$$
\left.\begin{aligned}
x_1 &= f_1(x_1, \ldots, x_n) \\
&\vdots \\
x_n &= f_n(x_1, \ldots, x_n)
\end{aligned}\right\}
$$

each x_i is the set of reachable states at location/line/node

---

concrete/collecting semantics

if, while, $x := E$, var x, y, z;

expression

if E then ... else ...

integer

concrete state $\quad CStates = Vars \rightarrow \mathbb{Z}$

$$[\![l_i]\!] \subseteq CStates$$

1: $x := 10$

2: $y := 7$

ℓ_1: is the set of states after executing line 1

$\{[x \mapsto 10, y \mapsto c] \mid c \in \mathbb{Z}\}$

Evaluating expressions

$$\text{ceval} : \text{CStates} \times E \to 2^{\mathbb{Z}}$$

$$\text{ceval}(s, x) = \{s(x)\}$$

$$\text{ceval}(s, c) = \{c\}$$

integer

$$\text{ceval}(s, \text{input}) = \mathbb{Z}$$

$$\text{ceval}(s, E_1 \text{ op } E_2) = \{v_1 \text{ op } v_2 \mid v_1 \in \text{ceval}(s, E_1) \wedge v_2 \in \text{ceval}(s, E_2)\}$$

$$\text{ceval} : 2^{\text{CStates}} \times E \to 2^{\mathbb{Z}}$$

$$\text{ceval}(S, E) = \bigcup_{s \in S} \text{ceval}(s, E)$$

$CSUCC : 2^{cstates} \times N \to 2^N$

locations in the program

$N : \{l_1, \dots, l_n\}$

next line after l_i

$CSUCC(S, l_i) = \{l_{i+1}\}$

for assignment/declaration

$CSUCC(S, l_i) = \{l_{then}, l_{else}\}$

l_i is an if statement with condition E

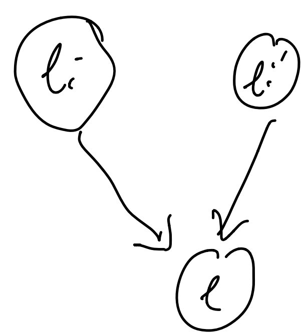
$$CJOIN(l) = \bigcup_{\substack{l_i \in N \\ l \in CSUCC([\![l_i]\!], l_i)}} [\![l_i]\!]$$

set of states reachable at l_i

$[\![l_i]\!] = \{[x \mapsto 0]\}$

$[\![l_i']\!] = \{[x \mapsto 1]\}$

```
if ...
    x := ...  ⟶ l_i
else
    x := ...  ⟶ l_i'
l:
```



$CJOIN(\ell) = \{[x \mapsto 0], [x \mapsto 1]\}$

$[\![\ell_i]\!] = \{ s[x \mapsto ceval(s,E)] \mid s \in CJOIN(\ell_i) \}$

where

ℓ_i is $x := E$

$[\![\ell_i]\!] : 2^{\underbrace{CStates}_{\text{infinite}}}$ powerset lattice

Scott-continuous

$f : L_1 \to L_2$ is continuous

if $f(\bigsqcup A) = \bigsqcup_{a \in A} f(a)$

if f is continuous, then it is monotone

```
var a, b, c

1: a = 42
2: b = 87
3: if (input)
4:     c = a + b
   else
5:     c = a - b
6: skip
```

$$[\![\ell_1]\!] = \{ [a \mapsto 42, b \mapsto c_1, c \mapsto c_2] \mid c_1, c_2 \in \mathbb{Z} \}$$

$$[\![\ell_2]\!] = \{ [a \mapsto 42, b \mapsto 87, c \mapsto c_2] \mid c_2 \in \mathbb{Z} \}$$

$$[\![\ell_6]\!] = \{ [a \mapsto 42, b \mapsto 87, c \mapsto 129], \; [a \mapsto '', b \mapsto '', c \mapsto -45] \}$$

