The Math Needed to Understand Image Processing^cont.^

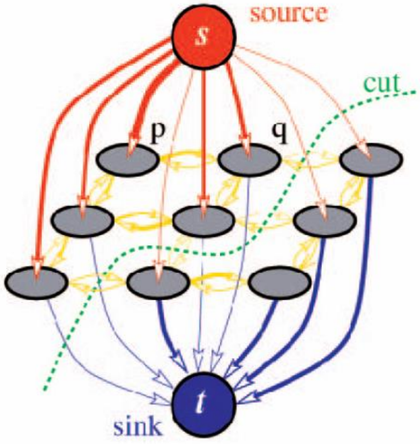
- **Two aspects of scale**
  - **Levels of detail**
  - **Gaussian apertures and spatial scale**
  - **Intensity noise vs. scale**

Little noise

noisy

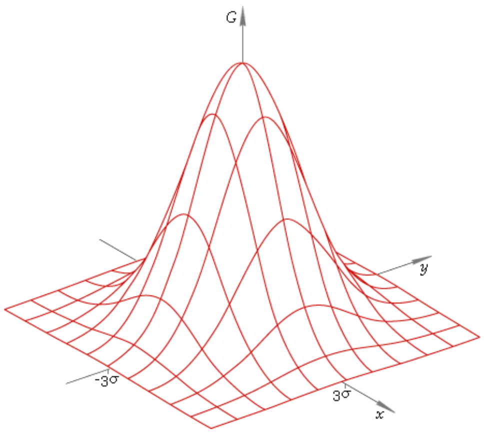
- Measures of edge and bar strength via derivatives
- Ridges in images,
  towards finding edges and bars

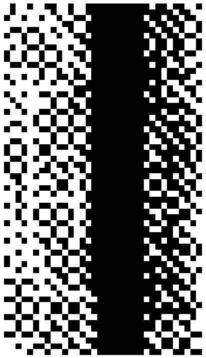
- Interpolation of discrete images
  - Via convolution; via orthogonal basis functions
  - Via splines
  - Via least-squares approximations
- Discrete images as algebraic graphs,
  with objects as graph cuts



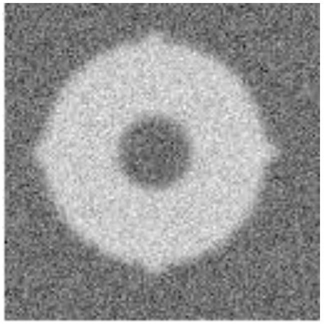
Scale and Locality

- Two different factors called *spatial scale* of a sample or a basis function
  - level of detail: basis functions $\psi^{lod}(\underline{u})$
    - So 1 basis function per lod; e.g., sinusoid wavelength
  - aperture (with locality): $\psi(\underline{u},\underline{u}_0,\sigma;\ lod)$,
    - Involves an aperture weighting function centered at a location $\underline{u}_0$
    - Determines interrelation distances, e.g., bar or disk widths
    - A whole set of basis functions at each scale σ
  - Both factors determine feature size on which to focus

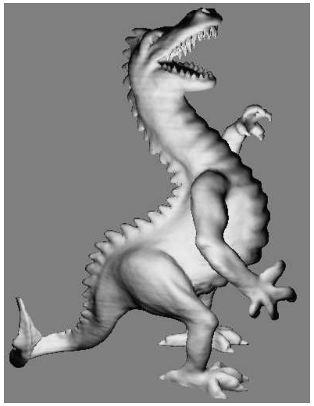


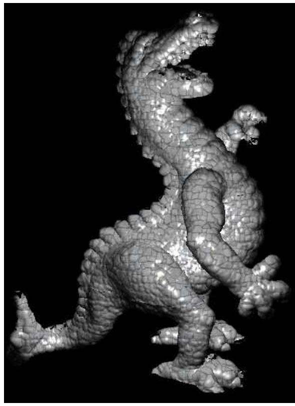


Focusing on the right scale

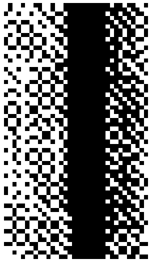


- Example: white noise in a blurred image
  - $I_{discrete}(x,y) = I_{discrete\ \&\ ideal}(x,y) + noise(x,y)$
- Choose aperture size to delete or attenuate undesired scales
- Choose level of detail to focus on lods with good signal-to-noise, i.e., large and moderate lods





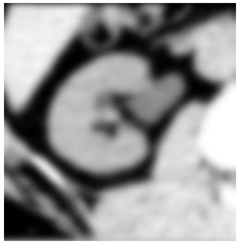
- Example: remove unwanted detail
- Example: bar or blob width

