

Singular Value Decomposition

Computational Intelligence, Lecture 4

by Sergei Savin

Spring 2022

- Singular Value Decomposition
- Rank and pseudoinverse
- SVD of a transpose
- Projectors

SINGULAR VALUE DECOMPOSITION

Given $\mathbf{A} \in \mathbb{R}^{n,m}$ we can find its Singular Value Decomposition (SVD):

$$\mathbf{A} = [\mathbf{C} \quad \mathbf{L}] \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\top \\ \mathbf{N}^\top \end{bmatrix} \quad (1)$$

$$\mathbf{A} = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top \quad (2)$$

where \mathbf{C} , \mathbf{L} , \mathbf{R} and \mathbf{N} are column, left null, row and null space bases (orthonormal), $\mathbf{\Sigma}$ is the diagonal matrix of singular values. The singular values are positive and softer in the decreasing order.

Rank of the matrix is computed as the size of Σ . Note that numeric tolerance applies when deciding if the singular value is non-zero.

Pseudoinverse \mathbf{A}^+ is defined as:

$$\mathbf{A}^+ = [\mathbf{R} \quad \mathbf{N}] \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}^\top \\ \mathbf{L}^\top \end{bmatrix} \quad (3)$$

$$\mathbf{A}^+ = \mathbf{R}\Sigma^{-1}\mathbf{C}^\top \quad (4)$$

Let's find SVD decomposition of a \mathbf{A}^\top :

$$\mathbf{A}^\top = [\mathbf{C}_t \quad \mathbf{L}_t] \begin{bmatrix} \boldsymbol{\Sigma}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}_t^\top \\ \mathbf{N}_t^\top \end{bmatrix} \quad (5)$$

Let us transpose it (remembering that transpose of a diagonal matrix the original matrix $\boldsymbol{\Sigma}_t^\top = \boldsymbol{\Sigma}_t$):

$$\mathbf{A} = [\mathbf{R}_t \quad \mathbf{N}_t] \begin{bmatrix} \boldsymbol{\Sigma}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}_t^\top \\ \mathbf{L}_t^\top \end{bmatrix} \quad (6)$$

Thus we can see that the row space of the original matrix \mathbf{A} is the column space of the transpose \mathbf{A}^\top . And the left null space of the original matrix \mathbf{A} is the null space of the transpose \mathbf{A}^\top .

Let's prove that $\mathbf{A}\mathbf{A}^+$ is equivalent to $\mathbf{C}\mathbf{C}^\top$:

$$\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (7)$$

$$\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (8)$$

$$\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{C}^\top \quad (9)$$

Let's prove that $\mathbf{A}^+\mathbf{A}$ is equivalent to $\mathbf{R}\mathbf{R}^\top$:

$$\mathbf{A}^+\mathbf{A} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top\mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top \quad (10)$$

$$\mathbf{A}^+\mathbf{A} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{R}^\top \quad (11)$$

$$\mathbf{A}^+\mathbf{A} = \mathbf{R}\mathbf{R}^\top \quad (12)$$

Let us denote $\mathbf{P} = \mathbf{A}\mathbf{A}^+$. Let's prove that $\mathbf{P}\mathbf{P} = \mathbf{P}$:

$$\mathbf{A}\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top\mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (13)$$

$$\mathbf{A}\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (14)$$

$$\mathbf{A}\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{C}^\top = \mathbf{A}\mathbf{A}^+ \quad (15)$$

The same is true for $\mathbf{P} = \mathbf{A}^+\mathbf{A}$: we can prove that $\mathbf{P}\mathbf{P} = \mathbf{P}$.

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022



Check Moodle for additional links, videos, textbook suggestions.