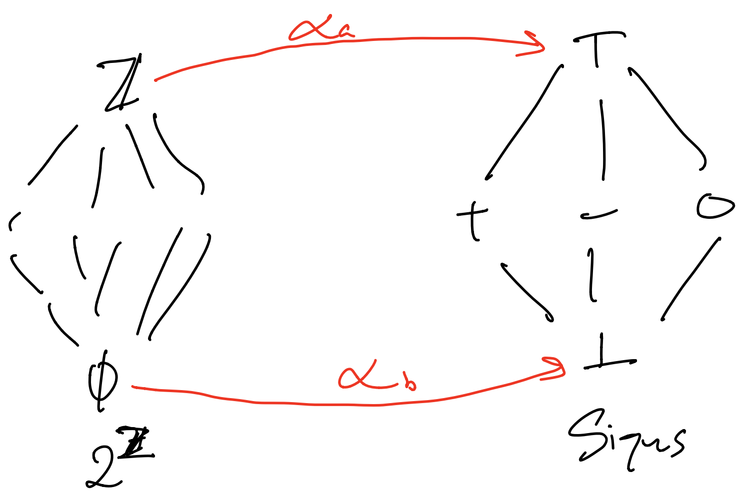
Abstraction

$Signs = \{+, -, \top, \bot, \emptyset\}$

$\alpha_a : 2^{\mathbb{Z}} \longrightarrow Signs$

E.g. $\alpha_a(\{1, 2\}) = +$

$\alpha_a(\mathbb{Z}) = \top$



$$\alpha_a(D) = \begin{cases} \bot & \text{if } D = \emptyset \\ 0 & \text{if } D = \{0\} \\ - & \text{if } D \text{ contains only neg int} \\ + & \text{" " " " pos "} \\ \top & \text{o/w} \end{cases}$$

$\alpha_b : 2^{CStates} \longrightarrow AState$

map lattice
Vars $\longrightarrow$ Signs

e.g. $\alpha_b(\{[x \mapsto 10, y \mapsto 0],$
$[x \mapsto 11, y \mapsto -10]\})$
$= [x \mapsto +, y \mapsto \top]$

$\alpha_b(R) = \sigma$ where $\sigma(x) = \alpha_a(\{s(x) \mid s \in R\})$
for all $x \in Vars$

set of conc states

a, A

$\alpha_c : \left(2^{CState}\right)^n \longrightarrow \widehat{AState}$

reachable states at every line in program

Concretization functions

$\gamma_a : Sign \longrightarrow 2^{\mathbb{Z}}$

$$\gamma_a(a) = \begin{cases} \emptyset & \text{if } a = \bot \\ \{1,2,3,\cdots\} & \text{if } a = + \\ \{-1,-2,\cdots\} & \text{if } a = -1 \\ \{0\} & \text{if } a = 0 \\ \mathbb{Z} & \text{if } a = \top \end{cases}$$

