Assignment 4

April 20, 2021

1 Interpolants

This question concerns the logical notion of *Craig interpolants*. You do not require prior knowledge about interpolation to answer this question.

Given two formulas A and B in first-order logic, such that $A \wedge B$ is unsatisfiable, there exists a formula I, called an interpolant, such that

1. $A \Rightarrow I$ is valid (recall that a formula ϕ is valid iff all models satisfy it)
2. $I \wedge B$ is unsatisfiable;
3. $\mathit{vars}(I) \subseteq \mathit{vars}(A) \cap \mathit{vars}(B)$, where $\mathit{vars}(\phi)$ is the set of all variables that appear in ϕ.

As an example, consider the following formulas in propositional logic (i.e., all variables are Boolean):

$$A \triangleq a \wedge b$$
$$B \triangleq \neg b \wedge c$$

We know that $A \wedge B$ is unsatisfiable. An interpolant I here is b. Observe that $A \Rightarrow I$ is valid, $I \wedge B$ is unsatisfiable, and I only contains variables that appear in A and B.

1.1

An alternative definition of an interpolant is as follows:

Suppose we have two formulas A and C such that $A \Rightarrow C$ is valid, then there exists a formula I such that

1. $A \Rightarrow I$ is valid;
2. $I \Rightarrow C$ is valid;
3. $\mathit{vars}(I) \subseteq \mathit{vars}(A) \cap \mathit{vars}(C)$.

Prove that the two definitions of an interpolant are equivalent.

1.2

Give two formulas A and B such that $A \wedge B$ is unsatisfiable, does there always exist a unique interpolant (up to logical equivalence)? If not, provide an example of two formulas A and B and two interpolants I_1 and I_2, such that $I_1 \neq I_2$.

1.3

Suppose you are working with formulas in quantifier-free linear integer arithmetic: meaning, formulas that are Boolean combinations (conjunctions, disjunctions, negations) of linear inequalities over integers of the form: $a_1x_1 + \dots a_nx_n \leq c$, where a_i, c are integer constants, and x_i are integer variables.

Consider the following two formulas in quantifier-free linear integer arithmetic:

$$A \triangleq x = 2y$$

$$B \triangleq x = 2z - 1$$

Is $A \wedge B$ satisfiable? If not, does there exist an interpolant for A and B that is also in quantifier-free linear integer arithmetic? If no such interpolant exists, explain why that is the case.

2 Galois connections

Consider these two definitions of Galois connections:

Definition 1: (L, α, γ, M) is a Galois connection between the complete lattices $(L, \sqsubseteq)$ and $(M, \sqsubseteq)$ if and only if $\alpha : L \to M$ and $\gamma : M \to L$ are monotone functions that satisfy the following conditions:

$$\text{forall } l \in L.\, \gamma(\alpha(l)) \sqsupseteq l$$

$$\text{forall } m \in M.\, \alpha(\gamma(m)) \sqsubseteq m$$

Definition 2: (L, α, γ, M) is a Galois connection between the complete lattices $(L, \sqsubseteq)$ and $(M, \sqsubseteq)$ if and only if $\alpha : L \to M$ and $\gamma : M \to L$ are total functions such that for all $l \in L, m \in M$,

$$\alpha(l) \sqsubseteq m \Leftrightarrow l \sqsubseteq \gamma(m)$$

Prove that the two definitions are equivalent.