UVA CS 6316: Machine Learning

Lecture 13: Maximum Likelihood Estimation (MLE)

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Last : Probability Review

- The big picture
- Events and Event spaces
- Random variables
- Joint probability, Marginalization, conditioning, chain rule, Bayes Rule, law of total probability, etc.
- Structural properties, e.g., Independence, conditional independence
- Maximum Likelihood Estimation

Sample space and Events

- O : Sample Space,
  - result of an experiment / set of all outcomes
  - If you toss a coin twice O = {HH,HT,TH,TT}
- Event: a subset of O
  - First toss is head = {HH,HT}
- S: event space, a set of events:
  - Contains the empty event and O

From Events to Random Variable

- Concise way of specifying attributes of outcomes
- Modeling students (Grade and Intelligence):
  - $\Omega =$ all possible students (sample space)
  - What are events (subset of sample space)
    - Grade_A = all students with grade A
    - Grade_B = all students with grade B
    - HardWorking_Yes = … who works hard
  - Very cumbersome

  - Need "functions" that maps from Ω to an attribute space T.
  - P(H = YES) = P({student ϵ Ω : H(student) = YES})

If hard to directly estimate from data, most likely we can estimate

- 1. Joint probability
  - Use Chain Rule

- 2. Marginal probability
  - Use the total law of probability

- 3. Conditional probability
  - Use the Bayes Rule

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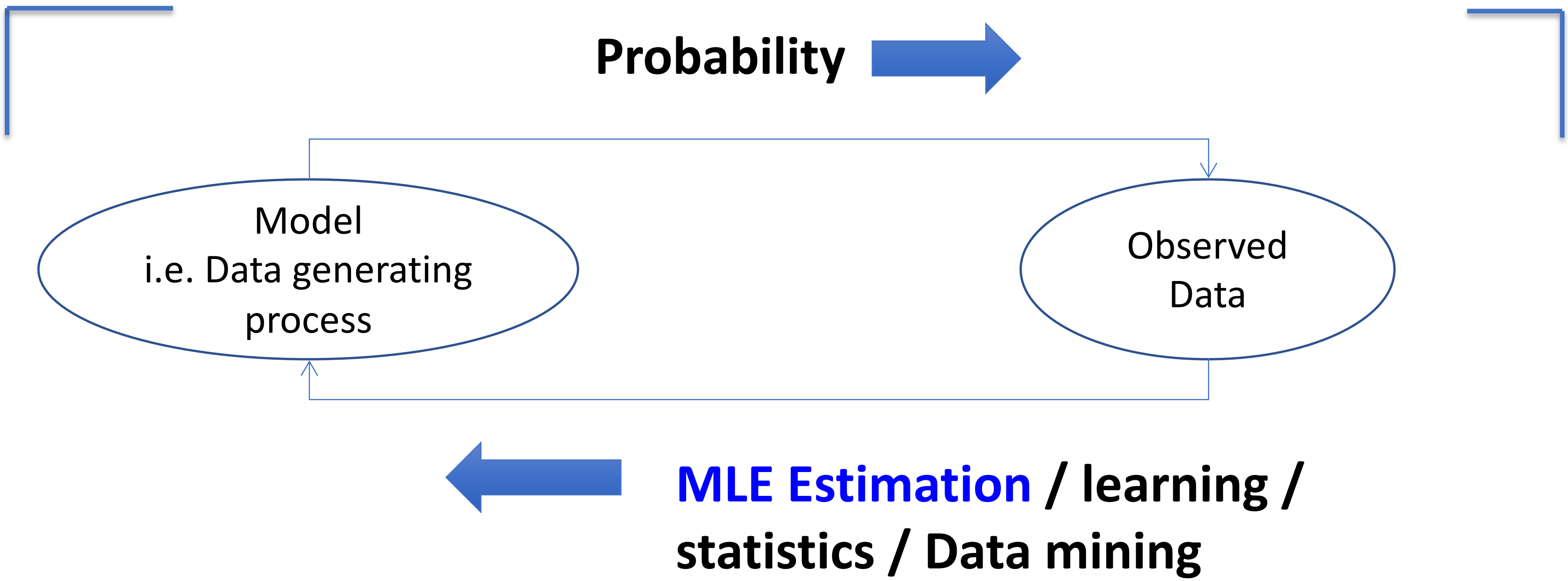
The Big Picture



Today

- ❑ Basic MLE
- ❑ MLE for Discrete RV
- ❑ MLE for Continuous RV (Gaussian)
- ❑ MLE connects to Normal Equation of LR
- ❑ More about Mean and Variance

Maximum Likelihood Estimation

A general Statement

Consider a sample set $T=(X_1...X_n)$ which is drawn from a probability distribution P(X|\theta) where \theta are parameters.

If the Xs are independent with probability density function $P(X_i|\theta)$, the joint probability of the whole set is

$$P(X_1 \dots X_n|\theta) = \prod_{i=1}^{n} P(X_i|\theta)$$

this may be maximised with respect to \theta to give the maximum likelihood estimates.

