Quadratically constrained quadratic programming, Second-order cone programming

Computational Intelligence, Lecture 9

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Remember the general form of a quadratic program:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{1}$$

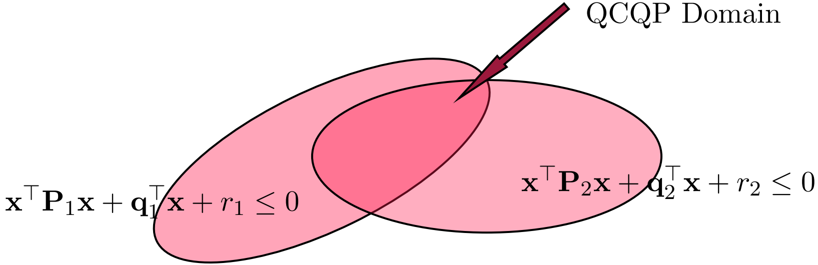
where $\mathbf{H}$ is positive-definite and $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ describe a convex region.

General form of a quadratically constrained quadratic program (QCQP) is given below:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{2}$$

where $\mathbf{P}_i$ are positive-definite.

Domain of a QCQP without equality constraints and with no degenerate inequality constraints is an intersection of ellipses:



QCQP to QP and LP

Set $\mathbf{P}_i = \mathbf{0}$ and you get a QP.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \begin{bmatrix} \mathbf{q}_1^\top \\ ... \\ \mathbf{q}_n^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} r_1 \\ ... \\ r_n \end{bmatrix} \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{3}$$

Set $\mathbf{P}_0 = \mathbf{0}$ and you get an LP.

The general form of a Second-order cone program (SOCP) is:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\
\text{subject to} \quad & \begin{cases} ||\mathbf{A}_i \mathbf{x} + \mathbf{b}_i||_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases}
\end{aligned}
\tag{4}
$$

LP, QP and QCQP are subsets of SOCP.

Consider the following SOC constraint:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}||_2 \leq \mathbf{c}^\top \mathbf{x} + d \tag{5}$$

Let us consider a special case when $\mathbf{x} \in \mathbb{R}^n$, rank $\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}\right) = n$.
Then we can introduce the following substitution:

$$\xi = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \mathbf{E} \\ \mathbf{e}^\top \end{bmatrix} \tag{6}$$

where $\mathbf{I} \in \mathbb{R}^{n,n}$ is an identity matrix. Then constraint (5) becomes:

$$||\mathbf{E}\xi||_2 \leq \mathbf{e}^\top \xi \tag{7}$$

SOC constraints, 2

Notice that $||\mathbf{E}\xi||_2 \leq \mathbf{e}^\top \xi$ is equivalent to:

$$\sum_{i=1}^{n-1} \xi_i^2 \leq \xi_n^2 \tag{8}$$

which is a standard form of a cone. A map back from ξ to $\mathbf{x}$ is given as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}^{-1} \left(\xi - \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix} \right) \tag{9}$$

We can write problem where our domain is a ball as SOCP:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & ||\mathbf{x}||_2 \leq d_i \end{aligned} \tag{10}$$

Same for ellipsoidal constraints:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & ||\mathbf{A}_i \mathbf{x}||_2 \leq d_i \end{aligned} \tag{11}$$

Set $\mathbf{c}_i = 0$ and recognize that $||\mathbf{A}_i\mathbf{x} + \mathbf{b}_i||_2 \leq d_i$ is the same as $(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i) \leq d_i^2$ (since the first implies that d_i is non-negative).

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top\mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x}^\top\mathbf{A}_i^\top\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^\top\mathbf{A}_i\mathbf{x} + \mathbf{b}_i^\top\mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{12}$$

SOCP to QCQP
Part 2

Now to make the cost quadratic:

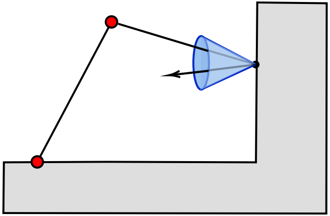
$$\begin{aligned} &\underset{\mathbf{x},t}{\text{minimize}} \quad t, \\ &\text{subject to} \quad \begin{cases} \mathbf{x}^\top \mathbf{A}_0^\top \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^\top \mathbf{A}_0 \mathbf{x} + \mathbf{b}_0^\top \mathbf{b}_0 \leq t \\ \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{13}$$

Which is the same as:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ &\text{subject to} \quad \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{14}$$

As long as $\mathbf{A}_0 = \sqrt{\mathbf{H}}$, and $\mathbf{b}_0 = 0.5\mathbf{A}_0^{-1}\mathbf{f}$.

Normal reaction force and friction



Let $\mathbf{f}$ be total reaction force, $\mathbf{f}_n$ be its normal component (perpendicular to the surface locally), also known as normal reaction; and let $\mathbf{f}_{fr}$ be its tangential component (a vector lying in the tangent plane to the surface, constructed at the contact point), or friction force. Let $\mathbf{e}_n$ be a unit vector, normal to the surface at the point of contact.

$$\mathbf{f} = \mathbf{f}_n + \mathbf{f}_{fr} \tag{15}$$

Defining $\mathbf{E}_t = [\mathbf{e}_{t,1},\ \mathbf{e}_{t,2}] = \mathcal{L}(\mathbf{e}_n)$ be an orthonormal basis in the tangential space to the surface, we can write:

$$\mathbf{f} = \mathbf{e}_n n + \mathbf{E}_t \mathbf{t}$$
$$\mathbf{f}_n = \mathbf{e}_n n$$
$$\mathbf{f}_{fr} = \mathbf{E}_t \mathbf{t}$$
$$\mathbf{t} = [t_1,\ t_2]$$

The friction cone conditions could be written in any of the following ways:

$$\sqrt{t_1^2 + t_2^2} < \mu n \tag{16}$$

$$||\mathbf{E}_t^\top \mathbf{f}|| \leq \mu \mathbf{e}_n^\top \mathbf{f} \tag{17}$$

where μ is a friction coefficient.

Implement a program that finds right-most point of an intersection of two ellipsoids; visualise the problem and the solution.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.