# Spatio-Temporal ETAS model sampling

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## Target Model

Hawkes process model with conditional intensity at time $t$ and location $\mathbf{s}$ given the history of the process at time $t$, $\mathcal{H}_t$, is given by:

$$\lambda(t, \mathbf{s}|\mathcal{H}_t) = \mu(\mathbf{s}) + K \sum_{i:t_i<t} g_m(m_i) g_t(t - t_i) g_s(\mathbf{s} - \mathbf{s}_i)$$

with expected number of point in a region $A \times [T_1, T_2]$

$$\Lambda(T_1, T_2, A) = \int_A \int_{T_1}^{T_2} \lambda(t, \mathbf{s}|\mathcal{H}_t) dt d\mathbf{s}$$

$$= (T_2 - T_1) \int_A \mu(\mathbf{s}) d\mathbf{s} + K \sum_i g_m(m_i) \int_A \int_{max(t_i, T_1)}^{T_2} g_t(t - t_i) g_s(|\mathbf{s} - \mathbf{s}_i|) dt d\mathbf{s}$$

$$= (T_2 - T_1) \int_A \mu(\mathbf{s}) d\mathbf{s} + K \sum_i g_m(m_i) I_t(t_i, T_1, T_2) I_s(\mathbf{s}_i, A)$$

where the summation is over $i : t_i \in H_{T_2}$ the history of the process up to time $T_2$ and

$$I_t(t_i, T_1, T_2) = \int_{max(t_i, T_1)}^{T_2} g_t(t - t_i) dt$$

$$I_s(s_i, A) = \int_A g_s(\mathbf{s} - \mathbf{s}_i) d\mathbf{s}$$

The log-likelihood is given by

$$\mathcal{L} = -\left( (T_2 - T_1) \int_A \mu(\mathbf{s}) d\mathbf{s} + K \sum_i g_m(m_i) I_t(t_i, T_1, T_2) I_s(\mathbf{s}_i, A) \right) + \sum_i \log \left( \mu(\mathbf{s}_i) + K \sum_{j:t_j<t_i} g_m(m_j) g_t(t_i - t_j) g_s(|\mathbf{s}_i - \mathbf{s}_j|) \right)$$

In our case

$$\mu(\mathbf{s}) = \exp(\theta_1) f(\mathbf{s})$$

We assume $f(\mathbf{s})$ is given or estimated independently from others parameters.

$$K = \exp(\theta_2)$$

Considering a magnitude of completeness $M_0$, we consider a specific form for each triggering function. The first, $g_m(\cdot)$ represents the influence of the magnitude of past events on the present. We suppose that events with higher magnitude have a stronger effect and, therefore, we consider $\exp\theta_3$ which is a non-negative quantity.

