The Boundary of the Mandelbrot set for a pair of linear maps

Stefano Silvestri

IUPUI

Setting: Iterated Function Systems

For each $\lambda \in \mathbb{D} \setminus \{0\}$ associate to it the compact sets

$$\mathsf{A}_\lambda := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j \;\middle|\; a_j \in \{-1, +1\} \right\}$$

$$\widetilde{\mathsf{A}}_\lambda := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j \;\middle|\; a_j \in \{-1, 0, +1\} \right\}$$

These are, respectively, the *attractors* of the IFS $\{\mathfrak{s}_-, \mathfrak{s}_+\}$ and $\{\mathfrak{s}_-, \mathfrak{s}_\mathsf{0}, \mathfrak{s}_+\}$ where

$$\mathfrak{s}_-(z) = \lambda z - 1 \qquad \mathfrak{s}_\mathsf{0}(z) = \lambda z \qquad \mathfrak{s}_+(z) = \lambda z + 1$$

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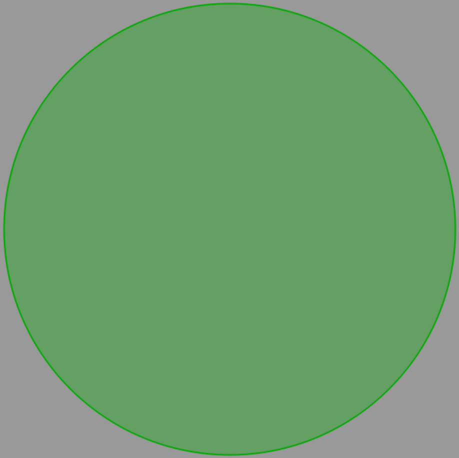
$$\mathfrak{s}_-(z) = \lambda z - 1 \qquad \mathfrak{s}_\mathsf{0}(z) = \lambda z \qquad \mathfrak{s}_+(z) = \lambda z + 1$$

In other words, they are the unique non-empty compact sets satisfying

$$\mathsf{A}_\lambda = \mathfrak{s}_-(\mathsf{A}_\lambda) \cup \mathfrak{s}_+(\mathsf{A}_\lambda) \quad \widetilde{\mathsf{A}}_\lambda = \mathfrak{s}_-(\widetilde{\mathsf{A}}_\lambda) \cup \mathfrak{s}_\mathsf{0}(\widetilde{\mathsf{A}}_\lambda) \cup \mathfrak{s}_+(\widetilde{\mathsf{A}}_\lambda)$$

Constructing A_λ for $\lambda \approx -0.366 + 0.52i$



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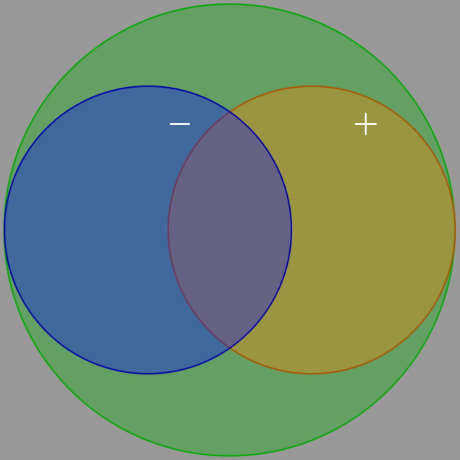

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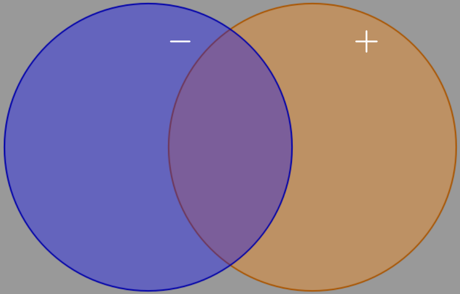

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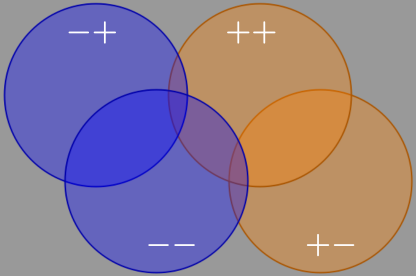

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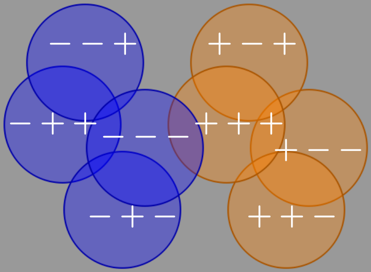

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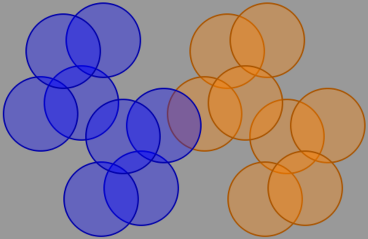

