Non Homeomorphic Julia Sets of Singularly Perturbed Rational Maps

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Abstract

In this paper we investigate the *Julia set* of singularly perturbed complex rational maps of the form

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}.$$

We will prove that for the case $n = d = 2$ two maps drawn from different main cardioids of accessible baby Mandelbrot sets containing a cycle of period 4 are not homeomorphic on their Julia sets, unless these cardioids are complex conjugates of one another.

1 Introduction

In the past few years, there has been a lot of interest in the dynamics of the family of rational maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}.$$

The reason for this is that the dynamics of these maps are enormously more complex than the well-known simple map $G_\lambda(z) = z^n$.

In this paper we will focus on the case where $n = d = 2$:

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}.$$

Note, first, that when $\lambda = 0$ we obtain the quadratic map, $z \mapsto z^2$. However, when $\lambda \neq 0$ the degree of the map doubles. When $|z|$ is large $F_\lambda(z) \approx z^2$ so there is a superattracting fixed point at ∞. In addition, the origin becomes a pole of order 2, so there is a neighborhood around 0 that is mapped into the basin of attraction at ∞, B_λ. Whenever the preimage of B_λ, surrounding the origin, is disjoint from B_λ, it is called the *trap door*, T_λ.

Besides 0 and ∞, F_λ has 4 additional critical points given by $c_\lambda = \lambda^{1/4}$ and they all lie on the circle of radius $\sqrt[4]{|\lambda|}$. However, F_λ has only two critical

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values given by $\nu_\lambda = F_\lambda(c_\lambda) = \pm 2\sqrt{\lambda}$. If we look at the second iterate of c_λ, $F_\lambda(\nu_\lambda) = F_\lambda^2(c_\lambda) = 4\lambda + \frac{1}{4}$, we notice there is only one free critical orbit for F_λ.

The parameter plane is symmetric under complex conjugation since $\overline{F_\lambda(z)} = F_{\bar{\lambda}}(\bar{z})$. While the dynamical plane is symmetric under rotation by roots of unity since $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$, where $\omega = \exp \frac{2\pi i}{2n} = \exp \frac{\pi i}{2}$.

2 The Julia Set

A numerically generated picture of the parameter plane for the case $n = 2$, is shown in Figure 1. Parameters drown from the black regions have the property that the critical orbit does not escape to ∞. It is known that there are infinitely many baby Mandelbrot sets i.e. the black regions. On the contrary, the white regions represent λ-values for which the critical orbits escape to ∞. The Mandelbrot sets situated on the outer boundary are called *accessible* Mandelbrot sets, while all other Mandelbrot sets are called *buried*. We are interested in the Julia sets generated by λ lying in the main cardioids of the accessible Mandelbrot sets.

The *Julia set* of F_λ, denoted by $J(F_\lambda)$, is defined to be the closure of the set of repelling periodic points of F_λ. The complement of $J(F_\lambda)$ is called the *Fatou set*. A *Fatou component* is a connected subset of the Fatou set: if the orbit of the points in the Fatou component escape to ∞ then that component is called a *white* Fatou component, otherwise if the points have bounded orbits, the Fatou component is denoted as a *black* Fatou component.

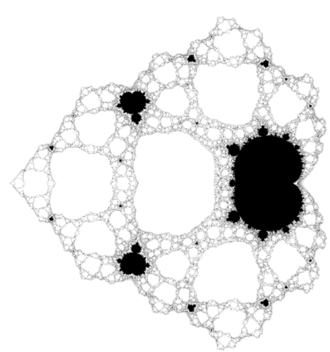

Figure 1: Parameter plane for the family of rational map $F_\lambda(z) = z^2 + \lambda/z^2$

In [1] it was proven that the topology of the $J(F_\lambda)$, for $F_\lambda(z) = z^n + \frac{\lambda}{z^n}$ with $n \in \mathbb{N}$, has the following trichotomy:

