

# Machine Learning (CE 40717)

## Fall 2024

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Sharif University of Technology

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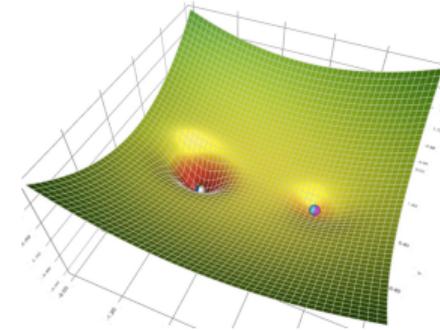
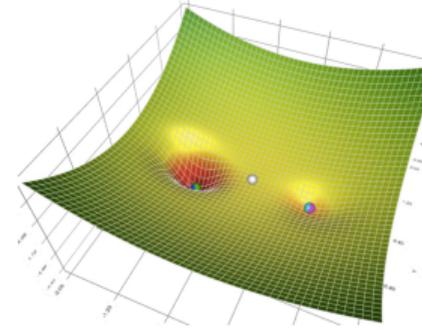
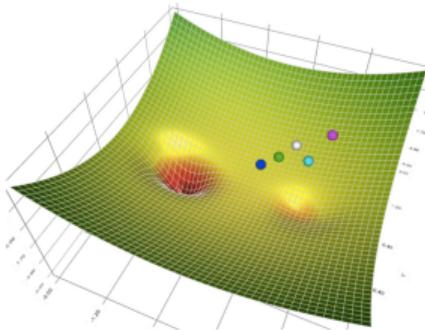
## 1 Gradient Descent

## ② Backpropagation

## 4 References

## How to Update Weights?

- Imagine training a large model like ChatGPT. It has billions of parameters that need to be adjusted.
  - If we used **random search** to update these weights, it would take an astronomical number of trials to find good parameters.
  - How to make training feasible at this massive scale?



## How to Update Weights?

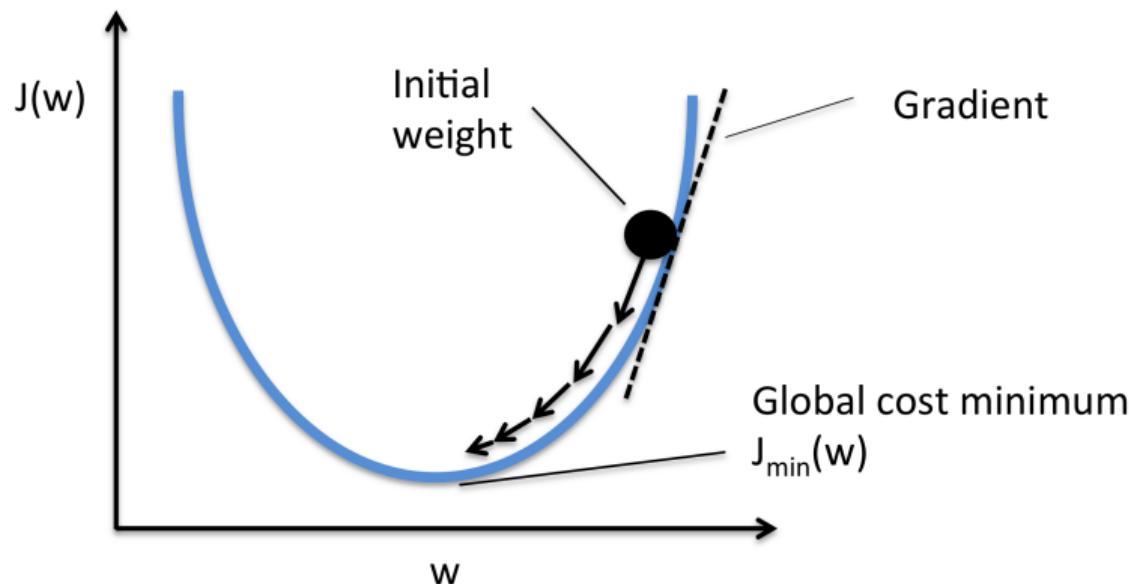
### **Options for Updating Weights:**

- **Random Search:** Tries values randomly—inefficient and impractical.
  - **Gradient Descent:** Follows the slope of the loss function—efficient and guided.

## Why Gradient Descent?

- It updates weights by following the slope, reducing error with each step.
  - Controlled, stepwise updates ensure we move closer to minimizing the loss effectively.

## How to Update Weights?



## Gradient Descent: Concept and Weight Updates

**Gradient Descent:** Minimizes the loss function by updating weights based on the gradient.

### **Weight Update Rule:**

$$w_{\text{new}} = w_{\text{old}} - \eta \cdot \frac{\partial L}{\partial w}$$

Where:

- $\eta$  is the learning rate (step size).
  - $\frac{\partial L}{\partial w}$  is the gradient of the loss function with respect to  $w$ .

## Example: Gradient Descent and Updating Weights

### **Example Problem:**

- Initial weight:  $w_0 = 2$
  - Learning rate:  $\eta = 0.1$
  - Loss function:  $L(w) = (\gamma - wx)^2$

## Gradient Calculation:

$$\frac{\partial L}{\partial w} = -2x(y - wx)$$

**Example:** For  $x = 3$ ,  $y = 10$ , and  $w_0 = 2$

$$\frac{\partial L}{\partial w} = -24, \quad w_{\text{new}} = 4.4$$

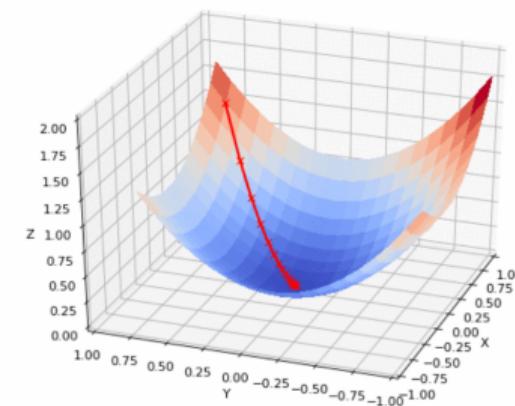
# Gradient Descent: Formula and Process

## Weight Update Formula:

$$w_{\text{new}} = w_{\text{old}} - \eta \cdot \frac{\partial L}{\partial w}$$

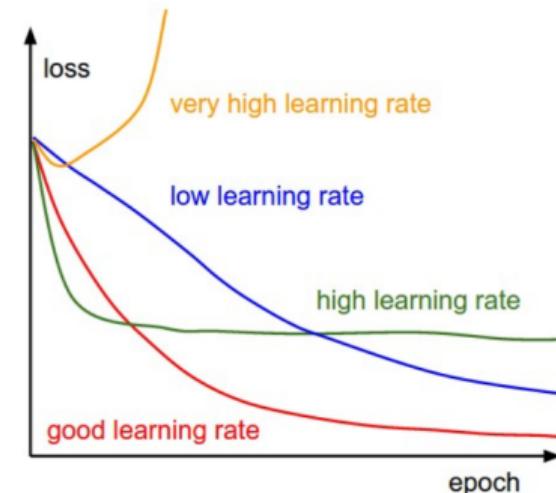
## Steps in Gradient Descent:

- Compute the gradient of the loss function.
- Update the weights using the update rule.
- Repeat until convergence.



# Identifying an Optimal Learning Rate

- Look for a **smooth, gradual** decrease in loss over time.
- Avoid a learning rate that causes **erratic fluctuations** in the loss function.
- Ensure that the learning rate is **not too small**, leading to **slow** convergence.
- A good learning rate often shows a **steady, reliable** trend without large spikes.
- More details later.



## 1 Gradient Descent

## 2 Backpropagation

Forward and Backward Passes

Vectorized Backpropagation

Chain Rule

## 3 Foundations in Detail: Initialization, Loss, and Activation

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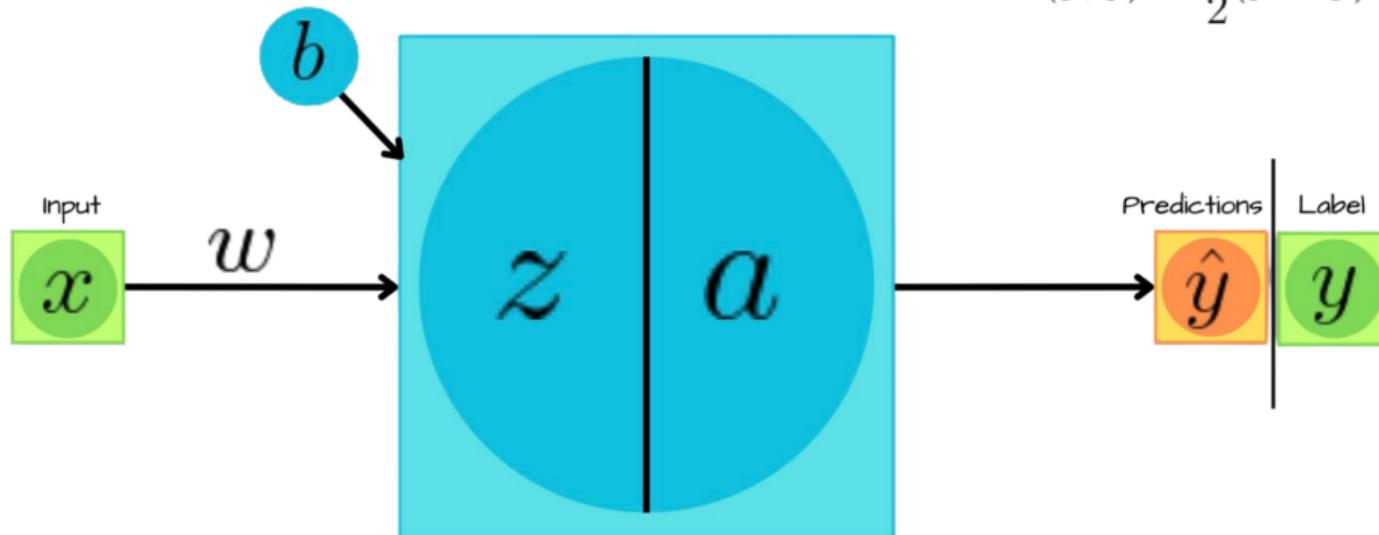
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## Example: One Neuron