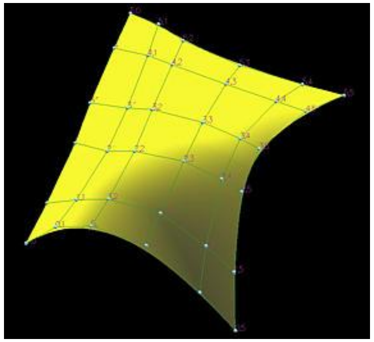


Apertures and Levels of Detail

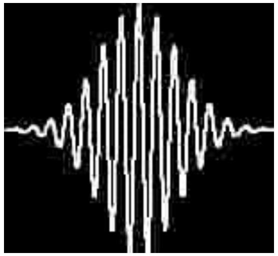
Large scale

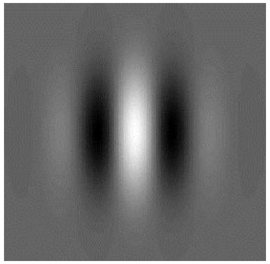
Smaller scale



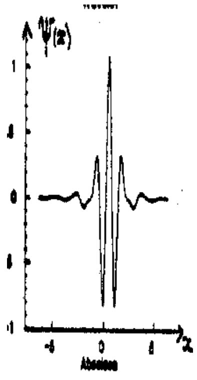


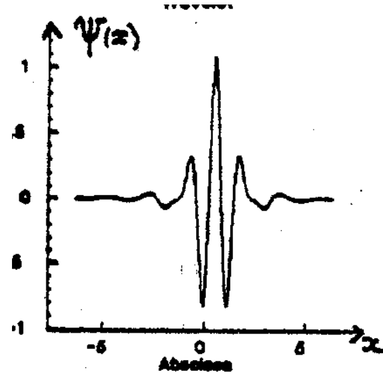
- Apertures
  - Global
  - Local
    - Gaussian and its derivatives
      - Aperture scale is σ
    - Splines
      - Aperture scale is size of data support for a patch
    - With orthogonality:
      - Gabor functions: sinusoid under Gaussian with aperture scale σ
      - Orthogonal wavelets





- Levels of detail
  - Sinusoid wavelength (also for Gabor)
  - Derivative order (of Gaussian)
  - Spline data grid spacing
  - Orthogonal wavelet binary decimation level



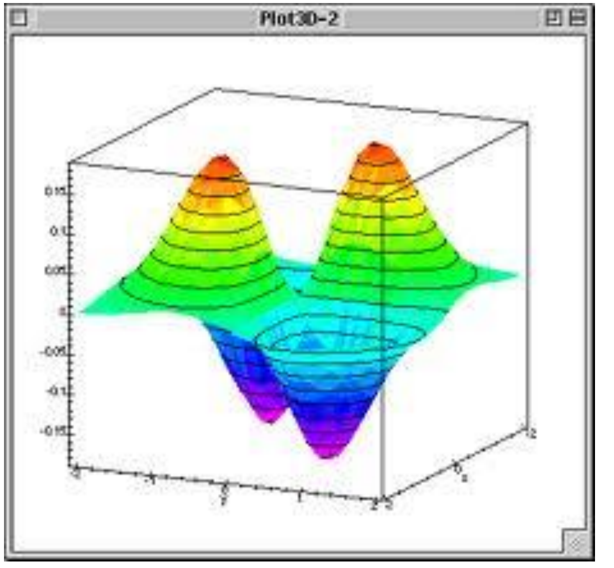


Properties desired of an aperture

- Unbiased re spatial scaling, translation, rotation
- Cascading apertures gives a legal aperture
- Do not create structure, only eliminate it
- Have finite integral

- The only continuous aperture that does all of that is the Gaussian!
  - The main reference on this: BM ter Haar Romeny, *Front-End Vision and Multi-Scale Image Analysis*. Kluwer (now Springer) 2003. Esp. Chs. 1-8
    - Book exists as a Mathematica program (can chg the figures)

Non-creation of structure



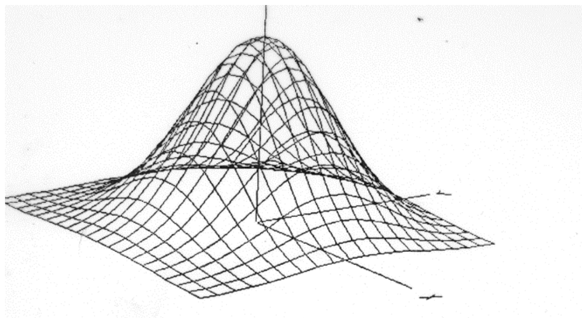
- No new level curves very nearby
  - Of intensity
  - Of derivatives of intensity
- Equivalently, upon application of aperture
  - local maxima disappear or decrease in intensity
  - local minima disappear or increase in intensity
- Not equivalent to no creation of local maxima or minima
  - Consider taut curtain between mountain-tops

