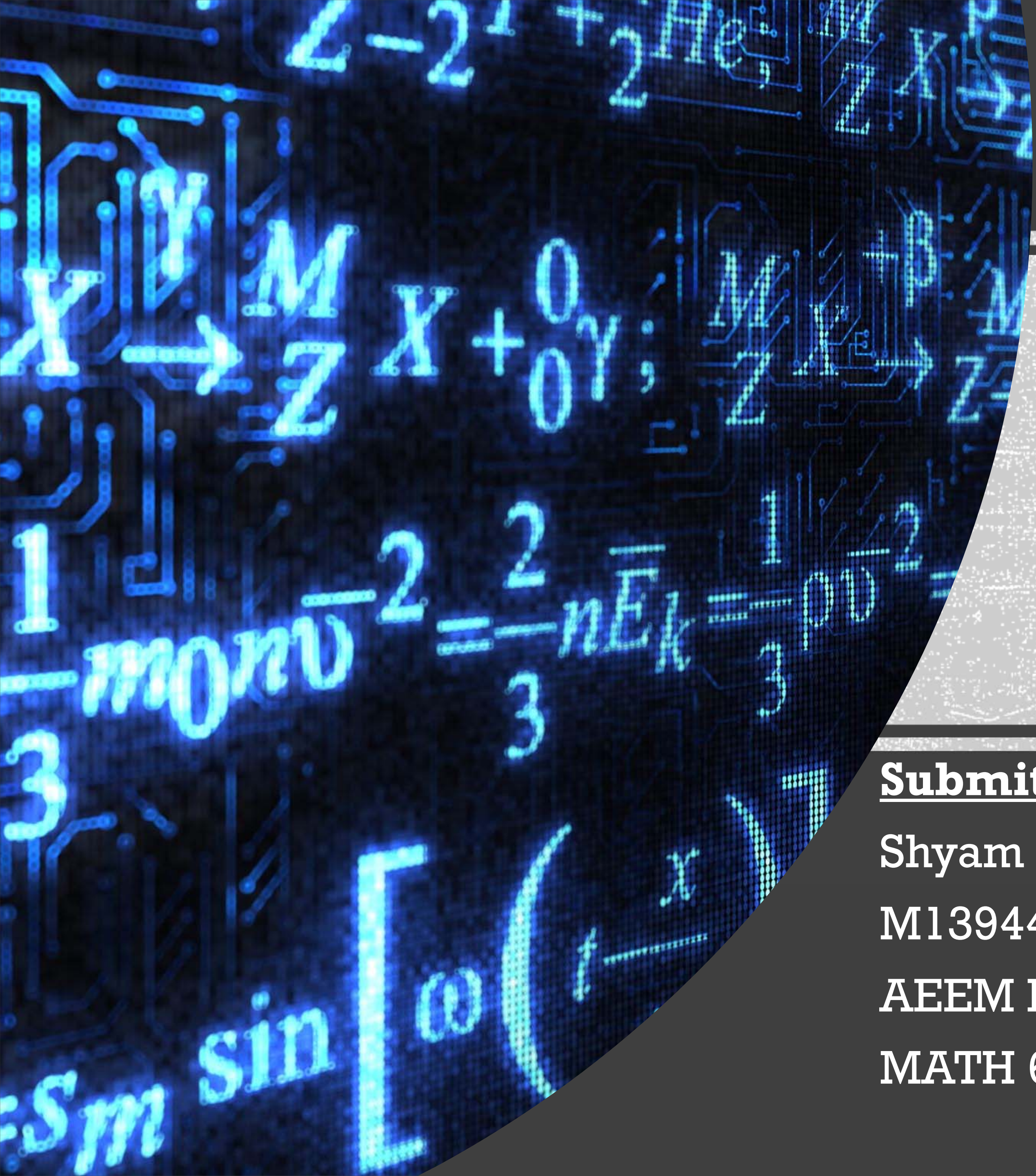


SIMULATIONS OF FRACTIONAL BROWNIAN MOTION AND LEVY BROWNIAN FIELD

Submitted By:

Shyam Rauniyar

M13944385

AEEM Fall 2020

MATH 6012

Submitted To:

Dr. Yizao Wang

Faculty of Applied Linear Algebra

MATH 6012

University of Cincinnati

OUTLINE OF PRESENTATION

- Goals of Project
- About Fractional Brownian Motion
- Plots of a sampled Fractional Brownian Motions
- Plots of independent samples of Fractional Brownian Motion
- About Complexity of Cholesky Decomposition
- Plots of Time vs. Resolution
- About Fractional Levy Brownian Field
- Heatmap of Sampled Levy Brownian Field

GOALS OF PROJECT

- Create a plots of a sampled Fractional Brownian Motion for following Hurst Index
    - $H = 0.1, 0.3, 0.5, 0.7, 0.9$
- Create plots of multivariate Fractional Brownian Motion to verify normal distribution along time domain
    - $H = 0.1, 0.5, 0.9$
- Empirical Test of Complexity of Cholesky Factorization
    - Plot $Log(T_n)$ vs $Log(n)$ graphs
    - Show t_n increases roughly with n^3 with slope of the graphs around ~ 3
- Create Heat Maps for a sampled Fractional Levy Brownian Field for:
    - $H = 0.1, 0.3, 0.5, 0.7, 0.9$

