Rational Maps
Julia Sets From Accessible Mandelbrot Sets Are Not Homeomorphic

Elizabeth L. Fitzgibbon & Stefano Silvestri
Department of Mathematics and Statistics; Boston University
Department of Mathematical Sciences; Indiana University Purdue University Indianapolis

Contact Information:
Email: `lizfitz@bu.edu`
Email: `ssilvest@iupui.edu`

Introduction

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}.$$

- Super-attracting critical point at ∞. Critical point at 0 maps onto ∞.
- 4 additional critical points, $c_\lambda = \lambda^{1/4}$.
- Two critical values, $\nu_\lambda = F_\lambda(c_\lambda) = \pm 2\sqrt{\lambda}$.
- $F_\lambda(\nu_\lambda) = F_\lambda^2(c_\lambda) = 4\lambda + \frac{1}{4} \Longrightarrow$ Only one free critical orbit.
- $\overline{F_\lambda(z)} = F_{\bar\lambda}(\bar z) \Longrightarrow$ Parameter plane symmetric under complex conjugation.
- $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$, where $\omega = \exp\frac{\pi i}{2} \Longrightarrow$ Dynamical plane has fourfold symmetry.

Parameter Plane

Infinitely many copies of the Mandelbrot set are located around the boundary of the connectedness locus. There exists a homeomorphism, via the Riemann Mapping Theorem, taking straight rays extending out from the unit disk to rays in the exterior region of the parameter plane. Each ray corresponds to an angle between 0 and 1. A ray whose angle is period-m under doubling lands on the cusp of the *accessible* Mandelbrot set whose main cardioid contains parameters whose maps have an attracting cycle of period-m.

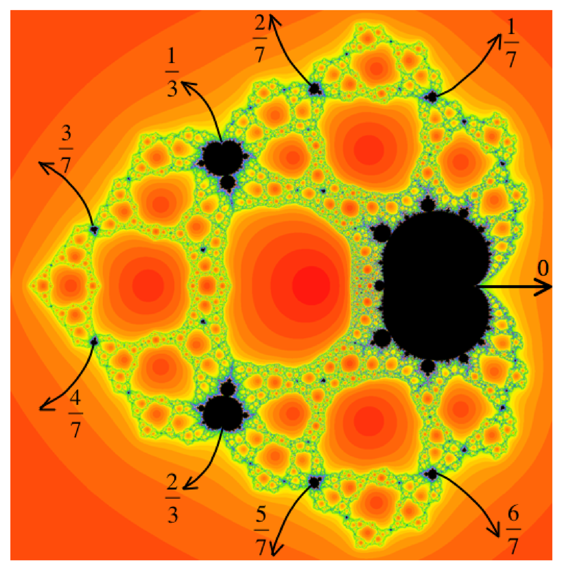

Figure 1: Parameter plane for $F_\lambda(z)$ with several external rays.

Objective

When do maps taken from accessible Mandelbrot sets in the parameter plane exhibit conjugate dynamics?

If F_λ and F_μ have conjugate dynamics on their Julia sets, we should be able to define a homeomorphism between their Julia sets, $J(F_\lambda)$ and $J(F_\mu)$. If B_λ denotes the immediate basin of attraction of ∞ under F_λ and T_λ denotes the preimage of B_λ containing the origin, then this homeomorphism should take ∂B_λ to ∂B_μ and ∂T_λ to ∂T_μ.

Theorem

Given parameters λ and μ from the centers of the main cardioids of distinct accessible Mandelbrot sets, there exists a homeomorphism between the Julia sets of F_λ and F_μ which takes ∂B_λ to ∂B_μ and ∂T_λ to ∂T_μ if and only if $\mu = \bar\lambda$.

Constructing a Julia Set Model

- Choose λ at the center of the main cardioid of the $\frac{2}{7}$-Mandelbrot set.
- F_λ is conjugate to $z \mapsto z^2$ on B_λ. ∂B_λ is homeomorphic to the unit circle and each point is assigned an angle between 0 and 1.
- The $\frac{16}{28}$-dynamical ray lands on the periodic critical point, since its second iterate has the same angle as the parameter ray.
- Critical points at quarter-turn intervals are the *corners* of T_λ.
- The Julia set is divided into 4 symmetry sectors. Each contains T_i^{-1}, a preimage of T_λ, dividing the sector into 4 subsectors.

