FIRST-ORDER LOGIC

$p \wedge q$

Boolean variable

variables

constant function

$\exists y . \forall x . \; x + y > 0 \; \wedge \; x > 5$

function

relation

relation

GT(a,b)

quantifiers

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a language of FOL is $L(C, F, R)$

const func relations

Basic term : any constant $c \in C$ or variable $(x, y, z, \dots)$

Composite terms : $f(t_1, \dots, t_k)$ where $f \in F$ with arity k

terms

e.g. age(mother(y))

Formulas: F

① atomic predicate: $p(t_1 \dots t_n)$

$p \in R$ where arity of p is n

e.g. older(x, y)

② $F_1 \wedge F_2$, $\neg F_1$, $F_1 \vee F_2$

③ given variable x, $\exists x. F$

$\forall x. \underbrace{F}_{\text{scope of } x}$

E.g. $\forall y. ((\forall x. p(x)) \Longrightarrow q(x, y))$

unbound/free

A formula is closed iff it doesn't have free variables

sentence

A formula with no variables is ground

$0 > 1$

E.g. $\forall x. \exists y. \text{friend}(x, y)$

$\exists x. \forall y. \text{friend}(x, y)$

Semantics

① Universe $\mathcal{U}$: a non-empty set of objects

e.g. $\mathcal{U} = \mathbb{Z}$

$\mathcal{U} = \emptyset$ ← has to be non-empty

$\mathcal{U} = \{ \square \}$

② An interpretation I is mapping from C, F, R to objects from $\mathcal{U}$

I maps $c \in C$ to $\mathcal{U}$, i.e., $I(c) \in \mathcal{U}$

I maps $f \in F$ to $I(f) \in \mathcal{U}^n \to \mathcal{U}$

where n is the arity of f

I maps $p \in R$ to $I(p) \subseteq \mathcal{U}^n$

n is the arity of p

E.g. $C = \{a, b, c\}$

f (unary)

r (ternary)

Universe $U = \{1, 2, 3\}$

Interp I s.t.

$I(a) = 1$, $I(b) = 2$, $I(c) = 3$

$I(f) = \{1 \to 2, 2 \to 3, 3 \to 1\}$

$I(r) = \{(1,2,3), (3,2,1)\}$

structure $S = (U, I)$

U: universe

I: interp

A variable assignment σ is mapping from variables to elements of $\mathcal{U}$

A FOL formula F is SAT iff there exist a structure S and assignment σ s.t.

$$S, \sigma \models F$$

E.g. $\forall x. \exists y. p(x, y)$

$\mathcal{U} = \{0\},\ I(p) = \{(0,0)\}$

$U = \{0\},\ I(p) = \{\ \}$

$(U, I) \not\models \forall x. \exists y. p(x, y)$

$$\forall x. \left(p(x,x) \implies \exists y. p(x,y) \right)$$

consider any object $o \in \mathcal{U}$

if $p(o,o)$ is false

if $p(o,o)$ is true then $\exists \underset{\substack{\uparrow \\ o}}{y}. p(o,y)$

FOL Theories

a theory T

① Σ_T : C_T, F_T, R_T (signature)

② Axioms A_T : A set FOL sentences over Σ_T

e.g.

signature $\Sigma_T = \{ \text{taller} \}$ $\boxed{\begin{array}{l} C = \emptyset \\ F = \emptyset \\ R = \{\text{taller}\} \end{array}}$

axiom : $\forall x, y. (\text{taller}(x, y) \Rightarrow \neg \text{taller}(y, x))$

$U = \{ A, B \}$

$I(\text{taller}) = \{ (A, B) \}$

$(B, A) \times$

(U, I) is a T-model

$T_=$ theory of equality

$$\Sigma_= = \{ = , \ldots \}$$

prediate

Axioms $A_= =$

$$\forall x.\ x = x$$

$$\forall x, y.\ x = y \Rightarrow y = x$$

$$\forall x, y, z.\ x = y \land y = z \Rightarrow x = z$$

e.g. $U = \{\circ, \bullet\}$

$$I(=) = \{(\circ,\circ), (\bullet,\bullet),$$
$$\underbrace{(\bullet,\circ)}, \underbrace{(\circ,\bullet)}\}$$

Theory of Peano Arithmetic

T_{PA} $\qquad \Sigma_{PA} = \{0, 1, +, \cdot, =\}$

Axioms

$\forall x.\ \neg(x+1 = 0)$

$\forall x.\ x + 0 = x$

$\forall x, y.\ x+1 = y+1 \Rightarrow x = y$

$\forall x, y.\ x + (y+1) = (x+y)+1$

$\forall x, y,\ x \cdot (y+1) = x \cdot y + x$

Presburger Arithmetic (T_{PA} without multiplication)

Theory of arrays

$$a[i]$$

$$a\langle i \lhd v\rangle$$

E.g. $\underbrace{a\langle 2 \lhd 5\rangle}[2] = 5$

$\forall a, i, j \quad i = j \implies a[i] = a[j]$

(congruence)

$\forall a, v, i, j \quad i = j \implies a\langle i \lhd v\rangle[j] = v$

$\forall a, v, i, j \quad i \neq j \implies a\langle i \lhd v\rangle[j] = a[j]$