$$g_m(m) = \exp\{\exp(\theta_3)(m - M_0)\}$$

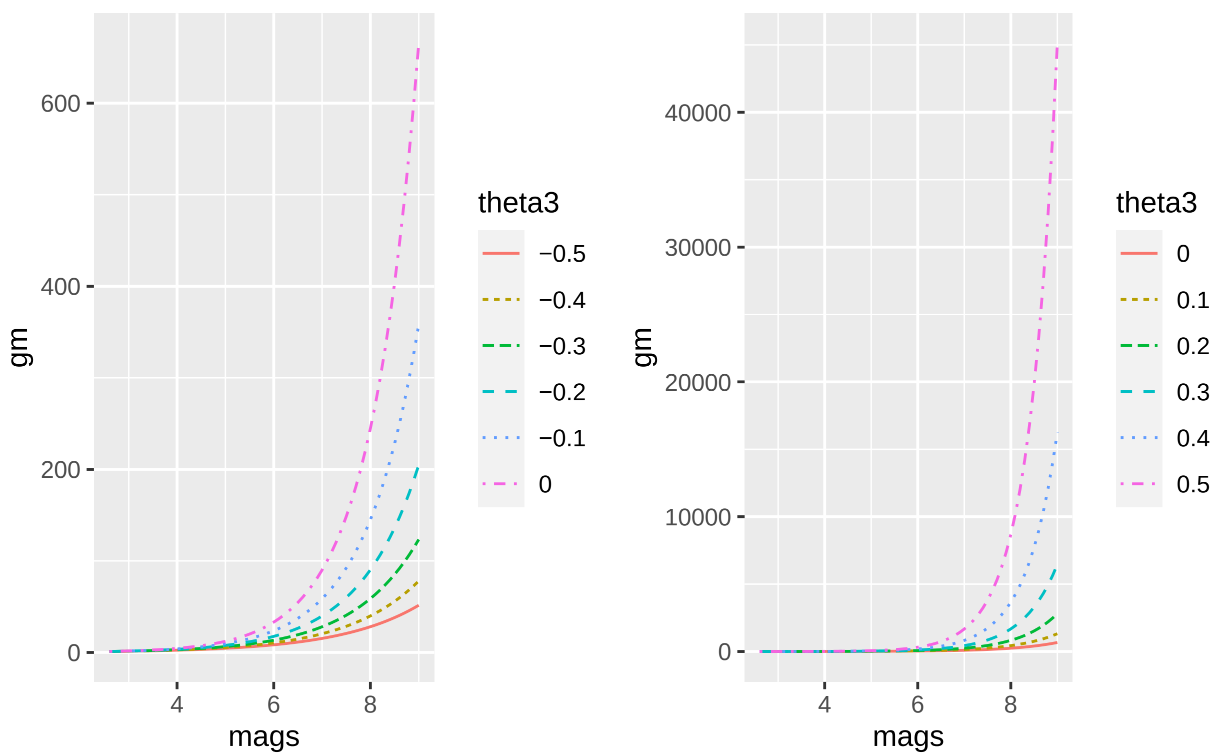


Figure 1: example of $g_m(m)$ considering different $\theta_3$ and $M_0 = 2.5$

The second function, $g_t(\cdot)$ represents the temporal decay of the influence of a past event on the present. It is given by the Omori's law and has two parameters $\theta_4$ and $\theta_5$

$$g_t(t - t_i) = \frac{1}{(t - t_i + \exp\theta_4)^{1+\exp\theta_5}}$$

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$$g_s(\mathbf{s} - \mathbf{s}_i) = \frac{1}{2\pi}(\det\Sigma)^{-1/2}\exp\{-(\mathbf{s} - \mathbf{s}_i)^T\Sigma^{-1}(\mathbf{s} - \mathbf{s}_i)\}$$

Where $\Sigma$ is a $2 \times 2$ covariance matrix. It has to be symmetric so, in this case, we need to specify three elements which will determine the shape of the triggering function. The location is governed by the mean which, in this case, is considered to be the observed point $\mathbf{s}_i$.

