

# Domain, Convex Domains

## Computational Intelligence, Lecture 5

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- Domain
- Bounded and unbounded domains
- Convex domains
- Examples of convex domains
- Examples of non-convex domains
- Convex functions
- Convex functions - examples
- Convex programming
- Homework

Problem 1. Find minimum of the function  $f = x^2 + 2y^2$  if  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ . Solution is  $x = 0, y = 0$ .

Problem 2. Find minimum of the function  $f = x^2 + 2y^2$  if  $x \in [1, 2]$  and  $y \in [2, 5]$ . Solution is  $x = 1, y = 2$ .

Note that solutions of problems 1 and 2 are different, and this is only due to the difference of the allowed values that the *decision variables*  $x$  and  $y$  can assume.

## Definition 1

Space of all allowed values that decision variables can assume is called the *domain* of optimization problem.

# BOUNDED AND UNBOUNDED DOMAINS

## Part 1

Problem 3. Find minimum of the function  $f = -x^2$  if  $x \in [-3, 2]$ . Solution is  $x = -3$ .

Problem 4. Find minimum of the function  $f = -x^2$  if  $x \in \mathbb{R}$ . The problem has no solution.

Problem 5. Find minimum of the function  $f = -x^2$  if  $x \in [-\infty, 2]$ . The problem has no solution.

The major difference between domains of the problems 2, 3 vs problems 1, 4 and 5 is that the later are *not bound* (i.e., you can construct a sequence of the values in the domain that would approach infinity).

We can see that in the case of problems 3-5, bounded domain allows the problem to have a solution.

# BOUNDED AND UNBOUNDED DOMAINS

## Part 2

Problem 6. Find maximum of the function  $f = x^2$  if  $1 \leq x < 2$ .  
It has no solution.

Problem 7. Find minimum of the function  $f = x^2$  if  $1 \leq x < 2$ .  
Solution is  $x = 1$ .

This time, it is the fact that one of the *boundaries* of the domain was not included in the domain that has lead the problem 6 to have no solution, while problem 7 had one. For the problem 6 we can pick a value arbitrary close to  $x = 2$ , approaching it from the left, but for any such value, there always will be other values of the decision variable closer to  $x = 2$  and hence producing larger values of  $f$ .

# CONVEX DOMAINS

## Definition 2

Domain is *convex* iff for any two points in the domain, the line segment connecting them is also in the domain.

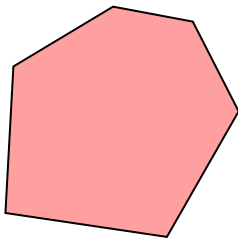


Figure 1: Convex domain

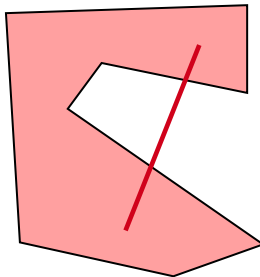


Figure 2: Non-convex domain

In the proofs it is convenient to remember that for any two points  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , all points in the line segment connecting them are given as  $\mathbf{x}_t = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$ , where  $\alpha \in [0, 1]$ . This is called *convex combination*.

# EXAMPLES OF CONVEX DOMAINS

$\mathbf{x} \in \mathcal{X}$ ,  $\mathcal{X} = \mathbb{R}^n$  is convex.

$\mathbf{x} \in \mathcal{X}$ ,  $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \leq \mathbf{h}\}$  is convex.

Proof: Note that  $\alpha \mathbf{x}_1 \leq \alpha \mathbf{h}$  and  $(1 - \alpha) \mathbf{x}_2 \leq (1 - \alpha) \mathbf{h}$ , hence,  
 $\alpha \mathbf{x}_1 + (1 - \alpha) \mathbf{x}_2 \leq \alpha \mathbf{h} + (1 - \alpha) \mathbf{h} = \mathbf{h}$ .  $\square$

$\mathbf{x} \in \mathcal{X}$ ,  $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq h^2\}$  is convex.

