

BOLIB: BILEVEL OPTIMIZATION LIBRARY OF TEST PROBLEMS

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ABSTRACT. To help accelerate the development of numerical solvers for bilevel optimization, BOLIB aims at presenting a collection of academic and real-world applications or case studies on the problem. This first version of the library is made of 124 academic examples of nonlinear bilevel optimization problems, tidied up from a wide range of publications. To the best of our knowledge, this is the first time that such a scale of examples are provided to render a uniform basis on which algorithms proposed to deal with nonlinear bilevel optimization can be tested and compared. All the collected examples are programmed via Matlab and the library will be made freely available online.

1. INTRODUCTION

The general bilevel optimization problem can take the form

$$\begin{aligned} \min_{x,y} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \leq 0, H(x, y) = 0, \\ & y \in S(x) := \arg \min_y \{f(x, y) : g(x, y) \leq 0, h(x, y) = 0\}, \end{aligned} \tag{1.1}$$

where the functions $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_g}$ and $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_h}$ define the upper-level constraints, while $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_g}$ and $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_h}$ describe the lower-level constraints. On the other hand, $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}$ and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}$ denote the upper- and lower-level objective/cost functions, respectively. The set-valued map $S : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_y}$ represents the optimal solution/argminimum mapping of the lower-level problem. Further recall that problem (1.1) as a whole is often called upper-level problem.

As the medium and long term goal of this library is to include various classes of bilevel optimization problems, in particular, simple, linear, nonlinear, and real-world applications or case studies, we intentionally consider our model (1.1) to be broad, as it is likely that the overwhelming majority of these problems will be of this form. However, we just focus on nonlinear academic examples here. Recall that problem (1.1) is linear if all the functions involved are linear; otherwise, it is nonlinear.

Another important restriction made in the codes of this version of the library is that equality constraint are not considered. This is because in the test set presented here, only four examples have equality constraints; see Table 3 or the detailed formulas of the problems in Appendix 3. Hence, we use the fact that $H(x, y) = 0$ (similarly to $h(x, y) = 0$) can be expressed as $H(x, y) \leq 0$ and $-H(x, y) \leq 0$. Changes will be introduced in future versions of the library, provided that a significant number of new

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examples include equality constraints in the upper or lower-level of the problem. Hence, our focus here will be on nonlinear bilevel optimization problems of the form

$$\begin{aligned} \min_{x,y} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \leq 0, \\ & y \in S(x) := \arg \min_y \{f(x, y) : g(x, y) \leq 0\}. \end{aligned} \quad (1.2)$$

This paper provides a unique platform for the development of numerical methods, as well as theoretical results for nonlinear bilevel optimization problems. The main contributions of the paper can be summarized as follows:

- (1) This first version of BOLIB provides codes for 124 examples of nonlinear bilevel optimization problems, ready to be used to test numerical algorithms.
- (2) BOLIB provides the true or best known solutions and the corresponding references for all the examples included in the library. Hence, can therefore serve as a benchmark for numerical accuracy for methods designed to solve (1.2).
- (3) All the mathematical formulas of the examples present in the library are also given, in order to allow researchers to test theoretical properties or build codes for the examples in software different from MATLAB, if necessary. For each example, the formulas of the functions F , G , f , and g , involved in (1.2), are put in Section 3, as well as some useful background details.

To the best of our knowledge, this is the largest library of test examples for bilevel optimization, especially for the nonlinear class of the problem. It includes problems from Benoit Colson's BIPA [11], Sven Leyffer's MacMPEC [33], as well as from Mitsos and Barton's technical report [36]. Special classes of examples from the latter test sets not in this version of BOLIB will be included in future versions, where corresponding classes of problems are expanded to a reasonable size.

The main goal that we hope to achieve with BOLIB is the acceleration of numerical software development for bilevel optimization, as it is our opinion that the level of expansion of applications of the problem has outpaced the development rate for numerical solvers, especially for the nonlinear class of the problem.

2. DESCRIPTIONS OF LIBRARY

This section mainly describes inputs and outputs of each example, and also lists all 124 examples, together with their true or best known solutions and the corresponding references. Before we proceed, note that each `m-file` contains information about the corresponding example, which include the first and second order derivatives of the input functions. For the upper-level objective function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}$, these derivatives are defined as follows

$$\begin{aligned} \nabla_x F(x, y) &= \begin{bmatrix} \nabla_{x_1} F \\ \vdots \\ \nabla_{x_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_x}, \\ \nabla_{xx}^2 F(x, y) &= \begin{bmatrix} \nabla_{x_1 x_1}^2 F & \cdots & \nabla_{x_{n_x} x_1}^2 F \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 x_{n_x}}^2 F & \cdots & \nabla_{x_{n_x} x_{n_x}}^2 F \end{bmatrix} \in \mathbb{R}^{n_x \times n_x}, \\ \nabla_{xy}^2 F(x, y) &= \begin{bmatrix} \nabla_{x_1 y_1}^2 F & \cdots & \nabla_{x_{n_x} y_1}^2 F \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 y_{n_x}}^2 F & \cdots & \nabla_{x_{n_x} y_{n_x}}^2 F \end{bmatrix} \in \mathbb{R}^{n_y \times n_x}. \end{aligned} \quad (2.1)$$

Similar expressions are valid for $\nabla_y F(x, y) \in \mathbb{R}^{n_y}$, $\nabla_{yy}^2 F(x, y) \in \mathbb{R}^{n_y \times n_y}$ and the lower-level objective function f . As the constraint functions are vector-valued, we have for instance, in the context of the upper-level constraint function $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_G}$, that

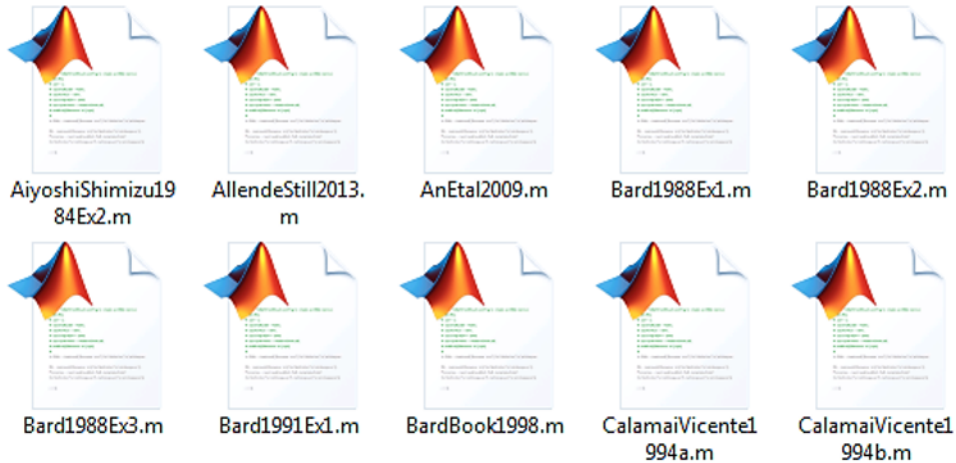
$$\nabla_x G(x, y) = \begin{bmatrix} \nabla_x G_1 \\ \vdots \\ \nabla_x G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla_{x_1} G_1 & \cdots & \nabla_{x_{n_x}} G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1} G_{n_G} & \cdots & \nabla_{x_{n_x}} G_{n_G} \end{bmatrix} \in \mathbb{R}^{n_G \times n_x}, \quad (2.2)$$

$$\nabla_{xx}^2 G(x, y) = \begin{bmatrix} \nabla_{xx}^2 G_1 \\ \vdots \\ \nabla_{xx}^2 G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla_{x_1 x_1}^2 G_1 & \cdots & \nabla_{x_{n_x} x_1}^2 G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 x_{n_x}}^2 G_1 & \cdots & \nabla_{x_{n_x} x_{n_x}}^2 G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 x_{n_x}}^2 G_{n_G} & \cdots & \nabla_{x_{n_x} x_{n_x}}^2 G_{n_G} \end{bmatrix} \in \mathbb{R}^{(n_G n_x) \times n_x}, \quad (2.3)$$

$$\nabla_{xy}^2 G(x, y) = \begin{bmatrix} \nabla_{xy}^2 G_1 \\ \vdots \\ \nabla_{xy}^2 G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla_{x_1 y_1}^2 G_1 & \cdots & \nabla_{x_{n_x} y_1}^2 G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 y_{n_y}}^2 G_1 & \cdots & \nabla_{x_{n_x} y_{n_y}}^2 G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 y_1}^2 G_{n_G} & \cdots & \nabla_{x_{n_x} y_1}^2 G_{n_G} \\ \vdots & \ddots & \vdots \\ \nabla_{x_1 y_{n_y}}^2 G_{n_G} & \cdots & \nabla_{x_{n_x} y_{n_y}}^2 G_{n_G} \end{bmatrix} \in \mathbb{R}^{(n_G n_y) \times n_x}. \quad (2.4)$$

Similar formulas will be valid for $\nabla_y G(x, y) \in \mathbb{R}^{n_G \times n_y}$, $\nabla_{yy}^2 G(x, y) \in \mathbb{R}^{n_G n_y \times n_y}$ and the lower-level constraint g . Here we need to emphasize that when $n_G = 1$, $\nabla_x G(x, y) \in \mathbb{R}^{1 \times n_x}$ which is a row vector whilst $\nabla_x F(x, y) \in \mathbb{R}^{n_x}$ or $\nabla_x f(x, y) \in \mathbb{R}^{n_x}$ are column vectors.

2.1. Inputs and outputs. Open folder `BOLIBExample` which contains 124 MATLAB m-files. Each one specifies a test example, named by a combination of authors' surnames, year of publication, and when necessary, the order of the example in the corresponding reference. For example, as in following figure (showing a partial list of the examples), `AiyoshiShimizu1984Ex2.m` stands for the Example 2 considered by Aiyoshi and Shimizu in 1984, see [1] for more details.



Now we describe the inputs and outputs for a given m-file example. All files have the uniform function handle as

$$w = \text{example_name}(x, y, \text{keyf}, \text{keyxy}).$$

For the inputs, we have

$$\begin{aligned} x &\in \mathbb{R}^{n_x}, \quad y \in \mathbb{R}^{n_y}, \\ \text{keyf} &\in \{ 'F', 'G', 'f', 'g' \}, \\ \text{keyxy} &\in \{ [], 'x', 'y', 'xx', 'xy', 'yy' \}, \end{aligned}$$

where 'F', 'G', 'f', and 'g' respectively stand for the four functions involved in (1.2). 'x' and 'y' represent the first order derivative with respect to x and y , respectively. Finally, 'xx', 'xy', and 'yy' correspond to the second order derivative of the function F , G , f , and g , with respect to xx , xy , and yy , respectively.

For the outputs, $w = \text{example_name}(x, y, \text{keyf})$ or $w = \text{example_name}(x, y, \text{keyf}, [])$ returns the function value of keyf , and $w = \text{example_name}(x, y, \text{keyf}, \text{keyxy})$ returns the first or second order derivative of keyf with respect to choice of keyxy as describe above. We can summarize the input-inputs scenarios in the following table:

keyfkeyxy	[]	'x'	'y'	'xx'	'xy'	'yy'
'F'	$F(x, y)$	$\nabla_x F(x, y)$	$\nabla_y F(x, y)$	$\nabla_{xx}^2 F(x, y)$	$\nabla_{xy}^2 F(x, y)$	$\nabla_{yy}^2 F(x, y)$
'G'	$G(x, y)$	$\nabla_x G(x, y)$	$\nabla_y G(x, y)$	$\nabla_{xx}^2 G(x, y)$	$\nabla_{xy}^2 G(x, y)$	$\nabla_{yy}^2 G(x, y)$
'f'	$f(x, y)$	$\nabla_x f(x, y)$	$\nabla_y f(x, y)$	$\nabla_{xx}^2 f(x, y)$	$\nabla_{xy}^2 f(x, y)$	$\nabla_{yy}^2 f(x, y)$
'g'	$g(x, y)$	$\nabla_x g(x, y)$	$\nabla_y g(x, y)$	$\nabla_{xx}^2 g(x, y)$	$\nabla_{xy}^2 g(x, y)$	$\nabla_{yy}^2 g(x, y)$

For the dimension of w in each scenario, see (2.1)–(2.4). If $n_G = 0$ (or $n_g = 0$), all outputs related to G (or g) should be empty, namely, $w = []$. Let us look at some specific usage:

- $w = \text{example_name}(x, y, 'F')$ or $w = \text{example_name}(x, y, 'F', [])$ returns the function value of F , i.e., $w = F(x, y)$; this is similar for G , f , and g ;
- $w = \text{example_name}(x, y, 'F', 'x')$ returns the partial derivative of F with respect to x , i.e., $w = \nabla_x F(x, y)$;
- $w = \text{example_name}(x, y, 'G', 'y')$ returns the Jacobian matrix of G with respect to y , i.e., $w = \nabla_y G(x, y)$;
- $w = \text{example_name}(x, y, 'f', 'xy')$ returns the Hessian matrix of f with respect to xy , i.e., $w = \nabla_{xy}^2 f(x, y)$;
- $w = \text{example_name}(x, y, 'g', 'yy')$ returns the second order derivative of g with respect to yy , i.e., $w = \nabla_{yy}^2 g(x, y)$.