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$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} P(X_1 \ldots X_n / \theta)$$

This is maximum likelihood. In most cases it is both consistent and efficient.

$$\log(L(\theta)) = \sum_{i=1}^{n} \log(P(X_i / \theta))$$

It is often convenient to work with the Log of the likelihood function.

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Likelihood

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Log-Likelihood

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Discrete Random Variables

- Random variables (RVs) which may take on only a **countable** number of **distinct** values
  - E.g. the total number of heads X you get if you flip 100 coins
- X is a RV with arity *k* if it can take on exactly one value out of
  - E.g. the possible values that X can take on are 0, 1, 2,…, 100

$$\left\{x_1, \ldots, x_k\right\}$$

e.g. Coin Flips cont.

$\{H, T\}$

- You flip a coin
  - Head with probability *p*
  - Binary random variable
  - Bernoulli trial with success probability *p*
- You flip *a* coin for *k* times
  - How many heads would you expect
  - Number of heads X is a discrete random variable
  - Binomial distribution with parameters *k* and *p*

Review: Bernoulli Distribution e.g. Coin Flips

- You flip *n* coins
  - How many heads would you expect
  - Head with probability *p*
  - Number of heads X out of n trial
  - Each Trial following Bernoulli distribution with parameters *p*

N trails, e.g. $\{H\ H\ T\ H\ H\ T\ H\ T\ \ldots\ H\}$
$X_1\ X_2\ X_3\ X_4\ \ldots\ X_n$

Calculating Likelihood

Given: $\{x_1, x_2, \ldots\ldots, x_n\}$

$\Downarrow$

$\{H, H, T, \ldots H\}$

$\Downarrow$ reformulate

$\{1, 1, 0, \ldots\ldots, 1\}$

$$p(x_i|\theta) = p^{x_i}(1-p)^{1-x_i}$$

(Here $x_i \in \{0, 1\}$)

Defining Likelihood for Bernoulli

- Likelihood = p(data | parameter)

➔ e.g., for n independent tosses of coins, with unknown parameter p

Observed data ➔ x heads-up from n trials

function of x_i

PMF: $$f(x_i \mid p) = p^{x_i}(1-p)^{1-x_i}$$

$$x = \sum_{i=1}^{n} x_i$$

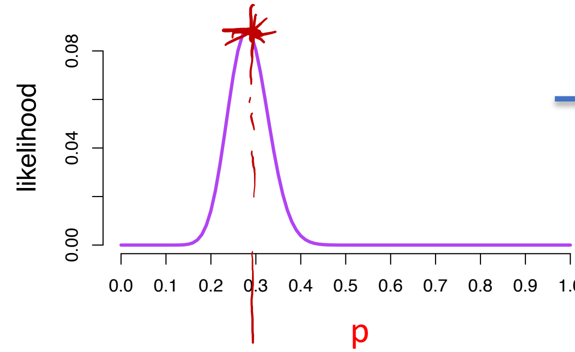
LIKELIHOOD:

$$L(p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{x}(1-p)^{n-x}$$

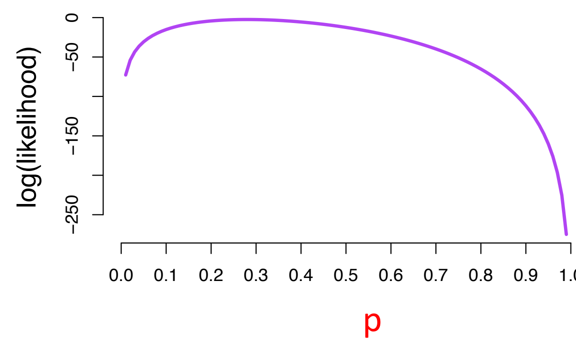
function of p

Deriving the Maximum Likelihood Estimate for Bernoulli

$$\overbrace{L(p)}^{\text{maximize}} = p^x(1-p)^{n-x}$$

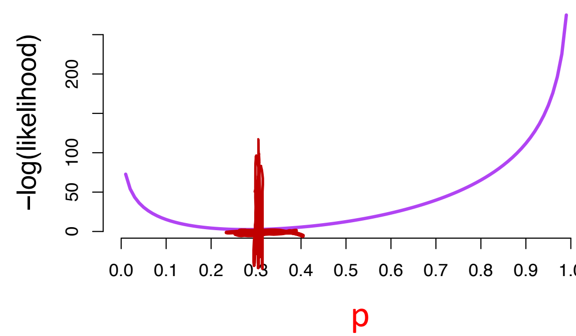


$$\overbrace{\log(L(p))}^{\text{maximize}} = \log\left[p^x(1-p)^{n-x}\right]$$



Minimize the negative log-likelihood

$$-l(p) = -\log\left[p^x(1-p)^{n-x}\right]$$



Deriving the Maximum Likelihood Estimate for Bernoulli

Minimize the negative log-likelihood

$$\underset{p}{\text{argmin}}\left\{-l(p)\right\}=-\log(L(p))=-\log\left[p^x(1-p)^{n-x}\right]$$

