Project 4: Zombies Outbreak

Chenxi Huang, Siyang Jing and Yingqiao Zhou

November 27, 2017

Introduction

This report introduces four zombies outbreak models, each of which is a system of first-order, nonlinear differential equations. We will use the Heun method to give numerical approximations of these models and analyze the behaviors of different classes in each model. Specifically, with one given initial condition of a specific class in a model, we will evaluate how the changes of initial conditions of other classes and the changes of parameters influence the decaying rate of that class.

Statement of Problem

The four zombie outbreak: Basic model, Latent infection model, Quarantine model and Treatment model is given as follows.

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S, \\ Z' &= \beta SZ + \zeta R - \alpha SZ, \\ R' &= \delta S + \alpha SZ - \zeta R. \end{aligned}$$

Part I: Basic model

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S, \\ I' &= \beta SZ - \rho I - \delta I \\ Z' &= \rho I + \zeta R - \alpha SZ, \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R. \end{aligned}$$

Part II: Latent infection model

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S, \\ I' &= \beta SZ - \rho I - \delta I - \kappa I \\ Z' &= \rho I + \zeta R - \alpha SZ - \sigma Z, \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R + \gamma Q. \\ Q' &= \kappa I + \sigma Z - \gamma Q \end{aligned}$$

Part III: Quarantine model

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S + cZ, \\ I' &= \beta SZ - \rho I - \delta I \\ Z' &= \rho I + \zeta R - \alpha SZ - cZ, \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R. \end{aligned}$$

Part IV: Treatment model

For all the models above, S, Z and R denote the susceptible, zombie and removed class respectively; I in Part II denotes the infected class and Q in Part III denotes the quarantine class; Π, β and δ represent the rate of birth, transmission from Q and non-zombie related death rate respectively. α, ζ and ρ represent the rate of zombies defeated, the rate of resurrection from R to Z and the rate of transmission from I to Z. κ, σ and γ represent the rates of entering at Q from I , from Z and the rate of I being killed. c represents the rate of zombies being cured. In this project, $\alpha = \rho = 0.005$, $\beta = 0.0095$, $\zeta = 0.0001$, $\Pi = \delta = 0.0001$ and $S(0) = S_0 = 3 \times 10^8$.

We need to evaluate under which initial condition (Z_0, R_0) would human kind (S) survive in Part I, the effect of the latent infection (I) to the model in Part II, the effect of quarantine (Q) to the model in Part III and at which rate (c) to treat zombies so that human kind (S) could survive in Part IV.

Numerical Method

To solve for the four models, we will use Heun's method given by the following formula:

$$u_{n+1} = u_n + \frac{h}{2}[f_n + f(t_{n+1}, u_n + hf_n)] \tag{1}$$

Heun's method is an explicit method, which demands less computational cost than implicit method. An explicit method could be evaluated directly in terms of known quantities at the previous time step, whereas an implicit method generally requires a matrix or iterative solution to compute the new quantities since unknowns are at the both sides of an equation. Besides, the Heun's method is a method of order 2, and thereby gives more accurate approximations than first-order methods. The estimate of truncation error τ_n is derived as follows.

$$
\begin{aligned}
\tau_n &= \tfrac{u(t_{n+1})-u(t_n)}{h} - \tfrac{1}{2}(f_n + f(t_{n+1}, u(t_n) + hu'(t_n))) \\
&= u'(t_n) + \tfrac{h}{2!}u^{(2)}(t_n) + \mathcal{O}(h^2) - \tfrac{1}{2}\left[u'(t_n) + u'(t_n) + hf'(t_n, u_n) + \mathcal{O}(h^2)\right] \\
&= u'(t_n) + \tfrac{h}{2}u^{(2)}(t_n) + \mathcal{O}(h^2) - \tfrac{1}{2}(2\cdot u'(t_n) + hu^{(2)}(t_n) + \mathcal{O}(h^2)) = \mathcal{O}(h^2)
\end{aligned}
$$

We are curious about the behavior of the error, such as convergence and stability, when applying Heun's method to these specific models. Since the models are non-linear, and difficult to solve analytically, it's hard to get a theoretical bound for the error and stability region for the time step. Nevertheless, we can still get some knowledge of the behavior of the error, by comparing the solutions obtained with different time steps. First we rewrite the four models as non-linear systems.

