}^*$ exists and is locally asymptotically stable.*

Remark 2. The proof of Lemma 3 is similar to that of Lemma 2 and we omit it.

Now we are ready to investigate the global stability of equilibria by using the method of Lyapunov functional.

Theorem 2. *Suppose that conditions (5) is satisfied.*

(i) *If $\widetilde{R}_0 > 1$, the positive equilibrium $\widetilde{E}^*$ is globally asymptotically stable;*
(ii) *If $\widetilde{R}_0 \le 1$, the disease-free equilibrium $\widetilde{E}_0$ is globally asymptotically stable.*

Proof: (i) Define

$$U^+ = \int_0^\omega \left\{ \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} - 1 - \ln \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} \right\} d\theta.$$

Now we calculate the derivative of U^+,

$$\begin{aligned}
\frac{dU^+}{dt} &= \frac{d}{dt}\int_0^\omega \left\{ \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} - 1 - \ln \frac{F(t-\theta)G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} \right\} d\theta \\
&= \int_0^\omega \frac{d}{dt}\left\{ \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} - 1 - \ln \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} \right\} d\theta \\
&= -\int_0^\omega \frac{d}{d\theta}\left\{ \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} - 1 - \ln \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} \right\} d\theta \\
&= -\left\{ \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} - 1 - \ln \frac{F(s(t-\theta))G(i(t-\theta))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} \right\}\Bigg|_{\theta=0}^{\omega} \\
&= -\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(\widetilde{i}^*)} + \ln \frac{F(s(t-\omega))G(i(t-\omega))}{F(s(t))G(i(t))} \\
&\quad + \frac{F(s(t))G(i(t))}{F(\widetilde{s}^*)G(\widetilde{i}^*)}.
\end{aligned}$$

Let us define the Lyapunov functional

$$W_1 = e^{-\mu\omega}\left(s(t) - \int_{\widetilde{s}^*}^{s(t)} \frac{F(\widetilde{s}^*)}{F(\eta)} d\eta \right) + i(t) - \int_{\widetilde{i}^*}^{i(t)} \frac{G(\widetilde{i}^*)}{G(\eta)} d\eta + (\sigma+\mu)\widetilde{i}^* U^+.$$

Since $\mu = F(\widetilde{s}^*)G(\widetilde{i}^*) + \mu\widetilde{s}^*$, the derivative of this functional along the positive solutions of system (15) satisfies

$$\begin{aligned}
\frac{dW_1}{dt} &= e^{-\mu\omega}\left(1 - \frac{F(\widetilde{s}^*)}{F(s)}\right)\dot{s}(t) + \left(1 - \frac{G(\widetilde{i}^*)}{G(i)}\right)\dot{i}(t) + (\sigma+\mu)\widetilde{i}^*\frac{dU^+}{dt} \\
&= e^{-\mu\omega}\left(1 - \frac{F(\widetilde{s}^*)}{F(s)}\right)(\mu\widetilde{s}^* - \mu s + F(\widetilde{s}^*)G(\widetilde{i}^*) - F(s)G(i)) \\
&\quad + \left(1 - \frac{G(\widetilde{i}^*)}{G(i)}\right)(e^{-\mu\omega}F(s(t-\omega))G(i(t-\omega)) - (\sigma+\mu)i(t)) \\
&\quad - (\sigma+\mu)\widetilde{i}^*\left(\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(\widetilde{i}^*)}\right. \\
&\quad \left. - \ln\frac{F(s(t-\omega))G(i(t-\omega))}{F(s)G(i)} - \frac{F(s)G(i)}{F(\widetilde{s}^*)G(\widetilde{i}^*)}\right).
\end{aligned}$$

Note that

$$\ln\frac{F(s(t-\omega))G(i(t-\omega))}{F(s)G(i)} = \ln\frac{F(\widetilde{s}^*)}{F(s)} + \ln\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(i)}.$$

Hence, we have that

$$\frac{dW_1}{dt} = \frac{\mu e^{-\mu\omega}}{F(s)}(s-\widetilde{s}^*)[F(\widetilde{s}^*) - F(s)] \tag{17}$$

$$+ (\sigma+\mu)\widetilde{i}^*\left(\frac{G(\widetilde{i}^*)}{G(i)} - 1\right)\left(\frac{i}{\widetilde{i}^*} - \frac{G(i)}{G(\widetilde{i}^*)}\right) \tag{18}$$

$$- (\sigma+\mu)\widetilde{i}^*\left(\frac{F(\widetilde{s}^*)}{F(s)} - 1 - \ln\frac{F(\widetilde{s}^*)}{F(s)}\right) \tag{19}$$

$$\begin{aligned}
&- (\sigma+\mu)\widetilde{i}^*\left(\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(i)} - 1\right. \\
&\left. - \ln\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(i)}\right).
\end{aligned} \tag{20}$$