Properties of the Gaussian

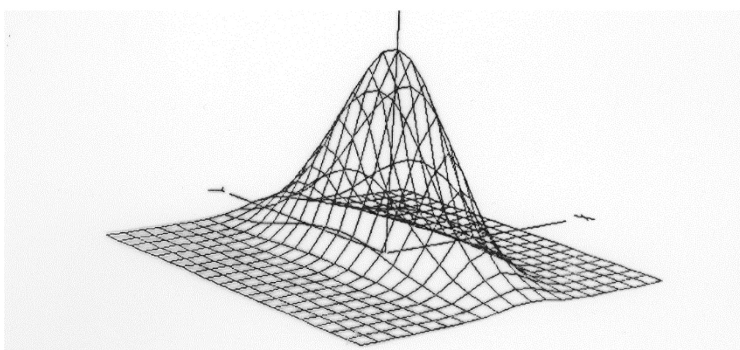
- Separable
- Convolution or product of 2 Gaussians is Gaussian
- Rotationally invariant; i.e., isotropic
  - Also ellipsoidal form is available
- Is its own Fourier transform, but reciprocal std deviation
- Central limit theorem: $*_{i=1}^{n} h_i$ is Gaussian in the limit
- Maximum entropy: most uncertain with fixed variance
- Diffusion (heat equation): $\partial f(\underline{x},t)/\partial t = \nabla^2 f$ with $f(\underline{x},0)=\delta(\underline{x})$ has Gaussian as solution (psf, convolution kernel)
- Scale (apertures) that
  - are agnostic re rotation, translation, and magnification
  - compose successive scale changes into a single scale change
  - do not create structure by increasing scale
- Result of Brownian motion

The Gaussian

- Formula:
  - Isotropic: $(2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2}[|\underline{x}-\underline{\mu}|/\sigma]^2\}$



  - General: $(2\pi)^{-n/2}\, |\det \Sigma|^{-1/2} \exp\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1}(\underline{x}-\underline{\mu})\}$



    - Eigenvectors of Σ are principal directions
    - (Eigenvalues of $\Sigma)^{1/2} \propto$ principal radii

