The Dynamics of Semigroups of Contraction Similarities of the Plane

Stefano Silvestri

IUPUI

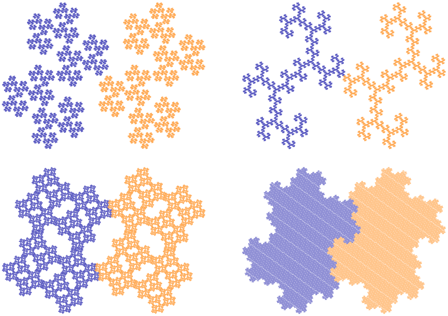
Setting: Iterated Function Systems

Given $\lambda \in \mathbb{D} \setminus \{0\}$ and contraction similarities

$$\mathfrak{s}_-(z) = \lambda z - 1 \qquad\qquad \mathfrak{s}_+(z) = \lambda z + 1$$

there exists a unique non-empty compact set satisfying

$$\mathsf{A}_\lambda = \mathfrak{s}_-(\mathsf{A}_\lambda) \cup \mathfrak{s}_+(\mathsf{A}_\lambda)$$



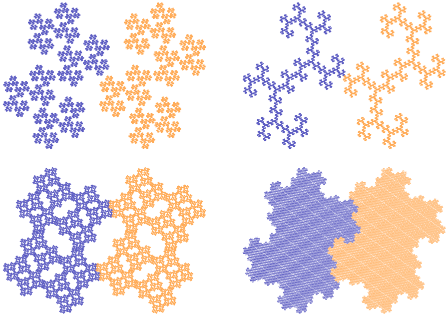
Setting: Iterated Function Systems

Given $\lambda \in \mathbb{D} \setminus \{0\}$ and contraction similarities

$$\mathfrak{s}_-(z) = \lambda z - 1 \qquad\qquad \mathfrak{s}_+(z) = \lambda z + 1$$

there exists a unique non-empty compact set satisfying

$$\mathsf{A}_\lambda \quad = \quad \mathfrak{s}_-(\mathsf{A}_\lambda) \cup \mathfrak{s}_+(\mathsf{A}_\lambda)$$



$$\mathcal{M} \quad = \quad \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\}$$

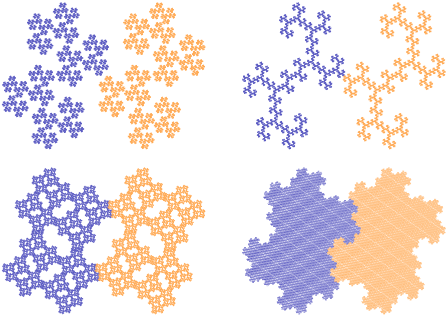
Setting: Iterated Function Systems

Given $\lambda \in \mathbb{D} \setminus \{0\}$ and contraction similarities

$$\mathfrak{s}_-(z) = \lambda z - 1 \qquad\qquad \mathfrak{s}_+(z) = \lambda z + 1$$

there exists a unique non-empty compact set satisfying

$$\mathsf{A}_\lambda = \mathfrak{s}_-(\mathsf{A}_\lambda) \cup \mathfrak{s}_+(\mathsf{A}_\lambda)$$



$$\mathcal{M} = \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\}$$

$$\mathcal{M}_0 = \{\lambda \in \mathbb{D} \mid 0 \in \mathsf{A}_\lambda\}$$

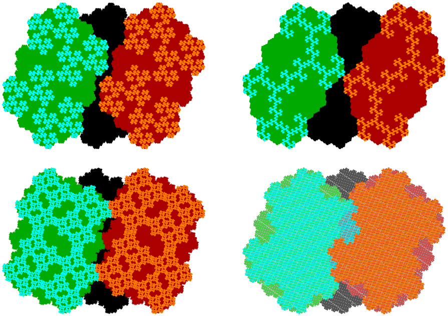
Setting: Iterated Function Systems

Given $\lambda \in \mathbb{D} \setminus \{0\}$ and contraction similarities

$$\mathfrak{s}_-(z) = \lambda z - 1 \qquad \mathfrak{s}_0(z) = \lambda z \qquad \mathfrak{s}_+(z) = \lambda z + 1$$

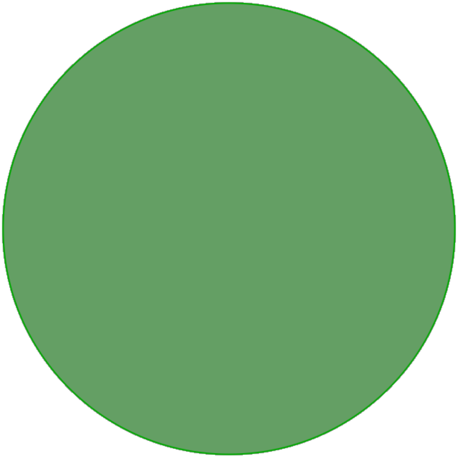
there exists unique non-empty compact sets satisfying

$$\begin{aligned} \mathsf{A}_\lambda &= \mathfrak{s}_-(\mathsf{A}_\lambda) \cup \mathfrak{s}_+(\mathsf{A}_\lambda) \\ \widetilde{\mathsf{A}}_\lambda &= \mathfrak{s}_-(\widetilde{\mathsf{A}}_\lambda) \cup \mathfrak{s}_0(\widetilde{\mathsf{A}}_\lambda) \cup \mathfrak{s}_+(\widetilde{\mathsf{A}}_\lambda) \end{aligned}$$

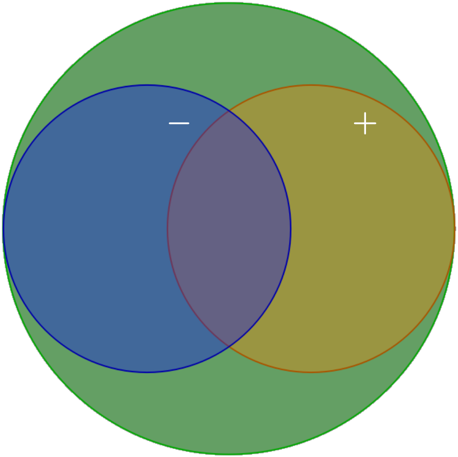


$$\begin{aligned} \mathcal{M} &= \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\} = \left\{\lambda \in \mathbb{D} \mid 0 \in \widetilde{\mathsf{A}}_\lambda\right\} \\ \mathcal{M}_0 &= \{\lambda \in \mathbb{D} \mid 0 \in \mathsf{A}_\lambda\} \end{aligned}$$

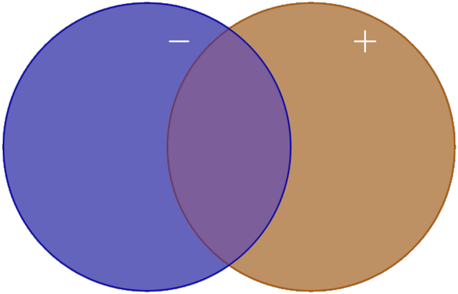
Constructing A_λ **for** $\lambda \approx -0.366 + 0.52i$



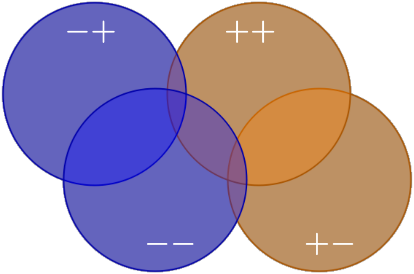
Constructing A_λ for $\lambda \approx -0.366 + 0.52\mathrm{i}$