Here,

$$\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(i)} - 1 - \ln\frac{F(s(t-\omega))G(i(t-\omega))}{F(\widetilde{s}^*)G(i)} \geq 0$$

and

$$\frac{F(\widetilde{s}^*)}{F(s)} - 1 - \ln\frac{F(\widetilde{s}^*)}{F(s)} \geq 0$$

for all $s(t-\omega), i(t-\omega) > 0, s(t), i(t) > 0$. From the monotonousness of the function $F(s)$ and the concaveness of the function $G(i)$, we also have that

$$(s-\widetilde{s}^*)[F(\widetilde{s}^*)-F(s)] \le 0 \quad \text{and} \quad \left(\frac{G(\widetilde{i}^*)}{G(i)}-1\right)\left(\frac{i}{\widetilde{i}^*}-\frac{G(i)}{G(\widetilde{i}^*)}\right) \le 0.$$

Hence, the functional W_1 satisfies all the conditions of Corollary 5.2 of Kuang (1993, p. 30). This proves that $E(\widetilde{s}^*, \widetilde{i}^*)$ is globally asymptotically stable under the condition $\widetilde{R}_0 > 1$.

(ii) We define a Lyapunov functional

$$W_2 = e^{-\mu\omega}\left(s(t) - \int_{s_0}^{s(t)} \frac{F(s_0)}{F(\eta)}d\eta\right) + i(t) + e^{-\mu\omega}\int_0^{\omega} F(s(t-\theta))G(i(t-\theta))d\theta. \tag{21}$$

Let

$$U^- = \int_0^{\omega} F(s(t-\theta))G(i(t-\theta))d\theta$$

and calculate the derivative of U^-:

$$\begin{aligned}
\frac{dU^-}{dt} &= \frac{d}{dt}\int_0^{\omega} F(s(t-\theta))G(i(t-\theta))d\theta \\
&= \int_0^{\omega} \frac{d}{dt}F(s(t-\theta))G(i(t-\theta))d\theta \\
&= -\int_0^{\omega} \frac{d}{d\theta}F(s(t-\theta))G(i(t-\theta))d\theta \\
&= -F(s(t-\omega))G(i(t-\omega)) + F(s(t))G(i(t)).
\end{aligned}$$

Hence, using $\mu = \mu s_0$, we obtain

$$\begin{aligned}
\frac{dW_2}{dt} &= e^{-\mu\omega}\left(1-\frac{F(s_0)}{F(s)}\right)\dot{s}(t) + \dot{i}(t) + e^{-\mu\omega}\frac{dU^-}{dt} \\
&= e^{-\mu\omega}\left(1-\frac{F(s_0)}{F(s)}\right)(\mu s_0 - \mu s - F(s)G(i)) + e^{-\mu\omega}F(s(t-\omega))G(i(t-\omega)) \\
&\quad - (\sigma+\mu)i(t) - e^{-\mu\omega}F(s(t-\omega))G(i(t-\omega)) + e^{-\mu\omega}F(s)G(i) \\
&= \frac{\mu e^{-\mu\omega}}{F(s)}(s-s_0)(F(s_0)-F(s)) + (\sigma+\mu)\left(\frac{F(s_0)e^{-\mu\omega}}{\sigma+\mu}G(i) - i\right).
\end{aligned}$$

Since

$$\frac{F(s_0)e^{-\mu\omega}}{\sigma+\mu}G(i) - i \le \left(\frac{F(s_0)}{\sigma+\mu}e^{-\mu\omega}\frac{\partial G(0)}{\partial i} - 1\right)i(t) = (\widetilde{R}_0 - 1)i(t),$$

similar to the proof Theorem 1, it is proved that the equilibrium $\widetilde{E}_0$ is globally asymptotically stable for any $\omega \ge 0$ when $\widetilde{R}_0 \le 1$.