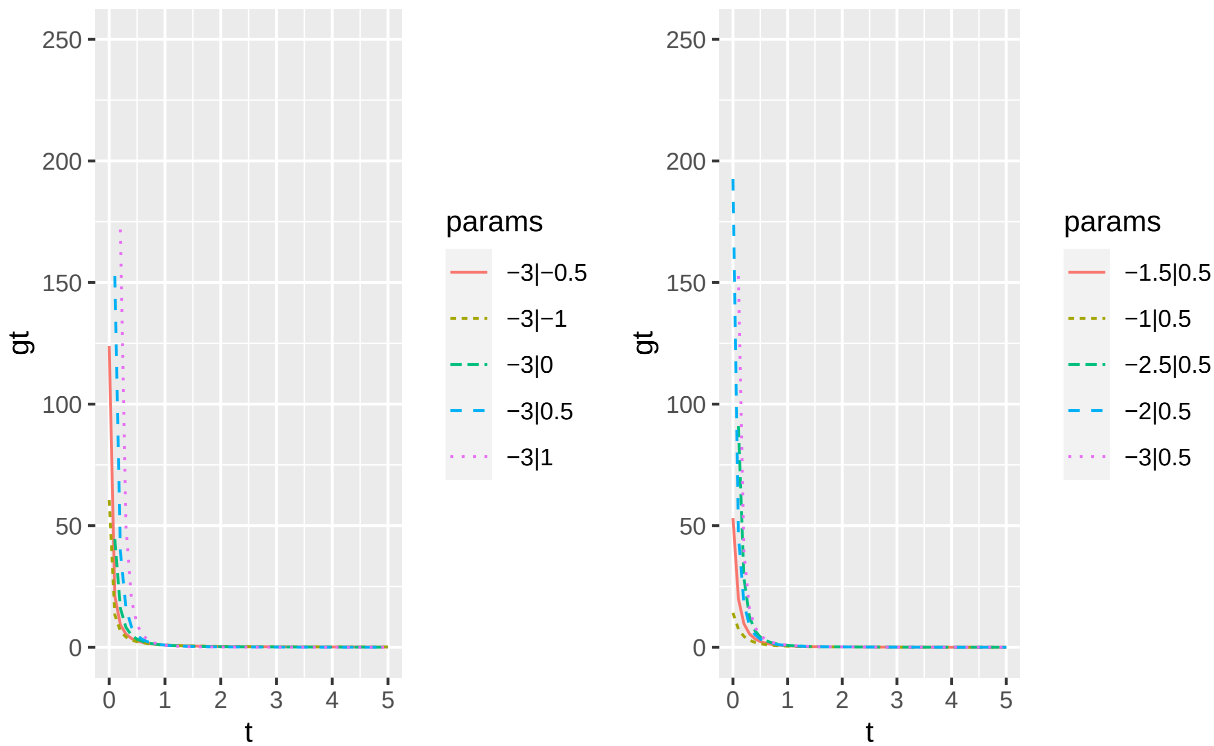


Figure 2: example of $g_t(t - t_i)$ considering different combinations of $\theta_4$ and $\theta_5$ assuming $t_i = 0$

