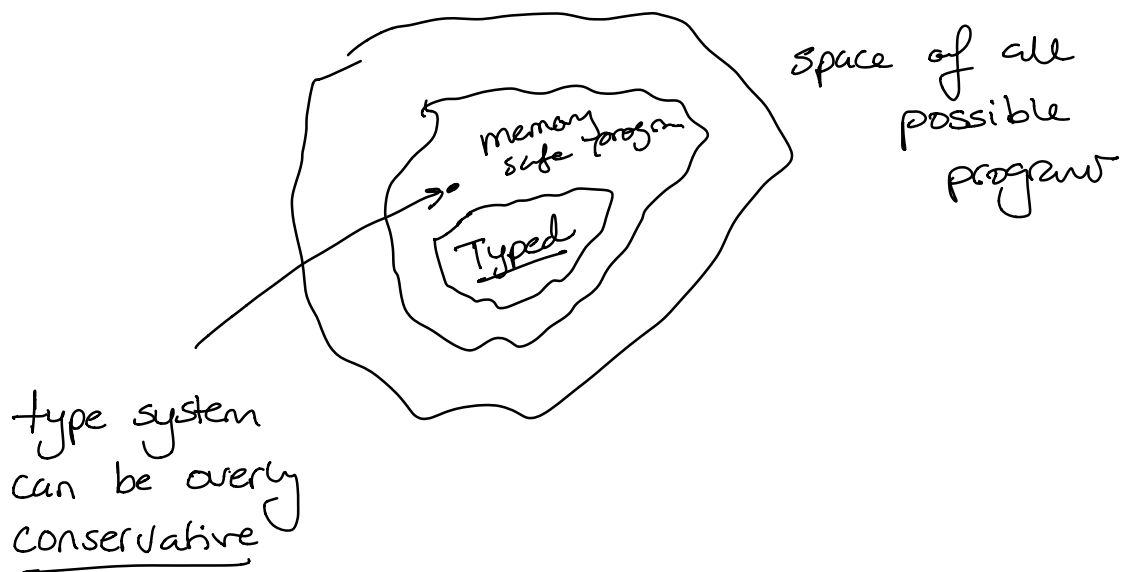


## TYPES

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

types ensure programs have desirable properties



Type checking

program  $P$   $\xrightarrow{\text{check}}$   $P$  has type  $T$ ?

Type inference

given program  $P \xrightarrow{\text{infer}} P$  has type  $T$ ?

$P$  "inhabits" type  $T$

$\mathbb{Z}$

$\mathbb{R}$

$1/\pi \in \mathbb{R}$

$\lambda$  calc subset

$\{ 0, \text{succ } 0, \text{isZero } n, \text{pred } n, \text{tru}, \text{fls}$   
 $\text{if } \boxed{\text{true}} \text{ then } t_1 \text{ else } t_2,$   
Boolean  
value

operational semantics  $\equiv$   $\beta$  reduction

$\triangleright$  define semantics

$\text{pred } 0 \rightarrow 0$

$\text{isZero } 0 \rightarrow \text{true}$

$\text{isZero } (\text{succ } n) \rightarrow \text{false}$

$\text{pred } (\text{succ } n) \rightarrow n$

$\text{if true then } t_1 \text{ else } t_2 \rightarrow t_1$

$\text{if false then } t_1 \text{ else } t_2 \rightarrow t_2$

$\boxed{t_1} \rightarrow \boxed{t_1'}$

$\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow \text{if } t_1' \text{ then } t_2 \text{ else } t_3$

$$\frac{t \rightarrow t'}{\text{succ } t \rightarrow \text{succ } t'}$$

$T := \text{Bool} \mid \text{Nat}$

or

$\underbrace{t : T}_{\text{more prevalent}} \text{ equiv. } t \in T$

$\left. \begin{array}{l} 0 : \text{Nat} \\ \text{true} : \text{Bool} \\ \text{false} : \text{Bool} \end{array} \right\} \text{Base cases}$

typing rules

$\frac{t : \text{Nat}}{\text{iszero } t : \text{Bool}} \quad \underline{T_{\text{iszero}}}$

same type

$\frac{t_1 : \text{Bool} \quad t_2 \in T \quad t_3 \in T}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in T} \quad \boxed{T_{\text{if}}}$