Theorem 1. *(The Escape Trichotomy) Let $\nu_\lambda = F_\lambda(c_\lambda)$ be a critical value, then*

1. *if $\nu_\lambda \in B_\lambda$, then $J(F_\lambda)$ is a Cantor set;*

2. *if $\nu_\lambda \in T_\lambda$ and $T_\lambda \neq B_\lambda$, then $J(F_\lambda)$ is a Cantor set of disjoint simple closed curves surrounding the origin (this does not occur when $n = 2$);*

3. *otherwise, $J(F_\lambda)$ is a connected set. In particular, if $F^i_\lambda(\nu_\lambda) \in T_\lambda$ and $T_\lambda \neq B_\lambda$ for some $i \geq 1$, then $J(F_\lambda)$ is a Sierpiński curve.*

As shown in [2], the Julia sets drawn from the main cardioids of "principal" Mandelbrot sets (the biggest black regions in the parameter plane) are called checkerboard Julia sets and they are all homeomorphic. In the case for $n = 2$, there is only one principal case, namely the main cardioid of the large black region on the right. Furthermore, the Julia sets drawn from the main cardioids of accessible Mandelbrot sets are checkerboard Julia sets (see Figure 2). However, the dynamics are not necessarily conjugate.

Because of the conjugacy $z \mapsto z^2$, we know there exists a homeomorphism taking the open unit disk to the external region in the parameter plane. This homeomorphism is the Böttcher map that sends rational rays extending out from the unit circle towards ∞ to rays in the parameter plane that land on the cusps of the Mandelbrot sets. Our intention is to use these *external* rays to determine the location, and count the number, of Mandelbrot sets whose main cardioid produces a Julia set with critical orbit of period k. Thus, we use the following formula to calculate the ray that lands on a Mandelbrot set of period $k > 1$: $\frac{t}{n^k-1}$ where $t = 1, \ldots, n^k - 2$ and $n = 2$.

For example, the two biggest Mandelbrot sets in the parameter plane, both contain attracting cycles of period two. In fact, their corresponding external rays are $\frac{1}{2^2-1} = \frac{1}{3}$ and $\frac{2}{2^2-1} = \frac{2}{3}$ which, under the doubling map, are sent to each other. Due to our previously defined symmetries, we know that these Mandelbrot sets in the parameter plane and their respective Julia sets are conjugate to each other by complex conjugation.

Now that we can find the location of the Mandelbrot sets with a certain period, how are the corresponding Julia sets created? Choose λ at the center of the main cardioid of an accessible Mandelbrot set. Let c^i_λ be the critical points of F_λ and call their immediate basin of attraction C^i_λ. These are the main black Fatou components and are called *connecting components* because they touch both T_λ and B_λ. Using the rotational symmetry we determine their positions. It should also be noted that each C^i_λ does not map into other C^i_λ's, which is quite different from the "principal" case. Instead, C^i_λ will map into one of the next smallest Fatou components which meet ∂B_λ at one point.

The following lemma was proved in [3]:

Lemma 1. *F_λ maps each I^i_λ univalently (except at the junction points) over the region that is the complement of the three sets B_λ, $F_\lambda(C^i_\lambda)$, and $F_\lambda(C^{i+1}_\lambda)$.*

More specifically, the portion of $\partial B_\lambda \cap I^i_\lambda$ is mapped to exactly half of ∂B_λ, while the portion of $\partial T_\lambda \cap I^i_\lambda$ is mapped to the remaining half of ∂B_λ. By this lemma it sufficies to focus on only one I^i_λ to understand the entire $J(F_\lambda)$.

