LAST TIME

$$x_1 = f_1(x_1, \ldots, x_n)$$
$$x_2 = f_2(\ldots)$$
$$\vdots$$
$$x_n = f_n(\ldots)$$

x_i is the set of reachable states at location/line in program

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concrete semantics / collecting semantics

PL: if, while, $x := E$, var x, y, z

expression

if E then ... else ...

integer

Concrete state $\quad CStates = Var \rightarrow \mathbb{Z}$

$[\![l_i]\!] \subseteq CStates$

E.g.:

1: $x := 10$
2: $y := 7$

$[\![l_1]\!] = \{ [x \mapsto 10, y \mapsto c] \mid c \in \mathbb{Z} \}$

Evaluating expressions

$$ceval : CStates \times E \rightarrow 2^{\mathbb{Z}}$$

$$ceval(s, x) = \{s(x)\}$$
$$ceval(s, c) = \{c\}$$
$$ceval(s, input) = \mathbb{Z}$$
$$ceval(s, E_1 \text{ op } E_2) = \{v_1 \text{ op } v_2 \mid v_1 \in ceval(s, E_1)$$

$\wedge \; v_2 \in ceval(S, \mathcal{E}_2) \}$

$$ceval : 2^{CStates} \times E \longrightarrow 2^{\mathbb{Z}}$$

$$ceval(S, E) = \bigcup_{s \in S} ceval(s, E)$$

set of states

$$CSUCC : 2^{CStates} \times N \longrightarrow 2^{N}$$

$N = \{ \ell_1, \dots \ell_n \}$

set of all program locations

$$csucc(S, \ell_i) = \{ \ell_{i+1} \}$$

when ℓ_i is an assignment/declaration

$$csucc(S, \ell_i) = \{ \ell_{then}, \ell_{else} \}$$

if stmt

ℓ_i is an if stmt with condition E

$$\begin{tikzcd} \ell_i \arrow[dr] & & \ell_j \arrow[dl] \\ & \ell & \end{tikzcd}$$

$$CJOIN(\ell) = \bigcup_{\substack{\ell_i \in N \\ \ell \in csucc([\![\ell_i]\!], \ell_i)}} [\![\ell_i]\!]$$

e.g.:

if ...

ℓ_i: x = ... $\longrightarrow$ $[\![\ell_i]\!] = \{ [x \mapsto 0] \}$

else

ℓ_j: x = ... $\quad [\![\ell_j]\!] = \{ [x \mapsto 1] \}$

ℓ: $\quad [\![\ell]\!] = \{ [x \mapsto 0], [x \mapsto 1] \}$

$$[\![l_i]\!] = \left\{ s[x \mapsto c \quad] \;\middle|\; \begin{array}{l} s \in CJOIN(l_i) \\ c \in ceval(s, E) \end{array} \right\}$$

l_i is $x := E$

What we now have is a definition of $[\![l_i]\!] \subseteq 2^{cstates}$

Powerset lattices

Scott continuous

$f : L_1 \to L_2$ is continuous

if $f(\bigsqcup A) = \bigsqcup_{a \in A} f(a)$

if f is continuous, then it is monotone

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e.g. var a, b, c

1: $a = 42$ $\xrightarrow{[\![l_1]\!]}$ $\{[a \mapsto 42, b \mapsto c, c \mapsto c'] \mid c, c' \in \mathbb{Z}\}$

2: $b = 87$

3: if (input)

4: $c = a + b$ $\longrightarrow$ $\{[a \mapsto 42, b \mapsto 87, c \mapsto 129]\}$

else

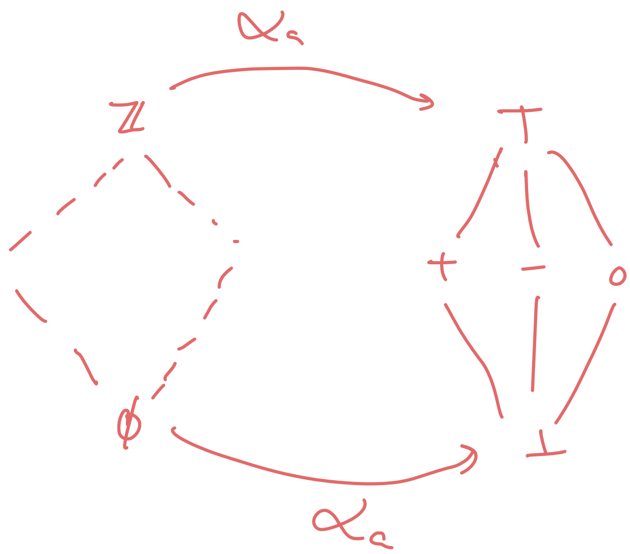
5: $c = a - b$ $\longrightarrow$ $\{[a \mapsto 42, b \mapsto 87, c \mapsto -45]\}$