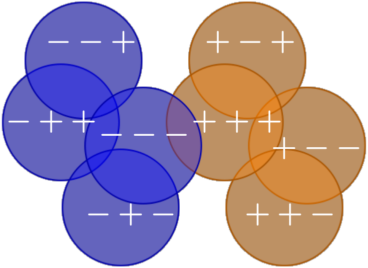
Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



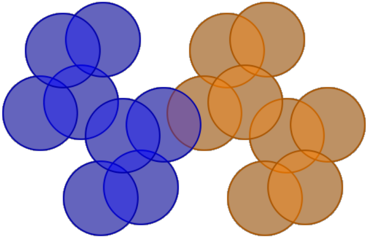
Constructing A_λ **for** $\lambda \approx -0.366 + 0.52i$



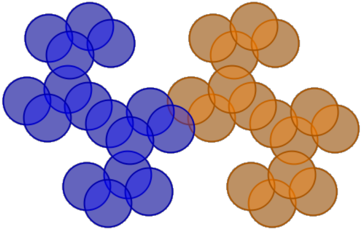
Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



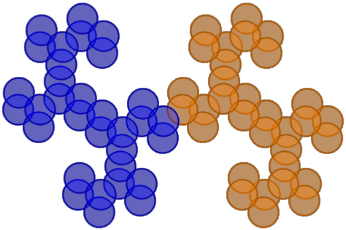
Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



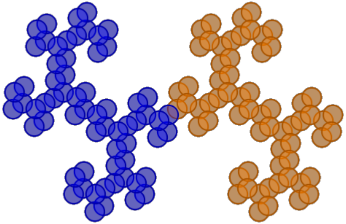
Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



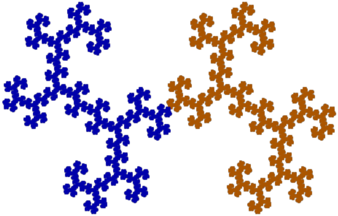
Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



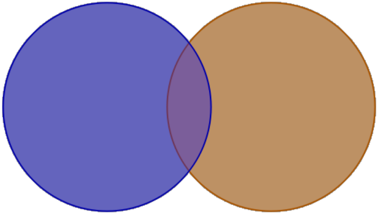
Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



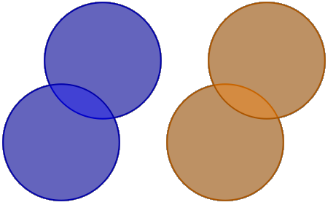
Constructing A_λ **for** $\lambda \approx -0.366 + 0.52i$



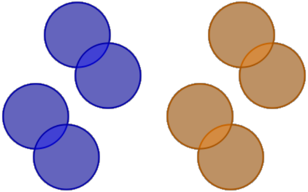
Constructing A_λ for $\lambda = 0.25 + 0.5i$



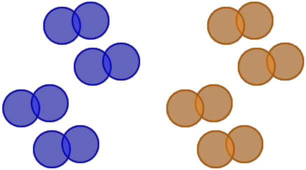
Constructing A_λ for $\lambda = 0.25 + 0.5i$



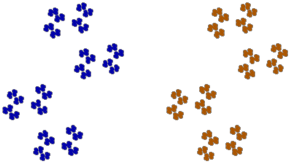
Constructing A_λ for $\lambda = 0.25 + 0.5i$



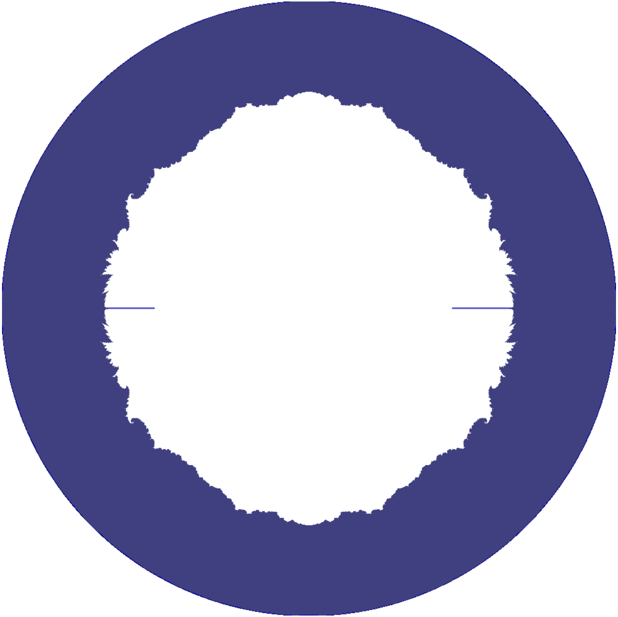
Constructing A_λ for $\lambda = 0.25 + 0.5i$



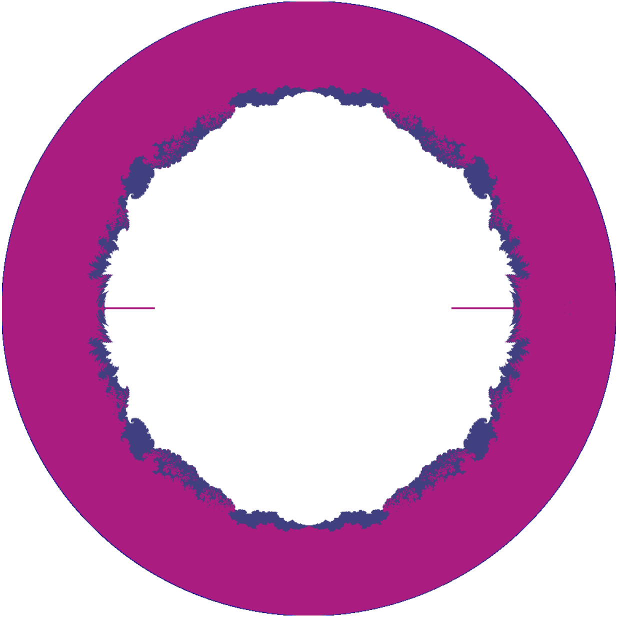
Constructing A_λ for $\lambda = 0.25 + 0.5\mathrm{i}$



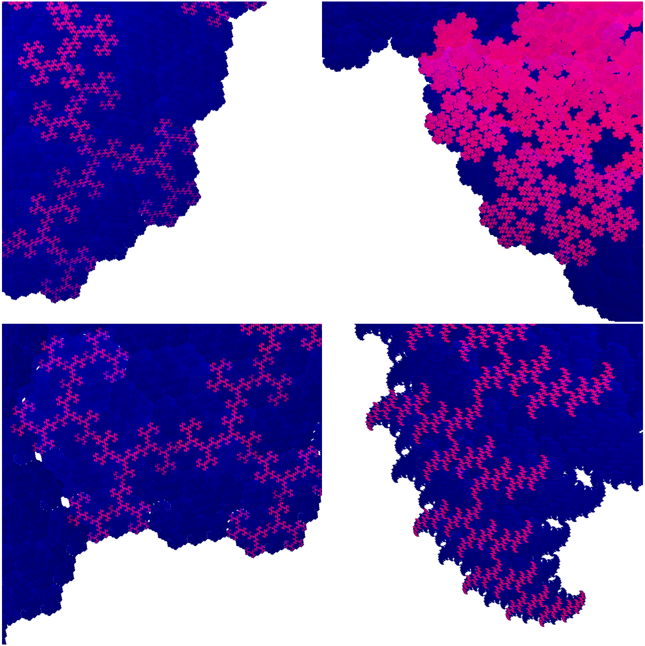
$\mathcal{M}$ **(in blue)**



$\mathcal{M}_0 \subset \mathcal{M}$



Interesting features of $\mathcal{M}$ and $\mathcal{M}_0$



History

1985 Barnsley & Harrington introduce $\mathcal{M}$.

History

1985 Barnsley & Harrington introduce $\mathcal{M}$.

1988 – 1992 Bousch proved both $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected.

