Elementary Analysis

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Selected Solutions

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EXERCISE 1.2

Claim:

$$P(n) = 3 + 11 + \cdots + (8n - 5) = 4n^2 - n \ \forall n \in \mathbb{N}$$

***Proof*:**

By induction. Let $n = 1$. Then $3 = 4(1)^2 - (1) = 3$, which will serve as the induction basis. Now for the induction step, we will assume $P(n)$ holds true and we need to show that $P(n+1)$ holds true. $P(n+1) = 3 + 11 + \cdots + (8n - 5) + (8(n+1) - 5) = 4(n+1)^2 - (n+1)$.

Now, $3 + 11 + \cdots + (8n - 5) + (8(n+1) - 5) = 4n^2 - n + 8(n+1) - 5$. So in order to show that $P(n+1)$ is true, we need to show that $4(n+1)^2 - (n+1) = 4n^2 - n + 8(n+1) - 5$, which, when evaluated is true, as desired. ♠

EXERCISE 1.3

Claim:

$$P(n) = 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \ \forall n \in \mathbb{N}$$

***Proof*:**

By induction. Let $n = 1$. Then, $P(1) = 1^3 = 1 = 1^2$, which will serve as the induction basis. Now for the induction step, we will assume $P(n)$ holds true and we need to show that $P(n+1)$ holds true.

$$\begin{aligned} P(n+1) = 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + \cdots + n)^2 + (n+1)^3 \\ &= (1 + 2 + \cdots + n)^2 + [(n+1)^3 - (n+1)^2] + (n+1)^2 \\ &= (1 + 2 + \cdots + n)^2 + (n+1)^2[n + 1 - 1] + (n+1)^2 \\ &= (1 + 2 + \cdots + n)^2 + 2(n+1)n(n+1)/2 + (n+1)^2 \\ &= (1 + 2 + \cdots + n)^2 + 2(n+1)(1 + 2 + \cdots + n) + (n+1)^2 \\ &=^{1} (1 + 2 + \cdots + (n+1))^2, \end{aligned}$$

as desired. ♠

[1] $(a+b)^2 = a^2 + 2ab + b^2$, where $a = 1 + 2 + \cdots + n$ and $b = n + 1$.

EXERCISE 1.4

(a) For

$n = 1$, the expression is 1.

$n = 2$, the expression is 4.

$n = 3$, the expression is 9.

$n = 4$, the expression is 16.

We note that $1 + 3 + \cdots (2n - 1) = n^2$, which is the proposed formula.

(b) *Claim*:

$$P(n) = 1 + 3 + \cdots (2n - 1) = n^2 \ \forall n \in \mathbb{N}$$

***Proof*:**

By induction. We already showed the case when $n = 1$. Now for the induction step, we will assume $P(n)$ holds true and we need to show that $P(n + 1)$ holds true.

$$P(n + 1) = 1 + 3 + \cdots (2(n + 1) - 1) = 1 + 3 + \cdots + (2n - 1) + (2n + 1)$$

$$= n^2 + 2n + 1 = (n + 1)^2,$$

as desired. ♠

EXERCISE 1.6

Claim:

$$P(n) = (11)^n - 4^n \text{ is divisible by 7 when } n \in \mathbb{N}$$

***Proof*:**

By induction. $P(1) = 11 - 4 = 7$ is divisible by 7. Now for the induction step, we will assume $P(n)$ holds true and we need to show that $P(n + 1)$ holds true.

$$P(n + 1) = (11)^{n+1} - 4^{n+1} = 11^{n+1} - 4 \cdot 11^n + 4 \cdot 11^n - 4^{n+1}$$

$$= 11^n(11 - 4) + 4(11^n - 4^n) = 7 \cdot 11^n + 4(11^n - 4^n).$$

Since we assumed that $11^n - 4^n$ was divisible by 7 because we assumed $P(n)$ was true, we can see that

$$11^{n+1} - 4^{n+1}$$

is divisible by 7, as desired. ♠

EXERCISE 1.9

(a) The inequality does not hold for $n = 2, 3, 4$. It holds true for all other $n \in \mathbb{N}$.

(b) It is true by by inspection for $n = 1$ and $2^4 = 4^2$ also holds for $n = 4$. Implement the induction step. For $n \geq 4$, if $2^n \geq n^2$, then $2^{(n+1)} > (n+1)^2$. But $2^{(n+1)} = 2 \cdot 2^n \geq 2n^2 > (n+1)^2$ iff $(n+1) < \sqrt{2}n$, for example when $n > \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$, which includes $n \in \mathbb{N} : n \geq 4$.

EXERCISE 1.10

Claim:

$$(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2 \ \forall n \in \mathbb{N}.$$

***Proof*:**

Using the results from 1.4, we can avoid induction. Observe that

$$\begin{aligned}(2n+1)+(2n+3)+\cdots+(4n-1) &= (1+3+\cdots+(4n-1)) - (1+3+\cdots+(2n-1)) \\ &= (1+3+\cdots+2(2n)-1) - (1+3+\cdots+(2n-1)) \\ &= (2n)^2 - n^2 = 4n^2 - n^2 = 3n^2,\end{aligned}$$

as desired. ♠

EXERCISE 1.12 **(b)** and **(c)**

(b) Observe the following:

$$\frac{1}{k} + \frac{1}{n-k+1} = \frac{(n-k+1)+k}{k(n-k+1)} = \frac{n+1}{k(n-k+1)}.$$

Hence,

$$\begin{aligned}\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} &= \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1}\right] \\ &= \frac{n!}{(k-1)!(n-k)!}\left[\frac{n+1}{k(n-k+1)}\right] = \frac{(n+1)!}{k!(n-k+1)!}.\end{aligned}$$

(c) For $n = 1 : (a+b)^n = a + b = \binom{1}{1}a + \binom{1}{1}b$. Now, for $n \geq 1$, we get

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Hence,

$$
\begin{aligned}
(a+b)^{(n+1)} &= (a+b)\sum_{l=0}^{n}\binom{n}{l}a^l b^{n-l} = \sum_{k=0}^{n+1}\left[\binom{n}{k-1}+\binom{n}{k}\right]a^k b^{n-k+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n-1+k}.
\end{aligned}
$$

EXERCISE 2.1

Show that $\sqrt{3}$ is not rational *(The exact same technique can be used to show the other numbers are not rational).*

Claim:

$$\sqrt{3} \notin \mathbb{R}$$

***Proof*:**

By Contradiction. If $\sqrt{3} \in \mathbb{R}$, then $\exists$ an $r \in \mathbb{Q} : \frac{p}{q} = \sqrt{3}$, where $r = \frac{p}{q} \Longrightarrow \left(\frac{p}{q}\right)^2 = 3 \Longrightarrow p^2 = 3q^2 \Longrightarrow p^2$ is divisible by $3 \Longrightarrow p$ is divisible by 3. Let $p = 3k$. Then $9k^2 = 3q^2 \Longrightarrow 3k^2 = q^2 \Longrightarrow q^2$ is divisible by $3 \Longrightarrow q$ is divisible by 3, which is a contradiction. ♠

EXERCISE 2.5

Let's assume $[3+\sqrt{2}]^{\frac{2}{3}}$ *does* represent a rational number. Further, let's call this rational number q. This implies $q^3 = [3+\sqrt{2}]^2 = 9 + 6\sqrt{2} + 2 = 11 + 6\sqrt{2}$, which in turn implies $\sqrt{2} = \frac{(q^3-11)}{6}$, which is a rational number. But we know $\sqrt{2} \notin \mathbb{Q}$ - a contradiction.

EXERCISE 3.3

This problem will be done in two parts.

Part I: Show that $(-a)(-b) = ab \ \forall \ (a, b) \in \mathbb{R}$.

Proof:

$$
\begin{aligned}
(-a)(-b) &= ab & (1)\\
\iff (-a)(-b) + (-ab) &= ab + (-ab) & (2)\\
\iff (-a)(-b) + (-a)b &= ab + (-ab) & (3)\\
\iff (-a)\left[(-b) + b\right] &= ab + (-ab) & (4)\\
\iff (-a)\left[b + (-b)\right] &= ab + (-ab) & (5)\\
\iff (-a)(0) &= ab + (-ab) & (6)\\
0 &= ab + (-ab), & (7)
\end{aligned}
$$

which is true by **A4**. Hence, $(-a)(-b) = ab$[2] ♠

Part II: $ac = bc$ and $c \neq 0$ imply $a = b \ \forall \ (a, b, c) \in \mathbb{R}$.

Proof:

$$
\begin{aligned}
ac(c^{-1}) &= bc(c^{-1}) & (8)\\
a(cc^{-1}) &= b(cc^{-1}) & (9)\\
a(1) &= b(1) & (10)
\end{aligned}
$$

Hence, $a = b$[3] ♠

EXERCISE 3.6

a. *Claim*: $|a + b + c| \leq |a| + |b| + |c|$

Proof: One iteration of the triangle inequality can be used to construct:

$$|a + b + c| = |(a + b) + c| \leq |a + b| + |c| \,.$$

A second iteration of the triangle inequality yields:

$$|a + b| + |c| \leq |a| + |b| + |c| \,,$$

[2] **(1)** used **(i)**, **(2)** used **(iii)**, **(3)** used **DL**, **(4)** used **A2**, **(5)** used **A4**, **(7)** reintroduced **(1)** and the conclusion reused **(i)**.

[3] **(8)** was built from multiplying both sides of the equality by the same element to preserve equality, **(9)** used **M1** and **(10)** used **M4**.

as desired. ♠

b. We want to show $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots |a_n|$

We will use the principal of mathematical induction to show the above is true.

Proof:

P_1 is certainly true, since $|a_1| = |a_1|$, and thus will serve as our basis for induction. Our induction hypothesis is:

Assume $\exists$ an $n \in \mathbb{R} : P_n$ holds, i.e. $P_n : |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots |a_n|$.

