Simply typed λ calculus

if true then 0 else false

× type systems are conservative

$T ::= Bool \mid T \to T$

E.g. $Bool \to Bool$

$Bool \to (Bool \to Bool)$

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int f (int x)

Intuition

arguments have types

Typed λ calc

$t = x$

$\mid \lambda x : T . \; t$

$\mid t \; t$

a value is either a Boolean or $(\lambda x \ldots)$

$(\lambda x : T . \; t) \; v \to [x \mapsto v] t$

$$\frac{t_1 \to t_1'}{t_1 \; t_2 \to t_1' \; t_2}$$

$$\frac{t_2 \longrightarrow t_2'}{v\ t_2 \longrightarrow v\ t_2'}$$

(Recall)

$$\frac{t \in Nat}{isZero\ t \in Bool}\ T_{iszero}$$

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typing context

$$\frac{x \in T_1 \vdash t \in T_2}{\lambda x{:}T_1 .\ t \ \in\ T_1 \rightarrow T_2}\ T_{ABS}$$

if x has type T_1 then I can prove $t \in T_2$

Intuition

$$x \in Int \vdash x + x \in Int$$

a typing context Γ is a sequence of the form $x_1 \in T_1, \ldots, x_n \in T_n$

Γ can be $\emptyset$ (empty)

$$\underbrace{\Gamma,}\ \underbrace{x_1 \in T_1}$$

T_{ABS}

$$\frac{\Gamma, x \in T_1 \quad \vdash \quad t \in T_2}{\Gamma \vdash \lambda x : T_1 . t \ \in T_1 \to T_2}$$

x is no longer free

T_{APP}

$$\frac{\Gamma \vdash t_1 \in T_1 \to T \qquad \Gamma \vdash t_2 \in T_1}{\Gamma \vdash t_1 \ t_2 : T}$$

T_{var}

$$\frac{x : T \in \Gamma}{\Gamma \vdash x \in T}$$

the typing context is a set

T_{IF}

$$\frac{\Gamma \vdash t_1 \in Bool \quad \Gamma \vdash t_2 \in T \quad \Gamma \vdash t_3 \in T}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in T}$$

if true the $(\lambda x : Bool . x)$ else $(\lambda x : Bool . \text{not } x)$

$\in (Bool \to Bool)$

$$x \in Bool \equiv x : Bool$$

typing context

$$\dfrac{\dfrac{\dfrac{x : Bool \in \{x \in Bool\}}{x : Bool \vdash x \in Bool}\,T_{VAR}}{\emptyset \vdash \lambda x : Bool . x \in Bool \to Bool}\,T_{ABS} \qquad \dfrac{}{\emptyset \vdash true \in Bool}}{\emptyset \vdash (\lambda x : Bool . x)\ true \in Bool}\,T_{APP}$$

Inversion lemma (Just reading rules upwards)

If $\Gamma \vdash x \in T$ then $x \in T \in \Gamma$

If $\Gamma \vdash \lambda x{:}T.t \in R$ then

$R = T \to R_2$ for some R_2

and $\Gamma, x{:}T \vdash t \in R_2$

$\vdots$

Uniqueness theorem

In a given typing context Γ, a term t, with free variables in Γ, has at most one type

$$\overbrace{x \in Bool}^{\Gamma} \vdash x \in Bool$$

$$\overbrace{x \in Bool \to Bool}^{\Gamma'} \vdash x \in Bool \to Bool$$

Canonical forms lemma

① if v has type Bool, then v is either true/false

② if v has type $T_1 \to T_2$ then

$$v = \lambda x : T_1 . t$$

PROGRESS THM

Suppose $\emptyset \vdash t \in T$

Then t is a value or $t \to t'$

PROOF Induction on typing derivation

The only interesting case T_{App}

imagine t is of form $t_1 \ t_2$

By hypothesis t_1 is a value or $t_1 \to t_1'$

case A – if $t_1 \to t_1'$ then $t \to t_1' \ t_2$

case B – if t_1 is a value

Case B1

t_2 can take a step

$t \longrightarrow t_1\, t_2'$

Case B2 t_1 and t_2 are values

By canonical forms

$t_1 = (\lambda x : t_1')$

this means

$$t \longrightarrow t' = [x \mapsto t_2]\, t_1'$$

PRESERVATION

If $\Gamma \vdash t \in T$ and

$t \longrightarrow t'$

then

$\Gamma \vdash t' \in T$

Erasure theorem

$$\text{erase}(x) = x$$
$$\text{erase}(\lambda x{:}T.\,t) = \lambda x.\,t$$
$$\text{erase}(t_1\ \ t_2) = \text{erase}(t_1)\ \text{erase}(t_2)$$

Thm

- If $t \to t'$ under typed evaluation then $\text{erase}(t) \longrightarrow \text{erase}(t')$

- If $\text{erase}(t) \longrightarrow m'$ then there is a typed term t' s.t. $t \to t'$ and $\text{erase}(t') = m'$