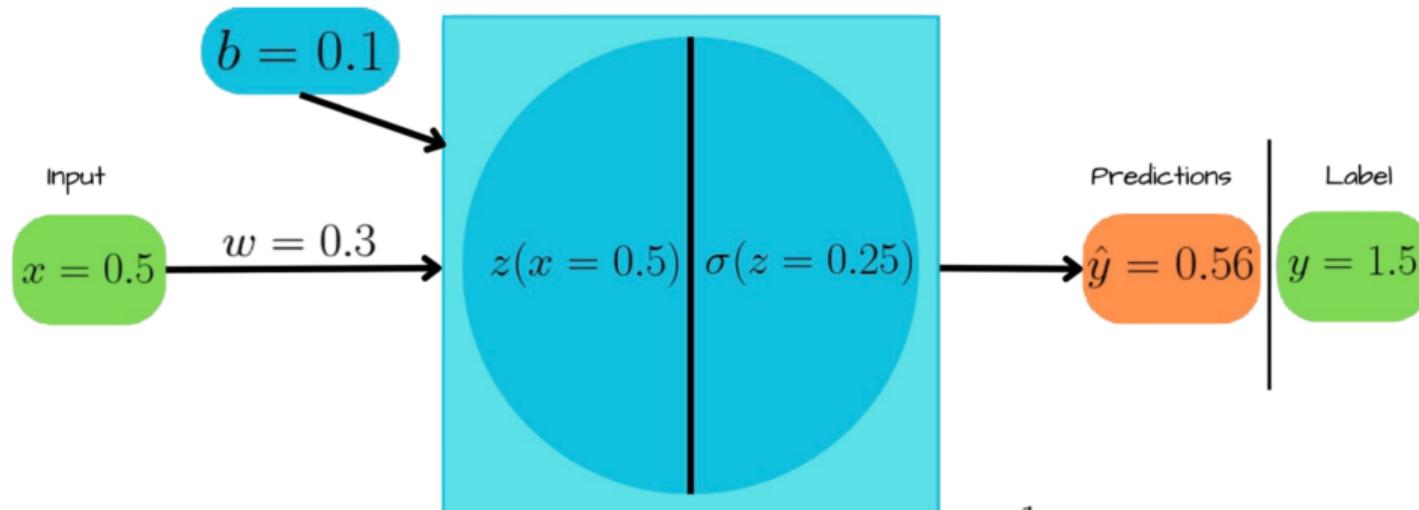
$$\text{Total Error} \quad L(y, \hat{y}) = \frac{1}{2}(y - \hat{y})^2$$



$$z(x) = wx + b \quad a = \sigma(z) = \frac{1}{1 + e^{-z}}$$

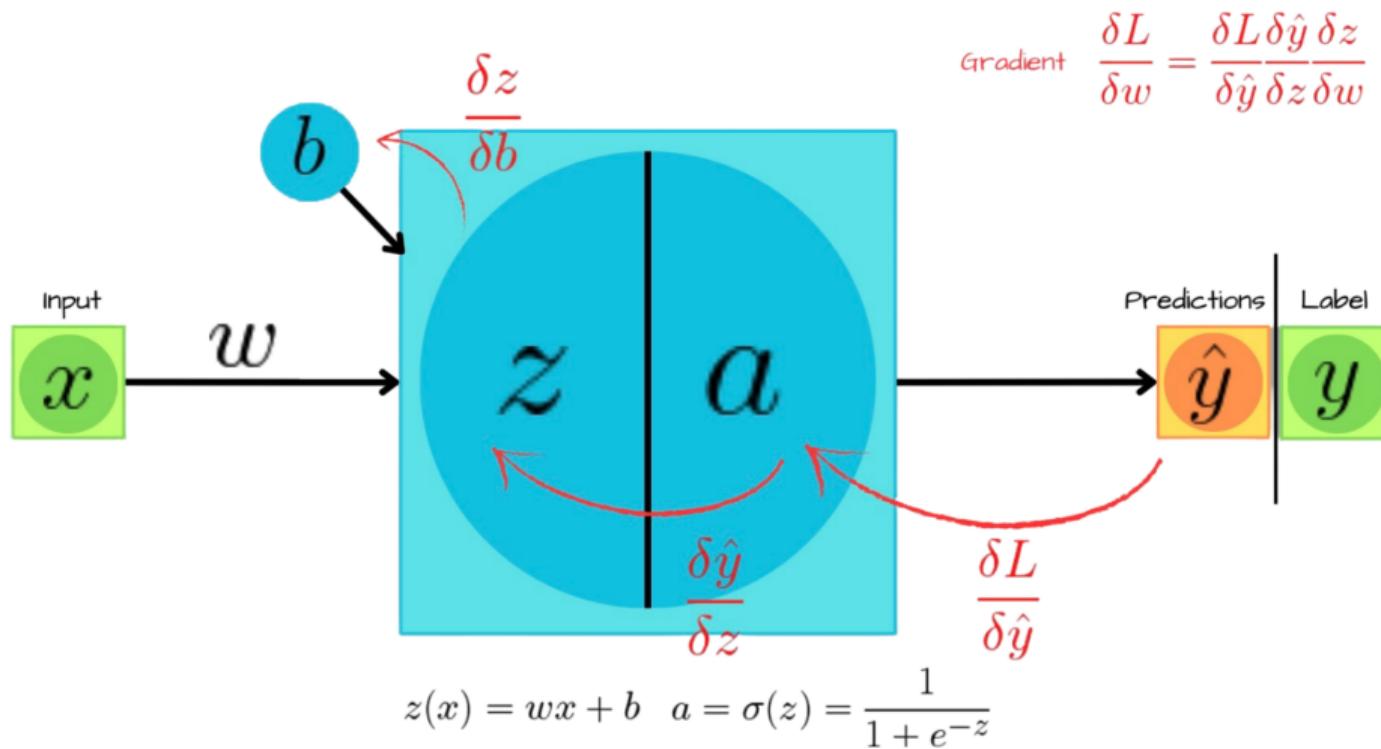
## Forward Pass

$$\text{Total Error } L(1.5, 0.56) = \frac{1}{2}(1.5 - 0.56)^2 = 0.44$$

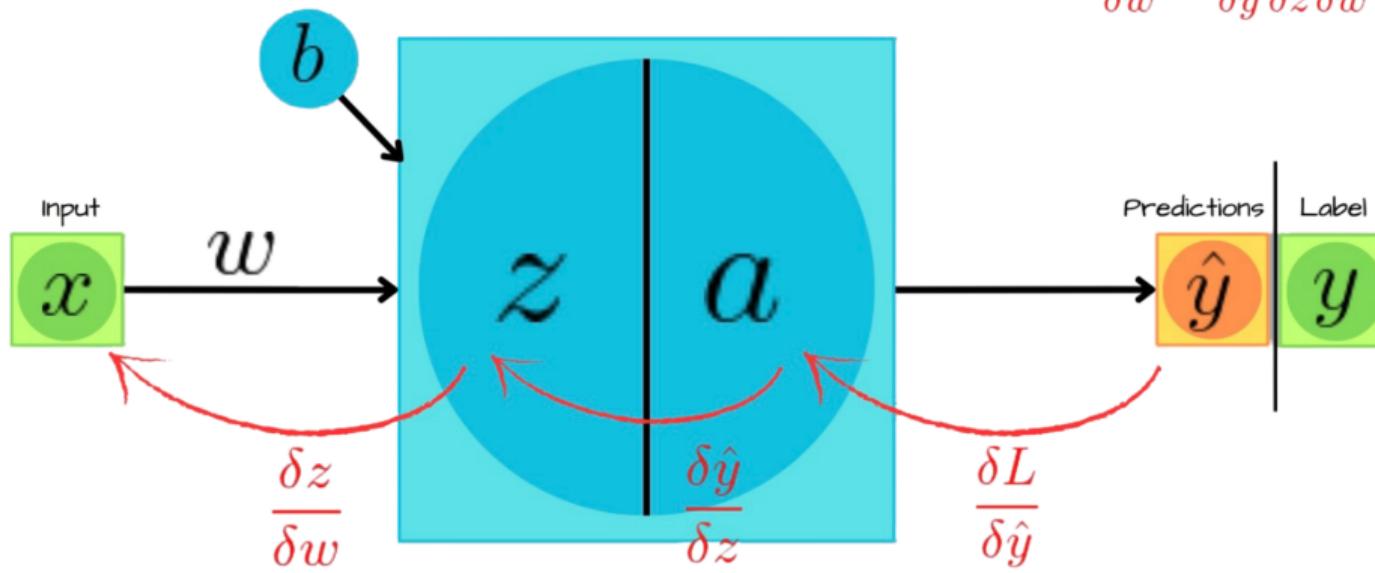


$$z(0.5) = 0.3 \cdot 0.5 + 0.1 = 0.25 \quad \sigma(z = 0.25) = \frac{1}{1 + e^{-0.25}} = 0.56$$

## Backward Pass



## Backward Pass

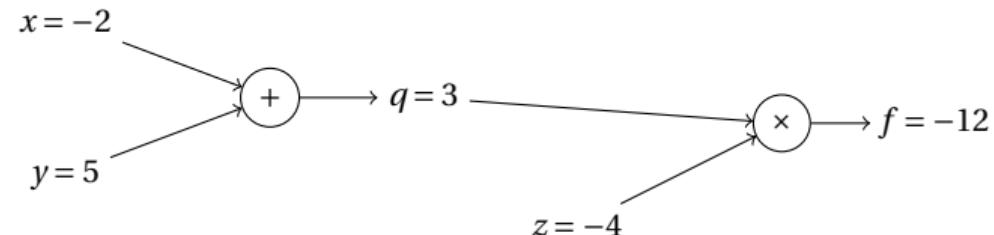


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# Backpropagation: a simple example