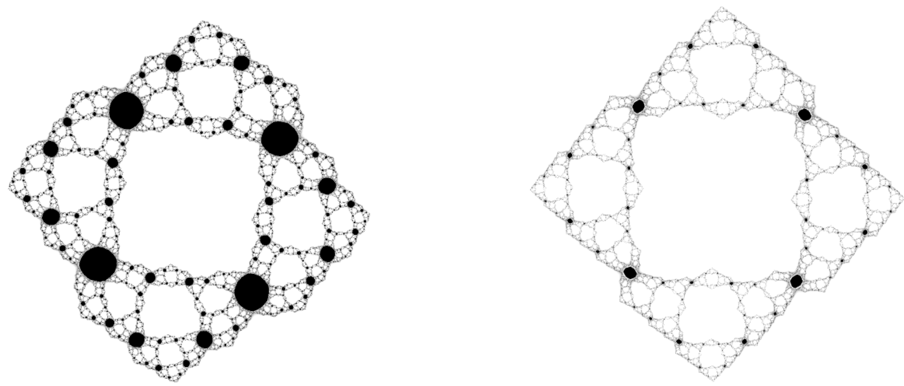

Figure 2: Two different $J(F_\lambda)$ for $\lambda = -0.157 + 0.138i$ on the left, and for $\lambda = -0.295 + 0.06038i$ on the right.

It is shown that the boundaries of B_λ and T_λ are simple closed curves which do not intersect, i.e., there are no critical points in $\partial B_\lambda \cap \partial T_\lambda$. We now denote by A_λ the closed annulus bounded by ∂T_λ and ∂B_λ. Let I^i_λ be the closed set in A_λ contained between the open disks C^i_λ and C^{i+1}_λ. Note that, I^i_λ intersects I^{i+1}_λ at exactly two points, q^{i+1}_λ and u^{i+1}_λ.

Call the arcs where I^j_λ meet the boundaries of C^j_λ and C^{j+1}_λ the internal boundary components of I^j_λ. There exists a preimage of T_λ in I^j_λ, call it $P_\lambda = F^{-1}_\lambda(T_\lambda)$. This preimage will be surrounded by four black Fatou components: two touching ∂B_λ, two touching ∂T_λ, and all four touching the preimage. From now on when we talk about the black Faotu components of $F^i_\lambda(T_\lambda)$ we imply that these components also touch the preimage. P_λ will not touch the internal boundary components of I^j_λ. There are four preimages of P_λ: one above, one below, and one each on the left and right side of P_λ. If we are in the case of period $k = 2$, the two side preimages of P_λ will touch the inner boundary components of I^j_λ. In general, in the Julia set containing a k-cycle, the kth side-preimage of P_λ will touch the C^j_λ. (See Figure 3)

Lemma 2. *In I^i_λ there is no preimage of T_λ with four black Fatou compononets touching ∂B_λ and the preimage.*

Proof. Suppose there is a preimage of T_λ in I^i_λ with four black Fatou components touching ∂B_λ. By the previous lemma we know that I^i_λ maps univalently over $J(F_\lambda)$ so $F_\lambda(\partial I^i_\lambda \cap \partial B_\lambda)$ contains exactly 2 of the C^i_λ. Therefore, the four black Fatou components touching ∂B_λ must map to the two C^i_λ, so the map is at best $2-$to-1. This is a contradiction. $\square$

Lemma 3. *In I^i_λ there is no preimage of T_λ with three black Fatou components touching ∂B_λ, one touching ∂T_λ.*

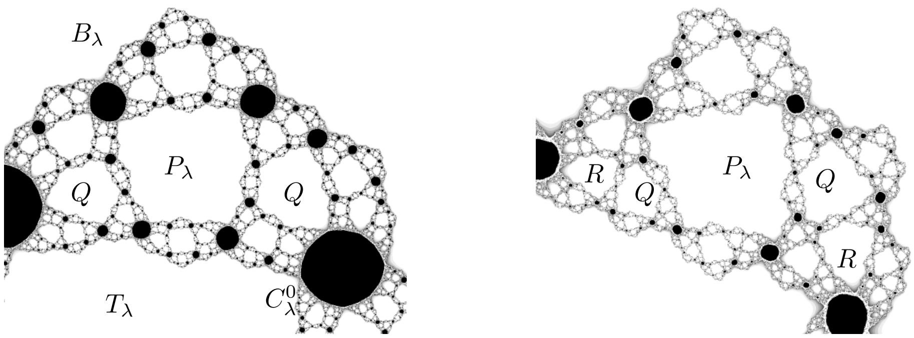

Figure 3: On the left: I_λ^1 of the Julia set containig a cycle of period 2. On the right: I_λ^1 of the Julia set containing a cycle of period 3. Q and R indicate $F_\lambda^{-2}(T_\lambda)$ and $F_\lambda^{-3}(T_\lambda)$ respectively.

