Barrier functions Computational Intelligence, Lecture 11

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CONTENT

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- Analytic center of linear inequalities
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LINEAR INEQUALITIES

Consider linear inequality constraints:

$$\mathbf{A}\mathbf{x} \le \mathbf{b} \tag{1}$$

Remember that we can rewrite it as:

$$\mathbf{a}_i^{\top} \mathbf{x} \le b_i \tag{2}$$

$$\mathbf{a}_i^{\mathsf{T}} \mathbf{x} - b_i \le 0 \tag{3}$$

Instead of $hard\ constraints$ in (3) we can turn these into a cost function component:

$$J = -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^{\mathsf{T}} \mathbf{x}) \tag{4}$$

Which is called a barrier function.

BARRIER FUNCTIONS

Let us consider barrier functions $J = -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^{\top} \mathbf{x})$:

- It removes the constraint, but modifies the cost.
- When $b_i \mathbf{a}_i^{\top} \mathbf{x}$ is a very small positive number, $\log(b_i \mathbf{a}_i^{\top} \mathbf{x})$ is a very big negative number, hence the minus sign in front.
- Barrier function does not behave well outside of the domain, when $b_i \mathbf{a}_i^{\top} \mathbf{x} < 0$.

Barrier functions for QPs

Hence the following QP:

minimize
$$\mathbf{x}^{\top} \mathbf{H} \mathbf{x} + \mathbf{f}^{\top} \mathbf{x}$$
,
subject to
$$\begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}(\mathbf{x}) = \mathbf{d}. \end{cases}$$
 (5)

...can be approximated as:

minimize
$$\mathbf{x}^{\top} \mathbf{H} \mathbf{x} + \mathbf{f}^{\top} \mathbf{x} - \sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^{\top} \mathbf{x}),$$
 subject to $\mathbf{C}(\mathbf{x}) = \mathbf{d}$ (6)

Analytic center of linear inequalities

We can define analytic center of linear inequalities as a minimum of the function $J = -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^{\top} \mathbf{x})$. And that can be solved as a convex optimization:

$$\mathbf{x}_a = \underset{\mathbf{x}}{\operatorname{argmin}} - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^{\mathsf{T}} \mathbf{x})$$

At the analytic center of linear inequalities the shape of contour lines can be analysed as a local quadratic approximation of the function J:

$$C = \{ \mathbf{x} : (\mathbf{x} - \mathbf{x}_a)^{\top} \frac{\partial^2 J}{\partial \mathbf{x}^2} (\mathbf{x} - \mathbf{x}_a) = \epsilon \}$$
 (7)

where ϵ is a small number.

Illustration of a barrier functions

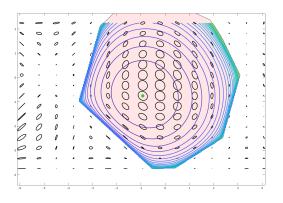


Figure 1: Barrier functions

Pink is the domain. The ellipsoids represent the shape of the hessian $\frac{\partial^2 J}{\partial \mathbf{x}^2}$ at different points on the domain. Green dot is \mathbf{x}_a .

HOMEWORK

Visualize contours of a quadratic program of your choice.

Lecture slides are available via Moodle.

 $You\ can\ help\ improve\ these\ slides\ at:$ github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022



Check Moodle for additional links, videos, textbook suggestions.