**Function:**

$$f(x, y, z) = (x + y)z$$



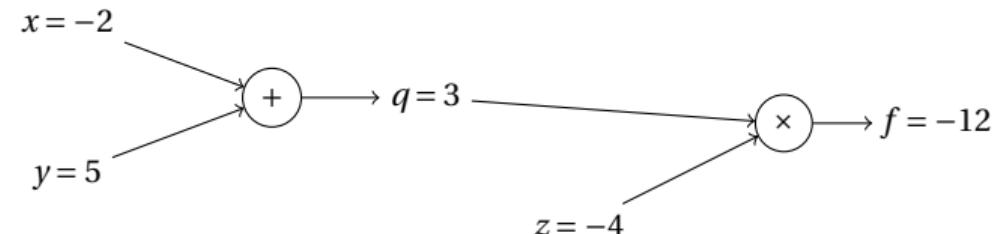
# Backpropagation: a simple example

**Function:**

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**Example:**

$$x = -2, \quad y = 5, \quad z = -4$$



## Backpropagation: a simple example

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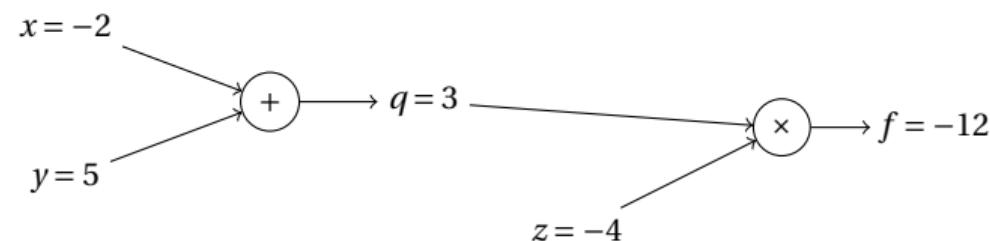
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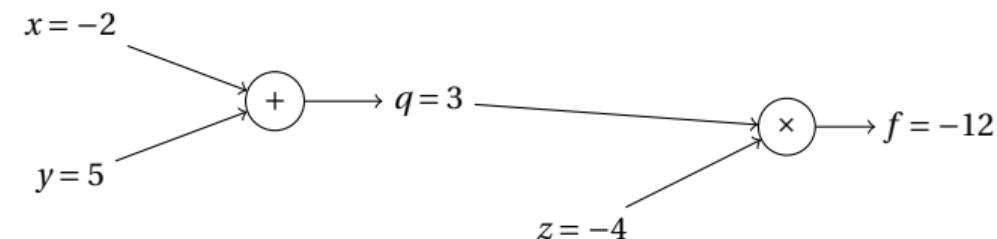
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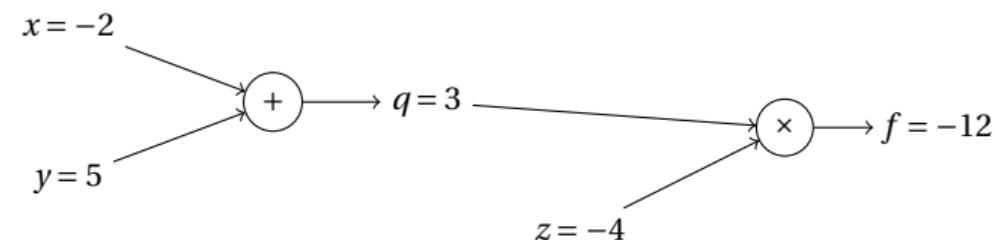
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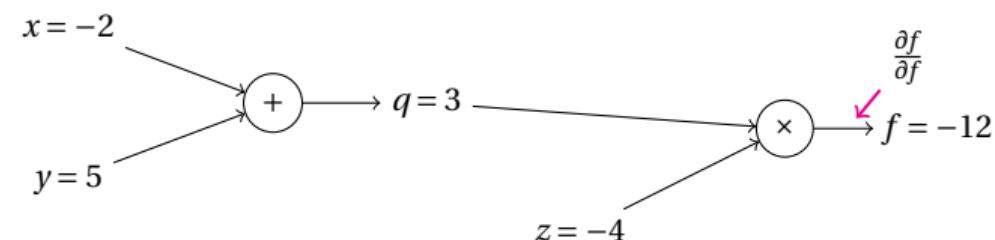
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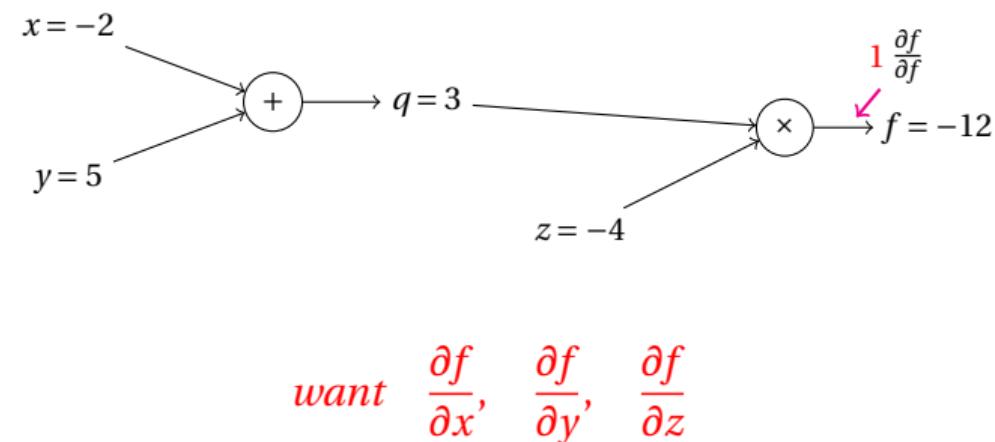
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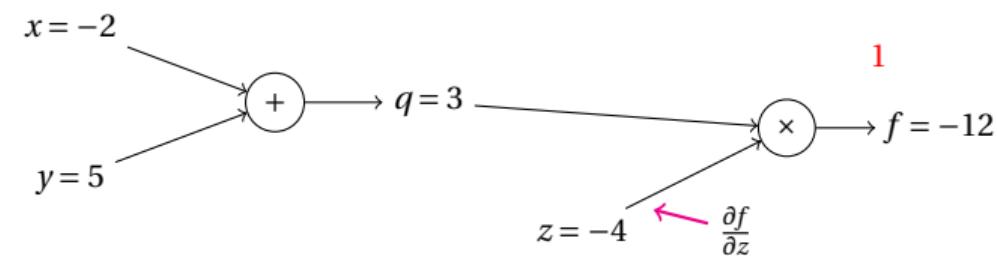
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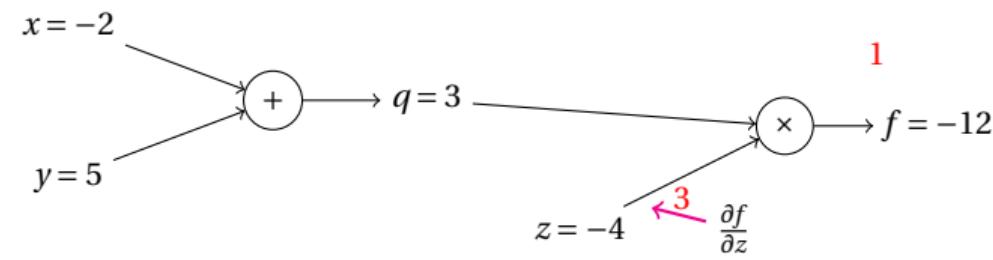
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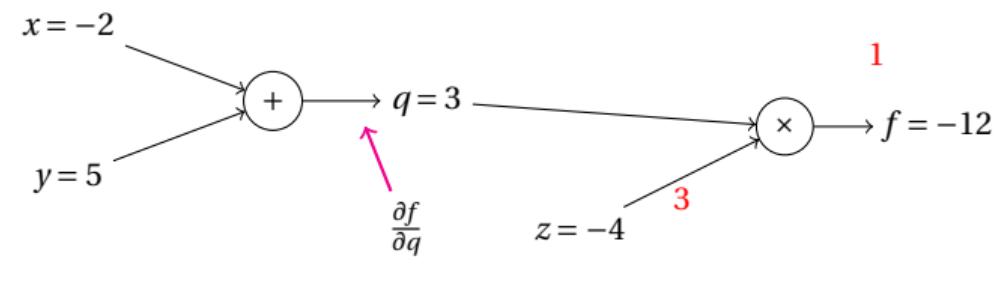
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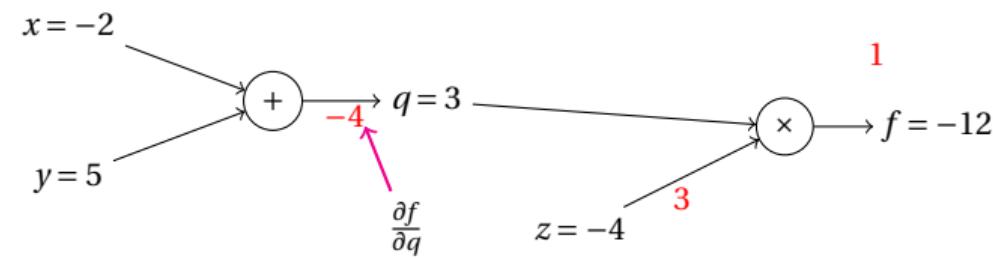
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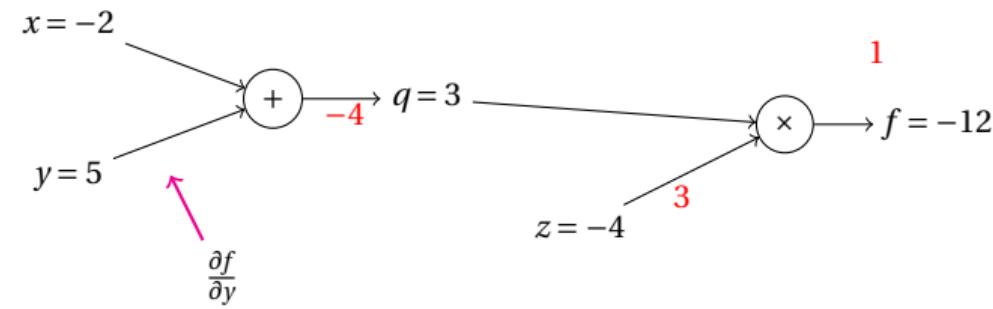
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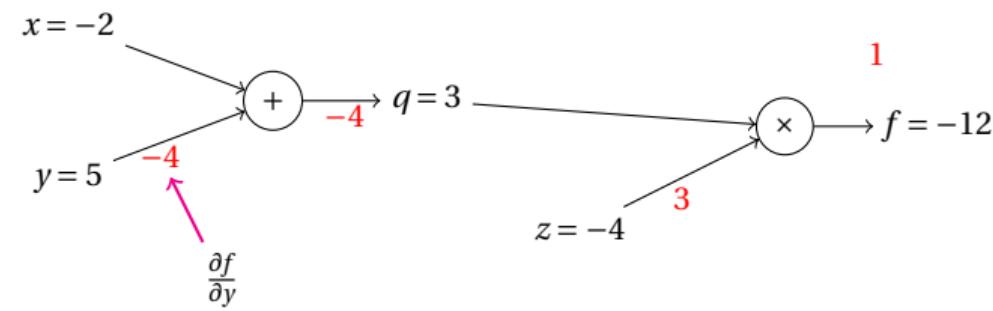
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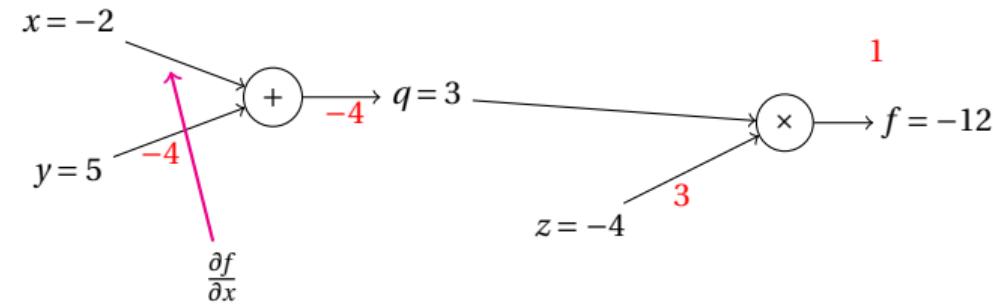
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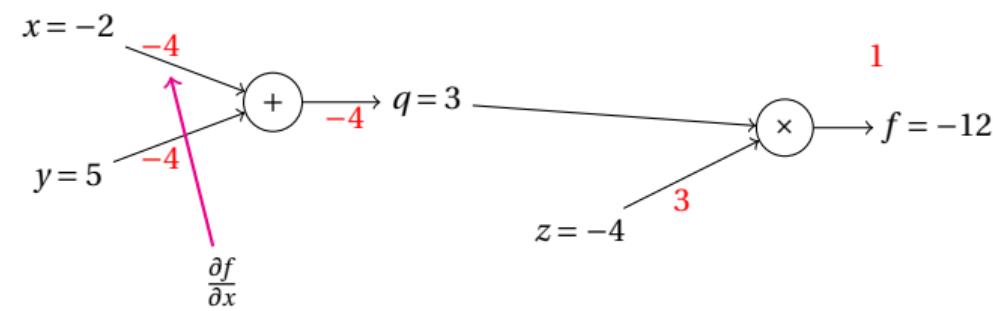
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Chain Rule

