Self-Organized Critical neural networks

Simone Gasperini
Physics and Astronomy Department, UNIBO - University of Bologna

Contents

1. Introduction

Information processing by a network of dynamical elements is a delicate matter: avalanches of activity can die out very fast if the network is not connected enough or if the elements are not sensitive enough; on the other hand, activity avalanches can grow and spread over the entire network and override information processing. Therefore, neural networks have to establish and maintain a certain intermediate level of activity, operating close to a dynamical phase transition that is naturally located between ordered and chaotic dynamics.

In the wake of self-organized criticality (SOC), it was asked if also neural systems can exhibit some form of self-organized behaviour towards criticality. In addition, actual observations of neural oscillations within the human brain were related to a possible SOC phenomenon.

Early examples of theoretical models for self-organized critical neural networks were able to exhibit self-regulation towards a critical system state, via simple local rewiring rules which are plausible in the biological context. After this first model approach, indeed strong experimental evidence for criticality in neural systems has been found in terms of spatio-temporal activity. These findings sparked intense research on dynamical models for criticality: while most models emphasized biological and neurophysiological details, the aim of this work is to revisit the thread of minimal SOC neural networks. The models described in this review were formulated looking for the simplest possible system able to dynamically adapt to perturbations and autonomously reach a stationary steady-state without external parameters tuning. Finally, in the last part of the work, the latest and more complete version of this class of neural network models is implemented to show and discuss some simulations results.

2. Models review

The experimental findings sparked a large number of theoretical studies about criticality and self-organization in neural networks, ranging from very simple toy models to detailed representations of biological functions.

Early works as Bornholdt and Rohlf (2000) [1] and Bornholdt and Röhl (2003) [2] were focused on simple mechanisms for self-organized critical dynamics in spin networks. These models aimed to formulate the simplest system as possible, quite similar to the universality viewpoint of statistical mechanics, rather than faithful representations of neurobiological and biochemical details. Nevertheless they are able to self-regulate towards and maintain a critical system state, manifested in features as a certain limit cycle scaling behavior. These models provide some of the groundwork for current more advanced versions. It is notable that they came up even before experimental evidence was found for the existence of neuronal avalanches by Beggs [3].

Understandably, extensive studies have been done on criticality in neural systems following his discovery. Most of them have their mechanisms of self-organization motivated by brain plasticity rules (short/long-term synaptic plasticity, spike timing dependent plasticity, Hebbian or anti-Hebbian like rules).

While the proposed organization mechanisms strongly differ between the individual models, the resulting evolved networks tend to be part of only a few fundamental universality classes, exhibiting statistical properties in a similar way as in the experimental data.

2.1 Random threshold spin network

The first very minimal model, by Bornholdt and Rohlf (2000) [1], consists in a random threshold network driven by a simple local mechanism for topology evolution. This mechanism is based on nodes average activity and it is capable of driving the network towards a critical state.

Consider a network composed of N randomly and asymmetrically connected spins $\sigma_i = \pm 1$, which are updated synchronously in discrete time steps via a threshold function of the inputs they receive:

$$\sigma_i(t+1) = \begin{cases} +1, & f_i(t) \geq 0 \\ -1, & f_i(t) < 0 \end{cases} \qquad (1)$$

where $f_i(t)$ is the incoming signal of node i

$$f_i(t) = \sum_{j=1}^{N} c_{ij}\sigma_j(t) + \theta_i \tag{2}$$

The link weights have discrete values $c_{ij} = \pm 1$ (or $c_{ij} = 0$ if nodes i and j are not connected). In the minimal model, activation thresholds are set to $\theta_i = 0$ for all nodes.
A network run is started with random initial configuration and iterated until a dynamics attractor is reached. The attractor of a network is where its dynamics ends up after a while: it can be either a fixed point (all nodes reach a constant state) or a limit cycle (finite cyclic sequence of active nodes).
For the network topology evolution, a random node i is selected and its activity during the attractor period is analyzed. In particular, the network is observed until such an attractor is reached and afterwards, the single node averaged activity A_i over the attractor period from T_1 to T_2 is measured as