$$X_I(t) = \begin{pmatrix} S(t) \\ Z(t) \\ R(t) \end{pmatrix} \; F_I(X_I) = \begin{pmatrix} \Pi - \beta SZ - \delta S \\ \beta SZ + \zeta R - \alpha SZ \\ \delta S + \alpha SZ - \zeta R \end{pmatrix} \; X_I' = F_I(X_I) \tag{2}$$

$$X_I(t) = \begin{pmatrix} S(t) \\ Z(t) \\ R(t) \end{pmatrix} \; F_{II}(X_{II}) = \begin{pmatrix} \Pi - \beta SZ - \delta S \\ \beta SZ - \rho I - \delta I \\ \rho I + \zeta R - \alpha SZ \\ \delta S + \delta I + \alpha SZ - \zeta R \end{pmatrix} \; X_{II}' = F_{II}(X_{II}) \tag{3}$$

$$X_{III}(t) = \begin{pmatrix} S(t) \\ I(t) \\ Z(t) \\ R(t) \\ Q(t) \end{pmatrix} \; F_{III}(X_{III}) = \begin{pmatrix} \Pi - \beta SZ - \delta S \\ \beta SZ - \rho I - \delta I - \kappa I \\ \rho I + \zeta R - \alpha SZ - \sigma Z \\ \delta S + \delta I + \alpha SZ - \zeta R \\ \kappa I + \sigma Z - \gamma Q \end{pmatrix} \; X_{III}' = F_{III}(X_{III}) \tag{4}$$

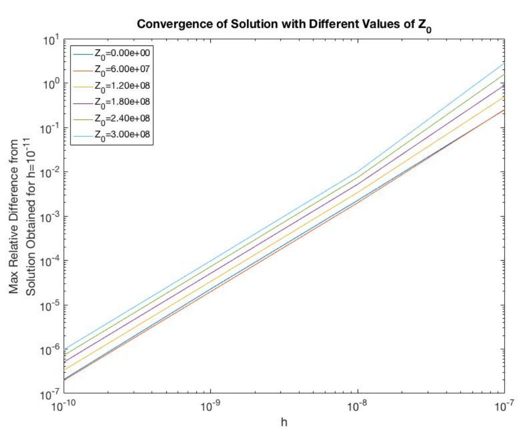
$$X_{IV}(t) = \begin{pmatrix} S(t) \\ I(t) \\ Z(t) \\ R(t) \end{pmatrix} \; F_{IV}(X_{IV}) = \begin{pmatrix} \Pi - \beta SZ - \delta S + cZ \\ \beta SZ - \rho I - \delta I \\ \rho I + \zeta R - \alpha SZ - cZ \\ \delta S + \delta I + \alpha SZ - \zeta R \end{pmatrix} \; X_{IV}' = F_{IV}(X_{IV}) \tag{5}$$

For each of the four models, we assign the initial condition Z_0 with 6 distinct values ranging from 0 to S_0 with equally-spaced interval, and set the other initial conditions I_0, R_0, and Q_0 to 0, and the other parameters to 0.1. We keep all the parameters, except Z_0, constant because in later discussions, we find that, for all the four models, the behavior of the solution $S(t)$ is dominated by Z_0, and the other parameters have considerably insignificant influence on the solution.

For each model and each Z_0, we assign time step sizes h with $10^{-7}, 10^{-8}, 10^{-9}, 10^{-10}, 10^{-11}$, and time interval $\left[0, 10^{-4}\right]$, after which the solutions stay considerably stable and roughly constant, and then apply Heun's method to get 5 solutions $S_i(t_j)$, $i = 1, 2, 3, 4, 5$, j standing for the time steps. Among the 5 solutions, the one obtained with the smallest time step, i.e. $S_5(t_j)$, is supposed to be the most accurate. We then calculate the maximum error of the other solutions relative to $S_5(t)$, defined as:

$$e_i = \max_j \left\{ \left| \frac{S_i(t_j) - S_5(t_j)}{S_5(t_j)} \right| \right\}, \; i = 1, 2, 3, 4 \tag{6}$$