We use one example to illustrate the definitions above.

Example 2.1. Shimizu et al. (1997), see [47], considered the bilevel program (1.2) with

$$\begin{aligned} F(x, y) &:= (x - 5)^2 + (2y + 1)^2, \\ f(x, y) &:= (y - 1)^2 - 1.5xy, \\ g(x, y) &:= \begin{bmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \end{bmatrix}. \end{aligned}$$

Clearly, $n_x = 1, n_y = 1, n_G = 0, n_g = 3$. The m-file is named by ShimizuEtal1997a (i.e., `exmaple_name = ShimizuEtal1997a`), which was coded through MATLAB as follows.

```

function w=ShimizuEtal1997a(x,y,keyf,keyxy)
if nargin<4 || isempty(keyxy)
    switch keyf
    case 'F'; w = (x-5)^2+(2*y+1)^2;
    case 'G'; w = [];
    case 'f'; w = (y-1)^2-1.5*x*y;
    case 'g'; w = [-3*x+y+3; x-0.5*y-4; x+y-7];
    end
else
    switch keyf
    case 'F'
        switch keyxy
        case 'x' ; w = 2*(x-5);
        case 'y' ; w = 4*(2*y+1);
        case 'xx'; w = 2;
        case 'xy'; w = 0;
        case 'yy'; w = 8;
        end
    case 'G'
        switch keyxy
        case 'x' ; w = [];
        case 'y' ; w = [];
        case 'xx'; w = [];
        case 'xy'; w = [];
        case 'yy'; w = [];
        end
    case 'f'
        switch keyxy
        case 'x' ; w = -1.5*y;
        case 'y' ; w = 2*(y-1)-1.5*x;
        case 'xx'; w = 0;
        case 'xy'; w = -1.5;
        case 'yy'; w = 2;
        end
    case 'g'
        switch keyxy
        case 'x' ; w = [-3; 1; 1];
        case 'y' ; w = [ 1;-0.5; 1];
        case 'xx'; w = [ 0; 0; 0];
        case 'xy'; w = [ 0; 0; 0];
        case 'yy'; w = [ 0; 0; 0];
        end
    end
end
end
end

```

If we are given some inputs (as in left column of the table below), then ShimizuEtal1997a will return us corresponding results as in the right column of the table:

Inputs	Outputs
$x = 4$	$x = 4$
$y = 0$	$y = 0$
$F = \text{ShimizuEtal1997a}(x, y, 'F')$	$F = 2$
$Fx = \text{ShimizuEtal1997a}(x, y, 'F', 'x')$	$Fx = -2$
$Gy = \text{ShimizuEtal1997a}(x, y, 'G', 'y')$	$Gy = []$
$fxy = \text{ShimizuEtal1997a}(x, y, 'f', 'xy')$	$fxy = -1.5$
$gyy = \text{ShimizuEtal1997a}(x, y, 'g', 'yy')$	$gyy = [0; 0; 0]$

2.2. List of test examples. For the examples presented here, at least one of the functions involved in the problem is nonlinear. The details related to each example presented in the BOLIB library are in a column of Table 3 below. The first column of the table provides the list of problems, as they appear in the BOLIBExample folder. The second column gives the reference in the literature where the example might have first appeared. The third column combines the labels corresponding to the nature of the functions involved in (1.2). Precisely, “N” and “L” will be used to indicate whether the functions F , G , f , and g are nonlinear (N) or linear (L), while “O” is used to symbolize that there is either no function G or g present in problem (1.2). Then follows the column with n_x and n_y for the upper and lower-level variables dimensions, as well as n_G (resp. n_g) to denote the the number of components of the upper (resp. lower)-level constraint function. On the other hand, F^* and f^* denote the best known optimal upper and lower-level objective function values, respectively, according to the literature that is listed in the last column **RefII**. Notice that example CalamaiVicente1994c has unknown lower-level objective function value, and Zlobec2001b has no optimal solutions. Moreover, 3 examples involve parameters. They are CalamaiVicente1994a with $\rho \geq 1$ (its F^* and f^* listed in the table are under $\rho = 1$, other cases can be found in Section 3), HenrionSurowiec2011 with $c \in \mathbb{R}$ and IshizukaAiyoshi1992a with $M > 1$.

TABLE 3. List of nonlinear bilevel programs with their labels.

Example name	RefI	F-G-f-g	n_x	n_y	n_G	n_g	F^*	f^*	RefII
AiyoshiShimizu1984Ex2	[1]	L-L-N-L	2	2	5	6	5	0	[1]
AllendeStill12013	[2]	N-L-N-N	2	2	5	2	1	-0.5	[2]
AnEtal2009	[3]	N-L-N-L	2	2	6	4	2251.6	165.2	[3]
Bard1988Ex1	[4]	N-L-N-L	1	1	1	4	17	1	[4]
Bard1988Ex2	[4]	N-L-N-L	4	4	9	12	-6600	54	[11]
Bard1988Ex3	[4]	N-N-N-N	2	2	3	4	-12.68	-1.02	[8]
Bard1991Ex1	[5]	L-L-N-L	1	2	2	3	2	12	[5]
BardBook1998	[6]	N-L-L-L	2	2	4	7	0	5	
CalamaiVicente1994a	[7]	N-O-N-L	1	1	0	3	0	0	[7]
CalamaiVicente1994b	[7]	N-O-N-L	4	2	0	6	0.3125	-0.4063	[7]
CalamaiVicente1994c	[7]	N-O-N-L	4	2	0	6	0.3125	-0.4063	[7]
CalveteGale1999P1	[9]	L-L-L-N	2	3	2	6	-29.2	0.31	[9, 22]
ClarkWesterberg1990a	[10]	N-L-N-L	1	1	2	3	5	4	[44]
Colson2002BIPA1	[11]	N-L-N-L	1	1	3	3	250	0	
Colson2002BIPA2	[11]	N-L-N-L	1	1	1	4	17	2	[8]
Colson2002BIPA3	[11]	N-L-N-L	1	1	2	2	2	24.02	[8]
Colson2002BIPA4	[11]	N-L-N-L	1	1	2	2	88.79	-0.77	[8]
Colson2002BIPA5	[11]	N-L-N-N	1	2	1	6	2.75	0.57	[8]

Dempe1992a	[12]	L-N-N-N	2	2	1	2	×	×	
Dempe1992b	[12]	N-O-N-N	1	1	0	1	31.25	4	[8]
DempeDutta2012Ex24	[13]	N-O-N-N	1	1	0	1	0	0	[13]
DempeDutta2012Ex31	[13]	L-N-N-N	2	2	4	2	-1	4	[13]
DempeEtal2012	[14]	L-L-N-L	1	1	2	2	-1	-1	[14]
DempeFranke2011Ex41	[15]	N-L-N-L	2	2	4	4	5	-2	[15]
DempeFranke2011Ex42	[15]	N-L-N-L	2	2	4	3	2.13	-3.5	[15]
DempeFranke2014Ex38	[16]	L-L-N-L	2	2	4	4	-1	-4	[16]
DempeLohse2011Ex31a	[17]	N-O-N-L	2	2	0	4	-6	1	[17]
DempeLohse2011Ex31b	[17]	N-O-N-L	3	3	0	5	-12	0	
DeSilva1978	[18]	N-O-N-L	2	2	0	4	-1	0	[8]
EdmundsBard1991	[19]	L-L-N-L	2	2	4	7	0	200	[49]
FalkLiu1995	[20]	N-O-N-L	2	2	0	4	-2.1962	0	[8]
FloudasZlobec1998	[21]	N-L-L-N	1	2	2	6	1	-1	[22, 36]
GumusFloudas2001Ex1	[22]	N-L-N-L	1	1	3	3	2250	197.75	[36]
GumusFloudas2001Ex3	[22]	L-L-N-L	2	3	4	9	-29.2	0.31	[36]
GumusFloudas2001Ex4	[22]	N-L-N-L	1	1	5	2	9	0	[36]
GumusFloudas2001Ex5	[22]	L-L-N-N	1	2	2	6	0.19	-7.23	[36]
HatzEtal2013	[23]	L-O-N-L	1	2	0	2	0	0	[23]
HendersonQuandt1958	[24]	N-L-N-L	1	1	2	1	-3266.7	-711.11	[24]
HenrionSurowiec2011	[25]	N-O-N-O	1	1	0	0	$-c^2/4$	$-c^2/8$	[26]
IshizukaAiyoshi1992a	[27]	N-L-L-L	1	2	1	5	0	-M	[27]
KleniatiAdjiman2014Ex3	[28]	L-L-N-L	1	1	2	2	-1	0	[28]
KleniatiAdjiman2014Ex4	[28]	N-N-N-N	5	5	13	11	-10	-3.1	[28]
LamparSagrat2017Ex23	[29]	L-L-N-L	1	2	2	2	-1	1	[29]
LamparSagrat2017Ex31	[30]	N-L-L-L	1	1	1	1	1	0	[30]
LamparSagrat2017Ex32	[30]	N-O-N-O	1	1	0	0	0.5	0	[30]
LamparSagrat2017Ex33	[30]	N-L-L-L	1	2	1	3	0.5	0	[30]
LamparSagrat2017Ex35	[30]	N-L-L-L	1	1	2	3	0.8	-0.4	[30]
LucchettiEtal1987	[31]	N-L-N-L	1	1	2	2	0	0	[31]
LuDebSinha2016a	[32]	N-L-N-O	1	1	4	0	1.94	1.96	[32]
LuDebSinha2016b	[32]	N-L-N-O	1	1	4	0	0	1.66	[32]
LuDebSinha2016c	[32]	N-L-N-O	1	1	4	0	1.12	0.06	[32]
LuDebSinha2016d	[32]	L-N-L-N	2	2	11	3	×	×	
LuDebSinha2016e	[32]	N-L-L-N	1	2	6	3	×	×	
LuDebSinha2016f	[32]	L-N-N-O	2	1	9	0	×	×	
MacalHurter1997	[34]	N-O-N-O	1	1	0	0	81.33	-0.33	[34]
Mirrlees1999	[35]	N-O-N-O	1	1	0	0	0.01	-1.04	[35]
MitsosBarton2006Ex38	[36]	N-L-N-L	1	1	4	2	0	0	[36]
MitsosBarton2006Ex39	[36]	L-L-N-L	1	1	3	2	-1	-1	[36]
MitsosBarton2006Ex310	[36]	L-L-N-L	1	1	2	2	0.5	-0.1	[36]
MitsosBarton2006Ex311	[36]	L-L-N-L	1	1	2	2	-0.8	0	[36]
MitsosBarton2006Ex312	[36]	N-L-N-L	1	1	2	2	0	0	[36]
MitsosBarton2006Ex313	[36]	L-L-N-L	1	1	2	2	-1	0	[36]
MitsosBarton2006Ex314	[36]	N-L-N-L	1	1	2	2	0.25	-0.08	[36]
MitsosBarton2006Ex315	[36]	L-L-N-L	1	1	2	2	0	-0.83	[36]
MitsosBarton2006Ex316	[36]	L-L-N-L	1	1	2	2	-2	0	[36]
MitsosBarton2006Ex317	[36]	N-L-N-L	1	1	2	2	0.19	-0.02	[36]
MitsosBarton2006Ex318	[36]	N-L-N-L	1	1	2	2	-0.25	0	[36]