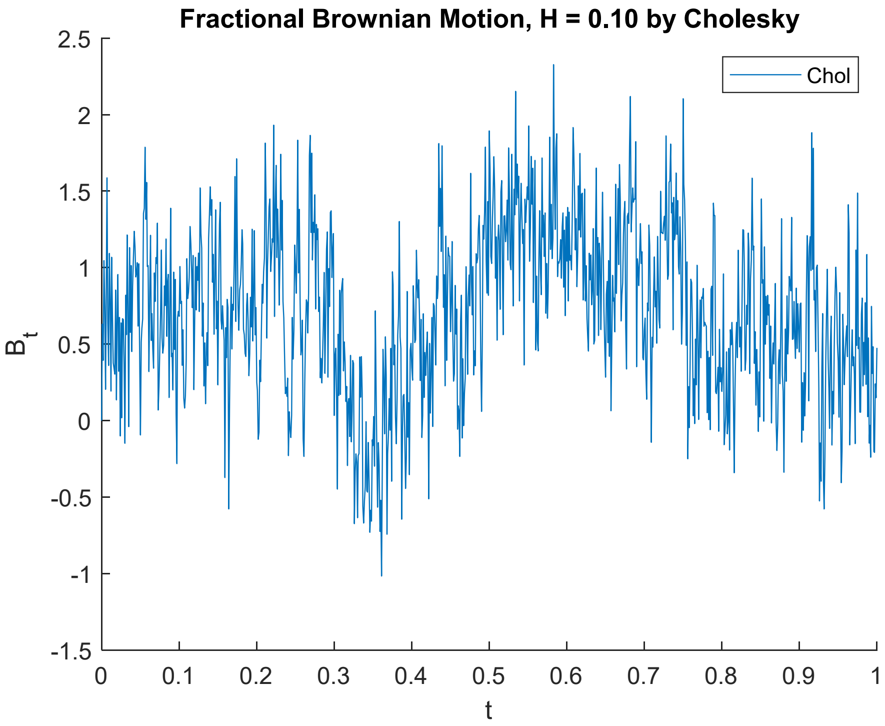
ABOUT FRACTIONAL BROWNIAN MOTION

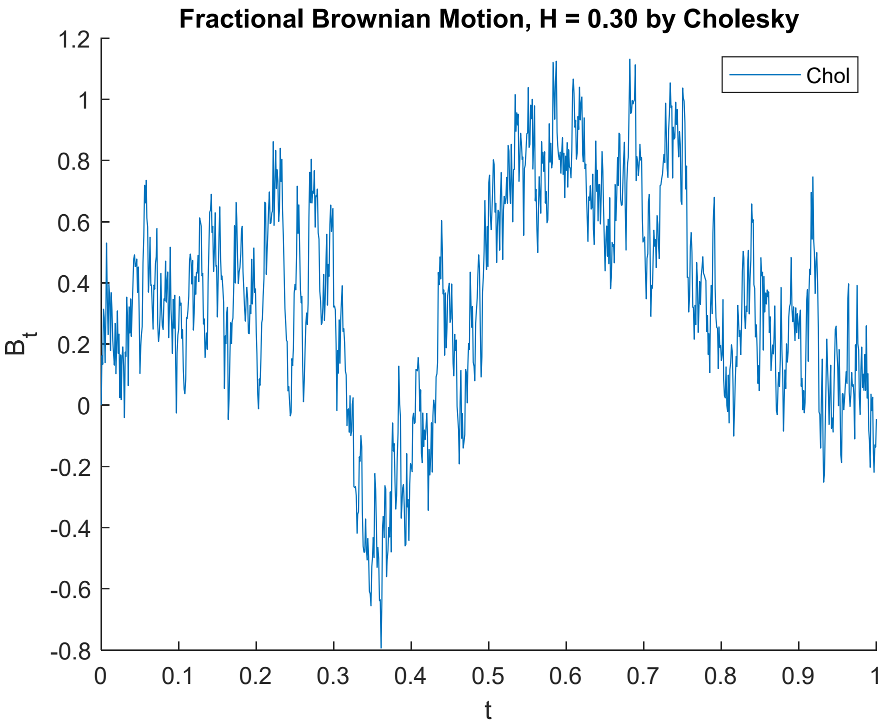
- A fractional Brownian motion with Hurst index $H \in (0,1)$, denoted by $\{\mathbb{B}_t{}^H\}_{t\geq 0}$, is a centered Gaussian process with covariance function
  - $Cov\left(\mathbb{B}_s{}^H, \mathbb{B}_t{}^H\right) = \frac{1}{2}\,(s^{2H} + t^{2H} - |t-s|^{2H}, \quad s,t \geq 0$ 1
- A fractional Brownian motion with Hurst index H = 1/2 is a Brownian motion $\mathbb{B} = \mathbb{B}^{\frac{1}{2}}$ $(with\ Cov(\mathbb{B}_s, \mathbb{B}_t) = \min\{s,t\})$

PLOTS OF SAMPLED FRACTIONAL BROWNIAN MOTIONS

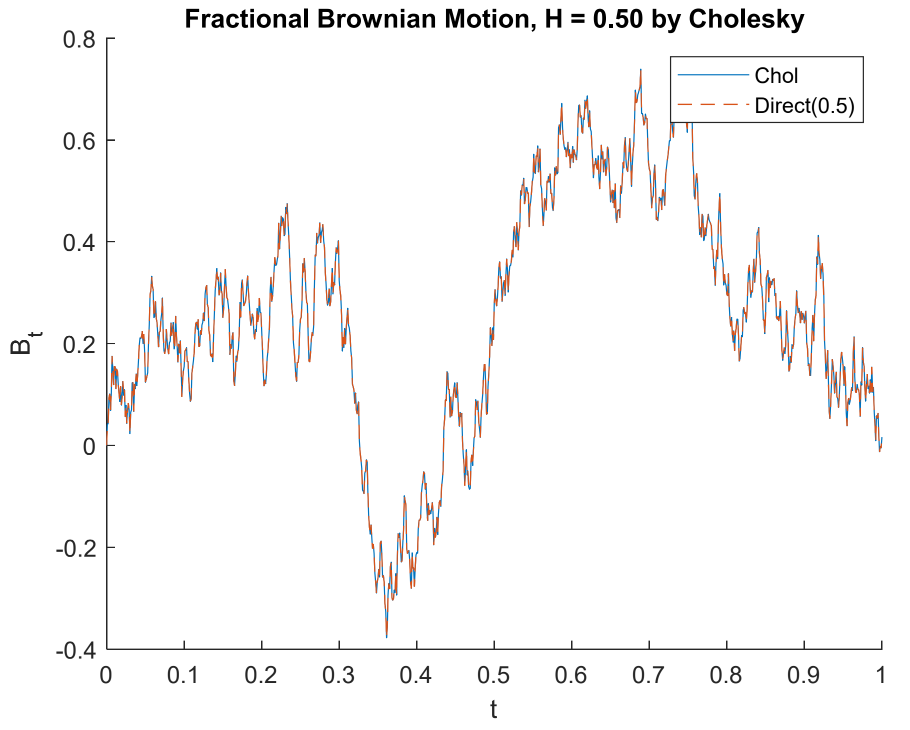
- Method of Calculation
  - Assumed a $Hurst\ Index\ (H)$
  - Created a $(n+1) \times (n+1)$ $Covariance\ Matrix\ (cov)$ for $n = 1000$ using $bsxfun()$ function that carries out element wise binary operations using Equation 1.
  - Created a $((n+1) \times 1)$ $random\ vector\ (Z)$ with normal distribution, $N(0,1/n)$, using $normrnd()$ function
  - Determined $Cholesky\ factor\ (M)$ using $chol()$ function
  - Determined a sampled fractional vector as $\mathbb{B} = MZ'$
  - Verified the above process with direct method of: $\mathbb{B} = sqrt(n).\backslash cumsum(Z')$, for $H = 0.5$
  - Determined a sampled fractional vector as $\mathbb{B} = mvnrnd()$ with normal distribution $N(0, cov)$
  - Plotted $\mathbb{B}$ vs. t graph for each $Hurst\ Index\ (H)$