Figure 1-4 show the log versus log plots of such max relative error against step sizes, for each Z_0 and each model, respectively. For each line, except for Z_0=0 in Part II, III, and IV, the slop is roughly 2, which means the error shows approximately quadratic convergence, as we expected from the order 2 convergence of Heun's method. For those exceptions, the error stays at a considerably small number, which does not undermine the general convergence of the method. Therefore, we can assume that with the time step h in the range of 10^{-7} to 10^{-11}, the numerical method is stable and convergent. Therefore, in solving all the differential equations below,we set the time step to $h = 10^{-10}$, which takes considerably less time than $h = 10^{-11}$.



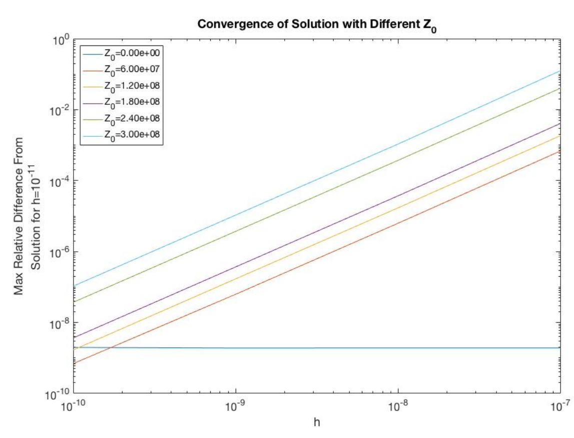
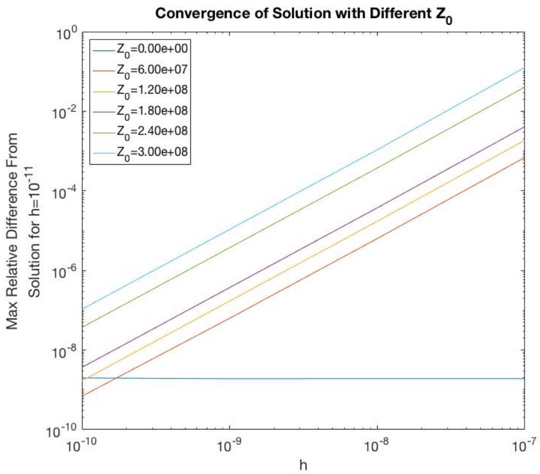


Figure 1: (Left) log-log of Estimated Relative Error versus Time Step, Part I
Figure 2: (Right) log-log of Estimated Relative Error versus Time Step, Part II



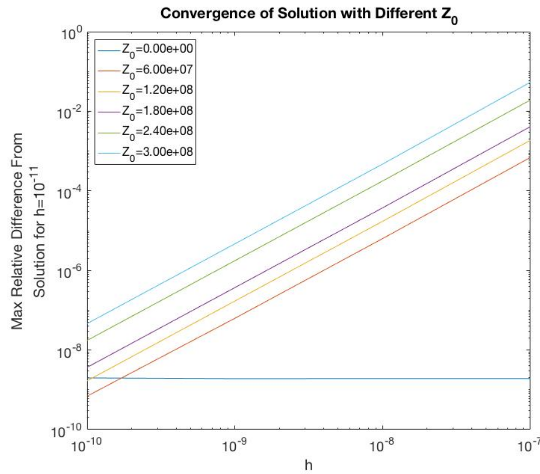


Figure 3: (Left) log-log of Estimated Relative Error versus Time Step, Part III

Figure 4: (Right) log-log of Estimated Relative Error versus Time Step, Part IV

Discussion

1. Basic Model

From equation $S^{'} = \Pi - \beta SZ - \delta S$, both Z and R have negative effect on the growth rate of the susceptible class. Consider an optimal case that minimizes the negative effect with $Z_0 = 0$ and $R_0 = 0$ as they are forced to be nonnegative. We try to numerically solve (2) with such initial condition by applying Heun's method (1) to get the solution. We set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_I(0) = (\ S_0, \ \ 0, \ \ 0 \ )^T$.