$$A_i = \frac{1}{T_2 - T_1} \sum_{t=T_1}^{T_2-1} \sigma_i(t) \tag{3}$$

Then, if $A_i = \pm 1$, node i receives a new in-link of random weight $c_{ij} = \pm 1$ from another random node j, otherwise (if the state of node i changes at least once during the attractor period), one of the existing in-links is set to zero.
The underlying principle, which facilitates self-organization, is based on the average activity of a node being closely connected to the frozen component of the network (the fraction of nodes which do not change their state along the attractor) which also undergoes a transition from a large to a vanishing frozen component at the critical connectivity. At low network connectivity, a large fraction of nodes will likely be frozen, and thus receive new in-links once they are selected for rewiring. On the other hand, at high connectivity, nodes will often change their state and lose in-links in the rewiring process. With a finite size scaling of the transition connectivities at the respective network sizes, it was shown that for $N \to \infty$, the transition occurs at the critical value of $\langle K_c \rangle \simeq 2$.

2.2 Spin network on 2D lattice

The model described in this section was originally proposed by Bornholdt and Röhl (2003) [2] and it captures self-organized critical behavior on a two-dimensional spin lattice. In contrast to the activity-regulated model previously discussed, this new version has the following features:

- the network topology is constrained to a squared lattice
- small thermal noise is added to the system
- link weights c_{ij} take continuous values in $[-1, +1]$
- activation thresholds θ_i may vary from 0
- the rewiring criterion is based on nodes i and j activity correlation Corr_{ij}

The model consists in a randomly and asymmetrically connected network of N spins, where links can only be established among the eight local neighbors of any lattice site. The continuous link weights c_{ij} can be activating or inhibiting and are chosen randomly from a uniform distribution $c_{ij} \in [-1, +1]$. All nodes are updated synchronously via a simple threshold function of their input signals from the previous time step:

$$\mathrm{Prob}[\sigma_i(t+1) = +1] = \frac{1}{1 + e^{-2\beta f_i(t)}} \tag{4}$$

where β is the inverse temperature and the signal $f_i(t)$ is defined as in the previously described model. The thresholds are here chosen as $\theta_i = -0.1 + \gamma$ where γ is a small random Gaussian noise of width ε.
On a larger time scale, the network topology is changed by a slow local rewiring mechanism according to the following principle: if the dynamics of two neighboring nodes is highly correlated or anti-correlated, they get a new link between them. Otherwise, if their activity shows low correlation, any present link between them is removed (similar to Hebbian learning rule). In this model, the correlation $\mathrm{Corr}_{ij}(\tau)$ of nodes i and j over a time interval τ is defined as

$$\mathrm{Corr}_{ij} = \frac{1}{\tau} \sum_{t=t_0}^{t_0+\tau-1} \sigma_i(t)\sigma_j(t) \tag{5}$$

choosing t_0 to exclude a possible transient period following the previous topology modification. If $\mathrm{Corr}_{ij}(\tau)$ is larger than a given threshold α, a new link from node j to i is inserted with random weight $c_{ij} \in [-1, +1]$. If else $\mathrm{Corr}_{ij}(\tau) < \alpha$ the weight c_{ij} is set to 0. Independently from the initial average connectivity $\langle K \rangle^{ini}$, one finds a slow convergence towards a specific critical value $\langle K_c \rangle$, which is characteristic for the respective network size N. It is also possible to measure that in the thermodynamics limit:

$$\lim_{N\to\infty} \langle K_c \rangle = 2.24 \pm 0.03 \tag{6}$$

This result is in agreement with the one of the previously described model. In addition, it was shown that the proposed adaptation mechanism works robustly in a wide range of thermal noise β, and that the specific choice of the correlation threshold α plays a minor role regarding the final critical state.

2.3 Random boolean network - 1

In the above sections, it was demonstrated how basic toy models can reproduce some of the observations made in real neuronal systems, even neglecting a lot of details. To move these models a step closer towards biological plausibility and, at the same time, construct an even simpler system, a transition to boolean state nodes ("on" : $\sigma = 1$, "off" : $\sigma = 0$) is needed, as done by Rybarsch and Bornholdt (2014) [4].
Note that in the previous spin versions a node with negative spin state $\sigma_j = -1$ transmits non-zero signals through its outgoing weights c_{ij}. In those systems, such signals arrive at