6:

Abstraction

$$\text{Signs} = \{+, -, \top, \bot, 0\}$$

$$\alpha_a : \underline{2^{\mathbb{Z}}} \longrightarrow \text{Signs}$$



e.g. $\alpha_a(\{1,2\}) = +$

$\alpha_a(\mathbb{Z}) = \top$

$\alpha_a(\emptyset) = \bot$

$\alpha_a(\{1,-2\}) = \top$

$D \in 2^{\mathbb{Z}}$

$$\alpha_a(D) = \begin{cases} \bot & \text{if } D = \emptyset \\ 0 & \text{if } D = \{0\} \\ + & \text{if } D \subseteq \mathbb{Z}^+ \\ - & \text{if. } D \subseteq \mathbb{Z}^- \\ \top & \text{o/w} \end{cases}$$

$$\alpha_b : 2^{\text{CStates}} \longrightarrow \text{AState}$$

map lattice $\text{Vars} \longrightarrow \text{Signs}$

e.g. $\alpha_b(\{[x \mapsto 10, y \mapsto 0], [x \mapsto 11, y \mapsto -16]\})$

$= [x \mapsto \underline{\quad + \quad}, \; y \mapsto \underline{\quad \top \quad}]$

$\alpha_b(R) = \sigma$ where $\sigma(x) = \alpha_a(\{s(x) \mid s \in R\})$ for all $x \in \text{Vars}$

set of concrete states

$$\alpha_c : \left(2^{CState}\right)^n \longrightarrow AState^n$$

\# of locations

Concretization

$$\gamma_a : Signs \longrightarrow 2^{\mathbb{Z}}$$

$$\gamma_a(a) = \begin{cases} \phi & \text{if } a = \bot \\ \{0\} & \text{if } a = 0 \\ \mathbb{Z} & \text{if } a = \top \\ \mathbb{Z}^+ & \text{if } a = + \\ \mathbb{Z}^- & \text{if } a = -1 \end{cases}$$

γ_b , γ_c

$$\gamma_b : AState \longrightarrow 2^{CState}$$

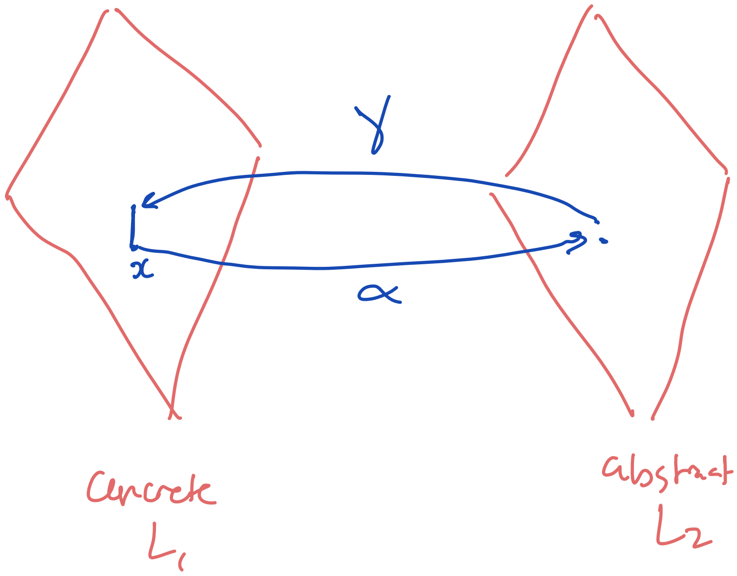
Galois connection

$L_1 \quad L_2$

$$L_1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} L_2$$

properties:

1. $x \sqsubseteq \gamma(\alpha(x))$ for all $x \in L_1$
2. $\alpha(\gamma(y)) \sqsubseteq y$ for all $y \in L_2$



2^{CState} $\qquad\qquad$ $AState$

$$x = \{1,2\} \xrightarrow{\alpha(x)} x = +$$

$$x = \{1,2,3,\dots\} \xleftarrow{\gamma(x)}$$

when $L_1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} L_2$

we know $\alpha(\bot_1) = \bot_2$

$\alpha(\top_1) = \top_2$

interval domain

$+$ $\qquad$ $\hat{+}$ $\qquad$ $x \longmapsto [a,b]$