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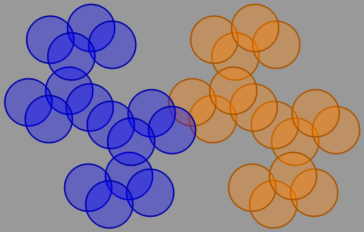

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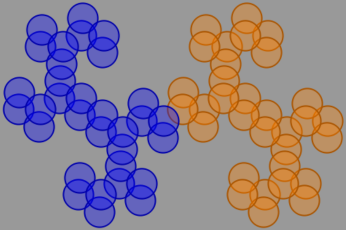

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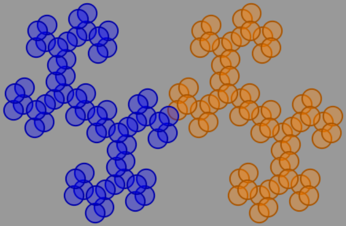

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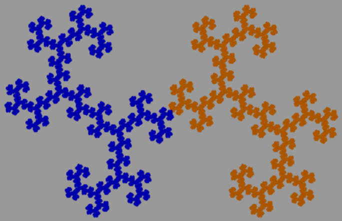

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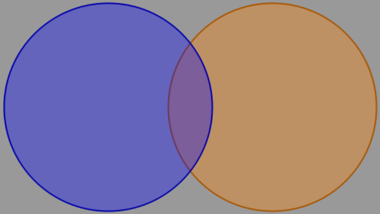

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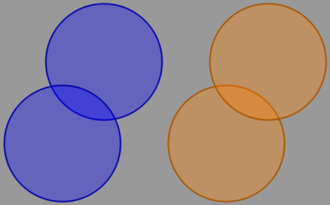

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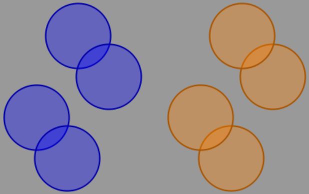

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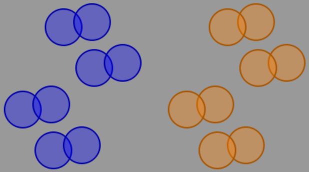

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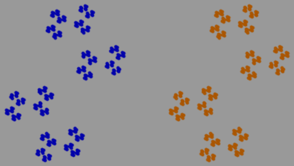

The Mandelbrot set for $\{\mathfrak{s}_-, \mathfrak{s}_+\}$

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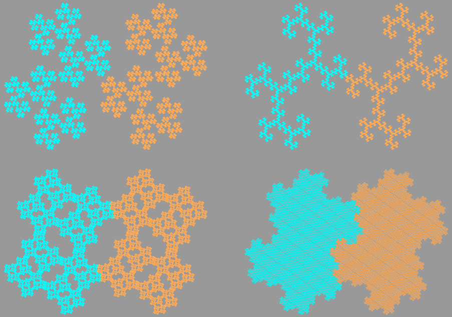

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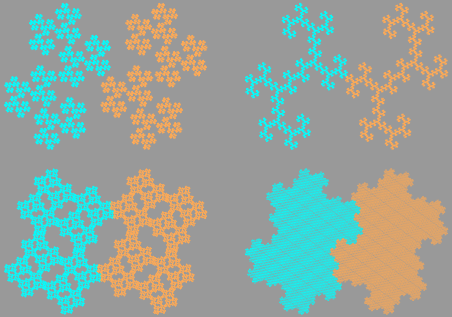

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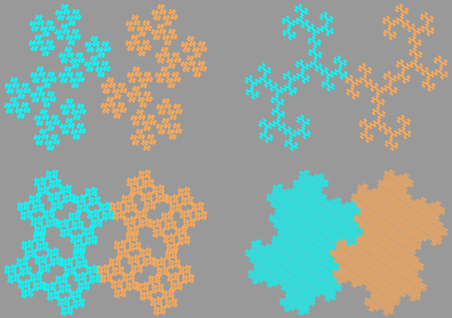

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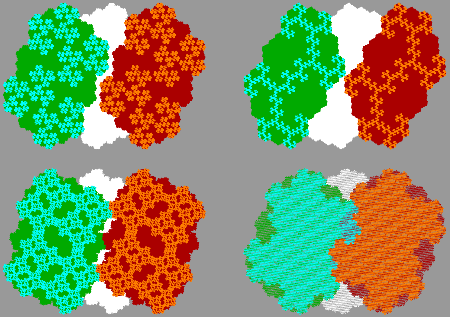

