ACCESSIBILITY OF THE BOUNDARY OF THE THURSTON SET

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Abstract. Consider two objects associated to the Iterated Function System (IFS) $\{1+\lambda z, -1+\lambda z\}$: the locus $\mathcal{M}$ of parameters $\lambda \in \mathbb{D}\setminus\{0\}$ for which the corresponding attractor is connected; and the locus $\mathcal{M}_0$ of parameters for which the related attractor contains 0. The set $\mathcal{M}$ can also be characterized as the locus of parameters for which the attractor of the IFS $\{1+\lambda z, \lambda z, -1+\lambda z\}$ contains λ^{-1}. Exploiting the asymptotic similarity of $\mathcal{M}$ and $\mathcal{M}_0$ with the respective associated attractors, we give sufficient conditions on $\lambda \in \partial\mathcal{M}$ or $\partial\mathcal{M}_0$ to guarantee it is path accessible from the complement $\mathbb{D}\setminus\mathcal{M}$.

1. Introduction

For any λ in the punctured disk $\mathbb{D}^* := \mathbb{D}\setminus\{0\}$ the maps

$$\mathfrak{s}_-(z) := -1+\lambda z, \qquad \mathfrak{s}_0(z) := \lambda z, \qquad \mathfrak{s}_+(z) := 1+\lambda z$$

are contraction similarities in all of $\mathbb{C}$. We associate to λ the compact sets

$$\mathsf{A}_\lambda := \left\{ \sum_{n=0}^{\infty} a_n\lambda^n \;\middle|\; a_n \in \{-1,+1\} \right\} \text{ and } \widetilde{\mathsf{A}}_\lambda := \left\{ \sum_{n=0}^{\infty} a_n\lambda^n \;\middle|\; a_n \in \{-1,0,+1\} \right\}.$$

These sets are the attractors of the iterated function systems or IFS $\{\mathfrak{s}_-, \mathfrak{s}_+\}$ and $\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$, respectively; that is, the unique non-empty compact sets satisfying

$$\mathsf{A}_\lambda = \mathfrak{s}_-(\mathsf{A}_\lambda)\cup\mathfrak{s}_+(\mathsf{A}_\lambda) \text{ and } \widetilde{\mathsf{A}}_\lambda = \mathfrak{s}_-(\widetilde{\mathsf{A}}_\lambda)\cup\mathfrak{s}_0(\widetilde{\mathsf{A}}_\lambda)\cup\mathfrak{s}_+(\widetilde{\mathsf{A}}_\lambda).$$

Both attractors are symmetric about 0, and clearly $\mathsf{A}_\lambda \subseteq \widetilde{\mathsf{A}}_\lambda$. It is advantageous to study them together because, in contrast to what happens with the Mandelbrot set, the parameter regions

$$\mathcal{M} := \{\lambda\in\mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected}\} \text{ and } \mathcal{M}_0 := \{\lambda\in\mathbb{D} \mid 0\in\mathsf{A}_\lambda\},$$

associated to the 2-map IFS $\{\mathfrak{s}_-, \mathfrak{s}_+\}$, do not coincide (see Figure 1). Rather, $\mathcal{M}$ equals with the set $\left\{\lambda\in\mathbb{D} \mid 0\in\mathfrak{s}_-(\widetilde{\mathsf{A}}_\lambda)\cap\mathfrak{s}_0(\widetilde{\mathsf{A}}_\lambda)\cap\mathfrak{s}_+(\widetilde{\mathsf{A}}_\lambda)\right\}$ associated to the 3-map IFS $\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$.

Interest in these IFS families spiked recently, after Tiozzo [Tio13] (inspired by a conjecture of Thurston [Thu14]) proved that $\mathcal{M}_0$ equals the closure of the set of Galois conjugates of entropies of superattracting real quadratic polynomials. Both $\mathcal{M}$ and $\mathcal{M}_0$ were first introduced by Barnsley and Harrington [BH85] in the mid 1980s. In 1988–92, Bousch [Bou88, Bou92] proved $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected. In 2002, Bandt [Ban02] proved the existence of a hole in $\mathcal{M}$, and conjectured that there are in fact infinitely many holes. In 2014 Calegari, Koch, and Walker [CKW17] gave a positive answer to the conjecture, for both $\mathcal{M}$ and $\mathcal{M}_0$, and in fact constructed infinite sequences of holes accumulating at

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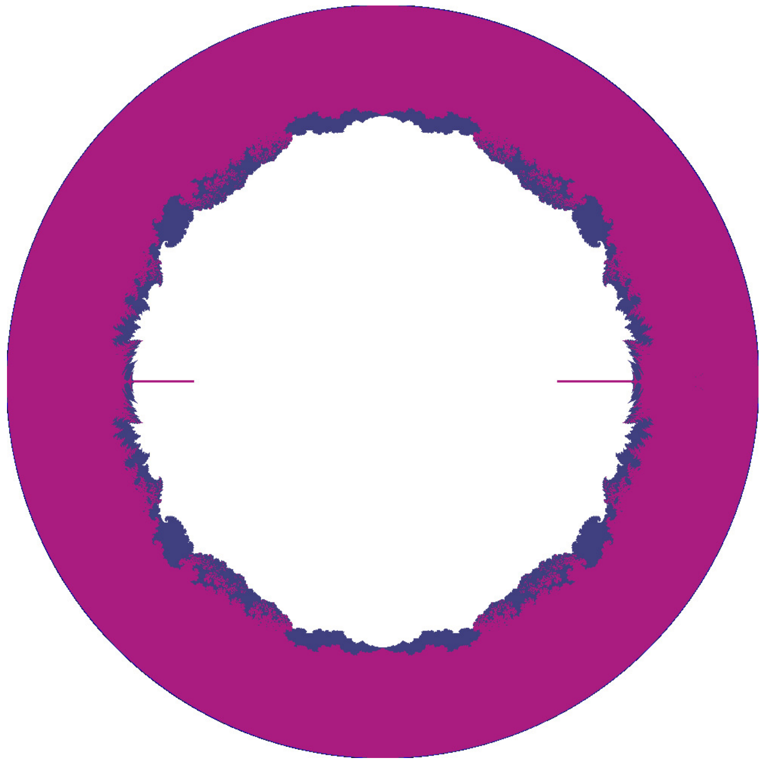

FIGURE 1. The connectivity locus $\mathcal{M}$ and its subset $\mathcal{M}_0$, in darker and lighter grays respectively (blue and fuchsia in the electronic version). The two spikes on the real line start at $\pm 1/2$.

certain parameters in $\partial\mathcal{M}$ (see Figure 2). These are buried points of $\partial\mathcal{M}$, i.e. not path accessible from the complement $\mathbb{D} \setminus \mathcal{M}$.

The question of classifying these connected components of $\mathbb{D} \setminus \mathcal{M}$ is still open; however our results are a step in that direction. Let $\mathcal{P}$ denote the set of all normalized power series with coefficients in $\{-1, 0, +1\}$, i.e.

$$\mathcal{P} := \left\{ f(z) = \sum_{j=0}^{\infty} c_j z^j \;\middle|\; c_j \in \{-1, 0, +1\},\ c_0 = 1 \right\}$$

then

Theorem 1. *Suppose f is the unique power series in $\mathcal{P}$ that vanishes at $\lambda \in \mathcal{M} \setminus \mathbb{R}$ with $|\lambda| \leq 2^{-1/2}$. If f has finitely many zero coefficients and its Taylor polynomials satisfy certain conditions then λ is on the boundary of a hole of $\mathcal{M}$.*

and, with minor adjustments to the conditions on the Taylor polynomials,

Theorem 2. *Suppose f is the unique power series in $\mathcal{P}$ that vanishes at $\lambda \in \mathcal{M} \setminus \mathbb{R}$ with $|\lambda| \leq 2^{-1/2}$. If f has no zero coefficients and its Taylor polynomials satisfy certain conditions then λ is on the boundary of a hole of $\mathcal{M}_0$.*

Further motivation to study points of $\partial\mathcal{M}_0 \cap \partial\mathcal{M}$ is provided by the results in [ERS10, HP12]. It is possible to derive a quadratic like map g from the maps $\mathfrak{s}_-$ and $\mathfrak{s}_+$, so that the dynamics of g on the limit set A_λ is quasisymmetrically conjugate to the one of $z^2 + c$ on the Julia set for Misiurewicz parameter c. In fact,

Conjecture 1. *For all parameters $\lambda \in (\partial\mathcal{M} \cap \partial\mathcal{M}_0) \setminus \mathbb{R}$ the dynamics of g on the limit set A_λ is quasisymmetrically conjugate to the one of $z^2 + c$ on the Julia set for Misiurewicz parameter c.*

This article is organized as follows. In Section 2 we set the notation, while in Section 3 we describe the sets $\mathcal{M}$ and $\mathcal{M}_0$ as the closure of the roots of power series with certain prescribed coefficients. we overview some of the known results about $\mathcal{M}$ and $\mathcal{M}_0$ in Section 4 and explain how our results fit in. The proofs are provided in Section 5. Finally, Section 6 is devoted to salient examples of parameters λ to which our theorems apply.

2. Notation

Let Σ^n be the set of all words $\mathtt{w} = a_0 \cdots a_{n-1}$ of length n from the alphabet $\{-, +\}$. Define $\mathfrak{s}_\mathtt{w} = \mathfrak{s}_{a_0} \circ \ldots \circ \mathfrak{s}_{a_{n-1}}$ so that $\mathsf{A}_\lambda^\mathtt{w} = \mathfrak{s}_\mathtt{w}(\mathsf{A}_\lambda)$ and

$$\mathsf{A}_\lambda = \bigcup_{\mathtt{w} \in \Sigma^n} \mathsf{A}_\lambda^\mathtt{w}.$$

The natural projection from the space Σ^∞ of infinite words, $\mathtt{w} = a_0 a_1 \cdots$ in $\{-, +\}$, onto the attractor, A_λ is given by

$$\pi_\lambda : \Sigma^\infty \to \mathsf{A}_\lambda \qquad \pi_\lambda(\mathtt{w}) = \sum_{n=0}^{\infty} a_n \lambda^n.$$

The word $\mathtt{w} \in \Sigma^\infty$ is called the *itinerary* of $\pi_\lambda(\mathtt{w})$ in A_λ. Remember that if A_λ is connected, then there could be multiple itineraries for a given point in A_λ.

Nevertheless, elements $\omega \in \mathsf{A}_\lambda$ that are in $\mathsf{A}_\lambda^{a_0 a_1 \cdots a_k}$ can be written as

$$\omega = a_0 + a_1\lambda + \ldots + a_k\lambda^k + \sum_{n=k+1}^{\infty} a_n \lambda^n, \qquad a_n \in \{-1, +1\}.$$

We will use $\mathtt{w}|k$ to denote the finite word $a_0 a_1 \cdots a_k$ coming from the truncation of the infinite word $\mathtt{w} = a_0 a_1 \cdots$. The notation $|\mathtt{w}|$ will indicate the length of the word $\mathtt{w}$.

