Institute of Robotics, University of Innopolis

Computational Intelligence
Subspaces and Projection

May 16, 2021

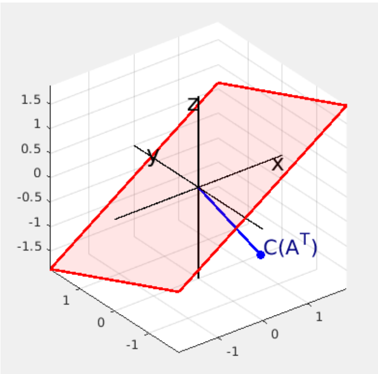
1 Four Fundamental Subspaces

1.1 Task 01

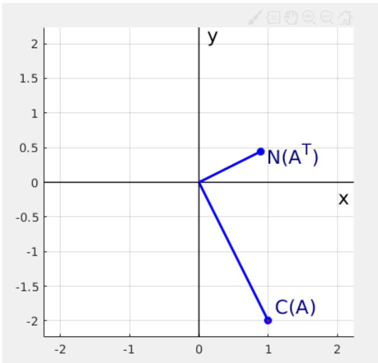
Let A be $\begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix}$. Calculate $C(A)$, $N(A)$, $C(A^T)$, and $N(A^T)$, then show that

1. Any member of row space is orthogonal to any member of nullspace
2. Any member of column space is orthogonal to any member of left nullspace

You may use null(A) and colspace(sym(A)) to calculate those listed above.



(a) ?



(b) ?

1.2 Task 02

A line containing the point $P_0 = x_0, y_0, z_0$ and parallel to the vector $\mathbf{v} =< a, b, c >$ can be described in the form of parametric equations

1. By looking at Figure **??**, define the parametric equations of a line?
2. If two lines, i.e., $\mathbf{r_1} = p_1 + t_1\mathbf{v_1}$ and $\mathbf{r_2} = p_2 + t_2\mathbf{v_2}$, are intersect each other, how can you verify it?

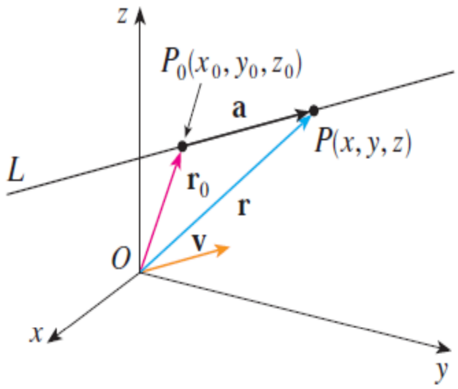


Figure 1

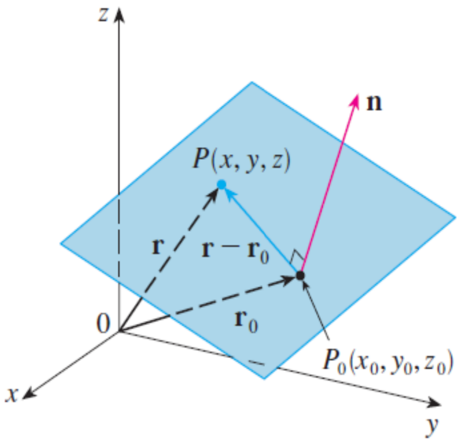


Figure 2

1.3 Task 03

A plane can be described by $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ where $\mathbf{n} =< a, b, c >$ (normal vector) and $\mathbf{r_0} =< x_0, y_0, z_0 >$ (position vector), refer to Fig.2

1. If two planes, i.e., $\mathbf{r_1} = p_1 + t_1\mathbf{v_1} + u_1\mathbf{w_1}$ and $\mathbf{r_2} = p_2 + t_2\mathbf{v_2} + u_2\mathbf{w_2}$, are intersect each other, how can you verify it?
2. Find the equation of the plane passing through origin if following two vectors are in the same plane, $\mathbf{v_1} =< 1, -5, 1 >$ and $\mathbf{v_2} =< 2, -1, -5 >$
3. Given $\mathbf{n}$ (or normal vector) how can you formulate the corresponding plane equation in the vector form?

