Domain, Convex Domains

Computational Intelligence, Lecture 5

by Sergei Savin

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Content

- Domain
- Bounded and unbounded domains
- Convex domains
- Examples of convex domains
- Examples of non-convex domains
- Convex functions
- Convex functions - examples
- Convex programming
- Homework

Domain

Problem 1. Find minimum of the function $f = x^2 + 2y^2$ if $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Solution is $x = 0$, $y = 0$.

Problem 2. Find minimum of the function $f = x^2 + 2y^2$ if $x \in [1\ 2]$ and $y \in [2\ 5]$. Solution is $x = 1$, $y = 2$.

Note that solutions of problems 1 and 2 are different, and this is only due to the difference of the allowed values that the *decision variables* x and y can assume.

Definition 1

Space of all allowed values that decision variables can assume is called the *domain* of optimization problem.

Problem 3. Find minimum of the function $f = -x^2$ if $x \in [-3\ 2]$. Solution is $x = -3$.

Problem 4. Find minimum of the function $f = -x^2$ if $x \in \mathbb{R}$. The problem has no solution.

Problem 5. Find minimum of the function $f = -x^2$ if $x \in [-\infty\ 2]$. The problem has no solution.

The major difference between domains of the problems 2, 3 vs problems 1, 4 and 5 is that the later are *not bound* (i.e., you can construct a sequence of the values in the domain that would approach infinity).

We can see that in the case of problems 3-5, bounded domain allows the problem to have a solution.

Problem 6. Find maximum of the function $f = x^2$ if $1 \leq x < 2$. It has no solution.

Problem 7. Find minimum of the function $f = x^2$ if $1 \leq x < 2$. Solution is $x = 1$.

This time, it is the fact that one of the *boundaries* of the domain was not included in the domain that has lead the problem 6 to have no solution, while problem 7 had one. For the problem 6 we can pick a value arbitrary close to $x = 2$, approaching it from the left, but for any such value, there always will be other values of the decision variable closer to $x = 2$ and hence producing larger values of f.

Definition 2

Domain is *convex* iff for any two points in the domain, the line segment connecting them is also in the domain.

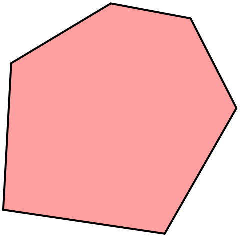

Figure 1: Convex domain

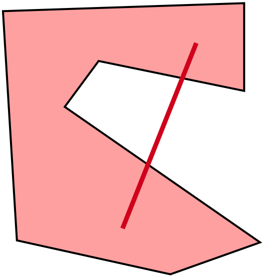

Figure 2: Non-convex domain

Convex combination

In the proofs it is convenient to remember that for any two points $\mathbf{x}_1$ and $\mathbf{x}_2$, all points in the line segment connecting them are given as $\mathbf{x}_l = \alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$, where $\alpha \in [0 \; 1]$. This is called *convex combination*.

Examples of convex domains

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \mathbb{R}^n$ is convex.

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} \leq \mathbf{h}\}$ is convex.

Proof: Note that $\alpha \mathbf{x}_1 \leq \alpha \mathbf{h}$ and $(1-\alpha)\mathbf{x}_2 \leq (1-\alpha)\mathbf{h}$, hence, $\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \leq \alpha \mathbf{h} + (1-\alpha)\mathbf{h} = \mathbf{h}$. $\square$

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \leq h^2\}$ is convex.

Proof: This is the same as $||\mathbf{x}|| \leq h$. Note that $||\alpha \mathbf{x}_1|| \leq \alpha h$ and $||(1-\alpha)\mathbf{x}_2|| \leq (1-\alpha)h$, also

$$||\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2|| \leq ||\alpha \mathbf{x}_1|| + ||(1-\alpha)\mathbf{x}_2||$$

$$||\alpha \mathbf{x}_1|| + ||(1-\alpha)\mathbf{x}_2|| \leq \alpha h + (1-\alpha)h = h$$

So the convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ is still in the domain. $\square$

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x}^\top \mathbf{H} \mathbf{x} \leq 1\}$, where $\mathbf{H} \succ 0$ is positive-definite symmetric matrix is convex.

For any positive-definite symmetric $\mathbf{H}$ it is true that $\mathbf{H} = \mathbf{D}^\top \mathbf{D}$, where $\mathbf{D} = \sqrt{\mathbf{H}}$ is called a matrix square root and it is full rank. With that $\mathbf{x}^\top \mathbf{H} \mathbf{x} \leq 1$ becomes $\mathbf{x}^\top \mathbf{D}^\top \mathbf{D} \mathbf{x} \leq 1$. Defining $\mathbf{y} = \mathbf{D}\mathbf{x}$ we get $\mathcal{X} = \{\mathbf{D}^{-1}\mathbf{y} : \ \mathbf{y}^\top \mathbf{y} \leq 1\}$. This is a linearly deformed previously covered domain, and as such it is also convex.

Examples of non-convex domains

$x \in \mathcal{X}$, $\mathcal{X} = [-1\ 2] \cup [3\ 7]$ is not convex..

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \geq h^2\}$ is not convex. Prove it.

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x}^\top \mathbf{H} \mathbf{x} \geq 1\}$, where $\mathbf{H}$ is positive-definite symmetric matrix is not convex. Prove it.

These proves simply require one counter-example to show that the defining property of convex domains does not hold.

Definition 3

Function $f(\mathbf{x})$ defined on a domain $\mathcal{D}$, for which it holds that $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, $f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2)$ is called *a convex function.*

Figure 3: Convex function

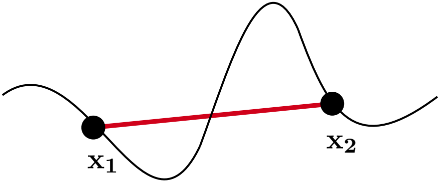

Figure 4: Non-convex function

Here are some single-variable convex functions:

- $f(x) = 1$
- $f(x) = x$, $f(x) = x + 1$, $f(x) = 6x + 3$
- $f(x) = x^2$, $f(x) = (x - 5)^2$, $f(x) = (x + 1)^2 - 10$
- $f(x) = x^3$, if $x > 0$

Here are some multi-variable convex functions:

- $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$
- $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{H}\mathbf{x}$, $\mathbf{H} \succ 0$

Definition 3

If the domain of the optimization problem is convex and the cost function is convex, it is called a *convex optimization problem*.

Additionally, we will always assume that the domain of the convex optimization problem contains its boundary. Also, without the loss of generality, we will consider only minimization problems.

There are a few important properties of convex optimization problems (with our additional assumption):

- If the domain is non-empty, there is a solution.
- The problem has no local minima. We can find a path from any point to the solution, along which the cost function will not increase.

Homework

- Make formal proofs asked for in this lecture.

- Convex Optimization, lecture 3, S. Boyd. Stanford. Convex functions.

Lecture slides are available via Moodle.

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