History

1985 Barnsley & Harrington introduce $\mathcal{M}$.

1988 – 1992 Bousch proved both $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected.

2002 Bandt proved $\mathbb{D} \setminus \mathcal{M}$ is disconnected.

History

1985 Barnsley & Harrington introduce $\mathcal{M}$.

1988 – 1992 Bousch proved both $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected.

2002 Bandt proved $\mathbb{D} \setminus \mathcal{M}$ is disconnected.

2014 Calegari-Koch-Walker proved that both $\mathbb{D} \setminus \mathcal{M}$ and $\mathbb{D} \setminus \mathcal{M}_0$ have infinitely many components.

History

1985 Barnsley & Harrington introduce $\mathcal{M}$.

1988 – 1992 Bousch proved both $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected.

2002 Bandt proved $\mathbb{D} \setminus \mathcal{M}$ is disconnected.

2013 Tiozzo showed $\mathcal{M}_0$ arises as the closure of the set (restricted to $\mathbb{D}$) of Galois conjugates of growth rates of postcritically finite real quadratic polynomials.

2014 Calegari-Koch-Walker proved that both $\mathbb{D} \setminus \mathcal{M}$ and $\mathbb{D} \setminus \mathcal{M}_0$ have infinitely many components.

History

1985 Barnsley & Harrington introduce $\mathcal{M}$.

1988 – 1992 Bousch proved both $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected.

2002 Bandt proved $\mathbb{D} \setminus \mathcal{M}$ is disconnected.

2009 Eroğlu-Rohde-Solomyak showed quasisymmetric conjugacy between quadratic dynamics and some IFS.

2013 Tiozzo showed $\mathcal{M}_0$ arises as the closure of the set (restricted to $\mathbb{D}$) of Galois conjugates of growth rates of postcritically finite real quadratic polynomials.

2014 Calegari-Koch-Walker proved that both $\mathbb{D} \setminus \mathcal{M}$ and $\mathbb{D} \setminus \mathcal{M}_0$ have infinitely many components.

Motivation

Is there a way to discern holes in $\mathcal{M}$? Given a $\lambda \in \partial\mathcal{M}$ is it possible to construct a path in $\mathbb{D} \setminus \mathcal{M}$ that converges to it?

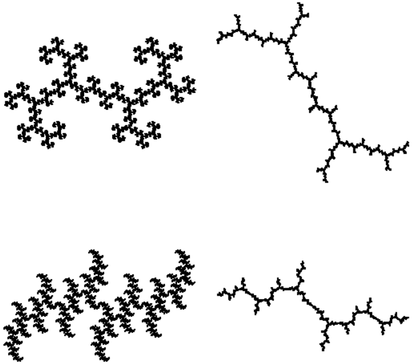
Motivation

Is there a way to discern holes in $\mathcal{M}_0$? Given a $\lambda \in \partial\mathcal{M}_0$ is it possible to construct a path in $\mathbb{D} \setminus \mathcal{M}_0$ that converges to it?

Motivation

Is there a way to discern holes in $\mathcal{M}_0$? Given a $\lambda \in \partial\mathcal{M}_0$ is it possible to construct a path in $\mathbb{D} \setminus \mathcal{M}_0$ that converges to it?

Conjecture: The parameters $\lambda \in (\partial\mathcal{M} \cap \partial\mathcal{M}_0) \setminus \mathbb{R}$ are those for which the dynamics on A_λ is quasisymmetrically conjugate to the dynamics of $z^2 + c$ for Misiurewicz c on its Julia set.



Other definition of $\mathcal{M}$

A_λ is also the closure of the set of fixed points of every finite composition of the maps $\mathfrak{s}_-$ and $\mathfrak{s}_+$.
For example, let $\mathfrak{s}_{ab} = \mathfrak{s}_a \circ \mathfrak{s}_b$ then

$$z = \mathfrak{s}_{-+}(z) = -1 + \lambda \ \ + \lambda^2 z \iff z = \frac{-1}{1+\lambda} = -\sum_{j=0}^{\infty} (-1)^j \lambda^j$$
$$= \lim_{n\to\infty} \mathfrak{s}_{(-+)^n}(0)$$

Other definition of $\mathcal{M}$

A_λ is also the closure of the set of fixed points of every finite composition of the maps $\mathfrak{s}_-$ and $\mathfrak{s}_+$.
For example, let $\mathfrak{s}_{ab} = \mathfrak{s}_a \circ \mathfrak{s}_b$ then

$$z = \mathfrak{s}_{-+}(z) = -1 + \lambda \ \ + \lambda^2 z \iff z = \frac{-1}{1+\lambda} = -\sum_{j=0}^{\infty}(-1)^j \lambda^j$$
$$= \lim_{n\to\infty} \mathfrak{s}_{(-+)^n}(0)$$

Define the surjection $\pi_\lambda : \Sigma = \{-,+\}^{\mathbb{N}} \to \mathsf{A}_\lambda$

$$\mathtt{a} = a_0 a_1 \cdots \mapsto \sum_{j=0}^{\infty} a_j \lambda^j$$

Other definition of $\mathcal{M}$

A_λ is also the closure of the set of fixed points of every finite composition of the maps $\mathfrak{s}_-$ and $\mathfrak{s}_+$.
For example, let $\mathfrak{s}_{ab} = \mathfrak{s}_a \circ \mathfrak{s}_b$ then

$$z = \mathfrak{s}_{-+}(z) = -1 + \lambda \ \ + \lambda^2 z \iff z = \frac{-1}{1+\lambda} = -\sum_{j=0}^{\infty}(-1)^j \lambda^j$$

$$= \lim_{n\to\infty} \mathfrak{s}_{(-+)^n}(0)$$

Define the surjection $\pi_\lambda : \Sigma = \{-,+\}^{\mathbb{N}} \to \mathsf{A}_\lambda$

$$\mathtt{a} = a_0 a_1 \cdots \mapsto \sum_{j=0}^{\infty} a_j \lambda^j$$

A_λ is connected $\iff$ $\mathfrak{s}_-(\mathsf{A}_\lambda) \cap \mathfrak{s}_+(\mathsf{A}_\lambda) \neq \varnothing$ $\iff$ $\exists \mathtt{a}, \mathtt{b} \in \Sigma$ with $a_0 = +, b_0 = -$ such that $\pi_\lambda(\mathtt{b}) = \pi_\lambda(\mathtt{a})$

$$\iff \sum_{j=0}^{\infty}(a_j - b_j)\lambda^j = 2\sum_{j=0}^{\infty} c_j \lambda^j = 0, \qquad c_j \in \{-1, 0, +1\}.$$

Other definition of $\mathcal{M}$

Let $\mathcal{F}_\lambda = \left\{ f(z) = 1 + \sum_{j=1}^{\infty} c_j z^j \;\middle|\; c_j \in \{-1, 0, +1\}\,;\; f(\lambda) = 0 \right\}$ then

$$\mathcal{M} = \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\} = \{\lambda \in \mathbb{D} \mid \ |\mathcal{F}_\lambda| \neq 0\}$$

Other definition of $\mathcal{M}$

Let $\mathcal{F}_\lambda = \left\{ f(z) = 1 + \sum_{j=1}^{\infty} c_j z^j \;\middle|\; c_j \in \{-1, 0, +1\}\,;\; f(\lambda) = 0 \right\}$
then

$$\mathcal{M} = \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\} = \{\lambda \in \mathbb{D} \mid \ |\mathcal{F}_\lambda| \neq 0\}$$