MitsosBarton2006Ex319	[36]	N-L-N-L	1	1	2	2	-0.26	0	[36]
MitsosBarton2006Ex320	[36]	N-L-N-L	1	1	2	2	0.31	-0.08	[36]
MitsosBarton2006Ex321	[36]	N-L-N-L	1	1	2	2	0.21	-0.07	[36]
MitsosBarton2006Ex322	[36]	N-L-N-N	1	1	2	3	0.21	-0.07	[36]
MitsosBarton2006Ex323	[36]	N-N-L-N	1	1	3	3	0.18	-1	[36]
MitsosBarton2006Ex324	[36]	N-L-N-L	1	1	2	2	-1.75	0	[36]
MitsosBarton2006Ex325	[36]	N-N-N-N	2	3	6	9	-1	-2	[36]
MitsosBarton2006Ex326	[36]	N-N-N-L	2	3	7	6	-2.35	-2	[36]
MitsosBarton2006Ex327	[36]	N-N-N-N	5	5	13	13	2	-1.1	[36]
MitsosBarton2006Ex328	[36]	N-N-N-N	5	5	13	13	-10	-3.1	[36]
MorganPatrone2006a	[37]	L-L-N-L	1	1	2	2	-1	0	[37]
MorganPatrone2006b	[37]	L-O-N-L	1	1	0	4	-1.25	0	[37]
MorganPatrone2006c	[37]	L-O-N-L	1	1	0	4	-1	-0.25	[37]
MuuQuy2003Ex1	[38]	N-L-N-L	1	2	2	3	-2.08	-0.59	[38]
MuuQuy2003Ex2	[38]	N-L-N-L	2	3	3	4	0.64	1.67	[38]
NieEtal2017Ex34	[39]	L-L-N-N	1	2	2	2	2	0	[39]
NieEtal2017Ex52	[39]	N-N-N-N	2	3	5	2	-1.71	-2.23	[39]
NieEtal2017Ex54	[39]	N-N-N-N	4	4	3	2	-0.44	-1.19	[39]
NieEtal2017Ex57	[39]	N-N-N-N	2	3	5	2	-2	-1	[39]
NieEtal2017Ex58	[39]	N-N-N-N	4	4	3	2	-3.49	-0.86	[39]
NieEtal2017Ex61	[39]	N-N-N-N	2	2	5	1	-1.02	-1.08	[39]
Outrata1990Ex1a	[40]	N-O-N-L	2	2	0	4	-8.92	-6.05	[40]
Outrata1990Ex1b	[40]	N-O-N-L	2	2	0	4	-7.56	-0.58	[40]
Outrata1990Ex1c	[40]	N-O-N-L	2	2	0	4	-12	-112.71	[40]
Outrata1990Ex1d	[40]	N-O-N-L	2	2	0	4	-3.6	-2	[40]
Outrata1990Ex1e	[40]	N-O-N-L	2	2	0	4	-3.15	-16.29	[40]
Outrata1990Ex2a	[40]	N-L-N-L	1	2	1	4	0.5	-14.53	[40]
Outrata1990Ex2b	[40]	N-L-N-L	1	2	1	4	0.5	-4.5	[40]
Outrata1990Ex2c	[40]	N-L-N-L	1	2	1	4	1.86	-10.93	[40]
Outrata1990Ex2d	[40]	N-L-N-N	1	2	1	4	0.92	-19.47	[40]
Outrata1990Ex2e	[40]	N-L-N-N	1	2	1	4	0.90	-14.94	[40]
Outrata1993Ex31	[41]	N-L-N-N	1	2	1	4	1.56	-11.67	[41]
Outrata1993Ex32	[41]	N-L-N-N	1	2	1	4	3.21	-20.53	[41]
Outrata1994Ex31	[42]	N-L-N-N	1	2	2	4	3.21	-20.53	[42]
OutrataCervinka2009	[43]	L-L-N-L	2	2	1	3	0	0	[43]
PaulaviciusEtal2017a	[44]	N-L-N-L	1	1	4	2	0.25	0	[44]
PaulaviciusEtal2017b	[44]	L-L-N-L	1	1	4	2	-2	-1.5	[44]
SahinCiric1998Ex2	[45]	N-L-N-L	1	1	2	3	5	4	[45]
ShimizuAiyoshi1981Ex1	[46]	N-L-N-L	1	1	3	3	100	0	[46]
ShimizuAiyoshi1981Ex2	[46]	N-L-N-L	2	2	3	4	225	100	[46]
ShimizuEtal1997a	[47]	N-O-N-L	1	1	0	3	×	×	
ShimizuEtal1997b	[47]	N-L-N-L	1	1	2	2	2250	197.75	[47]
SinhaMaloDeb2014TP3	[48]	N-N-N-N	2	2	3	4	-18.68	-1.02	[48]
SinhaMaloDeb2014TP6	[48]	N-L-N-L	1	2	1	6	-1.21	7.62	[48]
SinhaMaloDeb2014TP7	[48]	N-N-N-L	2	2	4	4	-1.96	1.96	[48]
SinhaMaloDeb2014TP8	[48]	N-L-N-L	2	2	5	6	0	100	[48]
SinhaMaloDeb2014TP9	[48]	N-O-N-L	10	10	0	20	0	1	[48]
SinhaMaloDeb2014TP10	[48]	N-O-N-L	10	10	0	20	0	1	[48]
TuyEtal2007	[49]	N-L-L-L	1	1	2	3	22.5	-1.52	[49]

Vogel2002	[50]	N-L-N-L	1 1 2 1	0 2	[50]
WanWangLv2011	[51]	N-O-L-L	2 3 0 8	10.63 -0.5	[51]
YeZhu2010Ex42	[52]	N-L-N-L	1 1 2 1	1 -2	[52]
YeZhu2010Ex43	[52]	N-L-N-L	1 1 2 1	1.25 -2	[52]
Yezza1996Ex31	[53]	N-L-N-L	1 1 2 2	1.5 -2.5	[53]
Yezza1996Ex41	[53]	N-O-N-L	1 2 0 2	0.5 2.5	[53]
Zlobec2001a	[54]	N-O-L-L	1 2 0 3	-1 -1	[54]
Zlobec2001b	[54]	L-L-L-N	1 1 2 4	× ×	[54]

It is worth mentioning that some examples contain equalities constraints $H(x, y) = 0$ or $h(x, y) = 0$. As we mentioned in the beginning of this manuscript, equalities constraints can be transformed to inequalities ones. However, for the sake of clarity, we list those examples whose constraints contain equalities below.

TABLE 4. List of nonlinear bilevel programs with equalities constraints.

Example name	Ref.	F/G/H/f/g/h	n_x	n_y	n_G	n_H	n_g	n_h
DempeDutta2012Ex31	[13]	L-L-N-N-N-O	2	2	2	1	2	0
DempeFranke2011Ex41	[15]	N-L-L-N-L-O	2	2	2	1	4	0
DempeFranke2011Ex42	[15]	N-L-L-N-L-O	2	2	2	1	3	0
Zlobec2001b	[54]	L-L-O-L-L-N	1	1	2	0	2	1

3. FORMULAS OF THE PROBLEMS

Here we provide formulas of the functions F , G , f , and g involved in problem (1.2) for all the 124 examples presented in this paper, together with true or best known values of the solutions, and some useful background information in some cases.

Problem name: AiyoshiShimizu1984Ex2

Source: [1]

Description: AiyoshiShimizu1984Ex2 is defined as follows

$$\begin{aligned}
 F(x, y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\
 G(x, y) &:= \begin{pmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ x - 50_2 \\ -x \end{pmatrix} \\
 f(x, y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\
 g(x, y) &:= \begin{pmatrix} 2y - x + 10_2 \\ -y - 10_2 \\ y - 20_2 \end{pmatrix}
 \end{aligned}$$

Comment: The global optimal solution of the problem is $(25, 30, 5, 10)$ according to [1]. A local optimal one is $(0, 0, -10, -10)$ by [27].

Problem name: AllendeStill2013

Source: [2]

Description: AllendeStill2013 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ -y \\ x_1 - 2 \end{pmatrix} \\ f(x, y) &:= y_1^2 - 2x_1y_1 + y_2^2 - 2x_2y_2 \\ g(x, y) &:= \begin{pmatrix} (y_1 - 1)^2 - 0.25 \\ (y_2 - 1)^2 - 0.25 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution from [2] is (0.5, 0.5, 0.5, 0.5).

Problem name: AnEtal2009

Source: [3]

Description: AnEtal2009 is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}(x^\top, y^\top)H(x^\top, y^\top)^\top + c_1^\top x + c_2^\top y \\ G(x, y) &:= \begin{pmatrix} -x \\ -y \\ Ax + By + d \end{pmatrix} \\ f(x, y) &:= y^\top Px + \frac{1}{2}y^\top Qy + q^\top y \\ g(x, y) &:= Dx + Ey + b \end{aligned}$$

with H , c_1 , c_2 , A , B , d , P , Q , q , D , E , and b , respectively defined as follows

$$\begin{aligned} H &:= \begin{bmatrix} -3.8 & 4.4 & 1.2 & -2.2 \\ 4.4 & -2.2 & 0.6 & 1.8 \\ 1.2 & 0.6 & 0.0 & 0.4 \\ -2.2 & 1.8 & 0.4 & 0.0 \end{bmatrix}, \quad c_1 := \begin{bmatrix} 935.74474 \\ 87.53654 \end{bmatrix}, \quad c_2 := \begin{bmatrix} 121.96196 \\ 299.24825 \end{bmatrix} \\ A &:= \begin{bmatrix} 0.00000 & 3.88889 \\ -2.00000 & 8.77778 \end{bmatrix}, \quad B := \begin{bmatrix} 4.88889 & 7.44444 \\ -5.11111 & 0.88889 \end{bmatrix}, \quad d := \begin{bmatrix} -61.57778 \\ -0.80000 \end{bmatrix} \\ P &:= \begin{bmatrix} -17.85000 & 6.57500 \\ 30.32500 & 30.32500 \end{bmatrix}, \quad Q := \begin{bmatrix} 21.10204 & 11.81633 \\ -5.11111 & -14.44898 \end{bmatrix}, \quad q := \begin{bmatrix} -18.21053 \\ 13.05263 \end{bmatrix} \\ D &:= \begin{bmatrix} 5.00000 & 7.44444 \\ -8.33333 & 3.00000 \\ -8.66667 & -8.55556 \\ 6.44444 & -5.11111 \end{bmatrix}, \quad E := \begin{bmatrix} 3.88889 & 1.77778 \\ 6.88889 & 6.11111 \\ -5.33333 & -7.00000 \\ 1.44444 & 4.44444 \end{bmatrix}, \quad b := \begin{bmatrix} -39.62222 \\ -60.00000 \\ 72.37778 \\ -17.28889 \end{bmatrix} \end{aligned}$$

Comment: (0.200001, 1.999997, 3.999998, 4.600005) is the best known approximation of the solution of the problem; cf. [3].

Problem name: Bard1988Ex1

Source: [4]

Description: Bard1988Ex1 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 5)^2 + (2y + 1)^2 \\ G(x, y) &:= -x \\ f(x, y) &:= (y - 1)^2 - 1.5xy \\ g(x, y) &:= \begin{pmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \\ -y \end{pmatrix} \end{aligned}$$

Comment: $(1, 0)$ is the global optimum and $(5, 2)$ is a local optimal point.

Problem name: Bard1988Ex2

Source: [4]

Description: Bard1988Ex2 is defined as follows

$$\begin{aligned}
 F(x, y) &:= (200 - y_1 - y_3)(y_1 + y_3) + (160 - y_2 - y_4)(y_2 + y_4) \\
 G(x, y) &:= \begin{pmatrix} x_1 + x_2 + x_3 + x_4 - 40 \\ x_1 - 10 \\ x_2 - 5 \\ x_3 - 15 \\ x_4 - 20 \\ -x \end{pmatrix} \\
 f(x, y) &:= (y_1 - 4)^2 + (y_2 - 13)^2 + (y_3 - 35)^2 + (y_4 - 2)^2 \\
 g(x, y) &:= \begin{pmatrix} 0.4y_1 + 0.7y_2 - x_1 \\ 0.6y_1 + 0.3y_2 - x_2 \\ 0.4y_3 + 0.7y_4 - x_3 \\ 0.6y_3 + 0.3y_4 - x_4 \\ y_1 - 20 \\ y_2 - 20 \\ y_3 - 40 \\ y_4 - 40 \\ -y \end{pmatrix}
 \end{aligned}$$

Comment: This version of the problem is taken from [11]. The original one in [4] has two lower-level problem. The upper-and lower-level optimal value are respectively obtained as -6600.00 and 57.48 in the former paper. However, the upper-and lower-level optimal value should be -6600.00 and 54 according to [4].

Problem name: Bard1988Ex3

Source: [4]

Description: Bard1988Ex3 is defined as follows

$$\begin{aligned}
 F(x, y) &:= -x_1^2 - 3x_2 - 4y_1 + y_2^2 \\
 G(x, y) &:= \begin{pmatrix} x_1^2 + 2x_2 - 4 \\ -x \end{pmatrix} \\
 f(x, y) &:= 2x_1^2 + y_1^2 - 5y_2 \\
 g(x, y) &:= \begin{pmatrix} -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -y \end{pmatrix}
 \end{aligned}$$

Comment: The upper-and lower-level optimal objective value are respectively obtained as -12.68 and -1.02 in the paper [8].

Problem name: Bard1991Ex1

Source: [5]

Description: Bard1991Ex1 is defined as follows

$$\begin{aligned} F(x, y) &:= x + y_2 \\ G(x, y) &:= \begin{pmatrix} -x + 2 \\ x - 4 \end{pmatrix} \\ f(x, y) &:= 2y_1 + xy_2 \\ g(x, y) &:= \begin{pmatrix} x - y_1 - y_2 + 4 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(2, 6, 0)$; cf. [5].