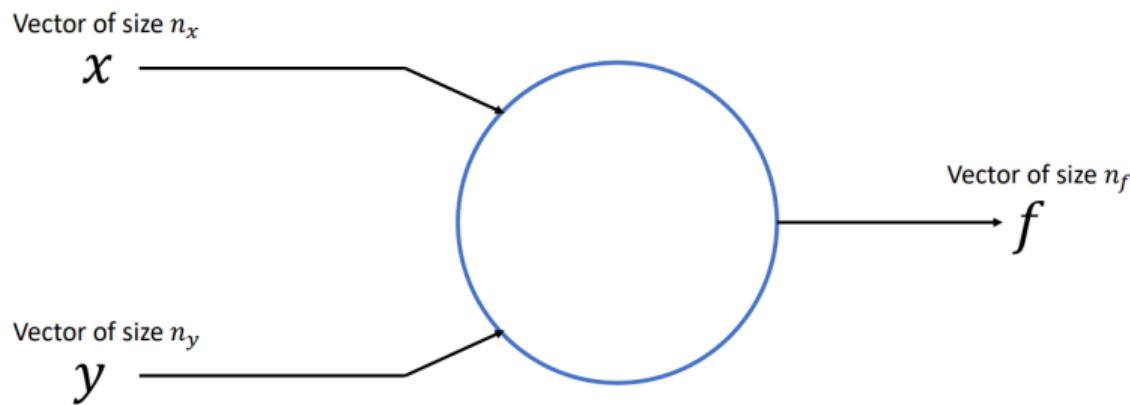
## 3 Foundations in Detail: Initialization, Loss, and Activation

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# Vectorized Backpropagation

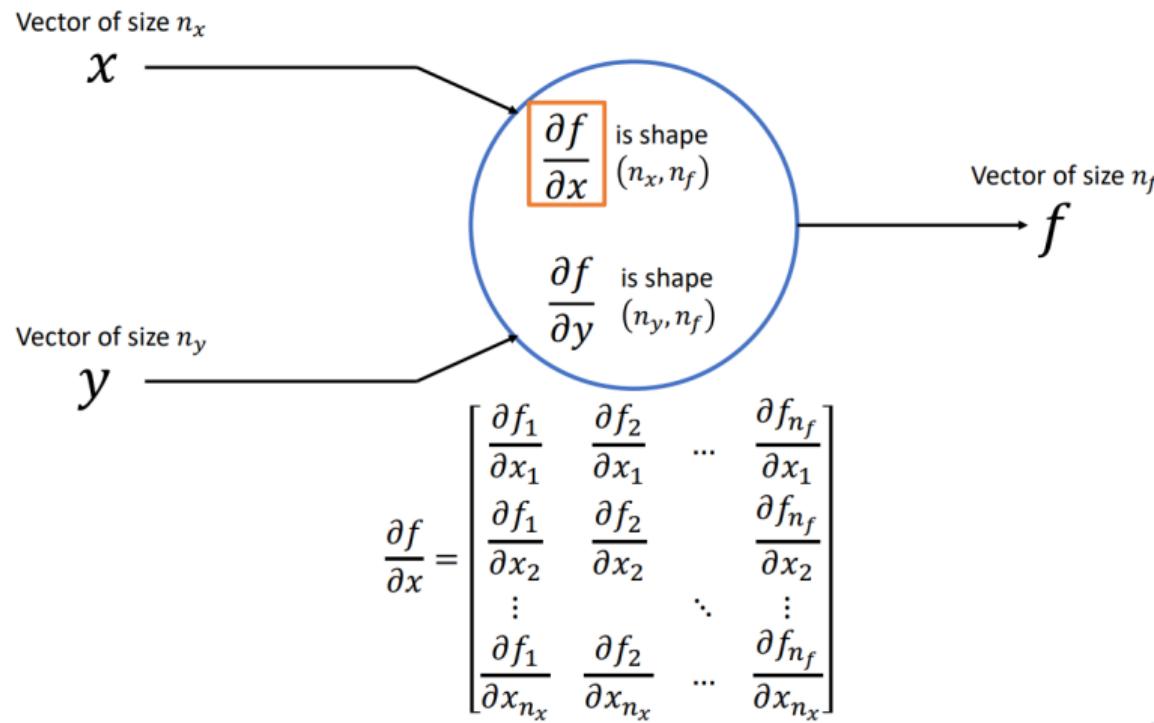
The faster you compute gradients, the quicker each parameter update in gradient descent.

**Derivative of a Vector by a Vector:** leveraging matrix operations for faster computation.

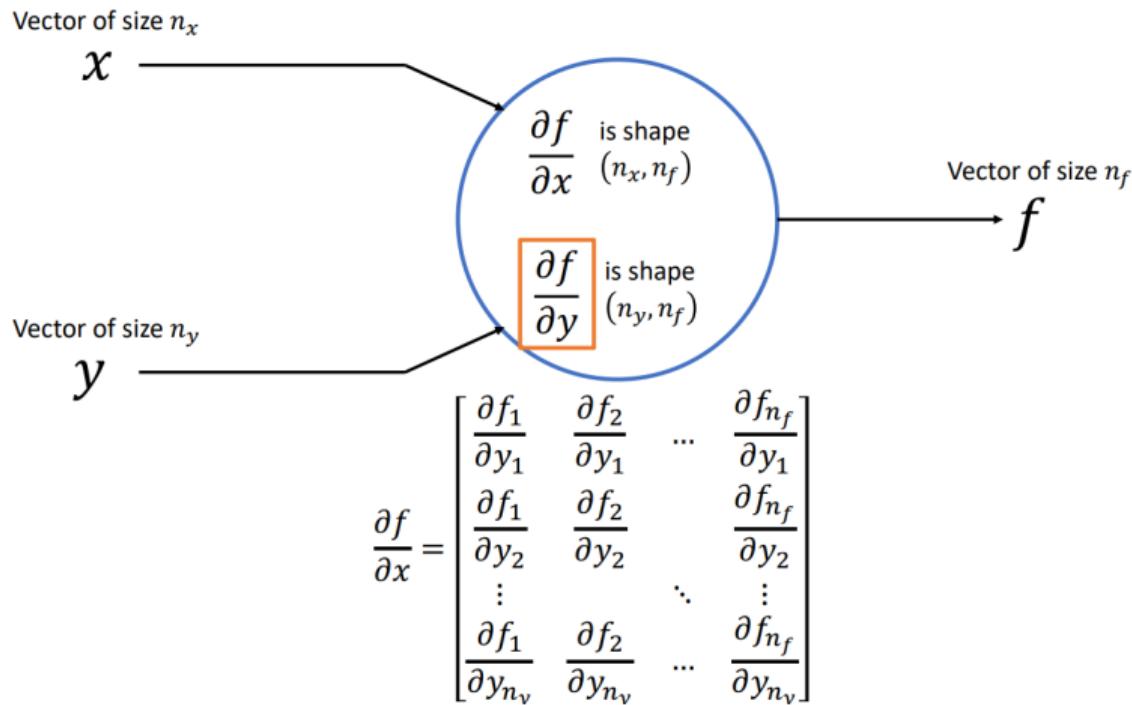


# Vectorized Backpropagation

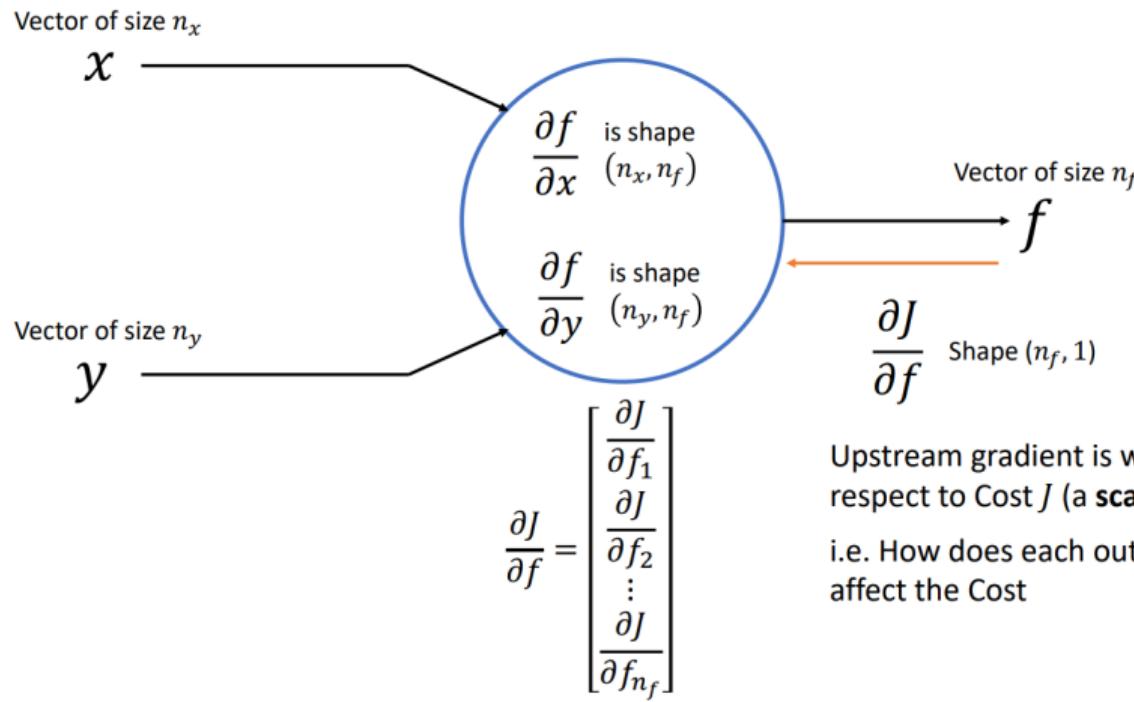
## Local Derivatives are Jacobian Matrices