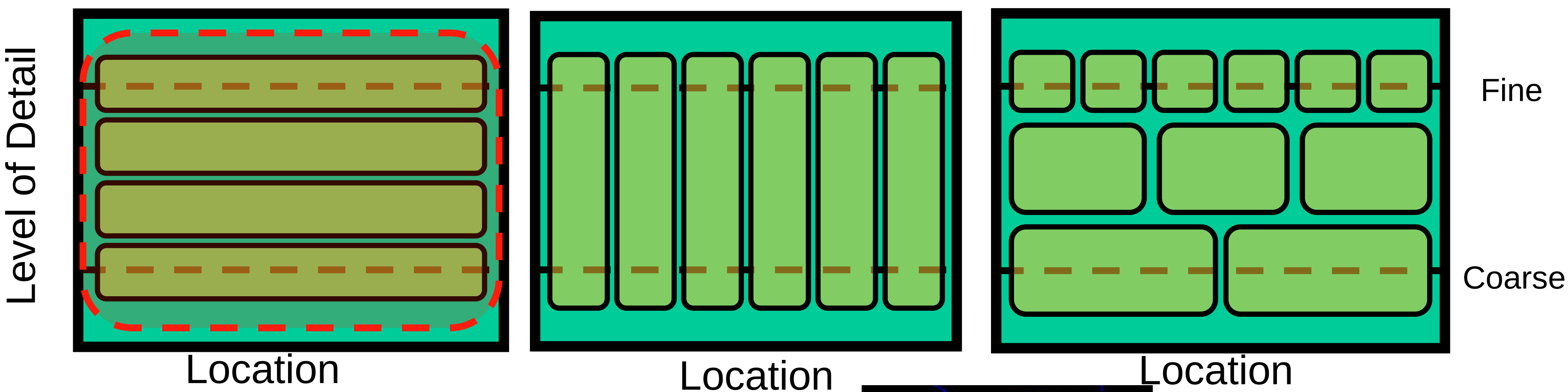
How to Compute a First Derivative

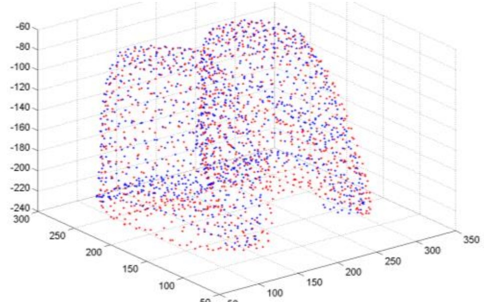
- Always via derivative of Gaussian. Let image be in M-D.
- In a non-cardinal direction
  - Compute the M cardinal derivatives in the gradient
    - If done via freq. domain (see below), multiply amplitudes (or both real and imaginary parts) by coefficientless M-D Gaussian once
  - Dot product result with direction
- In a cardinal direction, say x
  - If Gaussian's σ < 3 pixels, operate in space domain
    - Compute Gaussian kernel and apply that narrow (<8 pixels wide) weighting function pixel by pixel
  - Otherwise, take FFT of image and operate in frequency domain
    - Multiply Gaussian-updated amplitudes (or real and imaginary parts) by ν_x, and if necessary by the 2π that is part of $2\pi i$
    - To effect multiplication by $i=e^{i\pi/2}$, add $\pi/2$ to every phasee in FFT(I) (or change sign of imaginary party of FFT(I) and then swap real and imaginary parts of the result
- If in another cardinal dir., say y, only change ν_x to ν_y

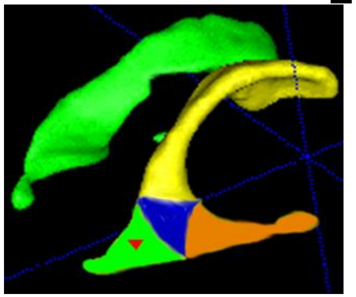
The Part of the Course Covered on the Midterm Ends Here

Scale Situations in Various Sampled Geometric Analysis Approaches

	Global coef for each level of detail	Multidetail feature	Detail residues
Examples:	Fourier coeffs, global principal components	image pixels boundary points, dense displacements	orthogonal wavelets Gabor, Gauss deriv, recursive splines

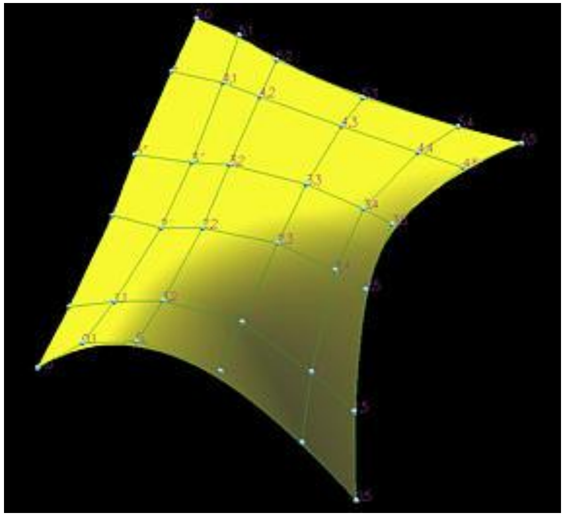






Splines

- Smoothly connected patches
- Typically a polynomial in each patch
- Local support by nearby grid elements



Polynomial Basis Functions w/ Locality

- Splines: patchwise fitting
  - Approximating
    - E.g., B-splines, related to wavelets

$$Q_i(t) = \frac{1}{6}\begin{bmatrix}(t-t_i)^3 & (t-t_i)^2 & (t-t_i) & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}\underline{P}_{i-1} \\ \underline{P}_i \\ \underline{P}_{i+1} \\ \underline{P}_{i+2}\end{bmatrix}$$

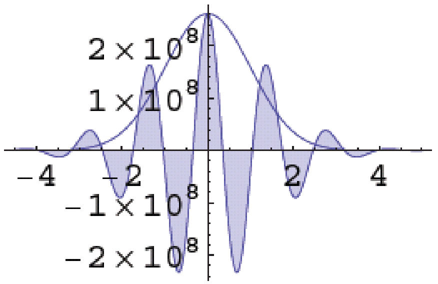
    - Above is in each dimension; separable
    - t_i integers, $(t-t_i) \in [0,1]$
  - Interpolating: Global
  - Approximating: Locality (aperture) by control point ($\underline{P}_i$) spacing

Some Uses of Splines

- Smooth bias fields for images
  - Subtract it
- Smooth sensitivity fields for images
  - Divide by it
- Smooth displacement fields for distortion
  - Separate splines for $\Delta x(\underline{x})$, $\Delta y(\underline{x})$, $\Delta z(\underline{x})$
  - Also used to compute deformable registrations
    - Optimize $\underline{\Delta x}$ control point sets