$\gamma_b : AState \longrightarrow 2^{CState}$

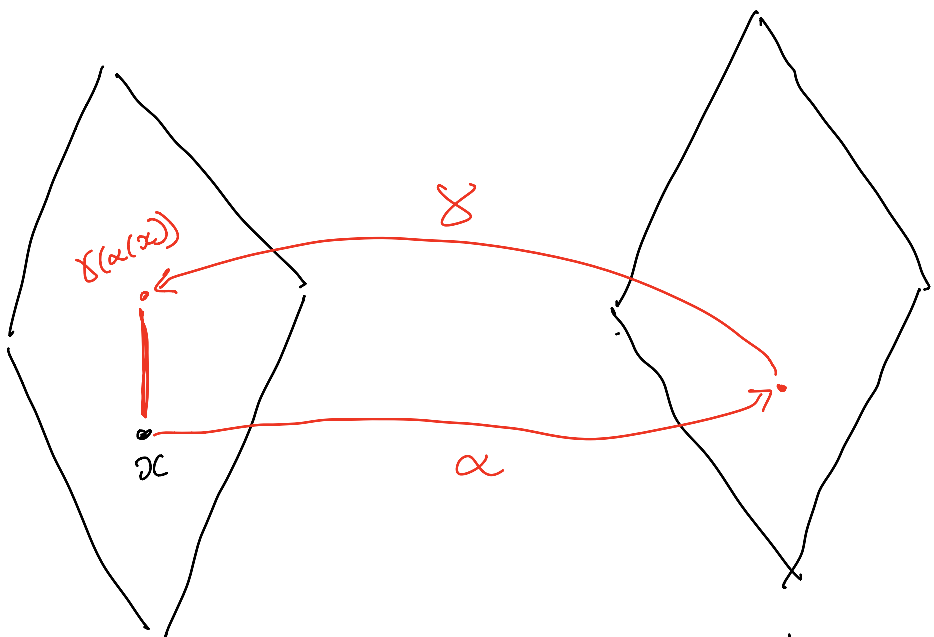
Galois connection

L_1 L_2 $\quad L_1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} L_2$

properties

① $x \sqsubseteq \gamma(\alpha(x))$ for all $x \in L_1$

② $\alpha(\gamma(y)) \sqsubseteq y$ for all $y \in L_2$

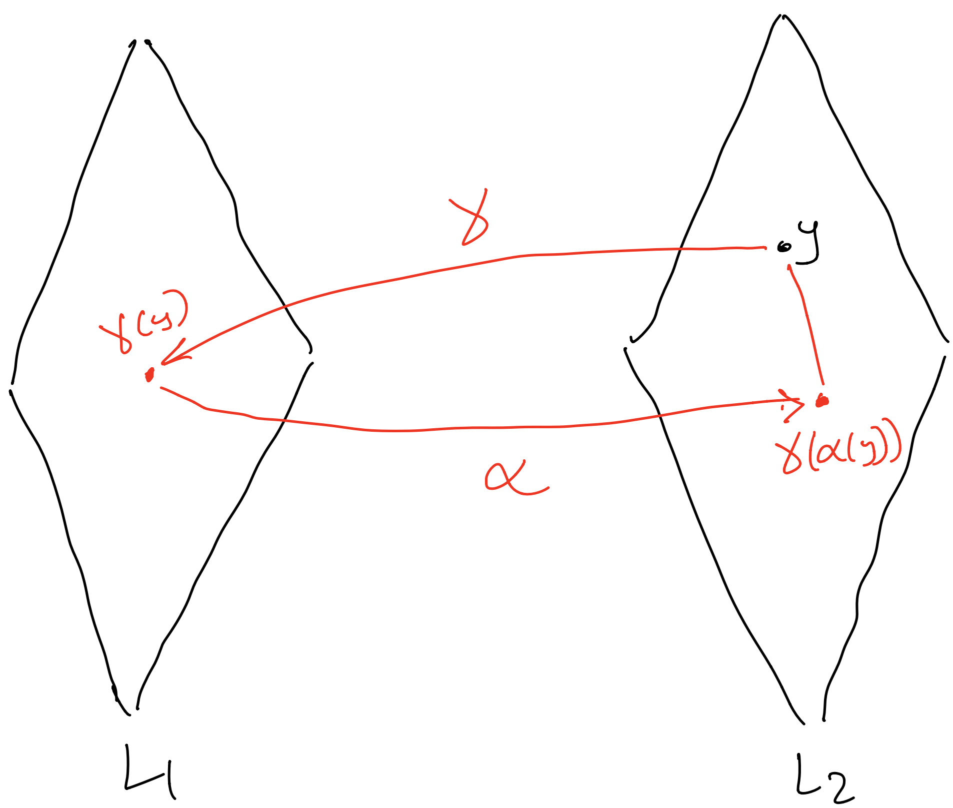


L_1 $\quad L_2$

2^{CState} $\quad$ AState

$\alpha(\{1,2\}) = +$

$\gamma(+) = \{1,2,3,\ldots\}$



γ
γ(y)
y
α
γ(α(y))
L1
L2

when we have a Galois connection

$$\alpha(\bot) = \bot$$

$$\gamma(\top) = \top$$

---

① soundness

② designing abstract transformers

f