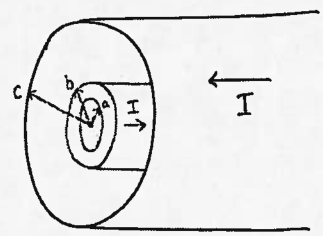
1. A long, ***hollow*** conducting pipe of inner radius a and outer radius b carries a current I parallel to its axis and disributed uniformly through the pipe. At radius $c > b$ is a concentric, long conducting ***shell*** carrying a current I in the direction opposite to the central hollow conducting pipe.



(a) Find the magnetic field everywhere. **[20]**

Use Ampère's Law for a loop of radius r in each relevant region:

For $r<a$: $I_{encl} = 0$ since this region is hollow. So: $B = 0$.

For $r \in (a,b)$: The enclosed current is a fraction of the total. Since the current is uniformly distributed, we may write:

$$J = \frac{I}{A_{pipe}} = \frac{I}{\pi b^2 - \pi a^2} = \frac{I}{\pi(b^2-a^2)}$$

Within the Ampèrian loop, the enclosed current is:

$$I_{encl} = \int_S \vec{J}\cdot d\vec{A} = J\int_S dA = J(\pi r^2 - \pi a^2) = I\left(\frac{r^2-a^2}{b^2-a^2}\right).$$

So: $$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} \Rightarrow B(2\pi r) = \mu_0 I\left(\frac{r^2-a^2}{b^2-a^2}\right) \Rightarrow B = \frac{\mu_0 I}{2\pi(b^2-a^2)}\left(\frac{r^2-a^2}{r}\right) = \frac{\mu_0 I}{2\pi(b^2-a^2)}\left(r - \frac{a^2}{r}\right)$$

For $r \in (b,c)$: Here, we have $I_{encl} = I$. So:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} \Rightarrow B(2\pi r) = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi}\frac{1}{r}.$$

For $r>c$: Here, we have complete cancellation of the currents. So: $I_{encl} = 0$

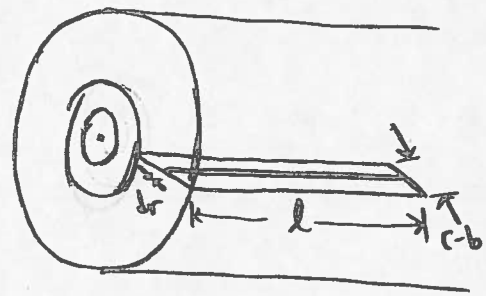
$$\Rightarrow B = 0.$$

(b) Find the self-inductance and magnetic energy *per unit length* of this system for $r \in [b, c]$. **[15]**

The magnetic field within this region is:

$$B = \frac{\mu_0 I}{2\pi} \frac{1}{r}$$

We must find the flux through the area perpendicular to the azimuthally circulating field bounded by $r=b$ and $r=c$, as below



However, the field strength changes, so we must integrate.

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} = \int B\,dA = \int_b^c \frac{\mu_0 I}{2\pi r} l\,dr = \frac{\mu_0 I}{2\pi} l \int_b^c \frac{dr}{r}$$

$$\Rightarrow \Phi_B = \frac{\mu_0 I}{2\pi} l \left[\ln(c) - \ln(b)\right]$$

So:

$$\Phi_B = \left[\frac{\mu_0}{2\pi} l \ln\left(\frac{c}{b}\right)\right] I \equiv L I$$

$$\therefore L = \frac{\mu_0}{2\pi} l \ln\left(\frac{c}{b}\right)$$

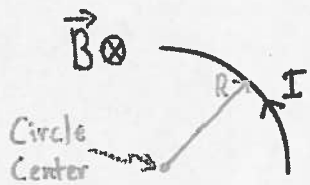
So, the inductance per unit length is:

$$L/l = \frac{\mu_0}{2\pi} \ln\left(\frac{c}{b}\right).$$

Thus, the magnetic energy per unit length is:

$$U_B = \frac{1}{2}\left(\frac{L}{l}\right) I^2 = \frac{\mu_0}{4\pi} I^2 \ln\left(\frac{c}{b}\right).$$

2. A wire carrying a current $I = 1.50$ A passes through a region containing a magnetic field of field strength $B = 4.80 \times 10^{-2}$ T. A segment of this wire is perpendicular to the field and makes a quarter-circle turn of radius $R = 21.0$ cm as it passes through the field region, as shown below. The remaining parts of the wire (not shown), from which the current runs into the segment and to which the current goes after running through the segment, are perpendicular to the segment and are *beneath* the plane of the arc shown.