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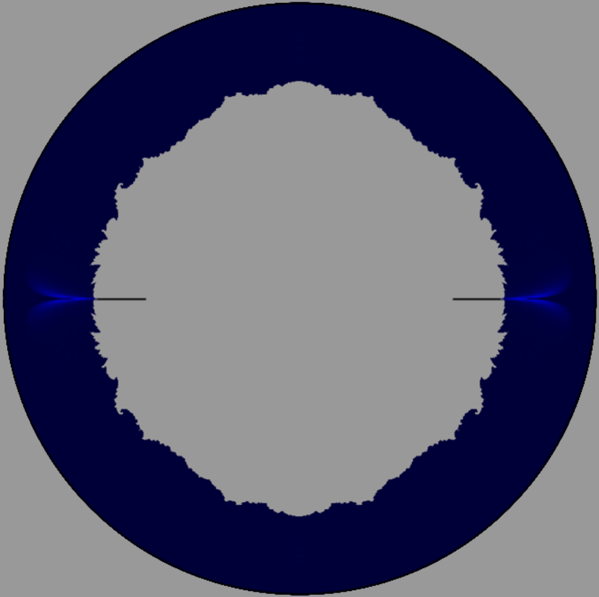
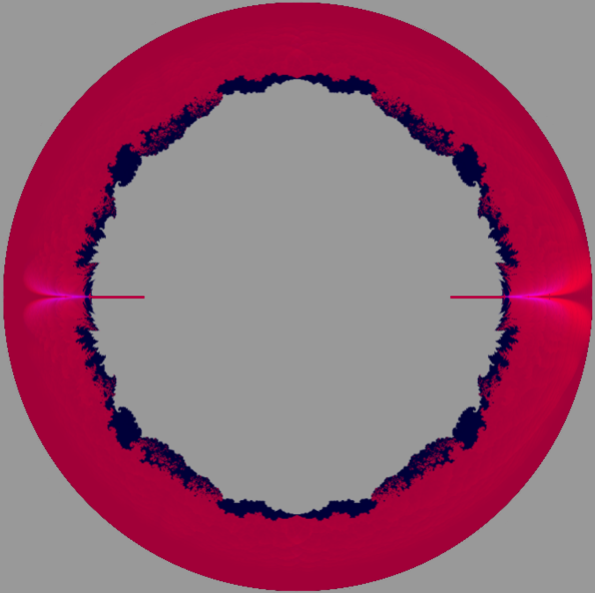
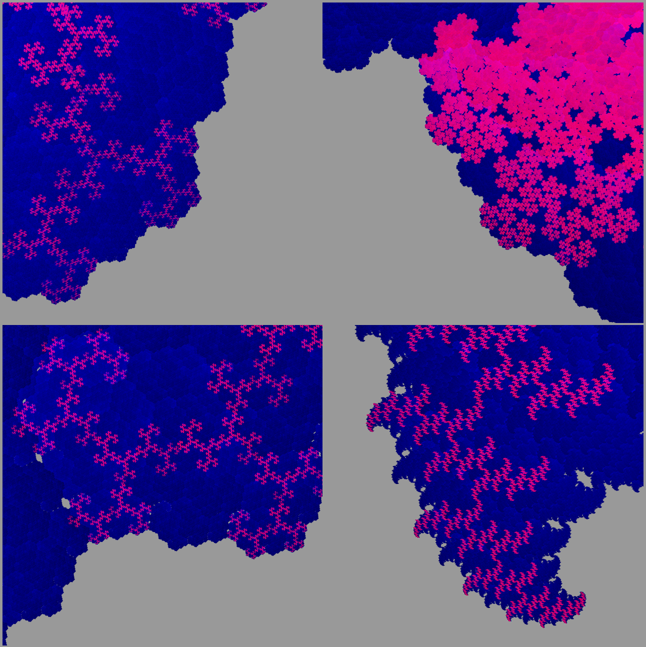
$\mathcal{M}$ **(in blue)**



$\mathcal{M}_0 \subset \mathcal{M}$



Interesting features of $\mathcal{M}$ and $\mathcal{M}_0$



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2009 Eroğlu-Rohde-Solomyak showed quasisymmetric conjugacy between quadratic dynamics and some IFS.

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Is there a way to discern holes in $\mathcal{M}$? Given a $\lambda \in \partial\mathcal{M}$ is it possible to construct a path in $\mathbb{D} \setminus \mathcal{M}$ that converges to it?

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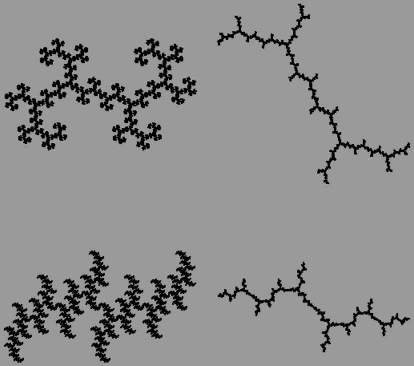
Conjecture: The parameters $\lambda \in (\partial\mathcal{M} \cap \partial\mathcal{M}_0) \setminus \mathbb{R}$ are those for which the dynamics on A_λ is quasisymmetric conjugate to the dynamics of $z^2 + c$ for Misiurewicz c on its Julia set.