Problem name: BardBook1998

Source: [6]

Description: BardBook1998 is defined as follows

$$\begin{aligned} F(x, y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ G(x, y) &:= \begin{pmatrix} x - 50_2 \\ -x \end{pmatrix} \\ f(x, y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\ g(x, y) &:= \begin{pmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ 2y - x + 10_2 \\ y - 20_2 \\ -y - 10_2 \end{pmatrix} \end{aligned}$$

Comment: The global optimal solution is $(25 \ 30 \ 5 \ 10)$.

Problem name: CalamaiVicente1994a

Source: [7]

Description: CalamaiVicente1994a is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}(x - 1)^2 + \frac{1}{2}y^2 \\ f(x, y) &:= \frac{1}{2}y - xy \\ g(x, y) &:= \begin{pmatrix} x - y - 1 \\ -x - y + 1 \\ x + y - \rho \end{pmatrix} \end{aligned}$$

Comment: It is assumed in [7] that the parameter $\rho \geq 1$. We consider the following scenarios studied in the latter reference:

- (i) For $\rho = 1$, the point $(1, 0)$ is global optimum of the problem.
- (ii) For $1 < \rho < 2$, the point $\frac{1}{2}(1 + \rho, -1 + \rho)$ is a global optimal solution, while $\frac{1}{2}(1, 1)$ is a local optimal solution of the problem.
- (iii) For $\rho = 2$, the points $\frac{1}{2}(1, 1)$ and $\frac{1}{2}(3, 1)$ are global optimal solution.
- (iv) For $\rho > 2$, the point $\frac{1}{2}(1, 1)$ is global optimum of the problem.

Problem name: CalamaiVicente1994b

Source: [7]

Description: CalamaiVicente1994b is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2} \sum_{i=1}^4 (x_i - 1)^2 + \sum_{i=1}^2 y_i^2 \\ f(x, y) &:= \sum_{i=1}^2 \left(\frac{1}{2} y_i^2 - x_i y_i \right) \\ g(x, y) &:= \begin{pmatrix} x - y - 1_2 \\ -x - y + 1_2 \\ x_1 + y_1 - 1.5 \\ x_1 + y_2 - 3 \end{pmatrix} \end{aligned}$$

Comment: The global optimal solution is (1.25 0.5 1 1 0.25 0.5) with upper-and lower-level optimal objective value being 0.3125 and -0.4063 .

Problem name: CalamaiVicente1994c

Source: [7]

Description: CalamaiVicente1994c is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2} x^T A x + \frac{1}{2} y^T B y + a^T x + 2 \\ f(x, y) &:= \frac{1}{2} y^T B y + x^T C y \\ g(x, y) &:= D x + E y + d \end{aligned}$$

with A, B, a, B and C respectively as follows

$$\begin{aligned} A &:= \begin{bmatrix} 197.2 & 32.4 & -129.6 & -43.2 \\ 32.4 & 110.8 & -43.2 & -14.4 \\ -129.6 & -43.2 & 302.8 & -32.4 \\ -43.2 & -14.4 & -32.4 & 289.2 \end{bmatrix}, \quad B := \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad a := \begin{bmatrix} -8.56 \\ -9.52 \\ -9.92 \\ -16.64 \end{bmatrix} \\ C &:= \begin{bmatrix} -132.4 & -10.8 \\ -10.8 & -103.6 \\ 43.2 & 14.4 \\ 14.4 & 4.8 \end{bmatrix}, \quad D := \begin{bmatrix} 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ -13.24 & -1.08 & +4.32 & +1.44 \\ -1.08 & -10.36 & +1.44 & +0.48 \end{bmatrix} \\ E &:= \begin{bmatrix} -10 & 0 \\ 0 & -10 \\ 10 & 0 \\ 0 & 10 \\ -10 & 0 \\ 0 & -10 \end{bmatrix}, \quad d := \begin{bmatrix} -1 \\ -1 \\ -1.5 \\ -3 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Comment: According to [7], the problem has a unique global optimal solution

$$(0.13085, 0.05195, 0.1022, 0.0674, 0.025, 0.05)$$

with the corresponding upper-level and lower-level objective function values being 0.3125 and -0.4063 .

Problem name: CalveteGale1999P1

Source: [9]

Description: CalveteGale1999P1 is defined as follows

$$\begin{aligned} F(x, y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\ G(x, y) &:= -x \\ f(x, y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\ g(x, y) &:= \begin{bmatrix} -y \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix} \end{aligned}$$

Comment: The best known value of the upper-level objective function is -29.2 and can be achieved at $(0.0, 0.9, 0.0, 0.6, 0.4)$, for example; cf. [22].

Problem name: ClarkWesterberg1990a

Source: [10]

Description: ClarkWesterberg1990a is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 3)^2 + (y - 2)^2 \\ G(x, y) &:= \begin{pmatrix} x - 8 \\ -x \end{pmatrix} \\ f(x, y) &:= (y - 5)^2 \\ g(x, y) &:= \begin{pmatrix} -2x + y - 1 \\ x - 2y - 2 \\ x + 2y - 14 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(1.0, 3.0)$; cf. [44].

Problem name: Colson2002BIPA1

Source: [11]

Description: Colson2002BIPA1 is defined as follows

$$\begin{aligned} F(x, y) &:= (10 - x)^3 + (10 - y)^3 \\ G(x, y) &:= \begin{pmatrix} x - 5 \\ -x + y \\ -x \end{pmatrix} \\ f(x, y) &:= (x + 2y - 15)^4 \\ g(x, y) &:= \begin{pmatrix} x + y - 20 \\ y - 20 \\ -y \end{pmatrix} \end{aligned}$$

Comment: A global optimal solution is $(5, 5)$.

Problem name: Colson2002BIPA2

Source: [11]

Description: Colson2002BIPA2 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 5)^2 + (2y + 1)^2 \\ G(x, y) &:= -x \\ f(x, y) &:= (y - 1)^2 - 1.5xy + x^3 \\ g(x, y) &:= \begin{pmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The global optimal solution is (1, 0); cf. [8].

Problem name: Colson2002BIPA3

Source: [11]

Description: Colson2002BIPA3 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 5)^4 + (2y + 1)^4 \\ G(x, y) &:= \begin{pmatrix} x + y - 4 \\ -x \end{pmatrix} \\ f(x, y) &:= \exp(-x + y) + x^2 + 2xy + y^2 + 2x + 6y \\ g(x, y) &:= \begin{pmatrix} -x + y - 2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is (4, 0); cf. [8].

Problem name: Colson2002BIPA4

Source: [11]

Description: Colson2002BIPA4 as defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + (y - 10)^2 \\ G(x, y) &:= \begin{pmatrix} x + 2y - 6 \\ -x \end{pmatrix} \\ f(x, y) &:= x^3 + 2y^3 + x - 2y - x^2 \\ g(x, y) &:= \begin{pmatrix} -x + 2y - 3 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is (0, 0.6039); cf. [8].

Problem name: Colson2002BIPA5

Source: [11]

Description: Colson2002BIPA5 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - y_2)^4 + (y_1 - 1)^2 + (y_1 - y_2)^2 \\ G(x, y) &:= -x \\ f(x, y) &:= 2x + \exp y_1 + y_1^2 + 4y_1 + 2y_2^2 - 6y_2 \\ g(x, y) &:= \begin{pmatrix} 6x + y_1^2 + \exp y_2 - 15 \\ 5x + y_1^4 - y_2 - 25 \\ y_1 - 4 \\ y_2 - 2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is (1.94, 0, 1.21); cf. [8].

Problem name: Dempe1992a

Source: [12]

Description: Dempe1992a is defined as follows

$$\begin{aligned} F(x, y) &:= y_2 \\ G(x, y) &:= x_1^2 + (x_2 + 1)^2 - 1 \\ f(x, y) &:= \frac{1}{2}(y_1 - 1)^2 + \frac{1}{2}y_2^2 \\ g(x, y) &:= \begin{pmatrix} y_1 + y_2x_1 + x_2 \\ y_1 \end{pmatrix} \end{aligned}$$

Comment: One possible solution is $(0, 0, 0, -0.5)$.

Problem name: Dempe1992b

Source: [12]

Description: Dempe1992b is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 3.5)^2 + (y + 4)^2 \\ f(x, y) &:= (y - 3)^2 \\ g(x, y) &:= y^2 - x \end{aligned}$$

Comment: The upper-and lower-level optimal values are respectively obtained as 31.25 and 4.00 in the paper [8].

Problem name: DempeDutta2012Ex24

Source: [13]

Description: DempeDutta2012Ex24 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 1)^2 + y^2 \\ f(x, y) &:= x^2y \\ g(x, y) &:= y^2 \end{aligned}$$

Comment: The global optimal solution of the problem is $(1, 0)$; cf. [13].

Problem name: DempeDutta2012Ex31

Source: [13]

Description: DempeDutta2012Ex31 is defined as follows

$$\begin{aligned} F(x, y) &:= -y_2 \\ G(x, y) &:= \begin{pmatrix} -x \\ y_1y_2 \\ -y_1y_2 \end{pmatrix} \\ f(x, y) &:= y_1^2 + (y_2 + 1)^2 \\ g(x, y) &:= \begin{pmatrix} (y_1 - x_1)^2 + (y_2 - x_1 - 1)^2 - 1 \\ (y_1 + x_2)^2 + (y_2 - x_2 - 1)^2 - 1 \end{pmatrix} \end{aligned}$$

Comment: The point $(0.71, 0.71, 0, 1)$ is the best known solution of the problem provided in [13, 39].

Problem name: DempeEtal2012

Source: [14]

Description: DempeEtal2012 is defined as follows

$$\begin{aligned} F(x, y) &:= x \\ G(x, y) &:= \begin{pmatrix} -1 - x \\ x - 1 \end{pmatrix} \\ f(x, y) &:= xy \\ g(x, y) &:= \begin{pmatrix} -y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(-1, 1)$; cf. [14].

Problem name: DempeFranke2011Ex41

Source: [15]

Description: DempeFranke2011Ex41 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1 + y_1^2 + y_2^2 \\ G(x, y) &:= \begin{pmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ 1 + x_2 \end{pmatrix} \\ f(x, y) &:= x^\top y \\ g(x, y) &:= \begin{pmatrix} -2y_1 + y_2 \\ y_1 - 2 \\ -y_2 \\ y_2 - 2 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(0, -1, 1, 2)$; cf. [15].

Problem name: DempeFranke2011Ex42

Source: [15]

Description: DempeFranke2011Ex42 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1 + (y_1 - 1)^2 + y_2^2 \\ G(x, y) &:= \begin{pmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ 1 + x_2 \end{pmatrix} \\ f(x, y) &:= x^\top y \\ g(x, y) &:= \begin{pmatrix} -y_1 + y_2 - 1 \\ y_1 + y_2 - 3.5 \\ y_2 - 2 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(1, -1, 0, 1)$; cf. [15].

Problem name: DempeFranke2014Ex38

Source: [16]

Description: DempeFranke2014Ex38 is defined as follows

$$\begin{aligned} F(x, y) &:= 2x_1 + x_2 + 2y_1 - y_2 \\ G(x, y) &:= \begin{pmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ x_2 + 0.75 \end{pmatrix} \\ f(x, y) &:= x_1^\top y \\ g(x, y) &:= \begin{pmatrix} -2y_1 + y_2 \\ y_1 - 2 \\ -y_2 \\ y_2 - 2 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(-1, -1, 2, 2)$; cf. [16].

Problem name: DempeLohse2011Ex31a

Source: [17]

Description: DempeLohse2011Ex31a is defined as follows

$$\begin{aligned} F(x, y) &:= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 3y_1 - 3y_2 \\ f(x, y) &:= x_1 y_1 + x_2 y_2 \\ g(x, y) &:= \begin{pmatrix} y_1 + y_2 - 2 \\ -y_1 + y_2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The point $(0.5, 0.5, 1, 1)$ is the unique global optimal solution of the problem according to [17].

Problem name: DempeLohse2011Ex31b

Source: [17]

Description: DempeLohse2011Ex31b is defined as follows

$$\begin{aligned} F(x, y) &:= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 + x_3^2 - 3y_1 - 3y_2 - 6x_3 \\ f(x, y) &:= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ g(x, y) &:= \begin{pmatrix} y_1 + y_2 + y_3 - 2 \\ -y_1 + y_2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The point $(0.5, 0.5, 0, 1, 1, 0)$ is a local optimal solution of the problem according to [17]. While a suggested the solution is $(0.5, 0.5, 0, 0, 0, 2)$.