Representations with Locality

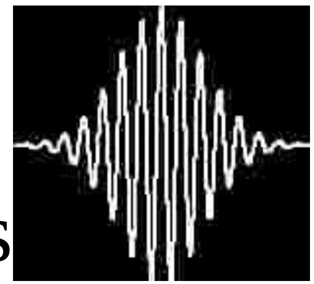
- With parametrized representation
  - With $\underline{u}$ as parameter(s)
  - With $\mathbf{f}$ = $\underline{x}$ or I or … as function of $\underline{u}$
- Need $\mathbf{f}(\underline{u},\sigma) = \Sigma_{lod} a(lod, \underline{u}_0,\sigma)\ \psi^{lod}(\underline{u},\underline{u}_0,\sigma)$
  - $\underline{u}$ is location
  - σ is aperture size (typically std dev of Gaussian), $\underline{u}_0$ is aperture center
  - lod is level of detail
    - For Fourier it is frequency ν
    - For derivative, it is order
      - Not too noisy if σ is well chosen and order < 6
    - For orthogonal wavelet it is level of decimation

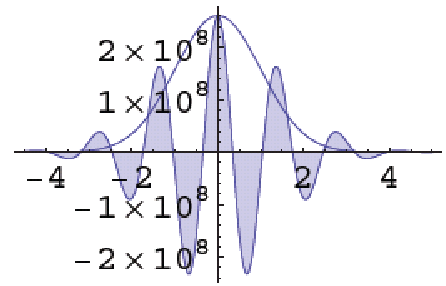


The 20th order Gaussian derivative's

Basis Functions **with Locality**

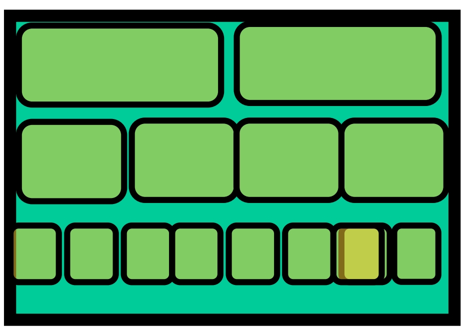
- Gabor functions: sinusoids under the Gaussian [ref Wechsler: *Comp'l Vision*]
  - Like derivatives of Gaussian, with $\nu \propto$ derivative order
  - Wavelength $1/\nu \propto \sigma$
  - Sampling $\propto \sigma$





The 20th order Gaussian derivative's

- Orthogonal wavelets
  - Interscale residues



  - Orthogonality across & within scales

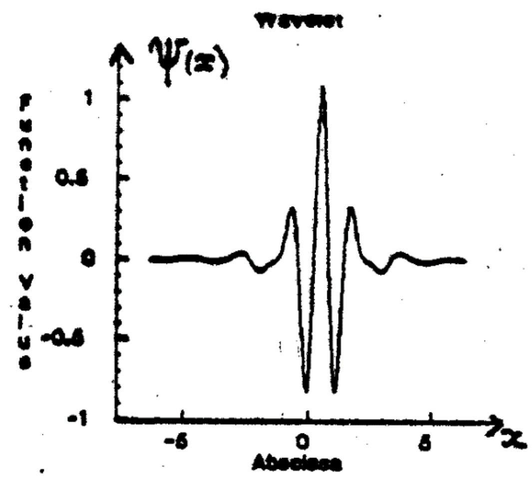
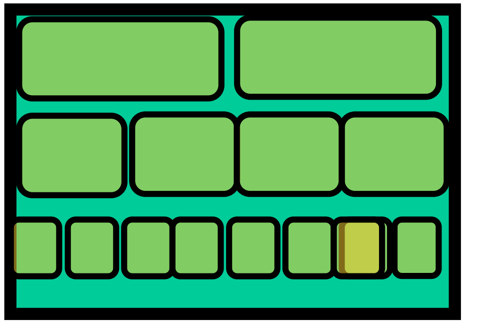


Fig. 1.4. ***Example of a wavelet with the***

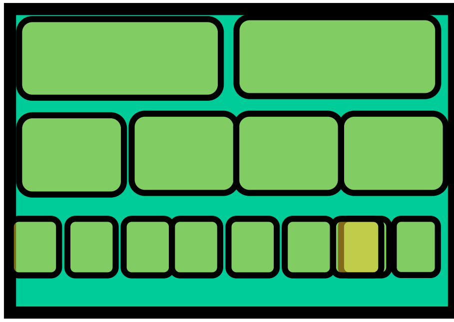
Basis Functions **with Locality**: Derivatives of Gaussian