Proof. Suppose in I_λ^i there is a preimage of T_λ with three black Fatou components touching ∂B_λ and one touching ∂T_λ then under F_λ it will be mapped to a region having four black Fatou components touching ∂B_λ since it sends B_λ to T_λ, but by the prevoius lemma there is no such region. $\square$

Lemma 4. *There is no other preimage of T_λ that has the same structure as $F_\lambda^{-1}(T_\lambda)$ in I_λ^i.*

Proof. Lemma 1 tells us that F_λ maps I_λ^i, excluding the junction points, one-to-one onto the complement of B_λ, $F_\lambda(C_\lambda^j)$, and $F_\lambda(C_\lambda^{j+1})$. Thus, there is exactly one first preimage of the trap door in I_λ with two black Fatou components touching ∂T_λ and two touching ∂B_λ. Call this region P_λ. Assume, now, there is another region with the same structure as P_λ, but it is not a first preimage of T_λ. Then, under F_λ, this region will be sent to a region with four black Fatou components touching ∂B_λ. However, by Lemma 2 this cannot happen. $\square$

Each I_λ^i can then be divided into four sectors: two sectors will be on the left and right side of P_λ, while the other two will be the regions enclosed by the two black Fatou components touching P_λ and touching either ∂B_λ or ∂T_λ. Because of Lemma 1, each of these sectors is mapped univalently onto I_λ^i. Therefore, there is a second preimage of the trap door in each sector. Let's assume we are in a Julia set with a cycle of period $k > 2$ and consider the right sector, then $F_\lambda^{-2}(T_\lambda)$ will have two black Fatou components touching ∂P_λ and the other two either: both touching ∂T_λ, both touching ∂B_λ, or one touching ∂T_λ and one touching ∂B_λ. If $k = 2$, then $F_\lambda^{-2}(T_\lambda)$ has two black Fatou components touching ∂P_λ, one either touching ∂B_λ or ∂T_λ, and the ramining one is actually C_λ^i.

Lemma 5. *There is no other preimage of T_λ that has the same structure as $F_\lambda^{-2}(T_\lambda)$ in each sector of I_λ^i.*

Proof. By Lemma 1 there exists one second preimages of the trap door in each sector of I_λ^i. Assume there exists in each sector of I_λ^i some other preimage of T_λ which is not a second preimage, but with the same structure as $F_\lambda^{-2}(T_\lambda)$. Then, by the dynamics of F_λ, this region will be the preimage of a region similar in structure to P_λ. However, by Lemma 4 the latter does not exist. □

Hence, by induction it is possible to prove that for each preimage of the trap door its structure is unique in each sector of I_λ^i, up to symmetry.

3 Period Four Julia Sets Are Not Homeomorphic.

In this section we prove the main subject of this paper:

Theorem 2. *Two maps drawn from different and non-conjugate main cardioids of (accessible) baby Mandelbrot sets containing a cycle of period $k = 4$, are not homeomorphic on their Julia set.*

As we mentioned before, for $k = 1, 2$ the above theorem fails. We will prove that if there exists a homeomorphism between two Julia sets then it will preserve some of their topological properties. Subsequently, we will show that the Julia sets have, in fact, different structures which therefore completes the proof. The case $k = 3$ can be proven similarly.

First, we need the following proposition.