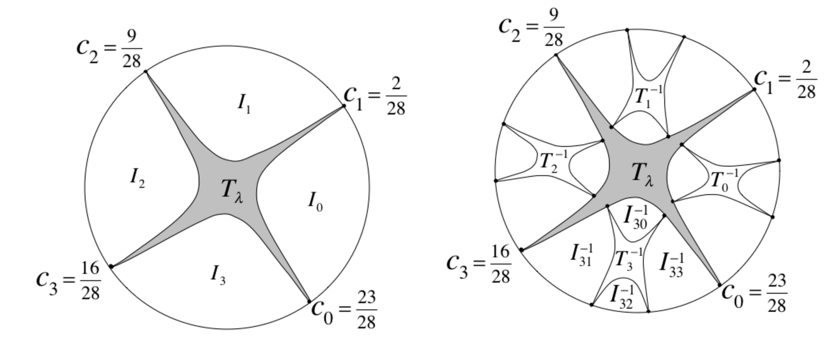

Figure 2: Locate 4 critical points and construct T_λ. Each symmetry sector contains a preimage of T_λ.

- The orbit of $\frac{16}{28}$ under doubling is $\frac{16}{28} \mapsto \frac{4}{28} \mapsto \frac{8}{28} \mapsto \frac{16}{28}$, thus the itinerary of the periodic critical point is $S(c_3) = \overline{311}$.
- c_3 lies on the boundary of I_{31}^{-1}, the preimage of I_1 within I_3.
- T_{31}^{-2} lies in I_{31}^{-1} with two corners on ∂T_3^{-1}. c_3 lies on ∂I_{311}^{-2}, so both free corners of T_{31}^{-2} lie on ∂B_λ.
- Two corners of T_{311}^{-3} lie on ∂T_{31}^{-2}. Another corner lies at c_3. The fourth corner lies on ∂T_λ, because its 3^{rd} iterate is c_2.

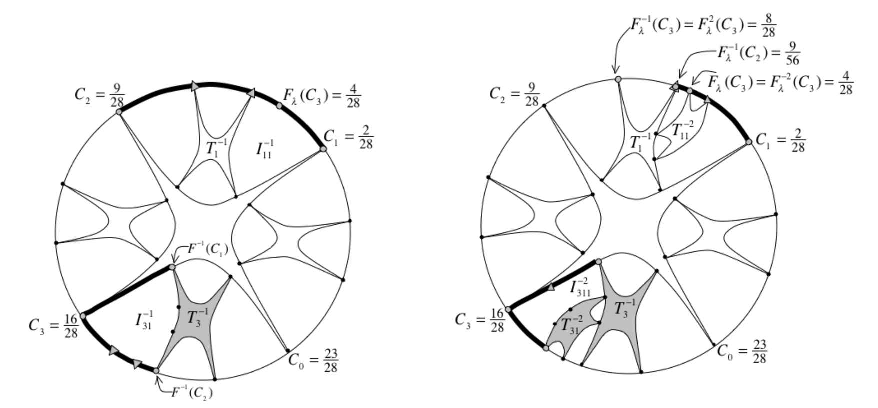

Figure 3: Locate the corners of T_{31}^{-2} and T_{311}^{-3} relative to the position of c_3.

Results

Constructing models for each of the period-3 cases in the upper half-plane, we deduce that the Julia Sets are not homeomorphic.