Intuitively, $\lambda \in \partial\mathcal{M} \iff O_\lambda := \mathfrak{s}_-(\mathsf{A}_\lambda) \cap \mathfrak{s}_+(\mathsf{A}_\lambda)$ is "thin"
Elements in $\mathcal{F}_\lambda$ give a complete description of O_λ

Other definition of $\mathcal{M}$

Let $\mathcal{F}_\lambda = \left\{ f(z) = 1 + \sum_{j=1}^{\infty} c_j z^j \;\middle|\; c_j \in \{-1, 0, +1\}\,;\; f(\lambda) = 0 \right\}$ then

$$\mathcal{M} = \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\} = \{\lambda \in \mathbb{D} \mid \; |\mathcal{F}_\lambda| \neq 0\}$$

Intuitively, $\lambda \in \partial\mathcal{M} \iff O_\lambda := \mathfrak{s}_-(\mathsf{A}_\lambda) \cap \mathfrak{s}_+(\mathsf{A}_\lambda)$ is "thin"
Elements in $\mathcal{F}_\lambda$ give a complete description of O_λ

Lemma (Solomyak, 2005)

If $|O_\lambda| \leq 2$ then $|\mathcal{F}_\lambda| = 1$:

(i.) $|O_\lambda| = 1 \iff f \in \mathcal{F}_\lambda$ *has no zero coefficients.*

(ii.) $|O_\lambda| = 2 \iff f \in \mathcal{F}_\lambda$ *has exactly one zero coefficient.*

Self & Asymptotic Similarity

Remember $\widetilde{\mathsf{A}}_\lambda$ is the attractor of the IFS$\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Theorem (Solomyak, 2005)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with*

$$f(z) = q(z) + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}, \qquad q(z) = \sum_{j=0}^{\ell} c_j z^j$$

then $f'(\lambda) \neq 0$ *and*

Self & Asymptotic Similarity

Remember $\widetilde{\mathsf{A}}_\lambda$ is the attractor of the IFS$\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Theorem (Solomyak, 2005)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with*

$$f(z) = q(z) + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}, \qquad q(z) = \sum_{j=0}^{\ell} c_j z^j$$

then $f'(\lambda) \neq 0$ *and*

(i.) $\widetilde{\mathsf{A}}_\lambda$ *is* $|\lambda^{-p}|$*-self similar about* $-q(\lambda)/\lambda^{\ell+1} =: \zeta$*;*

Self & Asymptotic Similarity

Remember $\widetilde{\mathsf{A}}_\lambda$ is the attractor of the IFS$\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Theorem (Solomyak, 2005)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with*

$$f(z) = q(z) + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}, \qquad q(z) = \sum_{j=0}^{\ell} c_j z^j$$

then $f'(\lambda) \neq 0$ *and*

(i.) $\widetilde{\mathsf{A}}_\lambda$ *is* $|\lambda^{-p}|$*-self similar about* $-q(\lambda)/\lambda^{\ell+1} =: \zeta$*;*

(ii.) $\mathcal{M}$ *about* λ *and* $\frac{\lambda^{\ell+1}}{f'(\lambda)}\widetilde{\mathsf{A}}_\lambda$ *about* $\frac{\lambda^{\ell+1}}{f'(\lambda)}\zeta$ *are asymptotically similar.*

Self & Asymptotic Similarity

Remember $\widetilde{\mathsf{A}}_\lambda$ is the attractor of the IFS$\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Theorem (Solomyak, 2005)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ and $\mathcal{F}_\lambda = \{f\}$ with

$$f(z) = q(z) + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}, \qquad q(z) = \sum_{j=0}^{\ell} c_j z^j$$

then $f'(\lambda) \neq 0$ and

(i.) $\widetilde{\mathsf{A}}_\lambda$ *is $|\lambda^{-p}|$-self similar about $-q(\lambda)/\lambda^{\ell+1} =: \zeta$;*

(ii.) $\mathcal{M}$ *about λ and $\frac{\lambda^{\ell+1}}{f'(\lambda)}\widetilde{\mathsf{A}}_\lambda$ about $\frac{\lambda^{\ell+1}}{f'(\lambda)}\zeta$ are asymptotically similar.*

(iii.) $\mathcal{M}$ *is asymptotically $|\lambda^{-p}|$-self similar about λ;*

Self & Asymptotic Similarity

Remember $\widetilde{\mathsf{A}}_\lambda$ is the attractor of the IFS$\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Theorem (Solomyak, 2005)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with*

$$f(z) = q(z) + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}, \qquad q(z) = \sum_{j=0}^{\ell} c_j z^j$$

then $f'(\lambda) \neq 0$ *and*

(i.) $\widetilde{\mathsf{A}}_\lambda$ *is* $|\lambda^{-p}|$*-self similar about* $-q(\lambda)/\lambda^{\ell+1} =: \zeta$*;*

(ii.) $\mathcal{M}$ *about* λ *and* $\frac{\lambda^{\ell+1}}{f'(\lambda)}\widetilde{\mathsf{A}}_\lambda$ *about* $\frac{\lambda^{\ell+1}}{f'(\lambda)}\zeta$ *are asymptotically similar.*

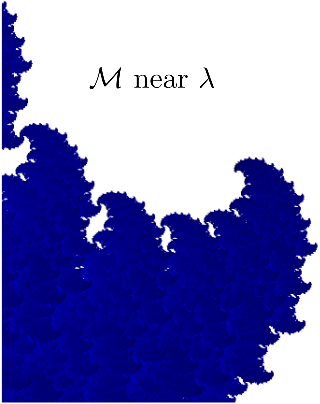
(iii.) $\mathcal{M}$ *is asymptotically* $|\lambda^{-p}|$*-self similar about* λ*;*

Remark. If f has no zero coefficients then we can also replace $\mathcal{M}$ with $\mathcal{M}_0$ and $\widetilde{\mathsf{A}}_\lambda$ with A_λ.