Implementing the induction step and the triangle inequality yield:

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leq |a_1 + a_2 + \cdots + a_n| + |a_{n+1}|.$$

The induction step and **O4** yield:

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leq |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|,$$

as desired. ♠

EXERCISE 4.2, **(a)** through **(n)**

This exercise is asking us to list three lower bounds for the set; if the set is not bounded below, it will be labeled "NOT BOUNDED BELOW" or "NBB". The rule outlined in the text we will follow is **Definition 4.2 (b)**:

If a real number m satisfies $m \leq s \; \forall \; s \in S$, then m is called a lower bound of S and the set S is said to be bounded below.

a. $S = [0, 1]$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

b. $S = [0, 1)$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

c. $S = \{2, 7\}$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

d. $S = \{\pi, e\}$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

e. $S = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

f. $S = \{0\}$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

g. $S = [0,1] \cup [2,3]$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

h. $S = \bigcup_{n=1}^{\infty} [2n, 2n+1]$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

i. $S = \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1+\frac{1}{n}\right]$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

j. $S = \left\{1 - \frac{1}{3^n} : n \in \mathbb{N}\right\}$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

k. $S = \left\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$; -1, -2 and -3 satisfy **Definition 4.2 (b)**.

l. $S = \{r \in \mathbb{Q} : r < 2\}$; **"NBB"**. The set **Q** has no maximum or minimum[4]. S has an upper bound, but since there is no lower bound outlined, this set is unbounded from below.

m. $S = \left\{r \in \mathbb{Q} : r^2 < 4\right\}$; $\frac{-2}{1}$, $\frac{-3}{1}$ and $\frac{-4}{1}$ satisfy **Definition 4.2 (b)**.

n. $S = \left\{r \in \mathbb{Q} : r^2 < 2\right\}$; $\frac{-2}{1}$, $\frac{-3}{1}$ and $\frac{-4}{1}$ satisfy **Definition 4.2 (b)**.

EXERCISE 4.4, **(a)** through **(n)**, ADDITIONALLY, DETERMINE IF THE MINIMUM OF THE SET EXISTS.

This exercise is asking us to give the infima of each set S. The rule outlined in the text we will follow is **Definition 4.3 (b)**:

If S is bounded below and S has a greatest lower bound, then we will call it the infimum of S and denote it by inf$S$.

Additionally, we are asked to determine if the minimum of the set (denoted min$S$) exists. Note that, unlike the minimum of the set S, inf$S$ need not belong to the set S[5]

a. $S = [0,1]$; inf$S$ = min$S$ = 0.

b. $S = (0,1)$; inf$S$ = 0, min$S$ D.N.E.

c. $S = \{2,7\}$; inf$S$ = min$S$ = 2.

d. $S = \{\pi, e\}$; inf$S$ = min$S$ = e.

[4] Page 20, **Example 1 (c)**.

[5] Page 21, first paragraph below **Definition 4.3**, first sentence.

e. $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$; $\inf S = 0$, however, $\min S$ does not exist.

f. $S = \{0\}$; $\inf S = \min S = 0$.

g. $S = [0,1] \cup [2,3]$; $\inf S = \min S = 0$.

h. $S = \bigcup_{n=1}^{\infty} [2n, 2n+1]$; $\inf S = \min S = 2$.

i. $S = \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$; $\inf S = 0$, $\min S$ D.N.E.

j. $S = \left\{1 - \frac{1}{3^n} : n \in \mathbf{N}\right\}$; $\inf S = \min S = \frac{2}{3}$.

k. $S = \left\{n + \frac{(-1)^n}{n} : n \in \mathbf{N}\right\}$; $\inf S = \min S = 0$.

l. $S = \{r \in \mathbb{Q} : r < 2\}$; both $\inf S$ and $\min S$ do not exist since S is **"NBB"**.

m. $S = \left\{r \in \mathbb{Q} : r^2 < 4\right\}$; $\inf S = -2$, however, $\min S$ does not exist.

n. $S = \left\{r \in \mathbb{Q} : r^2 < 2\right\}$; $\inf S = -\sqrt{2}$, however, $\min S$ does not exist.

EXERCISE 4.8

Let S and T be non-empty subsets of $\mathbb{R}$, with $s \le t$ $\forall s \in S$ and $t \in T$.

(a) Since it is given that $s \le t$ $\forall s \in S$ and $t \in T$, we know that any element of T will bound S from above, so $\exists \sup S$. Conversely, since it is given that $s \le t$ $\forall s \in S$ and $t \in T$, we know that any element of S will bound T from below, so $\exists \inf T$.

(b) *Claim*:

$$\sup S \le \inf T$$

This will be done in two parts.

***Proof*:**

Given $s \le t$ $\forall t \in T$, it follows that s is a lower bound for T. By definition, $\inf T$ is the greatest lower bound for T. Hence, $s \le \inf T$ $\forall s \in S$.

Since $s \le \inf T$ $\forall s \in S$, $\inf T$ is an upper bound for S. It follows that since $\sup S$ is the least upper bound for S,

$$\sup S \le \inf T,$$

as desired. ♠

(c) Let $S = (0, 1]$ and $T = [1, 2)$. It is readily observable that sup$S$ = inf$T$. It is also readily observable that

$$S \cap T \neq \emptyset.$$

(d) Let $S = (0, 1)$ and $T = (1, 2)$. It is readily observable that sup$S$ = inf$T$. It is also readily observable that

$$S \cap T = \emptyset.$$

EXERCISE 4.12

Claim:

Given $a < b$, $\exists\ x \in \mathbb{R} \setminus \mathbb{Q} : a < x < b$.

***Proof*:**

Following the hint, we know $r + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ when $r \in \mathbb{Q}$. By contradiction, if $r \in \mathbb{Q}$ and if the number $x = r + \sqrt{2} \in \mathbb{Q}$, then $\sqrt{2} = x - r$ would have been $\in \mathbb{Q}$, a contradiction. Now, due to the denseness of Q in R, we find an $r \in \mathbb{Q} : a - \sqrt{2} < r < b - \sqrt{2}$. Then we have $x = r + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ and we have $a < x < b$, as desired. ♠

EXERCISE 4.14 **(a)**

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let S be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

Claim: sup$S$ = sup$A$ + sup$B$.

***Proof*:**

The information we are given is that A and B are both subsets of $\mathbb{R}$ which are non-empty and bounded from above (since they are *bounded*, this implies they are bounded *from above and below*). Since both subsets A and B are both non-empty and bounded from above, they both possess a least upper bound, i.e., both sup$A$ and sup$B$ exist[6]. If S is to be defined as the set whose elements are all of the sums $a + b$, where $a \in A$ and $b \in B$ and we are given both A and B bounded, we know S is bounded; more specifically bounded from *below and above*. Hence, by the Completeness Axiom, S is bounded above and as a consequence sup$S$ exists. If sup$S$ exists, then $\exists$ a number $O \in \mathbb{R} : r \leq 0\ \forall\ r \in S$ and whenever $O_1 < O$, $\exists\ r_1 \in O : r_1 < O_1$..

[6] Otherwise, the "Completeness Axiom" wouldn't hold.

If both sup$A$ and sup$B$ exist, then there exists a number $M \in \mathbb{R} : s \leq M \; \forall \; s \in A$ and there exists a number $N \in \mathbb{R} : t \leq N \; \forall \; t \in B$; whenever $M_1 < M \; \exists \; s_1 \in A : s_1 < M_1$ and whenever $N_1 < N \; \exists \; t_1 \in B : t_1 < N_1$.

As a consequence of the above identities, it is obvious that since $S \equiv$ set of all sums $a + b$ where $a \in A$ and $b \in B$, $O \equiv M + N$, and hence sup$S$ = sup$A$ + sup$B$, as desired. ♠

EXERCISE 4.15

Claim:

Let $a, b \in \mathbb{R}$. If $a \leq b + \frac{1}{n} \; \forall n \in \mathbb{N}$, then $a \leq b$.

***Proof*:**

By contradiction. Assume $a > b$. This implies $a - b > 0$. By the "Archimedian Property" of $\mathbb{R} \; \exists \; n \in \mathbb{N} : a - b > \frac{1}{n}$. Using this specific n, we have $a > b + \frac{1}{n}$, a contradiction, as desired. ♠

EXERCISE 4.16

Claim: $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

***Proof*:**

Let $S \equiv \{r \in \mathbb{Q} : r < a\}$. From this, we can see that S is certainly bounded from above, since we can chose any number $a + n$ with $n \in \mathbb{N}$ and **Definition 4.2 (b)** will still be satisfied. Now by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, we know that[7] if $(a, b) \in \mathbb{R}$ with $a < b$, then $\exists$ an $r \in \mathbb{Q} : a < r < b$. So although we know S is bounded and we can certainly povide many upper bounds for this set, if we search for a *least upper bound* for this set, the search will go on indefinitely, since whatever rational upper bound is discovered, it is always possible to find one smaller. Thus, although $a \in \mathbb{R}$, but $a \notin S$, this does not disqualify it from being the sup$S$. However, since the search for a least upper bound for S goes on indefinitely due to the denseness of $\mathbb{Q}$, by the definition of the supremum of a set, $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$, as desired. ♠

EXERCISE 7.3 **(b)**, **(d)**, **(f)**, **(h)**, **(j)**, **(l)**, **(n)**, **(p)**, **(r)**, **(t)**

(b) $b_n = \frac{n^2+3}{n^2-3}$ converges to 1.

(d) $t_n = 1 + \frac{2}{n}$ converges to 1.

(f) $s_n = (2)^{\frac{1}{n}}$ converges to 1.

(h) $d_n = (-1)^n n$ diverges.

[7] Proof ommitted, as it was done in class

(j) $\frac{7n^3+8n}{2n^3-31}$ converges to $\frac{7}{2}$.

(l) $\sin\left(\frac{n\pi}{2}\right)$ diverges.

(n) $\sin\left(\frac{2n\pi}{3}\right)$ diverges.

