1. $x = y$
   $y = z$
   $z = 10$

   Solving equations over lattices

2. Reducing program analysis to solving such equations

   ↳ what is reachable at different program locations

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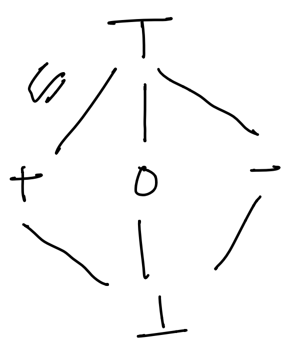
a lattice $L, \sqsubseteq$

$$Signs = \{\top, \bot, 0, +, -\}$$



$\sqcup$ LUB (join)

$\sqcap$ GLB (meet)

$$+ \sqcup 0 = \top \qquad \top \sqcap + = +$$

$$\bot \sqcup 0 = 0 \qquad + \sqcap - = \bot$$

powerset lattice

take any set of elements A

$$(2^A, \subseteq) \qquad \top = A$$
$$\bot = \emptyset$$

map lattice

set A

lattice L

$$A \to L = \left\{ [a_1 \mapsto x_1, \dots, a_n \mapsto x_n] \;\middle|\; \begin{array}{l} A = \{a_1 \dots a_n\} \\ x_i \in L \end{array} \right\}$$

set $A = Vars = \{x, y\}$

lattice $L = Signs$

$$A \to L$$

e.g. $[x \mapsto \top, y \mapsto -] \sqsubseteq [x \mapsto \top, y \mapsto \top]$

Equations

1/ var a, b;
$\rightsquigarrow x_1$
2/ a = 42;
$\rightsquigarrow x_2$
3/ b = a + input;
$\rightsquigarrow x_3$
4/ a = a - b
$\rightsquigarrow x_4$

$x_i \in Vars \to Signs$

$x_1 = f_1(x_1, \dots, x_4)$

where

$f_1(x_1, \dots, x_4) =$
$[a \mapsto \top, b \mapsto \top]$

$x_2 = f_2(x_1, \dots, x_4)$

$\vdots$

$x_1 = [a \mapsto \top, b \mapsto \top]$

$x_2 = x_1[a \mapsto 42]$

substitution

$x_3 = x_2[b \mapsto x_2(a) \hat{+} \top]$

input from user

$x_4 = x_3[a \mapsto x_3(a) \hat{-} x_3(b)]$

$+$	$\hat{+}$	$-$	$=$	$\top$
$+$	$\hat{+}$	$+$	$=$	$+$
$-$	$\hat{+}$	$-$	$=$	$-$
$\top$	$\hat{+}$	$\cdot$	$=$	$\top$
$\perp$	$\hat{+}$	$\cdot$	$=$	$\perp$

A function $f : L_1 \to L_2$ is <u>monotone</u>

when $\forall x, y .\ x \sqsubseteq y \Longrightarrow f(x) \sqsubseteq f(y)$

---

any constant function is monotone because

$f(x) \sqsubseteq f(y)$ for all x, y.

$\sqcup$ and $\sqcap$ are monotone

$f : L_1 \times L_2 \longrightarrow L_3$ is monotone iff

$\forall x_1, y_1 \in L_1 ,\ x_2 \in L_2$

$x_1 \sqsubseteq y_1 \Longrightarrow f(x_1, x_2) \sqsubseteq f(y_1, x_2)$

and symmetrically for x_2

$$\boxed{\begin{aligned} x_1 &= f_1(x_1 \dots x_n) \\ x_2 &= f_2(x_1 \dots x_n) \\ &\vdots \\ x_n &= f_n(x_1 \dots x_n) \end{aligned}}$$

x_i are variables over lattice L

f_i are functions in $L^n \longrightarrow L$

$\rightarrow$ Combine f_i into a single function

$$\boxed{\underbrace{f(x_1, \dots x_n)}_{\in L^n \to L^n} = \big(f_1(x_1 \dots x_n), \dots f_n(x_1, \dots x_n)\big)}$$