2 Projection

2.1 Task 01

1. Let's consider in 2D and higher dimensional projections separately. With help of Fig.4b, try to prove p is given by $a\frac{a^Tb}{a^Ta}$. If the projection matrix is defined by $p = Pb$, show that

$P = \frac{aa^T}{a^Ta}$. With that, we can define $A\hat{x} = p$ where we can find an approximate solution, i.e., $\hat{x}$

2. If a_1 and a_2 form a basis for the plane, and $A = [a_1\ a_2]$, show that projection is given by $P = A(A^TA)^{-1}A^T$
3. How can we find the projection onto row space of A?
4. What can you say about P^T, P and P^2, P, if A does happen to be square, invertible matrix?

2.2 Task 02

Let A be $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix}$.

1. Find the A^+ (left inverse of A)?
2. The projection onto the column space of A?
3. The projection onto the left nullspace of A (orthogonal complement of the column space)?

2.3 Task 03

$$V = \{\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 \mid x + y + z = 0\} = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

1. What does N(V) represent?
2. How can we find the projection of any vector in $\mathbb{R}^3$ on to V, namely $Proj_V\mathbf{x}$?
3. How can we find the orthogonal complement of V ($V^\perp$)
4. How can we find the projection of any vector in $\mathbb{R}^3$ on to $V^\perp$, namely $Proj_{V^\perp}\mathbf{x}$?
5. Find the projection that all $\mathbf{x}$s are not in the V?

2.4 Task 04

Consider the following system:

$$\dot{x} = f(x) = Ax,\ x(0) = x_0 \tag{1}$$

, where $x \in \mathbb{R}^n$ and $t \geq 0$. Let $x^* = \Theta(t_0, x^*)$ be an equilibrium point at t_0 for the considered system (1). Such a x^* said to be stable (in the sense of Lyapunov) [1] if and only if for a given $\epsilon > 0$, there exists a $\delta(t_0, \epsilon) > 0$ such that

$$\left\|\Theta(t_0, x^*)\right\| \leq \delta \Rightarrow \left\|\Theta(t, x^*)\right\| < \epsilon,\ \forall t \leq t_0. \tag{2}$$

and said to be attractive (in the sense of Lyapunov) if and only if there exists $\delta(t_0)$ such that:

$$\left\|\Theta(t_0, x^*)\right\| \leq \delta \Rightarrow \lim_{t\to inf} \left\|\Theta(t_0, x^*)\right\| = 0. \tag{3}$$

Along with that, considered system is asymptotically stable (sense of Lyapunov) if the system is stable and attractive. Let $V(x) = x^T P x$, where P is a positive definite matrix, be a Lyapunov function for the system (1). If the system is stable if there exists a P:

$$\dot{V}(x) = (Ax)^T P x + x^T P A x \; = -x^T Q x \tag{4}$$

Thus, Lyapunov conditions can be written as

$$\begin{aligned} P > 0 \\ A^T P + PA + Q = 0 \end{aligned} \tag{5}$$

Lyapunov equations for a continuous system has form: $A^T P + PA = -Q$. As long as there exists such positive definite P that Lyapunov equations holds for a positive definite Q, the system is stable. You are given the following information about system

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} -10 & 5 \\ -5 & -10 \end{bmatrix} \tag{6}$$

Check whether system is stable or not? These APIs may help

```
#Python
scipy.linalg.solve_continuous_lyapunov
scipy.linalg.solve_discrete_lyapunov
scipy.linalg.eig
#Matlab
lyap(A,Q)
dlyap(A,Q)
eig(X)
```

Now let's consider

$$A = \begin{bmatrix} -10.05 & -0.021 & -0.02 \\ 0 & 0 & 0 \\ -0.022 & 0.0032 & -10.055 \end{bmatrix} \tag{7}$$

where system state is defined by $x = x_1, x_2, x_3$. Let's denote the value of x_2 as c since x_2 is a constant. Can you rewrite the system in terms of remaining variables x_1 and x_3 and prove its stability using the Lyapunov equation?