(p) $\frac{2^{n+1}+5}{2^n-7} = \frac{2^n(2)+5}{2^n-7}$ converges to 2.

(r) $\left(1+\frac{1}{n}\right)^2$ converges to 1.

(t) $\frac{6n+4}{9n^2+7}$ converges to 0.

EXERCISE 8.4

If (s_n) is a sequence which converges to 0 and (t_n) is a bounded sequence, then the sequence $(s_n t_n)$ converges to 0.

We will argue the fact that we are given a sequence (s_n) which converges to 0 so we know that

$$\lim_{n\to\infty}(s_n) = 0,$$

and we are given a sequence (t_n) which is a bounded sequence, so we know that

$$\exists \text{ a constant } M : |t_n| \leq M \ \forall n \in \mathbb{N},$$

which means, in a geometric sense, that we can find an interval $[-M, M]$ that contains every term in the sequence t_n.

With the above given, we can almost assuredly argue with what we know about products that if one sequence which arbitrarily closes in on the value zero is multiplied by another sequence which is bounded by a constant we can conclude that their product will eventually close in on the value zero since "zero" $\times$ "a constant" = zero (eventually).

Claim: Given (s_n), a sequence which converges to 0 and (t_n), a bounded sequence,

$$\lim_{n\to\infty}(s_n t_n) \longrightarrow 0.$$

***Proof*:**

Let $\epsilon > 0$ be given. Since we know $\lim_{n\to\infty} s_n \to 0$, we can always find an $n > N : |s_n - 0| < \epsilon < \frac{\epsilon}{M}$[8], for some constant $M > 1$. Further, $\exists$ a constant $M : |t_n| \leq M \ \forall n \in \mathbb{N}$. Then, for $n > N$, it holds that

$$|s_n t_n - 0| = |s_n|\,|t_n| < \left(\frac{\epsilon}{M}\right)(M) = \epsilon. \spadesuit$$

[8]Proof ommitted since convergence to 0 is taken as given.

EXERCISE 8.7

Claim:

$s_n = \cos\left(\frac{n\pi}{3}\right)$ does not converge.

***Proof*:**

For $n = 1, 2, \cdots, 6$, the terms of the sequence are $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{1}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{1}, \ldots$. Hence, for any s and any N, we can come up with $n > N : s_n = 1 \Longrightarrow s_{n+3} = -1 \Longrightarrow |s_n - s| \geq 1$ or $|s_{n+3} - s| \geq 1$, by the triangle inequality, which proves this sequence does not converge, as desired. ♠

EXERCISE 8.8 **(c)**

Claim:

$\lim_{n\to\infty}[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$.

***Proof*:**

$$\sqrt{4n^2+n} - 2n = \frac{n}{\sqrt{4n^2+n} + 2n} = \frac{1}{2\sqrt{1+\frac{1}{4n}} + 2}$$

If $1 < a$, then $1 < a < a^2$ which then implies

$$1 \leq \lim_{n\to\infty} \sqrt{1 + \frac{1}{2n}} \leq \lim_{n\to\infty} \left[1 + \frac{1}{2n}\right] = 1 + 0 = 1.$$

Now, applying the limit theorems, we see that

$$\lim_{n\to\infty} \frac{1}{2\sqrt{1+\frac{1}{4n}} + 2} = \frac{1}{2 \cdot 1 + 2} = \frac{1}{4},$$

as desired. ♠

EXERCISE 8.10

We are given $s > a$. Let $\lim_{n\to\infty} s_n = s$.

Claim:

$\exists$ a number $N : n > N \Longrightarrow s_n > a$.

***Proof*:**

$s > a \Longrightarrow$ there is an $\epsilon > 0 : s - \epsilon > a$[9] For such chosen $\epsilon \ \exists \ N_\epsilon : n > N_\epsilon \Longrightarrow s_n \in (s - \epsilon, s + \epsilon) \Longrightarrow n > N_\epsilon \Longrightarrow s_n > s - \epsilon > a$, as desired. ♠

EXERCISE 9.4

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) The first four elements are $1, \sqrt{2}, \sqrt{\sqrt{2}+1}, \sqrt{\sqrt{\sqrt{2}+1}+1}$.

(b) Assume s_n converges.

Claim: $\lim_{n\to\infty} s_n = \frac{1}{2}(1+\sqrt{5})$.

***Proof*:**

We are given the assumption that s_n converges. We will name some σ as the element s_n converges to. If s_n converges, s_{n+1} inherently converges; more specifically, s_{n+1} converges to σ, as well. From the problem, we have

$$(s_{n+1})^2 = s_n + 1,$$

which when utilized with the assumption laid out will converge to

$$(\sigma)^{2\ 10} = \sigma + 1 \Longleftrightarrow \sigma^2 - \sigma - 1 = 0.$$

Implementing the quadratic formula[11] to find a pair of solutions to the equation yields

$$\sigma_1 = \frac{1+\sqrt{5}}{2} \text{ and } \sigma_2 = \frac{1-\sqrt{5}}{2}.$$

Since the greatest lower bound of the sequence s_n is 1, the solution we want to pick will reflect that the limit will trivially be positive, eliminating σ_2 as a solution.

Hence, $\lim_{n\to\infty} s_n = \sigma_1 = \frac{1}{2}(1+\sqrt{5})$. ♠

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[9] For example $\epsilon := \frac{s-a}{2}$.

[10] For this, we will use the fact that the limit of a product of sequences is the product of the limits, i.e. if $\lim s_n \to s$ and $\lim t_n \to t$, then $\lim(s_n t_n) \to st$.

[11] The quadratic formula, learned from principals, is $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$.

EXERCISE 9.5

Assume $\lim t_n$ exists and is defined to be t. Then $\lim t_{n+1} = t$ as well. For all n, we get $2t_n t_{n+1} = t_n^2 + 2$. Implementing the limit theorems, we see that

$$2t^2 = t^2 + 2 \Longrightarrow t = \pm\sqrt{2}.$$

Since we are given $t_1 = 1$ and the fact that the sequence stays positive, we can eliminate $-\sqrt{2}$ and conclude that the limit equals $\sqrt{2}$.

EXERCISE 9.11

(a) Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim (s_n + t_n) = +\infty$.

Let $M > 0$ and let $m = \inf\{t_n : n \in \mathbb{N}\}$. We want $s_n + t_n > M$ for $n > N$. This will be sufficed by $s_n + m > M$ or $s_n > M - m$ for $n > N$. So we will choose an $N : s_n > M - m$ for $n > N$. Let $\mu = M - m$. From the constraints above, we see that $0 \leq \mu < \infty$. Since we can always find an $N : s_n > \mu$ for $n > N$, we can conclude that $\lim (s_n + t_n) = +\infty$.

(b) Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim (s_n + t_n) = +\infty$.

If $\lim t_n > -\infty$, we can conclude that $\inf\{t_n : n \in \mathbb{N}\}$ exists. We can then use the same argument as above by letting $M > 0$ and $m = \inf\{t_n : n \in \mathbb{N}\}$. We want $s_n + t_n > M$ for $n > N$. This will be sufficed by $s_n + m > M$ or $s_n > M - m$ for $n > N$. So we will choose an $N : s_n > M - m$ for $n > N$. Let $\mu = M - m$. From the constraints above, we see that $0 \leq \mu < \infty$. Since we can always find an $N : s_n > \mu$ for $n > N$, we can conclude that $\lim (s_n + t_n) = +\infty$.

(c) Show that if $\lim s_n = +\infty$ and if (t_n) is a bounded sequence, then $\lim (s_n + t_n) = +\infty$.

Let $M > 0$. If (t_n) is a bounded sequence, then $\exists$ a constant $\omega : |t_n| \leq \omega \ \forall n \in \mathbb{N}$, which implies t_n is bounded from above, but more importantly for this proof, bounded from below by $-\omega$. We want $s_n + t_n > M$ for $n > N$. This will be sufficed by $s_n - \omega > M$ or $s_n > M + \omega$ for $n > N$. So we will choose an $N : s_n > M + \omega$ for $n > N$. Let $\mu = M + \omega$. From the constraints above, we see that $0 \leq \mu < \infty$. Since we can always find an $N : s_n > \mu$ for $n > N$, we can conclude that $\lim (s_n + t_n) = +\infty$.

EXERCISE 9.12 **(a)**

Choose $a : L < a < 1$. If $\epsilon = a - L > 0$, there exists an $N :\ \forall n \geq N \ |\frac{s_{n+1}}{s_n} - L| \leq \epsilon \left|\frac{s_{n+1}}{s_n}\right| < L + \epsilon = a < 1$. More specifically, $|s_{N+1}| < a\,|s_N|\,, |s_{N+2}| < a\,\left|s_{N+1}\right| < a^2\,|s_N| \ldots$. Thus, by induction, $|s_{N+n}| < a^n\,|s_N| \ \ \forall\ n \in N$. To summarize,

$$\lim_{n\to\infty} |s_n| = \lim_{n\to\infty} |s_{N+n}| \leq \lim_{n\to\infty} a^n\,|s_N| = |s_N| \lim_{n\to\infty} a^n = 0$$

given $|a| < 1$.

EXERCISE 9.15

Let $s_n = \frac{a^n}{n!} \Longrightarrow \frac{s_{n+1}}{s_n} = \frac{a}{(n+1)} \to 0$ as $n \to \infty \Longrightarrow \lim s_n = 0$

EXERCISE 10.1 **(b)**, **(d)** and **(f)**

(b) $\frac{(-1)^n}{n^2}$. The first term, -1, is less than the second term, $\frac{1}{4}$. Hence, the sequence is not nonincreasing. The second term is greater than the third term, $-\frac{1}{9}$. Hence, the sequence is not nondecreasing. Since $\left|\frac{(-1)^n}{n^2}\right| = \left|n^{-2}\right| \leq 1 \ \forall n \in \mathbb{N}$, the sequence is bounded.

