A Simple Introduction to Numerical Calculation in C++
third laboratory

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Gauss Elimination

The number of multiplications and divisions required for Gauss elimination is approximately $N = (n^3/3 - n/3)$ for matrix $\mathbf{A}$ and n^2 for each $\mathbf{b}$. For $n = 10$, $N = 430$, and for $n = 100$, $N = 343{,}300$. This is a considerable reduction compared to Cramer's rule.

The Gauss elimination procedure, in a format suitable for programming on a computer, is summarized as follows:

1. Define the $n \times n$ coefficient matrix $\mathbf{A}$, the $n \times 1$ column vector $\mathbf{b}$, and the $n \times 1$ order vector $\mathbf{o}$.
2. Starting with column 1, scale column k $(k = 1, 2, \ldots, n-1)$ and search for the element of largest magnitude in column k and pivot (interchange rows) to put that coefficient into the $a_{k,k}$ pivot position. This step is actually accomplished by interchanging the corresponding elements of the $n \times 1$ order vector $\mathbf{o}$.
3. For column k $(k = 1, 2, \ldots, n-1)$, apply the elimination procedure to rows i $(i = k+1, k+2, \ldots, n)$ to create zeros in column k below the pivot element, $a_{k,k}$. Do not actually calculate the zeros in column k. In fact, storing the elimination multipliers, em $= (a_{i,k}/a_{k,k})$, in place of the eliminated elements, $a_{i,k}$, creates the Doolittle LU factorization presented in Section 1.4. Thus,

$$a_{i,j} = a_{i,j} - \left(\frac{a_{i,k}}{a_{k,k}}\right) a_{k,j} \qquad (i, j = k+1, k+2, \ldots, n) \tag{1.100a}$$

$$b_i = b_i - \left(\frac{a_{i,k}}{a_{k,k}}\right) b_k \qquad (i = k+1, k+2, \ldots, n) \tag{1.100b}$$

After step 3 is applied to all k columns, $(k = 1, 2, \ldots, n-1)$, the original $\mathbf{A}$ matrix is upper triangular.

4. Solve for $\mathbf{x}$ using back substitution. If more than one $\mathbf{b}$ vector is present, solve for the corresponding $\mathbf{x}$ vectors one at a time. Thus,

$$x_n = \frac{b_n}{a_{n,n}} \tag{1.101a}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j} x_j}{a_{i,i}} \qquad (i = n-1, n-2, \ldots, 1) \tag{1.101b}$$

Example 1.7. Simple elimination.

Let's rework Example 1.6 using simple elimination. From Example 1.6, the **A** matrix augmented by the **b** vector is

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \tag{1.79}$$

Performing the row operations to accomplish the elimination process yields:

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j}x_j}{a_{i,i}} \qquad (i = n-1, n-2, \ldots, 1)$$

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \begin{array}{l} \\ R_2 - (-20/80)R_1 \\ R_3 - (-20/80)R_1 \end{array} \tag{1.80}$$

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ 0 & 35 & -25 & 25 \\ 0 & -25 & 125 & 25 \end{array}\right] \begin{array}{l} \\ \\ R_3 - (-25/35)R_2 \end{array} \tag{1.81}$$

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ 0 & 35 & -25 & 25 \\ 0 & 0 & 750/7 & 300/7 \end{array}\right] \rightarrow \begin{array}{l} x_1 = [20 - (-20)(1.00) - (-20)(0.40)]/80 \\ \quad = 0.60 \\ x_2 = [25 - (-25)(0.4)]/35 = 1.00 \\ x_3 = 300/750 = 0.40 \end{array} \tag{1.82}$$

The back substitution step is presented beside the triangularized augmented **A** matrix.

Gauss-Jordan Elimination

Gauss-Jordan elimination is a variation of Gauss elimination in which the elements above the major diagonal are eliminated (made zero) as well as the elements below the major diagonal. The **A** matrix is transformed to a diagonal matrix. The rows are usually scaled to yield unity diagonal elements, which transforms the **A** matrix to the identity matrix, **I**. The transformed **b** vector is then the solution vector **x**. Gauss-Jordan elimination can be used for single or multiple *b* vectors.

The number of multiplications and divisions for Gauss-Jordan elimination is approximately $N = (n^3/2 - n/2) + n^2$, which is approximately 50 percent larger than for Gauss elimination. Consequently, Gauss elimination is preferred.

Let's rework Example 1.7 using simple Gauss-Jordan elimination, that is, elimination without pivoting. The augmented $\mathbf{A}$ matrix is [see Eq. (1.79)]

$$\left[\begin{array}{rrr|r} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \begin{array}{l} R_1/80 \\ \\ \\ \end{array} \tag{1.102}$$

Scaling row 1 to give $a_{11} = 1$ gives

$$\left[\begin{array}{rrr|r} 1 & -1/4 & -1/4 & 1/4 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right] \begin{array}{l} \\ R_2 - (-20)R_1 \\ R_3 - (-20)R_1 \end{array} \tag{1.103}$$

Applying elimination below row 1 yields

$$\left[\begin{array}{rrr|r} 1 & -1/4 & -1/4 & 1/4 \\ 0 & 35 & -25 & 25 \\ 0 & -25 & 125 & 25 \end{array}\right] \begin{array}{l} \\ R_2/35 \\ \\ \end{array} \tag{1.104}$$

Scaling row 2 to give $a_{22} = 1$ gives

$$\left[\begin{array}{rrr|r} 1 & -1/4 & -1/4 & 1/4 \\ 0 & 1 & -5/7 & 5/7 \\ 0 & -25 & 125 & 25 \end{array}\right] \begin{array}{l} R_1 - (-1/4)R_2 \\ \\ R_3 - (-25)R_2 \end{array} \tag{1.105}$$

Applying elimination both above and below row 2 yields

$$\left[\begin{array}{rrr|r} 1 & 0 & -3/7 & 3/7 \\ 0 & 1 & -5/7 & 5/7 \\ 0 & 0 & 750/7 & 300/7 \end{array}\right] \begin{array}{l} \\ \\ R_3/(750/7) \end{array} \tag{1.106}$$

Scaling row 3 to give $a_{33} = 1$ gives

$$\left[\begin{array}{rrr|r} 1 & 0 & -3/7 & 3/7 \\ 0 & 1 & -5/7 & 5/7 \\ 0 & 0 & 1 & 215 \end{array}\right] \begin{array}{l} R_1 - (-3/7)R_3 \\ R_2 - (-5/7)R_3 \\ \\ \end{array} \tag{1.107}$$

Applying elimination above row 3 completes the process.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0.60 \\ 0 & 1 & 0 & 1.00 \\ 0 & 0 & 1 & 0.40 \end{array}\right]$$

The $\mathbf{A}$ matrix has been transformed to the identity matrix $\mathbf{I}$ and the $\mathbf{b}$ vector has been transformed to the solution vector, $\mathbf{x}$. Thus, $\mathbf{x}^T = [0.60 \quad 1.00 \quad 0.40]$.

Inwers matrix using applying Gauss-Jordan elimination

Gauss-Jordan elimination can be used to evaluate the inverse of matrix $\mathbf{A}$ by augmenting $\mathbf{A}$ with the identity matrix $\mathbf{I}$ and applying the Gauss-Jordan algorithm. The transformed $\mathbf{A}$ matrix is the identity matrix $\mathbf{I}$, and the transformed identity matrix is the matrix inverse, $\mathbf{A}^{-1}$. Thus, applying Gauss-Jordan elimination yields

$$\boxed{[\mathbf{A} \mid \mathbf{I}] \rightarrow [\mathbf{I} \mid \mathbf{A}^{-1}]}$$

Ax = b

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

from which

$$\boxed{\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}}$$