Proof: This is the same as  $\|\mathbf{x}\| \leq h$ . Note that  $\|\alpha \mathbf{x}_1\| \leq \alpha h$  and  
 $\|(1 - \alpha) \mathbf{x}_2\| \leq (1 - \alpha) h$ , also

$$\|\alpha \mathbf{x}_1 + (1 - \alpha) \mathbf{x}_2\| \leq \|\alpha \mathbf{x}_1\| + \|(1 - \alpha) \mathbf{x}_2\|$$

$$\|\alpha \mathbf{x}_1\| + \|(1 - \alpha) \mathbf{x}_2\| \leq \alpha h + (1 - \alpha) h = h$$

So the convex combination of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  is still in the domain.  $\square$



# EXAMPLES OF CONVEX DOMAINS

$\mathbf{x} \in \mathcal{X}$ ,  $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{H} \mathbf{x} \leq 1\}$ , where  $\mathbf{H} \succ 0$  is positive-definite symmetric matrix is convex.

For any positive-definite symmetric  $\mathbf{H}$  it is true that  $\mathbf{H} = \mathbf{D}^\top \mathbf{D}$ , where  $\mathbf{D} = \sqrt{\mathbf{H}}$  is called a matrix square root and it is full rank. With that  $\mathbf{x}^\top \mathbf{H} \mathbf{x} \leq 1$  becomes  $\mathbf{x}^\top \mathbf{D}^\top \mathbf{D} \mathbf{x} \leq 1$ . Defining  $\mathbf{y} = \mathbf{D} \mathbf{x}$  we get  $\mathcal{X} = \{\mathbf{D}^{-1} \mathbf{y} : \mathbf{y}^\top \mathbf{y} \leq 1\}$ . This is a linearly deformed previously covered domain, and as such it is also convex.

# EXAMPLES OF NON-CONVEX DOMAINS

$x \in \mathcal{X}$ ,  $\mathcal{X} = [-1 \ 2] \cup [3 \ 7]$  is not convex..

$\mathbf{x} \in \mathcal{X}$ ,  $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq h^2\}$  is not convex. Prove it.

$\mathbf{x} \in \mathcal{X}$ ,  $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{H} \mathbf{x} \geq 1\}$ , where  $\mathbf{H}$  is positive-definite symmetric matrix is not convex. Prove it.

These proves simply require one counter-example to show that the defining property of convex domains does not hold.

# CONVEX FUNCTIONS

## Definition 3

Function  $f(\mathbf{x})$  defined on a domain  $\mathcal{D}$ , for which it holds that  $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, f(\alpha \mathbf{x}_1 + (1 - \alpha) \mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1 - \alpha) f(\mathbf{x}_2)$  is called a *convex function*.



Figure 3: Convex function

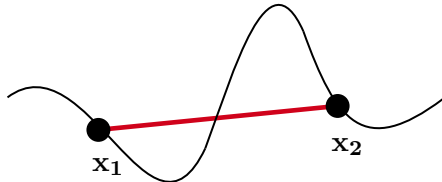


Figure 4: Non-convex function

# CONVEX FUNCTIONS - EXAMPLES

Here are some single-variable convex functions:

- $f(x) = 1$
- $f(x) = x, f(x) = x + 1, f(x) = 6x + 3$
- $f(x) = x^2, f(x) = (x - 5)^2, f(x) = (x + 1)^2 - 10$
- $f(x) = x^3, \text{ if } x > 0$

Here are some multi-variable convex functions:

- $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$
- $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{Hx}, \mathbf{H} \succ 0$

## Definition 3

If the domain of the optimization problem is convex and the cost function is convex, it is called a *convex optimization problem*.

Additionally, we will always assume that the domain of the convex optimization problem contains its boundary. Also, without the loss of generality, we will consider only minimization problems.

There are a few important properties of convex optimization problems (with our additional assumption):

- If the domain is non-empty, there is a solution.
- The problem has no local minima. We can find a path from any point to the solution, along which the cost function will not increase.

# HOMEWORK

- Make formal proofs asked for in this lecture.

- Convex Optimization, lecture 3, S. Boyd. Stanford.  
Convex functions.

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022)



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