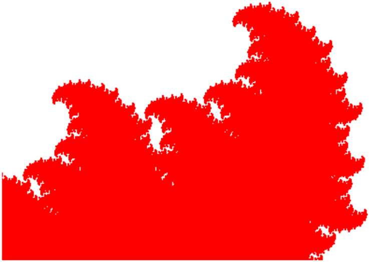
Self & Asymptotic Similarity

$$\lambda \approx 0.59574 + 0.25442\mathrm{i}$$

$\mathcal{M}$ near λ



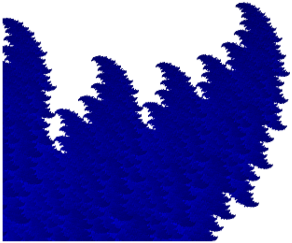
$\widetilde{\mathsf{A}}_\lambda$ near ζ



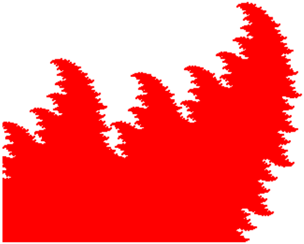
Self & Asymptotic Similarity

$$\lambda \approx 0.62196 + 0.18773\mathrm{i}$$

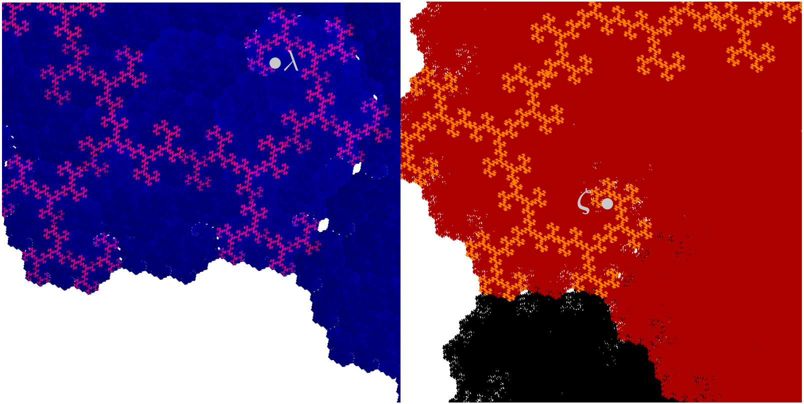
$\mathcal{M}$ near λ



$\widetilde{\mathsf{A}}_\lambda$ near ζ



Important Theorems

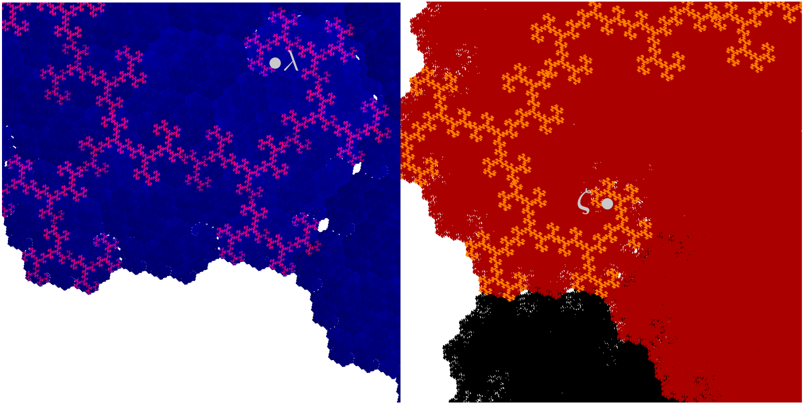


$$\lambda \approx 0.371859 + 0.519411i$$

Important Theorems

Theorem (Calegari-Koch-Walker, 2014)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ and $\mathcal{F}_\lambda = \{f\}$ with f rational as before. Then $\exists \delta > 0$ such that $\forall\ C \notin \frac{\lambda^{\ell+1}}{f'(\lambda)} \left(\widetilde{\mathsf{A}}_\lambda - \zeta \right)$ with $|C| < \delta$, the parameter $C\lambda^{pn} + \lambda$ is not in $\mathcal{M}$ for all sufficiently large n.



$$\lambda \approx 0.371859 + 0.519411i$$

Main Result

Theorem (Pérez-S.)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with* f *rational as before. If* f *has finitely many zero coefficients and its Taylor polynomials satisfy certain conditions then* $\lambda \in \partial\mathcal{M}$ *is accessible.*

Main Result

Theorem (Pérez-S.)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ and $\mathcal{F}_\lambda = \{f\}$ with f rational as before. If f has finitely many zero coefficients and its Taylor polynomials satisfy certain conditions then $\lambda \in \partial\mathcal{M}$ is accessible.

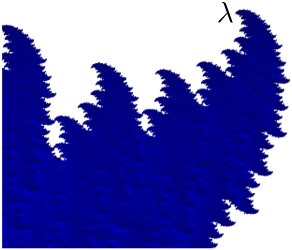
Minor adjustments to the conditions on the Taylor polynomials gives

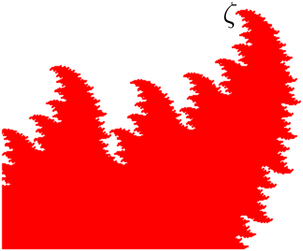
Theorem (Pérez-S.)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ and $\mathcal{F}_\lambda = \{f\}$ with f rational as before. If f has no zero coefficient and its Taylor polynomials satisfy certain conditions then $\lambda \in \partial\mathcal{M}_0$ is accessible.

Idea of proof

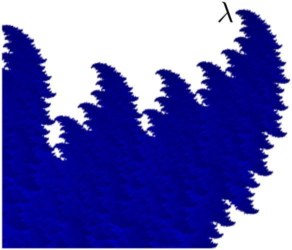
Use the asymptotic similarity:

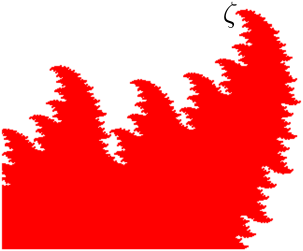




Idea of proof

Use the asymptotic similarity:

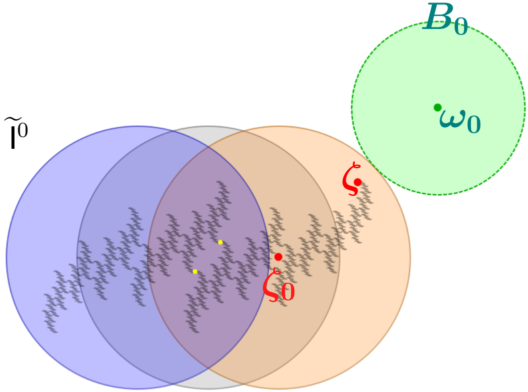




Show $\zeta \in \widetilde{\mathsf{A}}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \widetilde{\mathsf{A}}_\lambda$.

Idea of proof

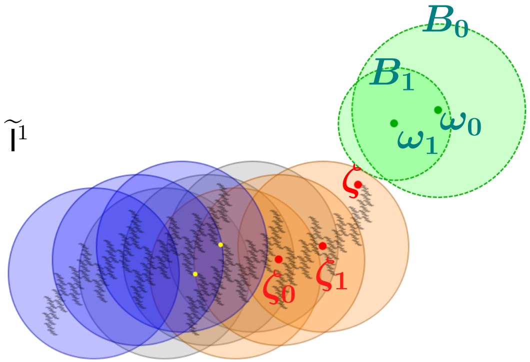
Show $\zeta \in \widetilde{\mathsf{A}}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \widetilde{\mathsf{A}}_\lambda$.
Use the iterative construction of $\widetilde{\mathsf{A}}_\lambda = \cap_{n\geq 0}\widetilde{\mathsf{I}}^n$ where $\widetilde{\mathsf{I}}^n$ is the finte union of closed discs covering $\widetilde{\mathsf{A}}_\lambda$ and the self-similarity of $\widetilde{\mathsf{A}}_\lambda$ at ζ.



Idea of proof

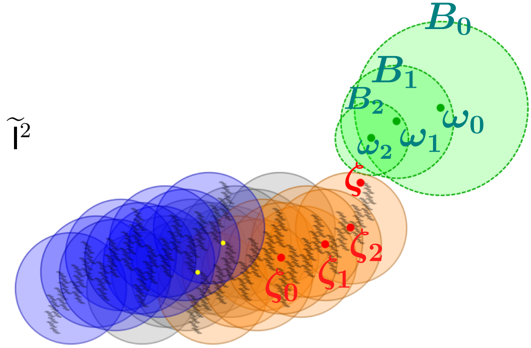
Show $\zeta \in \widetilde{\mathsf{A}}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \widetilde{\mathsf{A}}_\lambda$.

Use the iterative construction of $\widetilde{\mathsf{A}}_\lambda = \cap_{n \geq 0} \widetilde{\mathsf{I}}^n$ where $\widetilde{\mathsf{I}}^n$ is the finte union of closed discs covering $\widetilde{\mathsf{A}}_\lambda$ and the self-similarity of $\widetilde{\mathsf{A}}_\lambda$ at ζ.



