Barrier functions

Computational Intelligence, Lecture 11

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Consider linear inequality constraints:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{1}$$

Remember that we can rewrite it as:

$$\mathbf{a}_i^\top \mathbf{x} \leq b_i \tag{2}$$

$$\mathbf{a}_i^\top \mathbf{x} - b_i \leq 0 \tag{3}$$

Instead of *hard constraints* in (3) we can turn these into a cost function component:

$$J = -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^\top \mathbf{x}) \tag{4}$$

Which is called a *barrier function*.

Let us consider barrier functions $J = -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^\top \mathbf{x})$:

- It removes the constraint, but modifies the cost.
- When $b_i - \mathbf{a}_i^\top \mathbf{x}$ is a very small positive number, $\log(b_i - \mathbf{a}_i^\top \mathbf{x})$ is a very big negative number, hence the minus sign in front.
- Barrier function does not behave well outside of the domain, when $b_i - \mathbf{a}_i^\top \mathbf{x} < 0$.

Hence the following QP:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}(\mathbf{x}) = \mathbf{d}. \end{cases} \end{aligned} \tag{5}$$

...can be approximated as:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x} - \sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^\top \mathbf{x}), \\ \text{subject to} \quad & \mathbf{C}(\mathbf{x}) = \mathbf{d} \end{aligned} \tag{6}$$

Analytic center of linear inequalities

We can define *analytic center of linear inequalities* as a minimum of the function $J = -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^\top \mathbf{x})$. And that can be solved as a convex optimization:

$$\mathbf{x}_a = \underset{\mathbf{x}}{\text{argmin}} \quad -\sum_{i=1}^{n} \log(b_i - \mathbf{a}_i^\top \mathbf{x})$$

At the analytic center of linear inequalities the shape of contour lines can be analysed as a local quadratic approximation of the function J:

$$\mathcal{C} = \{\mathbf{x} : \ (\mathbf{x} - \mathbf{x}_a)^\top \frac{\partial^2 J}{\partial \mathbf{x}^2} (\mathbf{x} - \mathbf{x}_a) = \epsilon\} \tag{7}$$

where ϵ is a small number.

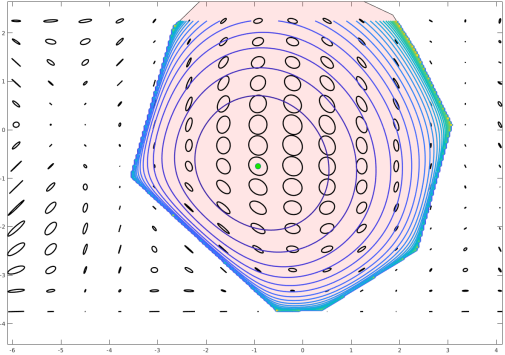

Figure 1: Barrier functions

Pink is the domain. The ellipsoids represent the shape of the hessian $\frac{\partial^2 J}{\partial \mathbf{x}^2}$ at different points on the domain. Green dot is $\mathbf{x}_a$.

Homework

Visualize contours of a quadratic program of your choice.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.