Institute of Robotics, University of Innopolis

Computational Intelligence
Semidefinite Programming

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1 Semidefinite Programming (SDP)

What is the suitable way to define semidefinite programming: a pecial case of convex programming or relaxation of quadratic programming or generalization of linear programming.

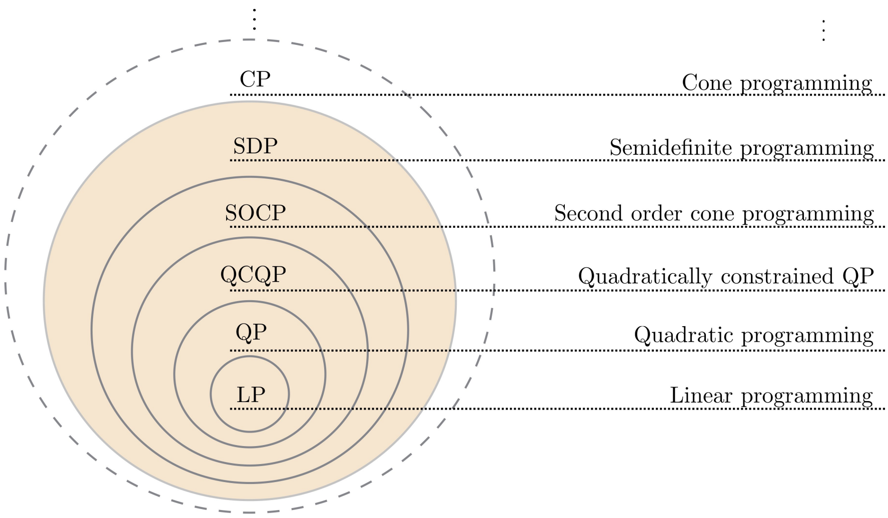

Figure 1: The hierarchy of disciplined convex optimization [1]

1.1 Task 01

In general, a SDP can be described in the following form:

$$\begin{aligned} \min \quad & C \cdot X \\ \text{s.t.} \quad & A_j \cdot X = b_j, j \in [m],\ X, A_1, \ldots, A_m \in \mathbb{R}^{n \times n},\ b_1, \ldots, b_m \in \mathbb{R} \\ & X \succeq 0, \end{aligned} \tag{1}$$

where both X and C be symmetric matrices. Consider the following example:

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{bmatrix},\ b_1 = 11, b_2 = 19 \tag{2}$$

1. Can you guess the values of n and m?
2. How many parameters to be optimized in this particular case?
3. Prove that $C \cdot X = x_{11} + 4x_{12} + 6x_{13} + 9x_{22} + 0x_{23} + 7x_{33}$
4. Assume x_{ij} denotes the elements of X and constraints are given by $x_{11} + 2x_{13} + 3x_{22} + 14x_{23} + 5x_{33} = 11$ and $4x_{12} + 16x_{13} + 6x_{22} + 4x_{33} = 19$. Aside this general case, what can you say about when all the entries of A_j and X become zeros except diagonal entries?

1.2 Task 02

A ellipsoid in $\mathbf{R}^n$ has the following form

$$Ellipsoid = \{x|(x-x_c)^T P^{-1}(x-x_c) \leq 1\} = \{x_c + Ax| \, \|x\|_2 \leq 1\}, \; P = P^T \succ 0 \tag{3}$$

where A is square and nonsingular ($A = P^{1/2}$). The matrix P determines the how far the ellipsoid extends in every direction from x_c whose length is determined by $\sqrt{\lambda_i}$, where λ_i is the i^{th} eiegen value. As given in (3), ellipsoid can be formulated in different ways,

$$Ellipsoid = \varepsilon = \{x|(x-x_c)^T P^{-1}(x-x_c) = \{x|(x-x_c)^T E(x-x_c) \leq 1\} \tag{4}$$

Intuition is to check given points x_i whether inside or outside of (5). In other words, the following condition must be satisfied

$$(x_i - x_c)^T E(x_i - x_c) \leq 1, \quad i = 1, \ldots, m \tag{5}$$

Hence, finding a optimal vector $x_c \in \mathbb{R}^n$ and positive definite symmetric matrix E which minimizes the volume subject to 5. The volume of an ellipsoid can be determined by[2]

$$Vol(\varepsilon) = \frac{v_0}{\sqrt{det(E)}} = v_0 det(E^{-1})^{1/2}, \tag{6}$$

where v_0 is the volume of the unit hypersphere in $\mathbb{R}^n$. Thus, minimizing the volume (or $det(E^{-1})^{1/2}$) can be formulated as an optimization problem as follows:

$$\begin{aligned} \min_{E,c} \quad & det(E^{-1})^{1/2} \\ \text{s.t.} \quad & (x_i - x_c)^T E(x_i - x_c) \leq 1 \; i = 1, \ldots, m \\ & E \succeq 0 \end{aligned} \tag{7}$$