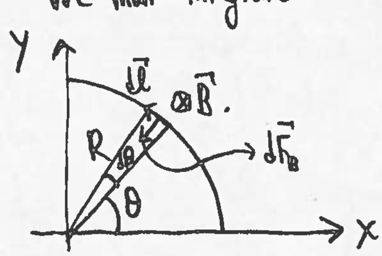


(a) Why is there no contribution to the magnetic force on this wire from the segments carrying the current to and from the circular arc? **[4]**

Since the field is into the page, the wire that feeds the arc with current has I out of the page, while the wire that receives the current from the arc has the current running into the page. In both cases, we have that $d\vec{l} \times \vec{B} = \vec{0}$. Thus, there is no magnetic-force contribution from these segments.

(b) Find the magnetic force (magnitude and direction) on the circular arc of wire. **[15]**

We must integrate.



$d\vec{F}_B$ is $\perp$ to both $d\vec{l}$ and $\vec{B}$.
From the right-hand rule of cross products, we have that $d\vec{F}_B = I\,d\vec{l} \times \vec{B}$ is radially inward:

$$d\vec{F}_B = -dF_B\,\hat{r} = -dF_B(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

with $dF_B = I\,dl\,B = IR\,d\theta\,B$.

So:
$$\vec{F}_B = \int d\vec{F}_B = -\hat{x}\int dF_B\cos\theta - \hat{y}\int dF_B\sin\theta$$
$$= -\hat{x}\int_0^{\pi/2} IRB\cos\theta\,d\theta - \hat{y}\int_0^{\pi/2} IRB\sin\theta\,d\theta$$
$$= -IRB\left[\hat{x}\left[\sin\theta\right]_0^{\pi/2} + \hat{y}\left[-\cos\theta\right]_0^{\pi/2}\right]$$
$$= -IRB(\hat{x}+\hat{y})$$

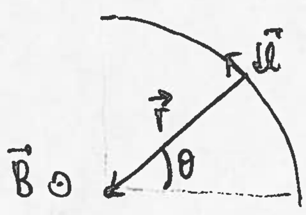
$\Rightarrow$ Direction is radially inward.

$\Rightarrow$ Magnitude is: $|\vec{F}_B| = \sqrt{2}\,IRB = 21.4$ mN $= 2.14 \times 10^{-2}$ N.

(c) Now, imagine that the external magnetic field is turned off.

i. Calculate the magnetic field (magnitude and direction) at the center of the circle (which is in the same plane as the arc) due to only the circular arc of wire. **[11]**

We must use the Biot-Savart Law:



$$d\vec{l} \times \hat{r} = dl\,\hat{z} = R\,d\theta\,\hat{z}$$

$$\frac{1}{r^2} = \frac{1}{R^2}$$

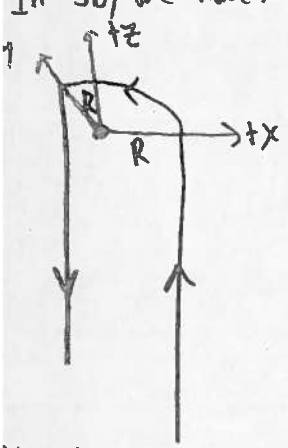
$$\text{So: } \vec{B}_{arc} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{r^2}$$

$$= \frac{\mu_0 I}{4\pi} \hat{z} \int_0^{\pi/2} \frac{R\,d\theta}{R^2} = \hat{z} \frac{\mu_0 I}{4\pi R} \frac{\pi}{2} = \hat{z} \frac{\mu_0 I}{8R} = (1.12\ \mu T)\,\hat{z}$$