Let $D_r(z)$ denote the closed disk centered at z with radius r and, when $z = 0$, we will abbreviate $D_r = D_r(0)$. The following lemma is straightforward.

Lemma 2.1. *Let $R \geq (1 - |\lambda|)^{-1}$, then $\mathsf{A}_\lambda \subset D_R$ and $\widetilde{\mathsf{A}}_\lambda \subset D_R$.*

From here on, we let $R := (1 - |\lambda|)^{-1}$. Let $\mathsf{I}^{-1} = \widetilde{\mathsf{I}}^{-1}$ be the disk D_R, and consider the recursive constructions $\mathsf{I}^n = \mathfrak{s}_-(\mathsf{I}^{n-1}) \cup \mathfrak{s}_+(\mathsf{I}^{n-1})$, and $\widetilde{\mathsf{I}}^n = \mathfrak{s}_-(\widetilde{\mathsf{I}}^{n-1}) \cup \mathfrak{s}_\mathtt{0}(\widetilde{\mathsf{I}}^{n-1}) \cup \mathfrak{s}_+(\widetilde{\mathsf{I}}^{n-1})$ for $n \in \mathbb{N}$. From Lemma 2.1, it is clear that

$$\mathsf{A}_\lambda = \bigcap_{n=0}^{\infty} \mathsf{I}^n \quad \text{and} \quad \widetilde{\mathsf{A}}_\lambda = \bigcap_{n=0}^{\infty} \widetilde{\mathsf{I}}^n.$$

As before, for any finite word $\mathtt{w} = a_0 a_1 \cdots a_k$ in Σ^{k+1} we can identify the disks in I^{k+1} as $\mathsf{D}^\mathtt{w} := \mathfrak{s}_\mathtt{w}(\mathsf{I}^{-1})$. Each of these disks will then be centered at $\mathfrak{s}_\mathtt{w}(0) = a_0 + a_1\lambda + \cdots + a_k\lambda^k$, and have a radius of $|\lambda|^{k+1} R$. If $\mathtt{w} \in \Sigma^\infty$ is an infinite word, then $\mathfrak{s}_\mathtt{w}(0) \in \mathsf{A}_\lambda^{\mathtt{w}|k} \subset \mathsf{D}^{\mathtt{w}|k}$ for all $k \geq 0$. Analogously, if $\mathtt{w} \in \widetilde{\Sigma}^\infty$, then $\mathfrak{s}_\mathtt{w}(0) \in \widetilde{\mathsf{A}}_\lambda^{\mathtt{w}|k} \subset \widetilde{\mathsf{D}}^{\mathtt{w}|k}$ for all $k \geq 0$.

From now on, we will refer to I^k and $\widetilde{\mathsf{I}}^k$ as the *instar*[1] at level k of the IFS $\{\mathfrak{s}_-, \mathfrak{s}_+\}$ and $\{\mathfrak{s}_-, \mathfrak{s}_0, \mathfrak{s}_+\}$, respectively. Given that I^k is the union of the copies of the instar at level $k-1$, $\mathfrak{s}_+(\mathsf{I}^{k-1})$ and $\mathfrak{s}_-(\mathsf{I}^{k-1})$ will be respectively called the *positive* and *negative instars* at level k. Similarly, we refer to $\mathfrak{s}_+(\widetilde{\mathsf{I}}^{k-1})$, $\mathfrak{s}_0(\widetilde{\mathsf{I}}^{k-1})$, and $\mathfrak{s}_-(\widetilde{\mathsf{I}}^{k-1})$ as the *positive*, *central*, and *negative instar* of level k.

It will also be useful to have a name for each of the disks and their centers in the instar. The center $\mathfrak{s}_{\mathtt{w}|k}(0) = \sum_{j=0}^{k} a_j \lambda^j$ will be called a *node* with itinerary $\mathtt{w}|k$ and generally denoted by $\nu_{\mathtt{w}|k}$, while the closed disk $\mathsf{D}^{\mathtt{w}|k}$ (or $\widetilde{\mathsf{D}}^{\mathtt{w}|k}$) centered there will be referred to as a *nodal disk* with itinerary $\mathtt{w}|k$.

3. The Overlap Set

The geometric structure of the attractor A_λ is determined in many ways by the overlap set $O_\lambda := \mathsf{A}_\lambda^- \cap \mathsf{A}_\lambda^+$. It is a standard exercise to show that if O_λ is empty then A_λ is simply a Cantor set and consequently $\lambda \in \mathbb{D} \setminus \mathcal{M}$. However, if O_λ is "large" then it becomes difficult to distinguish the smaller affine copies that constitute A_λ. Moreover, if the large size of the overlap is persistent through a small change of λ, then the parameter is in the interior of $\mathcal{M}$. Intuitively, $\lambda \in \partial\mathcal{M}$ whenever O_λ is in some sense "thin".

Whenever O_λ is nonempty there exist itineraries $\mathtt{a}, \mathtt{b} \in \Sigma^\infty$ with $a_0 = -$ and $b_0 = +$ such that

$$\pi_\lambda(\mathtt{w}) := \sum_{j=0}^{\infty} a_j \lambda^j = \sum_{j=0}^{\infty} b_j \lambda^j = \pi_\lambda(\mathtt{v}) \iff \sum_{j=0}^{\infty} (a_j - b_j)\lambda^j = 0$$

Observe that $a_j - b_j \in \{-2, 0, +2\}$ for every $j \geq 0$. Consequently, denote the set of all power series with coefficients from the set $\{-1, 0, +1\}$ as

$$\mathcal{P} = \left\{ f(z) = \sum_{j=0}^{\infty} c_j z^j \;\middle|\; c_j \in \{-1, 0, +1\},\ c_0 = 1 \right\}$$

and define the set of power series which have λ as a root by

$$\mathcal{F}_\lambda = \{ f \in \mathcal{P} \mid f(\lambda) = 0 \}.$$

Then for $\lambda \in \mathbb{D}$ the overlap set, O_λ is nonempty whenever there exists $f \in \mathcal{P}$ such that it has coefficients $c_j = (a_j - b_j)/2$ and $f(\lambda) = 0$. Conversely, if for a particular $\lambda \in \mathbb{D}$ the set $\mathcal{F}_\lambda$ is nonempty, then so is O_λ, and each element in it has an itinerary associated to some $f \in \mathcal{F}_\lambda$. We have just shown that

$$\mathcal{M} = \{ \lambda \in \mathbb{D} \mid \mathsf{A}_\lambda \text{ is connected} \} = \{ \lambda \in \mathbb{D} \mid |\mathcal{F}_\lambda| \neq 0 \}.$$

Moreover, since A_λ is symmetric with respect to 0, having the origin in the overlap implies that the coefficients c_j of at least one of the power series $f \in \mathcal{F}_\lambda$ must all be nonzero (see Lemma 3.1). It follows that

$$\mathcal{M}_0 = \{ \lambda \in \mathbb{D} \mid 0 \in \mathsf{A}_\lambda \} = \left\{ \lambda \in \mathbb{D} \;\middle|\; \exists f \in \mathcal{P},\ f(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^j = 0, \quad c_j \in \{-1, 1\} \right\}$$

from which it is clear that $\mathcal{M}_0 \subset \mathcal{M}$.

[1]The word "instar" is used in biology to describe the developmental stage of insects, between each molt until sexual maturity. We chose it because the limit set is obtained by going through (infinitely many) developmental stages.

The following result is important, as it gives more insight on the relationship between elements in O_λ and the power series which have λ as a root.

Lemma 3.1 (Solomyak [Sol05]). *$|O_\lambda| = 1$ or 2 if and only if $|\mathcal{F}_\lambda| = 1$. Moreover,*

(i.) *$|O_\lambda| = 1$ if and only if $f \in \mathcal{F}_\lambda$ has no zero coefficients.*
(ii.) *$|O_\lambda| = 2$ if and only if $f \in \mathcal{F}_\lambda$ has exactly one zero coefficient.*

Using Rouche's Theorem and careful estimates Solomyak was also able to prove that

Theorem 3 (Solomyak [Sol05]). *There exist uncountably many $\lambda \in \mathcal{M}$ for which $|O_\lambda| = 1$. The itinerary of $0 \in \mathsf{A}_\lambda$ is different for different λ.*

There is a historically important property of an IFS which ensures that there is not "too much" overlap. We say that an IFS of contraction similarities $\{\mathfrak{s}_j\}_{j=1}^m$ satisfies the *open set condition* (OSC) if there exists a nonempty open set $V \subset \mathbb{C}$ with

$$\bigcup_{j=1}^m \mathfrak{s}_j(V) \subseteq V \text{ and } \bigcap_{j=1}^m \mathfrak{s}_j(V) = \varnothing.$$

The example of $\lambda = \mathrm{i}/\sqrt{2}$ is useful in gaining intuition about the OSC. The attractor for A_λ is the rectangle with corners $\pm 2 \pm \sqrt{2}\mathrm{i}$, with side ratio $\sqrt{2}$. The two images $\mathfrak{s}_+(\mathsf{A}_\lambda)$, $\mathfrak{s}_-(\mathsf{A}_\lambda)$ cover the left and right halves of A_λ, much as a A4 sheet of paper folded in half. Their intersection is the middle fold, so a feasible open set is the interior of A_λ itself.

In general, it can be challenging to prove that the OSC holds for a given IFS with a connected attractor. Bandt and his collaborators have recently shown that for connected self-similar sets in the plane a finite overlap implies OSC [BR07]. Specifically, in our setting

Theorem 4 (Bandt-Hung [BH08]). *For every $m \in \mathbb{N}$ there are uncountably many $\lambda \in \mathcal{M}$ for which OSC holds, and the overlap set consists of 2^m points. For each λ there exists a unique and distinct $f \in \mathcal{P}$ such that $\mathcal{F}_\lambda = \{f\}$.*

Theorem 5 (Bandt-Hung [BH08]). *For every $\beta \in [0, 0.2]$ there are uncountably many $\lambda \in \mathcal{M}$ for which OSC holds, and the overlap set is a Cantor set of Hausdorff dimension β. For each λ there exists a unique and distinct $f \in \mathcal{P}$ such that $\mathcal{F}_\lambda = \{f\}$.*

It must be noted that the proof of the above lemma cannot be easily extended to the case of $|O_\lambda| = 2^m$ for $m \geq 2$. Indeed, Bandt and Hung used a different argument to show the uniqueness of the power series.

4. Self and Asymptotic Similarity

Before describing the old and new results about $\partial\mathcal{M}$, we recall some definitions which can be found in [Lei90]. Remember that $D_r(z)$ denotes a closed disk centered at z with radius r and $D_r = D_r(0)$. For compact sets $E, F \subset \mathbb{C}$ denote

$$[E]_r = (E \cap D_r) \cup \partial D_r; \qquad \mathrm{d}_r(E, F) = \mathrm{d}_{\mathrm{H}}([E]_r, [F]_r)$$

where d_{H} is the Hausdorff distance.