- [ref: ter Haar Romeny book]
- Order of derivative is LOD
  - Not orthogonal as basis functions
  - Sampling $\propto \sigma$ (see later slide on derivatives)
- Not too different in effect from Gabor wavelets
- Localized Taylor series
- Diffusion equation and Taylor series in t ($=\sigma^2/2$) yields Laplacian-based local multi-scale approximation



Pyramids: Images in Scale Space

- Gabor $\Delta_\sigma I = G(\underline{x};\sigma) * I(\underline{x}) - G(\underline{x};2\sigma) * I(\underline{x})$
  $= [G(\underline{x};\sigma) - G(\underline{x};2\sigma)] * I(\underline{x})$
  – As scale σ increases, the sampling distance can increase proportionally
- $G(\underline{x};\sigma) - G(\underline{x};2\sigma) \approx \nabla^2 G(\underline{x};\sigma)$
- As you increase scale by some constant factor, you produce an image as a function of $\underline{x}$ (with adjusted sampling) and σ: the Laplacian pyramid
- If you combine the Laplacian effect with the orthogonal wavelet, you get orthogonal wavelet pyramid



Uses of basis functions with locality

- Three dimensions
  - Location
  - Aperture size
  - LOD, often best $\propto$ aperture size
- Choosing the aperture size and LOD
  - PCA (SVD)
  - Biggest average response(s) per location
- Edgeness and barness operators
  - Edgeness: directional 1st derivative with appropriate aperture: cf. edge slope
  - Barness: directional 2nd derivative with appropriate aperture: cf. bar width

Loci as Height Ridges in Graphs

Ref: D. Eberly, *Ridges in Image and Data Analysis*, *Kluwer*

Examples: Edges, Bars (ridges will come in next course section)

The challenge: identify a point and direction (and for a bar, width) as being on an edge or bar

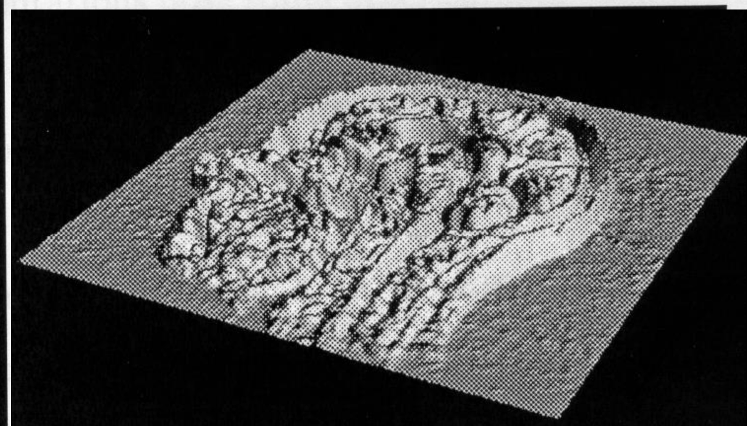
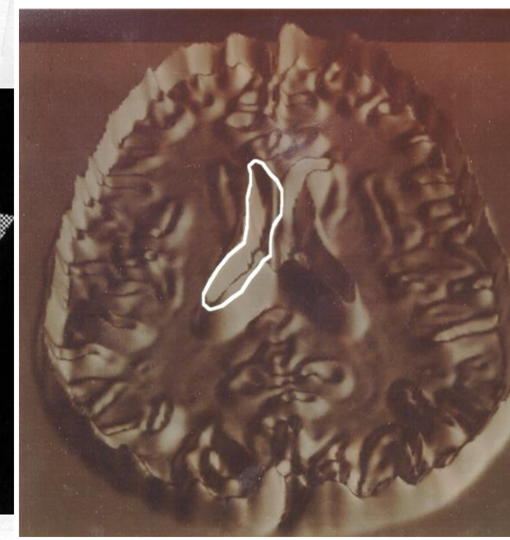


Figure 2.1: The MRI head image has an associated intensity surface.