$$x = f(x_1 \dots x_n)$$

L^n

$$\boxed{x = f(x)}$$

$x_1 \dots x_n \qquad x_1 \dots x_n$

GOAL:

find the least fixpoint

Solving LFP of $x = f(x)$

1. $x \in L^n$
2. f is monotone

Kleene's fixed point theorem

In a lattice L of finite height, every monotone function $f : L \to L$ has unique least fixed point

$$fix(f) = \bigsqcup_{i \geq 0} f^i(\bot)$$

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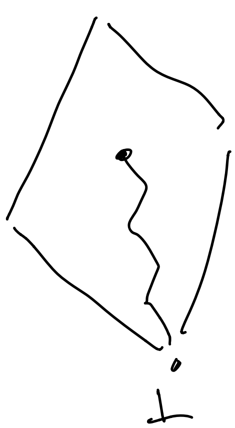
proof

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq f^3(\bot) \sqsubseteq \dots$$



Since L has finite height

$\exists k .\ f^k(\bot) = f^{k+1}(\bot)$

$fix(f) = f^k(\bot)$

Assume $x = f(x)$

$\bot \sqsubseteq x$

$f(\bot) \sqsubseteq f(x) = x$

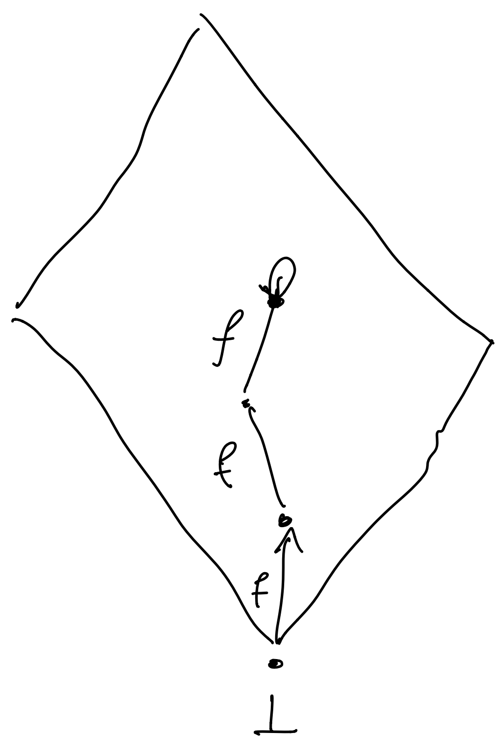
$\mathrm{fix}(f) = f^k(\bot) \sqsubseteq x$

Therefore $f^k(\bot)$ is a least fixedpoint

Fixpoint Algorithm (f)

```
x = ⊥
while x ≠ f(x)        fixpoint check
    x = f(x)

return x
```



$$\begin{cases} x_1 \sqsupseteq f_1(x_1, \ldots, x_n) \\ \vdots \\ x_n \sqsupseteq f_n(x_1, \ldots, x_n) \end{cases}$$

what is the relation between $\sqsupseteq$ and $=$

$$x \sqsupseteq y \quad \text{iff} \quad \underline{\lfloor x \rfloor} = \underline{\lfloor x \sqcup y \rfloor}$$

Intuition

$$\boxed{\begin{array}{l} 3 \geq 1 \quad \text{because} \quad 3 = 1 + 2 \\ \{1,2,3\} \supseteq \{1\} \quad \text{because} \quad \{1,2,3\} = \{1\} \cup \{2,3\} \end{array}}$$

$$x_1 = x_1 \sqcup f_1(x_1, \ldots, x_n)$$

$$\vdots$$

$$x_n = x_n \sqcup f_n(x_1, \ldots, x_n)$$

$$x_1 \sqsubseteq f_1(\ldots)$$

$$\vdots$$

$$x_n \sqsubseteq f_n(\ldots)$$

$$x \sqsubseteq y \quad \text{iff} \quad x = x \sqcap y$$

$$x_1 = x_1 \sqcap f_1(\ldots)$$

$$\vdots$$

$$x_n = x_n \sqcap f_n(\ldots)$$

$$x \mapsto \top$$

$$x \mapsto \sigma \quad \text{——} \quad z = 2000/x$$