Linear system of equations

1. [Programming] Use Gauss Elimination method to solve the following linear system of equations:

$$\begin{aligned}
x_1 - x_2 + 2x_3 - x_4 &= -8, \\
2x_1 - 2x_2 + 3x_3 - 3x_4 &= -20, \\
x_1 + x_2 + x_3 &= -2, \\
x_1 - x_2 + 4x_3 + 3x_4 &= 4.
\end{aligned}$$

$$\begin{aligned}
2x_1 + x_2 - x_3 + x_4 - 3x_5 &= 7, \\
x_1 + 2x_3 - x_4 + x_5 &= 2, \\
-2x_2 - x_3 + x_4 - x_5 &= -5, \\
3x_1 + x_2 - 4x_3 + 5x_5 &= 6, \\
x_1 - x_2 - x_3 - x_4 + x_5 &= 3.
\end{aligned}$$

2.

Given the linear system

$$\begin{aligned}
2x_1 - 6\alpha x_2 &= 3, \\
3\alpha x_1 - x_2 &= \tfrac{3}{2}.
\end{aligned}$$

a. Find value(s) of α for which the system has no solutions.

b. Find value(s) of α for which the system has an infinite number of solutions.

c. Assuming a unique solution exists for a given α, find the solution.

3. [Programming] (i)

Use Gaussian elimination and three-digit chopping arithmetic to solve the following linear systems, and compare the approximations to the actual solution.

a. $58.9x_1 + 0.03x_2 = 59.2,$
$-6.10x_1 + 5.31x_2 = 47.0.$

Actual solution $[1, 10]$.

b. $3.3330x_1 + 15920x_2 + 10.333x_3 = 7953,$
$2.2220x_1 + 16.710x_2 + 9.6120x_3 = 0.965,$
$-1.5611x_1 + 5.1792x_2 - 1.6855x_3 = 2.714.$

Actual solution $[1, 0.5, -1]$.

(ii) Repeat the above exercise using three digit rounding arithmatic.

(iii) Repeat the above exercise using GE with partial pivoting

4.

Suppose that in a biological system there are n species of animals and m sources of food. Let x_j represent the population of the jth species, for each $j = 1, \ldots, n$; b_i represent the available daily supply of the ith food; and a_{ij} represent the amount of the ith food consumed on the average by a member of the jth species. The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

represents an equilibrium where there is a daily supply of food to precisely meet the average daily consumption of each species.

a. Let

$$A = [a_{ij}] = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$\mathbf{x} = (x_j) = [1000, 500, 350, 400]$, and $\mathbf{b} = (b_i) = [3500, 2700, 900]$. Is there sufficient food to satisfy the average daily consumption?

b. What is the maximum number of animals of each species that could be individually added to the system with the supply of food still meeting the consumption?

c. If species 1 became extinct, how much of an individual increase of each of the remaining species could be supported?

d. If species 2 became extinct, how much of an individual increase of each of the remaining species could be supported?

5. [Programming] Use Gauss Jordan method to solve the following system of equations:

Gauss–Jordan Method: This method is described as follows. Use the ith equation to eliminate not only x_i from the equations $E_{i+1}, E_{i+2}, \ldots, E_n$, as was done in the Gaussian elimination method, but also from $E_1, E_2, \ldots, E_{i-1}$. Upon reducing $[A, \mathbf{b}]$ to:

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & 0 & \cdots & 0 & a_{1,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots & a_{2,n+1}^{(2)} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & a_{nn}^{(n)} & a_{n,n+1}^{(n)} \end{array}\right],$$

the solution is obtained by setting

$$x_i = \frac{a_{i,n+1}^{(i)}}{a_{ii}^{(i)}},$$

for each $i = 1, 2, \ldots, n$. This procedure circumvents the backward substitution in the Gaussian elimination. Construct an algorithm for the Gauss–Jordan procedure patterned after that of Algorithm 6.1.

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= -8, \\ 2x_1 - 2x_2 + 3x_3 - 3x_4 &= -20, \\ x_1 + x_2 + x_3 &= -2, \\ x_1 - x_2 + 4x_3 + 3x_4 &= 4. \end{aligned}$$

6. [Programming] Determine the Inverse of the following matrix using Gauss Elimination method

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix},$$

And

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -4 & -2 \\ 2 & 1 & 1 & 5 \\ -1 & 0 & -2 & -4 \end{bmatrix}$$