

Domain, Convex Domains

Computational Intelligence, Lecture 5

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- Convex domains
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Problem 1. Find minimum of the function $f = x^2 + 2y^2$ if $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Solution is $x = 0, y = 0$.

Problem 2. Find minimum of the function $f = x^2 + 2y^2$ if $x \in [1, 2]$ and $y \in [2, 5]$. Solution is $x = 1, y = 2$.

Note that solutions of problems 1 and 2 are different, and this is only due to the difference of the allowed values that the *decision variables* x and y can assume.

Definition 1

Space of all allowed values that decision variables can assume is called the *domain* of optimization problem.

BOUNDED AND UNBOUNDED DOMAINS

Part 1

Problem 3. Find minimum of the function $f = -x^2$ if $x \in [-3, 2]$. Solution is $x = -3$.

Problem 4. Find minimum of the function $f = -x^2$ if $x \in \mathbb{R}$. The problem has no solution.

Problem 5. Find minimum of the function $f = -x^2$ if $x \in [-\infty, 2]$. The problem has no solution.

The major difference between domains of the problems 2, 3 vs problems 1, 4 and 5 is that the later are *not bound* (i.e., you can construct a sequence of the values in the domain that would approach infinity).

We can see that in the case of problems 3-5, bounded domain allows the problem to have a solution.

BOUNDED AND UNBOUNDED DOMAINS

Part 2

Problem 6. Find maximum of the function $f = x^2$ if $1 \leq x < 2$.
It has no solution.

Problem 7. Find minimum of the function $f = x^2$ if $1 \leq x < 2$.
Solution is $x = 1$.

This time, it is the fact that one of the *boundaries* of the domain was not included in the domain that has lead the problem 6 to have no solution, while problem 7 had one. For the problem 6 we can pick a value arbitrary close to $x = 2$, approaching it from the left, but for any such value, there always will be other values of the decision variable closer to $x = 2$ and hence producing larger values of f .

CONVEX DOMAINS

Definition 2

Domain is *convex* iff for any two points in the domain, the line segment connecting them is also in the domain.

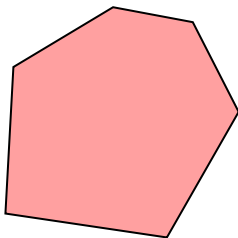


Figure 1: Convex domain



Figure 2: Non-convex domain

In the proofs it is convenient to remember that for any two points \mathbf{x}_1 and \mathbf{x}_2 , all points in the line segment connecting them are given as $\mathbf{x}_t = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$, where $\alpha \in [0, 1]$. This is called *convex combination*.

EXAMPLES OF CONVEX DOMAINS

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \mathbb{R}^n$ is convex.

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \leq \mathbf{h}\}$ is convex.

Proof: Note that $\alpha\mathbf{x}_1 \leq \alpha\mathbf{h}$ and $(1 - \alpha)\mathbf{x}_2 \leq (1 - \alpha)\mathbf{h}$, hence,
 $\alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2 \leq \alpha\mathbf{h} + (1 - \alpha)\mathbf{h} = \mathbf{h}$. \square

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq h^2\}$ is convex.

Proof: This is the same as $\|\mathbf{x}\| \leq h$. Note that $\|\alpha\mathbf{x}_1\| \leq \alpha h$ and $\|(1 - \alpha)\mathbf{x}_2\| \leq (1 - \alpha)h$, also

$$\|\alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2\| \leq \|\alpha\mathbf{x}_1\| + \|(1 - \alpha)\mathbf{x}_2\|$$

$$\|\alpha\mathbf{x}_1\| + \|(1 - \alpha)\mathbf{x}_2\| \leq \alpha h + (1 - \alpha)h = h$$

So the convex combination of \mathbf{x}_1 and \mathbf{x}_2 is still in the domain. \square

EXAMPLES OF CONVEX DOMAINS

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{H} \mathbf{x} \leq 1\}$, where $\mathbf{H} \succ 0$ is positive-definite symmetric matrix is convex.

For any positive-definite symmetric \mathbf{H} it is true that $\mathbf{H} = \mathbf{D}^\top \mathbf{D}$, where $\mathbf{D} = \sqrt{\mathbf{H}}$ is called a matrix square root and it is full rank. With that $\mathbf{x}^\top \mathbf{H} \mathbf{x} \leq 1$ becomes $\mathbf{x}^\top \mathbf{D}^\top \mathbf{D} \mathbf{x} \leq 1$. Defining $\mathbf{y} = \mathbf{D} \mathbf{x}$ we get $\mathcal{X} = \{\mathbf{D}^{-1} \mathbf{y} : \mathbf{y}^\top \mathbf{y} \leq 1\}$. This is a linearly deformed previously covered domain, and as such it is also convex.

EXAMPLES OF NON-CONVEX DOMAINS

$x \in \mathcal{X}$, $\mathcal{X} = [-1 \ 2] \cup [3 \ 7]$ is not convex..

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq h^2\}$ is not convex. Prove it.

$\mathbf{x} \in \mathcal{X}$, $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{H} \mathbf{x} \geq 1\}$, where \mathbf{H} is positive-definite symmetric matrix is not convex. Prove it.

These proves simply require one counter-example to show that the defining property of convex domains does not hold.

CONVEX FUNCTIONS

Definition 3

Function $f(\mathbf{x})$ defined on a domain \mathcal{D} , for which it holds that $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, f(\alpha \mathbf{x}_1 + (1 - \alpha) \mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1 - \alpha) f(\mathbf{x}_2)$ is called a *convex function*.



Figure 3: Convex function

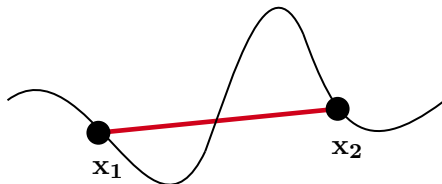


Figure 4: Non-convex function

CONVEX FUNCTIONS - EXAMPLES

Here are some single-variable convex functions:

- $f(x) = 1$
- $f(x) = x, f(x) = x + 1, f(x) = 6x + 3$
- $f(x) = x^2, f(x) = (x - 5)^2, f(x) = (x + 1)^2 - 10$
- $f(x) = x^3, \text{ if } x > 0$

Here are some multi-variable convex functions:

- $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$
- $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{Hx}, \mathbf{H} \succ 0$

Definition 3

If the domain of the optimization problem is convex and the cost function is convex, it is called a *convex optimization problem*.

Additionally, we will always assume that the domain of the convex optimization problem contains its boundary. Also, without the loss of generality, we will consider only minimization problems.

There are a few important properties of convex optimization problems (with our additional assumption):

- If the domain is non-empty, there is a solution.
- The problem has no local minima. We can find a path from any point to the solution, along which the cost function will not increase.

HOMEWORK

- Make formal proofs asked for in this lecture.

- Convex Optimization, lecture 3, S. Boyd. Stanford.
Convex functions.

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022



Check Moodle for additional links, videos, textbook suggestions.