PLOTS OF SAMPLED FRACTIONAL BROWNIAN MOTIONS

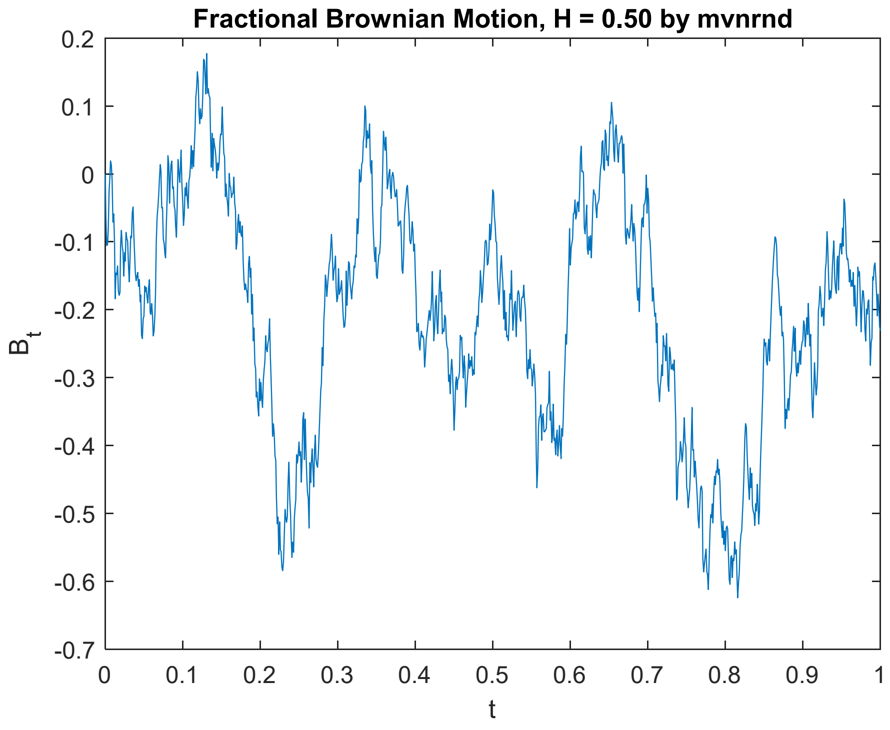




PLOTS OF SAMPLED FRACTIONAL BROWNIAN MOTIONS

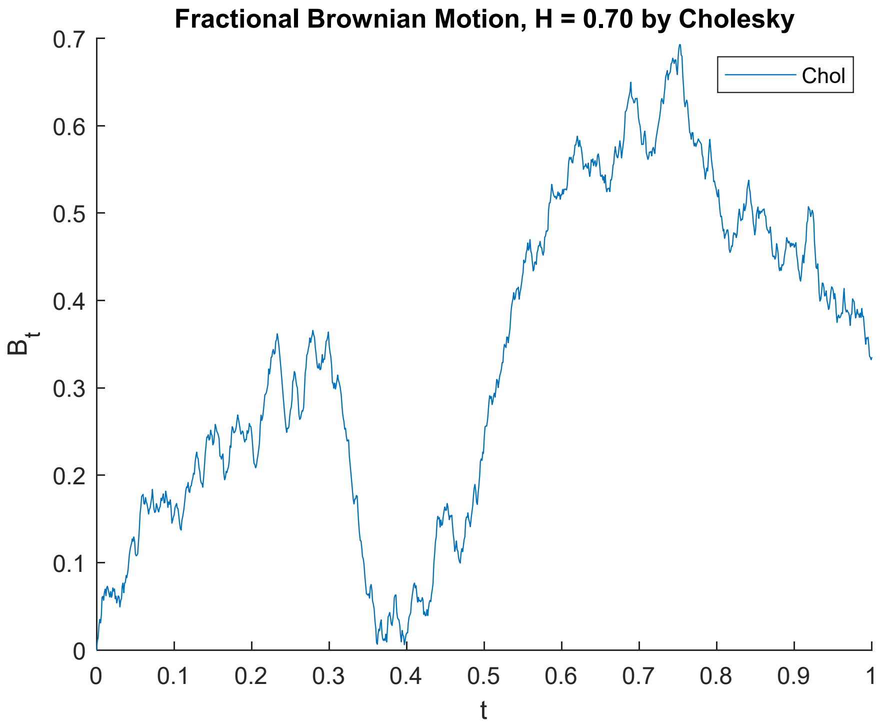


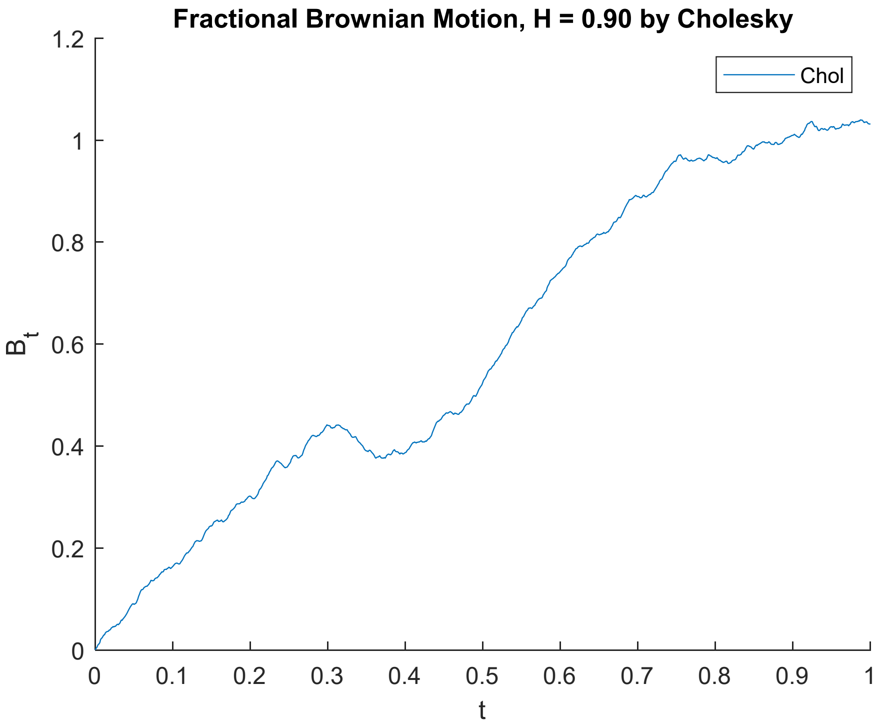
Confirming the method for Cholesky is Correct with Direct Method for $H = 0.5$