The solution to the Basic Model with optimal initial condition is given by Figure 5, which implies that the susceptible will eventually die out even for the extreme case in which initially there is no Zombies or the Removed. In that case, we are capable of deducing the conclusion that for any values of Z_0 and R_0, $S(t)$ will eventually decay to zero. Thus, under no initial conditions could human kind survive. Nevertheless, we still want to investigate the behavior of $S(t)$ with different values of Z_0 and R_0.

First, we fix the value of R_0 at 3×10^8. In Figure 6 below, solutions $S(t)$ are plotted with different colors corresponding to the 6 different values of Z_0 ranging from 0 to S_0. We can see that the speed of the decay of population varies, but all decay to sufficiently near zero eventually, again confirming our reasoning that human will extinct regardless of initial condition.

In addition, when $Z_0 = 0$, the behavior of $S(t)$ exhibits significant dependence on R_0. Figure 7 below shows the trend of $S(t)$ with fixed Z_0 at 0, in which different colors correspond to different R_0 values.

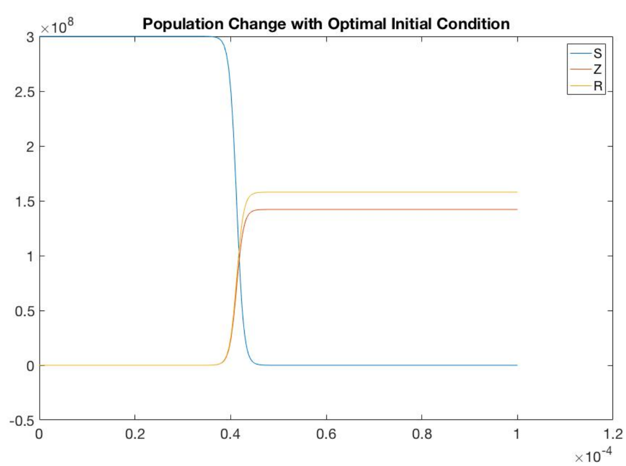
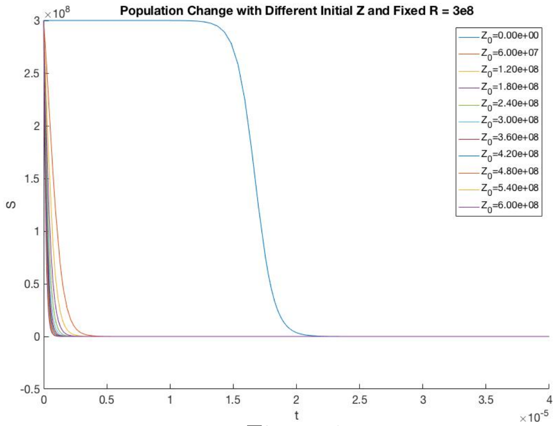


Figure 5: Population Change with $Z_0 = 0$, and $R_0 = 0$



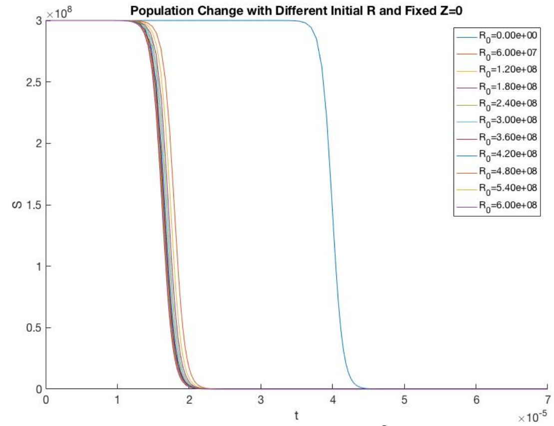


Figure 6: (Left) Population Change with Different Z_0 and $R_0 = 3 \times 10^8$

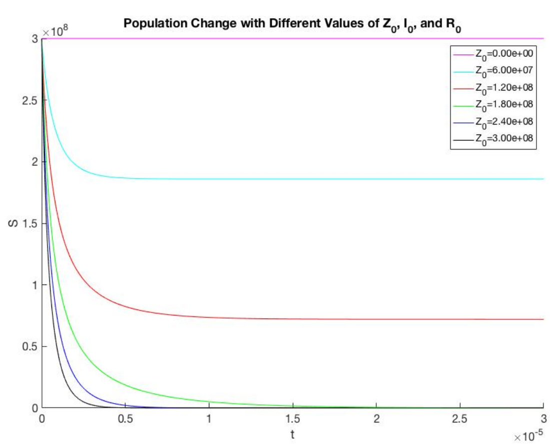
Figure 7: (Right) Population Change with Different Initial R and Fixed $Z = 0$