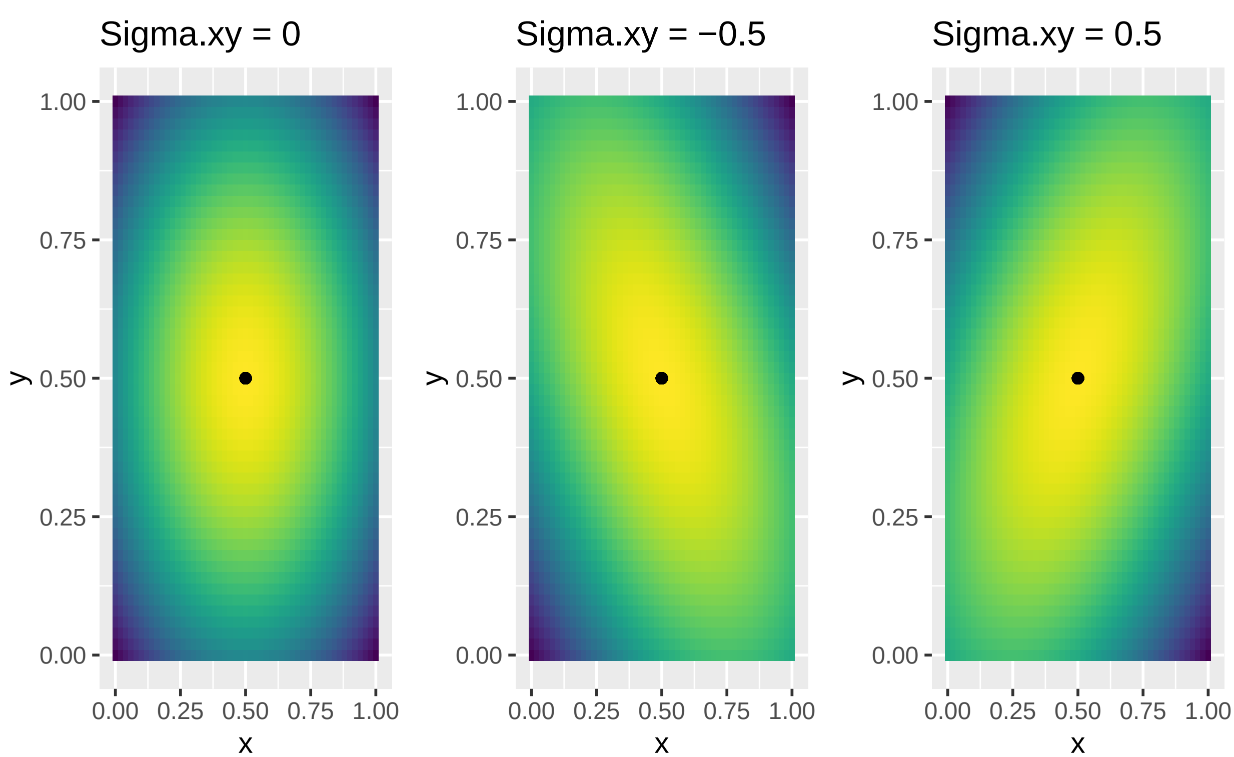


Figure 3: example of $g_s(\mathbf{s} - \mathbf{s}_i)$ considering different covariance matrices $\Sigma$ and assuming $\mathbf{s}_i = (0.5, 0.5)$

Given the above specification of the triggering functions, we need to integrate them to retrieve the expected number of points in a given region of the domain.

For the time triggering function:

$$I_t(t_i, T_1, T_2) = \int_{max(t_i, T_1)}^{T_2} g_t(t - t_i) dt$$

$$= \int_{max(t_i, T_1)}^{T_2} (t - t_i + \exp\theta_4)^{-1-\exp\theta_5} dt$$

$$= \frac{(\max(0, T_1 - t_i) + \exp\theta_4)^{-\exp\theta_5} - (T_2 - t_i + \exp\theta_4)^{-\exp\theta_5}}{\exp\theta_5}$$

For the spatial triggering function, being a multivariate Gaussian density function:

$$I_g(A) = \int_A g_s(\mathbf{s} - \mathbf{s}_i) d\mathbf{s}$$

$$= \Pr(\mathbf{Z} \in A)$$

Where $\mathbf{Z} \sim N(\mathbf{s}_i, \Sigma)$ it's a multivariate Gaussian distribution with mean given by the observed location $\mathbf{s}_i$ and covariance matrix $\Sigma$.

# ETAS sampling

Here, a small example of two sequences using paramers $\boldsymbol{\theta} = (-4.60, -2.99, 0.095, -4.60, -2.30, -6.21, -6.21, -6.26)$ which correspond to parameters $\mu = 10^{-2}, K = 0.05, \alpha = 1.1, c = 10^{-2}, p = 1.1, \sigma_x^2 = \sigma_y^2 = 0.002, \sigma_{xy} = 0.0019$. The points are generated in the square box $(0, 1) \times (0, 1)$ and time comprised between 0 and 1. The magnitude follow a GR law with parameter $\beta = 2.3$ which correspond to $b = \beta/\log(10) = 0.99$. Two spatial triggering functions are considered. One is isotropic, which correspond to $\sigma_{xy} = 0$ and the other one is not isotropic and has $\sigma_{xy} = 0.0019$.

The algorithm to sample from an ETAS model in the time interval $T_1, T_2$ and space region $W$ works as follows:

1. If we have knowledge of past events, generate aftershocks of all known past events $\mathbf{x}_0 = \{(t_{0j}, m_{0j}, x_{0j}, y_{0j}), j = 1, ..., J_0\}$ (using `sample.generation()`)
2. Generate background rate events $\mathbf{x}_1 = \{(t_{1j}, m_{1j}, x_{1j}, y_{1j}), j = 1, ..., J_1\}$ (using \texttt{points_sampler()})
3. Merge $\mathbf{x}_0$ and $\mathbf{x}_1$ in $parents = \mathbf{x}_0 \cup \mathbf{x}_1$
4. Set $i = 2$, generate aftershocks of all events in *parents*, namely $\mathbf{x}_i$ (using `sample.generation()`)
5. Set $parents = \mathbf{x}_i$ and $i = i + 1$ and repear point 4. Stop if there no generate aftershocks.