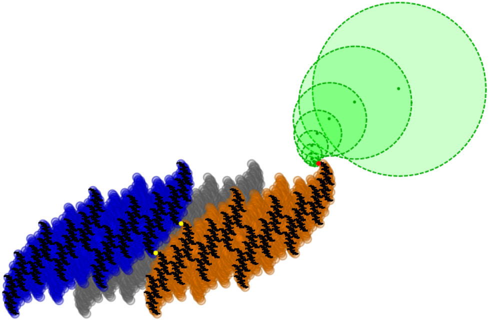
Idea of proof

Show $\zeta \in \widetilde{\mathsf{A}}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \widetilde{\mathsf{A}}_\lambda$.
Use the iterative construction of $\widetilde{\mathsf{A}}_\lambda = \cap_{n \geq 0} \widetilde{\mathsf{I}}^n$ where $\widetilde{\mathsf{I}}^n$ is the finte union of closed discs covering $\widetilde{\mathsf{A}}_\lambda$ and the self-similarity of $\widetilde{\mathsf{A}}_\lambda$ at ζ.



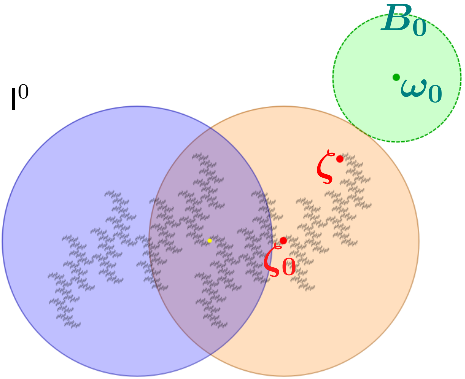
Idea of proof

Show $\zeta \in \widetilde{\mathsf{A}}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \widetilde{\mathsf{A}}_\lambda$.
Use the iterative construction of $\widetilde{\mathsf{A}}_\lambda = \cap_{n \geq 0} \widetilde{\mathsf{I}}^n$ where $\widetilde{\mathsf{I}}^n$ is the finte union of closed discs covering $\widetilde{\mathsf{A}}_\lambda$ and the self-similarity of $\widetilde{\mathsf{A}}_\lambda$ at ζ.



Idea of proof

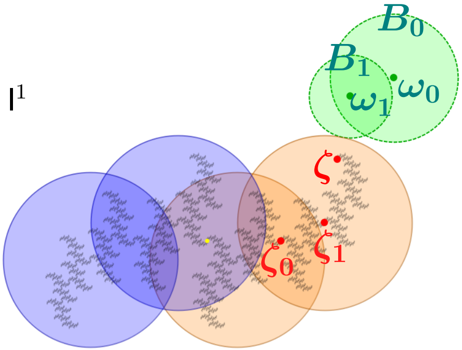
Show $\zeta \in \mathsf{A}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \mathsf{A}_\lambda$.
Use the iterative construction of $\mathsf{A}_\lambda = \cap_{n \geq 0} \mathsf{I}^n$ where I^n is the finte union of closed discs covering A_λ and the self-similarity of A_λ at ζ.



Idea of proof

Show $\zeta \in \mathsf{A}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \mathsf{A}_\lambda$.

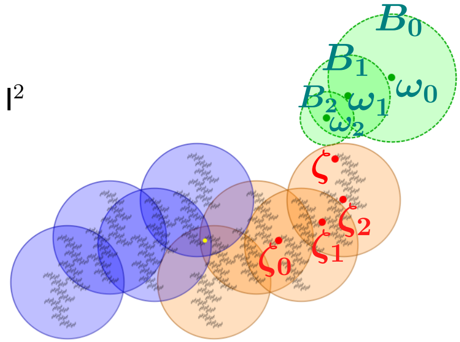
Use the iterative construction of $\mathsf{A}_\lambda = \cap_{n\geq 0} \mathsf{I}^n$ where I^n is the finte union of closed discs covering A_λ and the self-similarity of A_λ at ζ.



Idea of proof

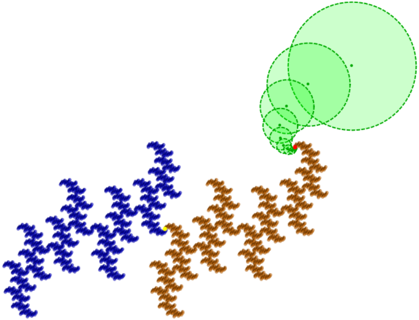
Show $\zeta \in \mathsf{A}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \mathsf{A}_\lambda$.

Use the iterative construction of $\mathsf{A}_\lambda = \cap_{n \geq 0} \mathsf{I}^n$ where I^n is the finte union of closed discs covering A_λ and the self-similarity of A_λ at ζ.



Idea of proof

Show $\zeta \in \mathsf{A}_\lambda$ is accessible (locally) from $\mathbb{C} \setminus \mathsf{A}_\lambda$.
Use the iterative construction of $\mathsf{A}_\lambda = \cap_{n \geq 0} \mathsf{I}^n$ where I^n is the finte union of closed discs covering A_λ and the self-similarity of A_λ at ζ.



Notation

$$\mathcal{F}_\lambda = \{f\} \text{ with } f(z) = \sum_{j=0}^{\ell} c_j z^j + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p} \text{ with}$$

$\mathsf{c} = c_0 \cdots c_\ell (c_{\ell+1} \cdots c_{\ell+p}) \in \Sigma$ then $O_\lambda = \{0\}$.

Notation

$$\mathcal{F}_\lambda = \{f\} \text{ with } f(z) = \sum_{j=0}^{\ell} c_j z^j + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p} \text{ with}$$

$\mathsf{c} = c_0 \cdots c_\ell (c_{\ell+1} \cdots c_{\ell+p}) \in \Sigma$ then $O_\lambda = \{0\}$.

Let $\mathsf{c}|n = c_0 \cdots c_n$ and set $f_n(\lambda) := \sum_{j=0}^{n} c_j \lambda^j$.

Notation

$\mathcal{F}_\lambda = \{f\}$ with $f(z) = \sum_{j=0}^{\ell} c_j z^j + \dfrac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}$ with

$\mathsf{c} = c_0 \cdots c_\ell (c_{\ell+1} \cdots c_{\ell+p}) \in \Sigma$ then $O_\lambda = \{0\}$.

Let $\mathsf{c}|n = c_0 \cdots c_n$ and set $f_n(\lambda) := \sum_{j=0}^{n} c_j \lambda^j$.

Then

$$\zeta = -\frac{f_\ell(\lambda)}{\lambda^{\ell+1}}, \qquad \zeta_n = \frac{f_{\ell+1+n}(\lambda) - f_\ell(\lambda)}{\lambda^{\ell+1}}.$$

Notation

$\mathcal{F}_\lambda = \{f\}$ with $f(z) = \sum_{j=0}^{\ell} c_j z^j + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}$ with

$\mathsf{c} = c_0 \cdots c_\ell (c_{\ell+1} \cdots c_{\ell+p}) \in \Sigma$ then $O_\lambda = \{0\}$.

Let $\mathsf{c}|n = c_0 \cdots c_n$ and set $f_n(\lambda) := \sum_{j=0}^{n} c_j \lambda^j$.