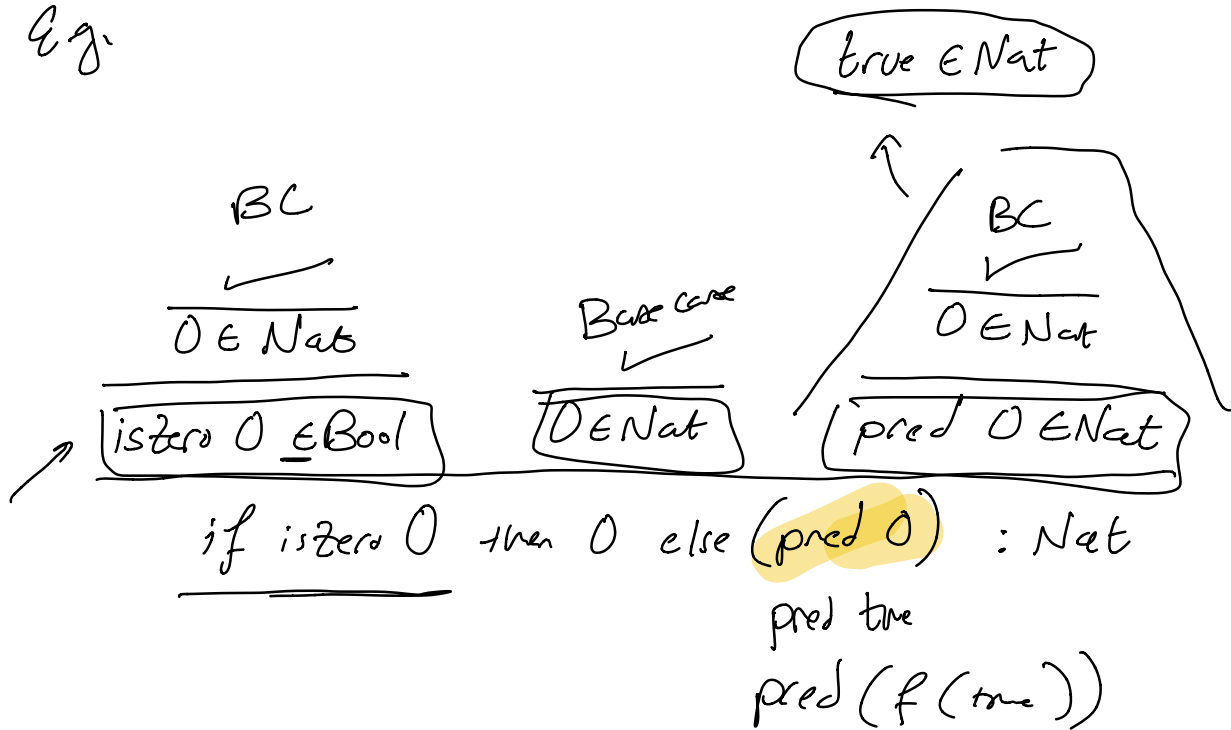
$\frac{t \in \text{Nat}}{\text{succ } t \in \text{Nat}} \quad T_{\text{succ}}$

$\frac{t \in \text{Nat}}{\text{pred } t \in \text{Nat}} \quad T_{\text{pred}}$

A term  $t$  is typable / well-typed  
if there is a type  $T$  s.t.  $t : T$

A typing relation is the smallest  
binary relation between terms and types  
satisfying our rules.

E.g.



## Theorem (Uniqueness)

Each term  $t$  has at most 1 type

Proof

Base cases

$0 \in \text{Nat}$ ,  $\text{true} \in \text{Bool}$ ,  $\text{false} \in \text{Bool}$  ✓

Hypothesis

any term of size  $\leq n$  has at most 1 type

Inductive step

take term of size  $n+1$

assume term has type  $T$

case 1  $\text{succ } t_1 \in T$

$$\frac{t_1 \in \text{Nat}}{\text{succ } t_1 \in T}$$

$\circ \circ$   $t_1$  has at most 1 type  
 $T = \text{Nat}$

case 2

$$\frac{t_1 \in \text{Bool} \quad t_2, t_3 \in T}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in T}$$

if  $t_1$ , then  $t_2$  else  $t_3 \in T$  ✓

Safety = Progress + preservation  
= Soundness

well-typed terms don't "get stuck"

Progress: A well-typed term is not stuck  
(either it is a value or can go one more step of evaluation)

Preservation: if a well-typed term takes a step of evaluation then the result is well-typed

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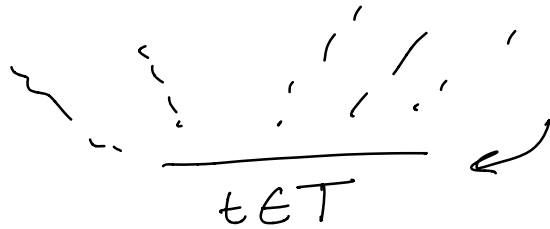
Thm (Progress)

suppose  $t \in T$ . Then either  $t$  is a value or  $t \rightarrow t'$

Thm (Preservation)

if  $t \in T$  and  $t \rightarrow t'$ ,  $t' \in T$

## Proof of preservation



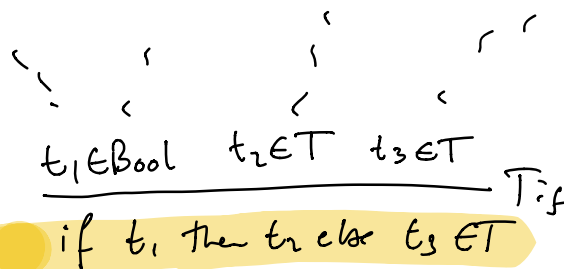
BC

Base case rules

$t = \text{true}$  or  $\text{false}$  or  $0$

theorem holds vacuously because  $t \rightarrow t'$

Induction



Case 1  $t_1 = \text{true}$

$t' = t_2$

and we know  $t_2 \in T$  so  $t' \in T$

Case 2  $t_1 = \text{false}$

$t' = t_3$

we know  $t_3 \in T$  so  $t' \in T$

Case 3  $t_1 \rightarrow t'_1$  By inductive hypothesis  
 $t'_1 \in \text{Bool}$


this means

$$t' = \text{if } b', \text{ then } t_2 \text{ else } t_3 \in T$$

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$$\text{if true then (succ 1) else true} \in \text{Nat}$$

$$\frac{b_1 \in T}{\text{if true then } t_1 \text{ else } t_2 \in T}$$

if  then  $t_1$  else  $t_2$