Proposition 1. *Let $F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$. Suppose there exists a homeomorphism, ϕ between two Julia sets contanining an attracting cycle of period k drawn from the main cardioid of accessible Mandelbrot sets. Furthermore, assume the boundary of the trap door, ∂T_λ and the basin of attraction, ∂B_λ are preserved under ϕ. Then $F_\lambda^{-1}(T_\lambda)$ and $F_\lambda^{-2}(T_\lambda)$ are also preserved under ϕ.*

Proof. Let $\lambda\,, \mu$ be the parameters for which we obtain two Julia sets containing a cycle of period k. By our assumption, ϕ preserves the outer basin and the inner trap door. In this way, the homeomorphism will map each of the main black Fatou components, C_λ^i, in the first Julia set, to the corresponding C_μ^j of the other Julia set, with i not necessarily equal to j. This implies that the annulus A_λ is preserved. Without loss of generality, we may assume $i = j$.

Now, we may consider only one I_λ^i since the arguments given for it will be equivalent for all the other i's. Thus, let $i = 0$.

Consider the first preimage of T_λ, $F_\lambda^{-1}(T_\lambda) = P_\lambda$ and denote its black Fatou components by $c_\lambda^{0,i}$ with $i = 0, 1, 2, 3$ going counterclockwise (see Figure 4). Since $\partial T_\lambda \to \partial T_\mu$ and $\partial B_\lambda \to \partial B_\mu$ under ϕ, then the two largest black Fatou components touching ∂B_λ, $c_\lambda^{0,x}$, where $x = 0, 1$, will be mapped to other black Fatou components touching ∂B_μ and, similarly, the two largest black Fatou

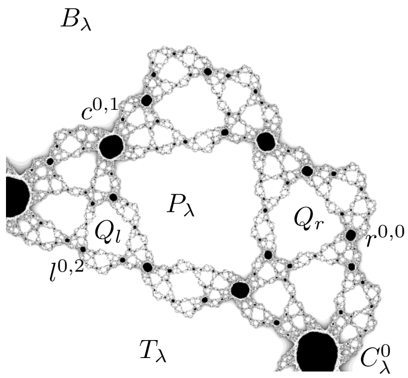

Figure 4: I^0_λ of the Julia set when $\lambda = 0.06456 + 0.199200i$

components touching ∂T_λ, $c_\lambda^{0,y}$, where $y = 2, 3$, will be mapped to other black Fatou components touching ∂T_μ. By Lemma 5 $\partial P_\lambda \to \partial P_\mu \implies P_\lambda \to P_\mu$ under ϕ.

Now, P_λ has four preimages, $F_\lambda^{-2}(T_\lambda)$. Call the one on the left and right side Q_l and Q_r respectively and the denote their four black Fatou components $l^{0,i}$ and $r^{0,i}$ with $i = 0, 1, 2, 3$ going counterclockwise (See Figure 4). By assumption ∂T_λ and ∂B_λ are preserved. Since also $\partial P_\lambda \to \partial P_\mu$ under the homeomorphism, the four black Fatou components, $r^{0,2}$, $r^{0,3}$, $l^{0,0}$, and $l^{0,1}$ are preserved as well by Lemma 4. The remaining two black Fatou components of each side preimage are mapped to the corresponding remaining two black Fatou components of Q_r and Q_l in I^0_μ. Therefore, the Q regions are preserved under ϕ. □

We can extend the above proposition to a more general one.

Definition 1. *A chain of Fatou components is the union of subsequent side-preimages of the trap door together with their corresponding black Fatou components.*

Proposition 2. *Let $F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$. Suppose there exists a homeomorphism, ϕ between two Julia sets contanining an attracting cycle of period 4 drawn from an exterior Mandelbrot set. Furthermore, assume the boundary of the trap door, ∂T_λ, and of the basin of attraction, ∂B_λ, are preserved under ϕ. Then the chain of preimages of the trap door is preserved under ϕ.*