Problem name: DeSilva1978

Source: [18]

Description: DeSilva1978 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2 \\ f(x, y) &:= (y_1 - x_1)^2 + (y_2 - x_2)^2 \\ g(x, y) &:= \begin{pmatrix} -y + (0.5)_2 \\ y - (1.5)_2 \end{pmatrix} \end{aligned}$$

Comment: The best known upper-and lower-level values obtained in [8] are -1.00 and 0.00 , respectively.

Problem name: EdmundsBard1991

Source: [19]

Description: EdmundsBard1991 is defined as follows

$$\begin{aligned} F(x, y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 50_2 \end{pmatrix} \\ f(x, y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ g(x, y) &:= \begin{pmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ 2y - x + 10_2 \\ -10_2 - y \\ y - 20_2 \end{pmatrix} \end{aligned}$$

Comment: The best know solution of the problem is $(0, 0, -10, -10)$; [49].

Problem name: FalkLiu1995

Source: [20]

Description: FalkLiu1995 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1^2 - 3x_1 + x_2^2 - 3x_2 + y_1^2 + y_2^2 \\ f(x, y) &:= (y_1 - x_1)^2 + (y_2 - x_2)^2 \\ g(x, y) &:= \begin{pmatrix} -y + (0.5)_2 \\ y - (1.5)_2 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(\sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2)$ according to [20].

Problem name: FloudasZlobec1998

Source: [21]

Description: FloudasZlobec1998 is defined as follows

$$\begin{aligned} F(x, y) &:= x^3 y_1 + y_2 \\ G(x, y) &:= \begin{pmatrix} x - 1 \\ -x \end{pmatrix} \\ f(x, y) &:= -y_2 \\ g(x, y) &:= \begin{pmatrix} -y_1 - 1 \\ y_1 - 1 \\ -y_2 \\ y_2 - 100 \\ xy_1 - 10 \\ y_1^2 + xy_2 - 1 \end{pmatrix} \end{aligned}$$

Comment: Notice that explicit bounds on the variable y were added. This is same as [36].
The global optimal solution is $(1, 0, 1)$ according to [22, 36].

Problem name: GumusFloudas2001Ex1

Source: [22]

Description: GumusFloudas2001Ex1 is defined as follows

$$\begin{aligned} F(x, y) &:= 16x^2 + 9y^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 12.5 \\ -4x + y \end{pmatrix} \\ f(x, y) &:= (x + y - 20)^4 \\ g(x, y) &:= \begin{pmatrix} -y \\ y - 50 \\ 4x + y - 50 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (11.25, 5); cf. [36].

Problem name: GumusFloudas2001Ex3

Source: [22]

Description: GumusFloudas2001Ex3 is defined as follows

$$\begin{aligned} F(x, y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\ G(x, y) &:= \begin{bmatrix} -x \\ x - 2_2 \end{bmatrix} \\ f(x, y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\ g(x, y) &:= \begin{bmatrix} -y \\ y - 2_3 \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (0, 0.9, 0, 0.6, 0.4); cf. [36].

Problem name: GumusFloudas2001Ex4

Source: [22]

Description: GumusFloudas2001Ex4 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 3)^2 + (y - 2)^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 8 \\ -2x + y - 1 \\ x - 2y + 2 \\ x + 2y - 14 \end{pmatrix} \\ f(x, y) &:= (y - 5)^2 \\ g(x, y) &:= \begin{pmatrix} -y \\ y - 10 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (3, 5); cf. [36].

Problem name: GumusFloudas2001Ex5

Source: [22]

Description: GumusFloudas2001Ex5 is defined as follows

$$\begin{aligned} F(x, y) &:= x \\ G(x, y) &:= \begin{pmatrix} -x + 0.1 \\ x - 10 \end{pmatrix} \\ f(x, y) &:= -y_1 + 0.5864y_1^{0.67} \\ g(x, y) &:= \begin{pmatrix} -y + (0.1)_2 \\ y - 10_2 \\ \frac{0.0332333}{y_2} + 0.1y_1 - 1 \\ 4\frac{x}{y_2} + 2\frac{x^{-0.71}}{y_2} + 0.0332333x^{-1.3} - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is (0.193616, 9.9667667, 10); cf. [36].

Problem name: HatzEtal2013

Source: [23]

Description: HatzEtal2013 is defined as follows

$$\begin{aligned} F(x, y) &:= -x + 2y_1 + y_2 \\ f(x, y) &:= (x - y_1)^2 + y_2^2 \\ g(x, y) &:= -y \end{aligned}$$

Comment: The optimal solution of the problem is (0, 0, 0); cf. [23].

Problem name: HendersonQuandt1958

Source: [24]

Description: HendersonQuandt1958 is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}x^2 + \frac{1}{2}xy - 95x \\ G(x, y) &:= \begin{pmatrix} x - 200 \\ -x \end{pmatrix} \\ f(x, y) &:= y^2 + (\frac{1}{2}x - 100)y \\ g(x, y) &:= -y \end{aligned}$$

Comment: The best solution from [26] is (93.33333, 26.667).

Problem name: HenrionSurowiec2011

Source: [25]

Description: HenrionSurowiec2011 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + cy \\ f(x, y) &:= 0.5y^2 - xy \end{aligned}$$

Comment: Here, c is a real-valued parameter. The optimal solution of the problem is $-0.5c(1, 1)$; cf. [25].

Problem name: IshizukaAiyoshi1992a

Source: [27]

Description: IshizukaAiyoshi1992a is defined as follows

$$\begin{aligned} F(x, y) &:= xy_2^2 \\ G(x, y) &:= -x - M \\ f(x, y) &:= y_1 \\ g(x, y) &:= \begin{pmatrix} -x \\ -x - y_1 \\ y_1 - x \\ -M - y_1 - y_2 \\ y_1 + y_2 - M \end{pmatrix} \end{aligned}$$

Comment: Here, M is assumed to be an arbitrarily large number such that $M > 1$. From [27], $(x^*, -M, 0)$ is the optimal solution where $x^* \in [0, M]$.

Problem name: KleniatiAdjiman2014Ex3

Source: [28]

Description: KleniatiAdjiman2014Ex3 is defined as follows

$$\begin{aligned} F(x, y) &:= x - y \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= 0.5xy^2 - xy^3 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: $(0, 1)$ is the optimal solution of the problem according to [28, 44].

Problem name: KleniatiAdjiman2014Ex4

Source: [28]

Description: KleniatiAdjiman2014Ex4 is defined as follows

$$\begin{aligned} F(x, y) &:= \sum_{j=1}^5 (x_j^2 + y_j^2) \\ G(x, y) &:= \begin{pmatrix} y_1 y_2 - x_1 \\ x_2 y_1^2 \\ x_1 - \exp x_2 + y_3 \\ -x - 1_5 \\ x - 1_5 \end{pmatrix} \\ f(x, y) &:= y_1^3 + y_2^2 x_1 + y_2^2 x_2 + 0.1 y_3 + (y_4^2 + y_5^2) x_3 x_4 x_5 \\ g(x, y) &:= \begin{pmatrix} x_1 - y_3^2 - 0.2 \\ -y - 1_5 \\ y - 1_5 \end{pmatrix} \end{aligned}$$

Comment: $(1.0, -(1.0)_9)$ is a possible solution of the problem according to [28, 44].

Problem name: LamparielloSagratella2017Ex23

Source: [29]

Description: LamparielloSagratella2017Ex23 is defined as follows

$$\begin{aligned} F(x, y) &:= x \\ G(x, y) &:= \begin{bmatrix} -x - 1 \\ x - 1 \end{bmatrix} \\ f(x, y) &:= (x - y_1)^2 + (y_2 + 1)^2 \\ g(x, y) &:= \begin{bmatrix} y_1^3 - y_2 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is $(-1, -1, 0)$; cf. [29].

Problem name: LamparielloSagratella2017Ex31

Source: [30]

Description: LamparielloSagratella2017Ex31 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + y^2 \\ G(x, y) &:= -x + 1 \\ f(x, y) &:= y \\ g(x, y) &:= -x - y + 1 \end{aligned}$$

Comment: The optimal solution is $(1, 0)$; cf. [30].

Problem name: LamparielloSagratella2017Ex32

Source: [30]

Description: LamparielloSagratella2017Ex32 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + y^2 \\ f(x, y) &:= (x + y - 1)^2 \end{aligned}$$

Comment: The optimal solution is $(0.5, 0.5)$; cf. [30].

Problem name: LamparielloSagratella2017Ex33

Source: [30]

Description: LamparielloSagratella2017Ex33 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + (y_1 + y_2)^2 \\ G(x, y) &:= -x + 0.5 \\ f(x, y) &:= y_1 \\ g(x, y) &:= \begin{pmatrix} -x - y_1 - y_2 + 1 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(0.5, 0, 0.5)$; cf. [30].

Problem name: LamparielloSagratella2017Ex35

Source: [30]

Description: LamparielloSagratella2017Ex35 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -1 - x \\ x - 1 \end{pmatrix} \\ f(x, y) &:= -y \\ g(x, y) &:= \begin{pmatrix} 2x + y - 2 \\ -y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(\frac{4}{5}, \frac{2}{5})$; cf. [30].

Problem name: LucchettiEtal1987

Source: [31]

Description: LucchettiEtal1987 is defined as follows

$$\begin{aligned} F(x, y) &:= 0.5(1 - x) + xy \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 1 \end{pmatrix} \\ f(x, y) &:= (x - 1)y \\ g(x, y) &:= \begin{pmatrix} -y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(1, 0)$; cf. [31].

Problem name: LuDebSinha2016a

Source: [32]

Description: LuDebSinha2016a is defined as follows

$$\begin{aligned} F(x, y) &:= 2 - \exp \left[- \left(\frac{0.2y - x + 0.6}{0.055} \right)^{0.4} \right] - 0.8 \exp \left[- \left(\frac{0.15y - 0.4 + x}{0.3} \right)^2 \right] \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 1 \\ -y \\ y - 2 \end{pmatrix} \\ f(x, y) &:= 2 - \exp \left[- \left(\frac{1.5y - x}{0.055} \right)^{0.4} \right] - 0.8 \exp \left[- \left(\frac{2y - 3 + x}{0.5} \right)^2 \right] \end{aligned}$$

Comment: $(1.4, 0.2)$ is a possible solution for the problem; [32]. Clearly, the $x^* = 1.4$ violates the constraints. One possible solution is $(0, 1.5)$.

Problem name: LuDebSinha2016b

Source: [32]

Description: LuDebSinha2016b is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 0.5)^2 + (y - 1)^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 1 \\ -y \\ y - 2 \end{pmatrix} \\ f(x, y) &:= 2 - \exp \left[- \left(\frac{1.5y - x}{0.055} \right)^{0.4} \right] - 0.8 \exp \left[- \left(\frac{2y - 3 + x}{0.5} \right)^2 \right] \end{aligned}$$

Comment: $(0.5, 1)$ is a possible solution for the problem; [32].

Problem name: LuDebSinha2016c

Source: [32]

Description: LuDebSinha2016c is defined as follows

$$\begin{aligned} F(x, y) &:= 2 - \exp \left[- \left(\frac{0.2y - x + 0.6}{0.055} \right)^{0.4} \right] - 0.8 \exp \left[- \left(\frac{0.15y - 0.4 + x}{0.3} \right)^2 \right] \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 1 \\ -y \\ y - 2 \end{pmatrix} \\ f(x, y) &:= (x - 0.5)^2 + (y - 1)^2 \end{aligned}$$

Comment: (0.26, 1) is a possible solution for the problem; [32].

Problem name: LuDebSinha2016d

Source: [32]

Description: LuDebSinha2016d is defined as follows

$$\begin{aligned} F(x, y) &:= -x_2 \\ g(x, y) &:= \begin{pmatrix} -(\frac{y_1}{14} + \frac{16}{7})(x_1 - 2)^2 + x_2 \\ -x_2 + (\frac{y_1}{14} + \frac{16}{7})(x_1 - 5) \\ -[x_1 + 4 - (\frac{y_1}{14} + \frac{16}{7})][x_1 + 8 - (\frac{y_1}{14} + \frac{16}{7})] + x_2 \\ -4 - x_1 \\ x_1 - 10 \\ -100 - x_2 \\ x_2 - 200 \\ -4 - y_1 \\ y_1 - 10 \\ -100 - y_2 \\ y_2 - 200 \end{pmatrix} \\ f(x, y) &:= -y_2 \\ g(x, y) &:= \begin{pmatrix} -(\frac{x_1}{14} + \frac{16}{7})(y_1 - 2)^2 + y_2 \\ -y_2 + 12.5(\frac{x_1}{14} + \frac{16}{7})(y_1 - 5) \\ -5[y_1 + 4 - (\frac{x_1}{14} + \frac{16}{7})][y_1 + 8 - (\frac{x_1}{14} + \frac{16}{7})] + y_2 \end{pmatrix} \end{aligned}$$

Comment: A possible solution is (10, 192, 10, 192).