Confirming the method for Cholesky is Correct with comparing results with in-built $mvnrnd()$ function

PLOTS OF SAMPLED FRACTIONAL BROWNIAN MOTIONS



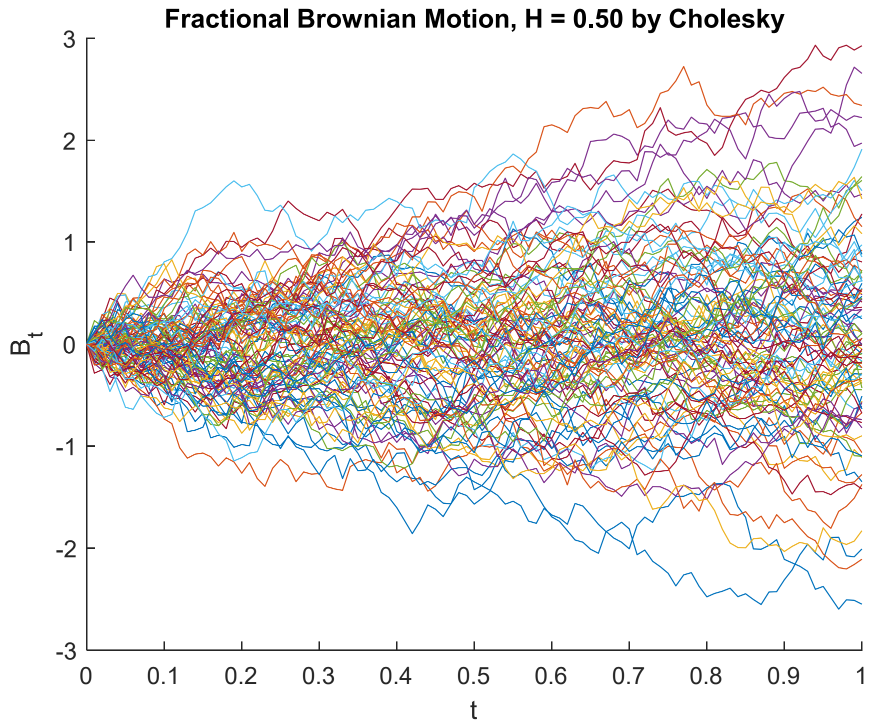


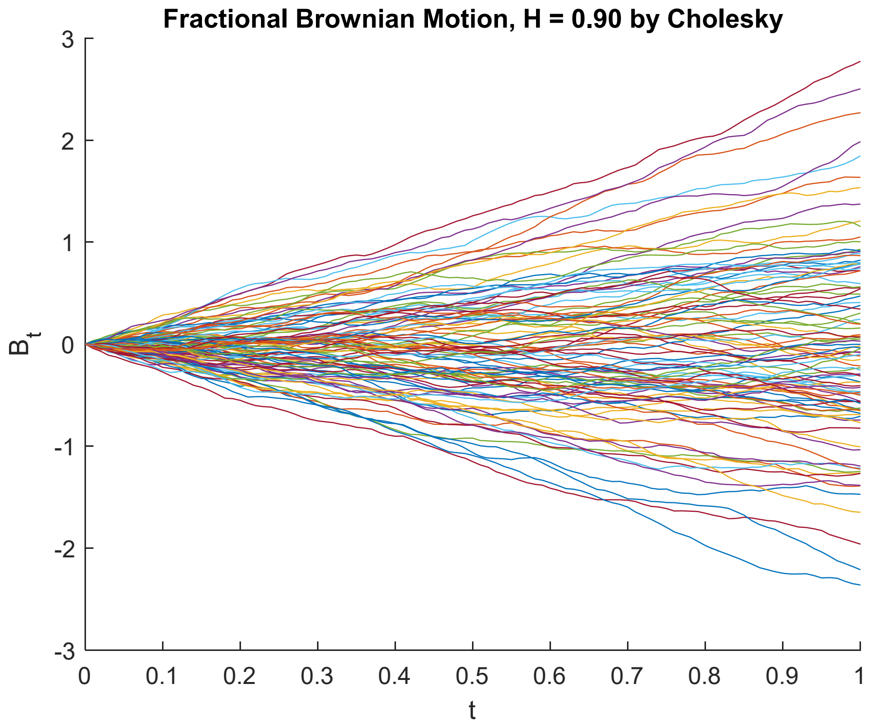
Shows that the Fractional Brownian Motion vector smoothens with increasing *Hurst Index* (H)