Orthonormal basis in the column space of the state matrix in this case is:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Motion of the system takes place in that column space. Let's denote $y = C^\top x$, and as long as x is in this column space, it is true that $x = Cy$. But if x is not in the column space, it is $x = Cy + x^*$

Notice that x^* is in the left null space of the state matrix, as long as $y = C^\top x$, because:

$$Cy = CC^\top x$$
$$x - x^* = CC^\top x$$
$$x^* = (I - CC^\top)x$$

where the last expression is a projection of x onto the left null space of the state matrix. Orthonormal basis in the left null space of the state matrix is:

$$L = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

And we know that $x_2 = c$, so $x^* = Lc$.

Variable y is the new coordinates in the column space basis.

Let's project the dynamics into the column space. First we multiply it by $C^\top$:

$$C^\top \dot{x} = C^\top \begin{pmatrix} -10.05 & -0.021 & -0.02 \\ 0 & 0 & 0 \\ -0.022 & 0.0032 & -10.055 \end{pmatrix} x$$

$$\dot{y} = \begin{pmatrix} -10.05 & -0.021 & -0.02 \\ -0.022 & 0.0032 & -10.055 \end{pmatrix} x$$

Then, since $x = Cy + Lc$ on the system trajectory, we get:

$$\dot{y} = \begin{pmatrix} -10.05 & -0.021 & -0.02 \\ -0.022 & 0.0032 & -10.055 \end{pmatrix} (Cy + Lc)$$

$$\dot{y} = \begin{pmatrix} -10.05 & -0.02 \\ -0.022 & -10.055 \end{pmatrix} y + \begin{pmatrix} -0.021 \\ 0.0032 \end{pmatrix} c$$

From here you apply Lyapunov eq. directly.

3 Appendix 1

Let A be a $m \times n$ matrix, then four sub spaces cab be defined as follows:

1. The column space $C(A) \in \mathbb{R}^m$, Its dimension is the Rank(A) = r
2. The nullspace $N(A) \in \mathbb{R}^n$, Its dimension is the Rank(A) = n - r
3. The row space $C(A^T) \in \mathbb{R}^n$, Its dimension is the Rank(A) = r
4. The left nullspace $N(A^T) \in \mathbb{R}^m$, Its dimension is the Rank(A) = r-m

The relationships among the fundamental subspaces are shown in Fig. 3.

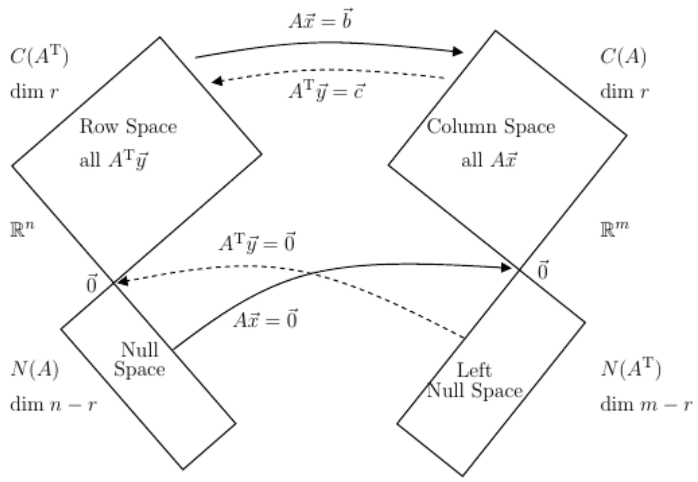
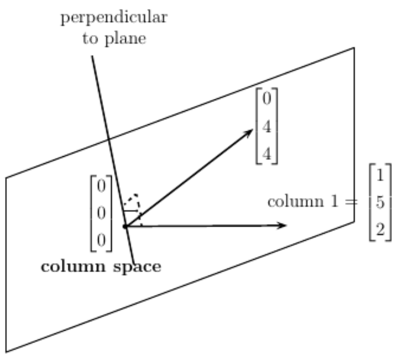
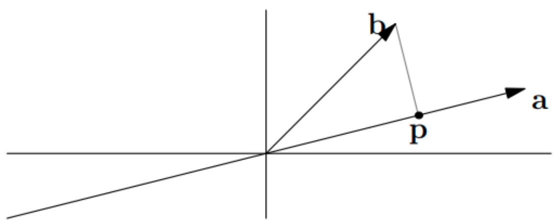


Figure 3: The fundamental subspaces [2]



(a) The column space of A, i.e., C(A) [2]