Problem name: LuDebSinha2016e

Source: [32]

Description: LuDebSinha2016e is defined as follows

$$\begin{aligned} F(x, y) &:= \left(\frac{y_2 - 50}{30} \right)^2 + \left(\frac{x - 2.5}{0.2} \right)^2 \\ G(x, y) &:= \begin{pmatrix} 2 - x \\ x - 3 \\ -4 - y_1 \\ y_1 - 10 \\ -100 - y_2 \\ y_2 - 200 \end{pmatrix} \\ f(x, y) &:= -y_2 \\ g(x, y) &:= \begin{pmatrix} -x(y_1 - 2)^2 + y_2 \\ -y_2 + 12.5x(y_1 - 5) \\ -5(y_1 + 4 - x)(y_1 + 8 - x) + y_2 \end{pmatrix} \end{aligned}$$

Comment: The solution is unknown.

Problem name: LuDebSinha2016f

Source: [32]

Description: LuDebSinha2016f is defined as follows

$$\begin{aligned}
 F(x, y) &:= -x_2 \\
 G(x, y) &:= \begin{pmatrix} 2 - y \\ y - 4 \\ -80 - x_1 \\ x_1 - 200 \\ -100 - x_2 \\ x_2 - 200 \\ -y \left(\frac{x_1}{20} - 2 \right) + x_2 \\ -x_2 + 12.5y \left(\frac{x_1}{20} - 5 \right) \\ -5 \left(\frac{x_1}{20} + 4 - y \right) \left(\frac{x_1}{20} + 8 - y \right) + x_2 \end{pmatrix} \\
 f(x, y) &:= \left(\frac{x_1 - 50}{28} \right)^2 + \left(\frac{y - 2.5}{0.2} \right)^2
 \end{aligned}$$

Comment: The solution is unknown.

Problem name: MacalHurter1997

Source: [34]

Description: MacalHurter1997 is defined as follows

$$\begin{aligned}
 F(x, y) &:= (x - 1)^2 + (y - 1)^2 \\
 f(x, y) &:= 0.5y^2 + 500y - 50xy
 \end{aligned}$$

Comment: The optimal solution is (10.0163, 0.8197); cf. [34].

Problem name: Mirrlees1999

Source: [35]

Description: Mirrlees1999 is defined as follows

$$\begin{aligned}
 F(x, y) &:= (x - 2)^2 + (y - 1)^2 \\
 f(x, y) &:= -x \exp [-(y + 1)^2] - \exp [-(y - 1)^2]
 \end{aligned}$$

Comment: This problem is known in the literature as Mirrlees problem. It is usually used to illustrate how the KKT reformulation of the bilevel optimization problem is not appropriate for problems with nonconvex lower-level problems. The optimal solution for the problem is (1, 0.95753); [35].

Problem name: MitsosBarton2006Ex38

Source: [36]

Description: MitsosBarton2006Ex38 is defined as follows

$$\begin{aligned} F(x, y) &:= y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \\ -y - 0.1 \\ y - 0.1 \end{pmatrix} \\ f(x, y) &:= (x + \exp x)y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(-0.567, 0)$; cf. [36].

Problem name: MitsosBarton2006Ex39

Source: [36]

Description: MitsosBarton2006Ex39 is defined as follows

$$\begin{aligned} F(x, y) &:= x \\ G(x, y) &:= \begin{pmatrix} -x + y \\ -x + 10 \\ x - 10 \end{pmatrix} \\ f(x, y) &:= y^3 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(-1, -1)$; cf. [36].

Problem name: MitsosBarton2006Ex310

Source: [36]

Description: MitsosBarton2006Ex310 is defined as follows

$$\begin{aligned} F(x, y) &:= y \\ G(x, y) &:= \begin{pmatrix} -x + 0.1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= x(16y^4 + 2y^3 - 8y^2 - 1.5y + 0.5) \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The set of all optimal solution is $[0.1, 1] \times \{0.5\}$; cf. [36].

Problem name: MitsosBarton2006Ex311

Source: [36]

Description: MitsosBarton2006Ex311 is defined as follows

$$\begin{aligned} F(x, y) &:= y \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= x(16y^4 + 2y^3 - 8y^2 - 1.5y + 0.5) \\ g(x, y) &:= \begin{pmatrix} -y - 0.8 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0, -0.8)$; cf. [36].

Problem name: MitsosBarton2006Ex312

Source: [36]

Description: MitsosBarton2006Ex312 is defined as follows

$$\begin{aligned} F(x, y) &:= -x + xy + 10y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= -xy^2 + 0.5y^4 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0, 0)$; cf. [36].

Problem name: MitsosBarton2006Ex313

Source: [36]

Description: MitsosBarton2006Ex313 is defined as follows

$$\begin{aligned} F(x, y) &:= x - y \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= 0.5xy^2 - x^3y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0, 1)$; cf. [36].

Problem name: MitsosBarton2006Ex314

Source: [36]

Description: MitsosBarton2006Ex314 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 0.25)^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= \frac{1}{3}y^3 - xy \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.25, 0.5)$; cf. [36].

Problem name: MitsosBarton2006Ex315

Source: [36]

Description: MitsosBarton2006Ex315 is defined as follows

$$\begin{aligned} F(x, y) &:= x + y \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= \frac{1}{2}xy^2 - \frac{1}{3}y^3 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(-1, 1)$; cf. [36].

Problem name: MitsosBarton2006Ex316

Source: [36]

Description: MitsosBarton2006Ex316 is defined as follows

$$\begin{aligned} F(x, y) &:= 2x + y \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= -\frac{1}{2}xy^2 - \frac{1}{4}y^4 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The points $(-1, 0)$ and $(-0.5, -1)$ are the two optimal solutions of the problem; cf. [36].

Problem name: MitsosBarton2006Ex317

Source: [36]

Description: MitsosBarton2006Ex317 is defined as follows

$$\begin{aligned} F(x, y) &:= (x + \frac{1}{2})^2 + \frac{1}{2}y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= \frac{1}{2}xy^2 + \frac{1}{4}y^4 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The points $(-0.25, 0.5)$ and $(-0.25, -0.5)$ are the two optimal solutions of the problem; cf. [36].

Problem name: MitsosBarton2006Ex318

Source: [36]

Description: MitsosBarton2006Ex318 is defined as follows

$$\begin{aligned} F(x, y) &:= -x^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= xy^2 - \frac{1}{2}y^4 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.5, 0)$; cf. [36].

Problem name: MitsosBarton06Ex319

Source: [36]

Description: MitsosBarton2006Ex319 is defined as follows

$$\begin{aligned} F(x, y) &:= xy - y + \frac{1}{2}y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= -xy^2 + \frac{1}{2}y^4 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.189, 0.4343)$; cf. [36].

Problem name: MitsosBarton2006Ex320

Source: [36]

Description: MitsosBarton2006Ex320 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - \frac{1}{4})^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= \frac{1}{3}y^3 - x^2y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (0.5, 0.5); cf. [36].

Problem name: MitsosBarton2006Ex321

Source: [36]

Description: MitsosBarton2006Ex321 is defined as follows

$$\begin{aligned} F(x, y) &:= (x + 0.6)^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= y^4 + \frac{4}{30}(-x + 1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2 \\ &\quad + (0.004x^3 - 0.036x^2 + 0.08x)y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (−0.5545, 0.4554); cf. [36].

Problem name: MitsosBarton2006Ex322

Source: [36]

Description: MitsosBarton2006Ex322 is defined as follows

$$\begin{aligned} F(x, y) &:= (x + 0.6)^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \end{pmatrix} \\ f(x, y) &:= y^4 + \frac{2}{15}(-x + 1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2 \\ &\quad + (0.004x^3 - 0.036x^2 + 0.08x)y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \\ 0.01(1 + x)^2 - y^2 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (−0.5545, 0.4554); cf. [36].

Problem name: MitsosBarton2006Ex323

Source: [36]

Description: MitsosBarton2006Ex323 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \\ 1 + x - 9x^2 - y \end{pmatrix} \\ f(x, y) &:= y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \\ y^2(x - 0.5) \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(-0.4191, -1)$; cf. [36].

Problem name: MitsosBarton2006Ex324

Source: [36]

Description: MitsosBarton2006Ex324 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 - y \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 1 \end{pmatrix} \\ f(x, y) &:= [(y - 1 - 0.1x)^2 - 0.5 - 0.5x]^2 \\ g(x, y) &:= \begin{pmatrix} -y \\ y - 3 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.2106, 1.799)$; cf. [36].

Problem name: MitsosBarton2006Ex325

Source: [36]

Description: MitsosBarton2006Ex325 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1 y_1 + x_2 y_1^2 - x_1 x_2 y_3 \\ G(x, y) &:= \begin{pmatrix} -x - 1_2 \\ x - 1_2 \\ 0.1 y_1 y_2 - x_1^2 \\ x_2 y_1^2 \end{pmatrix} \\ f(x, y) &:= x_1 y_1^2 + x_2 y_2 y_3 \\ g(x, y) &:= \begin{pmatrix} -y - 1_3 \\ y - 1_3 \\ y_1^2 - y_2 y_3 \\ y_2^2 y_3 - y_1 x_1 \\ -y_3^2 + 0.1 \end{pmatrix} \end{aligned}$$

Comment: The best known value for the upper-level objective function is -1 and a corresponding point is $(-1, -1, -1, 1, 1)$; cf. [36].

Problem name: MitsosBarton2006Ex326

Source: [36]

Description: MitsosBarton2006Ex326 is defined as follows

$$\begin{aligned}
 F(x, y) &:= x_1 y_1 + x_2 y_2^2 + x_1 x_2 y_3^3 \\
 G(x, y) &:= \begin{pmatrix} 0.1 - x_1^2 \\ 1.5 - y_1^2 - y_2^2 - y_3^2 \\ 2.5 + y_1^2 + y_2^2 + y_3^2 \\ -x - 1_2 \\ x - 1_2 \end{pmatrix} \\
 f(x, y) &:= x_1 y_1^2 + x_2 y_2^2 + (x_1 - x_2) y_3^2 \\
 g(x, y) &:= \begin{pmatrix} -y - 1_3 \\ y - 1_3 \end{pmatrix}
 \end{aligned}$$

Comment: The optimal solution of the problem is $(-1, -1, 1, 1, -0.707)$; cf. [36].

Problem name: MitsosBarton2006Ex327

Source: [36]

Description: MitsosBarton2006Ex327 is defined as follows

$$\begin{aligned}
 F(x, y) &:= \sum_{j=1}^5 (x_j^2 + y_j^2) \\
 G(x, y) &:= \begin{pmatrix} -x - 1_5 \\ x - 1_5 \\ y_1 y_2 - x_1 \\ x_2 y_1^2 \\ x_1 - \exp x_2 + y_3 \end{pmatrix} \\
 f(x, y) &:= y_1^3 + y_2^2 x_1 + y_2^2 x_2 + 0.1 y_3 + (y_4^2 + y_5^2) x_3 x_4 x_5 \\
 g(x, y) &:= \begin{pmatrix} -y - 1_5 \\ y - 1_5 \\ y_1 y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4 y_5 - 0.1 \end{pmatrix}
 \end{aligned}$$

Comment: The best known value for the upper-level objective function is 2 and a corresponding point is¹ $(0_5, -1, 0, -1, 0, 0)$; cf. [36].

Problem name: MitsosBarton2006Ex328

Source: [36]

¹ 0_5 is used to indicate the origin of \mathbb{R}^5

Description: MitsosBarton2006Ex328 is defined as follows

$$\begin{aligned} F(x, y) &:= -\sum_{j=1}^5 (x_j^2 + y_j^2) \\ G(x, y) &:= \begin{pmatrix} -x - 1_5 \\ x - 1_5 \\ y_1 y_2 - x_1 \\ x_2 y_1^2 \\ x_1 - \exp x_2 + y_3 \end{pmatrix} \\ f(x, y) &:= y_1^3 + y_2^2 x_1 + y_2^2 x_2 + 0.1 y_3 + (y_4^2 + y_5^2) x_3 x_4 x_5 \\ g(x, y) &:= \begin{pmatrix} -y - 1_5 \\ y - 1_5 \\ y_1 y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4 y_5 - 0.1 \end{pmatrix} \end{aligned}$$

Comment: The best known values for the upper and lower-level objective functions are -10 and -3.1 respectively, and a corresponding point is $(1, (-1)_5, 1, -1, -1, 1)$; cf. [36]. Another possible solution is $((-1)_5, 1, -1, -1, -1, 1)$ with the same the upper and lower-level objective function values.