PLOTS OF MULTIVARIATE FRACTIONAL BROWNIAN MOTIONS

- Method of Calculation
  - Similar to previous method of single sampled Fractional Brownian Vector
  - But random sample vector (Z) has several random vector in each of its row
  - Thus, the resultant $\mathbb{B}$ also has corresponding sampled Fractional Brownian Vector i.e. Fractional Brownian Motions all independent samples

PLOTS OF INDEPENDENT SAMPLES OF FRACTIONAL BROWNIAN MOTION



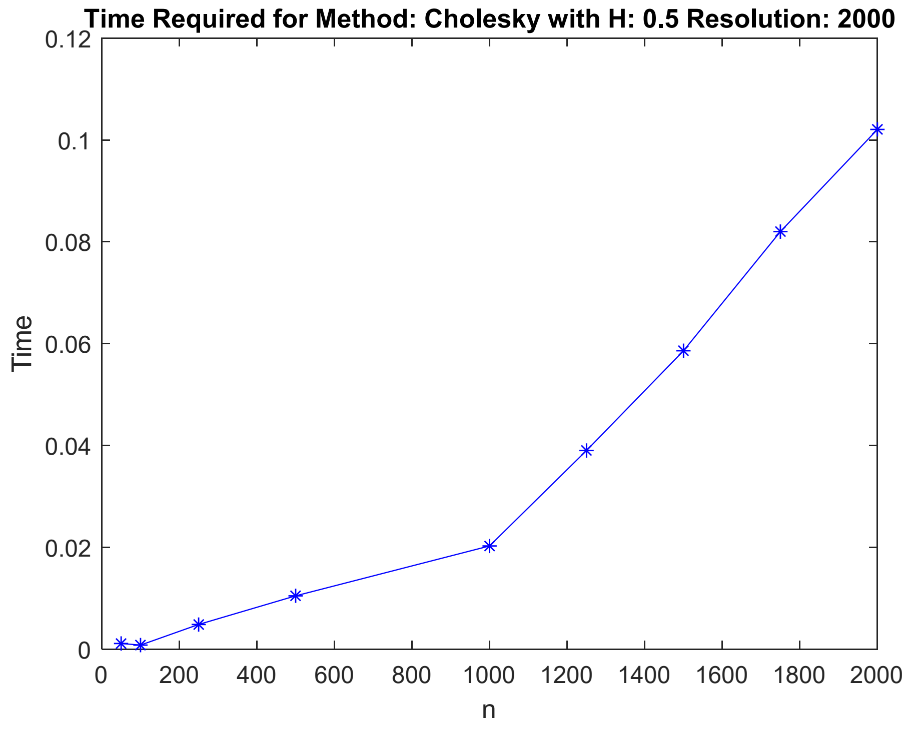


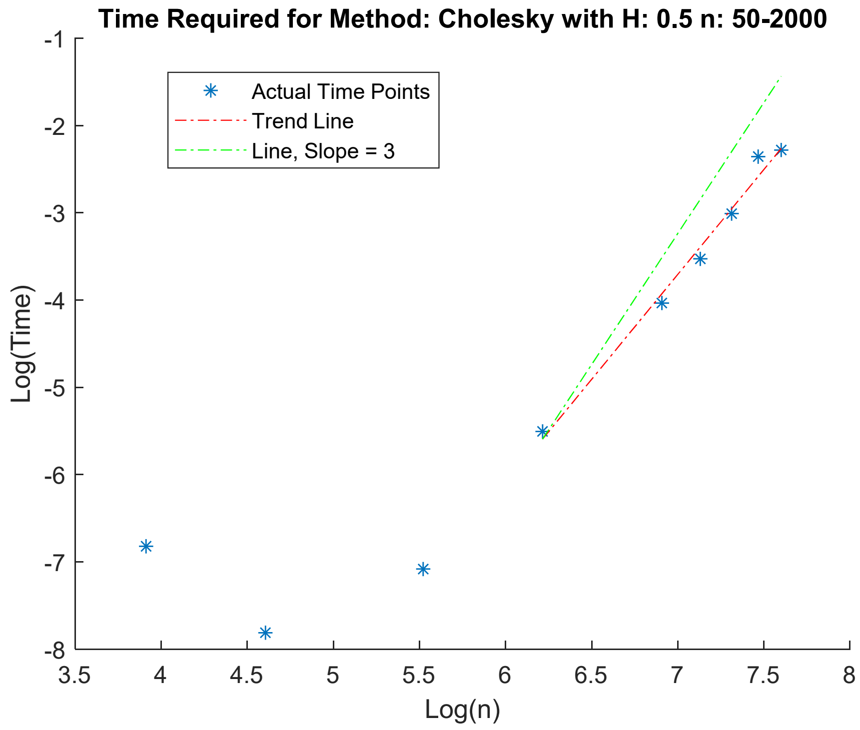
Shows that at each time instant the sampled Fractional Brownian Motion vectors follow a normal Distribution, $N(0,1/n)$