Gradient magnitude

Edgeness and barness operators

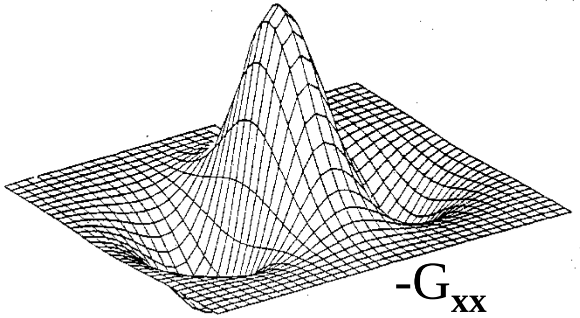
- Edgeness
  - Gradient with aperture: $D^1 f(\underline{x}, \sigma) = \nabla f(\underline{x}, \sigma) = [\partial G(\underline{x}, \sigma)/\partial x_1 * f, \ldots, \partial G(\underline{x}, \sigma)/\partial x_n * f]^T$
    - Gives direction of maximum edgness
    - Magnitude gives amount of edgeness in that dir.
  - Directional derivative with aperture = edgeness in the **v** direction = $\mathbf{v} \bullet \nabla f(\underline{x}, \sigma)$
- Barness
  - Hessian with aperture: $D^2 f(\underline{x}, \sigma)$ = M × M matrix $[\partial^2 G(\underline{x}, \sigma)/\partial x_i \, \partial x_j]$
  - Barness: directional 2nd derivative w/ appropriate aperture in the **v** direction = $-\mathbf{v}^T D^2 f(\underline{x}, \sigma)\, \mathbf{v}$

Optimal Barness Direction

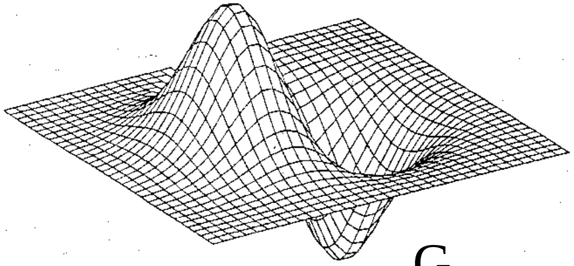
- $\mathrm{Max}_{\mathbf{v}}[- \mathbf{v}^T D^2 f(\underline{x}, \sigma)\, \mathbf{v}\ ]$
  - D^2f is symmetric
  - Thus $\mathbf{v}$ = eigenvector of D^2f with most negative eigenvalue
  - Thus, barness is the negative of the most negative eigenvalue of D^2f

Edgeness and Barness Seen as Matched Filters

- Edgness via $\nabla f(\underline{x}, \sigma)$ = $\mathbf{v} \bullet \nabla f(\underline{x}, \sigma)$ with $\mathbf{v}$ being unit vector in gradient direction
- $\mathrm{Max}_{|\mathbf{v}|=1}[- \mathbf{v}^T D^2 f(\underline{x}, \sigma)\, \mathbf{v}\,]$ attained with $\mathbf{v}$ being unit eigenvector with most negative eigenvalue
- Kernel corresponding locally to edge or bar respectively matches the edge itself
  - $\mathrm{Max}_{|h|=1} h(\underline{x}) * q(\underline{x})|_{\underline{x}=0}$ attained when $h(\underline{x}) = q(\underline{x})/|q(\underline{x})|$