target nodes i, providing their contribution to the sum f_i.
However, biological nodes (genes or neurons) usually do not transmit signals when inactive. Also in other contexts like biochemical network models, each node represents whether a specific chemical component is present ($=1$) or absent ($=0$).
The second aspect to be reviewed is the network topology the algorithm operates on. While original correlation-based rewiring mechanism was defined to simply operate on neighboring nodes on a 2D lattice, here the model is studied as an arbitrary self-organizing network, without specifying any underlying topology.
However, while on a lattice the number of possible neighbors of a node is strictly limited, on a large random network near critical connectivity there are far more unconnected pairs of nodes than connected ones. Thus, randomly selecting pairs of nodes for rewiring would introduce a strong bias towards connecting nodes which were previously unconnected. This bias would result in a strong increase of connectivity, far above any self-organized critical regime. Consequently, an adaptation of model definition is needed to include an arbitrary topology.
First, the nodes states update described by the function in Eq.4 has to be modified to take into account the shift due to the transition from spins to boolean states:

$$\text{Prob}[\sigma_i(t+1)=1] = \frac{1}{1+e^{-2\beta(f_i(t)-0.5)}} \tag{7}$$

The correlation-based rewiring mechanism has to be carefully revised as well, as the calculation of correlations is affected by the representation of inactive nodes by a value of 0 instead of -1. The adjusted algorithm operates as follows: after initializing the network with random links at a given initial average connectivity $\langle K\rangle^{ini}$ all nodes are synchronously updated in parallel according to Eq.7.
On a larger time scale, after τ updates, a rewiring is introduced at one randomly chosen, single node. The new element in this revised model is to test whether the addition or removal of one random in-link at the selected node will increase the average dynamical correlation to all existing inputs of that node. In this case, the dynamical correlation between a node i and one of its inputs j over the preceding τ time steps is represented by the Pearson correlation coefficient:

$$\text{Corr}_{ij} = \frac{\langle\sigma_i(t+1)\sigma_j(t)\rangle - \langle\sigma_i(t+1)\rangle\langle\sigma_j(t)\rangle}{S_i S_j} \tag{8}$$

where S_i and S_j denote the standard deviations of the states of nodes i and j in the time window τ. In case one or both nodes remain frozen in their state (yield a standard deviation of 0), zero correlation is assumed as the Pearson correlation coefficient would not be well defined. Note that, for node i, the state occurring one time step later than j is used, taking into account the signal transmission from one node to the next.
Finally, the average input correlation is defined as

$$\langle\text{Corr}_i\rangle = \frac{1}{k_i}\sum_{j=0}^{N}|c_{ij}|\,\text{Corr}_{ij} \tag{9}$$

where k_i is the in-degree of node i. The factor c_{ij} ensures that correlations are only measured where links are present between the nodes. If node i has no in-links ($k_i=0$) then $\langle\text{Corr}_i\rangle = 0$. In detail, the adaptive rewiring is now performed in the following steps:

1. select a random node i at which the next rewiring will take place
2. run network updates for τ simulation time steps and compute $\langle\text{Corr}_i\rangle$
3. with equal probability, either insert an additional in-link of random weight $c_{ij}=\pm1$ at a previously unlinked node j, or remove one of the existing in-links at node i
4. run again τ states updates and measure the new $\langle\text{Corr}_i\rangle$ after the local rewiring at this node
5. if $\langle\text{Corr}_i\rangle$ has increased after the insertion or removal of the in-link, the rewiring from step 3 is retained, otherwise, it is reverted

Note that the exact choice of τ is not critical, but it is done to ensure a minimum time scale separation between nodes dynamics (fast) and rewiring (slow).
It is also worth noting that even this updated model is solely based on locally available information at synapse level and takes into account both pre- and post-synaptic activity.
Even for this version, the self-organization behavior is independent from the initial conditions and robust to thermal noise. The average degree evolves towards a value $\langle K_c\rangle$ slightly below 2, which is near the expected critical connectivity.

2.4 Random boolean network - 2

The more recent and complete version of this class of self-organized neural networks is the one proposed by Landmann, Baumgarten, and Bornholdt (2021) [5]. This model is here described from a theoretical point of view, then, in the next section, simulations results will be shown and discussed.
The correlation-oriented mechanism presented in the section 2.3 was able to robustly drive the system towards criticality. Nevertheless, its algorithmic implementation is not fully autonomous: its adaptation rule still uses data from different simulation runs in order to determine the synaptic change to be performed. Therefore, an alternative fully autonomous implementation is needed, keeping nodes dynamics based solely on local information together with an arbitrary network topology and a certain degree of biological plausibility. This model attempts to achieve these goals formulating the simplest rules for the activity and network dynamics as possible, with the lowest number of free parameters.
The model consists in a random boolean network where the nodes states updating works exactly as in the previous system. The rewiring algorithm, instead, is based on a running average activity $A_i(t)$ which is updated for each node i at each time step t according to the equation:

$$A_i(t+1) = \alpha\cdot A_i(t) + (1-\alpha)\cdot\sigma_i(t) \tag{10}$$

where the parameter $\alpha \in [0,1]$ determines the temporal memory of the nodes. Note that as α approaches 1, the average activity of node i at time $t+1$ will depend more and more on the average activity itself up to time t, while as α goes to 0 the system will lose its memory and $A_i(t+1)$ will be close the last state $\sigma_i(t)$.
On a well-separated timescale (depending again on the model parameter τ) with respect to the fast nodes activity dynamics, a single node is chosen at random and its connectivity is changed on the basis of following rules (where $\varepsilon \ll 1$):

- if $A_i(t) < \varepsilon$, a new incoming link $c_{ij} = 1$ from another randomly chosen node j is added
- if $A_i(t) > (1-\varepsilon)$, a new incoming link $c_{ij} = -1$ from another randomly chosen node j is added
- otherwise, a randomly chosen existing link to node i is removed.

From these simple rules follows that non-switching nodes receive new links, while switching nodes lose links. This behaviour prevents the system from reaching both an ordered phase where all nodes are permanently frozen, as well as a chaotic regime with abundant switching activity. In particular, the system is driven towards a dynamical phase transition between a globally ordered and a globally chaotic phase.
It's also worth noting that the sign of the added link is determined by the nature of the frozen state ("on" or "off"), unlike the SOC networks described so far where the sign of new links had been chosen randomly.

3. Results and discussion

The final goal of this work was to implement the latest version (described in section 2.4) of this SOC model and to reproduce the expected behaviour of such kind of boolean neural network by the analysis of some simulations.

3.1 Network evolution

The first study consists in observing the evolution of an initially fully unconnected network ($c_{ij}^{ini} = 0 \;\; \forall i, j$) with all the nodes in a resting state ($\sigma_i^{ini} = 0 \;\; \forall i$).
As qualitative observables of its dynamical state, the average in-degrees $\langle K_+ \rangle$, $\langle K_- \rangle$ are tracked. Moreover, the global branching parameter $\langle \lambda \rangle$ (indicator about how activity propagates through the network) is measured as the number of active nodes at time step $t+1$ over the same quantity at the previous time step t and taking its average over the window spanned by τ (number of steps separating consecutive rewiring operations), namely:

$$\langle \lambda \rangle = \left\langle \frac{\sum_i \sigma_i(t+1)}{\sum_i \sigma_i(t)} \right\rangle_{t \in \tau} \tag{11}$$

There are three general regimes for $\langle \lambda \rangle$, as shown in Figure 1.

When the branching parameter is less than unity, the system is subcritical and activity dies out over time; when it is greater than unity, the system is supercritical and activity increases; finally, $\langle \lambda \rangle = 1$ is a good indicator of system criticality. Also note that when all the nodes are inactive, the denominator in Eq.11 is 0 and so $\langle \lambda \rangle$ is not well defined.

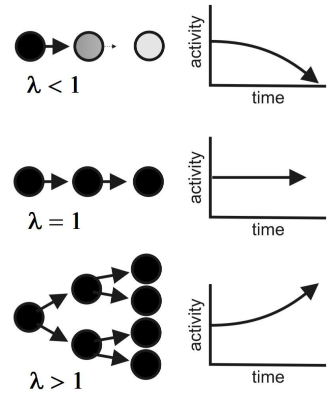

Figure 1. Role of the branching parameter at different regimes.

In Figure 2, the average inconnectivities and branching parameter time series are shown.
At the beginning, since the initial states are all set to 0, the activity can only be induced by noise and, due to the presence of very few links, it dies out very fast. Therefore, the activity is very low and only activating links are added. When the value of $\langle K_+ \rangle$ approaches 1, activity starts to propagate through the network and some nodes become permanently active. This causes the rewiring algorithm to insert inhibiting links, contributing to $\langle K_- \rangle$. After some transient time, the average connectivities become stationary and fluctuate around a mean value.