# ITALY

Here, we propose an example using Italy. The area of interest is the italy collection area provided by CSEP (Figure 5). We are going to assume a spatially varying background rate. Background rate determined using 1748 events from the Horus catalog. The events have occured between 01-01-2000 and 31-12-2020 and have

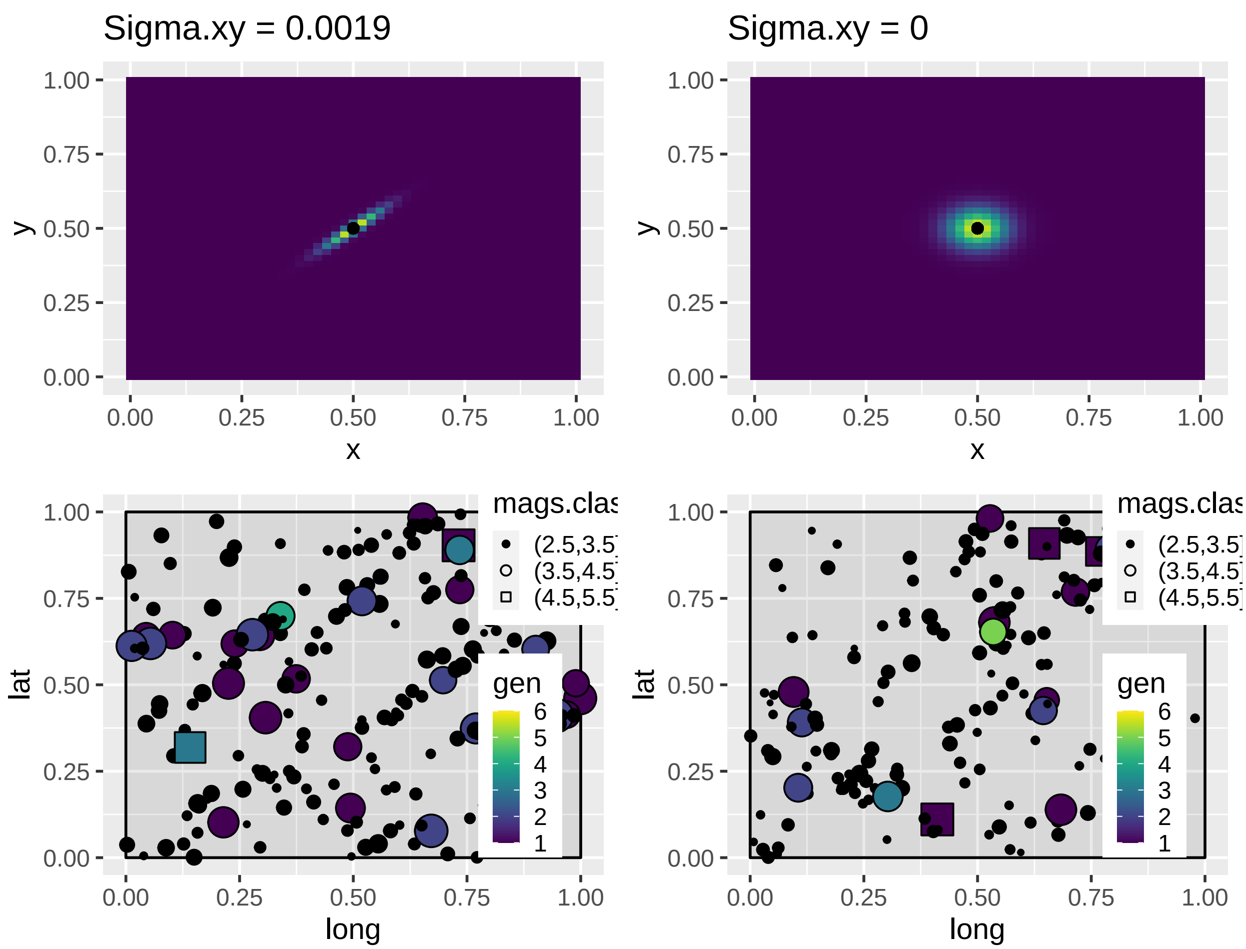


Figure 4: Two sample from two ETAS models with different spatial triggering functions characterized by $\sigma_{xy} = 0.0019$ (left) and $\sigma_{xy} = 0$ (right)

magnitude greater than 3.5, for each event we take an isotropic Gaussian kernel with center at the event and $\sigma_y^2 = 0.002 m_i$, where $m_i$ is the magnitude of the event. The backround rate is determined by the sum of the Gaussian kernels. The sum is normalized to integrate to one over the domain, in this way the number of points generated is controlled by $\mu = \exp\theta_1$ solely. Figure 5 shows the estimated background rate and a sample of 100 points placed in space accordingly.