Can we solve this as convex optimization problem? If not how can we solve this? By change of variables, (5) can be reformulated as follows:

$$\varepsilon = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 \leq 1\} \tag{8}$$

where $A = E^{1/2}$ and $b = E^{1/2}x_c$. Thus, (9) can be rewritten as:

$$\begin{aligned} \min_{A,b} \quad & -logdet(A) \\ \text{s.t.} \quad & \|Ax - b\|_2 \leq 1 \; i = 1, \ldots, m \\ & A \succeq 0 \end{aligned} \tag{9}$$

where norm constraints $\|Ax-b\|_2 \leq 1$, which are convex inequalities, can be expressed as Linear Matrix Inequalities (LMIs).

$$\|Ax-b\|_2 \leq 1 = \begin{bmatrix} I & Ax_i - b \\ (Ax_i - b)^T & I \end{bmatrix} \geq 0 \tag{10}$$

Where does this log comes form and what the reasoning behind this? When minimizing the $det(A^{-1})$, it might be fail due to non-convex nature. By applying log transformation the problem (9) guarantee to be strictly convex. This can be solved using any convex optimization solver. The following code can be used to design the problem in CVXPY,

1.3 Task 03

How can you estimate the maximum volume that encloses an inscribed ellipsoid of convex hull? Let's say you are given a set of inequalities that describes by a polytope:

$$P = \{x \in \mathbb{R}^n \; a_i^T x \leq b_i, \; i = 1, \ldots, m\} \tag{11}$$

So intention is to find inscribed ellipsoid with maximum volume. Let ε be the ellipsoid that is to be estimated:

$$\varepsilon = \{x \mid x = By + d, y \in \mathbb{R}^n, \|y\|_2 \leq 1, \; B = B^T \succ 0\}. \tag{12}$$

In order to maximize the volume, the condition $\varepsilon \subseteq P$ should be satisfied; this condition can be formed as the following way as a set of inequalities:

$$\|Ba_i\| + a_i^T d \leq b_i, \; i = 0, \ldots, m \tag{13}$$

Now we are ready to find the inscribed ellipsoid.

$$\begin{aligned} \min_{B,d} \quad & -logdet(B) \\ \text{s.t.} \quad & \|Ba_i\| + a_i^T d \leq b_i, \; i = 0, \ldots, m \\ & B \succeq 0 \end{aligned} \tag{14}$$

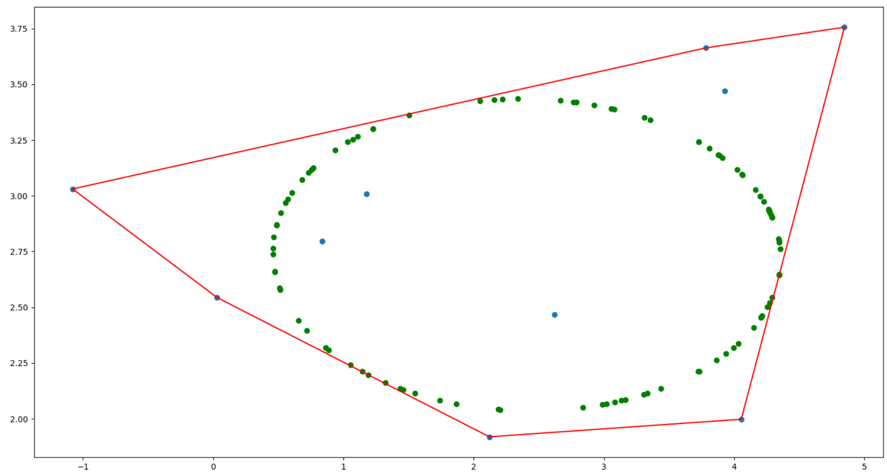

Figure 2: Maximum volume that encloses an inscribed ellipsoid of convex hull

References

[1] *https://twitter.com/CevherLIONS/status/1204918740392894464/photo/1*. 2020.

[2] Nima Moshtagh. *MINIMUM VOLUME ENCLOSING ELLIPSOIDS*.