(b) Projection b on a

3.1 Let's get to the point

Get back to the basic and see how we can solve Least-squares problem using linear algebra rather than using calculus. To get things clear, Let me formulate the problem in a bit different form.

$$\underbrace{\begin{bmatrix}1 & 0\\5 & 4\\2 & 4\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}_{b} \tag{8}$$

where $\mathbf{x} = [x\ y]^T$. In Eq. 8, the relationship between three points in $\mathbb{R}^2$ is given. If you are given $b_1, b_2,$ and b_3 can you try to find the solution for x and y? If there is exist a combination of columns of A that can be form exactly equal to b. In other words, to hold preceding condition b should be in the same column space where A's columns are belonged. Let's try to visualize and see in which condition should hold in order to b lies in the same column space, i.e., C(A).

When will b be either or both x and y equal to zero? Thus, $Ax = b$ can be solved if and only if b lies in the plane where columns of A lie or C(A). If $b = 0$ can you try to solve, $Ax = 0$? The solution to $Ax = 0$ form another vector space which we called the null space of A, i.e., N(A). So in this case, define the vectors that are belonged to N(A)? As we discussed, there is no exact solution for when b does not lie on the C(A). Hence,to solve these kind of problems we have to to find the projection b onto the C(A). In other word, projections gives the minimum error to b.

3.2 Homogeneous coordinates and some geometric primitives

A point (X) in $\mathbb{R}^3$ is presented by a vector with 4 entries in homgeneous representation, i.e., $x =< x_1, x_2, x3, x4 >$, where $x_4 \neq 0$. On the contrary, a point in $\mathbb{R}^3$, i.e., (x,y,z), what would be the homogeneous representation? Besides, the projective transformation on $\mathbb{P}^3$ can be defined as: $X' = HX$ where H is a linear transformation operator?

3.2.1 Points and Planes in 3D

A plane in $\mathbb{R}^3$ can be written as $P = aX + bY + cZ + d = 0$ with homogeneous representation $\mathbf{P} = (a, b, c, d)^T$ in $\mathbb{P}^3$. If a given point $\mathbf{X} = (x, y, z, 1)^T$ lies on P then $\mathbf{X}^T\mathbf{P} = 0$ or $\mathbf{P}^T\mathbf{X} = 0$. Now consider a set of n points are stacked into a single matrix as

$$A = \begin{bmatrix} \mathbf{X}^{1T} \\ \mathbf{X}^{2T} \\ \vdots \\ \mathbf{X}^{nT} \end{bmatrix} \in \mathbb{R}^{n\times 4}. \tag{9}$$

Based on the rank(A), configuration of the set of points can be described: Fig. 4, Fig. 5, Fig. 6 and Fig. 7. Similarly, a set of n planes are stacked into a single matrix as

$$\pi = \begin{bmatrix} \mathbf{P}^{1T} \\ \mathbf{P}^{2T} \\ \vdots \\ \mathbf{P}^{nT} \end{bmatrix} \in \mathbb{R}^{n\times 4}. \tag{10}$$

Based on the rank(π), configuration of the set of planes can be described: Fig. 4, Fig. 5, Fig. 6 and Fig. 7

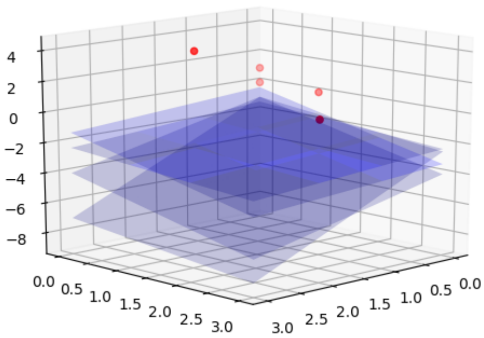


Figure 4: When rank(A) is 4 what will be the dimension of N(A)? or rank(π) is 4 what will be the dimension of N(π)?