COMPLEXITY OF CHOLESKY DECOMPOSITION

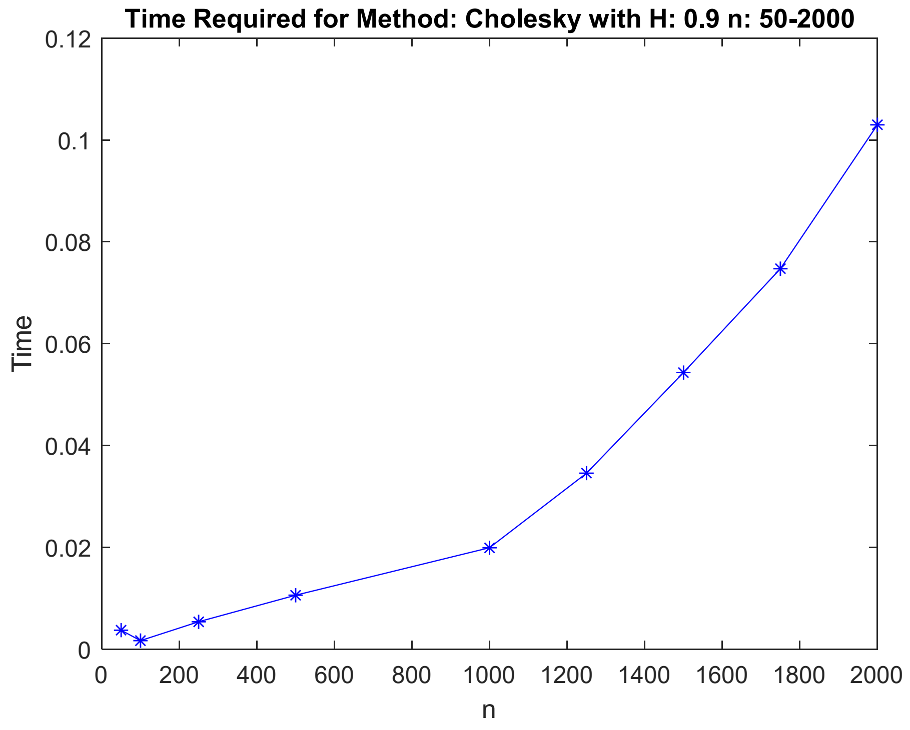
- Method of Calculation
  - For different resolutions, $n = [50,100,250,500,1000,1250,1500,1750,2000]$, the time of operating Cholesky factorization were estimated for few of the Hurst Indices
  - T vs. n graphs were plotted to see the trend
  - Slope was determined using the $Log(T_n)$ vs $Log(n)$ graph
  - Functions like $polyfit()$ and $slope()$ were used for curve fitting

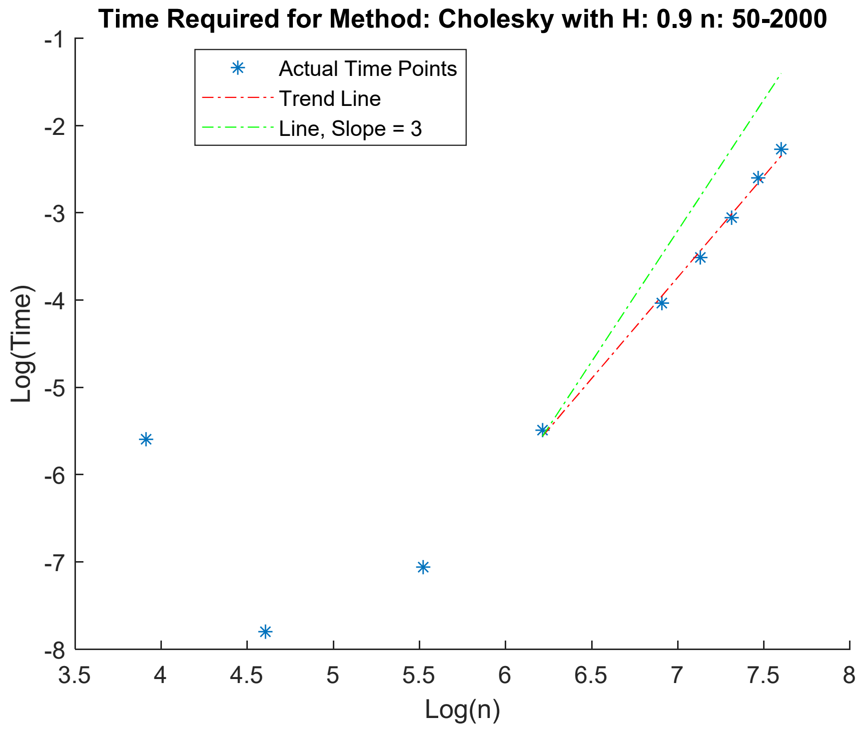
ABOUT COMPLEXITY OF CHOLESKY DECOMPOSITION





ABOUT COMPLEXITY OF CHOLESKY DECOMPOSITION





T vs n graph shows somewhat of n^3 behavior. Early points of n estimating time seem like outliers and are omitted to calculate slope in $Log(T_n)$ $vs.$ $Log(n)$ graph

COMPLEXITY OF CHOL. FAC. VIA SLOPE

Slope of Log(t_n) Vs Log(n) for Different Hurst Index (H), Via Chol					
H	0.10	0.30	0.50	0.70	0.90
Slope	2.52	2.38	2.40	2.36	2.32

Slope of the Line is determined to be around 2.4 which is roughly around 3 confirming the complexity of Cholesky Decomposition to be n^3.

Slope of Log(t_n) Vs Log(n) for Different Hurst Index (H), Via mvnrnd					
H	0.10	0.30	0.50	0.70	0.90
Slope	2.42	2.41	2.36	2.40	2.40