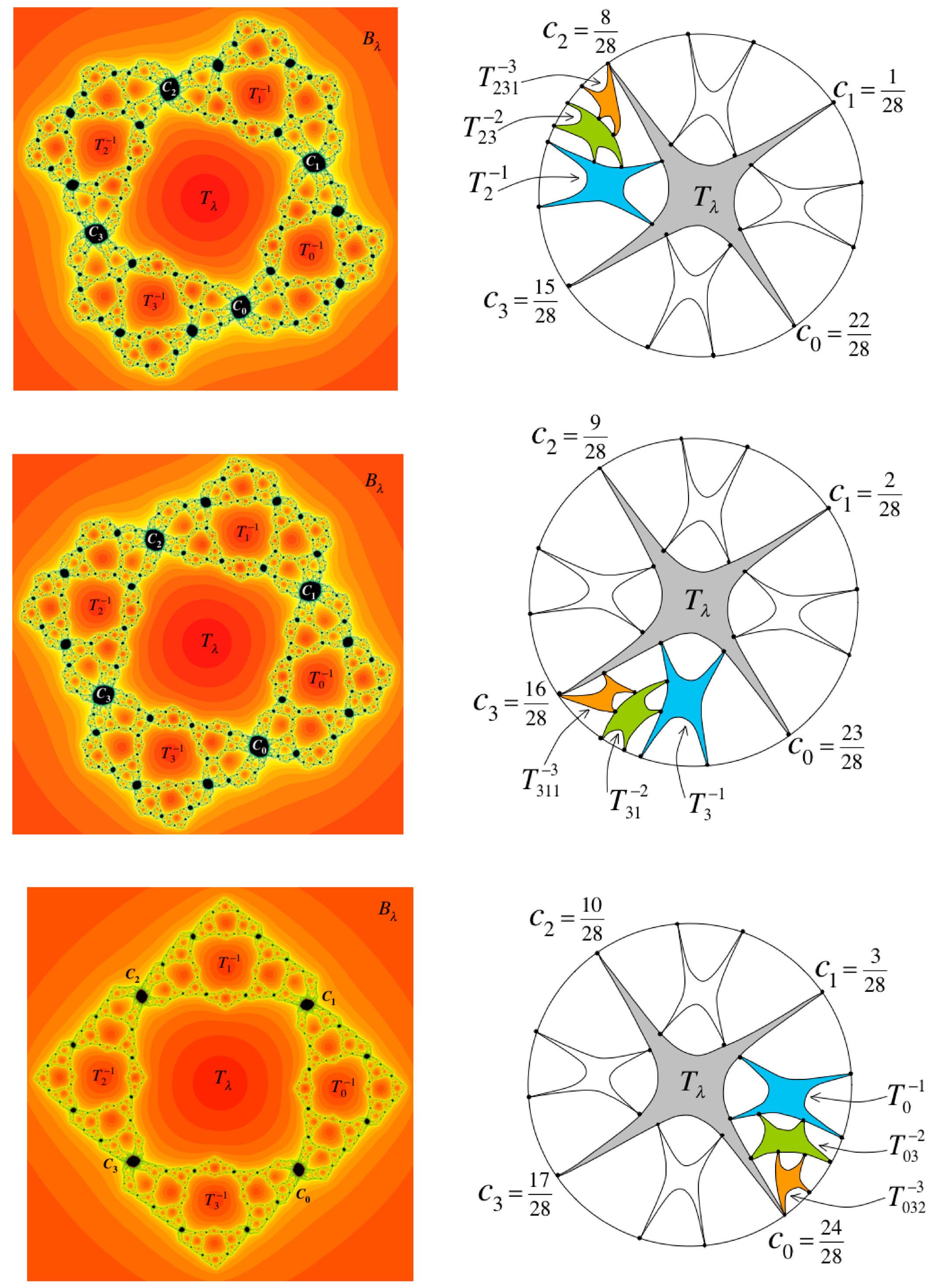

Figure 4: Julia sets and corresponding models for the centers of the $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$-Mandelbrot sets, respectively. Comparison of the models shows that these sets are not homeomorphic.

Conclusions

The rational parameter ray landing on each accessible Mandelbrot set corresponds to a unique itinerary for the periodic critical point of a map taken from center of the main cardioid of this set. The itinerary determines the model for the Julia set of the map. Thus, each ray defines a unique model. Models for distinct rays in the upper halfplane are not homeomorphic. Models for λ and $\bar\lambda$ are homeomorphic under $z \mapsto \bar z$.

Forthcoming Research

In ongoing work, we consider $F_\lambda(z) = z^n + \lambda/z^n$ with $n > 2$. We conjecture that the same result will hold. We have found that, when n is odd, the construction of the models can be more challenging because there are always two periodic critical points. Sometimes these points lie on the same orbit of period $2m$ and other times each periodic critical point has a unique orbit of period m. Construction of these models is made easier by considering a map, $S_\lambda = (z+\lambda)^{2n}/z^n$, which is semi-conjugate to F_λ, but has a single critical point.

