Fall 2022

Assignment 04

1. (a) $f(x) = x^3 - 7x^2 + 4x + 12 = 0$

$\Rightarrow 4x = 7x^2 - x^3 - 12$

$\Rightarrow x = \dfrac{7x^2 - x^3 - 12}{4} = g_1(x)$

$x^3 - 7x^2 + 4x + 12 = 0$

$\Rightarrow x(x^2 - 7x + 4) = -12$

$\Rightarrow x = -\dfrac{12}{x^2 - 7x + 4} = g_2(x)$

(b) From calculator,

$x_* = 6, 2, -1$

$g_1(x) = \dfrac{7x^2 - x^3 - 12}{4}$

$\Rightarrow g_1'(x) = \frac{1}{4}(14x - 3x^2)$

$$\Rightarrow \lambda = |g_1'(x_*)| = \begin{cases} |-6| = 6 & \text{for } x_* = 6 \\ 4 & \text{for } x_* = 2 \\ |-4.25| = 4.25 & \text{for } x_* = -1 \end{cases}$$

All of the $\lambda > 1$, so $g_1(x)$ is not converging to any root.

$$g_2(x) = \frac{-12}{x^2 - 7x + 4}$$

$$\Rightarrow g_2'(x) = -12\left[-\frac{1}{(x^2-7x+4)^2}\, d/dx\, (x^2 - 7x + 4)\right]$$

$$= \frac{12(2x-7)}{(x^2-7x+4)^2}$$

$$\Rightarrow \lambda = |g_2'(x_*)| = \begin{cases} 15 & \text{for } x_* = 6 \\ |-1| = 1 & \text{for } x_* = 2 \\ |-0.75| = 0.75 & \text{for } x_* = -1 \end{cases}$$

Only for $x_* = -1$, $0 < \lambda < 1$ is true. So. $g_2(x)$ is linearly converging to $x_* = -1$

(c)

Number of iterrations required to find the root,

$$n \geq \frac{\log(|b_0 - a_0|) - \log(\delta)}{\log(2)} - 1$$

$$\Rightarrow n \geq \frac{\log(|8.95 - 4.25|) - \log(1.4\times10^{-18})}{\log(2)} - 1$$

$$\Rightarrow n \geq 60.542$$

$\Rightarrow n \geq 61$ iterrations

(d) $a_0 = 4.25$ ; $f(a_0) = -20.67188$

$b_0 = 8.95$ ; $f(b_0) = 203.99988$

As $f(a_0) * f(b_0) < 0$, there is a solution within $I_0 = [a_0, b_0]$

| $k$ | $a_k$ | $m_k$ | $b_k$ | $f(a_k)$ | $f(m_k)$ | $f(b_k)$ | $x_k \in [\, , \,]$ |
|---|---|---|---|---|---|---|---|
| 0 | 4.25 | 6.6 | 8.95 | -20.672 | 20.976 | 203.99 | $[a_k, m_k]$ |
| 1 | 4.25 | 5.425 | 6.6 | -20.672 | -12.653 | 20.976 | $[m_k, b_k]$ |
| 2 | 5.425 | 6.013 | 6.6 | -12.653 | 0.3659 | 20.976 | $[a_k, m_k]$ |
| 3 | 5.425 | 5.719 | 6.013 | -12.653 | -7.0216 | 0.3659 | $[m_k, b_k]$ |
| 4 | 5.719 | 5.866 | 6.013 | -7.0216 | -3.5569 | 0.3659 | $[m_k, b_k]$ |

(e) $f(x) = x^3 - 7x^2 + 4x + 12 \quad \dots \text{(i)}$

$\Rightarrow f'(x) = 3x^2 - 14x + 4$

Newton's iteration method formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^3 - 7x_k^2 + 4x_k + 12}{3x_k^2 - 14x_k + 4} \quad \dots \text{(ii)}$$

Aitken Acceleration formula:

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} \quad \dots \text{(iii)}$$

| $k$ | $x_k$ | $f(x_k)$ |
|---|---|---|
| 0 | 2.26 | -3.17002 |
| 1 | 2.00263 | -0.03157 |
| 2 | 2.00000 | 0 |
| $\hat{2}$ | 2.00000 | 0 |
| 3 | 2.00000 | 0 |

$\therefore x_* = 2.00000$

02. (a)

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 3 & -2 \\ 3 & -1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$b = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 3 & -2 \\ 3 & -1 & -1 \end{bmatrix}$$

$$\det(A) = 2(-3-2) - (-2)(-1+5) + 1(-1-9)$$

$$= 2(-5) - (-2) * 5 + 1(-10)$$

$$= -10$$

$\therefore$ This system has unique solution as $\det(A) \neq 0$

(c)

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 1 & 3 & -2 & 1 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

$$r_2' = r_2 - \frac{1}{2} \times r_1$$

$$r_3' = r_3 - \frac{3}{2} \times r_1$$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{2}$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 2 & -5/2 & 13/2 \end{array}\right]$$

$$r_3' = r_3 - \frac{1}{2} \times r_2$$

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{2}{4} = \frac{1}{2}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 0 & -5/4 & 21/4 \end{array}\right]$$

(d)

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -3 \\ 0 & 4 & -5/2 & 5/2 \\ 0 & 0 & -5/4 & 21/4 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -5/2 \\ 0 & 0 & -5/4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 5/2 \\ 21/4 \end{bmatrix}$$

$-5/4\, c = 21/4$

$\Rightarrow c = -4 \cdot 2$

$4b - 5/2\, c = 5/2$

$\Rightarrow b = -2$

$2a - 2b + c = -3$

$\Rightarrow a = -1 \cdot 4$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \cdot 4 \\ -2 \\ -4 \cdot 2 \end{bmatrix}$$

03. (a)

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix}$$

There is 0 in the diagonal element. Because of that, multiplier for 2nd row operation will be undefined [divide by 0]. So, matrix A has pivoting problem & we will need to solve it before applying LU decomposition method.

Swapping row 2 & 3 :

$$A = \begin{bmatrix} 3 & 3 & 4 \\ \cancel{0} & \cancel{0} & \cancel{-} \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) $F^1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{0}{3} = 0$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{0}{3} = 0$$

$$F^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{0}{-1} = 0$$

(c) $U = \begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$

$L = F^{-1} = (F^1)^{-1} (F^2)^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d)

$$\underset{A}{\begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix}}$$

Solving for $Ly = b$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

$$y_1 = 1$$

$$y_2 = 4$$

$$y_3 = 6$$

Solving for $Ux = y$,

$$\begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cancel{a}\, x_1 \\ \cancel{b}\, x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

$-x_3 = 6$

$\Rightarrow x_3 = \cancel{6}\ -6$

$-x_2 - 3x_3 = 4$

$\Rightarrow x_2 = 14$

$3x_1 + 3x_2 + 4x_3 = 1$

$\Rightarrow x_1 = -\dfrac{17}{3}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -17/3 \\ 14 \\ -6 \end{bmatrix}$$