2. Latent infection Model

To check the effect of latent infection, we initialize Z_0, I_0 and R_0 with different values respectively. Assign Z_0 with 6 distinct values ranging from 0 to S_0 with equally-spaced interval; I_0 be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval; R_0 be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval. We apply Heun's method (1) to (3) to get the solution. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $x_{II}(0) = (\ S_0,\ \ I_0,\ \ Z_0,\ \ R_0\ )^T$. Figure 8 below gives solutions with different initial conditions, with different colors corresponding to different Z_0.

Notice that the lines of the same color basically overlap with each other, indicating that changes in I_0 and R_0 does not cause significant change. Thus, the behavior of $S(t)$ is dominated by Z_0. We can also see that for certain values of Z_0, $S(t)$ does not decay to zero, but rather seem to converge to a non-zero number. Such behavior is not observed in Part I. In addition, the final stable human population $S(t_n)$ seems to be dependent on Z_0. Specifically, the more initial zombies there are, the less final human population would be, which is certainly reasonable in this situation.

To further evaluate how latent infection affect the human population $S(t)$, fix I_0 and R_o as they are not significantly influencing the behavior of $S(t)$, and assign Z_0 with different values ranging from 1.5×10^8 to 3×10^8 with equally spaced interval. Applying (1) to (2) and(3) with these initial conditions, we get the solutions of $S(t)$ in different parts, as shown in Figure 9, with solid line representing Part II and dashed line representing Part I. Then we can see that latent infection reduced the rate of decay in $S(t)$.



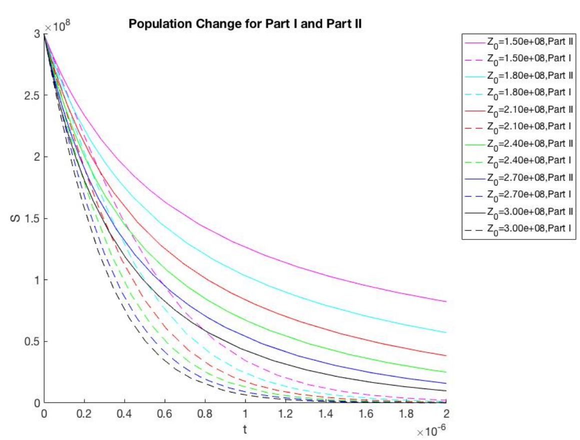


Figure 8: (Left) Population Change with Different Values of Z_0, I_0, and R_0

Figure 9: (Right) Population Change for Part I and Part II

3. Quarantine Model

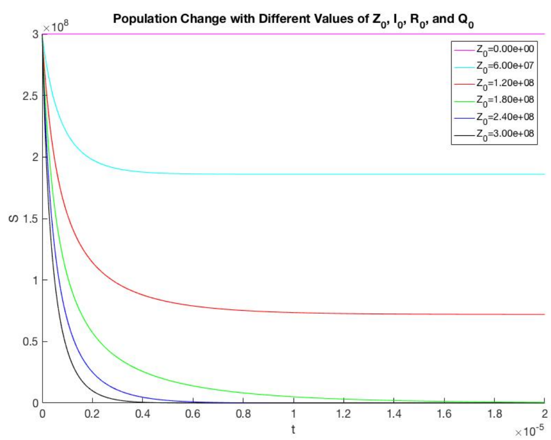
We need to investigate the dependence of S on initial conditions Z_0, I_0, R_0, Q_0, and parameters κ, γ, σ.

