Simply typed λ calculus (STLC)

$T ::= \text{Bool} \mid T \to T$

Bool , $\text{Bool} \to \text{Bool}$, $\text{Bool} \to (\text{Bool} \to \text{Bool})$, ...

a λ calculus term

int foo (bool x)

$t ::= x$ variable

$\mid \lambda x : T .\ t$ abstraction

$\mid t\ t$ application

operational semantics

a value is either a Boolean or $(\lambda x \ldots)$

① $(\lambda x : T .\ t)\ v \longrightarrow [x \to v]\ t$

② $$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2}$$

③ $$\frac{t_2 \longrightarrow t_2'}{v\ t_2 \longrightarrow v\ t_2'}$$

$$\frac{t \in \text{Nat}}{\text{isZero}\ t \in \text{Bool}}$$

Typing rule

$\in$:

Intuition

$\underbrace{x \in Int}_{\text{typing context}} \vdash \underbrace{x + x} \in Int$

a typing context Γ is a sequence of the form

$$x_1 \in T_1 \ , \ x_2 \in T_2 \ , \ x_3 \in T_3 \ , \ \ldots$$

Γ can be empty (denoted $\emptyset$)

$\Gamma, x \in T$

typing context

informal

$$\frac{x \in T_1 \vdash t \in T_2}{\lambda x{:}T_1 .\, t \in T_1 \to T_2} \; T_{ABS}$$

$$\frac{\Gamma, x \in T_1 \vdash t \in T_2}{\Gamma \vdash \lambda x{:}T_1 .\, t \in T_1 \to T_2} \; T_{ABS}$$

turnstile

vdash

$$\frac{\Gamma \vdash t_1 \in T' \to T \qquad \Gamma \vdash t_2 \in T'}{\Gamma \vdash t_1\ t_2 \in T} \; T_{APP}$$

$$\underset{\underset{T' \quad\ T}{int \to Bool}}{iszero} \quad \underset{\underset{T'}{int}}{0} \quad \in \quad \underset{T}{Bool}$$

$x \in T$ should be inside of sequence Γ

$$\frac{x \in T \in \Gamma}{\Gamma \vdash x \in T}\ T_{VAR} \qquad \frac{\Gamma' = \Gamma, x \in T}{\Gamma' \vdash x \in T}$$

$$\frac{\Gamma \vdash t_1 \in Bool \qquad \Gamma \vdash t_2 \in T \qquad \Gamma \vdash t_3 \in T}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in T}\ T_{IF}$$

$$\frac{\qquad}{\text{if true then } (\lambda x:Bool.\ x) \text{ else } (\lambda x:Bool.\ \text{not } x) \in}$$

$$Bool \rightarrow Bool$$

E.g.

$$\cfrac{\cfrac{\cfrac{\overset{BC}{\qquad}}{x \in Bool \vdash x \in Bool}\ T_{VAR}}{\emptyset \vdash \lambda x:Bool.\ x \in Bool \rightarrow Bool}\ T_{ABS} \qquad \cfrac{\overset{BC}{\qquad}}{\emptyset \vdash tru \in Bool}}{\emptyset \vdash \underbrace{(\lambda x:Bool.\ x)}\ \underbrace{tru} \in Bool}\ T_{APP}$$

Inversion lemma

- if $\Gamma \vdash x \in T$ then $x \in T \in \Gamma$
- if $\Gamma \vdash \lambda x{:}T.\, t \in R$ then $R = T \to R'$ for some R' s.t. $\Gamma, x \in T \vdash t \in R'$

⋮

Uniqueness

In a given Γ, a term t with free variables in Γ has at most a single type

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Canonical forms

1. if v has type Bool then v is either tru/fls
2. if v has type $T_1 \to T_2$ then v is of the form $\lambda x : T_1 .\, t$

PRESERVATION

If $\Gamma \vdash t \in T$ and $t \to t'$
then $\Gamma \vdash t' \in T$

PROGRESS

Suppose $\emptyset \vdash t \in T$

Then t is a value or $t \longrightarrow t'$ for some t'

PROOF Induction on typing derivation

assume that t is of the form $t_1 \ t_2$

By inductive hypothesis B t_1 is a value or
A $t_1 \longrightarrow t_1'$

CASE A if $t_1 \longrightarrow t_1'$ $\quad t \longrightarrow t_1' \ t_2$

CASE B if t_1 is a value

① t_2 can take a step $t_2 \longrightarrow t_2'$

$$\underbrace{t_1 \ t_2}_{t} \longrightarrow t_1 \ t_2'$$

② t_1 and t_2 are both values

$$t_1 \ t_2$$

By canonical forms $t_1 = \lambda x : \ldots$

this means

$$t \longrightarrow t'$$

through β-reduction

Erasure theorem

$$\text{erase}(t) = t'$$

STLC $\quad$ λ calc (untyped)

$$\text{erase}(x) = x$$
$$\text{erase}(t_1\ t_2) = \text{erase}(t_1)\ \text{erase}(t_2)$$
$$\text{erase}(\lambda x{:}T.\ t) = \lambda x.\ \text{erase}(t)$$

Thm

- If $t \longrightarrow t'$ under typed evaluation then $\text{erase}(t) \longrightarrow \text{erase}(t')$ under untyped evaluation
- if $\text{erase}(t) \longrightarrow m'$ then there is a typed term t' s.t. $t \longrightarrow t'$ and $\text{erase}(t') = m'$