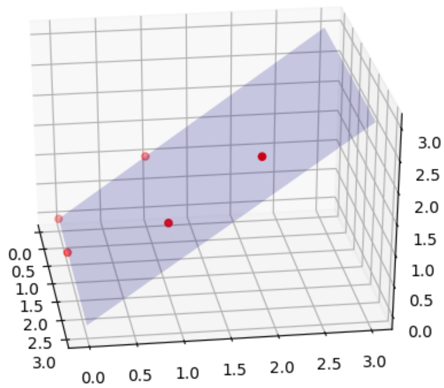


Figure 5: For rank(A) = 3, N(A) describes a plane that points lie on or for rank(π) = 1, N(π) describes three points on planes

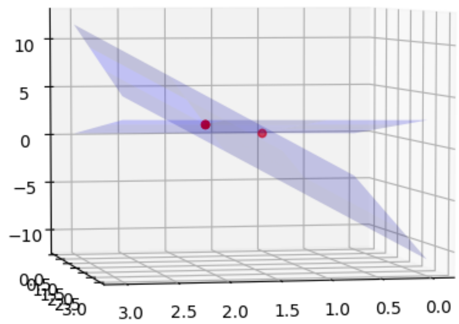


Figure 6: For rank(A) = 2, N(A) describes two planes intersecting at line that points lie on or for rank(π) = 2, N(π) describes two points in 3D line at intersection of planes

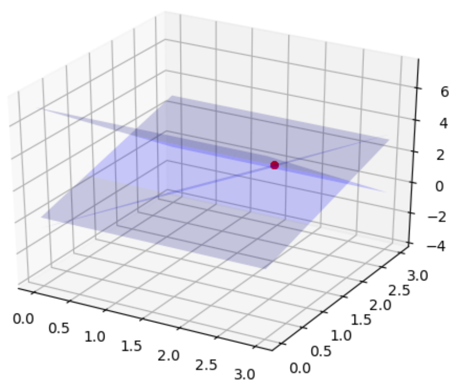


Figure 7: For rank(A) = 1, N(A) describes three planes intersecting at points or for rank(π) = 3, N(π) describes a point that intersection of planes

```
from numpy.linalg import svd
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

def rank(A, atol=1e-13):
    s = svd(A, compute_uv=False)
    rank = int((s >= atol).sum())
    return rank

def nullspace(A, atol=1e-13):
    _, s, vh = svd(A)
    nnz = (s >= atol).sum()
    ns = vh[nnz:].conj().T
    return ns

A = np.array([[2, 2, 2, 1], [4, 2, 2, 2], [4, 1, 1, 4]
, [1, 4, 4, 4], [1, 1, 1, 2]])
# A = np.array([[2, 2, 2, 1], [2, 2, 2, 1], [4, 1, 1, 4]
# , [4, 1, 1, 4], [4, 1, 1, 4]])
# A = np.array([[2, 2, 2, 1], [2, 2, 2, 1], [2, 2, 2, 1]
# , [2, 2, 2, 1], [2, 2, 2, 1]])
# A = np.array([[2, 1, 4, 1], [2, 3, 1, 1], [2, 3, 2, 3]
# , [2, 1, 2, 4], [2, 1, 4, 4]])

plane_ = nullspace(A)

xx, yy = np.meshgrid(range(4), range(4))
zz = [(-plane_[:,i][0]*xx - plane_[:,i][1]*yy - plane_[:,i][3])
                *1./plane_[:,i][2] for i in range(0, plane_.shape[1])]
if(plane_.shape[1]==0):
    zz = [(-A[:,i][0]*xx - A[:,i][1]*yy - A[:,i][3])*1./A[:,i][2]
            for i in range(0, A.shape[1])]
zz = np.array(zz)

plt3d = plt.figure()
ax = plt3d.add_subplot(111, projection='3d')
for i in range(0, zz.shape[0]):
    ax.plot_surface(xx, yy, zz[i], color='blue', alpha=0.2)
ax.scatter(A[:,0]/A[:,3], A[:,1]/A[:,3], A[:,2]/A[:,3]
```

```
                                                              , marker='o', color="red")
plt.show()
```

References

[1] Lecture Notes. *ReadingGroupControl*. `http://web.tuat.ac.jp/~gvlab/ronbun/ReadingGroupControl/mls-lyap.pdf`. [Online; accessed 2020-08-17].

[2] Nicholas J Rose. "Linear algebra and its applications (gilbert strang)". In: *SIAM Review* 24.4 (1982), pp. 499–501.