(d) $\sin\left(\frac{n\pi}{7}\right)$. The first term is positive, however, the 7th term is $0 \Longrightarrow$ the sequence is not nondecreasing. The second term is greater than the first term and hence the sequence is not nonincreasing. Since $|\sin x| \leq 1 \ \forall x$, the sequence is bounded.

(f) $\frac{n}{3^n}$. The sequence is nonincreasing since $\frac{a_{n+1}}{a_n} < 1$. As all terms are positive, the sequence is bounded from below and since the sequence is nonincreasing, it is bounded from above $\Longrightarrow$ the sequence is bounded.

EXERCISE 10.2

Claim:

All bounded monotone sequences converge.

***Proof*:**

Let (s_n) be a bounded nonincreasing sequence. Let S be defined as the set $\{s_n : n \in \mathbb{N}\}$ and let $v = \inf S$. S bounded $\Longrightarrow v$ exists and is real. We will now show that $\lim s_n = u$. Let $\epsilon > 0$. Since $v + \epsilon$ is not an upper bound for S, $\exists N : s_N < v + \epsilon$. Since s_n is nonincreasing, $\Longrightarrow s_N \geq s_n \ \forall n > N$. $s_n \geq v \ \forall n$ and hence, $n > N \Longrightarrow v + \epsilon > s_n \geq v \Longrightarrow |s_n - v| < \epsilon \Longrightarrow \lim s_n = v$, as desired. ♠

EXERCISE 10.5

Claim: If (s_n) is an unbounded non-increasing sequence, then $\lim s_n = -\infty$.

We want to show that

$$\lim s_n = -\infty \Longleftrightarrow \text{for each } M < 0 \ \exists \text{ a number } N : n > N \Longrightarrow s_n < M.$$

***Proof*:**
Let (s_n) be an unbounded non-decreasing sequence. Let $M < 0$. Since the set $\{s_n : n \in \mathbb{N}\}$ is unbounded and it is bounded above by s_1, it must be unbounded below, since for this sequence to be non-increasing, the condition $s_n \geq s_{n+1}$ must be fulfiilled. Hence, for some $N \in \mathbb{N}$ we have $s_N < M$. Clearly, $n > N \implies M > s_N \geq s_n$, so
$$\lim s_n = -\infty,$$
as desired. ♠

EXERCISE 10.6
(a) Let (s_n) be a sequence such that
$$|s_{n+1} - s_n| < 2^{-n} \ \forall n \in \mathbb{N}$$
Claim: (s_n) is a Cauchy sequence and hence a convergent sequence.

***Proof*:**
Given an arbitrary $\epsilon > 0$, for $n > N$, choose $N : 2^{-N} < \frac{\epsilon}{2}$. Then $\forall m > n > N$, we can see that
$$|s_m - s_n| \equiv \left| \sum_{k=n}^{m-1} s_{k+1} - s_k \right|.$$
Now,
$$\left| \sum_{k=n}^{m-1} s_{k+1} - s_k \right| \leq \sum_{k=n}^{m-1} |s_{k+1} - s_k| \leq \sum_{k=n}^{m-1} 2^{-k} \tag{11}$$
**Note: The second inequality from line (1) is justified by "Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n} \ \forall n \in \mathbb{N}$" in the first line of the problem.*
$$\sum_{k=n}^{m-1} 2^{-k} \leq \sum_{k=N}^{\infty} 2^{-k} = 2^{-N+1} < \epsilon, \tag{12}$$
as desired. ♠

(b) Is the result in **(a)** true if we only assume that $|s_{n+1} - s_n| < \frac{1}{n} \ \forall n \in \mathbb{N}$?

If we only assume that $|s_{n+1} - s_n| < \frac{1}{n} \ \forall n \in \mathbb{N}$, our immediate reaction is to say that the new constraint makes the result in **(a)** false, since the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, is taken to be divergent.

As a counter-example, let us choose $s_n = \sum_{n=2}^{\infty} \frac{1}{n-1}$, which will still fulfill $|s_{n+1} - s_n| \leq \frac{1}{n}$, but will inevitably diverge. Thus, as we travel down the sequence $(N \to \infty)$, arbitrary s_ms and s_ns are not encroaching towards eachother, violating the Cauchy criterion.

EXERCISE 10.7

Choose $n \in \mathbb{N}$. Construct $s_n \in S : \sup S - s_n < \frac{1}{n}$ and $s_n > s_{n-1}$ for $n > 1 \Longrightarrow$ $s_n \equiv$ an increasing sequence converging to $\sup S$.

Pick $s_1 \in S : \sup S - 1 < \sup S$ is not an upper bound of S. Implement induction. Assume $s_1 < \cdots < s_{n-1}$ exists. Since $\sup S \notin S \Longrightarrow s_{n-1} < \sup S \Longrightarrow$ $\exists\ s_n \in S : \sup S \geq s_n > s_{n-1}$ and $\sup S - s_n < \frac{1}{n}$. This is possible given that neither $s_{n-1} < \sup S$ nor $\sup S - \frac{1}{n} < \sup S$ is an upper bound for S.

EXERCISE 10.8

Let (s_n) be a non-decreasing sequence of positive numbers and define

$$\sigma_n = \frac{1}{n}\left(s_1 + s_2 + \cdots + s_n\right)$$

Claim: (σ_n) is a non-decreasing sequence.

***Proof*:**

From the assumption of s_n being a non-decreasing sequence, we will proclaim

$$s_n \leq s_{n+1}$$

Hence,

$$ns_n \leq ns_{n+1} \tag{13}$$

$$\Longrightarrow (s_1 + s_2 + \cdots s_n) \leq ns_n \leq n(s_{n+1}) \tag{14}$$

$$\Longrightarrow n(s_1 + s_2 + \cdots s_n) + (s_1 + s_2 + \cdots s_n) \leq n(s_1 + s_2 + \cdots s_n) + n(s_{n+1}) \tag{15}$$

$$\Longrightarrow (n+1)(s_1 + s_2 + \cdots s_n) \leq n(s_1 + s_2 + \cdots s_n + s_{n+1}) \tag{16}$$

$$\Longrightarrow \frac{1}{n}(s_1 + s_2 + \cdots s_n) \leq \frac{1}{n+1}(s_1 + s_2 + \cdots s_n + s_{n+1}), \tag{17}$$

Thus,

$$\sigma_n \leq \sigma_{n+1},$$

as desired. ♠

EXERCISE 11.8

(a) Use Definition 10.6 and Exercise 5.4 to prove that lim inf $s_n = -$lim sup$(-s_n)$.

It is readily observable that

lim sup$(-s_n) = \lim\limits_{N\to\infty} \sup\{(-s_n) : n > N\} = \lim\limits_{N\to\infty} -\inf\{(s_n) : n > N\} = -$lim inf$(s_n)$.

Hence,

$$\limsup(-s_n) = -\liminf(s_n) \Longleftrightarrow \liminf(s_n) = -\limsup(-s_n),$$

as desired.

(b) Let (t_k) be a monotonic subsequence of $(-s_n)$ converging to $\limsup(-s_n)$. Show that $(-t_k)$ is a monotonic subsequence of (s_n) converging to $\liminf s_n$. Observe that this completes the proof of Corollary 11.4.

We are given a monotonic subsequence of $(-s_n)$, denoted (t_k). Then

$$t_k = -s_{n_k}\ \forall k,\ (t_k \text{ monotonic}) \Longleftrightarrow -t_k = s_{n_k}\ \forall k,\ (-t_k \text{ monotonic}).$$

When taken with the results from part **(a)**, the desired result is apparent.

EXERCISE 11.9

(a) Let (x_n) be a convergent sequence such that $a \le x_n \le b\ \forall n \in \mathbb{N} \Longrightarrow$
$a \le \lim_{n\to\infty} x_n \le b \Longrightarrow [a, b]$ is a closed set.

(b) No. We know that the set of any subsequential limits of any set mus be closed. The interval $(0, 1)$ is not closed.

EXERCISE 11.10

Let (s_n) be the sequence of numbers in Figure 11.2 listed in the indicated order.

(a) Find the set S of subsequential limits of (s_n).

Laying out the numbers listed in the indicated order gives the sequence

$$1, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 1, ...$$

Rearranging,

$$\begin{array}{llllllll}
1/1, \\
1/2, & 1/1, \\
1/1, & 1/2, & 1/3, \\
1/4, & 1/3, & 1/2, & 1/1, \\
1/1, & 1/2, & 1/3, & 1/4, & 1/5, \\
1/6, & 1/5, & 1/4, & 1/3, & 1/2, & 1/1, \\
1/1, & 1/2, & 1/3, & 1/4, & 1/5, & 1/6, & 1/7, \\
1/8, & 1/7, & 1/6, & 1/5, & 1/4, & 1/3, & 1/2, & 1/1,
\end{array}$$

It seems that $\exists$ 2 sequences, both monotonic, one increasing the other decreasing. The even rows of the pyramid represent a monotonically increasing sequence and the odd rows correspond to a monotonically decreasing sequence. Upon further inspection, we see that with respect to the decreasing sequences, the denominator of the last entry corresponds to the row number of the pyramid for which the decreasing sequence is located, i.e., the last entry of row 3 has $\frac{1}{3}$ as its last entry. Conversely, with respect to the increasing sequences, the denominator of the entry for which the increasing sequence commences corresponds to the row number of the pyramid for which the increasing sequence is located, i.e., the first entry of row 6 is $\frac{1}{6}$. Since the odd rows all commence at $\frac{1}{1}$ and progressively converge to $\frac{1}{n}$, we can say that one subsequential limit is 0. Conversely, since the even rows all seem to converge to 1 from a fraction progressively closer to $\frac{1}{n}$, we could say the other subsquential limit is 1.

Hence the set of subsequential limits is

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \bigcup \left\{0\right\}$$

EXERCISE 12.1

Let (s_n) and (t_n) be sequences and suppose that there exists $N_0 : s_n \leq t_n \ \forall n > N_0$.

Claim:

$\liminf s_n \leq \liminf t_n$ and $\limsup s_n \leq \limsup t_n$.