Other definition of $\mathcal{M}$

A_λ is also the closure of the fixed points of finite composition of the maps $\mathfrak{s}_-$ and $\mathfrak{s}_+$. For example, let $\mathfrak{s}_{ab} = \mathfrak{s}_a \circ \mathfrak{s}_b$ then

$$z = \mathfrak{s}_{-+}(z) = -1 + \lambda + \lambda^2 z \iff z = \frac{-1}{1+\lambda} = -\sum_{j=0}^{\infty}(-1)^j \lambda^j$$

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A_λ is connected $\iff \mathfrak{s}_-(\mathsf{A}_\lambda) \cap \mathfrak{s}_+(\mathsf{A}_\lambda) \neq \varnothing \iff \exists \mathtt{a}, \mathtt{b} \in \Sigma$ with $a_0 = +, b_0 = -$ such that $\pi_\lambda(\mathtt{b}) = \pi_\lambda(\mathtt{a})$

$$\iff \sum_{j=0}^{\infty}(a_j - b_j)\lambda^j = 2\sum_{j=0}^{\infty} c_j \lambda^j = 0, \qquad c_j \in \{-1, 0, +1\}.$$

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Let $\mathcal{F}_\lambda = \left\{ f(z) = 1 + \sum_{j=1}^{\infty} c_j z^j \;\middle|\; c_j \in \{-1, 0, +1\}\,;\; f(\lambda) = 0 \right\}$
then

$$\mathcal{M} = \{\lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\} = \{\lambda \in \mathbb{D} \mid \; |\mathcal{F}_\lambda| \neq 0\}$$

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Intuitively $\lambda \in \partial\mathcal{M} \iff O_\lambda := \mathfrak{s}_-(\mathsf{A}_\lambda) \cap \mathfrak{s}_+(\mathsf{A}_\lambda)$ is "thin"
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Lemma (Solomyak, 2005)

If $|O_\lambda| \leq 2$ *then* $|\mathcal{F}_\lambda| = 1$*:*

(i.) $|O_\lambda| = 1 \iff f \in \mathcal{F}_\lambda$ *has no zero coefficients.*

(ii.) $|O_\lambda| = 2 \iff f \in \mathcal{F}_\lambda$ *has exactly one zero coefficient.*

Important Theorems

Remember $\widetilde{\mathsf{A}}_\lambda$ is the attractor of the IFS$\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Theorem (Solomyak, 2005)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with*

$$f(z) = q(z) + \frac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}, \qquad q(z) = \sum_{j=0}^{\ell} c_j z^j$$

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(ii.) $\mathcal{M}$ *about* λ *and* $\frac{\lambda^{\ell+1}}{f'(\lambda)} \widetilde{\mathsf{A}}_\lambda$ *about* $\frac{\lambda^{\ell+1}}{f'(\lambda)} \zeta$ *are asymptotically similar.*

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(iii.) $\mathcal{M}$ *is asymptotically $|\lambda^{-p}|$-self similar about λ;*

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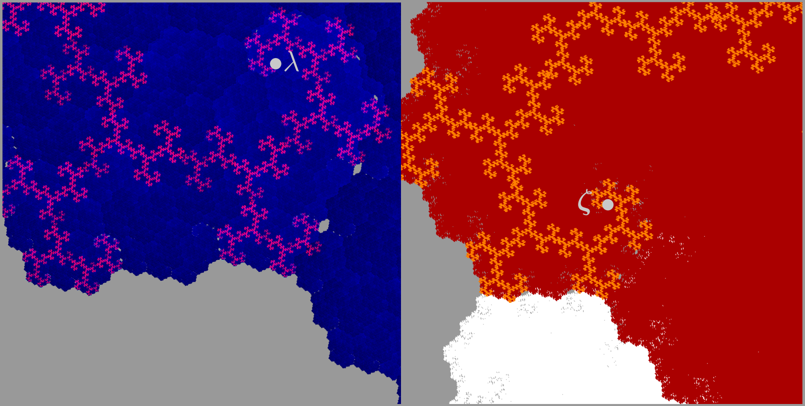
Remark. If f has no zero coefficients then we can also replace $\mathcal{M}$ with $\mathcal{M}_0$ and $\widetilde{\mathsf{A}}_\lambda$ with A_λ.