$$=-\log(p^x)-\log((1-p)^{n-x})$$

$$=-x\log(p)-(n-x)\log(1-p)$$

Deriving the Maximum Likelihood Estimate for Bernoulli

$$\arg\min_p \{-l(p)\} = \arg\min_p \{-x\log(p) - (n-x)\log(1-p)\}$$

$$\frac{dl(p)}{dp} = -\frac{x}{p} - \frac{-(n-x)}{1-p} = 0$$

$$0 = -\frac{x}{p} + \frac{n-x}{1-p}$$

$$0 = \frac{-x(1-p) + p(n-x)}{p(1-p)}$$

$$0 = -x + px + pn - px$$

$$0 = -x + pn$$

Minimize the negative log-likelihood

➔ MLE parameter estimation

$$\hat{p} = \frac{x}{n}$$

i.e. Relative frequency of a binary event

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Review: Continuous Random Variables

- Probability density function (pdf) instead of probability mass function (pmf)
  - For discrete RV: Probability mass function (pmf): $P(X = x_i)$

- A pdf (prob. Density func.) is any function $f(x)$ that describes the probability density in terms of the input variable x.

Review: Probability of Continuous RV

- Properties of pdf
  - $$f(x) \geq 0, \forall x$$
  - $$\int_{-\infty}^{+\infty} f(x) = 1$$

$$\longrightarrow \sum_{i=1}^{k_i} P(X = x_i) = 1$$

- Actual probability can be obtained by taking the integral of pdf
  - E.g. the probability of X being between 5 and 6 is

$$P(5 \leq X \leq 6) = \int_{5}^{6} f(x)dx$$

Review: Mean and Variance of RV

- Mean (Expectation): $\mu = E(\mathrm{X})$
  - Discrete RVs:

$$E(\mathrm{X}) = \sum_{v_i} v_i \mathrm{P}(\mathrm{X} = v_i)$$

$$E(g(X)) = \sum_{v_i} g(v_i) P(X = v_i)$$

  - Continuous RVs:

$$E(\mathrm{X}) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f(x) dx$$

Adapt From Carols' prob tutorial

Review: Mean and Variance of RV

- Variance: $Var(X) = E((X-\mu)^2)$

$\sigma_X = \sqrt{V(X)}$

  - Discrete RVs:

$$V(\mathrm{X}) = \sum_{v_i} (v_i - \mu)^2 \mathrm{P}(\mathrm{X} = v_i)$$

  - Continuous RVs:

$$V(\mathrm{X}) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx$$

Correlation

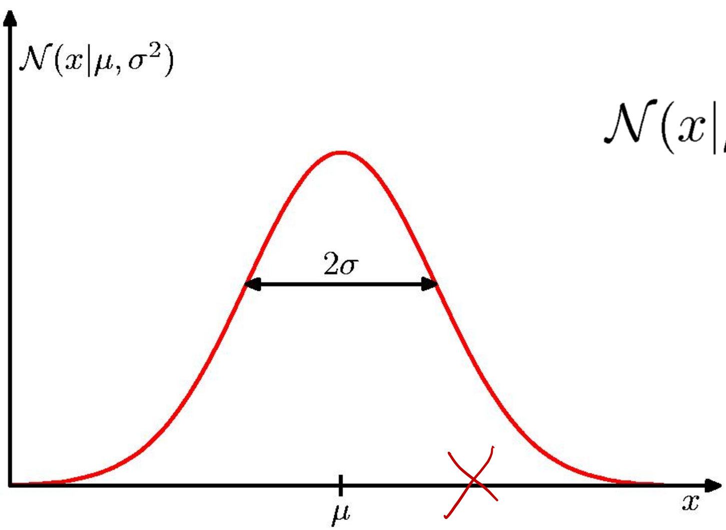
$$\rho_{X,Y} = Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

- Covariance:

$$Cov(X,Y) = E((X-\mu_x)(Y-\mu_y)) = E(XY) - \mu_x \mu_y$$

Adapt From Carols' prob tutorial

Single-Variate Gaussian Distribution

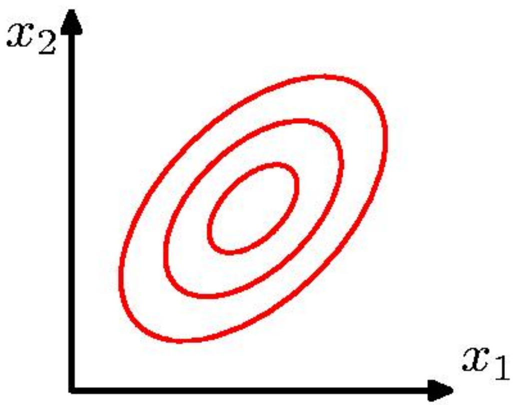


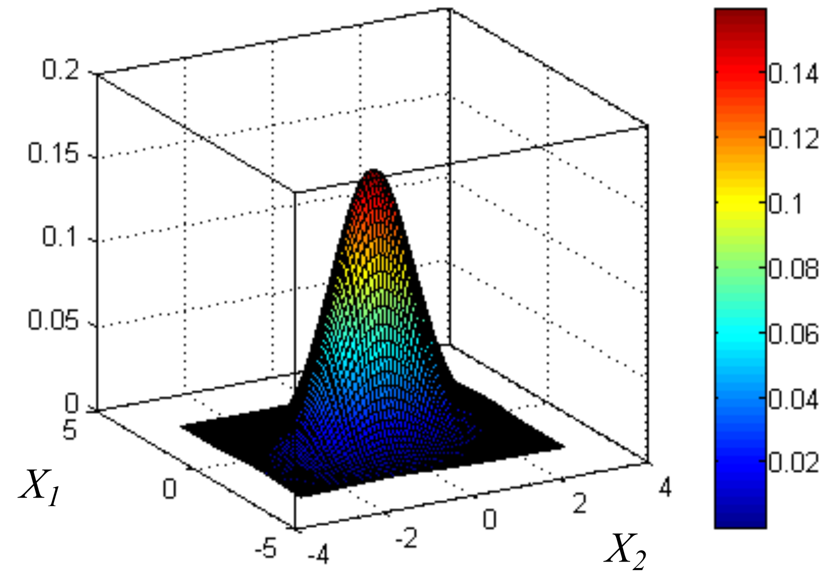
$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$X \sim N\left(\mu, \sigma^2\right)$$

Courtesy: http://research.microsoft.com/~cmbishop/PRML/index.htm

Bi-Variate Gaussian Distribution





Bivariate normal PDF:

- Mean of normal PDF is at peak value. Contours of equal PDF form ellipses.

- The covariance matrix captures linear dependencies among the variables

Courtesy: http://research.microsoft.com/~cmbishop/PRML/index.htm

Multivariate Normal (Gaussian) PDFs

The only widely used continuous joint PDF is the multivariate normal (or Gaussian):

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu}) \right\}$$

Mean

Covariance Matrix

Where |*| represents determinant

- Mean of normal PDF is at peak value. Contours of equal PDF form ellipses.

- The covariance matrix captures linear dependencies among the variables

Example: the Bivariate Normal distribution

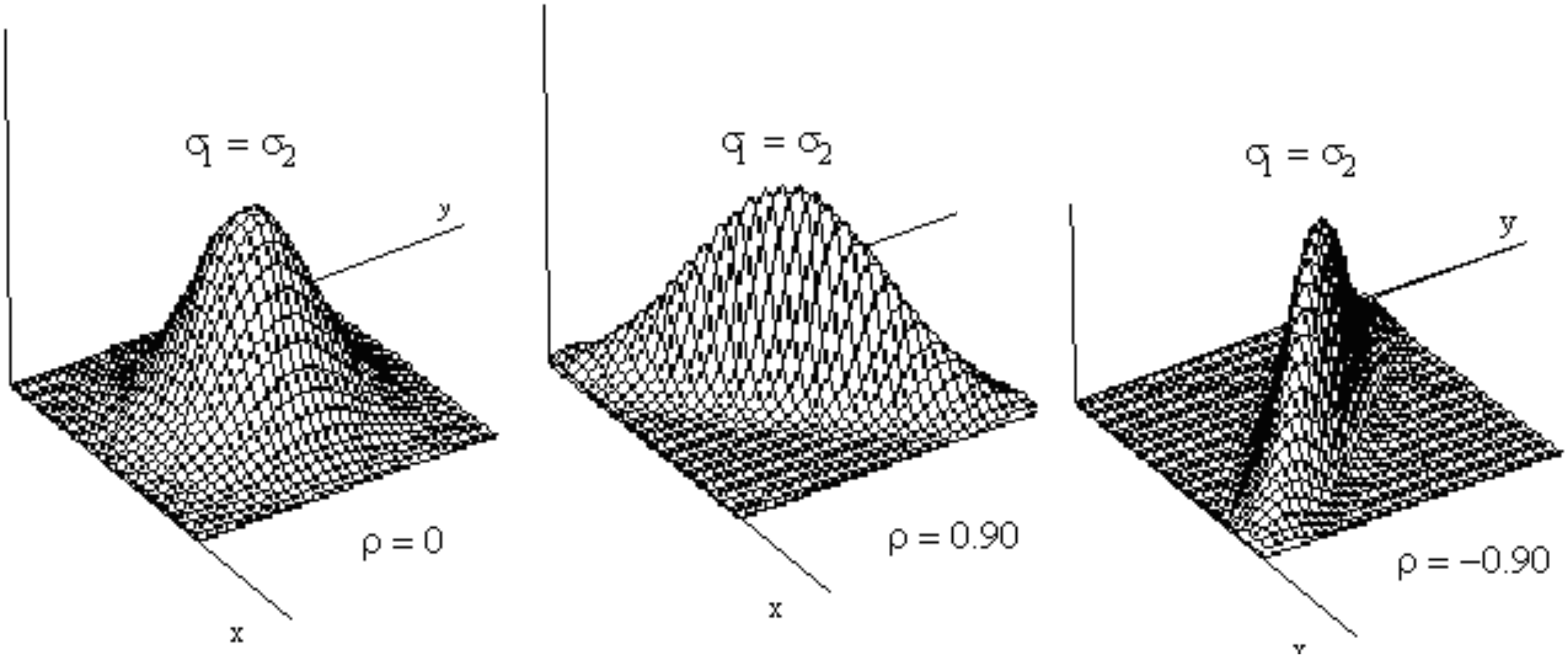
$$f\left(x_1, x_2\right) = \frac{1}{\left(2\pi\right)\left|\Sigma\right|^{1/2}} e^{-\frac{1}{2}\left(\vec{x}-\vec{\mu}\right)^T \Sigma^{-1}\left(\vec{x}-\vec{\mu}\right)}$$