***Proof*:**

Refer to Exercise 4.8 **(b)**. From there, we see that

$$\begin{aligned} \sup\{s_n : n > N\} \leq \inf\{t_n : n > N\} \ \ \forall N \geq N_0 \Longrightarrow \\ \lim_{N\to\infty} \sup\{s_n : n > N\} \leq \lim_{N\to\infty} \inf\{t_n : n > N\} \Longleftrightarrow \\ \limsup(s_n) \leq \liminf(t_n) \qquad (*) \end{aligned}$$

Since

$$\begin{aligned} \liminf(s_n) \leq \limsup(s_n) \text{ and} \\ \liminf(t_n) \leq \limsup(t_n), \end{aligned}$$

the desired inequalities are obtained implementing (*), as desired. ♠

EXERCISE 12.2

Prove that lim sup $|s_n| = 0$ if and only if lim $s_n = 0$.

We will use Theorem 11.7 throughout this proof.

Claim: lim sup $|s_n| = 0 \Longleftrightarrow \lim s_n = 0$.

***Proof*:**

" $\Longleftarrow$ " :

$$\lim s_n = 0 \Rightarrow \lim|s_n| = 0 \Rightarrow \text{lim sup} s_n = 0.$$[12]

" $\Longrightarrow$ " :

$$\limsup |s_n| = 0 \Rightarrow \liminf |s_n| = 0, \text{ since } |s_n| \geq 0.$$

As $N \to \infty$, $\limsup |s_n| \to \lim s_n$, since Theorem 11.7 tells us that $\limsup |s_n|$ is exactly the largest subsequential limit of s_n. Hence,

$$\limsup |s_n| = 0 \Longleftrightarrow \lim s_n = 0,$$

as desired. ♠

EXERCISE 12.3

(a) $\liminf s_n + \liminf t_n = 0 + 0 = 0$

(b) $\liminf(s_n + t_n) = 1$

(c) $\liminf s_m + \limsup t_n = 0 + 2 = 2$

(d) $\limsup(s_n + t_n) = 3$

(e) $\limsup s_n + \limsup t_n = 2 + 2 = 4$

(f) $\liminf s_n t_n = 0$

(g) $\limsup s_n t_n = 2$

EXERCISE 12.4

Claim:

$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for bounded (s_n) and (t_n).

***Proof*:**

For any $n > N$,

$$s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\} \Longrightarrow$$
$$\sup\{(s_n + t_n) : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

[12] By Theorem 11.7.

Hence,

$$\begin{aligned}
\lim_{N\to\infty} \sup\{(s_n + t_n) : n > N\} &\le \lim_{N\to\infty} (\sup\{s_n : n > N\} + \sup\{t_n : n > N\}) \\
&= \lim_{N\to\infty} \sup\{s_n : n > N\} + \lim_{N\to\infty} \sup\{t_n : n > N\},
\end{aligned}$$

where the last equality holds since

$$\big(\sup\{s_n : n > N\}\big)_{N=N_0}^{\infty} \text{ and } \big(\sup\{t_n : n > N\}\big)_{N=N_0}^{\infty}$$

are both convergent and monotonic, as desired. ♠

EXERCISE 12.8

Let (s_n) and (t_n) be bounded sequences of nonnegative numbers. Prove that

$$\text{lim sup } s_n t_n \le (\text{lim sup } s_n)(\text{lim sup } t_n).$$

Give an example where the strict inequality holds.

Claim:

$$\text{lim sup } s_n t_n \le (\text{lim sup } s_n)(\text{lim sup } t_n).$$

***Proof*:**

Since s_n and t_n are both bounded sequences of nonnegative numbers,

$$s_n \le \sup\{s_n : n > N\}\, \forall n \in \mathbb{N}$$

and

$$t_n \le \sup\{t_n : n > N\}\, \forall n \in \mathbb{N},$$

by definition. This implies

$$(s_n)(t_n) \le (\sup\{s_n : n > N\})(\sup\{t_n : n > N\})\ \forall n \in \mathbb{N},$$

by multiplicativity and since, once again, s_n and t_n are both bounded and nonnegative. Now since the right hand side of the inequality serves as an upper bound for the left hand side,

$$\sup\{s_n t_n : n > N\} \le (\sup\{s_n : n > N\})(\sup\{t_n : n > N\})\ \forall n \in \mathbb{N}$$

If s_n and t_n are both sequences which converge to s and t with $s_n \le t_n\ \forall n \in \mathbb{N}$, we can take limits and conclude

$$\text{lim sup } s_n t_n \le (\text{lim sup } s_n)(\text{lim sup } t_n),$$

as desired. ♠

An example where the strict inequality would hold would be the case when s_n and t_n both converge to s and t, respectively, but $s_n < t_n$ $\forall n \in \mathbb{N}$.

EXERCISE 12.10

Claim:

(s_n) is bounded $\iff \limsup |s_n| < +\infty$

***Proof*:**

$\limsup = +\infty \implies \exists$ a subsequence $(s_{n_k}) : \lim |s_{n_k}| = +\infty \implies$ this subsequence is unbounded. Conversely, (s_n) unbounded $\implies$ for any $k \in \mathbb{N}$ $\exists$ $s_{n_k} : |s_{n_k}| > k$. Assume that $n_1 < n_2 < \cdots < n_k < \cdots$ and thus get a subsequence (s_{n_k}) : $\lim |s_{n_k}| = +\infty$, i.e., $\limsup |s_n| = +\infty$. ♠

EXERCISE 12.11

Claim:

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{\frac{1}{n}}$$

***Proof*:**

Let $\alpha = \liminf |s_n|^{\frac{1}{n}}$. Let $L = \liminf \left|\frac{s_{n+1}}{s_n}\right|$. We need to show that $\alpha \geq L$ for any $L_1 < L$.

$$\begin{aligned} L = \liminf \left| \frac{s_{n+1}}{s_n} \right| &= \lim_{n\to\infty} \left(\inf \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n > N \right) \right\} > L_1 \\ &\implies \exists\ N \in \mathbb{N} : \inf \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n \geq N \right\} > L_1 \\ &\implies \forall n \geq N \text{ we have } \left| \frac{s_{n+1}}{s_n} \right| > L_1 \end{aligned}$$

It then follows that

$$\begin{aligned} |s_n| > L_1^{n-N} |s_n|, \ \forall n > N \\ |s_n| > L_1^n \underbrace{\left(L_1^{-N} |s_n|\right)}_{=\alpha}, \ \forall n > N \\ |s_n| > L_1^n \alpha, \ \forall n > N \implies |s_n|^{\frac{1}{n}} > L_1 \alpha^{\frac{1}{n}}, \ \forall n > N \end{aligned}$$

Since $\lim_{n\to\infty} \alpha^{\frac{1}{n}} = 1$, the result is that $\limsup |s_n|^{\frac{1}{n}} \geq L_1$. ♠

EXERCISE 13.3 **(a)**

Let B be the set of all bounded sequences $\mathbf{x} = (x_1, x_2, ...)$ and define $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, ...\}$.

Show that d is a metric for B.

To show that d is indeed a metric for B, we need to show that it satisfies the three conditions of Definition 13.1. As we will see the first 2 conditions are trivial.

D1. $d(x, x) = 0$, since $\sup\{|x_j - x_j| : j = 1, 2, ...\} = \sup\{0\} = 0$, and $d(x, y) > 0$, since for bounded sequences $\mathbf{x} = (x_1, x_2, ...)$ and $\mathbf{y} = (y_1, y_2, ...)$, $\sup\{|x_j - y_j| : j = 1, 2, ...\} > 0$.

D2. $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, ...\} = \sup\{|y_j - x_j| : j = 1, 2, ...\} = d(\mathbf{y}, \mathbf{x})$.

D3. If $x, y, z \in B$, then for each $j = 1, 2, ..., k$,

$$\begin{aligned} d(x, z) &= \sup\{|x_j - z_j| : j = 1, 2, ...\} && (18)\\ &= \sup\{|x_j - y_j + y_j - z_j| : j = 1, 2, ...\} && (19)\\ &= \sup\{|(x_j - y_j) + (y_j - z_j)| : j = 1, 2, ...\} && (20)\\ &\leq \sup\{|x_j - y_j| : j = 1, 2, ...\} + \sup\{|y_j - z_j| : j = 1, 2, ...\} && (21)\\ &\leq d(x, y) + d(y, z), && (22) \end{aligned}$$

where the inequality follows from the triangle inequality and the nature of the supremum. Hence d is a metric.

EXERCISE 13.6

We will prove each part of Proposition 13.9 in turn.

(a) *Claim:*

The set E is closed if and only if $E = E^-$.

***Proof*:**

"$\Longrightarrow$"

Assume E is closed $\Longrightarrow$ E is a closed set containing E $\Longrightarrow$ the intersection of all closed sets containing $E \equiv E \Longrightarrow E^- = E$.

"$\Longleftarrow$"

Assume $E^- = E$. By definition, E is a closed set since any intersection of closed sets is closed (by Definition 13.8). $E \equiv$ closed set $\Longrightarrow E$ is closed. ♠

(b) *Claim:*

The set E is closed if and only if it contains the limit of every convergent sequence of points in E.

***Proof*:**

Assume E is closed and for purposes of contradiction, assume the limit of every convergent sequence of points $\notin E \implies \exists$ a convergent sequence of points $\in E \implies$ for some arbitrary limit point in E $\exists$ some ϵ-ball around the limit point such that all points are inside the ball. But, it then becomes a problem (a contradiction) to have a sequence from E converge to its limit point if for some ϵ there does not exist elements of E contained in such a ball. Hence, all limit points of E must be contained in $E^{-} \equiv E$, as desired. ♠

(c) *Claim*:

An element is in E^{-} if and only if it is the limit of some sequence of points in E.

