LAST TIME

1. Review lattices
2. solving equations over lattices

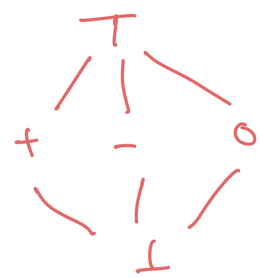
$x = y$
$y = z + 1$
$z = 10$

program analysis $\longrightarrow$ equations over lattices

a lattice $L, \sqsubseteq$

$Signs = \{\top, \bot, 0, +, -\}$

$+ \sqsubseteq \top$ $\quad$ $\bot \sqsubseteq +$
$- \sqsubseteq \top$ $\quad$ $\bot \sqsubseteq -$
$0 \sqsubseteq \top$ $\quad$ $\bot \sqsubseteq 0$



$\sqcup$ LUB/JOIN
$\sqcap$ GLB/MEET

$+ \sqcup 0 = \top$ $\qquad$ $\top \sqcap + = +$

$\bot \sqcup 0 = 0$ $\qquad$ $+ \sqcap - = \bot$

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powerset lattice

A = set of elements

$2^A, \subseteq$ $\qquad$ $\top = A$

subset $\qquad$ $\bot = \emptyset$

map lattice

set A $\qquad$ all functions from $A \rightarrow L$

lattice L

e.g. $A = Vars = \{x, y\}$

$L = Signs$

$A \rightarrow L$

$[x \mapsto \top, y \mapsto -] \sqsubseteq [x \mapsto \top, y \mapsto \top]$

Equations

1: var a, b; $\longrightarrow x_1 = [a \mapsto \top, b \mapsto \top]$

2: $a = 42$ $\longrightarrow x_2 = [a \mapsto +, b \mapsto \top]$

$x_i \in Vars \longrightarrow Signs$

3: $b = a + input$ $\longrightarrow x_3 = [a \mapsto +, b \mapsto \top]$

4: $a = a - b$ $\longrightarrow x_4 = [a \mapsto \top, b \mapsto \top]$

$$x_1 = [a \mapsto \top, b \mapsto \top]$$

$$x_2 = x_1[a \mapsto +]$$ substitution

$$x_3 = x_2[b \mapsto x_2(a) \hat{+} \top]$$

$$x_4 = x_3[a \mapsto x_3(a) \hat{-} x_3(b)]$$

if $x > 10$

$x \mapsto 0$

$x \mapsto \bot$

$+$	$\hat{+}$	$-$	$=$	$\top$
$+$	$\hat{+}$	0	$=$	$+$
$+$	$\hat{+}$	$\top$	$=$	$\top$
$+$	$\hat{+}$	$\bot$	$=$	$\bot$

$\vdots$

System of equations

$$x_1 = f_1(x_1, \dots, x_4)$$

$$x_2 = f_2(x_1, \dots, x_4)$$

$$x_3 = \dots$$

$$x_4 = \dots$$

$$f_1(x_1 \dots x_4) = [a \mapsto \top, b \mapsto \top]$$

A function $f : L_1 \to L_2$ is monotone if

$\forall x, y \in L_1$. if $x \sqsubseteq y$ then $f(x) \sqsubseteq f(y)$

any constant function is monotone

because $f(x) \sqsubseteq f(y)$ forall x, y.

$\sqcap, \sqcup$ are monotone

$\sqcap, \sqcup : L \times L \to L$

generally $f : L_1 \times L_2 \to L_3$ is monotone iff

$\forall x_1, y_1 \in L_1, x_2 \in L_2$

$x_1 \sqsubseteq y_1 \implies f(x_1, x_2) \sqsubseteq f(y_1, x_2)$

and same for 2^{nd} parameter

$$\boxed{\begin{aligned} x_1 &= f_1(x_1 \dots x_n) \\ x_2 &= f_2(x_1 \dots x_n) \\ &\vdots \\ x_n &= f_n(x_1 \dots x_n) \end{aligned}}$$

x_i are variables of some lattice L

f_i are elements of $L^n \to L$

$$\underset{\substack{\uparrow \\ (x_1, \dots, x_n) \\ \in L^n}}{x} = \underbrace{\left(f_1(x_1 \dots x_n), f_2(x_1, \dots, x_n), \dots, f_n(x_1 \dots x_n)\right)}_{f}$$

$x = f(x)$

GOAL: find the least fixpoint

SOLVE LFP $x = f(x)$

1) $x \in L^n$

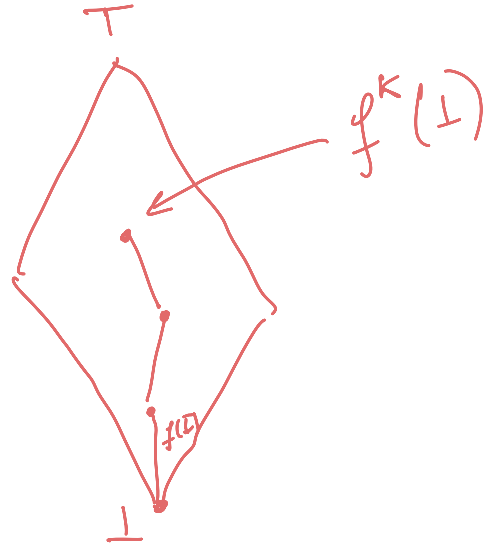
2) f is monotone

Kleene's fixed point theorem

In a lattice L of finite height, and monotone $f : L \to L$

f has a unique LFP

$$fix(f) = \bigsqcup_{i \geq 0} f^i(\bot)$$



Proof

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq \dots$$

Since L has finite height

$$\exists k.\ f^K(\bot) = f^{K+1}(\bot)$$

$$fix(f) = f^K(\bot)$$

Assume $x = f(x)$

for some value of x

$$\bot \sqsubseteq x$$

$$f(\bot) \sqsubseteq f(x) \qquad \text{because } f \text{ is monotone}$$

$$= x$$

$$fix(f) = f^K(\bot) \sqsubseteq x$$

Therefore $f^K(\bot)$ is a least fixpoint of f

Fixpoint algorithm (f)

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x = ⊥
while x ≠ f(x)
        x = f(x)
return x
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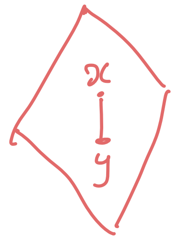
$x_1 \sqsupseteq f_1(x_1 \ldots x_n)$

$\vdots$

$x_n \sqsupseteq f_n(x_1 \ldots x_n)$



$x \sqsupseteq y$ iff $x = x \sqcup y$

$f' = x_1 \sqcup f_1(x_1 \ldots x_n)$

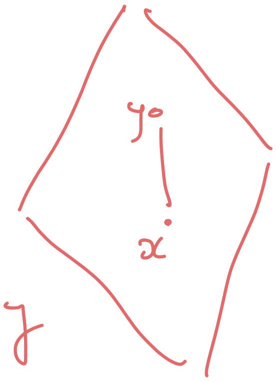
monotone

$x_1 = x_1 \sqcup f_1(x_1 \ldots x_n)$

$\vdots$

$x_n = x_n \sqcup f_n(x_1 \ldots x_n)$

$x \sqsubseteq f(x)$



FACT $x \sqsubseteq y$ iff $x = x \sqcap y$

$x = x \sqcap f(x)$

Galois connections