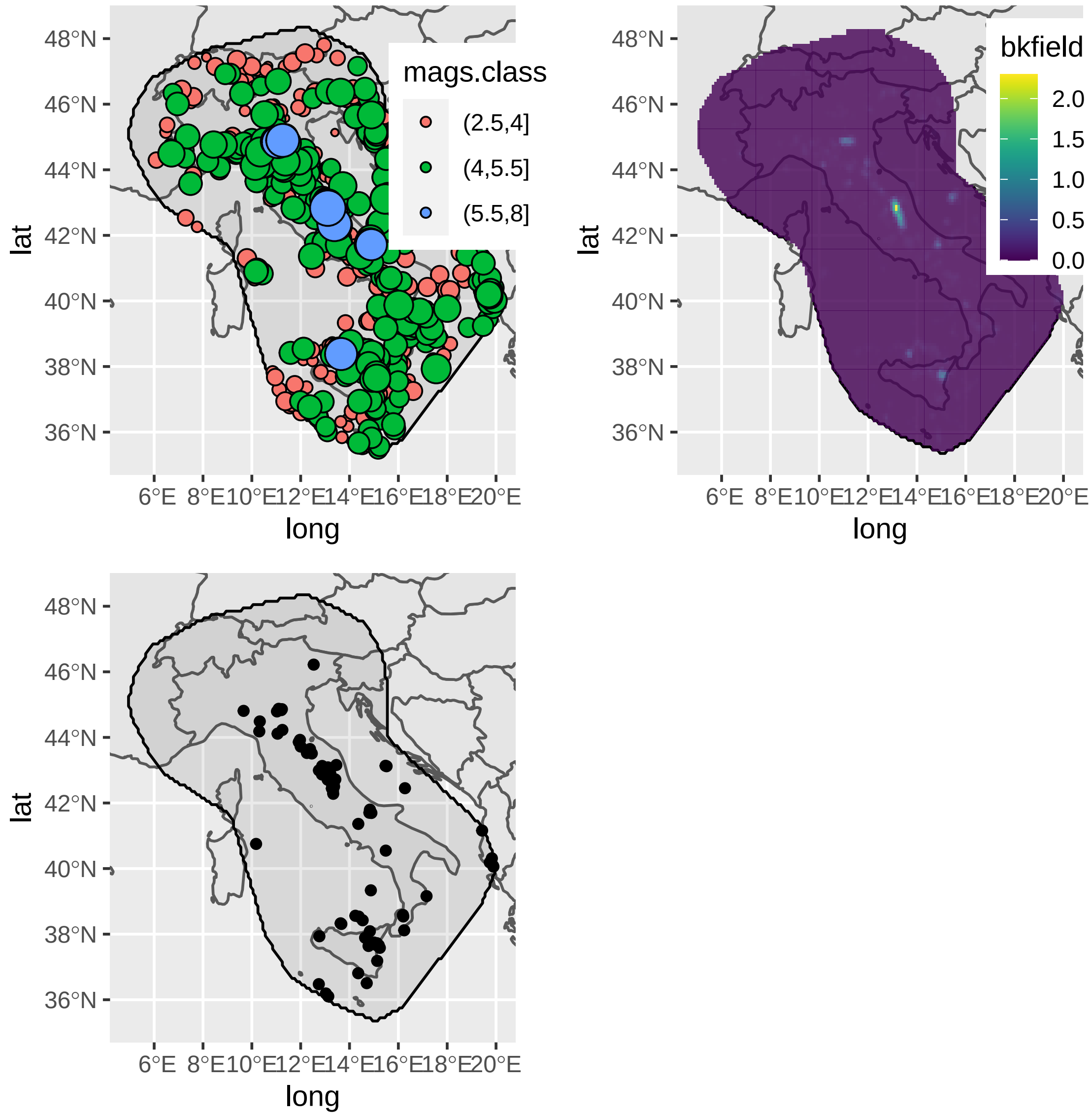


Figure 5: From left to right: Horus catalog for events between 01-01-2000 to 31-12-2020 with magnitude greater than 3.5; background intensity estimated from the Horus catalog; sample of 100 points placed according to the background intensity

Then, we assume parameters $\boldsymbol{\theta}$ equal to the one estimated here https://www.earth-prints.org/bitstream/2122/6419/1/ANNALI_Lombardi_and_Marzocchi_CSEP_ETAS.doc.pdf