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Physics-based animation

Tutorial: Rigid body dynamics

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Overview

- Rigid body dynamics
- Extra work
  - Using quaternions
  - Handling collisions
- Resources

Rigid body dynamics

- State variables

$$
\begin{gathered}
\mathbf{x}(t) \\
\mathbf{R}(t) \\
\mathbf{P}(t) \\
\mathbf{L}(t)
\end{gathered}
$$

Rigid body dynamics

- State variables

$$\mathbf{x}(t)$$ Position

$$\mathbf{R}(t)$$ Orientation

$$\mathbf{P}(t)$$ Linear momentum

$$\mathbf{L}(t)$$ Angular momentum

Rigid body dynamics

- Change of state variables w.r.t time

$$\frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ \mathrm{R}(t) \\ \mathrm{P}(t) \\ \mathrm{L}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \boldsymbol{\omega}^{\star}(t)\mathrm{R}(t) \\ \mathbf{f}(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

Change in position

Change in orientation

Change linear momentum

Change in angular momentum

Rigid body dynamics

- Change of state variables w.r.t time

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}(t)\\ \mathrm{R}(t)\\ \mathrm{P}(t)\\ \mathrm{L}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{v}(t)\\ \boldsymbol{\omega}^{\star}(t)\mathrm{R}(t)\\ \mathbf{f}(t)\\ \boldsymbol{\tau}(t)\end{pmatrix}$$

Rigid body dynamics

- Auxiliary variables

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M}, \qquad \mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_0\mathbf{R}(t)^T, \qquad \text{and} \qquad \boldsymbol{\omega}(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

  - *Variable computed from the state variables.*

Implementation (An example)

- Pre-computation

$$M = \sum_i m_i$$

$$\mathbf{x}_{cm} = \frac{1}{M} \sum_i \mathbf{x}_i m_i$$

$$\mathbf{r}_{0i} = \mathbf{x}_{0i} - \mathbf{x}_{cm}$$

$$\mathbf{I}_0^{-1} = \left(\sum_i^N m_i \mathbf{r}_{0i}^T \mathbf{r}_{0i} \delta - \mathbf{r}_{0i} \mathbf{r}_{0i}^T \right)^{-1}$$

Implementation (An example)

- Initialization

$$\mathbf{x}_{cm}, \mathbf{v}_{cm}, \mathbf{R}, \mathbf{L}$$

$$\mathbf{I}^{-1} = \mathbf{R}\mathbf{I}_0^{-1}\mathbf{R}^T$$

$$\omega = \mathbf{I}^{-1}\mathbf{L}$$

Implementation (An example)

- Initialization

Initial conditions of your simulation

$$\mathbf{x}_{cm}, \mathbf{v}_{cm}, \mathbf{R}, \mathbf{L}$$

$$\mathbf{I}^{-1} = \mathbf{R}\mathbf{I}_0^{-1}\mathbf{R}^T$$

$$\omega = \mathbf{I}^{-1}\mathbf{L}$$

Implementation (An example)

- Simulation step (1)
  - Per particle forces

$$\tau = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\mathbf{F} = \sum_i \mathbf{f}_i$$

Implementation (An example)

- Simulation step (2)
  - Rigid body state update

$$\mathbf{x}_{cm} = \mathbf{x}_{cm} + \Delta t \cdot \mathbf{v}_{cm}$$
$$\mathbf{v}_{cm} = \mathbf{v}_{cm} + \Delta t \cdot \mathbf{F}/M$$
$$\mathbf{R} = \mathbf{R} + \Delta t \cdot \omega^{\star}\mathbf{R}$$
$$\mathbf{L} = \mathbf{L} + \Delta t \cdot \tau$$
$$\mathbf{I}^{-1} = \mathbf{R}\mathbf{I}_0^{-1}\mathbf{R}^T$$
$$\omega = \mathbf{I}^{-1}\mathbf{L}$$

Implementation (An example)

- Simulation step (3)
  - Particle update

$$\mathbf{r}_i = \mathbf{R} \cdot \mathbf{r}_{0i}$$
$$\mathbf{x}_i = \mathbf{x}_{cm} + \mathbf{r}_i$$
$$\mathbf{v}_i = \mathbf{v}_{cm} + \omega \mathbf{r}_i$$

Issues with Rotation (matrix)

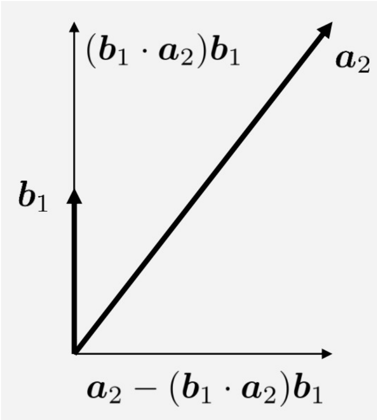
$$\mathbf{R} = \mathbf{R} + \Delta t \cdot \omega^{\star}\mathbf{R}$$

Issues with Rotation (matrix)

$$\mathbf{R} = \mathbf{R} + \Delta t \cdot \omega^{\star}\mathbf{R}$$

- Errors accumulate (numerical drift)
  - The columns of the rotation matrix become no longer orthornormal
- So we must orthornormalize the rotation matrix (occasionally)

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{a}_1/|\mathbf{a}_1| \\
\mathbf{b}_2 &= \mathbf{a}_2 - (\mathbf{b}_1 \cdot \mathbf{a}_2)\mathbf{b}_1 \\
\mathbf{b}_2 &= \mathbf{b}_2/|\mathbf{b}_2| \\
\mathbf{b}_3 &= \mathbf{a}_3 - (\mathbf{b}_1 \cdot \mathbf{a}_3)\mathbf{b}_1 - (\mathbf{b}_2 \cdot \mathbf{a}_3)\mathbf{b}_2 \\
\mathbf{b}_3 &= \mathbf{b}_3/|\mathbf{b}_3|
\end{aligned}$$



Getting started ...

- Open the zip file provided to get started.
- You need to
  - know C++ (a bit)
  - Know how to configure CMake
  - ...
  - Glance through the README file for instructions

Extras

- Quaternions for rotations...?
- Handling collisions...?
- More complex objects...?

Additional resources

- Adam W. Bargteil, Tamar Shinar, and Paul G. Kry. 2020. An introduction to physics-based animation. In SIGGRAPH Asia 2020 Courses (SA '20). Association for Computing Machinery, New York, NY, USA, Article 5, 1–57. DOI: https://doi.org/10.1145/3415263.3419147

- An Introduction to Physically Based Modeling: Rigid Body Simulation I—Unconstrained Rigid Body Dynamics. David Baraff: https://www.cs.cmu.edu/~baraff/sigcourse/notesd1.pdf

- GPU Gems 3: Chapter 29. Real-Time Rigid Body Simulation on GPUs. Takahiro Harada University of Tokyo: https://developer.nvidia.com/gpugems/gpugems3/part-v-physics-simulation/chapter-29-real-time-rigid-body-simulation-gpus