

# Singular Value Decomposition

## Computational Intelligence, Lecture 4

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Spring 2022

- Singular Value Decomposition
- Rank and pseudoinverse
- SVD of a transpose
- Projectors

# SINGULAR VALUE DECOMPOSITION

Given  $\mathbf{A} \in \mathbb{R}^{n,m}$  we can find its Singular Value Decomposition (SVD):

$$\mathbf{A} = [\mathbf{C} \quad \mathbf{L}] \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\top \\ \mathbf{N}^\top \end{bmatrix} \quad (1)$$

$$\mathbf{A} = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top \quad (2)$$

where  $\mathbf{C}$ ,  $\mathbf{L}$ ,  $\mathbf{R}$  and  $\mathbf{N}$  are column, left null, row and null space bases (orthonormal),  $\mathbf{\Sigma}$  is the diagonal matrix of singular values. The singular values are positive and softer in the decreasing order.

Rank of the matrix is computed as the size of  $\Sigma$ . Note that numeric tolerance applies when deciding if the singular value is non-zero.

Pseudoinverse  $\mathbf{A}^+$  is defined as:

$$\mathbf{A}^+ = [\mathbf{R} \quad \mathbf{N}] \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}^\top \\ \mathbf{L}^\top \end{bmatrix} \quad (3)$$

$$\mathbf{A}^+ = \mathbf{R}\Sigma^{-1}\mathbf{C}^\top \quad (4)$$

Let's find SVD decomposition of a  $\mathbf{A}^\top$ :

$$\mathbf{A}^\top = [\mathbf{C}_t \quad \mathbf{L}_t] \begin{bmatrix} \boldsymbol{\Sigma}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}_t^\top \\ \mathbf{N}_t^\top \end{bmatrix} \quad (5)$$

Let us transpose it (remembering that transpose of a diagonal matrix the original matrix  $\boldsymbol{\Sigma}_t^\top = \boldsymbol{\Sigma}_t$ ):

$$\mathbf{A} = [\mathbf{R}_t \quad \mathbf{N}_t] \begin{bmatrix} \boldsymbol{\Sigma}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}_t^\top \\ \mathbf{L}_t^\top \end{bmatrix} \quad (6)$$

Thus we can see that the row space of the original matrix  $\mathbf{A}$  is the column space of the transpose  $\mathbf{A}^\top$ . And the left null space of the original matrix  $\mathbf{A}$  is the null space of the transpose  $\mathbf{A}^\top$ .

Let's prove that  $\mathbf{A}\mathbf{A}^+$  is equivalent to  $\mathbf{C}\mathbf{C}^\top$ :

$$\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (7)$$

$$\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (8)$$

$$\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{C}^\top \quad (9)$$

Let's prove that  $\mathbf{A}^+\mathbf{A}$  is equivalent to  $\mathbf{R}\mathbf{R}^\top$ :

$$\mathbf{A}^+\mathbf{A} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top\mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top \quad (10)$$

$$\mathbf{A}^+\mathbf{A} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{R}^\top \quad (11)$$

$$\mathbf{A}^+\mathbf{A} = \mathbf{R}\mathbf{R}^\top \quad (12)$$

Let us denote  $\mathbf{P} = \mathbf{A}\mathbf{A}^+$ . Let's prove that  $\mathbf{P}\mathbf{P} = \mathbf{P}$ :

$$\mathbf{A}\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top\mathbf{C}\mathbf{\Sigma}\mathbf{R}^\top\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (13)$$

$$\mathbf{A}\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{C}^\top \quad (14)$$

$$\mathbf{A}\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{C}\mathbf{C}^\top = \mathbf{A}\mathbf{A}^+ \quad (15)$$

The same is true for  $\mathbf{P} = \mathbf{A}^+\mathbf{A}$ : we can prove that  $\mathbf{P}\mathbf{P} = \mathbf{P}$ .



Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2022)



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