Proof. Denote by R_r and S_r the third and fourth right side-preimage of T_λ (see Figure 5). By the previous proposition we know P_λ and Q_r are preserved. By similar arguments it is easy to show that both R_r and S_r in Figure 5 are also preserved. The lemma in Section 2 tell us that each preimage of the trap door has a unique structure up to symmetry. Therefore, under ϕ R_r and S_r are sent to regions with analogous structure, i.e. they are preserved. Hence, the chain of preimages is preserved under ϕ. □

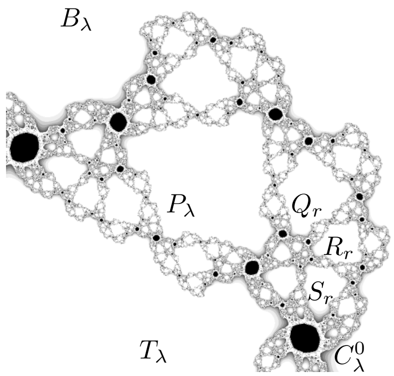

Figure 5: I^0_λ of the Julia set when $\lambda = 0.13697 + 0.12807i$. The right chain of Fatou components is $P_\lambda \cup Q_r \cup R_r \cup S_r$ and their corresponding black Faotu components

It now remains to show that the Julia sets from different (and accessible) Mandelbrot sets that are not complex conjugate with an attracting cycle of period 4 all have different structures. To see this, we will generate the Julia sets numerically, look at the chain of preimages of T_λ, and count the number of black Fatou components on ∂B_λ and ∂T_λ for each preimage in the chain. Therefore, if for some preimage of T_λ in the chain the number of black Fatou components on either ∂B_λ or ∂T_λ is different, then the corresponding Julia sets are not homeomorphic.

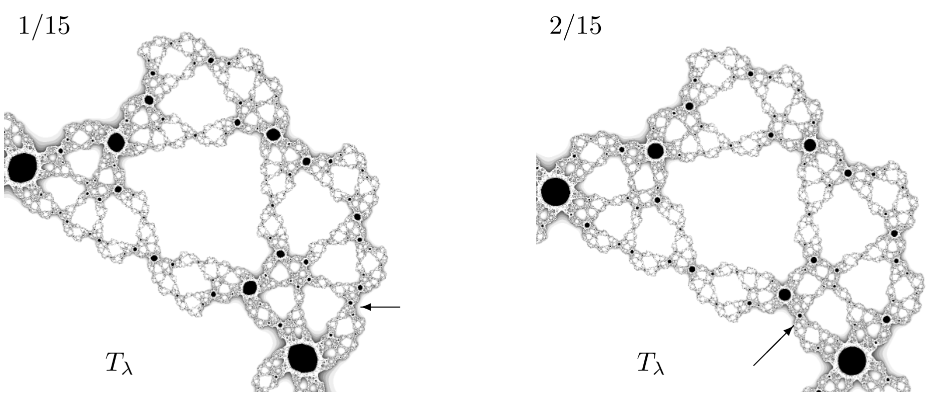

Figure 6: On the left: I^0_λ from the 1/15 Julia set. On the right: I^0_λ from the 2/15 Julia set. The Fatou component distinguishing the two Julia sets is indicated by the arrow.