***Proof*:**

"$\implies$"

Let $x \in E^{-}$ be arbitrary. If this x is not the limit point $\implies \exists$ some ϵ-ball around this x: x is the only element of E^{-} in this ball $\implies E^{o}$ is not the interior of E because any $(s \neq x \in S) \in$ the aforementioned ϵ-ball is not contained in E^{o}, which is a contradiction. Now if $x \in E^{-} \setminus E^{o} \implies$ we can create a similar ϵ-ball around x: some of the interior of E is missing, another contradiction $\implies$ any such point in E^{-} is a limit point.

"$\impliedby$"

see part **(b)** above, as desired, lol. ♠

(d) *Claim*:

A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.

***Proof*:**

Let $E \equiv E^{o} \implies S \setminus E$ closed $\implies S \setminus E$ contains its boundary $\implies$ any $s \in$ boundary of $S \setminus E$ must also be in the boundary of E, otherwise either $\exists$ some x: x is neither $\in E$ or $S \setminus E$, ***or*** $x \in$ both E and $S \setminus E$, which both contradict $S \setminus E \equiv E^{c}$. Let $E \equiv E^{-} \implies S \setminus E$ is open $\implies S \setminus E$ does not contain its boundary $\implies$ any $s \in$ boundary of $S \setminus E$ must also be $\in$ the boundary of E and $\in E \implies S \setminus E \equiv E^{c}$, as desired. ♠

EXERCISE 13.8 **(b)**

We will verify each assertion step by step.

1. In $\mathbb{R}^k$, open balls $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) < r\}$ are open sets. We can verify this by showing that every point in the set is interior to the set. Choose an arbitrary x_1 in the set. Then $|x_0|+r$ represents the boundary. We can then take $\frac{(|x_0|+r)-|x_1|}{2}$, which represents the distance between the boundary and x_1, but divided by 2. Hence all of the points in the set are interior to the set.

2. In $\mathbb{R}^k$, closed balls $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) \le r\}$ are closed sets. We want to show that the complement of closed balls are by definition open sets $\Longrightarrow$ closed balls are closed sets. WLOG define our closed ball to be

$$\overline{B}_\epsilon(0) := \{x : d(x, 0) \le \epsilon\},$$

WLOG, define the complement of the closed ball by

$$\overline{B}_\epsilon(0) := \{x : d(x, 0) > \epsilon\}.$$

To show the complement is open, let α be arbitrary. Define $\delta :=$ the magnitude of α - ϵ. Define a new ϵ'-ball, with radius:$= \frac{\delta}{2}$. All points in this new ball are contained in the complement of our closed ball.

3. The boundary of each of these sets is

$$\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) = r\}$$

The boundary of each of these sets is the boundary of the ball with radius r. **By Proposition 13.9 (d)**,

> *A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.*

Consider the "neighborhood" of this set

$$\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) < r\} = (\mathbf{x} - r, \mathbf{x} + r).$$

whose closure, *the intersection of all closed sets containing* $(\boldsymbol{x} - r, \boldsymbol{x} + r)$, is $[\mathbf{x} - r, \mathbf{x} + r]$. The complement of $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) < r\}$ is defined to be $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) \ge r\}$.

Hence,

$$[\mathbf{x} - r, \mathbf{x} + r] \cap \{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) \ge r\} = \{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) = r\}.$$

Assertion 1

$$\{(x_1, x_2) : x_1 > 0\} \text{ is open.}$$

Discussion: Noting that any element in this set is, by definition, interior to it, we see this is an open set.

Assertion 2

$$\{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > 0\} \text{ is open.}$$

Discussion: Once, again, noting the *strict inequalties*, this defines an open set. Conversely, if $\exists$ non-strict inequalities, i.e., $> \to \geq$, we can note that the set will include its limit points, hence *closing* the set.

Assertion 3

$$\{(x_1, x_2) : x_1 > 0 \text{ and } x_2 \geq 0\} \text{ is neither open nor closed.}$$

Discussion: This set fails to be open or closed. As counterexamples, consider $(1, 0)$ which is not interior to this set and $(0, 0)$, which is a limit point, but is not $\in$ the set.

EXERCISE 13.10

(a) We want to show that the interior of the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \emptyset$.

Claim:

$$int\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \emptyset$$

***Proof*:**

Define a "neighborhood" in this set as:

$$\frac{1}{n+1} \leq \frac{1}{n} \leq \frac{1}{n-1}.$$

Now define an ϵ-ball with radius

$$r = \frac{\frac{1}{n} - \frac{1}{n+1}}{2}.$$

This makes ϵ-balls around an arbitrary element in the set such that only the arbitrary element is contained in the ball, as desired. ♠

(b) Using the fact that $\mathbb{Q}$ contains "gaps",[13] we want to show that

$$int(\mathbb{Q}) = \emptyset$$

***Proof*:**

Let $q \in \mathbb{Q}$ and $r > 0$ be arbitrary. Consider the "neighborhood"

$$\{s \in \mathbb{R} : |s - q| < r\} = (s - r, s + r)$$

[13] This is resolved by the Completeness Axiom.

By the "denseness" of the irrationals[14], we know that $\exists$ an irrational number $z \in (s-r, s+r) \implies$ the "neighborhood" $(s-r, s+r)$ is not contained in $\mathbb{Q}$. But s was given as arbitrary $\implies q$ cannot be defined as an interior point. But q was given as arbitrary $\implies$ interior of $\mathbb{Q} = \emptyset$, as desired. ♠

(c) *Claim*:

The interior of the Cantor set is empty.

Proof:

By the definition of the Cantor Set outlined in the text, the Cantor Set $\equiv$ the intersection of closed sets $\implies$ the Cantor Set is closed. Now define the different "articulations" (labeled T_n) of the Cantor Set as follows:

$$T_0 = [0,1], T_1 = [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right), ..., T_\infty{}^{15} = \cap \text{ of all } T_n$$

Recalling the above, T_∞ is closed $\implies T_\infty \equiv \overline{T_\infty}$, the closure of the Cantor Set.

Now,
$$int(T_\infty) = \left(\overline{T_\infty^c}\right)^c$$

and note that we are still in the metric space $[0,1]$. Thus, $T_\infty^c = [0,1] \setminus T_\infty \implies \overline{T_\infty^c} = ([0,1] \setminus T_\infty) \cup \{x \in [0,1] : x \text{ is a limit point of } [0,1] \setminus T\infty\}$. Since $T_\infty \subseteq \{x \in [0,1] : x \text{ is a limit point of } [0,1] \setminus T\infty\} \subseteq [0,1]$, we know

$$\overline{T_\infty^c} = ([0,1] \setminus T_\infty) \cup T_\infty \equiv [0,1].$$

In other words, the closure of the Cantor Set is T_0, the interval from 0 to 1. Hence,
$$int(T_\infty) = \left(\overline{T_\infty^c}\right)^c \equiv [0,1]^c \equiv \emptyset,$$

as desired. ♠

EXERCISE 13.12

(a) Let (S,d) be any metric space.

Claim:

If E is a closed subset of a compact set F, then E is also compact.

***Proof*:**

Let E be a closed subset of a compact set F. Let a collection of open sets U be an open cover for E. E closed $\implies E^c$ open. Now, $E^c + U \equiv$ an open cover of F. Given F compact, $\exists$ a finite subcover that covers F. Now given $E \subseteq F$ and F covered by some finite subcover, $E \equiv$ compact. ♠

[14] For every real numbers x and y, with $x < y$, $\exists$ a rational α and an irrational β such that $x < \alpha < y$ and $x < \beta < y$.

[15] $T_\infty \equiv$ *THE* Cantor Set

(b) Let (S, d) be any metric space.

Claim:

The finite union of compact sets $\in S$ is compact.

***Proof*:**

Let $C_1, C_2, ..., C_n : n < \infty$ be compact sets, each with a finite subcover $S_i : i = 1, 2, ..., n$. Since each S_i is the union of open sets and contains finite open sets, it is open. Now consider the union of all of these finite subcovers $(S_1 \cup S_2 \cup \cdots \cup S_n)$, which trivially covers the union of $C_1, C_2, ..., C_n$. The union of all of these finite subcovers $(S_1 \cup S_2 \cup \cdots \cup S_n)$ is, by definition, open, since it is the union of open sets and further, it is a finite collection of open covers, since it is a union of finitely many finite subcovers. Hence, since $\exists$ a finite subcover of the union of $C_1, C_2, ..., C_n \Longrightarrow$ the union of $C_1, C_2, ..., C_n \equiv$ compact. ♠

EXERCISE 13.13

We will show that inf E belongs to E and the case for the sup E is similar.

Claim: If E is a nonempty subset of $\mathbb{R}$, then inf $E \in E$.

***Proof*:**

Assume, by contradiction, that inf $E \notin E$. Since E is nonempty and compact, we know, by the Heine-Borel Theorem, that *a subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.* Since E is closed and bounded, $\exists$ a sequence (s_n) in E where inf $E = \lim s_n =$ sup E. Furthermore, if the set E is closed, this implies that it contains the limit of every covergent sequence of points in E, including inf E, a contradiction. ♠

Claim: If E is a nonempty subset of $\mathbb{R}$, then sup $E \in E$.

***Proof*:**

Assume, by contradiction, that sup $E \notin E$. Since E is nonempty and compact, we know, by the Heine-Borel Theorem, that *a subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.* Since E is closed and bounded, $\exists$ a sequence (s_n) in E where inf $E = \lim s_n =$ sup E. Furthermore, if the set E is closed, this implies that it contains the limit of every covergent sequence of points in E, including sup E, a contradiction. ♠

Alternatively, both of these proofs can be combined into one proof.