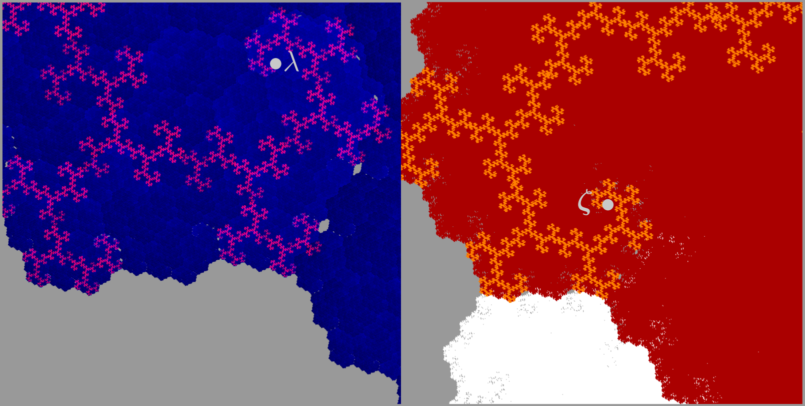
Important Theorems



Important Theorems

Theorem (Calegari-Koch-Walker, 2014)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ and $\mathcal{F}_\lambda = \{f\}$ with f rational as before. Then $\exists \delta > 0$ such that $\forall\ C \notin \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\widetilde{\mathsf{A}}_\lambda - \zeta\right)$ with $|C| < \delta$, the parameter $C\lambda^{pn} + \lambda$ is not in $\mathcal{M}$ for all sufficiently large n.



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Theorem (Pérez-S., 2018)

Suppose $\lambda \in \mathcal{M} \setminus \mathbb{R}$ *and* $\mathcal{F}_\lambda = \{f\}$ *with* f *rational as before. If* f *has finitely many zero coefficients and its Taylor polynomials satisfy certain conditions then* λ *is on the boundary of a hole of* $\mathcal{M}$*.*

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Minor adjustments to the conditions on the Taylor polynomials gives

Theorem (Pérez-S., 2018)

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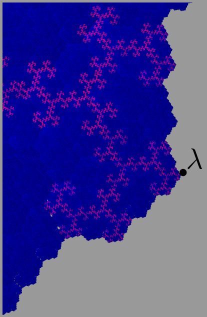
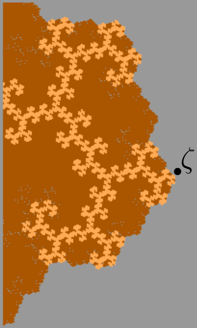
Idea of proof

Use the asymptotic similarity:





Idea of proof

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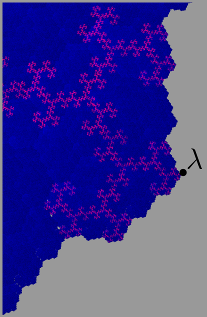

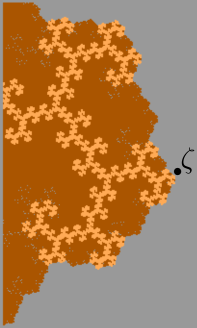

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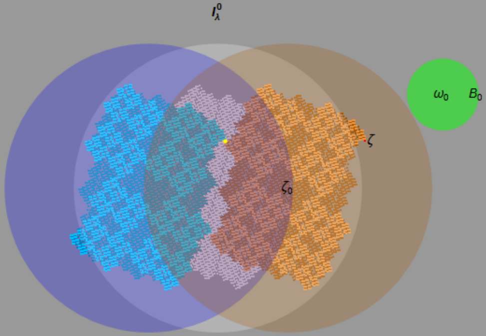

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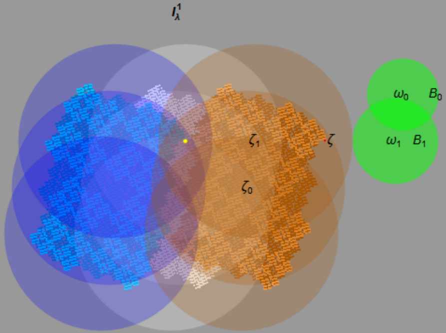

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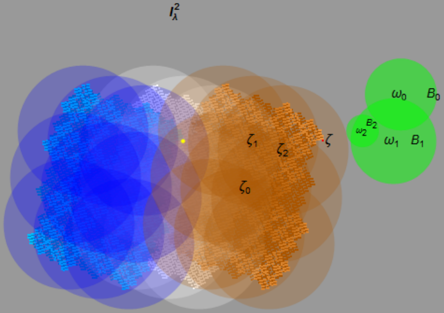

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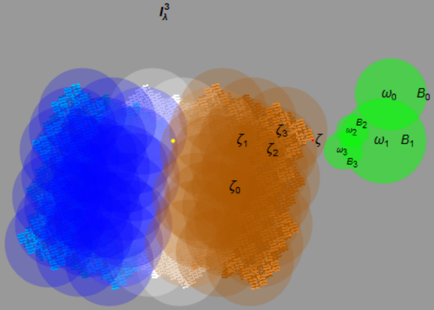