with $\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_{2\times 1}$ and

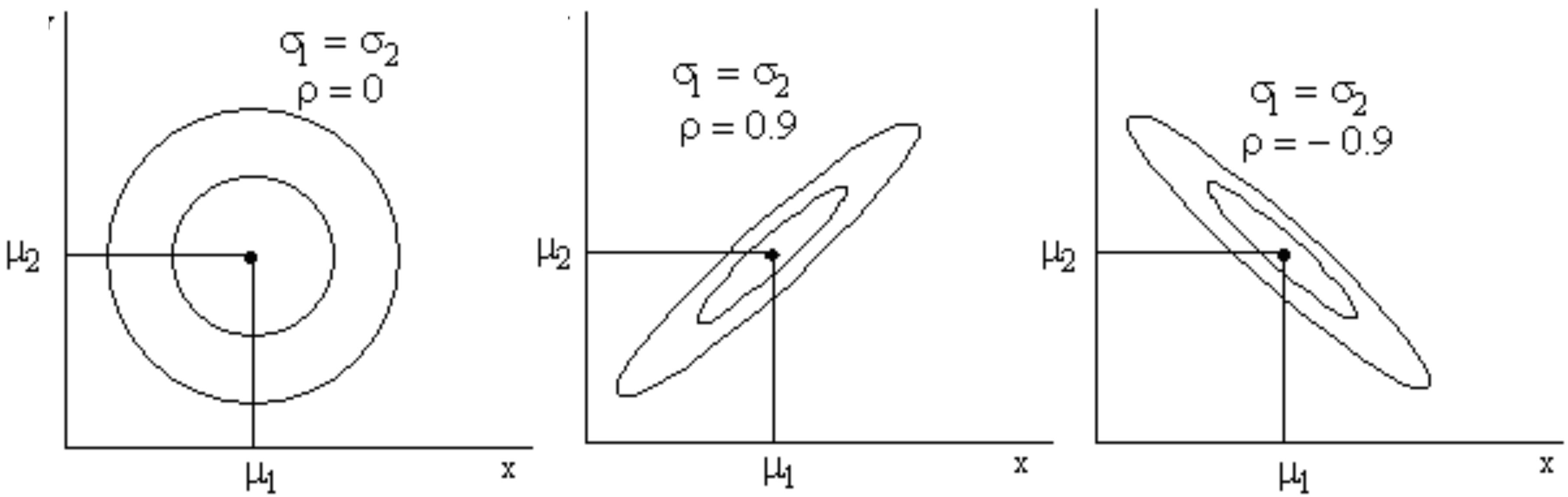
$$\underset{2\times 2}{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \overset{V(X_1)}{\sigma_1^2} & \overset{Cov(X_1,X_2)}{\underline{\rho\sigma_1\sigma_2}} \\ \underline{\rho\sigma_1\sigma_2} & \underset{V(X_2)}{\sigma_2^2} \end{bmatrix}_{2\times 2}$$

$$\left|\Sigma\right| = \sigma_{11}\sigma_{22} - \sigma_{12}^2 = \sigma_1^2\sigma_2^2\left(1-\rho^2\right)$$