Problem name: MorganPatrone2006a

Source: [37]

Description: MorganPatrone2006a is defined as follows

$$\begin{aligned} F(x, y) &:= -(x + y) \\ G(x, y) &:= \begin{pmatrix} -0.5 - x \\ x - 0.5 \end{pmatrix} \\ f(x, y) &:= xy \\ g(x, y) &:= \begin{pmatrix} -1 - y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(0, 1)$; cf. [37].

Problem name: MorganPatrone2006b

Source: [37]

Description: MorganPatrone2006b is defined as follows

$$\begin{aligned} F(x, y) &:= -(x + y) \\ f(x, y) &:= \begin{cases} (x + 0.25)y & \text{if } x \in [-0.5, -0.25] \\ 0 & \text{if } x \in [-0.25, 0.25] \\ (x - 0.25)y & \text{if } x \in [0.25, 0.5] \end{cases} \\ g(x, y) &:= \begin{pmatrix} -0.5 - x \\ x - 0.5 \\ -1 - y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(0.25, 1)$; cf. [37].

Problem name: MorganPatrone2006c

Source: [37]

Description: MorganPatrone2006c is defined as follows

$$\begin{aligned} F(x, y) &:= -(x + y) \\ f(x, y) &:= \begin{cases} (x + \frac{7}{4})y & \text{if } x \in [-2, -\frac{7}{4}] \\ 0 & \text{if } x \in [-\frac{7}{4}, \frac{7}{4}] \\ (x - \frac{7}{4})y & \text{if } x \in [\frac{7}{4}, 2] \end{cases} \\ g(x, y) &:= \begin{pmatrix} -2 - x \\ x - 2 \\ -1 - y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(2, -1)$; cf. [37].

Problem name: MuuQuy2003Ex1

Source: [38]

Description: MuuQuy2003Ex1 is defined as follows

$$\begin{aligned} F(x, y) &:= y_1^2 + y_2^2 + x^2 - 4x \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 2 \end{pmatrix} \\ f(x, y) &:= y_1^2 + \frac{1}{2}y_2^2 + y_1y_2 + (1 - 3x)y_1 + (1 + x)y_2 \\ g(x, y) &:= \begin{pmatrix} 2y_1 + y_2 - 2x - 1 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is $(0.8438, 0.7657, 0)$ according to [38].

Problem name: MuuQuy2003Ex2

Source: [38]

Description: MuuQuy2003Ex2 is defined as follows

$$\begin{aligned} F(x, y) &:= y_1^2 + y_3^2 - y_1y_3 - 4y_2 - 7x_1 + 4x_2 \\ G(x, y) &:= \begin{pmatrix} -x \\ x_1 + x_2 - 1 \end{pmatrix} \\ f(x, y) &:= y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2 + y_1y_2 + (1 - 3x_1)y_1 + (1 + x_2)y_2 \\ g(x, y) &:= \begin{pmatrix} 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is $(0.609, 0.391, 0.000, 0.000, 1.828)$; cf. [38].

Problem name: NieEtal2017Ex34

Source: [39]

Description: NieEtal2017Ex34 is defined as follows

$$\begin{aligned} F(x, y) &:= x + y_1 + y_2 \\ G(x, y) &:= \begin{pmatrix} -x + 2 \\ -3 + x \end{pmatrix} \\ f(x, y) &:= x(y_1 + y_2) \\ g(x, y) &:= \begin{pmatrix} -y_1^2 + y_2^2 + (y_1^2 + y_2^2)^2 \\ -y_1 \end{pmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(2, 0, 0)$; cf. [39].

Problem name: NieEtal2017Ex52

Source: [39]

Description: NieEtal2017Ex52 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1 y_1 + x_2 y_2 + x_1 x_2 y_1 y_2 y_3 \\ G(x, y) &:= \begin{pmatrix} -1_2 - x \\ x - 1_2 \\ y_1 y_2 - x_1^2 \end{pmatrix} \\ f(x, y) &:= x_1 y_1^2 + x_2^2 y_2 y_3 - y_1 y_3^2 \\ g(x, y) &:= \begin{pmatrix} 1 - y_1^2 - y_2^2 - y_3^2 \\ y_1^2 + y_2^2 + y_3^2 - 2 \end{pmatrix} \end{aligned}$$

Comment: The point $(-1, -1, 1.1097, 0.3143, -0.8184)$ is the best known solution of the problem provided in [39].

Problem name: NieEtal2017Ex54

Source: [39]

Description: NieEtal2017Ex54 is defined as follows

$$\begin{aligned} F(x, y) &:= x_1^2 y_1 + x_2 y_2 + x_3 y_3^2 + x_4 y_4^2 \\ G(x, y) &:= \begin{pmatrix} \|x\|^2 - 1 \\ y_1 y_2 - x_1 \\ y_3 y_4 - x_3^2 \end{pmatrix} \\ f(x, y) &:= y_1^2 - y_2(x_1 + x_2) - (y_3 + y_4)(x_3 + x_4) \\ g(x, y) &:= \begin{pmatrix} \|y\|^2 - 1 \\ y_2^2 + y_3^2 + y_4^2 - y_1 \end{pmatrix} \end{aligned}$$

Comment: $(0, -0, -0.7071, -0.7071, 0.6180, 0, -0.5559, -0.5559)$ is the best known solution of the problem obtained in [39].

Problem name: NieEtal2017Ex57

Source: [39]

Description: NieEtal2017Ex57 is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2} x_1^2 y_1 + x_2 y_2^2 - (x_1 + x_2^2) y_3 \\ G(x, y) &:= \begin{pmatrix} -1_2 - x \\ x - 1_2 \\ -x_1 - x_2 + x_1^2 + y_1^2 + y_2^2 \end{pmatrix} \\ f(x, y) &:= x_2(y_1 y_2 y_3 + y_2^2 - y_3^3) \\ g(x, y) &:= \begin{pmatrix} -x_1 + y_1^2 + y_2^2 + y_3^2 \\ -1 + 2y_2 y_3 \end{pmatrix} \end{aligned}$$

Comment: The point $(1, 1, 0, 0, 1)$ is the best known solution of the problem provided in [39].

Problem name: NieEtal2017Ex58

Source: [39]

Description: NieEtal2017Ex58 is defined as follows

$$\begin{aligned} F(x, y) &:= (x_1 + x_2 + x_3 + x_4)(y_1 + y_2 + y_3 + y_4) \\ G(x, y) &:= \begin{pmatrix} \|x\|^2 - 1 \\ y_3^2 - x_4 \\ y_2 y_4 - x_1 \end{pmatrix} \\ f(x, y) &:= x_1 y_1 + x_2 y_2 + 0.1 y_3 + 0.5 y_4 - y_3 y_4 \\ g(x, y) &:= \begin{pmatrix} y_1^2 + 2y_2^2 + 3y_3^2 + 4y_4^2 - x_1^2 - x_3^2 - x_2 - x_4 \\ -y_2 y_3 + y_1 y_4 \end{pmatrix} \end{aligned}$$

Comment: (0.5135, 0.5050, 0.4882, 0.4929, -0.8346, -0.4104, -0.2106, -0.2887) is the best known solution of the problem obtained in [39].

Problem name: NieEtal2017Ex61

Source: [39]

Description: NieEtal2017Ex61 is defined as follows

$$\begin{aligned} F(x, y) &:= y_1^3(x_1^2 - 3x_1 x_2) - y_1^2 y_2 + y_2 x_2^3 \\ G(x, y) &:= \begin{pmatrix} -x - 1_2 \\ x - 1_2 \\ -y_2 - y_1(1 - x_1^2) \end{pmatrix} \\ f(x, y) &:= y_1 y_2^2 - y_2^3 - y_1^2(x_2 - x_1^2) \\ g(x, y) &:= y_1^2 + y_2^2 - 1 \end{aligned}$$

Comment: The point (0.5708, -1, -0.1639, 0.9865) is the best known solution of the problem provided in [39].

Problem name: Outrata1990Ex1a

Source: [40]

Description: Outrata1990Ex1a is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}(y_1^2 + y_2^2) - 3y_1 - 4y_2 + r(x_1^2 + x_2^2) \\ f(x, y) &:= \frac{1}{2}\langle y, Hy \rangle - \langle b(x), y \rangle \\ g(x, y) &:= \begin{pmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{pmatrix} \end{aligned}$$

with $r := 0.1$, $H := \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $b(x) := x$.

Comment: Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are obtained by respectively replacing r , H , and $b(x)$ in the lower-level objective function of Outrata1990a by

$$\begin{aligned} r &:= 1, \quad H := \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad \text{and } b(x) := x, \\ r &:= 0, \quad H := \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad \text{and } b(x) := x, \\ r &:= 0.1, \quad H := \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad \text{and } b(x) := x, \\ r &:= 0.1, \quad H := \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad \text{and } b(x) := \begin{bmatrix} -1 & 2 \\ 3 & -3 \end{bmatrix} x. \end{aligned}$$

According to [40], the best known solutions for problems Outrata1990Ex1a, Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are respectively

$$(0.97, 3.14, 2.6, 1.8), (0.28, 0.48, 2.34, 1.03), \\ (20.26, 42.81, 3, 3), (2, 0.06, 2, 0), \text{ and } (2.42, -3.65, 0, 1.58).$$

Note that for Outrata1990Ex1b and Outrata1990Ex1c, the solutions above change with a different starting point for the algorithm used in [40].

Problem name: Outrata1990Ex2a

Source: [40]

Description: Outrata1990Ex2a is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2} [(y_1 - 3)^2 + (y_2 - 4)^2] \\ G(x, y) &:= -x \\ f(x, y) &:= \frac{1}{2} \langle y, H(x)y \rangle - (3 + 1.333x)y_1 - xy_2 \\ g(x, y) &:= \begin{pmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{pmatrix} \end{aligned}$$

with the matrix $H(x)$ defined by $H(x) := I$.

Comment: Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d and Outrata1990Ex2e are respectively obtained by performing some changes on the terms $H(x)$ and $g(x, y)$ in the lower-level objective function of Outrata1990Ex2a:

$$\begin{aligned} H(x) &:= \begin{bmatrix} 1+x & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } g(x, y) := \begin{pmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{pmatrix}, \\ H(x) &:= \begin{bmatrix} 1+x & 0 \\ 0 & 1+0.1x \end{bmatrix}, \text{ and } g(x, y) := \begin{pmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{pmatrix}, \\ H(x) &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } g(x, y) := \begin{pmatrix} (-0.333 + 0.1x)y_1 + y_2 - x \\ y_1 + (-0.333 - 0.1x)y_2 - 2 \\ -y \end{pmatrix}, \\ H(x) &:= \begin{bmatrix} 1+x & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } g(x, y) := \begin{pmatrix} (-0.333 + 0.1x)y_1 + y_2 - x \\ y_1 + (-0.333 - 0.1x)y_2 - 2 \\ -y \end{pmatrix}, \end{aligned}$$

According to [40], the best known solutions for problems Outrata1990Ex2a, Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d, Outrata1990Ex2e are respectively

$$(2.07, 3, 3), (0, 3, 3), (3.456, 1.707, 2.569), \\ (2.498, 3.632, 2.8) \text{ and } (3.999, 1.665, 3.887), .$$

Problem name: Outrata1993Ex31

Source: [41]

Description: Outrata1993Ex31 is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}(y_1 - 3)^2 + \frac{1}{2}(y_2 - 4)^2 \\ G(x, y) &:= -x \\ f(x, y) &:= \frac{1}{2}(1 + 0.2x)y_1^2 + \frac{1}{2}(1 + 0.1x)y_2^2 - (3 + 1.33x)y_1 - xy_2 \\ g(x, y) &:= \begin{pmatrix} (-0.333 + 0.1x)y_1 + y_2 + 0.1x - 2 \\ y_1 + (-0.333 - 0.1x)y_2 + 0.1x - 2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: Outrata1993Ex32 is obtained by replacing the lower-level constraint by

$$g(x, y) := \begin{pmatrix} -0.333y_1 + y_2 + 0.1x - 1 \\ y_1^2 + y_2^2 - 0.1x - 9 \\ -y \end{pmatrix}.$$

For Outrata1993Ex1 and Outrata1993Ex2, the best known solutions from [41] are (1.90910, 2.97836, 2.23182) and (4.06095, 2.68227, 1.48710), respectively.

Problem name: Outrata1994Ex31

Source: [42]

Description: Outrata1994Ex31 is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}(y_1 - 3)^2 + \frac{1}{2}(y_2 - 4)^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 10 \end{pmatrix} \\ f(x, y) &:= \frac{1}{2}(1 + 0.2x)y_1^2 + \frac{1}{2}(1 + 0.1x)y_2^2 - (3 + 1.333x)y_1 - xy_2 \\ g(x, y) &:= \begin{pmatrix} -0.333y_1 + y_2 + 0.1x - 1 \\ y_1^2 + y_2^2 - 0.1x - 9 \\ -y \end{pmatrix} \end{aligned}$$

Comment: According to [42], the best known solution is (4.0604, 2.6822, 1.4871).