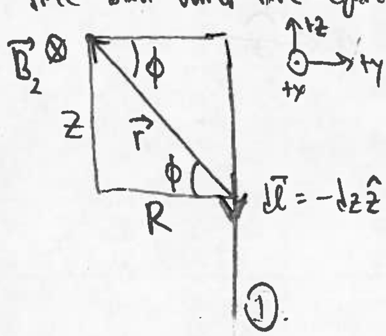
$$= (1.12 \times 10^{-6}\ T)\,\hat{z}$$

ii. If the segments carrying current to and from this circular arc were taken to be infinitely long beneath the plane of the arc, calculate their contribution to the field at the center of the circle described in Part (ci). **[15]**

In 3D, we have:



So, we must calculate, using the Biot-Savart Law, the field contribution from a semi-infinite wire a distance R from the very edge of the wire. Since $|\vec{B}_1| = |\vec{B}_2|$ since both wires are equidistant, we may just focus on one of them:



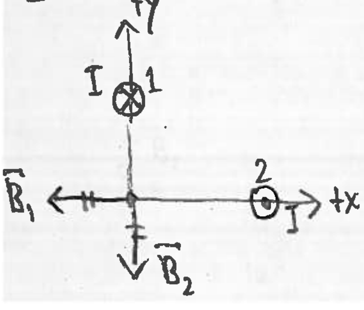
$$d\vec{l} \times \hat{r} = -\hat{x}\,dz \sin\left(\phi + \frac{\pi}{2}\right) = -\hat{x}\,dz \cos\phi$$

$$\frac{1}{r^2} = \frac{1}{R^2 + z^2}$$

$$\tan\phi = \frac{z}{R} \Rightarrow z = R\tan\phi \Rightarrow dz = \frac{R}{\cos^2\phi}\,d\phi$$

$$\cos\phi = \frac{R}{\sqrt{z^2+R^2}} \Rightarrow \frac{1}{z^2+R^2} = \frac{\cos^2\phi}{R^2}$$

We have two semi-infinite wires located a distance R from the observation point. Looking down from +z:



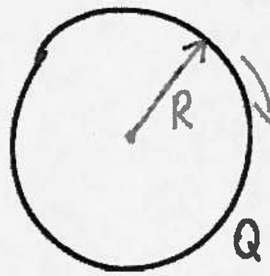
$$\vec{B}_1 = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{-\hat{x}\,dz\cos\phi}{R^2+z^2} = -\hat{x}\frac{\mu_0 I}{4\pi}\int_0^{\pi/2} \frac{\cos^2\phi}{R^2}\,\frac{R}{\cos^2\phi}\cos\phi\,d\phi$$

$$= -\hat{x}\frac{\mu_0 I}{4\pi R}\int_0^{\pi/2}\cos\phi\,d\phi = -\hat{x}\frac{\mu_0 I}{4\pi R}\left[\sin\phi\right]_0^{\pi/2} = -\hat{x}\frac{\mu_0 I}{4\pi R}$$

From inspection: $\vec{B}_2 = -\hat{y}\,\frac{\mu_0 I}{4\pi R}$

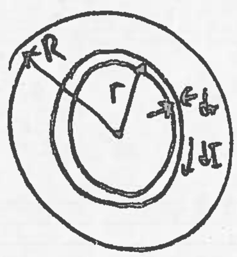
$$\text{So: } \vec{B}_{wires} = -\frac{\mu_0 I}{4\pi R}(\hat{x} + \hat{y}) \Rightarrow |\vec{B}_{wires}| = \sqrt{2}\frac{\mu_0 I}{4\pi R} = 1.01\ \mu T = 1.01 \times 10^{-6}\ T$$

3. Imagine an insulating *disk* of radius R carrying a total charge Q and rotating with angular frequency ω about its symmetric axis, as shown below. The rotating disk creates the effect of a bunch of concentric current loops.



(a) If the disk has a uniform surface charge density, determine its magnetic dipole moment (magnitude and direction) in terms of the given quantities. **[12]**

Let's focus on a single disk of rotating charge.