Assume E is a compact subset of $\mathbb{R}$. We know, by the Heine-Borel theorem E is closed and bounded. Call L_u and G_l the least upper and greatest lower bound, respectively. Then we know $L_u \equiv \sup E$ and $G_l \equiv \inf E$. Now consider the sequences $L_u - \frac{1}{n}$ and $G_l + \frac{1}{n}$ which are clearly $\in E$. Furthermore, $\lim L_u - \frac{1}{n} = L_u$ and $\lim G_l + \frac{1}{n} = G_l$. E closed $\Longrightarrow L_u, G_l \in E$. But then by the definition of

L_u and G_l, (sup$E$, inf$E$) $\in E$. ♠

EXERCISE 14.2

(a) $\sum \frac{n-1}{n^2} =$ [16] $\sum \frac{n}{n^2} - \sum \frac{1}{n^2} = \sum \frac{1}{n} - \sum \frac{1}{n^2}$. Since $\sum \frac{1}{n}$ diverges, the entire series diverges.

(b) The series $\sum(-1)^n$ fails to converge because it doesn't satisfy the Cauchy criterion[17]. In other words, the terms of the sequence a_n do not arbitrarily grow closer to eachother as $n \to \infty$.

(c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2} = 3 \sum \frac{1}{n^2}$, which we know is 3 times a convergent series, thus the series converges.

(d) If $a_n = \frac{n^3}{3^n}$, then $a_{n+1}/a_n = \frac{(n+1)^3 3^n}{3^{n+1} n^3}$, so $\lim|a_{n+1}/a_n| = \frac{1}{3}$. Hence the series converges by the Ratio Test.

(e) If $a_n = \frac{n^2}{n!}$, then $a_{n+1}/a_n = \frac{(n+1)^2 n!}{(n+1)! n^2}$, so $\lim|a_{n+1}/a_n| = 0$. Hence the series converges by the Ratio Test.

(f) If $a_n = \frac{1}{n^n} = \left(\frac{1}{n}\right)^n$, then $\lim \sup|a_n|^{\frac{1}{n}} = 0$. Hence the series converges by the Root Test.

(g) If $a_n = \frac{n}{2^n}$, then $a_{n+1}/a_n = \frac{(n+1)2^n}{2^{n+1} n}$, so $\lim|a_{n+1}/a_n| = \frac{1}{2}$. Hence the series converges by the Ratio Test.

EXERCISE 14.4

(a) We will use induction and the Comparison Test to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{[n + (-1)^n]^2}$$

converges.

To accomplish this task we need to show

$$n + (-1)^n \geq \frac{1}{2} n,$$

so that

$$\sum_{n=2}^{\infty} \frac{1}{[n + (-1)^n]^2} \leq \sum_{n=2}^{\infty} \frac{4}{n^2} = 4 \sum_{n=2}^{\infty} \frac{1}{n^2},$$

since, by the Comparison Test, this would show the series is convergent.

[16] This single series has been split into 2 seperate series. The rule I am following is that the sum of two series will converge is both of the sums converge. Hence the series will diverge, if we can show that one of the sums diverges. Reference: **http://www.sosmath.com/calculus/series/examples/examples.html**

[17] Proof ommitted, as it wasn't required.

The summand index commences at $n = 2$, so this will serve as the induction basis.

$$2 + (-1)^2 \geq \frac{1}{2}(2) \Longrightarrow 3 \geq 1,$$

which is trivial.

Let us now implement the induction step, $n + 1$, and show the inequality still holds.

$$(n+1) + (-1)^{n+1} \geq \frac{1}{2}(n+1) \tag{23}$$

$$\Longleftrightarrow n + 1 + (-1)^n(-1) \geq \frac{1}{2}n + \frac{1}{2} \tag{24}$$

$$\Longleftrightarrow n + 1 - (-1)^n \geq \frac{1}{2}n + \frac{1}{2} \tag{25}$$

$$\Longleftrightarrow \frac{1}{2}n + \frac{1}{2} - (-1)^n \geq 0 \tag{26}$$

$$\frac{1}{2}n + \frac{1}{2} \geq (-1)^n, \tag{27}$$

which holds for $n \geq 2$, as desired. Hence,

$$n + (-1)^n \geq \frac{1}{2}n \tag{28}$$

$$\Longleftrightarrow (n + (-1)^n)^2 \geq \left(\frac{1}{2}n\right)^2 \tag{29}$$

$$\Longleftrightarrow \frac{1}{(n+(-1)^n)^2} \leq \frac{1}{\left(\frac{1}{2}n\right)^2} = \frac{4}{n^2} \tag{30}$$

$$\Longleftrightarrow \sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2} \leq \sum_{n=2}^{\infty} \frac{4}{n^2} = 4\sum_{n=2}^{\infty} \frac{1}{n^2}, \tag{31}$$

as desired.

(b) Since

$$\sum \left[\sqrt{n+1} - \sqrt{n}\right] = \sum \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \sum \frac{1}{2\sqrt{n+1}} \geq \sum \frac{1}{2\sqrt{2n}} = \frac{1}{2\sqrt{2}} \sum \frac{1}{\sqrt{n}},$$

which is a divergent "p-series"[18], the series diverges.

(c) We will show the series

$$\sum \frac{n!}{n^n}$$

[18] A "p-series" is a series of the form $\sum \frac{1}{n^p}$. Such a series converges if $p > 1$ and diverges if $p \leq 1$.

converges via the Ratio Test. We want to show

$$\lim \left|\frac{a_{n+1}}{a_n}\right| < 1.$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \lim \left|\frac{n!(n+1)n^n}{(n+1)^n(n+1)n!}\right| = \lim \left|\frac{n^n}{(n+1)^n}\right| = \lim \left|\left(\frac{n}{n+1}\right)^n\right|.$$

Since

$$\lim \left|\left(\frac{n+1}{n}\right)^n\right| = \lim \left|\left(1+\frac{1}{n}\right)^n\right| \Longrightarrow e,$$

this suffices to show that

$$\lim \left|\left(\frac{n}{n+1}\right)^n\right| \Longrightarrow \frac{1}{e} < 1.$$

Hence, the series converges by the Ratio Test.

EXERCISE 14.7; assume $p \in \mathbb{Z} : p > 1$

We want to show that if we have a known convergent series $\sum a_n$ and raise it to a power $p > 1$, it will simply converge quicker. An example would be the convergent p-series. We know that

$$\sum \frac{1}{n^p}$$

converges for values of $p > 1$. Now if the series is *further* raised by a power of $p > 1$ the original p will be *even greater*, and thus wll still be convergent.

Claim:

$\sum a_n :=$ convergent and $a_n \geq 0 \ \forall n \in \mathbb{N} \Longrightarrow \sum (a_n)^p$ converges.

***Proof*:**

If we know that $\sum a_n$ is a convergent series, then

$$\lim_{n\to\infty} a_n = 0 \Longrightarrow \exists N \in \mathbb{N} : a_n \in (0,1) \ \forall n \geq N \Longrightarrow (a_n)^p < a_n \ \forall p > 1 \text{ and } \forall n \geq N$$

$$(a_n)^p < a_n \ \forall p > 1 \text{ and } \forall n \geq N \Longrightarrow \sum (a_n)^p < \sum a_n \ \forall p > 1 \text{ and } \forall n \geq N.$$

Hence, $\sum (a_n)^p$ converges by the Comparison Test, as desired. ♠

EXERCISE 14.10

Consider *this* series:

$$\sum_{n=0}^{\infty} 2^{(-1)^n+n}$$

EXERCISE 15.4

Determine which of the following series converge.

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$. Since $\log n < \sqrt{n}$ for large $n \implies \frac{1}{\sqrt{n}} < \frac{1}{\log n} \implies \frac{1}{n} < \frac{1}{\sqrt{n}\log n} \implies \sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$, and hence is divergent by comparison with the harmonic series.

(b) $\sum_{n=2}^{\infty} \frac{\log n}{n}$. This problem trivially diverges when compared[19] to the harmonic series for values of $n > 1$.

(c) $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$. Implement the integral Test. $\int_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)} dn$ can be evaluated with a substitution. Let $u = \log\log n$. The integral now becomes

$$\int_{\log\log 4}^{\log\log\infty} \frac{1}{u} du = \Big[\log u\Big]_{\log\log 4}^{\log\log\infty} = \log\log\log\infty - \log\log\log 4 = \infty,$$

hence the series diverges by the integral test.

(d) $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$. The integral of $\frac{\log n}{n^2}$ is $\frac{-(\log n+1)}{n}$, so implementing the integral test yields:

$$\lim_{n\to\infty} \frac{-(\log x + 1)}{x}\Bigg|_2^n = \frac{-\log n}{n} - \frac{1}{n} + \frac{\log 2}{2} + \frac{1}{2},$$

which converges to $\frac{\log 2}{2} + \frac{1}{2}$ using L'Hospital's Rule. Hence, since the integral converges to an existent finite number, the series converges.

EXERCISE 15.6

(a) A divergent series $\sum a_n$ for which $\sum a_n^2$ converges is the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

(b) If $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges. This can be shown with a similar method as the proof done in Problem No. 3.

[19]Note: the index on the series begins at $n = 2 \implies \log n > 1$ for $n \geq 2$.

Claim:

$\sum a_n :=$ convergent and $a_n \geq 0 \ \forall n \in \mathbb{N} \Longrightarrow \sum(a_n)^2$ converges.

***Proof*:**

If we know that $\sum a_n$ is a convergent series, then

$$\lim_{n\to\infty} a_n = 0 \Longrightarrow \exists N \in \mathbb{N} : a_n \in (0,1) \ \forall n \geq N \Longrightarrow (a_n)^2 < a_n \ \forall n \geq N$$

$$(a_n)^2 < a_n \ \forall n \geq N \Longrightarrow \sum(a_n)^2 < \sum a_n \ \forall n \geq N.$$

Hence, $\sum(a_n)^2$ converges by the Comparison Test, as desired. ♠

(c) Consider *this* series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

EXERCISE 17.5

(a) We will use induction to prove $f(x) = x^m$ is continuous. This will be done by first showing that when $m = 1$ we are dealing with $f(x) = x$, which will be taken as continuous (proof is provided in the Appendix). Then by assuming that $f(x) = x^m$ is continuous for some $m \in \mathbb{N}$, we will show our induction step, $f(x) = x^{m+1}$ is continuous, by noting that $f(x) = x^{m+1} = x^m x$ is hence continuous given Theorem 17.4 (ii), which will then imply that $f(x) = x^m$ is continuous for $m \in \mathbb{N}$.