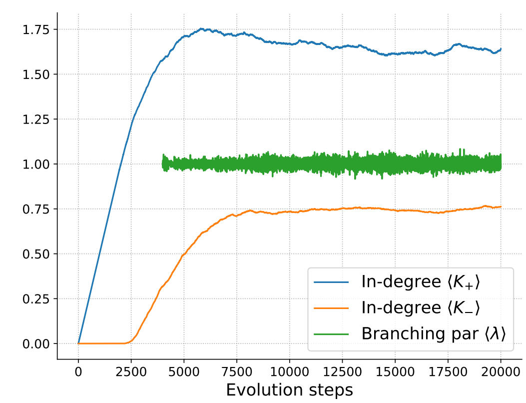

Figure 2. Evolution of the average in-degrees and branching parameter for a system with $N = 2000$, $\alpha = 0.2$, $\beta = 10$, $\tau = 10$, starting from a completely unconnected network. The branching parameter is plotted only for the critical regime due to its wide oscillations in the transient time.

The branching parameter also fluctuates around the critical value of 1 indicating a possible critical behavior.
It is also possible to show that the total in-degree $K = K_+ + K_-$ in the evolved stationary state is Poisson-distributed, as depicted in Figure 3, and its average value is around 2.4.

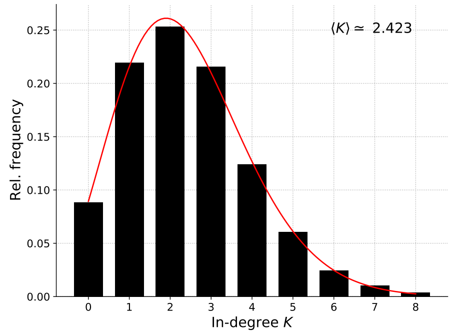

Figure 3. In-degree K distribution and Poisson fit (in red).

3.2 Criticality robustness

The critical behaviour of neural network models presented in this work is independent from the system initial conditions.

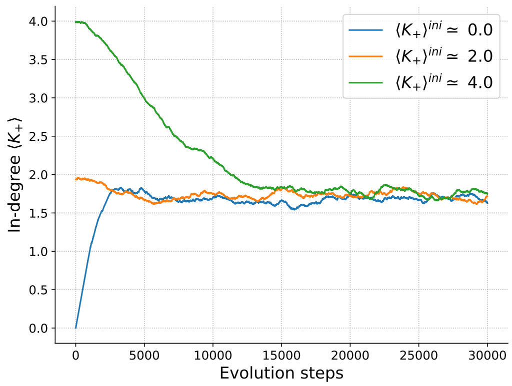

Figure 4. Average activating in-degree $\langle K_+ \rangle$ time series for different initial conditions. Here, the system parameters are $N = 1000$, $\alpha = 0.2$, $\beta = 10$, $\tau = 10$.

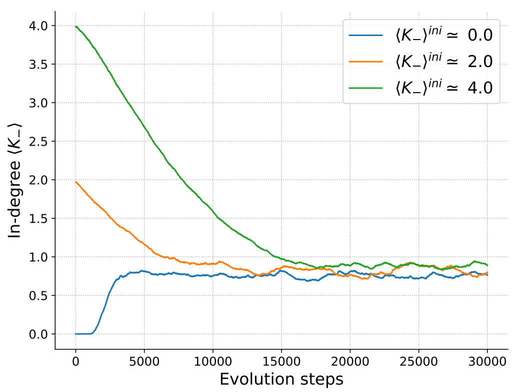

Figure 5. Average inhibiting in-degree $\langle K_- \rangle$ time series for different initial conditions. Here, the system parameters are $N = 1000$, $\alpha = 0.2$, $\beta = 10$, $\tau = 10$.

In Figure 4 and 5, $\langle K_+ \rangle$ and $\langle K_- \rangle$ time series are respectively shown starting from different initial average connectivities. The networks are left free to evolve and the rewiring algorithm is able to make the final connectivities slowly but robustly converge to the same values. Note that this happens both starting with the network in a subcritical regime (for $\langle K_\pm \rangle^{ini} = 0$) and in a supercritical regime (for $\langle K_\pm \rangle^{ini} = 4$). In the latter case the network starts its evolution in a densely connected configuration and the nodes change their states often. Since they rarely stay in the same state, links are preferentially deleted in the beginning but, after a transient period, the system finally reaches the stationary steady state.

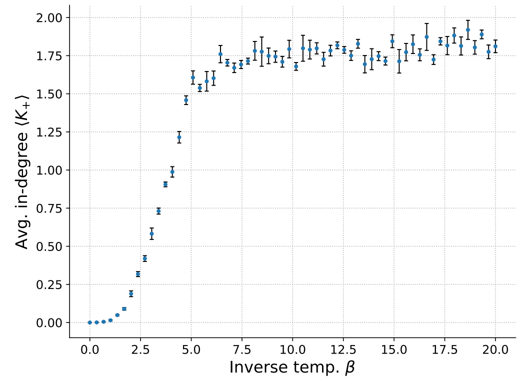

Figure 6. Evolved connectivity $\langle K_+ \rangle$ as a function of the inverse temperature β. Each point is averaged over 1000 time steps in a network of size $N = 400$, $\alpha = 0.2$, $\tau = 10$.

.

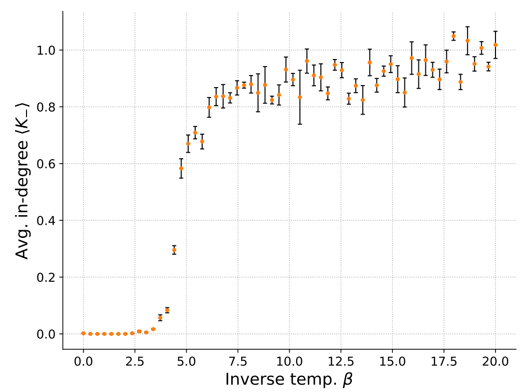

Figure 7. Evolved connectivity $\langle K_- \rangle$ as a function of the inverse temperature β. Each point is averaged over 1000 time steps in a network of size $N = 400$, $\alpha = 0.2$, $\tau = 10$.

As already mentioned, the self-organization behaviour towards a specific average connectivity is also largely insensitive to thermal noise of the network dynamics. This indicates that the structure of a given dynamical attractor is robust against a large degree of noise.
Figure 6 and 7 show the evolved average connectivities as a function of the inverse temperature β. It's interesting to note how this plot exhibits the typical order parameter shape highlighting a system phase transition for a temperature value which corresponds to about $\beta = 5$.
For values above this threshold, the system always reaches the same stationary state, characterised by the same averaged in-degrees.

4. Conclusion

In this work, very minimalistic binary neural network models have been shown, which are capable of self-organized critical

behavior.
While older models show some drawbacks regarding biological plausibility originating from their nature as spin networks, it is possible to perform a transition to a self-organized critical, randomly wired network of boolean states with emerging dynamical patterns, as observed in real neuronal systems. This is possible via a simple, locally realizable, and completely autonomous rewiring mechanism which uses a running average activity as its regulation criterion.
The model involves only four parameters, the system size N, the inverse temperature β, determining the amount of noise in the model, the neurons temporal memory α and the network evolution timescale τ, representing the separation between fast neurons dynamics and slow change in the network topology. None of them needs a fine tuning and they can be varied over a considerable range.
While it is obvious that there are far more details involved in self-organization of real neuronal networks (some of which are reflected in other existing models), even in this minimal formulation is possible to observe a fundamental organization mechanism leading to a critical system that exhibits common statistical properties, regardless of the details of a specific biological implementation.

Supplementary material

A Python implementation of the model described in section 2.4 is fully available here: SOCmodel GitHub repository.
The code relies on *Numpy* and *Scipy* libraries for numerical computation as well as on *Numba* library for a ”just-in-time” compilation, ensuring good performances in the simulations execution.
For an example about how to create an instance of the model and run a simulation see the *Jupyter Notebook* here: SOCmodel example.

References

[1] Stefan Bornholdt and Thimo Rohlf. Topological evolution of dynamical networks: Global criticality from local dynamics. *Phys. Rev. Lett.*, Jun 2000.

[2] Stefan Bornholdt and Torsten Röhl. Self-organized critical neural networks. *Phys. Rev. E*, Jun 2003.

[3] J. M. Beggs. Neuronal avalanche. *Scholarpedia*, 2007.

[4] Matthias Rybarsch and Stefan Bornholdt. Avalanches in self-organized critical neural networks: A minimal model for the neural soc universality class. *Plos One*, Apr 2014.

[5] Stefan Landmann, Lorenz Baumgarten, and Stefan Bornholdt. Self-organized criticality in neural networks from activity-based rewiring. *Physical Review E*, Mar 2021.