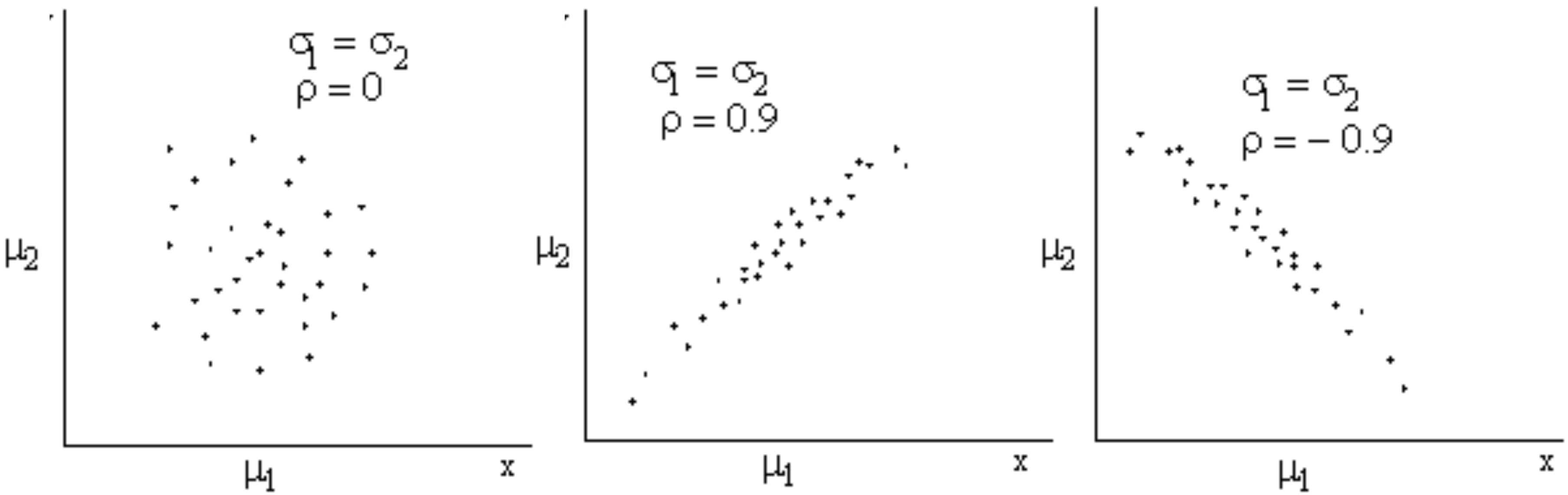
Surface Plots of the bivariate Normal distribution



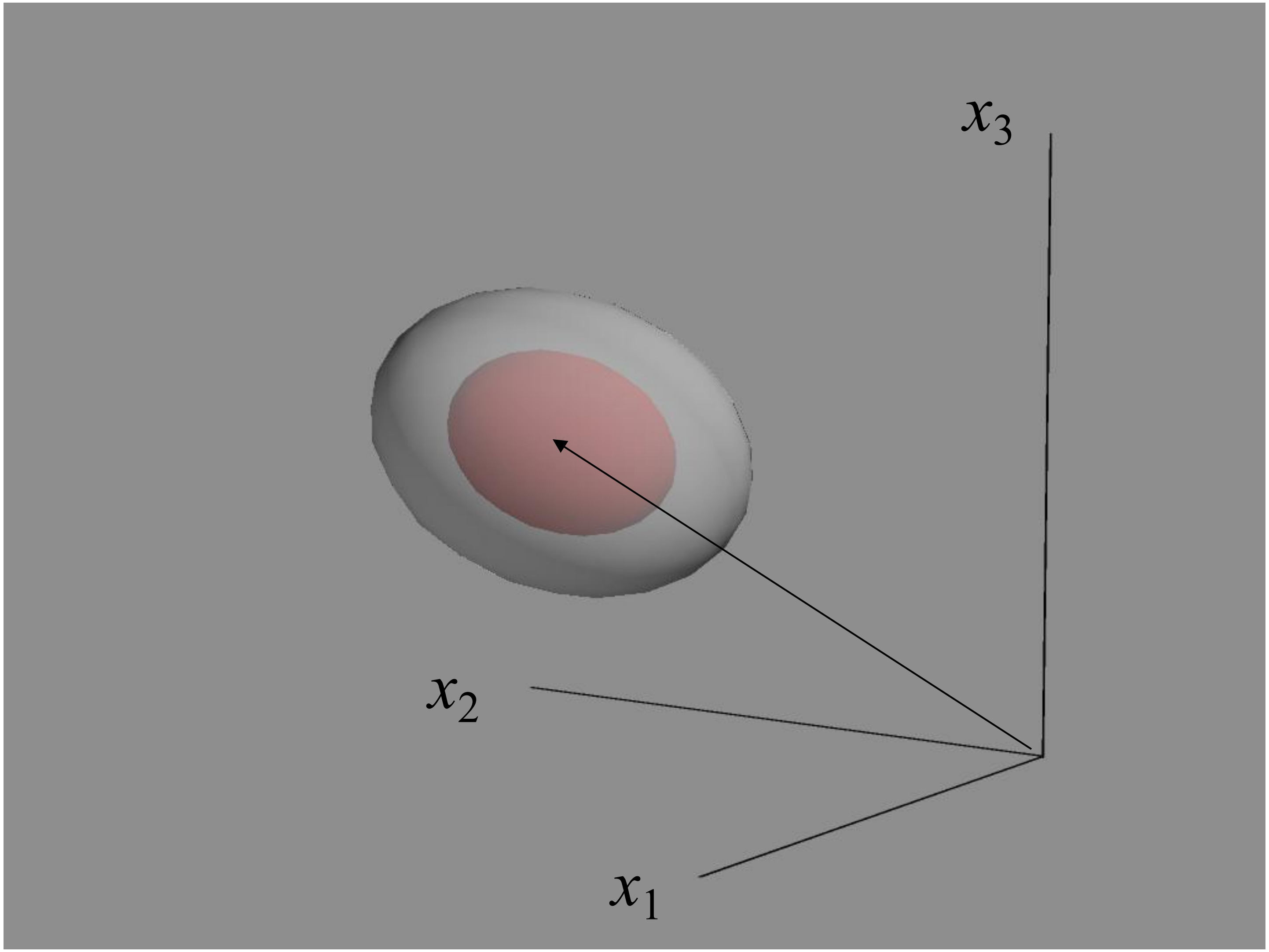
Contour Plots of the bivariate Normal distribution