Claim:

If $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.

***Proof*:**

This proof will use mathematical induction. For $m = 1$, $f(x) = x$, which will be taken as continuous (proof provided in the Appendix). $f(x) = x$ will then serve as our induction basis. We can now assume our induction hypothesis, $f(x) = x^m$ is continuous. But we need to show continuity holds for the induction step, $m + 1$. But we know f(x) = $\mathrm{x}^{m+1} \equiv x^m \cdot x$, which is the product of two continuous functions, our inductions hypothesis, and our induction basis, and is hence continuous by Theorem 17.4 (ii). ♠

(b) *Claim*:

Every polynomial function $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

***Proof*:**

Given the solution from part **(a)**, $p(x)$ is simply the sum and product of continuous functions and is hence continuous, by Theorem 17.4 (i) and Theorem 17.3, as desired. ♠

EXERCISE 17.6

Claim:

Every rational function is continuous.

***Proof*:**

A rational function is composed of constants, $f(x) = c$ and the continuous function $f(x) = x$ by multiplication, addition and division. Since $f(x) = x$ and $f(x) = c$ are trivially continuous $\Longrightarrow$ rational functions are continuous by the continuity theorems of sums, products and quotients of continuous functions, as desired. ♠

EXERCISE 17.7 **(b)**

Claim:

$|x|$ is a continuous function on $\mathbb{R}$.

***Proof*:**

$|x|$ is continuous at any $x0$ since it coincides with x for $x > 0$ and $-x$ for $x < 0$. At $x = 0$, the function $f(x) = |x|$ is continuous because for any $\delta > 0$ we have: $|x - 0| < \epsilon \Longrightarrow |f(x) - f(0)| = |x| < \epsilon$. ♠

EXERCISE 17.8

(a) *Claim*:

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

***Proof*:**

Case 1: Let $f(x) \leq g(x)$. Then

$$\min(f, g) = f(x) = \frac{1}{2}(f + g) - \frac{1}{2}(g - f) \tag{32}$$

$$= \frac{1}{2}(f + g) - \frac{1}{2}|f - g| \tag{33}$$

Case 2: Let $f(x) \geq g(x)$. Then

$$\min(f, g) = g(x) = \frac{1}{2}(f + g) - \frac{1}{2}(f - g) \tag{34}$$

$$= \frac{1}{2}(f + g) - \frac{1}{2}|f - g|. \tag{35}$$

Hence,

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|,$$

as desired. ♠

(b) *Claim*:
$$\min(f,g) = -\max(-f,-g)$$

***Proof*:**

Case 1: Let $f(x) \leq g(x) \Longrightarrow -f(x) \geq -g(x)$
$\Longrightarrow \min(f,g) = f(x) = -(-f(x)) = -\max(-f,-g)$

Case 2: Let $f(x) \geq g(x) \Longrightarrow -f(x) \leq -g(x)$
$\Longrightarrow \min(f,g) = g(x) = -(-g(x)) = -\max(-f,-g)$.

Hence,
$$\min(f,g) = -\max(-f,-g),$$
as desired. ♠

(c) *Claim*:
$$f \text{ and } g \text{ continuous at } x_0 \in \mathbb{R} \Longrightarrow \min(f,g) \text{ is continuous at } x_0$$

***Proof*:**

Recall Theorems 17.3 and 17.4 (i):

> **Theorem 17.3:** Let f be a real-valued function with $\text{dom}(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $\text{dom}(f)$, then $|f|$ and kf, $k \in \mathbb{R}$, are continuous at x_0.
>
> **Theorem 17.4 (i):** Let f and g be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then $f+g$ is continuous at x_0.

In combination with the results from part **(a)**, i.e.,
$$\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|,$$
we see that $\min(f,g)$ is simply the sum, difference and composition of functions which are continuous at x_0, and hence is itself continuous at x_0, as desired. ♠.

EXERCISE 17.9

(a) *Claim*:
$$f(x) = x^2 \text{ is continuous at } x_0 = 2$$

***Proof*:**

Let $\epsilon > 0$ be given. We want to show that $|x^2 - 4| < \epsilon$ provided $|x-2|$ is sufficiently small, i.e., less than some δ. We observe that $|x^2-4| = |(x+2)(x-2)| \leq |x+2| \cdot |x-2|$. We need to get a bound for $|x-2|$ that doesn't depend on x. We notice that if $|x-2| < 1$, say, then $|x+2| < 5$, so it suffices to get $|x-2| \cdot 5 < \epsilon$. So by setting $\delta = \min\left\{1, \frac{\epsilon}{5}\right\}$, we see that $f(x) = x^2$ is continuous at $x_0 = 2$, as desired. ♠

(b) *Claim*:

$$f(x) = \sqrt{x} \text{ is continuous at } x_0 = 0$$

***Proof*:**

Let $\epsilon > 0$ be given. We want to show that $|\sqrt{x} - 0| < \epsilon$ provided $|x - 0|$ is sufficiently small, i.e., less than some δ. We observe that $|\sqrt{x} - 0| = \sqrt{x}$. Since we want this to be less than ϵ, we set $\delta = \epsilon^2$. Then $|x - 0| < \delta$ implies $\sqrt{x} < \sqrt{\delta} = \epsilon$, so

$$|x - 0| < \delta \Longrightarrow |f(x) - f(0)| < \epsilon,$$

as desired. ♠

(c) *Claim*:

$$f(x) = x\sin\left(\frac{1}{x}\right) \text{ for } x \neq 0 \text{ and } f(0) = 0 \text{ is continuous at } x_0 = 0$$

***Proof*:**

Let $\epsilon > 0$ be given. We want to show that $|x\sin\left(\frac{1}{x}\right) - 0| < \epsilon$ provided $|x - 0|$ is sufficiently small, i.e., less than some δ. We see that $|x\sin\left(\frac{1}{x}\right) - 0| \leq x \ \forall x$. Since we want this to be less than ϵ, we set $\delta = \epsilon$. Then $|x - 0| < \delta$ implies $x < \delta = \epsilon$, so

$$|x - 0| < \delta \Longrightarrow \left|x\sin\left(\frac{1}{x}\right) - 0\right| < \epsilon,$$

as desired. ♠

(d) *Claim*:

$$g(x) = x^3 \text{ is continuous at } x_0 \text{ arbitrary}$$

***Proof*:**

By the hint given,

$$x^3 - x_0^3 = (x - x_0)(x^2 + x_0x + x_0^2) = (x - x_0)\left[(x^2 - x_0)^2 + 3xx_0)\right]$$

and we know that

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0|,$$

by the triangle inequality. Hence,

$$|g(x) - g(x_0)| = |x^3 - x_0^3| \leq |x - x_0|(|x - x_0|^2 + 3|x - x_0||x_0| + 3x_0^2).$$

Now let $\epsilon > 0$ be given. We show that $\exists$ a $\delta = \delta(x_0, \epsilon) > 0$ such that

$$|x - x_0| < \delta \Longrightarrow |g(x) - g(x_0)| < \epsilon.$$

Let $\delta = \min\left(1, \frac{\epsilon}{3}, \frac{\epsilon}{g|x_0|+1}, \frac{\epsilon}{gx_0^2+1}\right)$. Then $|x - x_0| < \delta$ implies

$$
\begin{aligned}
|g(x) - g(x_0)| &\leq \delta(\delta^2 + 3\delta|x_0| + 3x_0^2) \leq {}^{20}\delta(1 + 3|x_0| + 3x_0^2) \\
&= \delta + 3\delta|x_0| + 3\delta x_0^2 < \frac{\epsilon}{2} + 3|x_0|\frac{\epsilon}{g|x_0| + 1} + 3x_0^2\frac{\epsilon}{g|x_0|^2 + 1}, \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}
$$

as desired. ♠.

Alternatively,

Assume $g(x) = x^2$, x_0 arbitrary. Observe that $(x^3 - x_0^3) = (x - x_0)(x^2 + x_0x + x_0^2)$. Now we will make the assumption that $|x - x_0| < 1 \implies |x| < |x_0| + 1$, which enables us to see

$$|x^2 + x_0x + x_0^2| \leq |x^2| + |x_0x| + |x_0^2| < |x_0|^2 + 2|x_0| + 1 + |x_0| \cdot ||x_0| + 1| + |x_0^2| \equiv k,$$

where the material to the right of the last inequality is all greater than 0.

We can then set $\delta = \min\left\{1, \frac{\epsilon}{k}\right\}$. Hence,

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| = |(x - x_0)(x^2 + x_0x + x_0^2)| < |x - x_0||k| < \frac{\epsilon}{k} \cdot k = \epsilon,$$

as desired. ♠

EXERCISE 17.12

(a) Let f be a continuous real-valued function with domain (a, b).

Claim:

If $f(r) = 0$ for each rational number $r \in (a.b)$, then $f(x) = 0 \ \forall x \in (a.b)$.

***Proof*:**

We are given $f(x) = 0 \ \forall x \in \mathbb{Q}$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $\exists$ a sequence of rational numbers, (r_n), which converges to x. Hence, by continuity, $r_n \to x \implies f(r_n) \to f(x)$. But $f(r_n) = 0 \ \forall n$, given the conditions in the claim, so $0 \to f(x) \implies f(x) = 0 \ \forall x \in (a, b)$, as desired. ♠

(b) Let f and g be continuous real-valued functions on $(a, b) : f(r) = g(r)$ for each rational number $r \in (a, b)$.

Claim:

$f(x) = g(x) \ \forall x \in (a, b)$.

***Proof*:**

Using the limit concept of sequences, for any $x \in (a, b)$ $\exists$ a sequence of rational numbers, $(r_n) : \lim f_n = x \implies f(x) = \lim f(r_n) = \lim g(r_n) = g(x)$.

[20] $\delta \leq 1$

Hence, $f(x) = g(x)$, as desired. ