

program \rightarrow turn this into
 property \rightarrow a formula
 $\{P\} \subseteq \{Q\}$ OR constraint system

$$\begin{array}{ll}
 \min & x+y \\
 \text{s.t.} & x+y \geq 10
 \end{array}$$

propositional logic $(p \wedge q \wedge \neg s) \Rightarrow r$

Atoms

T	F
t	f
true	false
1	0

 OR p, q, r, s, \dots

Literals atom or its negation $p, \neg p$

Formula F is either a literal

or $\neg F_1$

or $F_1 \wedge F_2$ (conjunction, AND)

or $F_1 \vee F_2$ (disjunction)

$$\text{or } F_1 \Rightarrow F_2 \quad \neg F_1 \vee F_2$$

$$\text{or } F_1 \Leftrightarrow F_2 \quad ((F_1 \Rightarrow F_2) \wedge (F_2 \Rightarrow F_1))$$

An interpretation I is a map from variables to $\{T, \perp\}$

a formula has 2^n interpretations where n is the number of variables

$I \models F$ if F evaluates to T under I
 I is a "model" of F

$I \not\models F$ if F evaluates to \perp under I

e.g. $F \triangleq (p \wedge q) \Rightarrow (p \vee \neg q)$

$I = \{p \mapsto \perp, q \mapsto \perp\}$

$$(1 \wedge 1) \Rightarrow (1 \vee \neg 1)$$

$$\perp \Rightarrow T \equiv \neg \perp \vee T \equiv T$$

Satisfiability and validity

F is satisfiable (SAT) iff there is $I \models F$

F is valid (VALID) iff for all I , $I \models F$

F is VALID iff $\neg F$ is unsatisfiable (UNSAT)

- ① Truth table
- ② Binary decision diagrams (BDDs)
- ③ SAT solvers
- ④ deductive proof

semantic argument method

$$\begin{array}{c} \text{neg} \\ \hline I \models \neg F \\ \hline I \not\models F \end{array}$$

$$\begin{array}{c} I \not\models \neg F \\ \hline I \models F \end{array}$$

$$\begin{array}{c} \text{conj} \\ \hline I \models F \wedge G \\ \hline I \models F \quad I \models G \end{array}$$

$$\begin{array}{c} I \not\models F \wedge G \\ \hline I \not\models F \text{ or } I \not\models G \end{array}$$

$$\begin{array}{c} \text{disj} \\ \hline I \models F \vee G \\ \hline I \models F \text{ or } I \models G \end{array}$$

$$\begin{array}{c} I \not\models F \vee G \\ \hline I \not\models F \quad I \not\models G \end{array}$$

$$\begin{array}{c} \neg F \vee G \\ \text{impl} \\ \hline I \models F \Rightarrow G \\ \hline I \models \neg F \text{ or } I \models G \end{array}$$

$$\begin{array}{c} I \not\models F \Rightarrow G \\ \hline I \models F \quad I \not\models G \end{array}$$

Contradiction

$$\begin{array}{c} I \models F \quad I \not\models F \\ \hline I \models \perp \end{array}$$

assume F is VALID

$$\downarrow$$
$$I \not\models F$$

$$I \not\models (p \wedge q) \implies (p \vee \neg q)$$

$$I \models p \wedge q$$

$$I \not\models p \vee \neg q$$

$$I \models p \quad I \models q$$

$$I \not\models p \quad I \not\models \neg q$$



Contradiction

no I exists therefore formula is VALID

Two formulas are equivalent $F_1 \equiv F_2$

iff for all I , $I \models F_1$ iff $I \models F_2$

$F_1 \equiv F_2$ iff $F_1 \iff F_2$ is VALID

$\perp \iff \perp$ is VALID

$T \iff T$ is VALID

Normal forms

Negation Normal form (NNF)

Disjunctive Normal form (DNF)

Conjunctive Normal form (CNF)

$$\underline{\text{NNF}} \rightarrow -(x + y) = -x - y$$

$$\begin{array}{c} \underline{\text{DNF}} \\ \downarrow \quad \downarrow \quad \downarrow \\ \boxed{(p \wedge q \wedge \neg r)} \\ \vee (\neg p \wedge q \wedge \neg r) \\ \vee, \\ \vee, \\ \vee \end{array}$$

CNF

$(p \vee q \vee r)$ clause

$\wedge (\neg p \vee q \vee r)$

$\wedge \dots$

$\wedge \dots$

eg. $r \vee (p \wedge q)$

distribute \vee over \wedge

$(r \vee p) \wedge (r \vee q)$

Tseitin's transformation

$$F \xrightarrow{\text{translate}} F' \text{ is in CNF}$$

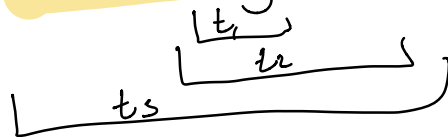
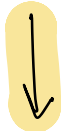
properties:

- ① if F' is UNSAT then F is UNSAT
- ② Any model of F' is a model of F if we disregard the additional variables

Inhibition

def $f(x, y, z)$

return $x + (2*y + 3)$



def $f(x, y, z)$

$t_1 = 2*y$

$t_2 = t_1 + 3$

$t_3 = t_2 + x$

return t_3

$$F \triangleq (p \wedge q) \vee (q \wedge \neg r \wedge s)$$

$\underbrace{(p \wedge q)}_{F_1} \quad \underbrace{(q \wedge \neg r \wedge s)}_{F_2}$
 $\underbrace{\qquad\qquad\qquad}_{F_3}$
 $\underbrace{\qquad\qquad\qquad}_{F_4 \triangleq F_1 \vee F_3}$

- ① for every F_i create a new variable t_i
- ② for every F_i (starting with most-deeply nested)

$$F_i' \triangleq t_i \iff (l_i \circ l_i')$$

\nearrow LHS of F_i \nearrow \wedge, \vee \nearrow RHS of F_i

$$F_1' \triangleq t_1 \iff (p \wedge q)$$

$$F_2' \triangleq t_2 \iff (q \wedge \neg r)$$

$$F_3' \triangleq t_3 \iff (t_2 \wedge s)$$

$$F_4' \triangleq t_4 \iff (t_3 \vee t_1)$$

use De Morgan's laws

$$F' = t_4 \wedge F_1' \wedge F_2' \wedge F_3' \wedge F_4'$$