TYPE

Reconstruction

Inference

$\lambda x . x\ 0 \in \underline{Y_1 \to Y_2}$

type of $x \in \underline{int \to X_1}$

System of constraints

$$\left\{ \begin{array}{l} Y_1 = int \to X_1 \\ Y_2 = \underline{X_1} \end{array} \right\}$$

$$C = \{ S_i = T_i \}_{i \in [1,n]}$$

these are types
may contain variables
$X, Y, Z, \ldots$

e.g. $\text{int} \to X$

1. How to solve constraints C
2. How to construct C

Solving constraints

a type substitution σ

e.g. $\sigma = [X \mapsto T, Y \mapsto U, \ldots]$

$\text{dom}(\sigma) = \{X, Y, \ldots\}$

$\text{range}(\sigma) = \{T, U, \ldots\}$

e.g. $\sigma = [X \mapsto \text{Bool}]$

$\sigma(X \to X) = \text{Bool} \to \text{Bool}$

$\sigma(Y \to Y) = Y \to Y$

A substitution σ unifies $S = T$ if $\sigma S = \sigma T$
syntactically

σ unifies C if $\sigma S_i = \sigma T_i$
for all $S_i = T_i \in C$

Unification algorithm (Robinson's)

$\{int = Bool\}$

Goal: check if there are solutions
find the "best" possible solution

Def'n σ is more general than σ' $(\sigma \le \sigma')$
if $\sigma' = \gamma \circ \sigma$ for some substitution γ

Composition

$\gamma \circ \sigma =$

$X \mapsto \gamma T$ for each $X \mapsto T \in \sigma$

$X \mapsto T$ for each $X \mapsto T \in \gamma$
and $X \notin dom(\sigma)$

A principal unifier σ for C is s.t.
for all σ' unifying C, $\sigma \le \sigma'$
σ is more general

E.g. $\{X = Y\}$

$\sigma = [X \mapsto Bool, Y \mapsto Bool]$

$\sigma X = \sigma Y$ ✓

$\sigma = [X \mapsto int, Y \mapsto int]$

$\sigma = [X \mapsto Y]$ principal unifier

$\sigma X = Y = \sigma Y$

E.g.

$\{ Nat \to Nat = X \to Y \}$

$\sigma = [X \mapsto Nat, Y \mapsto Nat]$

E.g. $\{ Y = Nat \to Y \}$ no solution

---

unify (C):

match C with

1. $\emptyset \to [\,]$

2. $\{S = T\} \cup C' \longrightarrow$

if $S = T$ (syntactically) then unify (C')

if S is of the form X and $X \notin \underbrace{FV(T)}_{\text{free variables}}$

$X = \underbrace{\ldots}_{T}$

symmetric cases

$\text{unify}([X \mapsto T]C') \circ [X \mapsto T]$

if T is of the form X and $X \notin FV(S)$

$\text{unify}([X \mapsto S]C') \circ [X \mapsto S]$

if S of the form $S_1 \to S_2$

T of the form $T_1 \to T_2$

$\text{unify}(C' \cup \{S_1 = T_1, S_2 = T_2\})$

else fail (no solution)

E.g. $S = T$

$$(\mathrm{Int} \to \mathrm{Int}) \to \mathrm{Int} \;=\; X \to \mathrm{Int}$$

case ③

$$\boxed{\mathrm{Int} \to \mathrm{Int} = X}$$

$\mathrm{Int} = \mathrm{Int}$ ✓

$X \mapsto \mathrm{Int} \to \mathrm{Int}$

$$\mathrm{Int} \to (\mathrm{Int} \to \mathrm{Int}) \;=\; X \to \mathrm{Int}$$

Typing relation

$$\Gamma \vdash t \in T$$

typing context

λ term

type

$$\Gamma \vdash t \in T \;\big|\; C$$

E.g. $\emptyset \vdash (\lambda x{:}T.\ x)\ \emptyset : T' \;\Big|\; \left\{ \begin{array}{l} T = T' \\ T' = \mathrm{Nat} \end{array} \right\}$

$$\frac{\Gamma \vdash t_1 \in T \mid C \qquad C' = C \cup \{T = Nat\}}{\Gamma \vdash \text{succ } t_1 : Nat \mid C'}$$

$$\frac{\Gamma \vdash t_1 \in T_1 \mid C_1 \qquad \Gamma \vdash t_2 \in T_2 \mid C_2 \qquad C' = C_1 \cup C_2 \cup \{T_1 = T_2 \rightarrow X\}}{\Gamma \vdash t_1\ t_2 : X \mid C'}$$

Context

$\Gamma = x_1 \in T_1 \ , \ x_2 \in T_2 \ , \ x_3 \in T_3 \ \ldots$

$\emptyset$