First, we check dependence on the initial conditions by initializing Z_0, I_0, R_0 and Q_0 with different values and fixing the parameters κ, γ, and σ. Assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval; I_0, R_0, and Q_0 be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval. Fix

κ, γ, and σ to be 0.1. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_{III}(0) = (\ S_0,\ \ I_0,\ \ Z_0,\ \ R_0,\ \ Q_0\ )^T$. Apply Heun's method (1) to equation (4) to get the solutions with different initial conditions, which is shown in Figure 10, with different colors corresponding to different Z_0. Notice that the lines of the same color basically overlap with each other, indicating that changes in I_0, R_0, and Q_0 do not cause significant change. Thus, the behavior of $S(t)$ is dominated by Z_0.

Second, we check dependence on the parameters κ, γ, and σ by initializing Z_0, κ, γ, and σ with different values and fixing the parameters I_0, R_0, and Q_0. Assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval. For each Z_0, assign κ, γ, and σ to be 6 distinct values ranging from 0 to 1, with equally-spaced interval. Fix I_0, R_0 and Q_0 to be 3×10^8. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_{III}(0) = (\ S_0,\ \ S_0,\ \ Z_0,\ \ S_0,\ \ S_0\ )^T$. Apply Heun's method (1) to (4) to get the solutions with different parameters, which is shown in Figure 11, with different colors corresponding to different Z_0. Notice that the lines of the same color basically overlap with each other, indicating that changes in κ, γ, and σ do not cause significant change. Thus, the behavior of $S(t)$ is dependent of parameters κ, γ, and σ.

To further evaluate how quarantine affects the human population $S(t)$, fix I_0, R_0 and Q_0, respectively, to be S_0; κ, γ, and σ, respectively, be 0.2 as they are not significantly influencing the behavior of $S(t)$, and assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_I(0) = (\ S_0,\ \ Z_0,\ \ S_0\ )^T$, $X_{II}(0) = (\ S_0,\ \ S_0,\ \ Z_0,\ \ S_0\ )^T$, $X_{III}(0) = (\ S_0,\ \ S_0,\ \ Z_0,\ \ S_0,\ \ S_0\ )^T$. Apply Heun's method (1) to (2), (3), and (4), respectively, to get the solution to each part, as shown in Figure 12, with solid line representing Part III, dashed line representing Part II, and dotted line representing Part I. From the figure we can see the dashed line and solid line basically overlap with each other, thus quarantine does not have a huge influence.



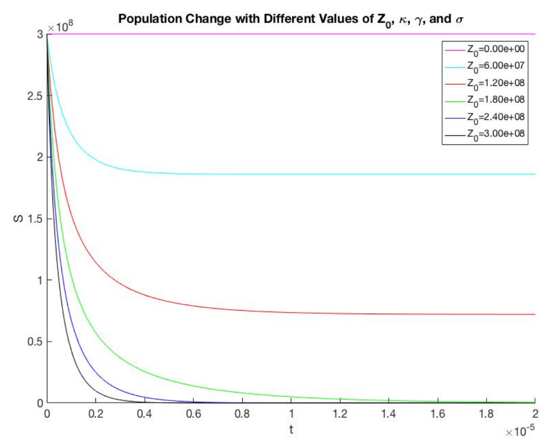


Figure 10: (Left) Population Change with Different Values of Z_0, I_0, R_0, and Q_0

Figure 11: (Right) Population Change with Different Values of Z_0, κ, γ, and σ

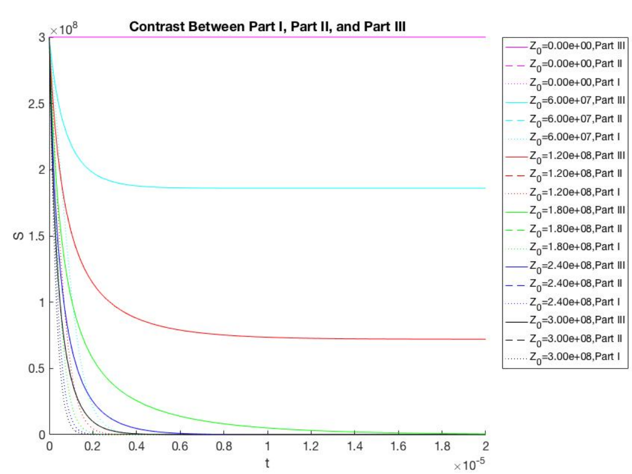
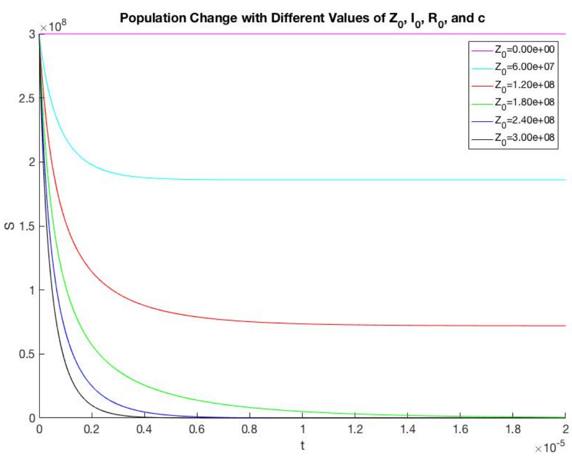