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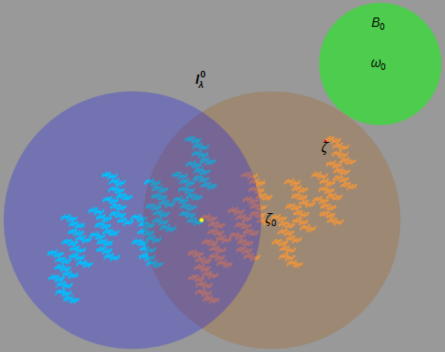

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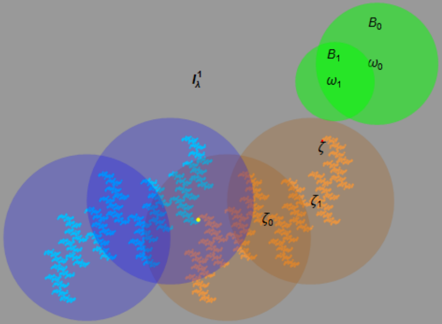

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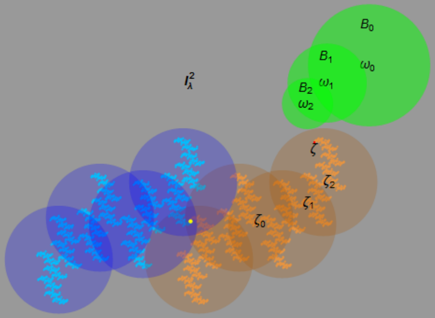

Notation

$\mathcal{F}_\lambda = \{f\}$ with $f(z) = \sum_{j=0}^{\ell} c_j z^j + \dfrac{c_{\ell+1} z^{\ell+1} + \ldots + c_{\ell+p} z^p}{1 - z^p}$ with $\mathsf{c} = c_0 \cdots c_\ell (c_{\ell+1} \cdots c_{\ell+p}) \in \Sigma$ then $O_\lambda = \{0\}$.

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So B_n is centered at

$$\omega_n = -(\zeta_n - \zeta) + \zeta = -\zeta_n + 2\zeta = -\frac{1}{\lambda^{\ell+1}} f_{\ell+1+n}(\lambda) + \zeta$$

with a radius

$$r_n = |\zeta_n - \omega_n| - \left|\lambda^{n+1}\right| R = \frac{2}{|\lambda^{\ell+1}|} \left|f_{\ell+1+n}(\lambda)\right| - \left|\lambda^{n+1}\right| R$$

where $R = (1 - |\lambda|)^{-1}$.

Conditions on Taylor polynomials of f

For all $n \geq 0$
B_n exists if and only if $r_n > 0$

$$|f_{\ell+1+n}(\lambda)| > \frac{1}{2}\frac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}.$$

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$B_n \cap \mathsf{I}_\lambda^n = \varnothing$ if and only if $\left|\omega_n - \nu_{\mathtt{a}|n}\right| > r_n + \left|\lambda^{n+1}\right| R$ for any $\mathtt{a} \in \Sigma$ and $\nu_{\mathtt{a}|n} = \sum_{j \leq n} a_j \lambda^j$

$$2\,|f_{\ell+1+n}(\lambda)| < \left|f_\ell(\lambda) + f_{\ell+1+n}(\lambda) + \lambda^{\ell+1}\nu_{\mathtt{a}|n}\right|$$

Self-similarity of I_λ^n at 0

Since

$$f(\lambda) = 0 \implies \sum_{j=\ell+1}^{\ell+p} c_j \lambda^j = (\lambda^p - 1) \sum_{j=0}^{\ell} c_j \lambda^j$$

then for any $0 \le n \le p$

$$\frac{1}{\lambda^p} \left(\mathsf{I}_\lambda^{\overline{\mathsf{c}}|\ell+p+n} \cup \mathsf{I}_\lambda^{\mathsf{c}|\ell+p+n} \right) = \mathsf{I}_\lambda^{\overline{\mathsf{c}}|\ell+n} \cup \mathsf{I}_\lambda^{\mathsf{c}|\ell+n}$$

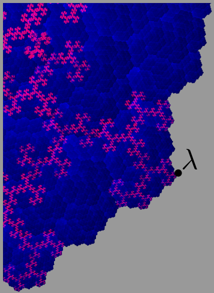
where $\overline{\mathsf{c}}$ is the opposite of c.

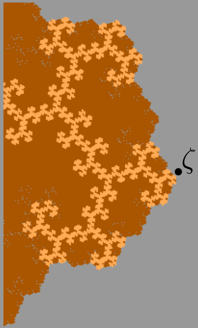
We need to check finitely many inequalities to certify $\lambda \in \partial \mathcal{M}_0$!

$+(++-)$

Proposition

Let $\lambda \approx -0.366 + 0.520i$ be the root of $f(z) = 1 + \frac{z+z^2-z^3}{1-z^3}$ then λ is on the boundary of a hole of both $\mathcal{M}_0$ and $\mathcal{M}$.