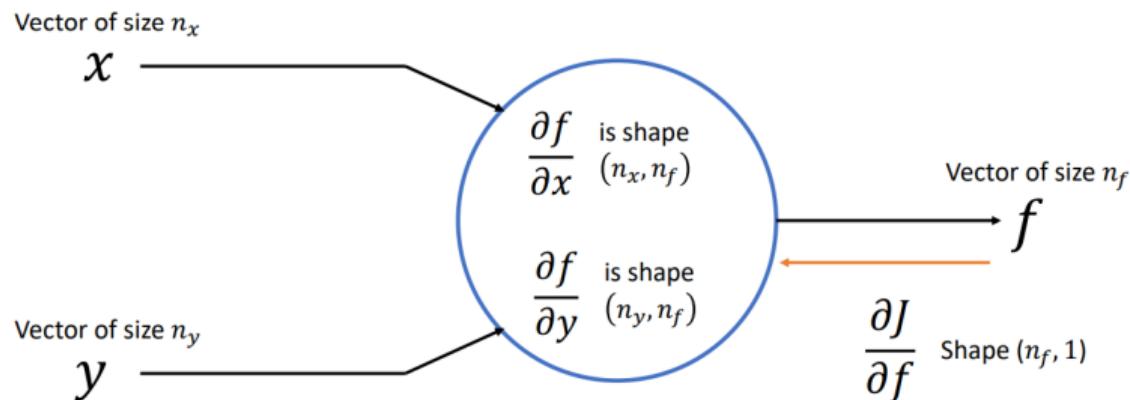
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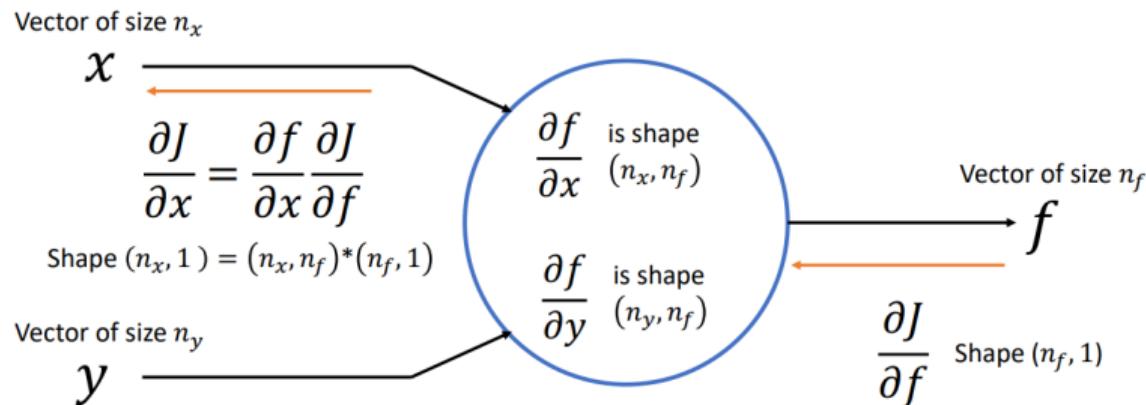


# Vectorized Backpropagation



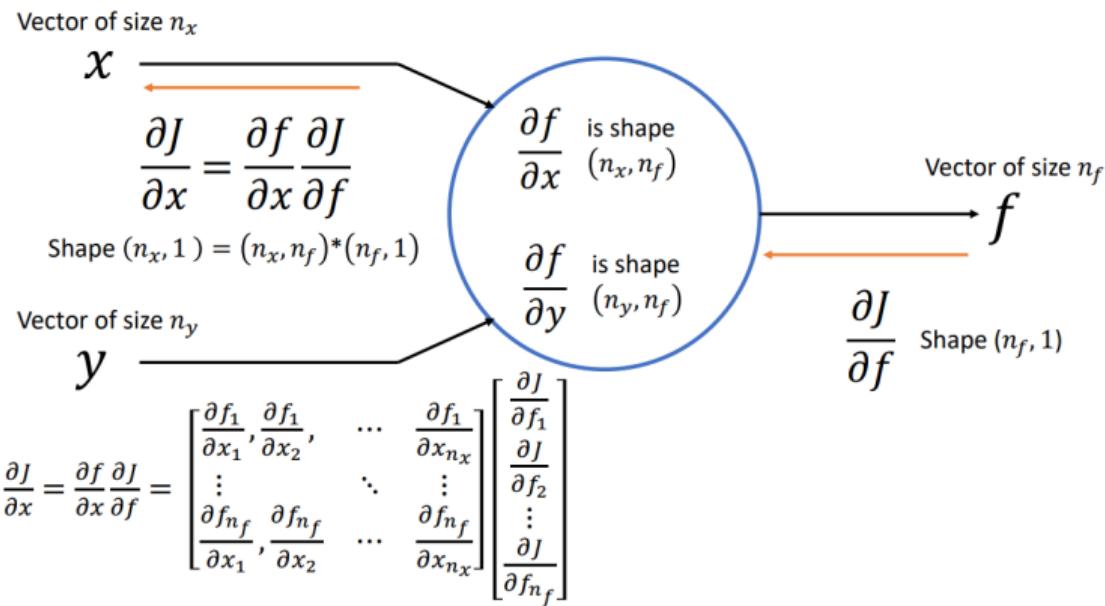
Apply chain rule like before!

# Vectorized Backpropagation

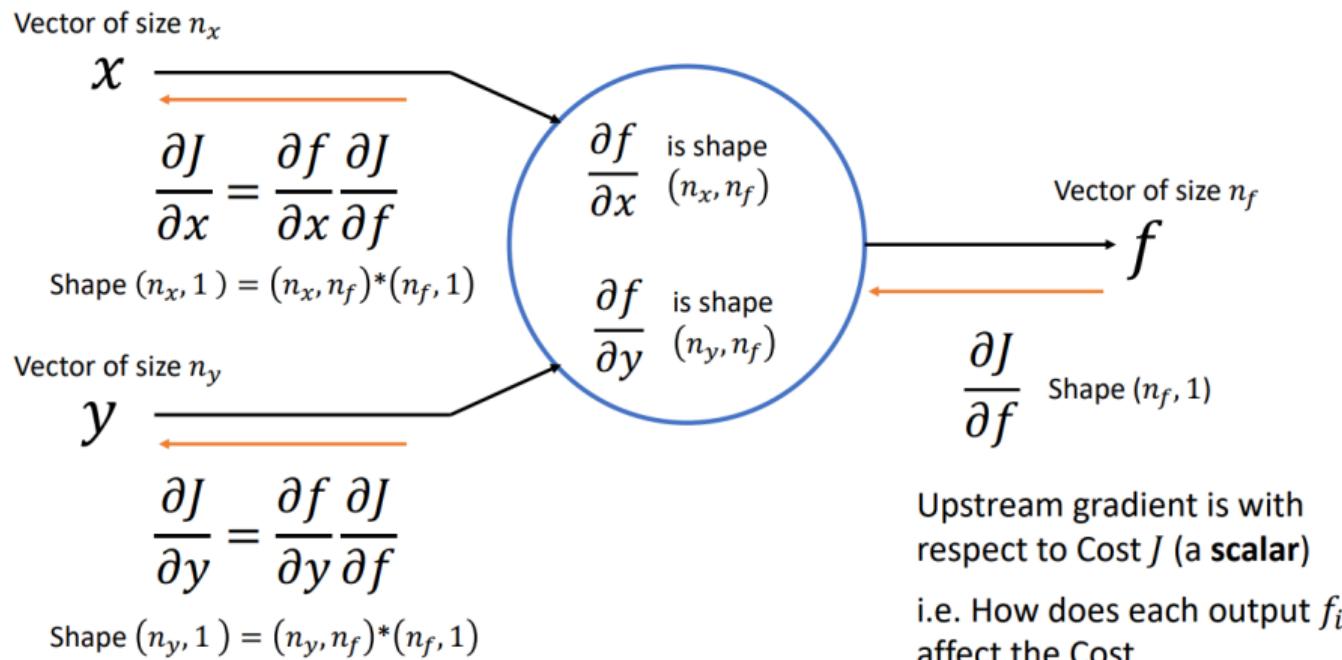


Applying the chain rule involves matrix-vector multiplication

# Vectorized Backpropagation



# Chain Rule – Matrix-Vector Multiply



Chain Rule application is Matrix-Vector Multiply

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# Chain Rule – Matrix-Vector Multiply

$$\frac{\partial J}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial J}{\partial f} \rightarrow \begin{bmatrix} \frac{\partial J}{\partial x_1} \\ \frac{\partial J}{\partial x_2} \\ \vdots \\ \frac{\partial J}{\partial x_{n_x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, & \dots & \frac{\partial f_{n_f}}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_{n_x}}, \frac{\partial f_2}{\partial x_{n_x}}, & \dots & \frac{\partial f_{n_f}}{\partial x_{n_x}} \end{bmatrix} \begin{bmatrix} \frac{\partial J}{\partial f_1} \\ \frac{\partial J}{\partial f_2} \\ \vdots \\ \frac{\partial J}{\partial f_{n_f}} \end{bmatrix}$$

$$\text{Shape } (n_x, 1) = (n_x, n_f) * (n_f, 1)$$

# Chain Rule – Matrix-Vector Multiply

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Jacobian

Shape  $(n_x, 1) = (n_x, n_f) * (n_f, 1)$

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Jacobian

Upstream Gradient

Shape  $(n_x, 1) = (n_x, n_f) * (n_f, 1)$

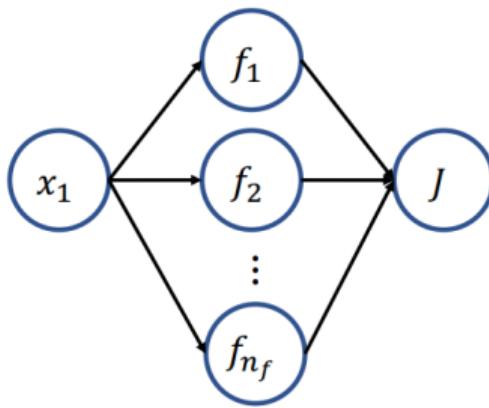
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Shape  $(n_x, 1) = (n_x, n_f) * (n_f, 1)$

$$\frac{\partial J}{\partial x_1} = \frac{\partial f_1}{\partial x_1} \frac{\partial J}{\partial f_1} + \frac{\partial f_2}{\partial x_1} \frac{\partial J}{\partial f_2} + \dots + \frac{\partial f_{n_f}}{\partial x_1} \frac{\partial J}{\partial f_{n_f}}$$

# Chain Rule – Matrix-Vector Multiply



$$\frac{\partial J}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial J}{\partial f} \rightarrow \begin{bmatrix} \frac{\partial J}{\partial x_1} \\ \frac{\partial J}{\partial x_2} \\ \vdots \\ \frac{\partial J}{\partial x_{n_x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \dots, \frac{\partial f_{n_f}}{\partial x_1} \\ \vdots \\ \frac{\partial f_1}{\partial x_{n_x}}, \frac{\partial f_2}{\partial x_{n_x}}, \dots, \frac{\partial f_{n_f}}{\partial x_{n_x}} \end{bmatrix} \begin{bmatrix} \frac{\partial J}{\partial f_1} \\ \frac{\partial J}{\partial f_2} \\ \vdots \\ \frac{\partial J}{\partial f_{n_f}} \end{bmatrix}$$

Shape  $(n_x, 1) = (n_x, n_f) * (n_f, 1)$

$$\frac{\partial J}{\partial x_1} = \frac{\partial f_1}{\partial x_1} \frac{\partial J}{\partial f_1} + \frac{\partial f_2}{\partial x_1} \frac{\partial J}{\partial f_2} + \dots + \frac{\partial f_{n_f}}{\partial x_1} \frac{\partial J}{\partial f_{n_f}}$$

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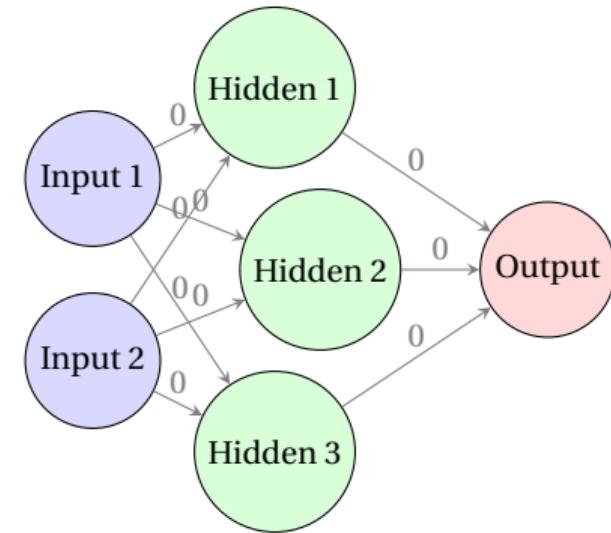
## 4 References

# Weight Initialization

**Example:** Imagine a network where all weights are initialized to zero.

**Issue:** If all weights are zero, each neuron in a layer will produce **identical** outputs. This symmetry prevents the network from learning **distinct features**, as every neuron updates identically.

**Solution:** To break this symmetry, weights need to be initialized with small random values, allowing neurons to learn unique features and avoid identical updates.



All weights initialized to zero

# Why Weight Initialization Matters

## Importance:

- Proper **initialization** ensures **faster convergence** and improves **training stability**.
- Prevents issues like **vanishing** or **exploding gradients**, which can make training slow or unstable.

**Question:** How can we initialize weights to maximize **learning efficiency** and prevent gradient problems?

# Zero Initialization and Random Initialization

## Zero Initialization

- **Description:** Set all weights to zero.
- **Key Point:** Rarely used, as it leads to identical updates for all neurons, preventing the network from learning distinct features.

## Random Initialization

- **Description:** Assign small random values to weights.
- **Distribution:** Typically, weights are initialized using a uniform or normal distribution.

$$w \sim \mathcal{U}(-\epsilon, \epsilon) \quad \text{or} \quad w \sim \mathcal{N}(0, \sigma^2)$$

- **Key Point:** Helps break symmetry but can still cause issues with gradient magnitudes.

# Xavier Initialization

**Description:** Xavier Initialization is designed to keep the variance of activations consistent across layers, ideal for sigmoid and tanh activations.

**Objective:** Prevents the shrinking or exploding of signal magnitudes during forward and backward propagation.

**Condition:**

$$\frac{1}{n_l} \text{Var}[w] = 1$$

**Initialization Scheme:**

$$w \sim \mathcal{U}\left(-\sqrt{\frac{1}{n_l}}, \sqrt{\frac{1}{n_l}}\right)$$

This results in a uniform distribution within the range  $-\sqrt{\frac{1}{n_l}}$  to  $\sqrt{\frac{1}{n_l}}$ , ensuring stable signal variance across layers.

# He Initialization

**Description:** He Initialization (or Kaiming Initialization) is designed for neural networks with ReLU activations, considering the non-linearity of these functions.

**Objective:** Aims to prevent the exponential growth or reduction of input signal magnitudes through layers.

**Condition:**

$$\frac{1}{2} n_l \text{Var}[w] = 1$$

**Initialization Scheme:**

$$w_l \sim \mathcal{N}\left(0, \frac{2}{n_l}\right)$$

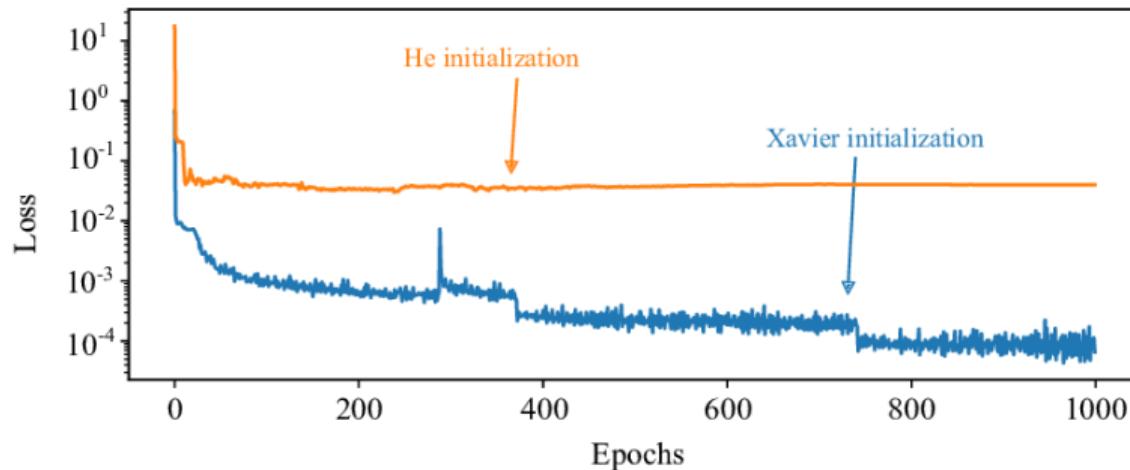
This implies a zero-centered Gaussian distribution with a standard deviation of  $\sqrt{\frac{2}{n_l}}$ , where biases are initialized to 0.

# Key Point Summary

Proper initialization:

- Reduces the risk of gradient issues (vanishing/exploding gradients).
- Helps the network converge faster.

# Xavier vs He



Evolution of loss term for Xavier weight initialization and He weight initialization.

# Choosing the Right Initialization – Examples

- **Scenario 1:** Using ReLU activation functions in a deep network.
  - **Best Choice:** He Initialization.
  - **Reason:** Helps maintain gradient flow through the layers.
- **Scenario 2:** Using Sigmoid activation functions in a shallow network.
  - **Best Choice:** Xavier Initialization.
  - **Reason:** Keeps variance balanced, which is crucial for non-ReLU activations.
- **Experiment:** Try initializing with zeros and random weights to see how it impacts training speed and performance.

# Transition to Loss and Activation Functions

**Recap:** Proper weight initialization:

- Ensures stability during training by maintaining gradient magnitudes.
- Helps the network converge faster and learn more effectively.

**Next Steps:**

- Once weights are initialized, the network needs a measure of error — this is where **loss functions** come in.
- After initializing weights, **activation functions** determine the output of each neuron, enabling the network to learn complex patterns.

**Question:** How do we measure the error in predictions and adjust our weights to minimize it?

## 1 Gradient Descent

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# Types of Loss Functions

- **Mean Squared Error (MSE):** Used in regression to minimize squared differences between predicted and true values.
- **Mean Absolute Error (MAE):** Minimizes absolute differences, also for regression tasks.
- **Binary Cross-Entropy:** Used for binary classification to compare predicted probabilities with binary labels.
- **Categorical Cross-Entropy:** For multi-class classification, comparing predicted probabilities across multiple classes.

# Mean Squared Error (MSE)

## Definition:

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (y_i - \hat{y}_i)^2$$

## Characteristics:

- Amplifies larger errors due to squaring, making it sensitive to outliers.

# Example of MSE Calculation

## Example:

- Predicted values:  $\hat{y} = [4.2, 3.8, 5.1]$
- True values:  $y = [5.0, 4.0, 4.9]$
- Calculation:

$$\begin{aligned}\text{MSE} &= \frac{1}{3} [(5.0 - 4.2)^2 + (4.0 - 3.8)^2 + (4.9 - 5.1)^2] \\ &= \frac{1}{3} [0.64 + 0.04 + 0.04] = \frac{0.72}{3} = 0.24\end{aligned}$$

# Mean Absolute Error (MAE)

## Definition:

$$\text{MAE} = \frac{1}{N} \sum_{i=1}^N |y_i - \hat{y}_i|$$

## Characteristics:

- Provides a linear measure of error, treating all deviations equally.

# Example of MAE Calculation

## Example:

- Predicted values:  $\hat{y} = [4.2, 3.8, 5.1]$
- True values:  $y = [5.0, 4.0, 4.9]$
- Calculation:

$$\begin{aligned}\text{MAE} &= \frac{1}{3} (|5.0 - 4.2| + |4.0 - 3.8| + |4.9 - 5.1|) \\ &= \frac{1}{3} (0.8 + 0.2 + 0.2) = \frac{1.2}{3} \approx 0.4\end{aligned}$$

# Impact on Model Outputs

## MSE Impact:

- Penalizes large errors heavily, leading to smoother outputs, useful in cases like refining blurry autoencoder images.

## MAE Impact:

- Treats errors uniformly, often resulting in sharper outputs and better handling of outliers.

# Gradient and Optimization Differences

- **MSE** promotes faster convergence for large errors due to its quadratic gradient.
- **MAE** provides a constant gradient, making optimization stable but slower with large errors.

# Summary

- **MSE** is suitable for models where large errors need higher penalization.
- **MAE** is better for robust models that handle outliers and avoid smoothing effects.
- Choice of loss function affects model behavior and output characteristics.

# Binary Classification Loss – Binary Cross-Entropy

## Binary Cross-Entropy:

$$\mathcal{L}_{\text{BCE}} = -\frac{1}{n} \sum_{i=1}^n [y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i)]$$

### Example:

- Predicted probabilities:  $\hat{y} = [0.7, 0.3, 0.9]$
- True labels:  $y = [1, 0, 1]$
- Calculation:

$$\begin{aligned}\mathcal{L}_{\text{BCE}} &= -\frac{1}{3} [1 \cdot \log(0.7) + (1 - 1) \cdot \log(1 - 0.7) \\ &\quad + 0 \cdot \log(0.3) + (1 - 0) \cdot \log(1 - 0.3) \\ &\quad + 1 \cdot \log(0.9) + (1 - 1) \cdot \log(1 - 0.9)] \\ &\approx -\frac{1}{3} (\log(0.7) + \log(0.7) + \log(0.9)) \\ &\approx -\frac{1}{3} (-0.357 + -0.357 + -0.105) \approx 0.273\end{aligned}$$

# Categorical Cross-Entropy

## Categorical Cross-Entropy Formula:

$$L_{CCE} = -\frac{1}{n} \sum_{i=1}^n \sum_{c=1}^C y_{i,c} \log(\hat{y}_{i,c})$$

# One-Hot Encoding

One-hot encoding represents categorical variables as binary vectors, with a 1 indicating the actual class and 0s elsewhere.

## Example:

- Class 1: [1, 0, 0]
- Class 2: [0, 1, 0]
- Class 3: [0, 0, 1]

# Example Calculation (3-Class)

**Given:**

- True labels (One-hot): Class 2, Class 1, Class 3
- Predicted probabilities:

$$\hat{y} = \begin{bmatrix} 0.1 & 0.7 & 0.2 \\ 0.6 & 0.3 & 0.1 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}$$

# Solution

① True labels:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② Calculation:

$$L_{CCE} = -\frac{1}{3} (\log(0.7) + \log(0.6) + \log(0.3))$$

# Solution Continued

## Compute Log Terms:

$$\log(0.7) \approx -0.357, \quad \log(0.6) \approx -0.511, \quad \log(0.3) \approx -1.204$$

The categorical cross-entropy loss for the given data is:

$$L_{CCE} = \frac{1}{3} \times 2.072 \approx 0.691$$

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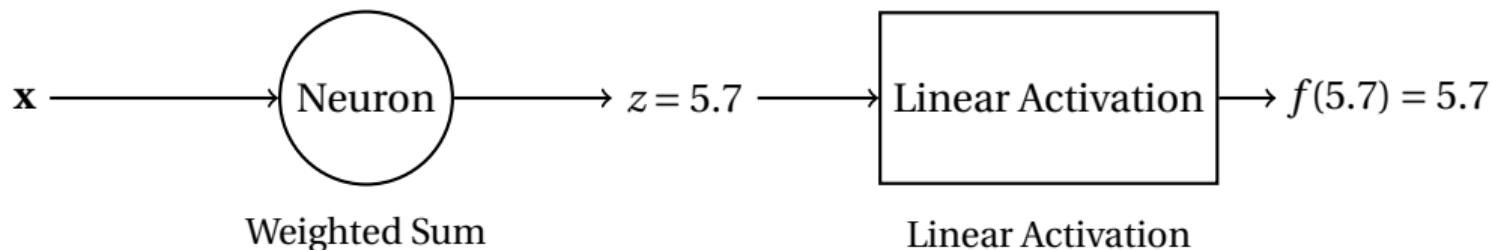
## 4 References

# Linear Activation - A Limitation

## Linear Activation:

$$f(z) = z$$

- Example: If a neuron produces a raw output  $z = 5.7$ , linear activation would pass this unchanged.



# Limitation of Linear Activation

**Why Transform Outputs?** Raw outputs need to be transformed into meaningful values, such as probabilities.

**The Problem:** Linear activation lacks non-linearity, restricting the model to simple linear relationships.

# Neural Networks: Why is the Max Operator Important?

- **Before:** Linear score function:

$$f = Wx$$

- **Now:** 2-layer Neural Network:

$$f = W_2 \max(0, W_1 x)$$

- The function  $\max(0, z)$  is called an activation function (in this case, ReLU).
- **Q: What if we try to build a neural network without an activation function?**

$$f = W_2 W_1 x$$

$$W_3 = W_2 W_1 \in \mathbb{R}^{C \times H}, \quad f = W_3 x$$

- **A:** We end up with a linear classifier again!

# ReLU

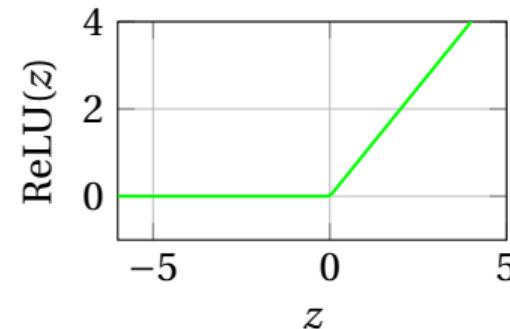
## Characteristics of ReLU:

$$\text{ReLU}(z) = \max(0, z)$$

- Faster convergence: Efficient computation, especially for deep networks.

## Advantages of ReLU:

- Does not saturate for positive values, helping to avoid the vanishing gradient problem.
- Computationally efficient (simpler than Sigmoid/Tanh).



# ReLU

## Limitation:

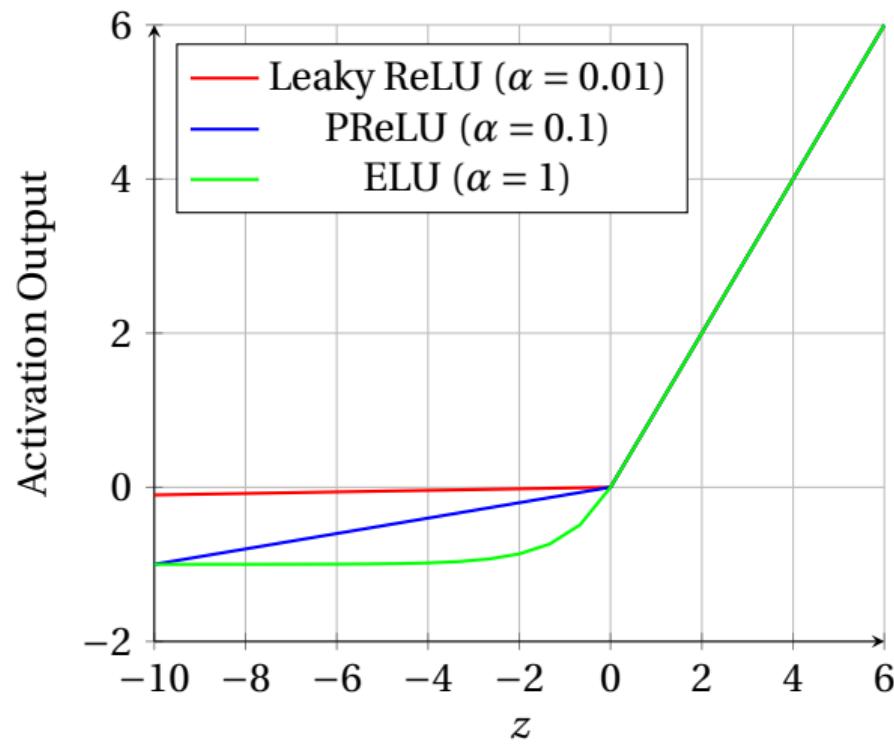
- **Dead ReLU Problem:** Neurons can become inactive during training, outputting 0 for all inputs if they receive negative values consistently.

## Question:

- Why does ReLU lead to faster training in deep networks?

# Variants of ReLU: Leaky ReLU, PReLU, ELU

## Leaky ReLU, PReLU, and ELU Activation Functions

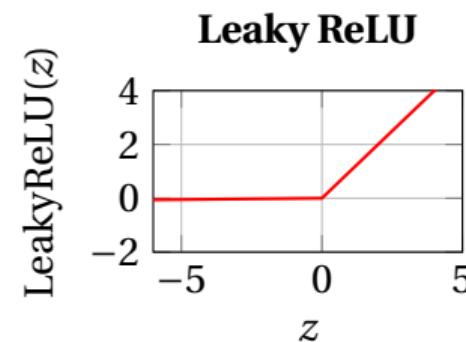


## Variants of ReLU: Leaky ReLU

- Allows a small, non-zero gradient for negative inputs.

$$\text{LeakyReLU}(z) = \max(\alpha z, z), \quad \alpha = 0.01$$

- Helps prevent the "dead ReLU" problem, where neurons stop updating.



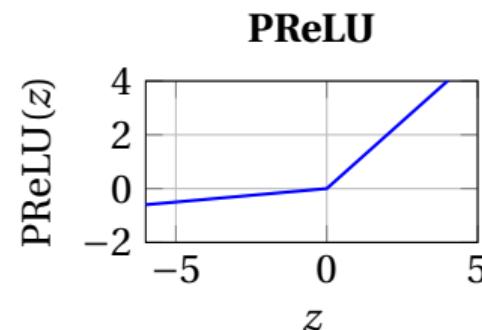
**Leaky ReLU:** Allows a small, non-zero gradient for negative inputs.

## Variants of ReLU: PReLU (Parametric ReLU)

- Similar to Leaky ReLU, but the slope for negative inputs ( $\alpha$ ) is learned during training.

$$\text{PReLU}(z) = \max(\alpha z, z), \quad \alpha \text{ is learned}$$

- Provides more flexibility by adjusting the slope for negative inputs based on data.



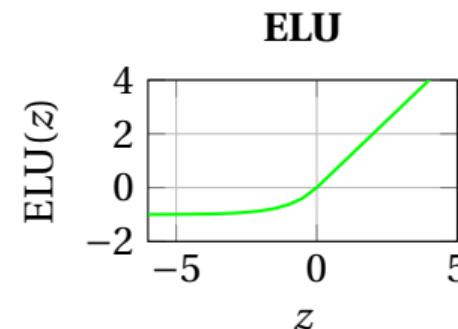
**PReLU:** Similar to Leaky ReLU, with a learnable slope.

## Variants of ReLU: ELU (Exponential Linear Unit)

- Similar to ReLU for positive values but smoother for negative inputs.

$$\text{ELU}(z) = \begin{cases} z, & \text{if } z > 0 \\ \alpha(e^z - 1), & \text{if } z \leq 0 \end{cases}, \quad \alpha = 1$$

- Provides faster convergence and reduces bias shift by smoothing negative values.



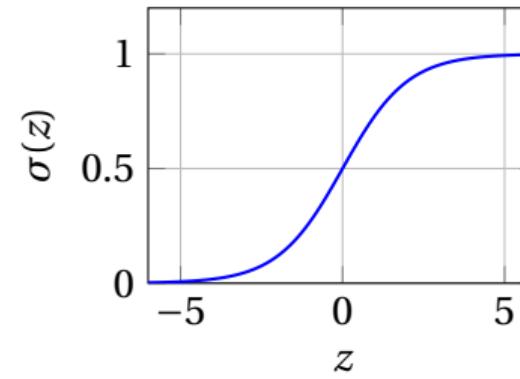
**ELU:** Smoother than ReLU for negative values.

# Sigmoid

## Characteristics of Sigmoid:

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

- Squashes the input between 0 and 1, which makes it useful in probabilistic interpretations (e.g., logistic regression).
- Often used in output layers for binary classification problems.

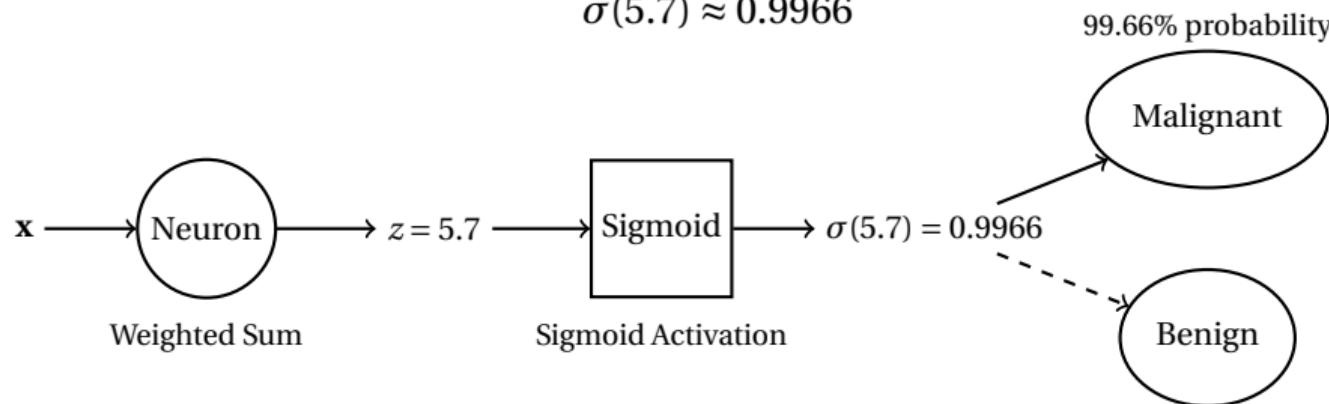


# Classification: Tumor Detection (Malignant vs. Benign)

**Sigmoid Activation:** Useful for binary classification!

- Example: For  $z = 5.7$ ,

$$\sigma(5.7) \approx 0.9966$$



# Sigmoid

## Limitations of Sigmoid:

- **Gradient Saturation:** When  $z$  is very large or very small, the gradient becomes nearly zero, causing slow learning (vanishing gradient problem).
- **Not Zero-Centered:** The output is not zero-centered, which can make optimization more difficult.

## Question:

- Why does the vanishing gradient problem occur with Sigmoid during backpropagation? (To be discussed in more detail later)

# Tanh (Hyperbolic Tangent)

## Characteristics of Tanh:

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

- Squashes input between -1 and 1, making it zero-centered (Balanced Updates → Reduced Bias in Gradient Descent → Faster Convergence)

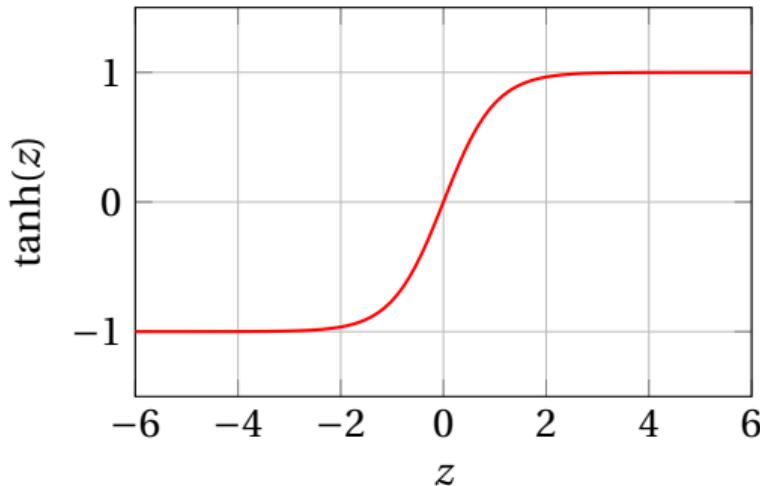
## Advantages of Tanh:

- **Zero-Centered:** Output ranges from -1 to 1, making optimization easier.
- Better for **hidden layers** than Sigmoid due to zero-centered output.

## Limitations:

- Similar saturation issues as Sigmoid: large input values push gradients towards zero (vanishing gradient problem).

# Tanh (Hyperbolic Tangent)

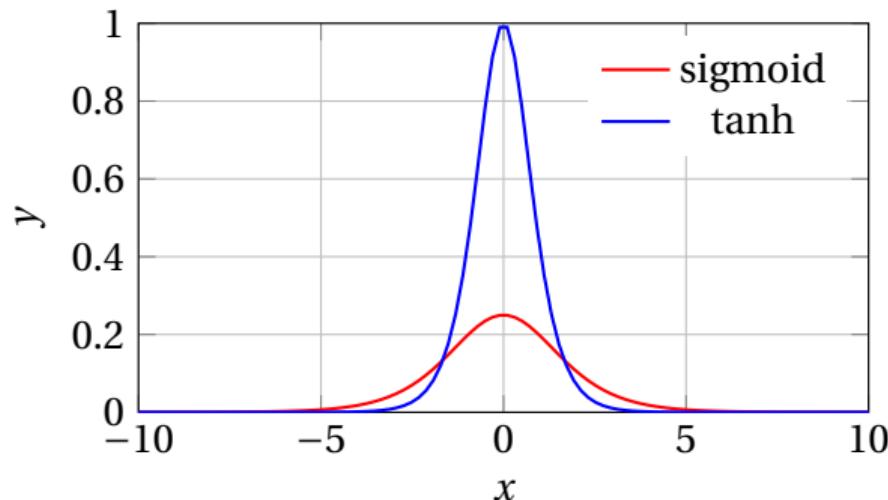


## Question:

- How does Tanh help with faster convergence compared to Sigmoid?

# Comparison: Sigmoid vs Tanh

Derivative of activation functions

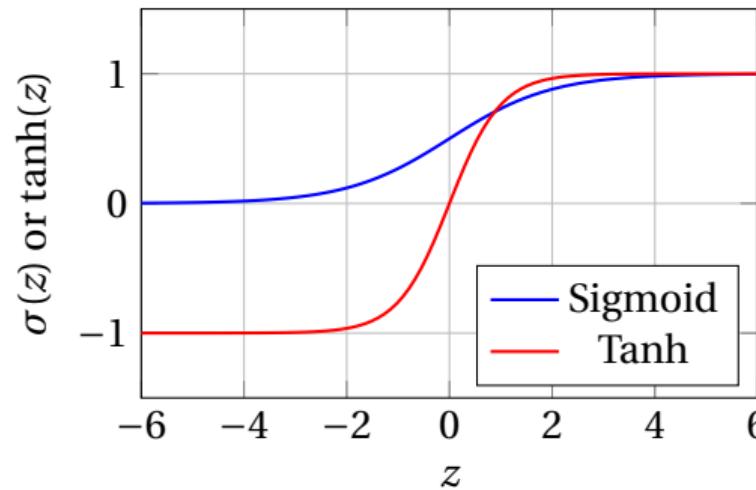


- The derivative of the Tanh function has a much steeper slope at  $x = 0$ , meaning it provides a larger gradient for backpropagation compared to the Sigmoid function.

# Comparison: Sigmoid vs Tanh

## Key Differences:

- **Sigmoid:** Maps input to  $[0, 1]$ . Output is not zero-centered.
- **Tanh:** Maps input to  $[-1, 1]$ . Output is zero-centered, leading to easier optimization.



# Comparison: Sigmoid vs Tanh

## When to Use:

- **Sigmoid:** Best for binary classification tasks, particularly in the output layer.
- **Tanh:** More suitable for hidden layers due to its centered output, allowing faster training.

## Question:

- In what scenario might Sigmoid be preferred over Tanh, despite its limitations?

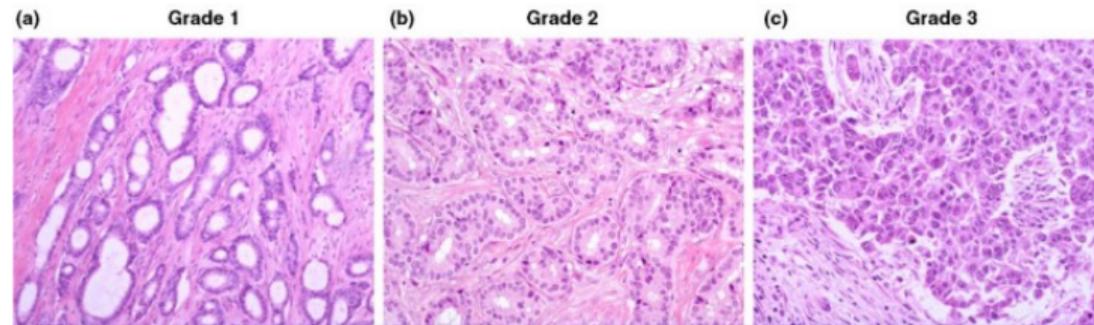
# Problem: Multi-Class Tumor Classification

**Scenario:** We want to classify a tumor into one of three categories:

- **Class 0:** Benign
- **Class 1:** Malignant
- **Class 2:** Pre-cancerous

**Goal:** Given a set of tumor features, predict which class the tumor belongs to.

This is a **multi-class classification problem**, and we will use the **Softmax activation function** to assign probabilities to each class.



## Softmax Activation: The Model

In multi-class classification, Softmax is used to convert raw outputs (logits) into probabilities for each class.

### Softmax Function:

$$P(y = i|X) = \frac{e^{z_i}}{\sum_{j=1}^K e^{z_j}}$$

Where:

- $z_i$  is the raw output (logit) for class  $i$ .
- $K$  is the number of classes (in this case, 3: benign, malignant, pre-cancerous).

The Softmax function ensures that the sum of the probabilities for all classes is 1, and the class with the highest probability is chosen as the prediction.

## Example: Softmax Calculation

Consider a tumor with the following logits from a neural network:

- Logit for Benign (Class 0):  $z_0 = 1.5$
- Logit for Malignant (Class 1):  $z_1 = 0.8$
- Logit for Pre-Cancerous (Class 2):  $z_2 = -0.5$

### Step 1: Exponentiate the logits

$$e^{z_0} = e^{1.5} \approx 4.48, \quad e^{z_1} = e^{0.8} \approx 2.23, \quad e^{z_2} = e^{-0.5} \approx 0.61$$

### Step 2: Compute the sum of exponentials

$$\text{Sum} = e^{z_0} + e^{z_1} + e^{z_2} = 4.48 + 2.23 + 0.61 = 7.32$$

## Example: Softmax Probabilities

### Step 3: Calculate Softmax probabilities for each class

$$P(\text{Benign}) = \frac{4.48}{7.32} \approx 0.612, \quad P(\text{Malignant}) = \frac{2.23}{7.32} \approx 0.305, \quad P(\text{Pre-Cancerous}) = \frac{0.61}{7.32} \approx 0.083$$

### Step 4: Make a classification decision

- The highest probability is 0.612 for the **Benign** class.
- Therefore, the model predicts that the tumor is **Benign** (Class 0).

# Conclusion: Softmax Activation for Classification

## Key Points:

- Softmax is used in the output layer for **multi-class classification**.
- It converts logits into a **probability distribution** across classes.
- The class with the highest probability is selected as the prediction.

**Main Idea:** Softmax ensures all outputs sum to 1, making it ideal for choosing one class out of multiple options.

## 1 Gradient Descent

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## 4 References

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# Any Questions?