Using the external rays, we find the location of the Mandelbrot sets with an

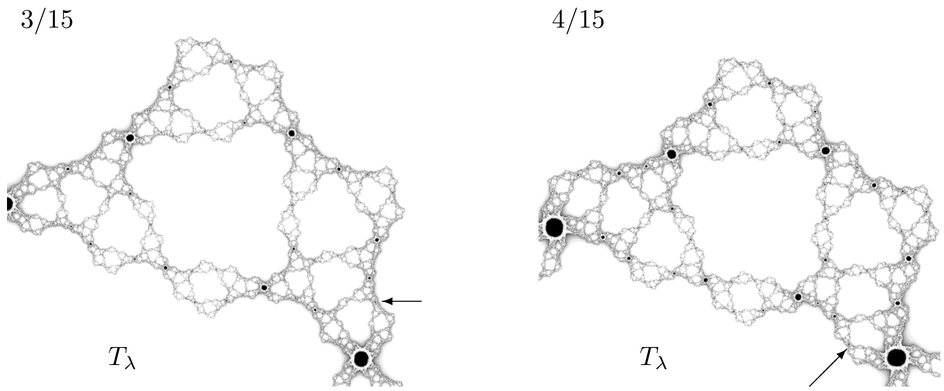

Figure 7: On the left: I_λ^0 from the 3/15 Julia set. On the right: I_λ^0 from the 4/15 Julia set. The Fatou component distinguishing the two Julia sets is indicated by the arrow.

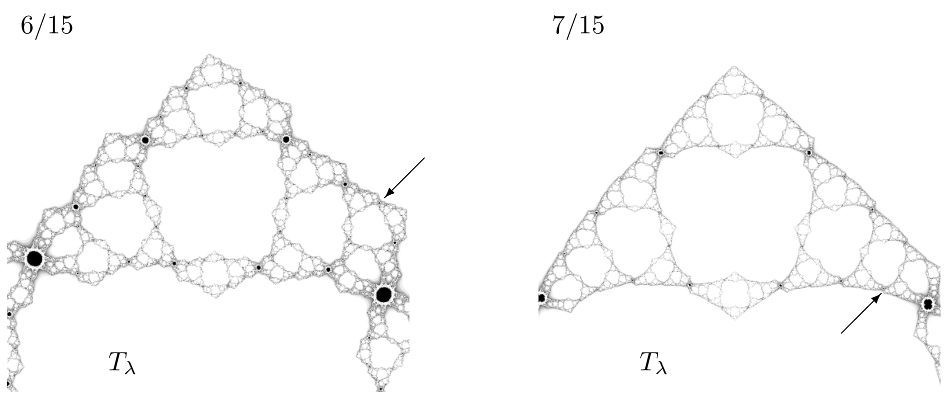

Figure 8: On the left: I_λ^0 from the 6/15 Julia set. On the right: I_λ^0 from the 7/15 Julia set. The Fatou component distinguishing the two Julia sets is indicated by the arrow.

attracting cycle of period 4 in the main cardiod. These are: 1/15, 2/15, 3/15, 4/15, 6/15, and 7/15. Let $i = 1, 2, 3, 4, 6, 7$. Without loss of generality, let's look at I_λ^0. The Julia sets in Figure 6 correspond to the 1/15 and 2/15 bulbs in the parametr plane. As we can see, the last preimage of the trap door in the chain, $F_\lambda^{-4}(T_\lambda)$ for 1/15 has one black Fatou component on ∂B_λ and none on ∂T_λ (not counting C_λ^0), while for 2/15 there are none on ∂B_λ and one on ∂T_λ. Hence, these two Julia sets are not homeomorphic.

Similarly from Figure 7, we note that the position of the black Fatou components of the last preimage of T_λ is different from the 3/15 and 4/15: for one is on ∂B_λ while for the other it's on ∂T_λ. Furthermore, it can be seen that these $J(F_\lambda)$ differ from the previous two on their third preimage of the trap door.

For the last two Julia sets we do not have to go further than $F_\lambda^{-3}(T_\lambda)$ to notice that they are not homeomorphic. In addition, the black Fatou components of $F_\lambda^{-2}(T_\lambda)$, from the 6/15 and 7/15 Julia set, are positioned differently than in the other four Julia sets. Hence, we conclude our proof of Theorem 2.

4 Conclusion

We have proven that Julia sets with an attracting cycle of period 4 drawn from the center of non-conjugate accessible Mandelbrot sets for the family of rational maps $F_\lambda(z) = z^2 + \lambda/z^2$ are not homeomorphic by constructing a topological invariant and showing that it is not preserved between them. Since, in the last step of our proof, we have given a numerical argument, further research is being carried out to see if its possible to give a more analytical proof and if it can be extended for Julia sets with cycles of higher periods.

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