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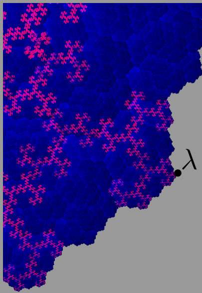

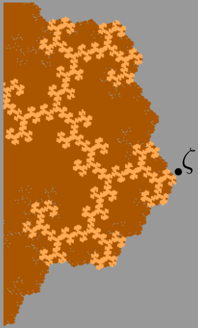

To show the inequalities are satisfied use estimates on $|\lambda|$ and $\arg(\lambda)$, and the fact that $1 + \lambda + \lambda^2 = 2\lambda^3$.

$+(++-)^{\infty}$

Proposition (Pérez-S., 2019)

Let $\lambda \approx -0.366 + 0.520i$ *be the root of* $f(z) = 1 + \frac{z+z^2-z^3}{1-z^3}$ *then* λ *is on the boundary of the main connected component of* $\mathbb{D} \setminus \mathcal{M}$.

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$\implies (1-\lambda^{3k})f(\lambda) = f_{3k-1}(\lambda) - 2\lambda^{3k} = 0$ for every $k \in \mathbb{N}$.

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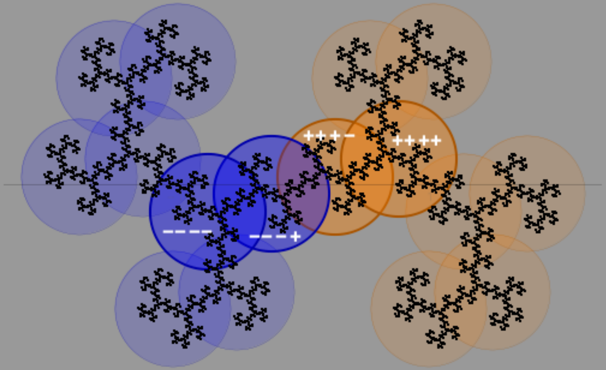
$\implies (1-\lambda^{3k})f(\lambda) = f_{3k-1}(\lambda) - 2\lambda^{3k} = 0$ for every $k \in \mathbb{N}$.

Geometrically, it means that the discs with centers

$$\pm f_{3k-1}(\lambda) \pm \lambda^{3k}$$

are aligned.

Geometry of A_λ for λ associated to $+(++-)^\infty$



$k = 1$

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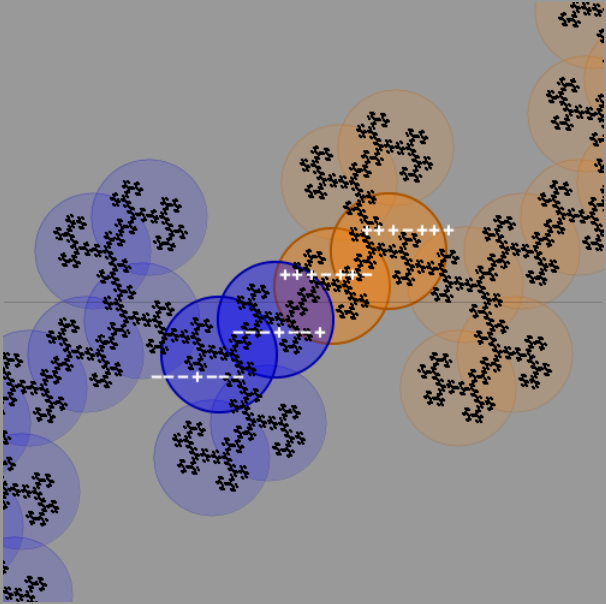
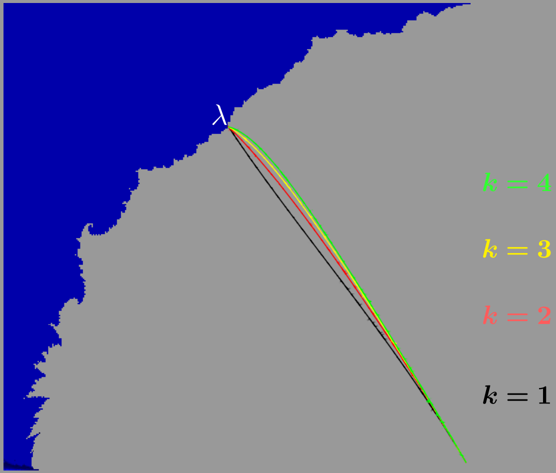


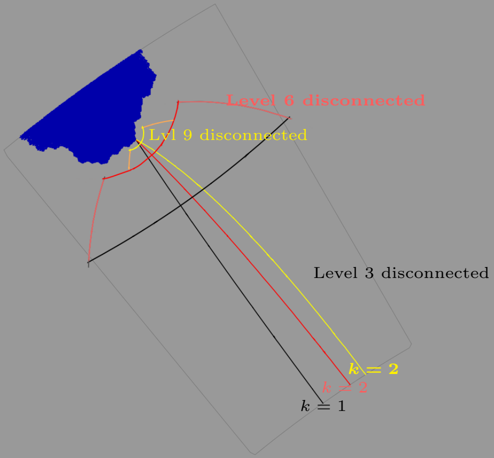
$k = 2$

Level Curves

λ is on the level curve $\mathrm{Im}\left(\dfrac{f_{3k-1}(z)}{z^{3k}}\right) = 0$ for every $k \in \mathbb{N}$.