Definition 1. **(i.)** *A compact set F is* ρ-self-similar *about $z \in F$, for $\rho \in \mathbb{C} \setminus \overline{\mathbb{D}}$, if there is $r > 0$ such that $[\rho(F - z)]_r = [F - z]_r$.*

(ii.) *Two compact sets E and F are* asymptotically similar *about $z \in E$ and $w \in F$ if there is $r > 0$ such that*

$$\lim_{t\in\mathbb{C},\ |t|\to\infty} \mathrm{d}_r(t(E-z), t(F-w)) = 0.$$

(iii.) *A compact set E is* asymptotically ρ-self similar *about a point $z \in E$ if there is $r > 0$ and a compact set F such that*

$$\mathrm{d}_r(\rho^n(E-z), F) \to 0 \quad n \to \infty.$$

We can now state the result of Solomyak:

Theorem 6 (Solomyak [Sol05])**.** *Suppose $\lambda \in \mathcal{M}\setminus\mathbb{R}$, with $|\lambda| \leq 2^{-1/2}$, is such that $\mathcal{F}_\lambda = \{f\}$ with*

$$f(z) = \sum_{n=0}^{\ell} c_j z^j + \frac{c_{\ell+1}z^{\ell+1} + \ldots + c_{\ell+p}z^{\ell+p}}{1-z^p}.$$

Then $f'(\lambda) \neq 0$ and

(i.) *$\widetilde{\mathsf{A}}_\lambda$ is λ^{-p}-self similar about $-\lambda^{-(\ell+1)}\sum_{n=0}^{\ell} c_j\lambda^j =: \zeta$.*
(ii.) *$\mathcal{M}$ about λ is asymptotically similar to $\frac{\lambda^{\ell+1}}{f'(\lambda)}\widetilde{\mathsf{A}}_\lambda$ about $\frac{\lambda^{\ell+1}}{f'(\lambda)}\zeta$.*
(iii.) *$\mathcal{M}$ is asymptotically λ^{-p}-self similar about λ.*

Notice that if the coefficients of f are all non zero, then the theorem holds true if we substitute $\widetilde{\mathsf{A}}_\lambda$ with A_λ and $\mathcal{M}$ with $\mathcal{M}_0$. However, Theorem 6 is not enough to certify that parameters λ satisfying the hypothesis lie on $\partial\mathcal{M}$, because points in a neighborhood of ζ, not in $\widetilde{\mathsf{A}}_\lambda$, are not necessarily also outside of $\mathcal{M}$ in a neighborhood of λ.

Theorem 7 (Calegari-Koch-Walker [CKW17])**.** *Suppose $\lambda \in \mathcal{M}\setminus\mathbb{R}$, with $|\lambda| \leq 2^{-1/2}$, is a root of*

$$f(z) = \sum_{n=0}^{\ell} c_j z^j + \frac{c_{\ell+1}z^{\ell+1} + \ldots + c_{\ell+p}z^{\ell+p}}{1-z^p}, \qquad c_j \in \{-1, 0, +1\}.$$

(i.) *If $C \in \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\widetilde{\mathsf{A}}_\lambda - \zeta\right)$, then for every $\varepsilon > 0$, there is a C' such that $|C - C'| < \varepsilon$ and for all sufficiently large n, a neighborhood of $C'\lambda^{pn} + \lambda$ is contained in $\mathcal{M}$.*
(ii.) *If $\mathcal{F}_\lambda = \{f\}$, then there is $\delta > 0$ such that for every $C \notin \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\widetilde{\mathsf{A}}_\lambda - \zeta\right)$ with $|C| < \delta$, the parameter $C\lambda^{pn} + \lambda$ is not in $\mathcal{M}$ for all sufficiently large n.*

Observe that this theorem is more descriptive than Theorem 6**(ii.)** as it describes explicitly which neighborhoods of $\zeta \in \widetilde{\mathsf{A}}_\lambda$ converge in the Hausdorff metric to neighborhoods of $\lambda \in \mathcal{M}$ (see Figure 2). However, this result gives little information about the local topology of $\mathcal{M}$ around λ. In particular, the question of recognizing points of $\partial\mathcal{M}$ path accessible from $\mathbb{D}\setminus\mathcal{M}$ remains open. Our main result gives a partial answer.

Remark 4.1. *In the following statements it is assumed that λ is not real with $|\lambda| \leq 2^{-1/2}$. We will call* accessible *those points of $\partial\mathcal{M}$ (respectively $\partial\mathcal{M}_0$) path accessible from $\mathbb{D}\setminus\mathcal{M}$ (respectively $\mathbb{D}\setminus\partial\mathcal{M}_0$).*

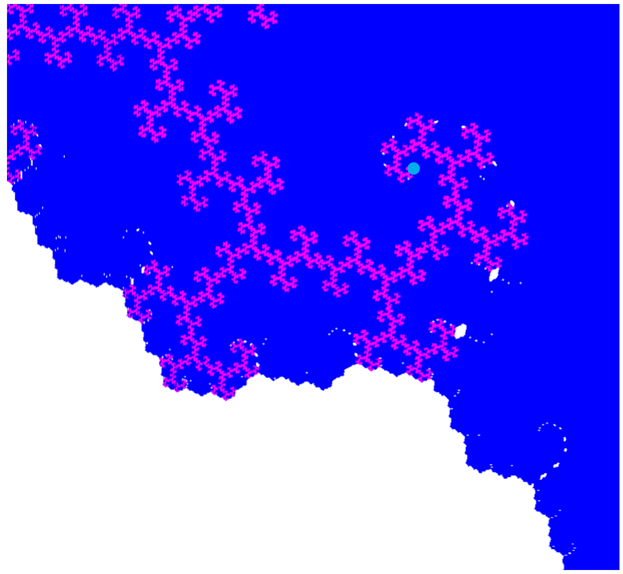

FIGURE 2. Calegari et al. showed that $\lambda \approx 0.371859+0.519411i$ (at the tip of the spiral, marked in cyan in the electronic version) is a point of $\partial\mathcal{M}\cap\partial\mathcal{M}_0$ not path accessible from $\mathbb{D}\setminus\mathcal{M}$.

Theorem 8. *Let $\lambda\in\mathcal{M}$ be such that $\mathcal{F}_\lambda=\{f\}$ with*

$$f(z)=\sum_{n=0}^{\ell} c_j z^j+\frac{c_{\ell+1}z^{\ell+1}+\ldots+c_{\ell+p}z^{\ell+p}}{1-z^p}.$$

Assume also that $f(z)$ has finitely many zero coefficients and that its Taylor polynomials, $f_k(z)=\sum_{j=0}^k c_j z^j$ satisfy the following conditions for every $0\le n\le p-1$:

- **(i.)** $|f_{\ell+1+n}(\lambda)|>\dfrac{1}{2}\dfrac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}$;
- **(ii.)** $|f_{\ell+1+n}(\lambda)|+|f_{\ell+1+n+1}(\lambda)|>\dfrac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}$;
- **(iii.)** $2\,|f_{\ell+1+n}(\lambda)|<\left|2f_\ell(\lambda)+\lambda^{\ell+1}P(\lambda)\right|$, *where $P\neq Q$ is any polynomial of degree less or equal than n with coefficients in $\{-2,-1,0,+1,+2\}$, and Q is the solution of $2f_\ell(z)+z^{\ell+1}Q(z)=2f_{\ell+1+n}(z)$.*

Then $\lambda\in\partial\mathcal{M}$ is accessible from a connected component of $\mathbb{D}\setminus\mathcal{M}$.

An immediate corollary is then

Corollary 9. *Let $\lambda\in\mathcal{M}$ be such that $\mathcal{F}_\lambda=\{f\}$ where f satisfies the hypothesis of Theorem 8 and has no zero coefficients. Then $\lambda\in\partial\mathcal{M}\cap\partial\mathcal{M}_0$ and it is accessible from a connected component of $\mathbb{D}\setminus\mathcal{M}$.*

Restricting the assumptions on the coefficients on the power series, f we also obtain

Theorem 10. *Let $\lambda\in\mathcal{M}$ be such that $\mathcal{F}_\lambda=\{f\}$ with*

$$f(z)=\sum_{n=0}^{\ell} c_j z^j+\frac{c_{\ell+1}z^{\ell+1}+\ldots+c_{\ell+p}z^{\ell+p}}{1-z^p}.$$

Assume also that $f(z)$ has no zero coefficients and that its Taylor polynomials, $f_k(z) = \sum_{j=0}^{k} c_j z^j$ satisfy the conditions **(i.)** *and* **(ii.)** *of Theorem 8 and*

(iii'.) $|f_{\ell+1+n}(\lambda)| < \left|f_\ell(\lambda) + \lambda^{\ell+1}P(\lambda)\right|$, *where $P \neq Q$ is any polynomial of degree less or equal than n with coefficients in $\{-1, 0, +1\}$., and Q is the solution of $f_\ell(z) + z^{\ell+1}Q(z) = f_{\ell+1+n}(z)$.*

Then $\lambda \in \partial\mathcal{M}$, is accessible from a connected component of $\mathbb{D} \setminus \mathcal{M}_0$.

In view of Corollary 9 and experimental evidence, it seems reasonable that the assumptions of Theorem 10 imply condition **(iii.)** (Theorem 8) must hold. Even better:

Conjecture 2. *Every parameter $\lambda \in \partial\mathcal{M} \cap \partial\mathcal{M}_0$ belongs to the boundary of the connected component of $\mathbb{D} \setminus \mathcal{M}_0$ containing 0.*

5. Proof of the Main Theorems

Here we will prove Theorems 8 and 10 in this section. In both cases, the idea of the proof is to construct locally, a connected chain of open disks outside $\widetilde{\mathsf{A}}_\lambda$ (or A_λ) that converges to $\zeta = -\lambda^{-(\ell+1)} \sum_{n=0}^{\ell} c_j \lambda^j$, and conclude by Theorem 7**(ii.)** that λ is accessible, hence on the boundary of a hole. The restrictions on the Taylor polynomials show up when obtaining the conditions for such a chain to exists.

We first need a lemma

Lemma 5.1. *Let $\lambda \in \mathcal{M}$ be such that $\mathcal{F}_\lambda = \{f\}$ with*

$$f(z) = \sum_{n=0}^{\ell} c_j z^j + \frac{c_{\ell+1}z^{\ell+1} + \ldots + c_{\ell+p}z^{\ell+p}}{1 - z^p},$$

where $c_j = 0$ only for some $0 < j \leq \ell$. Let $\xi \in O_\lambda$ have itineraries $\mathsf{a} = a_0 a_1 a_2 \cdots$ and $\bar{\mathsf{a}} = \bar{a}_0 \bar{a}_1 \bar{a}_2 \cdots$ where $a_j = -\bar{a}_j = c_j$ if $c_j \neq 0$, and otherwise $a_j = \bar{a}_j = -$ or $+$.
Then for every $0 \leq n \leq p-1$ the set $\mathsf{D}^{\mathsf{a}|\ell+n+p} \cup \mathsf{D}^{\bar{\mathsf{a}}|\ell+n+p}$ about ξ is λ^{-p}-self similar to $\mathsf{D}^{\mathsf{a}|\ell+n} \cup \mathsf{D}^{\bar{\mathsf{a}}|\ell+n}$ about ξ.