This completes the proof of Theorem 2. □

4. Discussion and conclusion

Time delays are used to model several different mechanisms in the dynamics of epidemics. Incubation period, maturation times, age structure, seasonal or diurnal variations, interactions across spatial distances or through complicated paths, as well as other mechanisms have been modeled by the introduction of time delays in dynamic models. In this paper, two classes of more general SIR and SEIR epidemic models with time delays and nonlinear incidence rate are studied. Based on epidemic models with nonlinear incidence rate $F(s)G(i)$ which was considered by Korobeinikov and Maini (2005), we incorporate discrete time delays to these models, which describe the latent period in a vector and in an infected host. By constructing Lyapunov functionals and using Lyapunov–LaSalle invariance principle, the global stability of the endemic equilibrium and the disease-free equilibrium are established. Comparing the conclusions obtained for the ODE models in Korobeinikov and Maini (2005), the results in this paper show that, the time delay which is a latent period in vector does not affect the global asymptotic properties of the model: that is if a model without time delay is global asymptotically stable, then the model with time delay is also global asymptotically stable. On the other hand, for system (15) with latent period in a host, the basic reproductive ratio $\widetilde{R}_0$ is a decreasing function on time delay ω. When all other parameters are fixed and delay ω is sufficiently large, $\widetilde{R}_0$ becomes to be less than one, which makes the decease free equilibrium globally asymptotically stable. By biological meanings, latent period plays a positive role in epidemic infection process in order to eliminate disease. Sufficiently large time delay in infected host makes the disease development slower and the disease has been controlled and disappeared. This gives us some suggestions on new drugs to prolong the latent period of infected host.

As pointed by Korobeinikov and Maini (2005), the concavity of the function $G(i)$ for i is a sufficient but not necessary condition for the global stability of system (1) and (15) without time delay. It is obvious that a weaker conditions (11) suffices for the global stability. From the proof of theorems in this paper, (11) is also sufficient for the global stability of the systems with time delays. The biologically feasible condition of monotonicity of the function $G(i)$ is not necessary to be satisfied for the condition (11) either.

And we would also like to point out here that the function of the form $h(x) = x - x^* - x^* \ln(x/x^*)$ which was successfully applied for ODE epidemics models by Korobenikov (2005, 2006, 2007, 2009a, 2009b), appears to be a sound basis to construct Lyapunov functions for more advanced models, including the models with time delay. In particular, in this paper, we construct the functional

$$H(t) = \int_0^\tau \left\{ \frac{x(t-\sigma)}{x^*} - 1 - \ln \frac{x(t-\sigma)}{x^*} \right\} d\sigma,$$

where x^* is an equilibrium value. Obviously, here $H(t) \geq 0$ for all $x \geq 0$, and $H(t) = 0$ if and only if $x(t-\sigma) = x^*$. This latter form of the Lyapunov functional was firstly introduced by McCluskey (2009a, 2009b) in order to investigate the global stability of endemic equilibrium of SEIR epidemic model with infinite delay. In this paper, we generalize this functional to delay models with a rather general nonlinear transmission $F(s)G(i)$. It is obvious that this type of Lyapunov functional can be also applied to specific delay models such as those given in Meng et al. (2009), Xu and Ma (2009), Zhao et al. (2008). Further, Korobeinikov considered epidemic and virus dynamics ODE models with nonlinear

transmission $F(s,i)$ in Korobeinikov, (2009a, 2009b). We expect to study the global stability of delay SIR and SEIR epidemic models with more general incidence rate $F(s,i)$ and leave it as a future work.

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Appendix: Proof of Lemma 2

(i) The characteristic equation of the linearized system of (1) is

$$\lambda^2+\left(\sigma+2\mu+\frac{\partial F(s)}{\partial s}G(i)\right)\lambda+(\sigma+\mu)\left(\mu+\frac{\partial F(s)}{\partial s}G(i)\right)$$
$$-(\lambda+\mu)F(s)\frac{\partial G(i)}{\partial i}e^{-\lambda\tau}=0. \tag{A.1}$$

It is clear that the associated transcendental characteristic Eq. (A.1) at $E_0=(s_0,0)$ becomes

$$(\lambda+\mu)\left(\lambda+\sigma+\mu-F(s_0)\frac{\partial G(0)}{\partial i}e^{-\lambda\tau}\right)=0. \tag{A.2}$$

For $\tau=0$, we have that

$$(\lambda+\mu)\left(\lambda+\sigma+\mu-F(s_0)\frac{\partial G(0)}{\partial i}\right)=0.$$

Obviously, $\lambda_1=-\mu<0$, and $\lambda_2=F(s_0)\frac{\partial G(0)}{\partial i}-(\sigma+\mu)=(\sigma+\mu)(R_0-1)$. Hence, for $R_0<1$ the roots of (A.2) have negative real parts for $\tau=0$.