which are $\mu = 237(\text{years}^{-1}), K = 0.011, \alpha = 1.3, c = 0.00004, p = 1.16$. We consider a different space-triggering function with different parametrization. We are going to use as $\Sigma$ the one considered to determine the sample in Figure 4 (left).

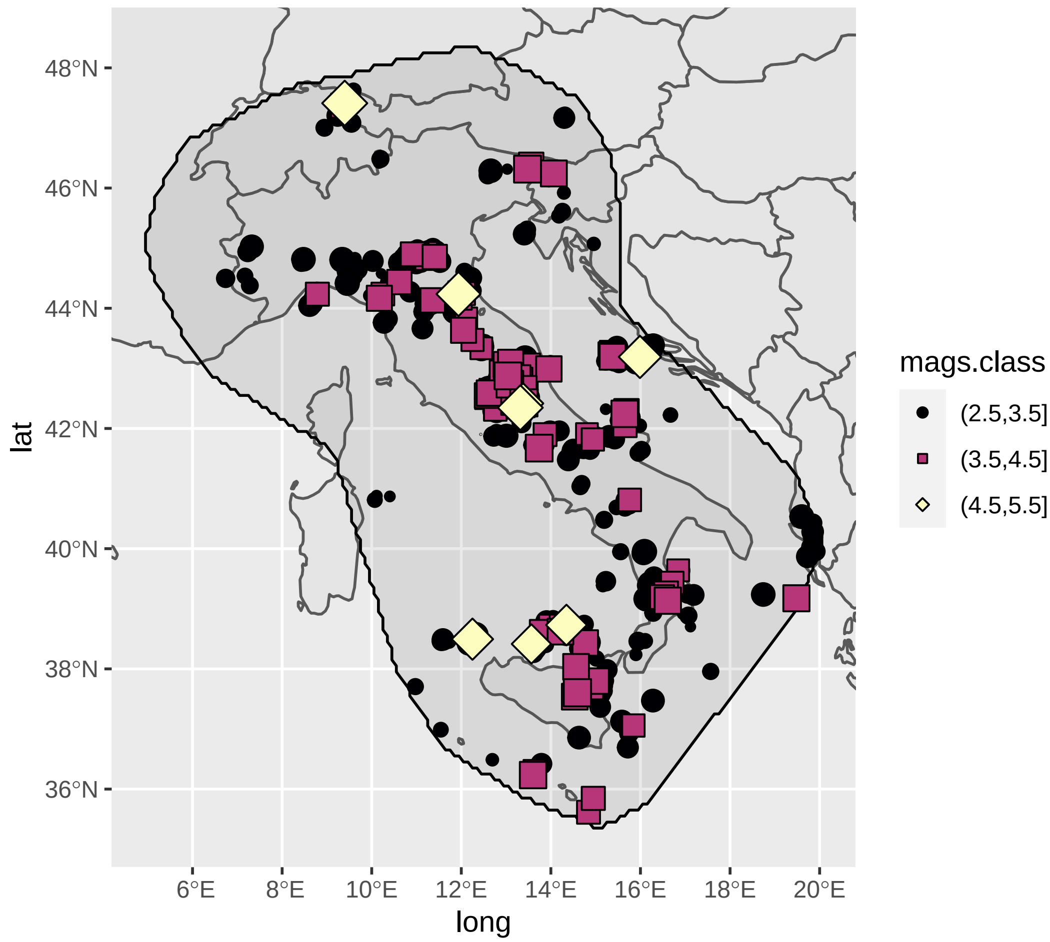


Figure 6: Example of ETAS sample with spatially varying background rate.