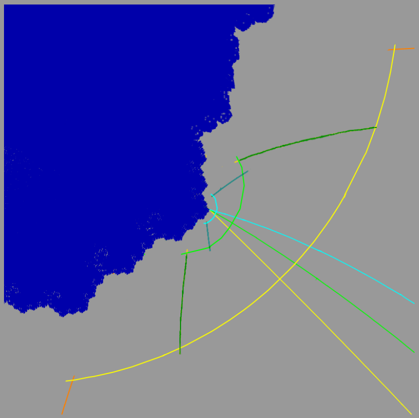


Block Tower

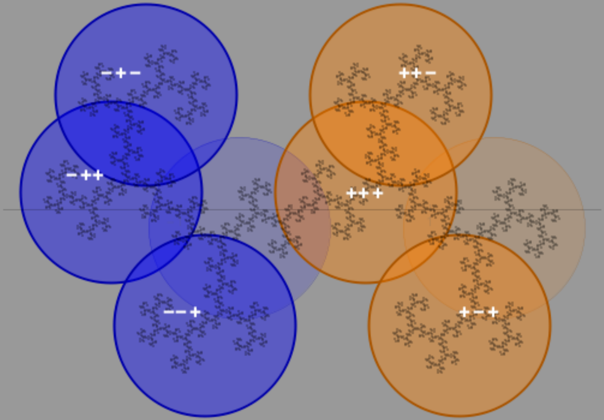


Level 6 disconnected
Lvl 9 disconnected
Level 3 disconnected
$k = 2$
$k = 2$
$k = 1$

Block Tower

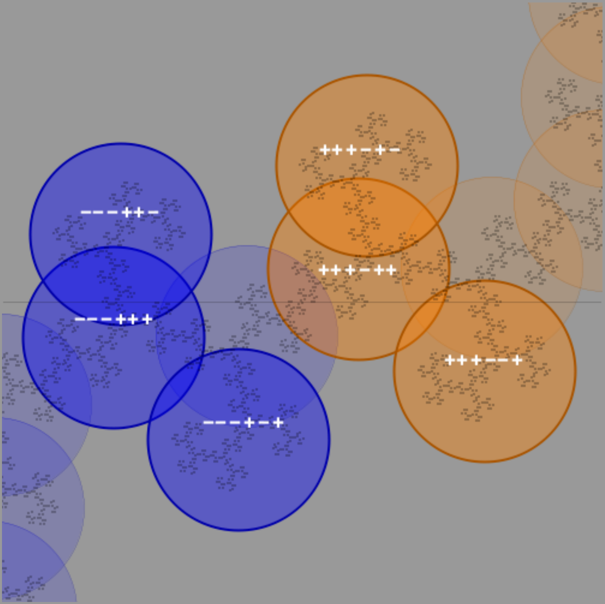


Shape Conditions



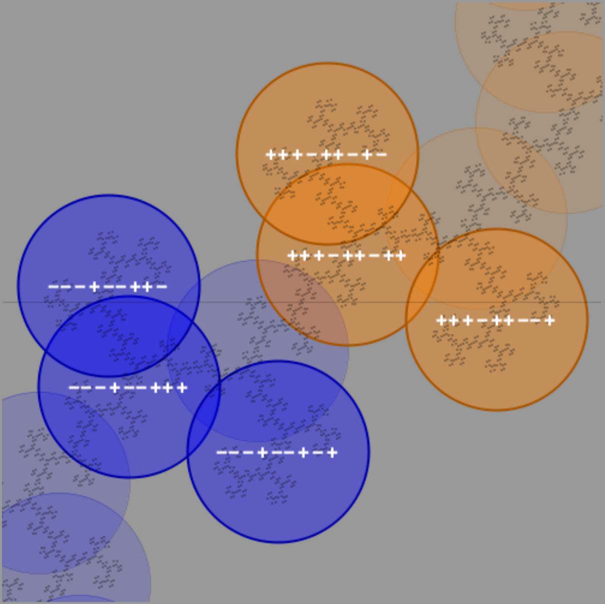
Level 2

Shape Condtions



Level 5

Shape Condtions



Level 8

Thank You!