Proof. Since the set in question is the union of two intersecting closed disks, proving the lemma is equivalent to showing that

$$\frac{1}{\lambda^p}\left(\left(\mathsf{D}^{\mathsf{a}|\ell+n+p} \cup \mathsf{D}^{\bar{\mathsf{a}}|\ell+n+p}\right) - \xi\right) = \left(\mathsf{D}^{\mathsf{a}|\ell+n} \cup \mathsf{D}^{\bar{\mathsf{a}}|\ell+n}\right) - \xi.$$

In order to simplify the expressions, we only prove this in the case $n = 0$; but the argument is identical for $0 < n < p$.

Observe that since λ is a root of $f(z)$ then

$$c_{\ell+1}\lambda^{\ell+1} + \ldots + c_{\ell+p}\lambda^{\ell+p} = (\lambda^p - 1)(c_0 + c_1\lambda + \ldots + c_\ell\lambda^\ell).$$

Consequently,

$$\begin{aligned}\left|(a_0 + a_1\lambda + \ldots + a_{\ell+p}\lambda^{\ell+p}) - (\bar{a}_0 + \bar{a}_1\lambda + \ldots + \bar{a}_{\ell+p}\lambda^{\ell+p})\right| &= 2\left|c_0 + c_1\lambda + \ldots + c_{\ell+p}\lambda^{\ell+p}\right| \\ &= 2\left|\lambda^p\right|\left|c_0 + c_1\lambda + \ldots + c_\ell\lambda^\ell\right|\end{aligned}$$

and

$$\left|(a_0 + a_1\lambda + \ldots + a_\ell\lambda^\ell) - (\bar{a}_0 + \bar{a}_1\lambda + \ldots + \bar{a}_\ell\lambda^\ell)\right| \quad = \quad 2\left|c_0 + c_1\lambda + \ldots + c_\ell\lambda^\ell\right|$$

which shows that the distance of the nodes at levels $\ell + p$ and ℓ, are multiples of each other.

Finally, the center of the disk $\mathsf{D}^{a_0a_1\cdots a_{\ell+p}}$ is

$$\begin{aligned}\sum_{j=0}^{\ell+p} a_j\lambda^j &= \sum_{0<j\leq\ell\;:\;c_j=0} a_j\lambda^j + c_0 + c_1\lambda + \ldots + c_{\ell+p}\lambda^{\ell+p} \\ &= \sum_{0<j\leq\ell\;:c_j=0} a_j\lambda^j + \lambda^p(c_0 + c_1\lambda + \ldots + c_\ell\lambda^\ell).\end{aligned}$$

In other words,

$$\sum_{j=0}^{\ell+p} a_j\lambda^j - \sum_{0<j\leq\ell\;:\;c_j=0} a_j\lambda^j = \lambda^p\sum_{j=0}^{\ell} c_j\lambda^j = \lambda^p\left(\sum_{j=0}^{\ell} a_j\lambda^j - \sum_{0<j\leq\ell\;:\;c_j=0} a_j\lambda^j\right).$$

Notice that by definition $\xi = \sum_{0<j\leq\ell\;:\;c_j=0} a_j\lambda^j$, so the above equation becomes

$$a_0 + a_1\lambda + \ldots + a_{\ell+p}\lambda^{\ell+p} - \xi = \lambda^p(a_0 + a_1\lambda + \ldots + a_\ell\lambda^\ell - \xi)$$

that is

$$\frac{1}{\lambda^p}\left(\mathsf{D}^{\mathsf{a}|\ell+p} - \xi\right) = \left(\mathsf{D}^{\mathsf{a}|\ell} - \xi\right).$$

Now, the disk $\mathsf{D}^{\bar{\mathsf{a}}|\ell+k}$ is symmetric to $\mathsf{D}^{\mathsf{a}|\ell+k}$ relative to ξ for any $k \geq 0$. Hence, analogous arguments holds for the disks $\mathsf{D}^{\bar{\mathsf{a}}|\ell+p}$ and $\mathsf{D}^{\bar{\mathsf{a}}|\ell}$. $\square$

Recall from Lemma 2.1 that $\mathsf{A}_\lambda \subset D_R$ and $\mathsf{A}_\lambda^{\mathtt{w}} \subset \mathsf{D}^{\mathtt{w}}$ for any finite word $\mathtt{w} \in \Sigma^n$. Consequently, the above lemma proves the self-similarity of the attractor A_λ at its overlap:

Corollary 11. *Let $\lambda \in \mathcal{M}$ be such that $\mathcal{F}_\lambda = \{f\}$ with*

$$f(z) = \sum_{n=0}^{\ell} c_jz^j + \frac{c_{\ell+1}z^{\ell+1} + \ldots + c_{\ell+p}z^{\ell+p}}{1 - z^p},$$

where $c_j = 0$ only for some $0 < j \leq \ell$. Let $\xi \in O_\lambda$ have itineraries $\mathsf{a} = a_0a_1a_2\cdots$ and $\bar{\mathsf{a}} = \bar{a}_0\bar{a}_1\bar{a}_2\cdots$ where $a_j = -\bar{a}_j = c_j$ if $c_j \neq 0$, and otherwise $a_j = \bar{a}_j = -$ or $+$.
Then for every $0 \leq n \leq p-1$ the set $\mathsf{A}_\lambda^{\mathsf{a}|\ell+p+n} \cup \mathsf{A}_\lambda^{\bar{\mathsf{a}}|\ell+p+n}$ about ξ is λ^{-p}-self similar to $\mathsf{A}_\lambda^{\mathsf{a}|\ell+n} \cup \mathsf{A}_\lambda^{\bar{\mathsf{a}}|\ell+n}$ about ξ.

We now proceed to the construction of the chain in the complement of $\widetilde{\mathsf{A}}_\lambda$. We exploit the recursive construction of $\widetilde{\mathsf{A}}_\lambda$ to find each disk in the chain: for each $n \geq 0$ we find an open disk tangent to the instar $\widetilde{\mathsf{I}}^n$. Moreover, two consecutive disks in the chain must intersect non trivially. Finally, this chain must converge to $\zeta \in \widetilde{\mathsf{A}}_\lambda$.

The parameter λ is the root of a unique power series $f \in \mathcal{P}$ whose non-zero coefficients eventually repeat: say

$$f(z) = \sum_{j=0}^{\ell} c_jz^j + \frac{c_{\ell+1}z^{\ell+1} + \ldots + c_{\ell+p}z^{\ell+p}}{1 - z^p}.$$

Now, ζ is defined to be $-\lambda^{-(\ell+1)}\sum_{j=0}^{\ell} c_j\lambda^j$ which means it can be described with the periodic itinerary $(c_{\ell+1}\cdots c_{\ell+p})^\infty \in \Sigma^\infty$. This itinerary will be the unique one associated to ζ as long as $|\mathcal{F}_\lambda| = 1$. Since we assume so in the statement of the theorems, there is a unique sequence

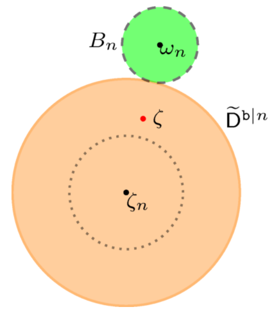

FIGURE 3. Construction of B_n, an element in the connected chain of open sets in $\mathbb{C} \setminus \widetilde{\mathsf{A}}_\lambda$. The dotted circle has half the radius of $\widetilde{\mathsf{D}}^{\mathsf{b}|n}$. The distance between ζ and ζ_n must be more than half of the radius of the instar $\widetilde{\mathsf{D}}^{\mathsf{b}|n}$ to allow for the existence of B_n.

of nodes converging to ζ, namely the ones whose itinerary is the truncation of $(c_{\ell+1} \cdots c_{\ell+p})^\infty$ at some index.

Set $\mathsf{b} = b_0 b_1 \cdots \in \widetilde{\Sigma}^\infty$ where $b_j = c_{\ell+1+j}$ and let $\zeta_n := \nu_{\mathsf{b}|n}$ be the node in $\widetilde{\mathsf{I}}^n$. Observe that ζ_n is by definition the center of the disk $\widetilde{\mathsf{D}}^{\mathsf{b}|n}$ and ζ is a point inside such disk. Therefore, if ζ_n is far enough from ζ, then $\omega_n = -\zeta_n + 2\zeta$, i.e. the reflection of ζ_n about ζ, will be outside $\widetilde{\mathsf{D}}^{\mathsf{b}|n}$. We can then find an open disk, B_n centered at ω_n tangent to $\widetilde{\mathsf{D}}^{\mathsf{b}|n}$ (see Figure 3).

Recall that each nodal disk in $\widetilde{\mathsf{I}}^n$ has a radius of $|\lambda^{n+1}|\,(1-|\lambda|)^{-1}$. Thus, the radius, r_n of B_n is easily found to be

$$r_n := |\omega_n - \zeta_n| - \frac{|\lambda^{n+1}|}{1-|\lambda|} = 2\,|\zeta - \zeta_n| - \frac{|\lambda^{n+1}|}{1-|\lambda|}.$$

Proof of Theorem 8. : We will show that the chain of open disks $\bigcup_{n\geq 0} B_n$ is a connected subset of $\mathbb{C}\setminus\widetilde{\mathsf{A}}_\lambda$. In fact, it is enough to prove $\widetilde{\mathsf{I}}^{p-1}\cap\bigcup_{n=0}^{p-1} B_n = \varnothing$, since by Theorem 6 the attractor $\widetilde{\mathsf{A}}_\lambda$ is λ^{-p}-self similar about ζ. Moreover, we will prove that $C\lambda^{pm} \in \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\bigcup_{n\geq 0} B_n - \zeta\right)$ for every $C \in \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\bigcup_{n\geq 0} B_n - \zeta\right)$ and m large enough. The claim of the theorem will then follow from Theorem 7**(ii.)**.

The condition of $|O_\lambda|$ being finite implies that all zero coefficients of the power series appear in the first $\ell+1$ terms. The choice of which $\xi \in O_\lambda$ to consider is arbitrary but ζ is always unique because $|\mathcal{F}_\lambda| = 1$.

We consider only the case $|O_\lambda| = 2$ to simplify notation. By assumption, the coefficients of f are strictly preperiodic, and exactly one of them must be zero. Hence, there exists $0 < k \leq \ell$ such that $c_k = 0$ which implies $O_\lambda = \{\pm\lambda^k\} = \{\pm\xi\}$. We deduce that the itinerary of ζ is a word in Σ^∞ and, thus, $\zeta \in \mathsf{A}_\lambda \subset \widetilde{\mathsf{A}}_\lambda$.

Let $\mathsf{a}, \bar{\mathsf{a}} \in \Sigma^\infty$ be such that $\pi_\lambda(\mathsf{a}) = \pi_\lambda(\bar{\mathsf{a}}) = \xi$. Therefore, $a_j = -\bar{a}_j = c_j$ for all $j \neq k$ and $a_k = \bar{a}_k = +1$ or -1 (since $c_k = 0$). In particular, $\xi = f(\lambda) + a_k\lambda^k$. Denote by ξ_n the nodes with itinerary $\mathsf{a}|n$, i.e. $\xi_n = \nu_{\mathsf{a}|n} = \sum_{j=0}^n a_j\lambda^j$. The Taylor polynomial $f_{\ell+1+n}(\lambda)$ for $n \geq 0$ can then be written as $\xi_{\ell+1+n} - \xi$.