Then

$$\zeta = -\frac{f_\ell(\lambda)}{\lambda^{\ell+1}}, \qquad \zeta_n = \frac{f_{\ell+1+n}(\lambda) - f_\ell(\lambda)}{\lambda^{\ell+1}}.$$

So $\boldsymbol{B_n}$ is centered at

$$\boxed{\omega_n} = -(\zeta_n - \zeta) + \zeta = -\zeta_n + 2\zeta = \boxed{-\frac{1}{\lambda^{\ell+1}} f_{\ell+1+n}(\lambda) + \zeta}$$

with a radius

$$\boxed{r_n} = |\zeta_n - \omega_n| - \left|\lambda^{n+1}\right| R = \boxed{\frac{2}{|\lambda^{\ell+1}|} \left|f_{\ell+1+n}(\lambda)\right| - \left|\lambda^{n+1}\right| R}$$

where $R = (1 - |\lambda|)^{-1}$.

Conditions on Taylor polynomials of f

For all $n \geq 0$
B_n exists if and only if $r_n > 0$

$$|f_{\ell+1+n}(\lambda)| > \frac{1}{2}\frac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}.$$

Conditions on Taylor polynomials of f

For all $n \geq 0$

B_n exists if and only if $r_n > 0$

$$|f_{\ell+1+n}(\lambda)| > \frac{1}{2}\frac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}.$$

$B_n \cap B_{n+1} \neq \varnothing$ if and only if $r_n + r_{n+1} > |\omega_n - \omega_{n+1}|$

$$|f_{\ell+1+n}(\lambda)| + |f_{\ell+1+n+1}(\lambda)| > \frac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}$$

Conditions on Taylor polynomials of f

For all $n \geq 0$

B_n exists if and only if $r_n > 0$

$$|f_{\ell+1+n}(\lambda)| > \frac{1}{2} \frac{\left|\lambda^{\ell+1+n+1}\right|}{1 - |\lambda|}.$$

$B_n \cap B_{n+1} \neq \varnothing$ if and only if $r_n + r_{n+1} > |\omega_n - \omega_{n+1}|$

$$|f_{\ell+1+n}(\lambda)| + |f_{\ell+1+n+1}(\lambda)| > \frac{\left|\lambda^{\ell+1+n+1}\right|}{1 - |\lambda|}$$

$B_n \cap \mathsf{I}^n = \varnothing$ if and only if $\left|\omega_n - \nu_{\mathtt{a}|n}\right| > r_n + \left|\lambda^{n+1}\right| R$ for any $\mathtt{a} \in \Sigma$ and $\nu_{\mathtt{a}|n} = \sum_{j \leq n} a_j \lambda^j$

$$2\,|f_{\ell+1+n}(\lambda)| < \left|f_\ell(\lambda) + f_{\ell+1+n}(\lambda) + \lambda^{\ell+1} \nu_{\mathtt{a}|n}\right|$$

Self-similarity of I^n at 0

Since

$$f(\lambda) = 0 \implies \sum_{j=\ell+1}^{\ell+p} c_j \lambda^j = (\lambda^p - 1) \sum_{j=0}^{\ell} c_j \lambda^j$$

then for any $0 \leq n \leq p$

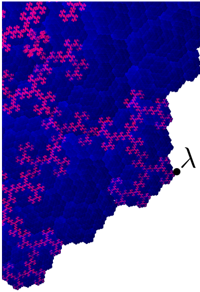
$$\frac{1}{\lambda^p} \left(\mathsf{I}^{\overline{\mathsf{c}}|\ell+p+n} \cup \mathsf{I}^{\mathsf{c}|\ell+p+n} \right) = \mathsf{I}^{\overline{\mathsf{c}}|\ell+n} \cup \mathsf{I}^{\mathsf{c}|\ell+n}$$

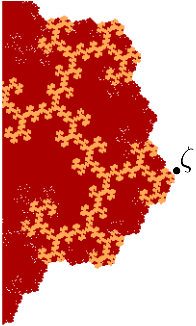
where $\overline{\mathsf{c}}$ is the opposite of c.

We need to check finitely many inequalities to certify $\lambda \in \partial\mathcal{M}_0$!

Example

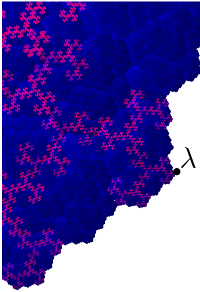
Let $\lambda \approx -0.366 + 0.520i$ *be the root of* $f(z) = \frac{1+z+z^2-2z^3}{1-z^3}$ *then* λ *is on the boundary of a hole of both* $\mathcal{M}_0$ *and* $\mathcal{M}$.

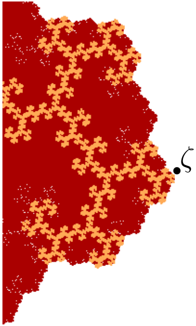




Example

Let $\lambda \approx -0.366 + 0.520i$ *be the root of* $f(z) = \frac{1+z+z^2-2z^3}{1-z^3}$ *then* λ *is on the boundary of a hole of both* $\mathcal{M}_0$ *and* $\mathcal{M}$*.*





Solomyak already proved $\mathcal{F}_\lambda = \{f\}$. To show the inequalities are satisfied use estimates on $|\lambda|$ and $\arg(\lambda)$, and the fact that $1 + \lambda + \lambda^2 = 2\lambda^3$.

$+(++-)^{\infty}$

Proposition (Pérez-S.)

Let $\lambda \approx -0.366 + 0.520\mathrm{i}$ *be the root of* $f(z) = \frac{1+z+z^2-2z^3}{1-z^3}$ *then* λ *is on the boundary of the largest connected component of* $\mathbb{D} \setminus \mathcal{M}$*.*

$+(++-)^\infty$

Proposition (Pérez-S.)

Let $\lambda \approx -0.366 + 0.520\mathrm{i}$ *be the root of* $f(z) = \frac{1+z+z^2-2z^3}{1-z^3}$ *then* λ *is on the boundary of the largest connected component of* $\mathbb{D} \setminus \mathcal{M}$*.*

λ is a root of the polynomial $1 + z + z^2 - 2z^3$.

$+(++-)^{\infty}$

Proposition (Pérez-S.)

Let $\lambda \approx -0.366 + 0.520\mathrm{i}$ *be the root of* $f(z) = \frac{1+z+z^2-2z^3}{1-z^3}$ *then* λ *is on the boundary of the largest connected component of* $\mathbb{D} \setminus \mathcal{M}$*.*

λ is a root of the polynomial $1 + z + z^2 - 2z^3$.
$\implies (1 - \lambda^{3k}) f(\lambda) = f_{3k-1}(\lambda) - 2\lambda^{3k} = 0$ for every $k \in \mathbb{N}$.

Proposition (Pérez-S.)

Let $\lambda \approx -0.366 + 0.520\mathrm{i}$ *be the root of* $f(z) = \frac{1+z+z^2-2z^3}{1-z^3}$ *then* λ *is on the boundary of the largest connected component of* $\mathbb{D} \setminus \mathcal{M}$.

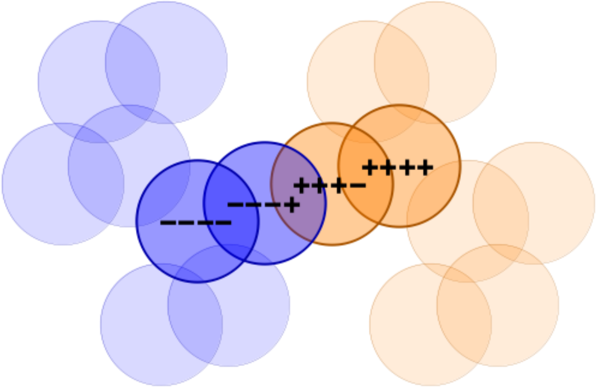
λ is a root of the polynomial $1 + z + z^2 - 2z^3$.
$\implies (1-\lambda^{3k})f(\lambda) = f_{3k-1}(\lambda) - 2\lambda^{3k} = 0$ for every $k \in \mathbb{N}$.