The current in this ring can be written as the charge over the period ($\mathcal{T}$). So:

areal charge density.

$$dI = \frac{dq}{\mathcal{T}} = \frac{\omega}{2\pi}\,dq = \frac{\omega}{2\pi}\,\sigma\,dA = \frac{\omega}{2\pi}\,\sigma\,(2\pi r\,dr) = \omega\sigma r\,dr$$

But since σ is uniform, $\sigma = \dfrac{Q}{\pi R^2}$. So: $dI = \omega \dfrac{Q}{\pi R^2} r\,dr$

By the right-hand rule for $\vec{\mu}$, the dipole moment points into the page, which we will call $+\hat{z}$. So:

$$\vec{\mu} = \hat{z}\int A\,dI = \hat{z}\int_0^R \pi r^2 \omega \frac{Q}{\pi R^2}\, r\,dr$$

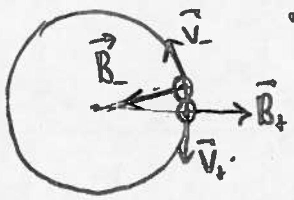
$$= \hat{z}\,\frac{Q\omega}{R^2}\int_0^R r^3\,dr$$

$$= \hat{z}\,\frac{Q\omega}{R^2}\frac{r^4}{4}\Big|_0^R = \hat{z}\,\frac{1}{4}Q\omega R^2.$$

(b) Now, suppose that the disk is made to be conducting and is discharged (i.e., it has NO NET CHARGE). It is then placed into a uniform magnetic field that points *into the page* with the field perpendicular to the plane of the disk with respect to the picture above.

i. Provide a qualitative description of what happens in this system. In particular, explain what is going on when the system reaches a steady-state under these circumstances. **[7]**

Since the disk is conducting, there are free charges that can be moved around if they feel a net external force. Because these charges are moving in the presence of a magnetic field, they will be deflected. Since the velocity is tangent to the circular path of each ring in the disk, then the charges will be deflected radially by the magnetic force law, with "-" charges pushed radially inward and "+" charges pushed radially outward. There will be a charge separation with an electric field pointing radially inward. When a steady-state is reached, the magnetic forces on the charges caused by the external magnetic field will balance the internal electric field caused by the charge separation.

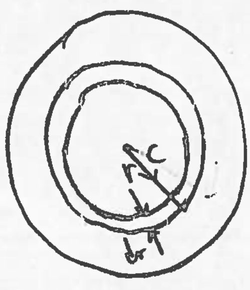


ii. Determine the emf (magnitude and direction) generated within the disk when the system reaches a steady state. **[11]**

Since the electric and magnetic forces balance on a given charge q, then:

$$q\vec{v}\times\vec{B} = q\vec{E} \Rightarrow \vec{E} = \vec{v}\times\vec{B} \Rightarrow |\vec{E}| = vB \text{ since } \vec{v}\perp\vec{B}.$$

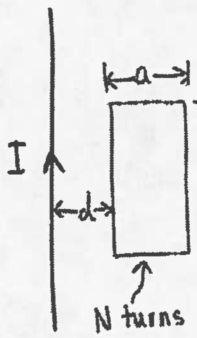
From above, the emf will be radially outward since the potential difference is positive in this direction. However, $|\vec{v}|$ is larger the farther out one goes from the center. So, there will be an integration. We have:



$d\vec{l} = \hat{r}\,dr$

$$\mathcal{E} = \int_C \vec{E}\cdot d\vec{l} = \int vB\,dr = \int_0^R \omega r B\,dr = \frac{1}{2}\omega B R^2$$

4. In the figure below, the wire on the left is infinitely long and carries a current I. The conducting, rectangular loop on the right has N turns with width a and length b. The left end of the loop is a distance d away from the infinitely long wire. In terms of the given quantities, calculate the mutual inductance between the wire-loop system. **[25]**

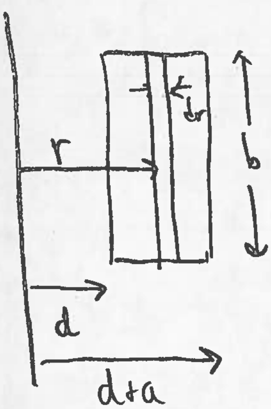


From Ampère's Law, we may calculate the field due to the infinitely long wire as a function of the radial distance, r, from the wire:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

$$B(2\pi r) = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi} \frac{1}{r}$$

Where the loop is, the field due to the infinitely long wire is directed into the page, so that $\vec{B} \perp \hat{n}$ (with $\hat{n}$ the unit normal to the loop).



