STOR 435 Homework 22

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Notes on Bivariate Normal Distribution

$$f(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)-2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)}$$

$$f(\boldsymbol{x})=|2\pi\boldsymbol{\Sigma}|^{-\frac{1}{2}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})},$$

$$\boldsymbol{x}=\begin{pmatrix} x \\ y \end{pmatrix},\ \boldsymbol{\mu}=\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix},\ \boldsymbol{\Sigma}=\begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix},\ \boldsymbol{\Sigma}^{-1}=\frac{1}{1-\rho^2}\begin{pmatrix} \sigma_X^{-2} & \frac{-\rho}{\sigma_X\sigma_Y} \\ \frac{-\rho}{\sigma_X\sigma_Y} & \sigma_Y^{-2} \end{pmatrix},\ |2\pi\boldsymbol{\Sigma}|=2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}$$

$$f(x,y)=\frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}\frac{1}{\sqrt{1-\rho^2}\sigma_Y\sqrt{2\pi}}e^{-\frac{\left(y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2}{2\sigma_Y^2(1-\rho^2)}}$$

$$f_X(x)=\frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}=\mathcal{N}(\mu_X,\sigma_X),$$

$$f_{Y|X}(y|x)=\frac{1}{\sqrt{1-\rho^2}\sigma_Y\sqrt{2\pi}}e^{-\frac{\left(y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2}{2\sigma_Y^2(1-\rho^2)}}=\mathcal{N}\left(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X),\sqrt{1-\rho^2}\sigma_Y\right)$$

Notes on Joint Probability and Mappings

Given f_{X_1,X_2}, consider $Y_1=g_1(X_1,X_2)$, $Y_2=g_2(X_1,X_2)$. Assume nice properties, i.e. continuous, differentiable, and invertible, Y_1, Y_2, (Y_1,Y_2)?

We consider a more general case: $\boldsymbol{X}\colon\mathbb{R}^n\to\mathbb{R}, \boldsymbol{x}\mapsto F(\boldsymbol{x})$, $G\colon\{\boldsymbol{F}\colon\mathbb{R}^n\to\mathbb{R}\}\to\{\boldsymbol{F}\colon\mathbb{R}^n\to\mathbb{R}\}$,

s.t. $G[\boldsymbol{X}](g(\boldsymbol{x}))=\boldsymbol{X}(\boldsymbol{x})$, assume suffiently nice properties of G and g, let $h:=g^{-1}$

$G[\boldsymbol{X}](\boldsymbol{y})=\boldsymbol{X}(g^{-1}(\boldsymbol{y}))$

//////////////TODO

$f_{\boldsymbol{Y}}(\boldsymbol{y})=f_{g(\boldsymbol{X})}(g(\boldsymbol{x}))=f_{\boldsymbol{X}}(\boldsymbol{x})\left|\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{y}}\right|$, which is analogous to the one dimensional case.

A definition for $\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{y}}$, $\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{y}}=\left(\frac{\partial\boldsymbol{y}}{\partial\boldsymbol{x}}\right)^{-1}=\det\begin{pmatrix} \frac{\partial g_1(\boldsymbol{x})}{\partial x_1} & \frac{\partial g_2(\boldsymbol{x})}{\partial x_1} \\ \frac{\partial g_1(\boldsymbol{x})}{\partial x_2} & \frac{\partial g_2(\boldsymbol{x})}{\partial x_2} \end{pmatrix}^{-1}=\det(\mathcal{J}[g](\boldsymbol{x}))^{-1}=\det(\mathcal{J}[g](h(\boldsymbol{y})))^{-1}$

We also have $\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{y}}=\det\begin{pmatrix} \frac{\partial h_1(\boldsymbol{y})}{\partial y_1} & \frac{\partial h_2(\boldsymbol{y})}{\partial y_1} \\ \frac{\partial h_1(\boldsymbol{y})}{\partial y_2} & \frac{\partial h_2(\boldsymbol{y})}{\partial y_2} \end{pmatrix}=\det(\mathcal{J}[h](\boldsymbol{y}))=\det(\mathcal{J}[h](g(\boldsymbol{x})))$

Questions:

1. Still need to figure out a rigorous proof.

2. What about CDF?

3. Decreasing? Increasing?
4. Do we have a higher perspective to look at this?

Homework

1. $Y|X=80 \sim \mathcal{N}\left(\mu = 75 + 0.8 \times \frac{16}{10}(80-85), \sigma = \sqrt{1-0.8^2} \times 16\right) \sim \mathcal{N}(\mu = 68.6, \sigma = 9.6)$

   $P(Y > 80|X=80) = 1 - F_{Y|X}(80|80) = 1 - \Phi\left(\frac{80-68.6}{9.6}\right) \approx 0.1170$ (table), 0.1175 (exact)

2.
   a) $f_{U,V}(u,v) = f_{X,Y}(uv, v)\left|\det\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}\right| = f_X(uv) f_Y(v) v = \alpha\beta v e^{-\alpha uv - \beta v}$

   $U \in [0, \infty)$, $V \in (0, \infty)$

   // Question: Can $V = 0$?

   // If $V = 0$, then $U = X/0$ is not really well-defined

   // If $V = 0$, then $f_{U,V}(u,v) = 0$.

   // From this point of view, V cannot equal 0

   // However, $V = Y$ means V have the same domain as Y, which includes 0

   // Further question: We have assumed g is invertible, but here it's probably not.

   // Consider a simpler case, $W := 1/Y$, what can we say about W?

   // However, for U, $0/0$ could be defined maybe? As a limit?

   // Need a more mathematically rigorous definition for things like X/Y

   // In fact, we haven't seen any mathematically rigorous definition for anything.

   b) $f_V(v) = f_Y(v) = \beta e^{-\beta v}$

3.
   a) $\frac{\partial(x,y)}{\partial(R,\theta)} = \det\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -R\sin(\theta) & R\cos(\theta) \end{pmatrix} = R$

   $f_{R,\theta}(R,\theta) = f_{X,Y}(x,y)R = f_X(x) f_Y(y) R = (2\pi)^{-1} e^{-\frac{1}{2}(x^2+y^2)} R = (2\pi)^{-1} e^{-\frac{1}{2}R^2} R$

   b) $f_\theta(\theta) = \int_0^\infty f_{R,\theta}(R,\theta) dR = (2\pi)^{-1}$

   θ is a uniform distribution on $[0, 2\pi]$,

   which confirms the isometric symmetry of Gaussian.

4.
   a) $\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = -1/2$

   $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)/2 = (2\pi)^{-1} e^{-\frac{1}{2}(x_1^2+x_2^2)}/2 = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\left(\frac{y_1}{\sqrt{2}}\right)^2} \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\left(\frac{y_2}{\sqrt{2}}\right)^2}$

   b) We identify this as the product of two normal pdf's, which means Y_1 and Y_2 are separable, and $Y_1 \sim Y_2 \sim \mathcal{N}(\mu = 0, \sigma^2 = 2)$