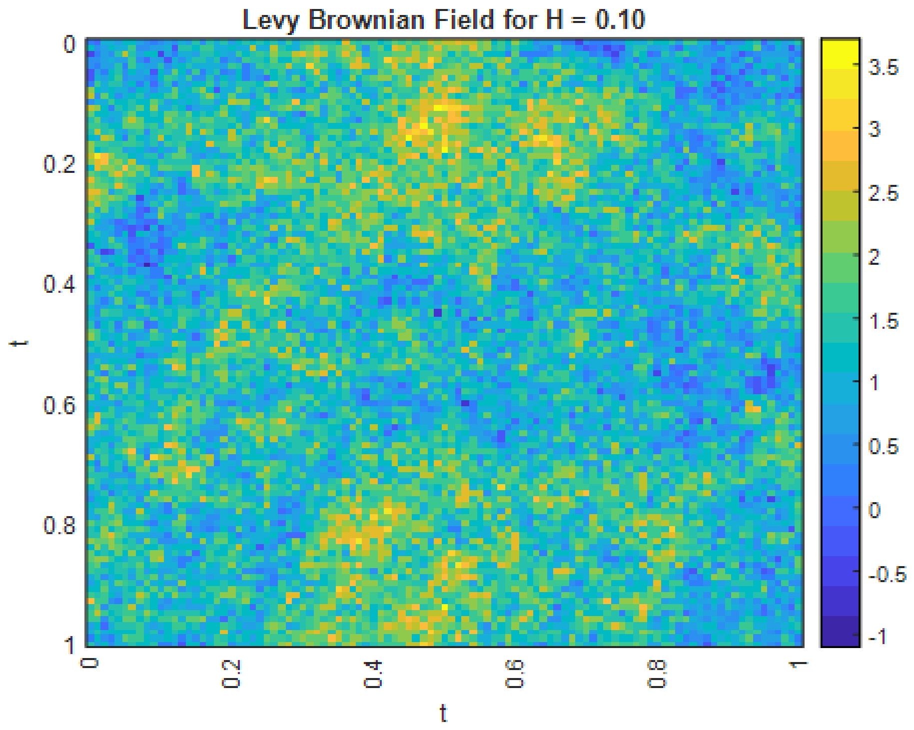
ABOUT FRACTIONAL LEVY BROWNIAN FIELD

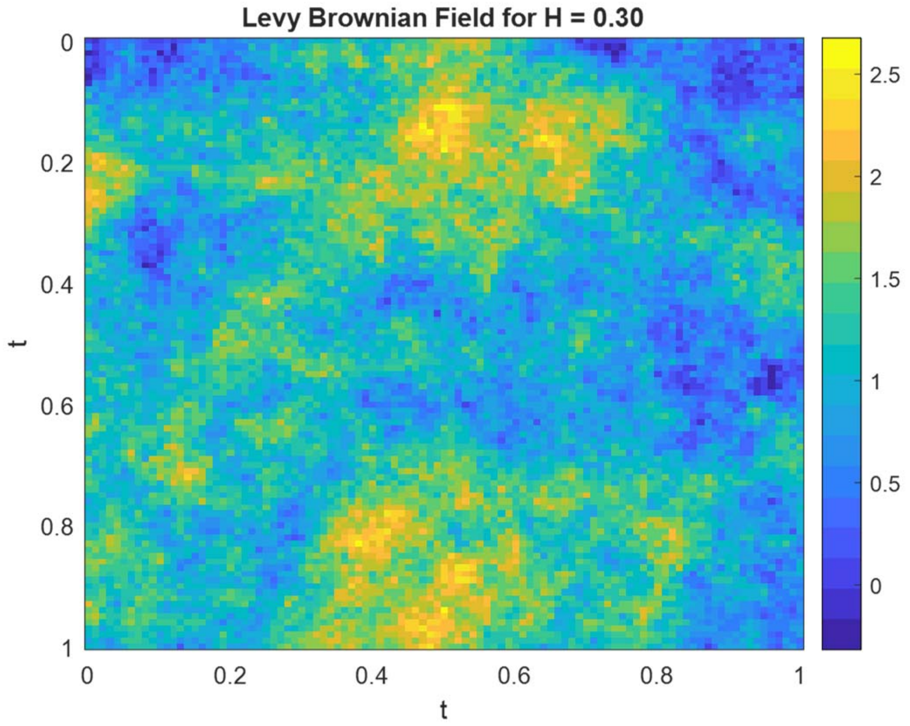
- A random field is a collection of random variables $\{X_{\boldsymbol{t}}\}_{\boldsymbol{t}\in\mathbb{R}^2}$ where each $\boldsymbol{t}$ represents a spatial location
- A fractional Levy Brownian Fields with Hurst index $H \in (0,1)$, denoted by $\{\mathbb{G}_{\boldsymbol{t}}{}^{H}\}_{\boldsymbol{t}\in\mathbb{R}^2}$, is a centered Gaussian process with covariance function
  - $Cov\left(\mathbb{B}_{\boldsymbol{s}}{}^{H}, \mathbb{B}_{\boldsymbol{t}}{}^{H}\right) = \frac{1}{2}\,(\|\boldsymbol{s}\|^{2H} + \|t\|^{2H} - \|\boldsymbol{s}-\boldsymbol{t}\|^{2H}, \;\; s,t \in \mathbb{R}^2$ 2

PLOTS OF SAMPLED FRACTIONAL BROWNIAN MOTIONS

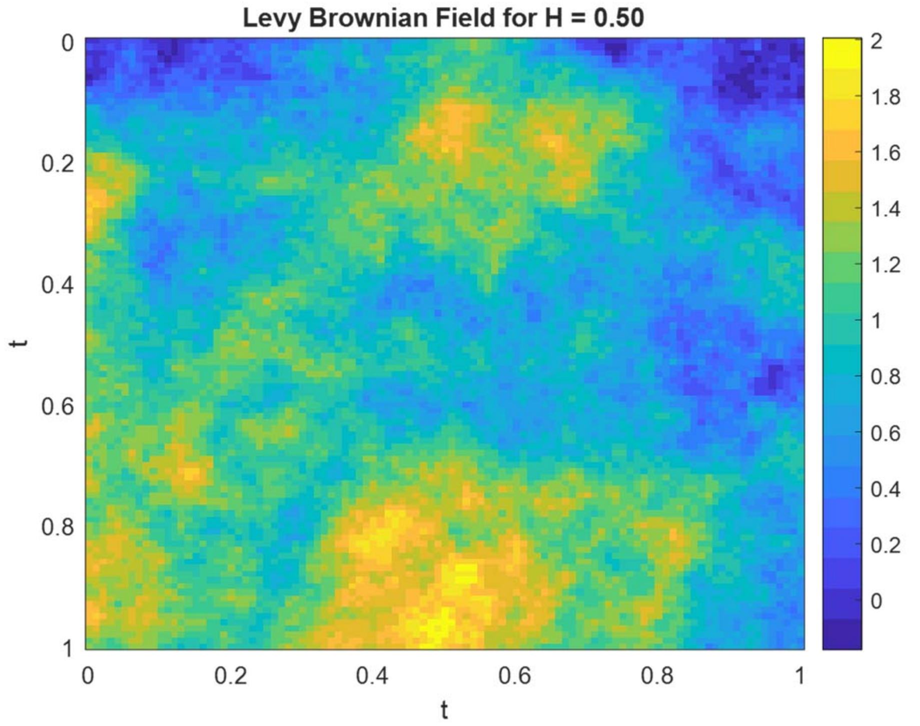
- Method of Calculation
  - Created a time field with $\left((n+1)^2 \times 2\right)$ matrix with each row giving a location in a $2D$ time field
  - Created a vector with each row being $\|\boldsymbol{t}\|$ and a $(n+1)^2 \times (n+1)^2$ matrix for $\|\boldsymbol{s}-\boldsymbol{t}\|$
  - Assumed a $Hurst\ Index\ (H)$
  - Created a $(n+1)^2 \times (n+1)^2\ Covariance\ Matrix\ (cov)$ for $n\ =\ 100$ using $bsxfun()$ function that carries out element wise binary operations using Equation 1.
  - Created a $\left((n+1)^2 \times 1\right)\ random\ vector\ (Z)$ with normal distribution, $N(0{,}1/n)$, using $normrnd()$ function
  - Determined $Cholesky\ factor\ (M)$ using $chol()$ function
  - Determined a sampled fractional vector as $\mathbb{G} = MZ'$ for the same Z to compare heatmaps
  - Reshaped $\mathbb{G}$ to a $(n+1) \times (n+1)$ matrix using reshape() function to get magnitude of $\mathbb{G}_{\boldsymbol{t}}^{H}$
  - Plotted heatmap for $\mathbb{G}_{\boldsymbol{t}}^{H}$ vs. $\boldsymbol{t}$ for each $Hurst\ Index\ (H)$

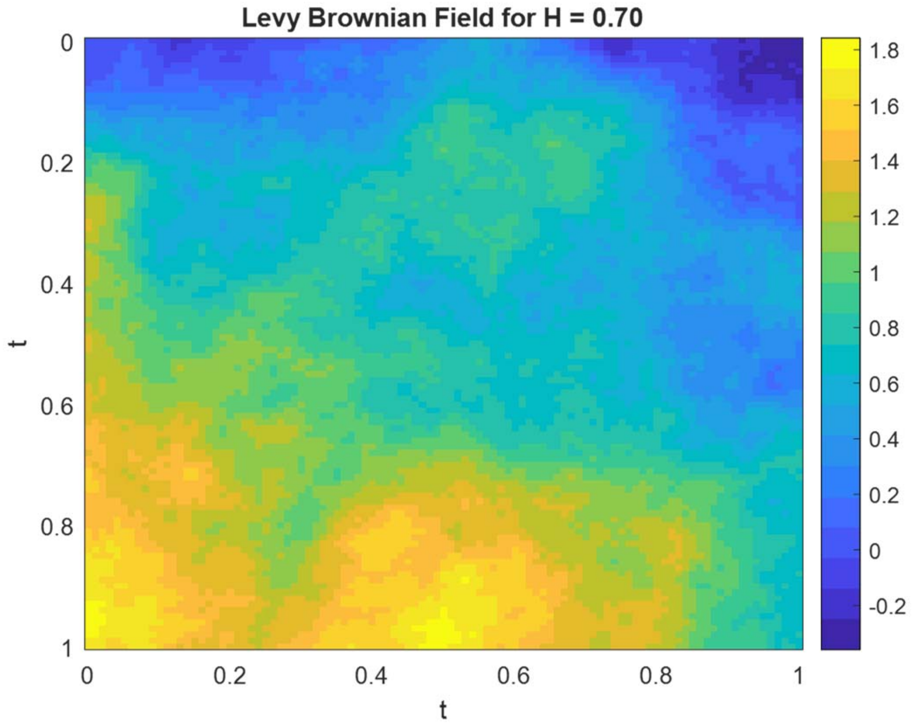
PLOTS OF SAMPLED FRACTIONAL LEVY BROWNIAN FIELDS



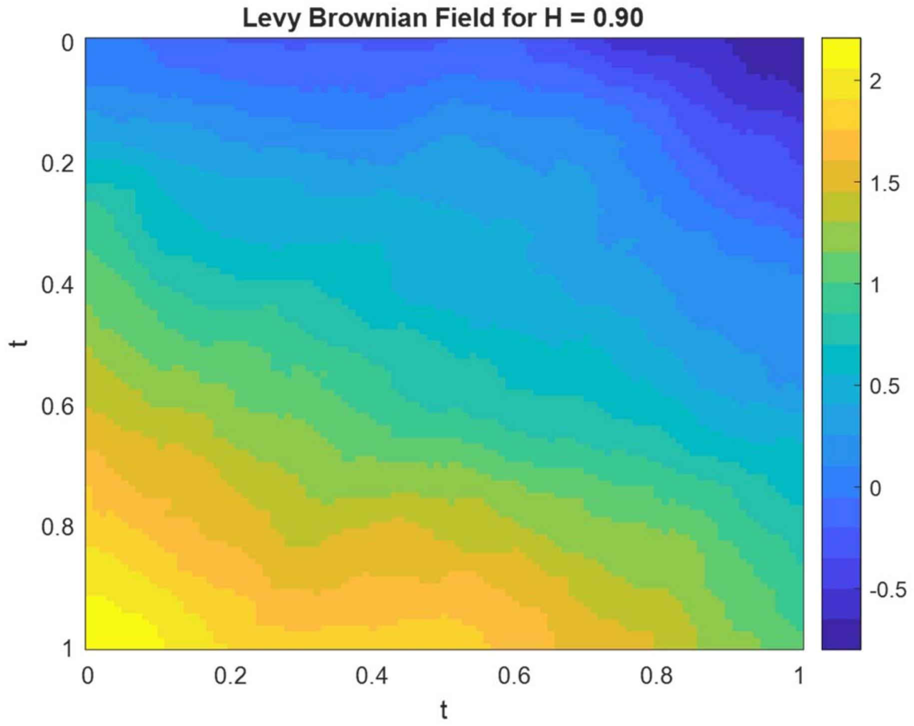


PLOTS OF SAMPLED FRACTIONAL LEVY BROWNIAN FIELDS





PLOTS OF SAMPLED FRACTIONAL LEVY BROWNIAN FIELDS



- We see that the Field is smoother as the Hurst Index increases

CONCLUDING REMARKS

- Increment of Hurst Index in both the Fractional Brownian Motion and Levy Fields increase the smoothness of the magnitude of the vectors along the time vector/field
- There is more local oscillation in case of lower $Hurst\ Index$ $(H = 0.1, 0.3)$ than the Higher $Hurst\ Index$ $(H = 0.7, 0.9)$.
- Time for the Cholesky Decomposition is a function of n^3 showing the time of computation of the sampled vectors increases with the cube of the chosen resolution of time intervals

21 THANK YOU!!