We must find the flux through the loop via integration since the field strength changes radially across the loop.

The flux through 1 loop is:

$$\Phi_1 = \int_S \vec{B} \cdot d\vec{A} = \int_d^{a+d} \frac{\mu_0 I}{2\pi r} b\,dr = \frac{\mu_0 I b}{2\pi} \int_d^{a+d} \frac{dr}{r} = \frac{\mu_0 I}{2\pi} b \ln\left(\frac{a+d}{d}\right).$$

The flux through the N loops is:

$$\Phi_B = N\Phi_1 = \frac{\mu_0 N I}{2\pi} b \ln\left(\frac{a+d}{d}\right).$$

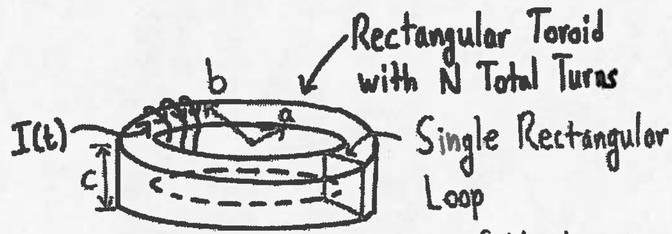
OR:

$$\Phi_B = \left[\frac{\mu_0 N b}{2\pi} \ln\left(\frac{a+d}{d}\right)\right] I \equiv MI.$$

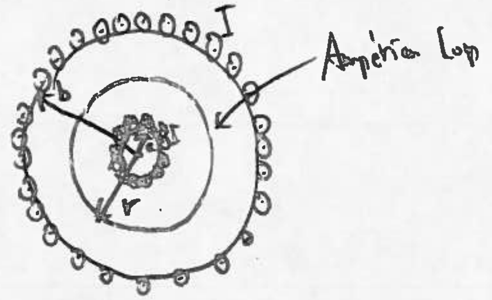
So:

$$M = \frac{\mu_0 N b}{2\pi} \ln\left(\frac{a+d}{d}\right).$$

5. A toroidal coil of rectangular cross-section has inner radius a, outer radius b, and height c. It consists of N total turns of wire and carries a time-varying current $I = I_0 \sin(\omega t)$. A single-turn rectangular loop encircles the toroid, outlining its cross section, as shown below. Find an expression for the peak emf induced in the rectangular loop. **[25]**



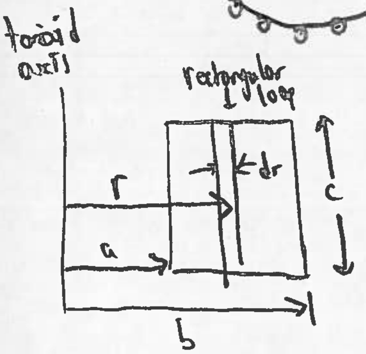
We may use Ampère's Law to find the field due to the toroid. We have:



$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl}$$

$$B(2\pi r) = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{2\pi} \frac{1}{r}.$$



Since the field strength changes radially along the rectangular loop, we must integrate to find the flux. So, we have:

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} = \int_a^b \frac{\mu_0 N I}{2\pi} \frac{1}{r} c\,dr$$

$$= \frac{\mu_0 N I}{2\pi} c \int_a^b \frac{dr}{r} = \frac{\mu_0 N}{2\pi} c \ln\left(\frac{b}{a}\right) I(t).$$

From Faraday's Law:

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -\frac{\mu_0 N}{2\pi} c \ln\left(\frac{b}{a}\right) \frac{d}{dt}\left[I_0 \sin(\omega t)\right]$$

$$= -\frac{\mu_0 N}{2\pi} c \ln\left(\frac{b}{a}\right) I_0 \omega \cos(\omega t)$$

$$= -\frac{\mu_0 N}{2\pi} c \omega I_0 \ln\left(\frac{b}{a}\right) \cos(\omega t) \Rightarrow$$

Since $|\cos(x)| \leq 1$, then,

$$|\mathcal{E}_{peak}| = \frac{\mu_0 N}{2\pi} c \omega I_0 \ln\left(\frac{b}{a}\right).$$