$-G_{\mathbf{xx}}$



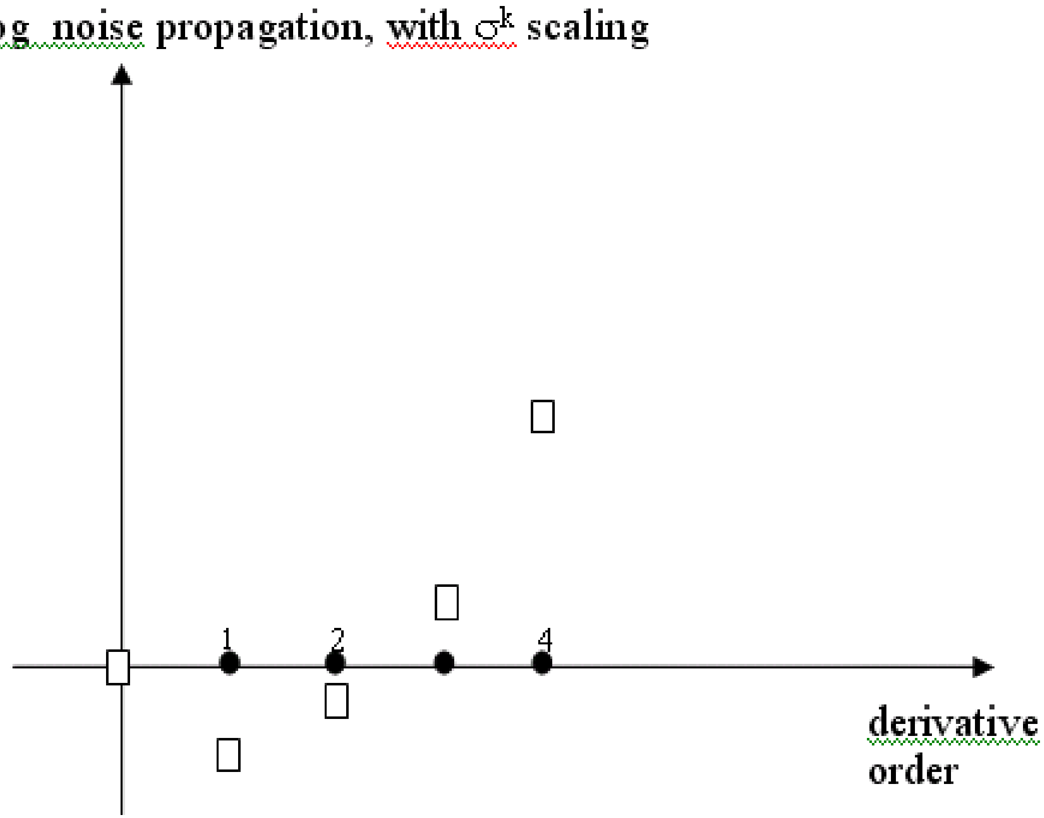
$G_{\mathbf{x}}$

Aperture size for edgeness and barness operators (and other derivatives)

- Derivatives are not commensurable
  - 1^{st} derivatives have units of intensity/mm
  - 2^{nd} derivatives have units of intensity/mm^2
  - Etc.
- Make them commensurable by multiplying k^{th} derivative by aperture's σ^k
  - (or $(c\sigma)^k$)
  - Make them comparable via error propagation behavior (see next slide)

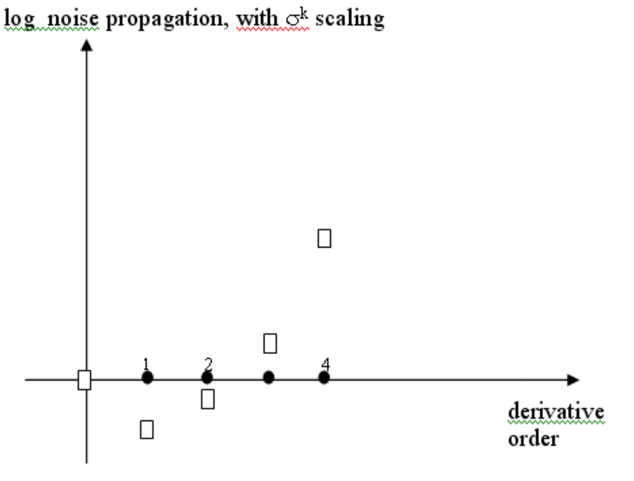
Error Propagation of σ^k-Scaled Derivatives

- 1D
- Relative to error in 0th derivative
- Displayed as log (so at order 0, is zero)



Error Propagation under Convolution with Gaussian

- Noise level (standard deviation) is multiplied by $[\int h^2(\underline{x})\, d\underline{x}]^{½}$, with h the Gaussian
  - That is, in M dimensions output noise level is divided by $\sigma^{M/2}$
- Thus change propagation by choosing σ for each derivative order to achieve the desired level of propagation



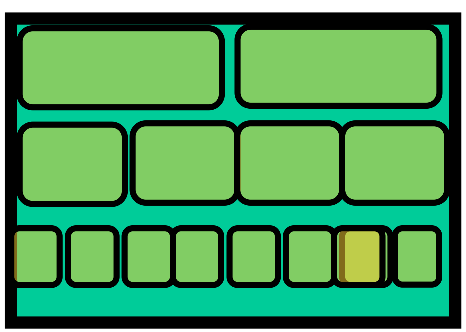
White Intensity Noise vs. Level of Detail

- $I_{noisy}(\underline{x}) = I_{noise\text{-}free}(\underline{x}) + noise(\underline{x})$
- In any orthonormal function basis, noise that is uncorrelated between pixels has constant variance in every basis function coefficient
- In that basis, as lod increases, the coefficient $a_{noise\text{-}free}(lod)$ of $I_{noise\text{-}free}(\underline{x})$ roughly falls like a Gaussian
- Thus signal-to-noise = $a_{noise\text{-}free}(lod) / var(a(lod))^{1/2}$ falls as lod increases, eventually becoming < 1

Uses of Spatial Scale

- Measurements at an appropriate chosen scale
- Optimality in scale space: best scale at each location
- Decomposition into residues at various scales

LOD



location

One orthonormal basis function coefficient per box

Comparative Properties of Main Parametrized Decompositions

	Locality	Invariances	Speed
• Fourier	--	+	++
• Gaussian derivatives	+	+	-
• Wavelets & Splines	+	--	+