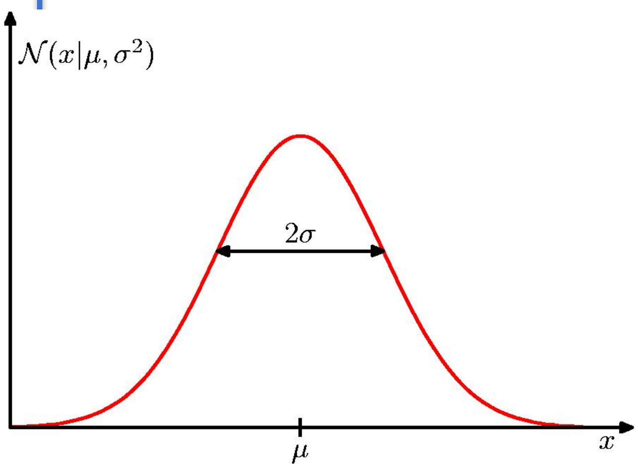
Scatter Plots of data from the bivariate Normal distribution



Trivariate Normal distribution



How to Estimate 1D Gaussian: MLE



- In the 1D Gaussian case, we simply set the mean and the variance to the sample mean and the sample variance:

$$\bar{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\overline{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{\mu}\right)^2$$

How to Estimate p-D Gaussian: MLE

$$< X_1, X_2 \cdots, X_p > \sim N\left(\vec{\mu}, \Sigma\right)$$

$$\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}_{p \times 1}$$

$$\mu_i = \frac{1}{n} \sum_{j=1}^{N} \underbrace{X_j^{(i)}}_{j\text{th sample}}$$

i-th feature $\in \{1, 2, \cdots, p\}$

jth sample $\in \{1, 2, \cdots, N\}$

$$\underset{p \times p}{\Sigma} = \begin{array}{cc} & -i- \\ \begin{bmatrix} Var(X_1) & & Cov(X_i, X_j) \\ & \ddots & \\ & & Var(X_p) \end{bmatrix} & -j- \end{array}$$

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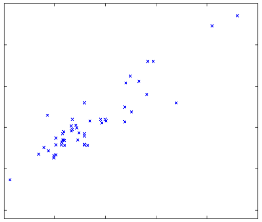
DETOUR: Probabilistic Interpretation of Linear Regression

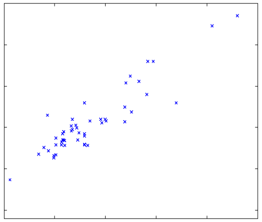


- Let us assume that the target variable and the inputs are related by the equation:

$$y_i = \theta^T \mathbf{x}_i + \varepsilon_i$$

where ε is an error term of unmodeled effects or random noise

DETOUR: Probabilistic Interpretation of Linear Regression



- Let us assume that the target variable and the inputs are related by the equation:

RV $\varepsilon \sim N(0, \sigma^2)$

$$y_i = \theta^T \mathbf{x}_i + \varepsilon_i$$

where ε is an error term of unmodeled effects or random noise

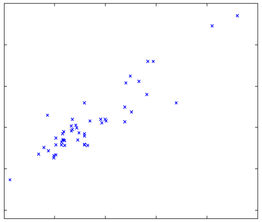
2

- Now assume that ε follows a Gaussian $N(0,\sigma)$, then we have:

$$p(y_i \mid x_i; \theta) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y_i - \theta^T \mathbf{x}_i)^2}{2\sigma^2} \right)$$