Geometrically, it means that the discs with centers

$$\pm f_{3k-1}(\lambda) \pm \lambda^{3k}$$

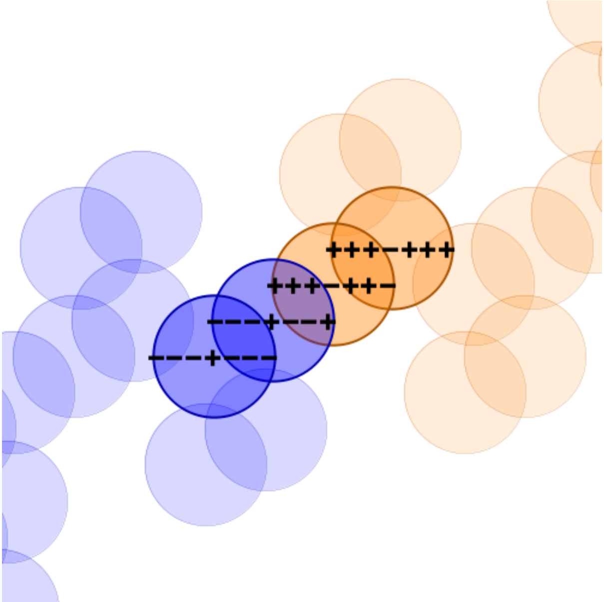
are aligned.

Alignment of Discs



$k = 1$ Level 3

Alignment of Discs

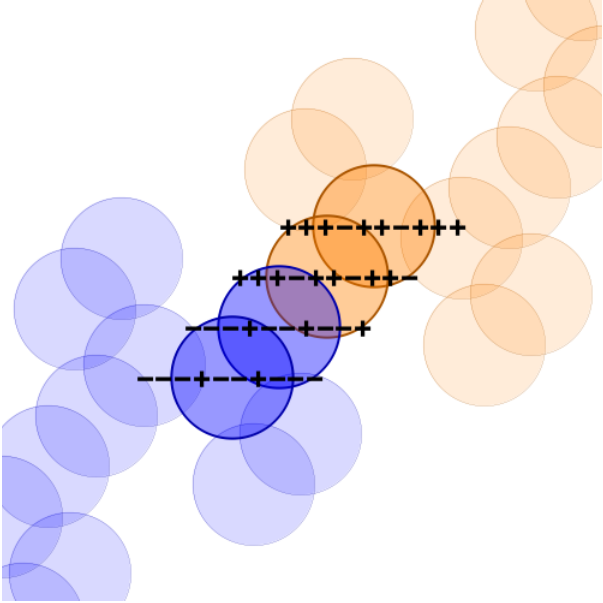


$k = 2$

Level 6

Alignment of Discs

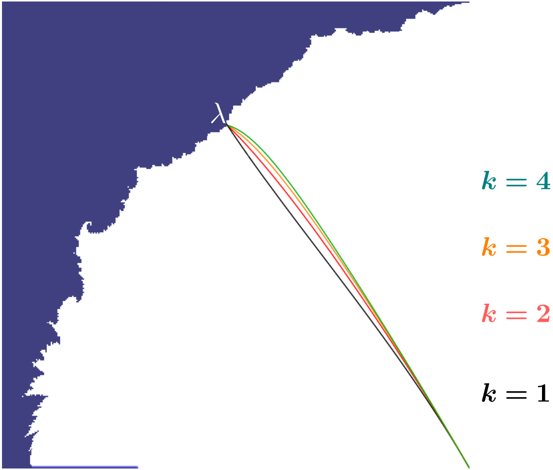
$k = 3$



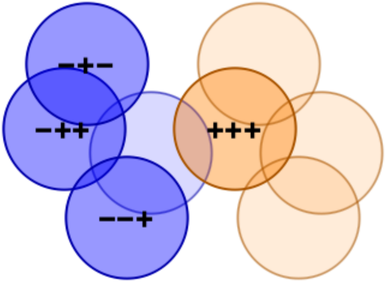
Level 9

Level Curves

λ is on the level curve $\mathrm{Im}\left(\frac{f_{3k-1}(z)}{z^{3k}}\right) = 0$ for every $k \geq 1$.

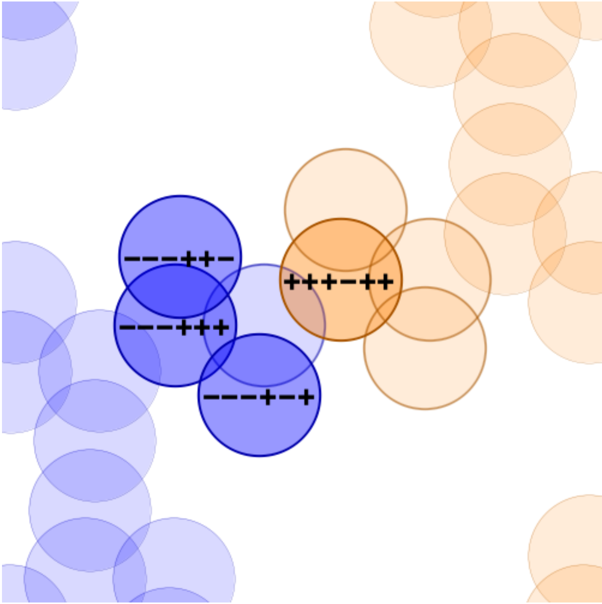


Shape Conditions



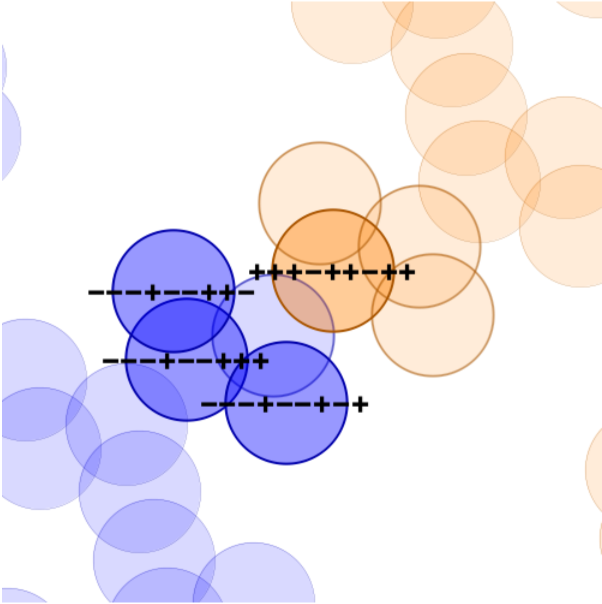
Level 2

Shape Conditions



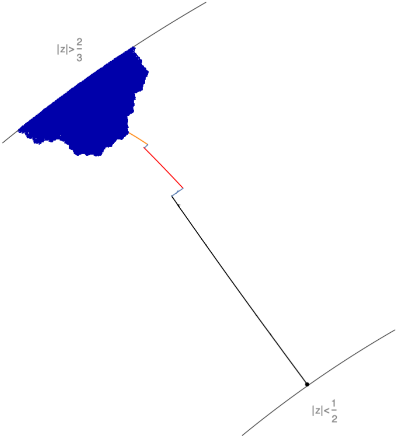
Level 5

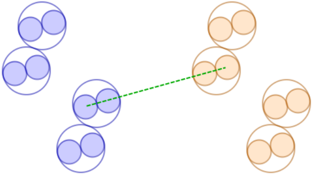
Shape Conditions



Level 8

Path Construction





Questions?

Thank You!