Figure 12: Population Change for Part I, Part II, and Part III

4. Treatment Model

Again, we need to investigate the dependence of $S(t)$ on the parameters Z_0, I_0, R_0, and $c$by initializing Z_0, I_0, R_0, and c with different values respectively. Assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval; I_0 and R_0, respectively, be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval; c be 11 distinct values ranging from 0 to 1, with equally spaced interval. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_{IV}(0) = (\; S_0, \quad I_0, \quad Z_0, \quad R_0 \;)^T$. Apply Heun's method (1) to equation (5) to get the solutions of different initial conditions, with different colors corresponding to different Z_0. The result is shown in Figure 13. Notice that the lines of the same color basically overlap with each other, indicating that changes in I_0, R_0, and $c$do not cause significant change. Thus, the behavior of $S(t)$ is dominated by Z_0 and barely dependent of the parameter c.

To further evaluate how treatment affects the human population $S(t)$, we plot the human population $S(t)$ in different parts on the same figure. Fix $I_0 = S_0$, $R_0 = S_0$, $Q_0 = S_0$, $\kappa = 0.1$, $\gamma = 0.1$, $\sigma = 0.1$, and $c = 0.1$, as they are not significantly influencing the behavior of $S(t)$, and assign Z_0 with different values. The solution to each part is shown in Figure 14, with solid line representing Part IV, dashed line representing Part II, dash-dot line representing Part IV, and dotted line representing Part I. From the figure we can see the dashed line, dash-dot line, and solid line basically overlap with each other, indicating treatment does not have a huge influence on $S(t)$.



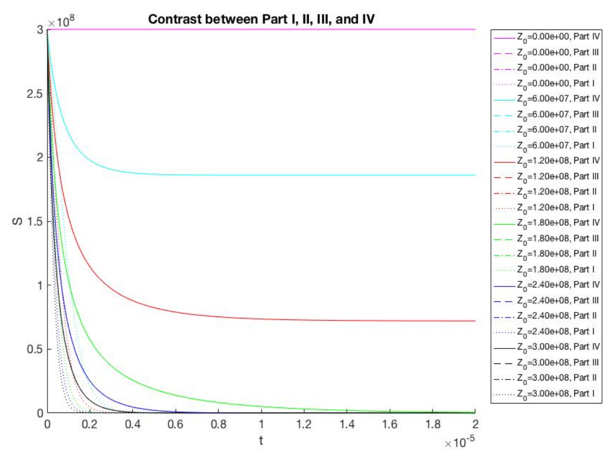


Figure 13: (Left) Population Change with Different Values of Z_0, I_0, R_0, and c

Figure 14: (Right) Population Change for Part I, II, III, and IV

Conclusion

We used Heun's method to solve the non-linear system of ordinary differential equations (1)-(4). We find that in Part I, human will extinct regardless of initial condition, as shown in Figure 6 and 7. In Part II, with latent infection, human population will not decay to zero with small values of initial zombie population, and the initial population of the removed does not have a significant influence, as shown in Figure 8; in the situation where human will extinct, the rate of decay of human population is slowed, compared with Part I, as shown in Figure 9. In Part III and IV, we find that the situation is not significantly different from that of Part II, and the dominant factor is still initial zombie population, and other parameters do not play an important role.