Singular Value Decomposition

Computational Intelligence, Lecture 4

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Spring 2022

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Singular Value Decomposition

Given $\mathbf{A} \in \mathbb{R}^{n,m}$ we can find its Singular Value Decomposition (SVD):

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\top \\ \mathbf{N}^\top \end{bmatrix} \tag{1}$$

$$\mathbf{A} = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top \tag{2}$$

where $\mathbf{C}$, $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{N}$ are column, left null, row and null space bases (orthonormal), $\mathbf{\Sigma}$ is the diagonal matrix of singular values. The singular values are positive and softer in the decreasing order.

Rank and pseudoinverse

Rank of the matrix is computed as the size of $\mathbf{\Sigma}$. Note that numeric tolerance applies when deciding if the singular value is non-zero.

Pseudoinverse $\mathbf{A}^+$ is defined as:

$$\mathbf{A}^+ = \begin{bmatrix} \mathbf{R} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}^\top \\ \mathbf{L}^\top \end{bmatrix} \tag{3}$$

$$\mathbf{A}^+ = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \tag{4}$$

SVD of a transpose

Let's find SVD decomposition of a $\mathbf{A}^\top$:

$$\mathbf{A}^\top = \begin{bmatrix} \mathbf{C}_t & \mathbf{L}_t \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}_t^\top \\ \mathbf{N}_t^\top \end{bmatrix} \tag{5}$$

Let us transpose it (remembering that transpose of a diagonal matrix the original matrix $\mathbf{\Sigma}_t^\top = \mathbf{\Sigma}_t$):

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_t & \mathbf{N}_t \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}_t^\top \\ \mathbf{L}_t^\top \end{bmatrix} \tag{6}$$

Thus we can see that the row space of the original matrix $\mathbf{A}$ is the column space of the transpose $\mathbf{A}^\top$. And the left null space of the original matrix $\mathbf{A}$ is the null space of the transpose $\mathbf{A}^\top$.

Let's prove that $\mathbf{A}\mathbf{A}^{+}$ is equivalent to $\mathbf{C}\mathbf{C}^{\top}$:

$$\mathbf{A}\mathbf{A}^{+} = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^{\top}\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^{\top} \tag{7}$$

$$\mathbf{A}\mathbf{A}^{+} = \mathbf{C}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{C}^{\top} \tag{8}$$

$$\mathbf{A}\mathbf{A}^{+} = \mathbf{C}\mathbf{C}^{\top} \tag{9}$$

Projectors (2)

Let's prove that $\mathbf{A}^{+}\mathbf{A}$ is equivalent to $\mathbf{R}\mathbf{R}^{\top}$:

$$\mathbf{A}^{+}\mathbf{A} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^{\top}\mathbf{C}\mathbf{\Sigma}\mathbf{R}^{\top} \tag{10}$$

$$\mathbf{A}^{+}\mathbf{A} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{R}^{\top} \tag{11}$$

$$\mathbf{A}^{+}\mathbf{A} = \mathbf{R}\mathbf{R}^{\top} \tag{12}$$

Projectors (3)

Let us denote $\mathbf{P} = \mathbf{A}\mathbf{A}^{+}$. Let's prove that $\mathbf{P}\mathbf{P} = \mathbf{P}$:

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{C}\boldsymbol{\Sigma}\mathbf{R}^{\top}\mathbf{R}\boldsymbol{\Sigma}^{-1}\mathbf{C}^{\top}\mathbf{C}\boldsymbol{\Sigma}\mathbf{R}^{\top}\mathbf{R}\boldsymbol{\Sigma}^{-1}\mathbf{C}^{\top} \tag{13}$$

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{C}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\mathbf{C}^{\top} \tag{14}$$

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{C}\mathbf{C}^{\top} = \mathbf{A}\mathbf{A}^{+} \tag{15}$$

The same is true for $\mathbf{P} = \mathbf{A}^{+}\mathbf{A}$: we can prove that $\mathbf{P}\mathbf{P} = \mathbf{P}$.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.