Note that $\lambda=0$ is not a root of (A.2) while $R_0<1$. If (A.2) has pure imaginary roots $\lambda=\pm i\omega$ (where $\omega>0$) for some $\tau>0$, then we have from (A.2) that

$$\sigma+\mu=F(s_0)\frac{\partial G(0)}{\partial i}\cos\omega\tau,$$
$$-\omega=F(s_0)\frac{\partial G(0)}{\partial i}\sin\omega\tau.$$

This implies that

$$\omega^2=\left(F(s_0)\frac{\partial G(0)}{\partial i}\right)^2-(\sigma+\mu)^2=(\sigma+\mu)^2\left(R_0^2-1\right)<0.$$

Therefore, any root of (A.2) must have a negative real part, and hence the disease-free equilibrium E_0 is locally asymptotically stable for any time delay $\tau\geq 0$. This proves (i).

(ii) It is clear that at $E^* = (s^*, i^*)$ the associated transcendental characteristic Eq. (A.1) becomes

$$\lambda^2 + A\lambda + B - (C\lambda + D)e^{-\lambda\tau} = 0, \tag{A.3}$$

where

$$A = \sigma + 2\mu + \frac{\partial F(s^*)}{\partial s}G(i^*),$$

$$B = (\sigma + \mu)\left(\mu + \frac{\partial F(s^*)}{\partial s}G(i^*)\right),$$

$$C = F(s^*)\frac{\partial G(i^*)}{\partial i},$$

$$D = \mu F(s^*)\frac{\partial G(i^*)}{\partial i}.$$

For $\tau = 0$, we have from (A.3) that

$$\lambda^2 + (A - C)\lambda + B - D = 0. \tag{A.4}$$

As the function $G(i)$ is concave and $G(i^*) > \frac{\partial G(i^*)}{\partial i}i^*$, it follows from (3) that

$$A - C = \mu + \frac{\partial F(s^*)}{\partial s}G(i^*) + \frac{\sigma + \mu}{G(i^*)}\left(G(i^*) - \frac{\partial G(i^*)}{\partial i}i^*\right) > 0, \tag{A.5}$$

$$B - D = (\sigma + \mu)\frac{\partial F(s^*)}{\partial s}G(i^*) + \frac{\mu(\sigma + \mu)}{G(i^*)}\left(G(i^*) - \frac{\partial G(i^*)}{\partial i}i^*\right) > 0. \tag{A.6}$$

From Routh–Hurwitz criterion it follows that for $\tau = 0$ both roots of (A.3) have negative real parts.

Since $B > D$, $\lambda = 0$ is not a root of (A.4). If Eq. (A.3) has pure imaginary roots $\lambda = \pm i\omega$ (where $\omega > 0$) for some $\tau > 0$, then we have from (A.3) that

$$-\omega^2 + (\sigma + \mu)\left(\mu + \frac{\partial F(s^*)}{\partial s}G(i^*)\right) = F(s^*)\frac{\partial G(i^*)}{\partial i}(\mu\cos\omega\tau + \omega\sin\omega\tau),$$

$$\omega\left(\sigma + 2\mu + \frac{\partial F(s^*)}{\partial s}G(i^*)\right) = F(s^*)\frac{\partial G(i^*)}{\partial i}(\omega\cos\omega\tau - \mu\sin\omega\tau).$$

From these equalities, we have that

$$\omega^4 + (A^2 - C^2 - 2B)\omega^2 + B^2 - D^2 = 0.$$

Denoting $r = \omega^2$, we obtain that

$$f(r) = r^2 + (A^2 - C^2 - 2B)r + B^2 - D^2 = 0.$$

Since the function $G(i)$ is concave with respect to the variable i, we obtain

$$A^2 - C^2 - 2B = (\sigma + \mu)^2 + \left(\mu + \frac{\partial F(s^*)}{\partial s}G(i^*)\right)^2 - \left(\frac{\partial F(s^*)}{\partial s}G(i^*)\right)^2$$

$$= \left(\mu + \frac{\partial F(s^*)}{\partial s} G(i^*)\right)^2$$
$$+ \frac{\sigma + \mu}{G(i^*)} \left(\sigma + \mu + F(s^*) \frac{\partial G(i^*)}{\partial i}\right) \left(G(i^*) - \frac{\partial G(i^*)}{\partial i} i^*\right) > 0.$$

Since $B > D$ by (A.6), we have $B^2 - D^2 > 0$. Hence, we have that $f(r) > 0$ for any $r > 0$, which contradicts to $f(r) = 0$. This shows that for any time delay $\tau \geq 0$ both roots of the characteristic Eq. (A.3) have negative real parts.

This completes the proof of Lemma 2.

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