RV $y \mid x; \theta \sim N(\theta^T x, \sigma)$

DETOUR: Probabilistic Interpretation of Linear Regression



- By IID (independent and identically distributed) assumption, we have data likelihood

$$L(\theta) = \prod_{i=1}^{n} p(y_i | x_i; \theta) = \left(\frac{1}{\sqrt{2\pi\sigma}} \right)^n \exp\left(-\frac{\sum_{i=1}^{n} (y_i - \theta^T \mathbf{x}_i)^2}{2\sigma^2} \right)$$

$$l(\theta) = \log(L(\theta)) = n \log \frac{1}{\sqrt{2\pi\sigma}} - \frac{1}{\sigma^2} \frac{1}{2} \sum_{i=1}^{n} (y_i - \theta^T \mathbf{x}_i)^2$$

$$L(\theta) = \prod_{i=1}^{n} p(y_i \mid x_i; \theta) = \left(\frac{1}{\sqrt{2\pi\sigma}} \right)^n \exp\left(- \frac{\sum_{i=1}^{n} (y_i - \theta^T \mathbf{x}_i)^2}{2\sigma^2} \right)$$

We can learn \theta by maximizing the probability / likelihood of generating the observed samples:

$$P\{(\vec{x}_1, y_1) \wedge (\vec{x}_2, y_2) \wedge \ldots (\vec{x}_N, y_N)\}$$

$$= \prod_{i=1}^{N} p(y_i, \vec{x}_i) = \prod_{i=1}^{N} p(y_i \mid \vec{x}_i; \theta) p(\vec{x}_i) \quad \text{IID}$$

$$\theta^* = \underset{\theta}{\operatorname{argmax}} \prod_{i=1}^{N} p(y_i \mid \vec{x}_i; \theta)$$

Thus under independence Gaussian residual assumption, residual square error is equivalent to MLE of ϑ !

$y|x;\theta \sim N(\theta^T x, \sigma)$

$\Downarrow$

Two Unknown parameters : $\{\theta, \sigma\}$

$$l(\theta) = \log(L(\theta)) = n\log\frac{1}{\sqrt{2\pi\sigma}} - \frac{1}{\sigma^2}\frac{1}{2}\sum_{i=1}^{n}(y_i - \theta^T \mathbf{x}_i)^2$$

$\operatorname{argmax} l(\theta) \Rightarrow \operatorname{argmin} J(\theta)$

$\Downarrow$

$$J(\theta) = \frac{1}{2}\sum_{i=1}^{n}(\mathbf{x}_i^T\theta - y_i)^2$$

$\boxed{y_i \sim N(exp(wx_i), 1)}$

(b) (6 points) (no explanation required) Suppose you decide to do a maximum likelihood estimation of w. You do the math and figure out that you need w to satisfy one of the following equations. Which one?

A. $\Sigma_i x_i exp(wx_i) = \Sigma_i x_i y_i exp(wx_i)$
B. $\Sigma_i x_i exp(2wx_i) = \Sigma_i x_i y_i exp(wx_i)$
C. $\Sigma_i x_i^2 exp(wx_i) = \Sigma_i x_i y_i exp(wx_i)$
D. $\Sigma_i x_i^2 exp(wx_i) = \Sigma_i x_i y_i exp(wx_i/2)$
E. $\Sigma_i exp(wx_i) = \Sigma_i y_i exp(wx_i)$

$y_i \sim N(exp(wx_i), 1)$

Answer: B (this is an extra credit question.)

$L(\theta)$

$\downarrow$

$LL(\theta)$

$\downarrow$

$\frac{\partial LL(\theta)}{\partial \theta} = 0 \Rightarrow (B)$

References

- Prof. Andrew Moore's review tutorial
- Prof. Nando de Freitas's review slides
- Prof. Carlos Guestrin recitation slides

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- ❑ Extra: about Mean and Variance

Mean and Variance

- Correlation:

$$\rho(X,Y) = Cov(X,Y)/\sigma_x\sigma_y$$

$$-1 \leq \rho(X,Y) \leq 1$$

Properties

- Mean
$$E(\mathrm{X}+\mathrm{Y})=E(\mathrm{X})+E(\mathrm{Y})$$
$$E(a\mathrm{X})=aE(\mathrm{X})$$
  - If X and Y are independent, $E(\mathrm{XY})=E(\mathrm{X})\cdot E(\mathrm{Y})$

- Variance
$$V(a\mathrm{X}+b)=a^2V(\mathrm{X})$$
  - If X and Y are independent, $V(\mathrm{X}+\mathrm{Y})=V(\mathrm{X})+V(\mathrm{Y})$

Some more properties

- The conditional expectation of Y given X when the value of X = x is:

$$E\left(Y \mid X = x\right) = \int y * p(y \mid x) dy$$

- The Law of Total Expectation or Law of Iterated Expectation:

$$E(Y) = E\left[E(Y \mid X)\right] = \int E(Y \mid X = x) p_X(x) dx$$

Some more properties

- The law of Total Variance:

$$Var(Y) = Var\left[E(Y \mid X)\right] + E\left[Var(Y \mid X)\right]$$