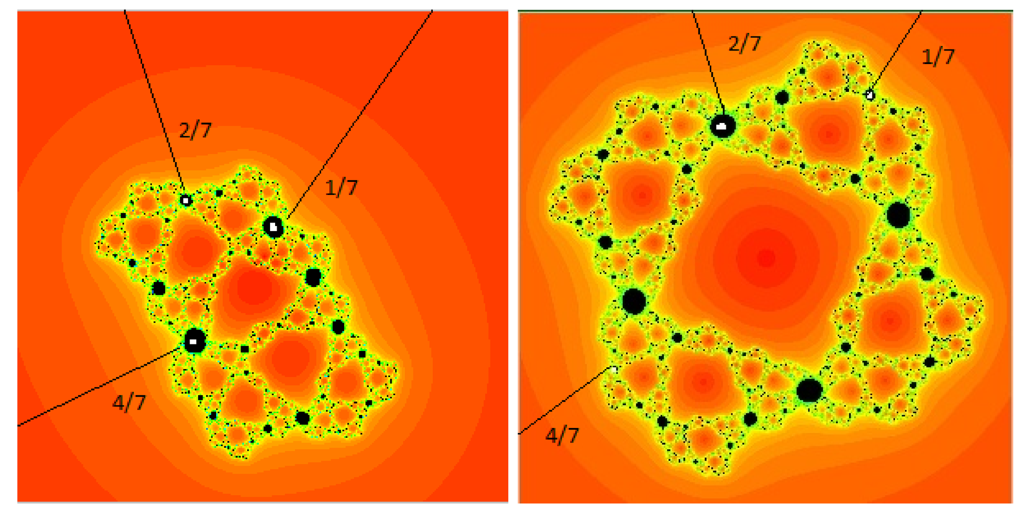

Figure 5: On the left is $J(S_\lambda)$ and on the right is $J(F_\lambda)$, for $n = 2$ and λ in the $\frac{1}{7}$-Mandelbrot set

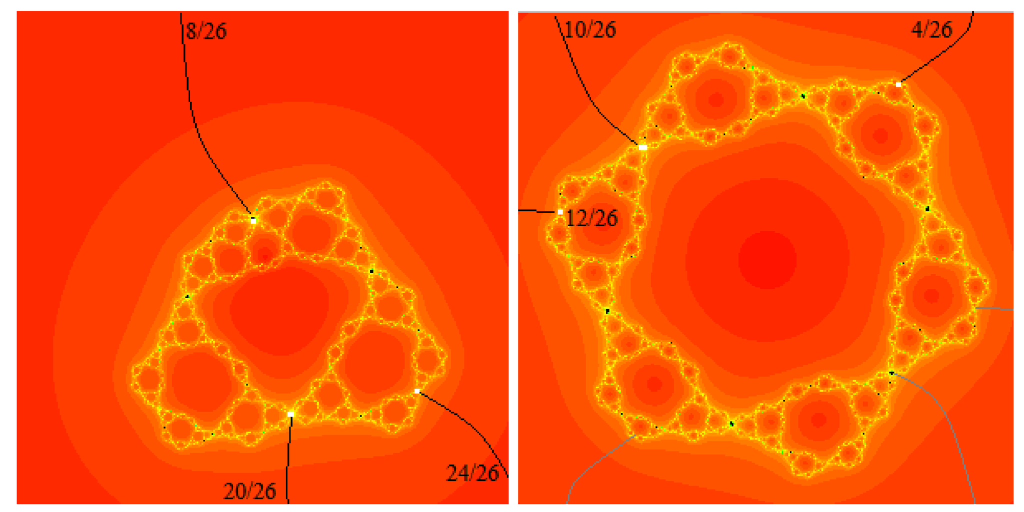

Figure 6: On the left is $J(S_\lambda)$ and on the right is $J(F_\lambda)$, for $n = 3$ and λ in the $\frac{2}{26}$-Mandelbrot set

References

[1] P. Blanchard, F. Cilingir, D. Cuzzocreo, R. L. Devaney, D. Look, E. D. Russell. Checkerboard Julia sets for rational maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **23**, 2013

[2] R. L. Devaney, D. Look, D. Uminsky. The Escape Trichotomy for singularly perturbed rational maps. *Indiana Univ. Math. J.* **54**, 1621-1634, 2005

[3] J. Milnor. *Dynamics In One Complex Variable*. Princeton University Press (2006).

[4] E. Fitzgibbon, S. Silvestri. Rational Maps: Julia sets of accessible Mandelbrot sets are not homeomorphic. submitted to Topology Proceedings, 2013

[5] S.Silvestri. Non homeomorphic Julia sets of singularly perturbed rational maps. submitted to Pi Mu Epsilon, 2013