Problem name: OutrataCervinka2009

Source: [43]

Description: OutrataCervinka2009 is defined as follows

$$\begin{aligned} F(x, y) &:= -2x_1 - 0.5x_2 - y_2 \\ G(x, y) &:= x_1 \\ f(x, y) &:= y_1 - y_2 + x^\top y + \frac{1}{2}y^\top y \\ g(x, y) &:= \begin{pmatrix} y_2 \\ y_2 - y_1 \\ y_2 + y_1 \end{pmatrix} \end{aligned}$$

Comment: The point 0_4 is an optimal solution of the problem according to [43].

Problem name: PaulaviciusEtal2017a

Source: [44]

Description: PaulaviciusEtal2017a is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \\ -y - 1 \\ y - 1 \end{pmatrix} \\ f(x, y) &:= xy^2 - \frac{1}{2}y^4 \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: This problem a slight modification of MitsosBarton2006Ex318, just with the upper-level objective function there replaced by $x^2 + y^2$. Doing so, the point $(0.5, 0)$ remains optimal for the new problem [44].

Problem name: PaulaviciusEtal2017b

Source: [44]

Description: PaulaviciusEtal2017b is defined as follows

$$\begin{aligned} F(x, y) &:= x + y \\ G(x, y) &:= \begin{pmatrix} -x - 1 \\ x - 1 \\ -y - 1 \\ y - 1 \end{pmatrix} \\ f(x, y) &:= 0.5xy^2 - x^3y \\ g(x, y) &:= \begin{pmatrix} -y - 1 \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: This problem a slight modification of MitsosBarton2006Ex313, just with the *minus* in upper-level objective function replaced by a *plus*. Doing so, the optimal solution the new problem above is $(-1, -1)$ according to [44].

Problem name: SahinCircic1998Ex2

Source: [45]

Description: SahinCircic1998Ex2 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 3)^2 + (y - 2)^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 8 \end{pmatrix} \\ f(x, y) &:= (y - 5)^2 \\ g(x, y) &:= \begin{pmatrix} -2x + y - 1 \\ x - 2y + 2 \\ x + 2y - 14 \end{pmatrix} \end{aligned}$$

Comment: The optimal value for the upper-level objective function is 5 and a corresponding optimal point is $(1, 3)$; cf. [45].

Problem name: ShimizuAiyoshi1981Ex1

Source: [46]

Description: ShimizuAiyoshi1981Ex1 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + (y - 10)^2 \\ G(x, y) &:= \begin{pmatrix} x - 15 \\ -x + y \\ -x \end{pmatrix} \\ f(x, y) &:= (x + 2y - 30)^2 \\ g(x, y) &:= \begin{pmatrix} x + y - 20 \\ y - 20 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (10, 10) according to [46].

Problem name: ShimizuAiyoshi1981Ex2

Source: [46]

Description: ShimizuAiyoshi1981Ex2 is defined as follows

$$\begin{aligned} F(x, y) &:= (x_1 - 30)^2 + (x_2 - 20)^2 - 20y_1 + 20y_2 \\ G(x, y) &:= \begin{pmatrix} -x_1 - 2x_2 + 30 \\ x_1 + x_2 - 25 \\ x_2 - 15 \end{pmatrix} \\ f(x, y) &:= (x_1 - y_1)^2 + (x_2 - y_2)^2 \\ g(x, y) &:= \begin{pmatrix} y - 10_2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is (20, 5, 10, 5) according to [46].

Problem name: ShimizuEtal1997a

Source: [47]

Description: ShimizuEtal1997a is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 5)^2 + (2y + 1)^2 \\ f(x, y) &:= (y - 1)^2 - 1.5xy \\ g(x, y) &:= \begin{bmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \end{bmatrix} \end{aligned}$$

Comment: A possible solution is (5, 2).

Problem name: ShimizuEtal1997b

Source: [47]

Description: ShimizuEtal1997b is defined as follows

$$\begin{aligned} F(x, y) &:= 16x^2 + 9y^2 \\ G(x, y) &:= \begin{pmatrix} -4x + y \\ -x \end{pmatrix} \\ f(x, y) &:= (x + y - 20)^4 \\ g(x, y) &:= \begin{pmatrix} 4x + y - 50 \\ -y \end{pmatrix} \end{aligned}$$

Comment: (11.25, 5) is the global optimal solution of the problem and (7.2, 12.8) is a local optimal solution [47].

Problem name: SinhaMaloDeb2014TP3

Source: [48]

Description: SinhaMaloDeb2014TP3 is defined as follows

$$\begin{aligned} F(x, y) &:= -x_1^2 - 3x_2^2 - 4y_1 + y_2^2 \\ G(x, y) &:= \begin{pmatrix} x_1^2 + 2x_2 - 4 \\ -x \end{pmatrix} \\ f(x, y) &:= 2x_1^2 + y_1^2 - 5y_2 \\ g(x, y) &:= \begin{pmatrix} -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -18.6787 and -1.0156 , respectively; cf. [48].

Problem name: SinhaMaloDeb2014TP6

Source: [48]

Description: SinhaMaloDeb2014TP6 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 1)^2 + 2y_1 - 2x \\ G(x, y) &:= -x \\ f(x, y) &:= (2y_1 - 4)^2 + (2y_2 - 1)^2 + xy_1 \\ g(x, y) &:= \begin{pmatrix} 4x + 5y_1 + 4y_2 - 12 \\ 4y_2 - 4x - 5y_1 + 4 \\ 4x - 4y_1 + 5y_2 - 4 \\ 4y_1 - 4x + 5y_2 - 4 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.2091 and 7.6145 , respectively; cf. [48].

Problem name: SinhaMaloDeb2014TP7

Source: [48]

Description: SinhaMaloDeb2014TP7 is defined as follows

$$\begin{aligned} F(x, y) &:= -\frac{(x_1 + y_1)(x_2 + y_2)}{1 + x_1 y_1 + x_2 y_2} \\ G(x, y) &:= \begin{pmatrix} x_1^2 + x_2^2 - 100 \\ x_1 - x_2 \\ -x \end{pmatrix} \\ f(x, y) &:= \frac{(x_1 + y_1)(x_2 + y_2)}{1 + x_1 y_1 + x_2 y_2} \\ g(x, y) &:= \begin{pmatrix} y - x \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.96 and 1.96 , respectively; cf. [48].

Problem name: SinhaMaloDeb2014TP8

Source: [48]

Description: SinhaMaloDeb2014TP8 is defined as follows

$$\begin{aligned} F(x, y) &:= |2x_1 + 2x_2 - 3y_1 - 3y_2 - 60| \\ g(x, y) &:= \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ x - 50_2 \\ -x \end{bmatrix} \\ f(x, y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ g(x, y) &:= \begin{bmatrix} 2y - x + 10_2 \\ y - 20_2 \\ -y - 10_2 \end{bmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are 0 and 100.0, respectively; cf. [48].

Problem name: SinhaMaloDeb2014TP9

Source: [48]

Description: SinhaMaloDeb2014TP9 is defined as follows

$$\begin{aligned} F(x, y) &:= \sum_{i=1}^{10} [(x_i - 1)^2 + y_i^2] \\ f(x, y) &:= \exp \left\{ \left[1 + \frac{1}{400} \sum_{i=1}^{10} y_i^2 - \prod_{i=1}^{10} \cos \left(\frac{y_i}{\sqrt{i}} \right) \right] \sum_{i=1}^{10} x_i^2 \right\} \\ g(x, y) &:= \begin{pmatrix} y - \pi_{10} \\ -y - \pi_{10} \end{pmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [48].

Problem name: SinhaMaloDeb2014TP10

Source: [48]

Description: SinhaMaloDeb2014TP10 is defined as follows

$$\begin{aligned} F(x, y) &:= \sum_{i=1}^{10} [(x_i - 1)^2 + y_i^2] \\ f(x, y) &:= \exp \left[1 + \frac{1}{4000} \sum_{i=1}^{10} x_i^2 y_i^2 - \prod_{i=1}^{10} \cos \left(\frac{x_i y_i}{\sqrt{i}} \right) \right] \\ g(x, y) &:= \begin{pmatrix} y - \pi_{10} \\ -y - \pi_{10} \end{pmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [48].

Problem name: TuyEtal2007

Source: [49]

Description: TuyEtal2007 is defined as follows

$$\begin{aligned} F(x, y) &:= x^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x \\ -y \end{pmatrix} \\ f(x, y) &:= -y \\ g(x, y) &:= \begin{pmatrix} 3x + y - 15 \\ x + y - 7 \\ x + 3y - 15 \end{pmatrix} \end{aligned}$$

Comment: The best known solution of the problem is (4.492188, 1.523438); cf. [49].

Problem name: Vogel2002

Source: [50]

Description: Vogel2002 is defined as follows

$$\begin{aligned} F(x, y) &:= (y + 1)^2 \\ G(x, y) &:= \begin{pmatrix} -3 - x \\ x - 2 \end{pmatrix} \\ f(x, y) &:= y^3 - 3y \\ g(x, y) &:= x - y \end{aligned}$$

Comment: The point $(-2, -2)$ is the optimal solution of the problem; cf. [50].

Problem name: WanWangLv2011

Source: [51]

Description: WanWangLv2011 is defined as follows

$$\begin{aligned} F(x, y) &:= (1 + x_1 - x_2 + 2y_2)(8 - x_1 - 2y_1 + y_2 + 5y_3) \\ f(x, y) &:= 2y_1 - y_2 + y_3 \\ g(x, y) &:= \begin{pmatrix} -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - 0.5y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - 0.5y_3 - 1 \\ -x \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution is $(0, 0.75, 0, 0.5, 0)$ according to [51].

Problem name: YeZhu2010Ex42

Source: [52]

Description: YeZhu2010Ex42 is defined as follows

$$\begin{aligned} F(x, y) &:= (x - 1)^2 + y^2 \\ G(x, y) &:= \begin{pmatrix} -x - 3 \\ x - 2 \end{pmatrix} \\ f(x, y) &:= y^3 - 3y \\ g(x, y) &:= x - y \end{aligned}$$

Comment: YeZhu2010Ex43 is obtained by replacing the upper-level objective function by

$$(x - 1)^2 + (y - 2)^2$$

Note that YeZhu2010Ex42 is a slightly modified version of Vogel2002, with the term y^2 added to the upper-level objective function. The point $(1, 1)$ is the optimal solution for both YeZhu2010Ex42 and YeZhu2010Ex43; cf. [52].

Problem name: Yezza1996Ex31

Source: [53]

Description: Yezza1996Ex31 is defined as follows

$$\begin{aligned} F(x, y) &:= -(4x - 3)y + 2x + 1 \\ G(x, y) &:= \begin{pmatrix} -x \\ x - 1 \end{pmatrix} \\ f(x, y) &:= -(1 - 4x)y - 2x - 2 \\ g(x, y) &:= \begin{pmatrix} -y \\ y - 1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.25, 0)$; cf. [53].

Problem name: Yezza1996Ex41

Source: [53]

Description: Yezza1996Ex41 is defined as follows

$$\begin{aligned} F(x, y) &:= \frac{1}{2}(y-2)^2 + \frac{1}{2}(x-y-2)^2 \\ f(x, y) &:= \frac{1}{2}y^2 + x - y \\ g(x, y) &:= \begin{pmatrix} -y \\ y - x \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(3, 1)$; cf. [53].

Problem name: Zlobec2001a

Source: [54]

Description: Zlobec2001a is defined as follows

$$\begin{aligned} F(x, y) &:= -y_1/x \\ f(x, y) &:= -y_1 - y_2 \\ g(x, y) &:= \begin{pmatrix} x + y_1 \\ y_2 - 1 \\ -y \end{pmatrix} \end{aligned}$$

Comment: This example is used in [54] to illustrate that the objective function of the problem can be discontinuous. As stated in [54], an optimal solution is $(1, 1, 0)$.

Problem name: Zlobec2001b

Source: [54]

Description: Zlobec2001b is defined as follows

$$\begin{aligned} F(x, y) &:= x + y \\ G(x, y) &:= \begin{pmatrix} x - 1 \\ -x \end{pmatrix} \\ f(x, y) &:= -y \\ g(x, y) &:= \begin{pmatrix} y - 1 \\ -y \\ xy \\ -xy \end{pmatrix} \end{aligned}$$

Comment: This example is used in [54] to illustrate that the feasible set of a bilevel optimization problem is not necessarily closed. As stated in [54], this problem does not have an optimal solution.

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