Observe that the itinerary of ζ is the (left) shift of a by ℓ terms. Indeed, we claim that

$$\zeta = \mathfrak{s}_{\mathsf{a}|\ell}^{-1}(\xi) \quad \text{ and } \quad \zeta_n = \mathfrak{s}_{\mathsf{a}|\ell}^{-1}(\xi_{\ell+1+n}).$$

Since $\mathfrak{s}_{a_0a_1}(z) = \mathfrak{s}_{a_0}(\mathfrak{s}_{a_1}(z)) = \nu_{a_0a_1} + \lambda^2 z$ then $\mathfrak{s}^{-1}_{a_0a_1}(z) = \mathfrak{s}^{-1}_{a_1}(\mathfrak{s}^{-1}_{a_0}(z)) = \frac{1}{\lambda^2}(z - \nu_{a_0a_1})$. Consequently,

$$\begin{aligned}
\mathfrak{s}^{-1}_{\mathtt{a}|\ell}(\xi) = \frac{1}{\lambda^{\ell+1}}(\xi - \xi_\ell) &= -\frac{1}{\lambda^{\ell+1}} f_\ell(\lambda) = \zeta \\
\mathfrak{s}^{-1}_{\mathtt{a}|\ell}(\xi_{\ell+1+n}) = \frac{1}{\lambda^{\ell+1}}(\xi_{\ell+1+n} - \xi_\ell) &= \frac{1}{\lambda^{\ell+1}}(f_{\ell+1+n}(\lambda) - f_\ell(\lambda)) = \zeta_n.
\end{aligned}$$

The centers of the disks B_n were defined in terms of ζ_n and ζ, but we can now rewrite them in terms of the Taylor polynomials

$$\omega_n = -\zeta_n + 2\zeta = -\frac{1}{\lambda^{\ell+1}}(f_{\ell+1+n}(\lambda) + f_\ell(\lambda)).$$

Using the above equations, we also rewrite the radius of B_n in terms of a Taylor polynomial:

$$r_n = |\omega_n - \zeta_n| - \frac{|\lambda^{n+1}|}{1-|\lambda|} = \frac{2}{|\lambda^{\ell+1}|}|f_{\ell+1+n}(\lambda)| - \frac{|\lambda^{n+1}|}{1-|\lambda|}.$$

We are now practically done: for each $0 \leq n \leq p-1$

(i.) the disk B_n exists if and only if $r_n > 0$, namely

$$\frac{2}{|\lambda^{\ell+1}|}|f_{\ell+1+n}(\lambda)| - \frac{|\lambda^{n+1}|}{1-|\lambda|} > 0 \iff |f_{\ell+1+n}(\lambda)| > \frac{1}{2}\frac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}$$

which is true by assumption;

(ii.) $B_n \cap B_{n+1} \neq \varnothing$ if and only if $r_n + r_{n+1} > |\omega_n - \omega_{n+1}| = |\lambda^{n+1}|$, namely

$$\begin{aligned}
\frac{2}{|\lambda^{\ell+1}|}(|f_{\ell+1+n}(\lambda)| + |f_{\ell+1+n+1}(\lambda)|) - \frac{|\lambda^{n+1}|}{1-|\lambda|} - \frac{|\lambda^{n+2}|}{1-|\lambda|} &> \left|\lambda^{n+1}\right| \\
\iff |f_{\ell+1+n}(\lambda)| + |f_{\ell+1+n+1}(\lambda)| &> \frac{\left|\lambda^{\ell+1+n+1}\right|}{1-|\lambda|}
\end{aligned}$$

which is true by assumption;

(iii.) the disk B_n is tangent to $\widetilde{\Gamma}^n$ if and only if for every node $\nu_{\mathtt{w}|n} \in \widetilde{\Gamma}^n$ with $\mathtt{w} \in \widetilde{\Sigma}^\infty$ we have $r_n + |\lambda^{n+1}|(1-|\lambda|)^{-1} < |\omega_n - \nu_{\mathtt{w}}|$, namely

$$\begin{aligned}
\frac{2}{|\lambda^{\ell+1}|}|f_{\ell+1+n}(\lambda)| &< \left|-\frac{1}{\lambda^{\ell+1}}(f_{\ell+1+n}(\lambda) + f_\ell(\lambda)) - \nu_{\mathtt{w}|n}\right| \\
\iff 2|f_{\ell+1+n}(\lambda)| &< \left|f_\ell(\lambda) + f_{\ell+1+n}(\lambda) + \lambda^{\ell+1}\nu_{\mathtt{w}|n}\right| \\
&= \left|2f_\ell(\lambda) + \lambda^{\ell+1}P(\lambda)\right|
\end{aligned}$$

where P is a polynomial of degree at most n with coefficients taken from the set $\{-2,-1,0,+1,+2\}$. Again, the above inequality is true by assumption.

Finally, we claim that $C\lambda^{pm} \in \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\bigcup_{n\geq 0} B_n - \zeta\right)$ for every $C \in \frac{\lambda^{\ell+1}}{f'(\lambda)}\left(\bigcup_{n\geq 0} B_n - \zeta\right)$ and $m \geq 1$. We will show that, after a translation by $-\zeta$, the chain is forward invariant under $z \mapsto \lambda^p z$.

Observe that $\omega_n - \zeta = -\lambda^{-\ell-1} f_{\ell+1+n}(\lambda)$ and since $(\lambda^p - 1) f_\ell(\lambda) = \sum_{j=\ell+1}^{\ell+p} c_j \lambda^j$, then

$$\begin{aligned}
\lambda^p(\omega_n - \zeta) &= -\frac{1}{\lambda^{\ell+1}} \lambda^p f_{\ell+1+n}(\lambda) \\
&= -\frac{1}{\lambda^{\ell+1}} \lambda^p \left(f_\ell(\lambda) + \sum_{j=\ell+1}^{\ell+1+n} c_j \lambda^j \right) \\
&= -\frac{1}{\lambda^{\ell+1}} \left(\lambda^p f_\ell(\lambda) + \lambda^p \sum_{j=\ell+1}^{\ell+1+n} c_j \lambda^j \right) \\
&= -\frac{1}{\lambda^{\ell+1}} \left(f_\ell(\lambda) + \sum_{j=\ell+1}^{\ell+p} c_j \lambda^j + \sum_{j=\ell+1+p}^{\ell+1+n+p} c_j \lambda^j \right) \\
&= -\frac{1}{\lambda^{\ell+1}} f_{\ell+1+n+p}(\lambda) = \omega_{n+p} - \zeta
\end{aligned}$$

where the second to last equality is due to the fact that $c_{\ell+k} = c_{\ell+k+p}$ for every $k \geq 1$. Hence, the claim follows. $\square$

The proof of Theorem 10 is analogous, except we only have to show $\bigcup_{n\geq 0} B_n$ is outside A_λ. Therefore, in step **(iii.)** we need to check that B_n does not intersect the instar I^n, rather than $\widetilde{\mathsf{I}}^n$.

Remark 5.1. *The conditions of both Theorems 8 and 10 can be weakened as follows: there exists integers $2 \leq m \leq p$ and $\{k_1, k_2, \cdots, , k_m\}$ where $0 \leq k_1 < k_2 < \cdots < k_m \leq p-1$ such that for all $1 \leq j \leq m$*

- **(i.)** $\left|f_{\ell+1+k_j}(\lambda)\right| > \frac{1}{2} \frac{\left|\lambda^{\ell+1+k_j+1}\right|}{1-|\lambda|}$
- **(ii.)** $\left|f_{\ell+1+k_j}(\lambda)\right| + \left|f_{\ell+1+k_{j+1}}(\lambda)\right| - \frac{1}{2}\left|f_{\ell+1+k_j}(\lambda) - f_{\ell+1+k_{j+1}}(\lambda)\right| > \frac{1}{2} \frac{\left|\lambda^{\ell+2+k_j}\right| + \left|\lambda^{\ell+2+k_{j+1}}\right|}{1-|\lambda|}$
- **(iii.)** $\left|f_{\ell+1+k_j}(\lambda)\right| < \left|f_\ell(\lambda) + \lambda^{\ell+1} P(\lambda)\right|$*, where $P \neq Q$ is any polynomial of degree less or equal to k_j with coefficients in $\{-1, 0, +1\}$, and Q is the solution of $f_\ell(z) + z^{\ell+1} Q(z) = f_{\ell+1+k_j}(z)$.*

These conditions allow the possibility of intersection between two non-consecutive chain disks.

6. Examples: Landmark Points

In [Sol05], Solomyak discussed six sample parameters that satisfy $\left|\mathcal{F}_{\lambda_j}\right| = 1$, and named them *landmark points*. As a note of caution, we changed Solomyak's nomenclature to tie in with our exposition, so that our $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ correspond to his $\lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_1, \lambda_2$. In this section we will prove that λ_j for $j = 1, \ldots, 5$ satisfy the conditions of Theorems 8 and 10 and are therefore, accessible.

6.1. **Period One.** The landmark points λ_j for $j = 1, \ldots, 4$ are all inside the following sector:

$$\boldsymbol{S} := \left\{ z \in \mathbb{D} \;\middle|\; \frac{\sqrt{5}-1}{2} < |z| < \frac{2}{3} \text{ and } 0 < \arg(z) < \frac{5\pi}{32} \right\}$$

and have itineraries of various preperiods, but all with period 1. By Proposition 6.1 they are all accessible. Later in Section 6.2 we present a method to circumvent the lengthier computations that would be required to establish the inequalities for λ_5 (which has period 3).

Proposition 6.1. *Let $\lambda \in \boldsymbol{S}$ and assume $\mathcal{F}_\lambda = \{f\}$ where*

$$f(z) = \sum_{j=0}^{\ell} c_j z^j + \frac{z^{\ell+1}}{1-z}.$$

Then λ satisfies the conditions of Theorem 8 (and therefore $\lambda \in \partial\mathcal{M}$ is accessible).

The proof will use the following properties of $\boldsymbol{S}$:

Lemma 6.1. *For all $\lambda \in \boldsymbol{S}$ the following holds:*
(a.) $1 - |\lambda| > \frac{1}{2}|1-\lambda|$;
(b.) $1 - |\lambda|^2 > |1-\lambda|$;
(c.) $|\lambda| < |2-\lambda|$;
(d.) $2|\lambda| < |3-\lambda|$;
(e.) $2|\lambda| < |1+\lambda|$.

Proof. From the law of cosines we obtain

$$\begin{aligned} 0.541 &> \sqrt{1^2 + \left(\frac{\sqrt{5}-1}{2}\right)^2 - 2\left(\frac{\sqrt{5}-1}{2}\right)\cos\left(\frac{5\pi}{32}\right)} \\ &> |1-\lambda| > \sqrt{1^2 + \left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)\cos(0)} = \frac{1}{3}. \end{aligned}$$

It follows that

$$1 - |\lambda| > 1 - \frac{2}{3} > \frac{1}{2}(0.541) > \frac{1}{2}|1-\lambda|$$

and

$$1 - |\lambda|^2 > 1 - \frac{4}{9} > 0.541 > |1-\lambda|,$$

which gives **(a.)** and **(b.)**.