## Example space-time forecasting

Here, we present an example of a forecasting experiment. We generate data for two years using the above values of $\boldsymbol{\theta}$ and we use this sample as observed data. Then, we split the time in 2-week intervals. For each interval, we simulate 1000 sequences from a target model, assuming to have knowledge of the events before the forecasted interval. The plot below shows the observed number of events in each bin and the intervals containing the 95% of the numbers of events of the simulated sequences.

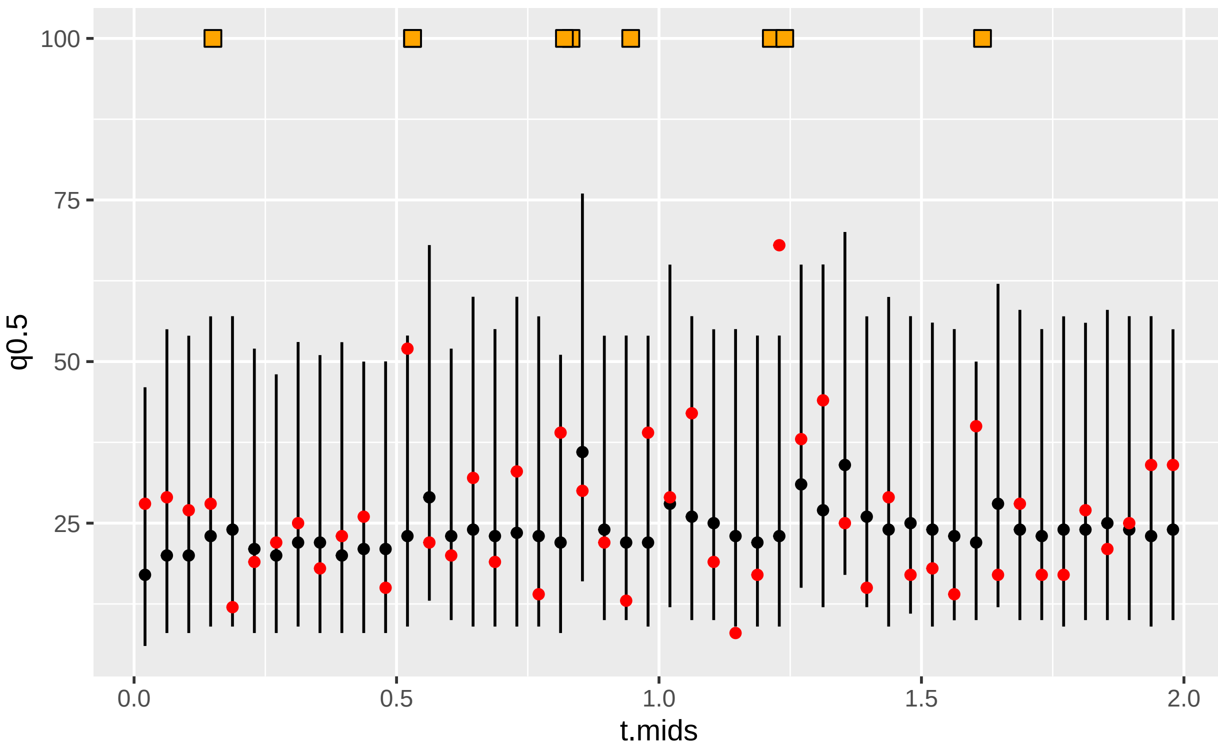


Figure 7: Simulated predictive intervals for each time bin. Segments represents 95% predictive intervals, red dots represents observed values and orange squares indicates events with magnitude greater than 4.5. Each simulated interval is obtained simulating 1000 sequences in the respective time bin assuming to know the past events.