Similarly,

$$|2-\lambda| > \sqrt{2^2 + \left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)2\cos(0)} = \frac{4}{3} > 2|\lambda| > |\lambda|$$

$$|3-\lambda| > \sqrt{3^2 + \left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)3\cos(0)} = \frac{7}{3} > 2|\lambda|,$$

which gives **(c.)** and **(d.)**.

Finally,

$$|1+\lambda| > \sqrt{1^2 + \left(\frac{\sqrt{5}-1}{2}\right)^2 + 2\left(\frac{\sqrt{5}-1}{2}\right)\cos\left(\frac{5\pi}{32}\right)} > 1.572 > \frac{4}{3} > 2|\lambda|,$$

giving **(e.)** and concluding the proof of the lemma. $\square$

Armed with these inequalities, we proceed to prove the Proposition:

Proof of Proposition 6.1. Since λ is the root of the power series f, we can write Taylor polynomials as follows

$$f_\ell(\lambda) = \sum_{j=0}^{\ell} c_j \lambda^j = -\frac{\lambda^{\ell+1}}{1-\lambda},$$

$$f_{\ell+1}(\lambda) = \sum_{j=0}^{\ell} c_j \lambda^j + \lambda^{\ell+1} = -\frac{\lambda^{\ell+2}}{1-\lambda},$$

$$f_{\ell+2}(\lambda_1) = 1\sum_{j=0}^{\ell} c_j \lambda^j + \lambda^{\ell+1} + \lambda^{\ell+2} = -\frac{\lambda^{\ell+3}}{1-\lambda}.$$

Condition **(i.)** in Theorem 8 is satisfied since

$$|f_{\ell+1}(\lambda)| > \frac{1}{2}\frac{\left|\lambda^{\ell+2}\right|}{1-|\lambda|} \iff \left|\frac{1}{1-\lambda}\right| > \frac{1}{2}\frac{1}{1-|\lambda|} \iff 1-|\lambda| > \frac{1}{2}|1-\lambda|$$

holds by Lemma 6.1 part **(a.)**.
Condition **(ii.)** in Theorem 8 is satisfied since

$$\begin{aligned} |f_{\ell+1}(\lambda)| + |f_{\ell+2}(\lambda)| > \frac{\left|\lambda^{\ell+2}\right|}{1-|\lambda|} &\iff \left|\frac{1}{1-\lambda}\right| + \left|\frac{\lambda}{1-\lambda}\right| > \frac{1}{1-|\lambda|} \\ &\iff 1-|\lambda|^2 > |1-\lambda| \end{aligned}$$

holds by Lemma 6.1 part **(b.)**.
Condition **(iii.)** in Theorem 8 has four cases since the polynomial P can only be either $-2, -1, 0$, or 1. The case $P(z) = -2$ is satisfied because

$$\begin{aligned} |2f_{\ell+1}(\lambda)| < \left|2f_\ell(\lambda) + \lambda^{\ell+1}(-2)\right| &\iff \left|\frac{-2\lambda^{\ell+2}}{1-\lambda}\right| < \left|\frac{-4\lambda^{\ell+1} + 2\lambda^{\ell+2}}{1-\lambda}\right| \\ &\iff |\lambda| < |2-\lambda| \end{aligned}$$

holds by Lemma 6.1 part **(c.)**.
The case $P(z) = -1$ is satisfied because

$$\begin{aligned} |2f_{\ell+1}(\lambda)| < \left|2f_\ell(\lambda) + \lambda^{\ell+1}(-1)\right| &\iff \left|\frac{-2\lambda^{\ell+2}}{1-\lambda}\right| < \left|\frac{-3\lambda^{\ell+1} + \lambda^{\ell+2}}{1-\lambda}\right| \\ &\iff |2\lambda| < |3-\lambda| \end{aligned}$$

holds by Lemma 6.1 part **(d.)**.
The case $P(z) = 0$ is trivial since

$$|2f_{\ell+1}(\lambda)| < \left|2f_\ell(\lambda) + \lambda^{\ell+1}(0)\right| \iff \left|\frac{-2\lambda^{\ell+2}}{1-\lambda}\right| < \left|\frac{-2\lambda^{\ell+1}}{1-\lambda}\right| \iff |\lambda| < 1$$

The case $P(z) = 1$ is satisfied because

$$\begin{aligned} |2f_{\ell+1}(\lambda)| < \left|2f_\ell(\lambda) + \lambda^{\ell+1}(1)\right| &\iff \left|\frac{-2\lambda^{\ell+2}}{1-\lambda}\right| < \left|\frac{-\lambda^{\ell+1} - \lambda^{\ell+2}}{1-\lambda}\right| \\ &\iff |2\lambda| < |1+\lambda| \end{aligned}$$

holds by Lemma 6.1 part **(e.)**. $\square$

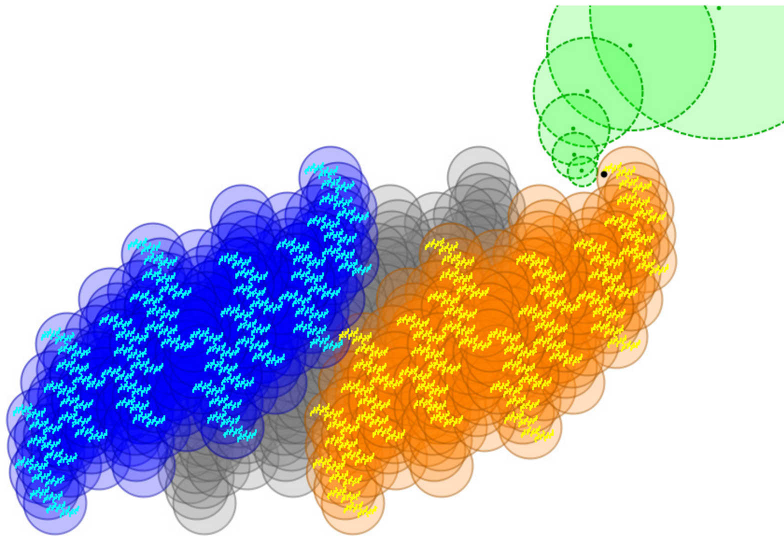

FIGURE 4. The attractor A_{λ_1} inside $\widetilde{\mathsf{I}}^5$. The disks in the top-right corner (green in the electronic version) are part of the chain constructed in Theorem 8.

In the following examples recall that the uniqueness of the power series was proved by Solomyak in [Sol05].

Example 1. Let $\mathsf{c} = c_0c_1\cdots = + - -(+)^\infty \in \Sigma^\infty$ and set $f(z) := \sum_{k=0}^{\infty} c_k z^k = \dfrac{1-2z+2z^3}{1-z}$. The associated parameter in $\mathcal{M}\cap \boldsymbol{S}$ is $\lambda_1 \approx 0.5957439 + 0.2544259\mathrm{i}$. It follows from Proposition 6.1 that the parameter λ_1, is an accessible point of $\partial\mathcal{M}$ (see Figure 4). In fact, by Corollary 9, we can conclude that $\lambda_1 \in \partial\mathcal{M}\cap\partial\mathcal{M}_0$.

Example 2. Let $\mathsf{c} = c_0c_1\cdots = + - -\mathsf{0}(+)^\infty \in \widetilde{\Sigma}^\infty$ and set $f(z) := \sum_{k=0}^{\infty} c_k z^k = \dfrac{1-2z+z^3+z^4}{1-z}$. The associated parameter in $\mathcal{M}\cap \boldsymbol{S}$ is $\lambda_2 \approx 0.6219644 + 0.1877304\mathrm{i}$. It follows from Proposition 6.1 that the parameter λ_2, is an accessible point of $\partial\mathcal{M}$ (see Figure 5).

Example 3. Let $\mathsf{c} = c_0c_1\cdots = + - -\mathsf{00}(+)^\infty \in \widetilde{\Sigma}^\infty$ and set $f(z) := \sum_{k=0}^{\infty} c_k z^k = \dfrac{1-2z+z^3+z^5}{1-z}$. The associated parameter in $\mathcal{M}\cap \boldsymbol{S}$ is $\lambda_3 \approx 0.643703 + 0.140749\mathrm{i}$. It follows from Proposition 6.1 that the parameter λ_3, is an accessible point of $\partial\mathcal{M}$ (see Figure 6).

Example 4. Let $\mathsf{c} = c_0c_1\cdots = + - - - (+)^\infty \in \Sigma^\infty$ and set $f(z) := \sum_{k=0}^{\infty} c_k z^k = \dfrac{1-2z+2z^4}{1-z}$. The associated parameter in $\mathcal{M}\cap \boldsymbol{S}$ is $\lambda_4 \approx 0.63601 + 0.106924\mathrm{i}$. It follows from Proposition 6.1 that the parameter λ_4, is an accessible point of $\partial\mathcal{M}$ (see Figure 7). In fact, by Corollary 9, we can conclude that $\lambda_4 \in \partial\mathcal{M}\cap\partial\mathcal{M}_0$.

6.2. **Higher period.** Here we consider the landmark point λ_5 whose associated itinerary has period 3. Observe that in this case the number of inequalities to be checked before applying Theorem 8 becomes quite large. To overcome this difficulty we rely on the argument and modulus of λ_5, and the geometric structure of the instar at each level. We will construct each of the disks of the connected chain that lies in the complement of $\widetilde{\mathsf{A}}_{\lambda_5}$.

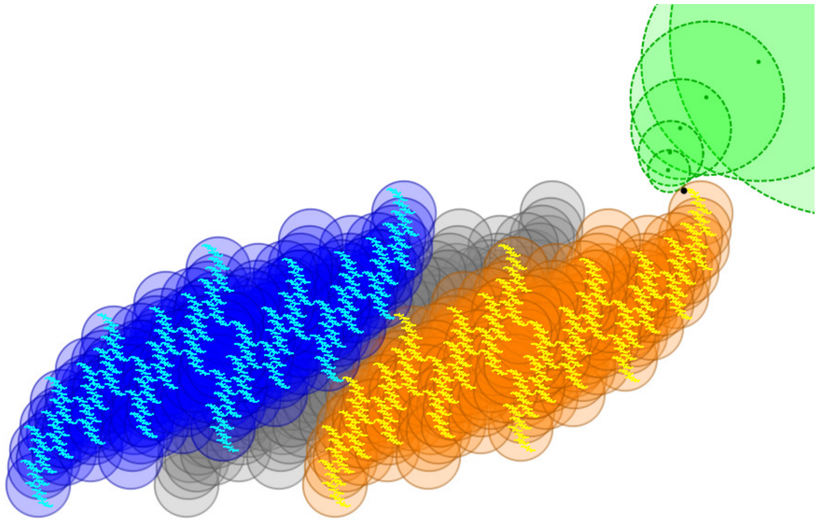

FIGURE 5. The attractor A_{λ_2} inside $\widetilde{\mathsf{I}}^5$. Compare description in Figure 4.

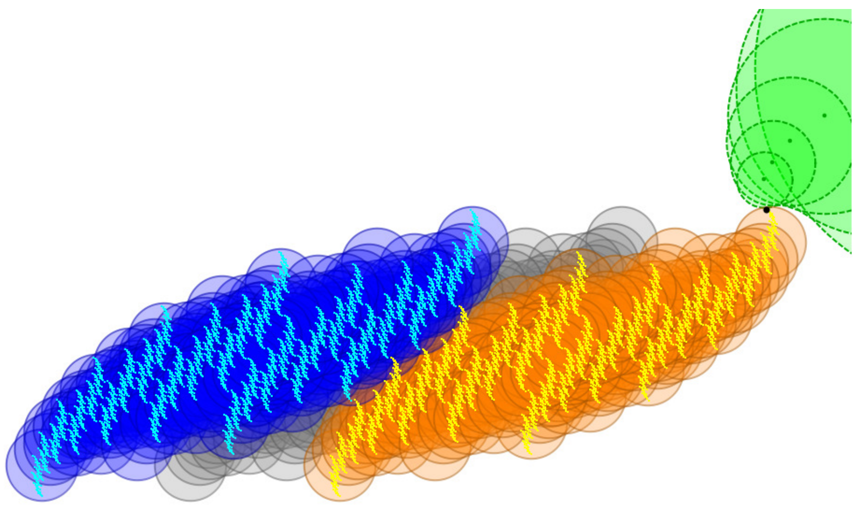

FIGURE 6. The attractor A_{λ_3} inside $\widetilde{\mathsf{I}}^5$. Compare description in Figure 4.

Example 5. Let $\mathsf{c} = c_0c_1\cdots = +(++-)^\infty \in \Sigma^\infty$ and consider the associated power series $f(z) := \sum_{k=0}^\infty c_k z^k = \dfrac{1+z+z^2-2z^3}{1-z^3}$. Solomyak proved in [Sol05] that[2] f is the unique power series for which $\lambda_5 \approx -0.366+0.520i$ is a root. Using the notation of Theorem 8 we have $\ell = 0$, $p = 3$, $\xi = 0$, $\xi_n = f_n(\lambda_5)$, and $\zeta = -\frac{1}{\lambda_5} \in \mathsf{A}_{\lambda_5} \subset \widetilde{\mathsf{A}}_{\lambda_5}$ with itinerary $\mathsf{b} = (++-)^\infty$.

We begin by proving that hypothesis **(i.)** of Theorem 8 is satisfied:

Lemma 6.2. *Let λ_5 and $f(z)$ be as above. Then for every $0 \le n \le p-1$*

$$2\,|f_n(\lambda_5)| > \frac{\left|\lambda_5^{n+1}\right|}{1-|\lambda_5|}.$$

[2]Solomyak considered the word $+(-+++--)^\infty$ which gives the power series $\frac{1-z+z^2+2z^3}{1+z^3}$; in other words, he considered $f(-z)$. The symmetry of $\mathcal{M}$ allows us to consider $f(z)$ instead.

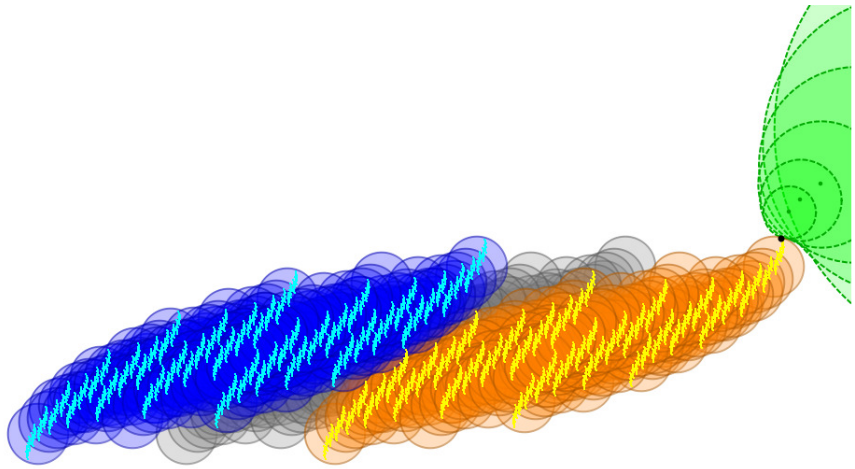

FIGURE 7. The attractor A_{λ_4} inside $\widetilde{\mathsf{I}}^5$. Compare description in Figure 4.

Proof. First notice that
(i.) $\frac{\sqrt{5}-1}{2} < |\lambda_5| < \frac{2}{3}$ and
(ii.) $\frac{2}{3}\pi < \arg(\lambda_5) < \frac{23}{32}\pi$.
Then, from (i.) we get

$$2\,|f_0(\lambda_5)| = 2\,|1| > \frac{|\lambda_5|}{1-|\lambda_5|}.$$

From (ii.) and the Law of cosines we get

$$\begin{aligned} 2\,|f_1(\lambda_5)| = 2\,|1+\lambda_5| &> 2\sqrt{1+|\lambda_5|^2 - 2\,|\lambda_5|\cos\left(\frac{9\pi}{32}\right)} \\ &> 2\sqrt{1+\frac{(\sqrt{5}-1)^2}{4} - (\sqrt{5}-1)\cos\left(\frac{9\pi}{32}\right)} \\ &> 2\,\frac{1}{\sqrt{2}} > 2\,|\lambda_5| > \frac{|\lambda_5^2|}{1-|\lambda_5|}. \end{aligned}$$

Since $f(\lambda_5) = 0$ then $f_2(\lambda_5) = 1 + \lambda_5 + \lambda_5^2 = 2\lambda_5^3$ and consequently

$$2\,|f_2(\lambda_5)| = 2\left|2\lambda_5^3\right| > \frac{|\lambda_5^3|}{1-|\lambda_5|}$$

because $4 > (1-|\lambda_5|)^{-1}$. □

Lemma 6.2 guarantees that the disks B_n in the chain exists for every $n \geq 0$. We can show now that $B_n \cap B_{n+1} \neq \varnothing$ and that $B_n \subset \mathbb{C} \setminus \widetilde{\mathsf{I}}^n$, namely the remaining two hypothesis of Theorem 8. Recall that each disk B_n is centered at ω_n and has radius r_n:

$$\omega_n = -\frac{1}{\lambda_5}(f_{n+1}(\lambda_5) + f_0(\lambda_5)), \qquad r_n = \frac{2}{|\lambda_5|}\,|f_{n+1}(\lambda_5)| - \frac{\left|\lambda_5^{n+1}\right|}{1-|\lambda_5|}$$

Lemma 6.3. *For $\lambda_5 \approx -0.366 + 0.520\mathrm{i}$, the set $\bigcup_{n\geq 0} B_n$ is connected and lies in the complement of $\widetilde{\mathsf{A}}_{\lambda_5}$.*

Proof. By the self-similarity of the attractor A_{λ_5} at ζ it is enough to prove that $\cup_{0\le n\le 2} B_n$ is connected and lies outside the instar $\widetilde{\mathsf{I}}^2$. We will first show that B_n lies outside the instar $\widetilde{\mathsf{I}}^n$ and then prove that $B_n \cap B_{n+1} \neq \varnothing$ for each $n = 0, 1, 2$.

$n = 0$: The instar $\widetilde{\mathsf{I}}^0$ is the union of the disks $\mathsf{D}^+, \mathsf{D}^0$, and D^-. We already know that B_0 is outside D^+, we want to show that the distances between ω_0 and the nodes $\nu_0 = 0$ and $\nu_- = -1$ is larger than the sum of the radii of B_0 and the nodal disk of level 0. In other words, we need that $r_0 + |\lambda_5|\,(1-|\lambda_5|)^{-1} < |\omega_0 - 0|$ and $r_0 + |\lambda_5|\,(1-|\lambda_5|)^{-1} < |\omega_0 - (-1)|$. Using that $2^{-1}(\sqrt{5}-1) < |\lambda_5| < 2/3$ and $\arg(\lambda_5) > 2\pi/3$, the Law of cosines tells us

$$|1+\lambda_5| < \sqrt{1+|\lambda_5|^2 - 2\,|\lambda_5|\cos\left(\frac{\pi}{3}\right)} < \sqrt{1+\frac{4}{9}-\frac{4}{3}\cos\left(\frac{\pi}{3}\right)} < 1$$

so

$$|\omega_0 - (-1)| = 2\left|\frac{1}{\lambda_5}\right| > 2\left|\frac{1+\lambda_5}{\lambda_5}\right| = r_0 + \frac{|\lambda_5|}{1-|\lambda_5|}.$$

Moreover, using the better estimate $\frac{11}{16}\pi < \arg(\lambda_5) < \frac{45}{64}\pi$,

$$2\,|1+\lambda_5| < 2\sqrt{1+\frac{4}{9}-\frac{4}{3}\cos\left(\frac{5\pi}{16}\right)} < \sqrt{2^2+\frac{4}{9}-2\,\frac{4}{3}\cos\left(\frac{19\pi}{64}\right)} < |2+\lambda_5|$$

implying that

$$|\omega_0 - 0| = \left|\frac{2+\lambda_5}{\lambda_5}\right| > 2\left|\frac{1+\lambda_5}{\lambda_5}\right| = r_0 + \frac{|\lambda_5|}{1-|\lambda_5|}.$$

$n = 1$: The instar $\widetilde{\mathsf{I}}^1$ is the union of the disks $\widetilde{\mathsf{D}}^{w_0+}, \widetilde{\mathsf{D}}^{w_0 0}$, and $\widetilde{\mathsf{D}}^{w_0-}$ where $w_0 \in \{+, 0, -\}$. Instead of checking that eight more inequalities are satisfied, we use the fact that $\arg(\lambda_5) \in (2\pi/3, 23\pi/32)$ to show $0 < \mathrm{Re}(\zeta_1) < \mathrm{Re}(\zeta)$. Given that $\zeta_1 = \frac{1}{\lambda_5}(f_{1+1}(\lambda_5) - f_0(\lambda_5)) = 1 + \lambda_5$ we need to check

$$-\frac{1}{|\lambda_5|}\cos(\arg(\lambda_5)) > 1 + |\lambda_5|\cos(\arg(\lambda_5)) \iff -\cos(\arg(\lambda_5)) > \frac{|\lambda_5|}{1+|\lambda_5^2|}$$

Indeed, $-\cos(\arg(\lambda_5)) \in (0.55, 0.64)$ and $\frac{|\lambda_5|}{1+|\lambda_5^2|} \in (0.44, 0.47)$. Moreover, we have $\mathrm{Re}(\zeta_1) > 0$ because $\cos(\arg(\lambda_5)) > -3/2 > -|\lambda_5|^{-1}$. Consequently, ω_1 is in the first quadrant with $\mathrm{Re}(\omega_1) > \mathrm{Re}(\zeta_1)$. It follows that B_1 has a chance to intersect only the disks $\widetilde{\mathsf{D}}^{+0}$ and $\widetilde{\mathsf{D}}^{+-}$. However, in Lemma 6.2 we showed that $|1+\lambda_5| > 2^{-1/2}$, therefore, since $2\,|\lambda_5^3| < 2^{-1/2}$ we have

$$|\omega_1 - (1-\lambda_5)| \;=\; 2\left|\frac{1+\lambda_5}{\lambda_5}\right| > 4\,|\lambda_5^2| = r_1 + \frac{|\lambda_5^2|}{1-|\lambda_5|}.$$

so $B_1 \cap \widetilde{\mathsf{D}}^{+-} = \varnothing$.

Finally, B_1 does not intersect $\widetilde{\mathsf{D}}^{+0}$ because

$$\begin{aligned} |\omega_1 - (1+0\cdot\lambda_5)| &= \left|\frac{-\lambda_5^2+4\lambda_5^3}{\lambda_5}\right| = |\lambda_5|\,|1-4\lambda_5| > |\lambda_5|\,\big(1+4\,|\lambda_5|\cos(\pi/4)\big) \\ &> |\lambda_5|\,\frac{8}{3} > 4\,|\lambda_5^2| = r_1 + \frac{|\lambda_5^2|}{1-|\lambda_5|}. \end{aligned}$$

In the first equality we used the fact that $1+\lambda_5 = -\lambda_5^2 + 2\lambda_5^3$ and in the second line that $1+\sqrt{2}\big(\sqrt{5}-1\big) > 8/3 > 4\,|\lambda_5|$.

$n=2$: In this case we prove that $B_2 \subset B_1$ which implies that B_2 is outside $\widetilde{\mathsf{I}}^2$ because of the containment $\widetilde{\mathsf{I}}^2 \subset \widetilde{\mathsf{I}}^1$. Firstly, we have $|\omega_1 - \omega_2| = |\lambda_5^2|$ and that the radius of B_1 is $r_1 = 4\,|\lambda_5^2| - \frac{|\lambda_5^2|}{1-|\lambda_5|} > |\lambda_5^2|$ because $3 > (1-|\lambda_5|)^{-1}$. Secondly, the radius of B_2 is $r_2 = 2\,|\lambda_5^2| - \frac{|\lambda_5^3|}{1-|\lambda_5|}$, so for the containment to hold we must have

$$4\,|\lambda_5^2| - \frac{|\lambda_5^2|}{1-|\lambda_5|} > 2\left(2\,|\lambda_5^2| - \frac{|\lambda_5^3|}{1-|\lambda_5|}\right),$$

which holds since $2\,|\lambda_5| > 1$.

We have shown that B_0, B_1, and B_2 do not intersect their respective instar and, therefore, their union lies outside the instar $\widetilde{\mathsf{I}}^2 \supset \widetilde{\mathsf{A}}_{\lambda_5}$. We have also proved that $B_2 \subset B_1$ so it remains to show that $B_1 \cap B_0 \neq \varnothing$ and $B_2 \cap B_3 \neq \varnothing$.

We will prove that $\omega_1 \in B_0$, i.e. $|\omega_0 - \omega_1| = |\lambda_5| < r_0$: recall that $r_0 = 2\left|\frac{1+\lambda_5}{\lambda_5}\right| - \frac{|\lambda_5|}{1-|\lambda_5|}$ and $0.63 < |\lambda_5| < 0.64$ then

$$2\,|1-2\lambda_5| > 2\big(1+2(0.63)\cos(\pi/4)\big) > \frac{1}{1-(0.64)} + 1 > \frac{1}{1-|\lambda_5|} + 1$$

which implies

$$2\left|\frac{1+\lambda_5}{\lambda_5}\right| = 2\left|\frac{-\lambda_5^2+2\lambda_5^3}{\lambda_5}\right| = 2\,|\lambda_5|\,|1-2\lambda_5| > |\lambda_5|\left(\frac{1}{1-|\lambda_5|}+1\right).$$

Finally, we want to show that $|\omega_2 - \omega_3|\,|\lambda_5^3| = |\lambda_5^3| < r_2 + r_3$. By definition $r_3 = 2\,|\lambda_5^2|\,|1+\lambda_5| - \frac{|\lambda_5^4|}{1-|\lambda_5|}$ and since

$$\begin{aligned} |\lambda_5| + \frac{|\lambda_5^2|}{1-|\lambda_5|} < 0.639 + \frac{(0.639)^2}{1-0.639} \quad &< \quad 1 + \sqrt{1 + \frac{(\sqrt{5}-1)^2}{4} - (\sqrt{5}-1)\cos\left(\frac{9\pi}{32}\right)} \\ &< \quad 1 + |1+\lambda_5| \end{aligned}$$

then

$$\begin{aligned} |\lambda_5| \quad &< \quad 1 + |1+\lambda_5| - \frac{|\lambda_5^2|}{1-|\lambda_5|} \\ \iff |\lambda_5| \quad &< \quad 2 + 2\,|1+\lambda_5| - |\lambda_5| - \frac{2\,|\lambda_5^2|}{1-|\lambda_5|} \\ &= \quad 2 - \frac{|\lambda_5|}{1-|\lambda_5|} + 2\,|1+\lambda_5| - \frac{|\lambda_5^2|}{1-|\lambda_5|} \\ \iff \left|\lambda_5^3\right| \quad &< \quad r_2 + r_3. \end{aligned}$$

The equality in the third line holds because $\frac{|\lambda_5|}{1-|\lambda_5|} = |\lambda_5| + \frac{|\lambda_5^2|}{1-|\lambda_5|}$.

□

By Lemma 6.3 the chain $\bigcup_{n\geq 0} B_n$ is connected and lies in the complement of $\widetilde{\mathsf{A}}_{\lambda_5}$ (see Figure 8). Then, by Theorem 8, $\lambda_5 \in \partial\mathcal{M}$ is an accessible point. Because f is unique and has no zero coefficients, by Corollary 9, λ_5 is an accessible point of $\partial\mathcal{M} \cap \partial\mathcal{M}_0$.

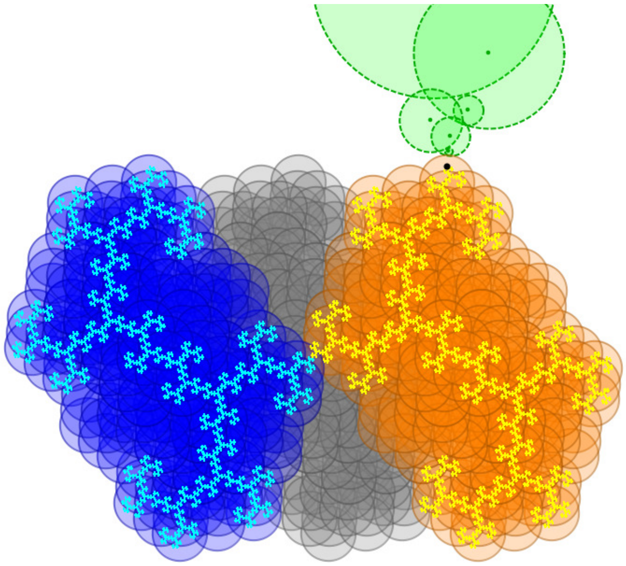

FIGURE 8. The attractor A_{λ_5} inside $\widetilde{\mathsf{I}}^5$. Compare description in Figure 4.

It must be noted that the uncountable set $\mathcal{T}$ of Theorem 4 contains $\lambda_5 \approx -0.366 + 0.520i$. In fact, such set was found by perturbing the number of repeating coefficients in the power series f and by allowing that some of the perturbed repeating coefficients to be 0. The method Bandt and Hung used to prove that no other power series $g \in \mathcal{P}$ has a root in $\mathcal{T}$, entailed finding a uniform lower bound in a neighborhood of $\mathcal{T}$ on the normalized difference

$$\frac{g(z) - h(z)}{z^k} = \sum_{j=0}^{\infty} \epsilon_j z^j, \quad \epsilon_j \in \{-2, -1, 0, +1, +2\}; \ \epsilon_0 \neq 0$$

where $k \geq 1$ and $h(z)$ a power series with a root in $\mathcal{T}$. It is unclear whether all parameters in $\mathcal{T}$ are accessible, since the associated itineraries are not necessarily preperiodic and hence our theorems do not apply.

Remark 6.1. *We want to emphasize that the assumptions of Theorem 8 are not meant to be necessary for accessibility . The landmark point* $\lambda_6 \approx 0.57395 + 0.368989\mathrm{i}$ *satisfies* $\mathcal{F}_\lambda = \{1 - z + z^3/(1-z)\}$. *Based on computer pictures, Bandt [Ban02] describes* λ_6 *as the "tip of the largest (visible) spiral of* $\mathcal{M}$*". It is likely that* λ_6 *is accessible, but even though it is possible to prove that the disks of the chain exist, they are all disconnected and intersect the instar at every level.*

References

[Ban02] Christoph Bandt. On the Mandelbrot set for pairs of linear maps. *Nonlinearity*, 15(4):1127–1147, 2002.

[BH85] Michael F Barnsley and Andrew N Harrington. A Mandelbrot set for pairs of linear maps. *Physica D: Nonlinear Phenomena*, 15(3):421–432, 1985.

[BH08] Christoph Bandt and Nguyen Hung. Self-similar sets with an open set condition and great variety of overlaps. *Proceedings of the American Mathematical Society*, 136(11):3895–3903, 2008.

[Bou88] Thierry Bousch. Paires de similitudes, 1988.

[Bou92] Thierry Bousch. Connexité locale et par chemins hölderiens pour les systemes itérés de fonctions, 1992.

[BR07] Christoph Bandt and Hui Rao. Topology and separation of self-similar fractals in the plane. *Nonlinearity*, 20(6):1463–1474, 2007.

[CKW17] Danny Calegari, Sarah Koch, and Alden Walker. Roots, Schottky semigroups, and a proof of Bandt's conjecture. *Ergodic Theory and Dynamical Systems*, 37(8):2487–2555, 2017.

[ERS10] Kemal Ilgar Eroğlu, Steffen Rohde, and Boris Solomyak. Quasisymmetric conjugacy between quadratic dynamics and iterated function systems. *Ergodic Theory and Dynamical Systems*, 30(6):1665–1684, 2010.

[HP12] Peter Haïssinsky and Kevin M. Pilgrim. Examples of coarse expanding conformal maps. *Discrete Contin. Dyn. Syst.*, 32(7):2403–2416, 2012.

[Lei90] Tan Lei. Similarity between the Mandelbrot set and Julia sets. *Communications in mathematical physics*, 134(3):587–617, 1990.

[Sol05] Boris Solomyak. On the 'Mandelbrot set' for pairs of linear maps: asymptotic self-similarity. *Nonlinearity*, 18(5):1927–1943, 2005.

[Thu14] William Thurston. Entropy in dimension one. *Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday*, pages 339–384, 02 2014.

[Tio13] Giulio Tiozzo. Galois conjugates of entropies of real unimodal maps. *International Mathematics Research Notices*, 2013.

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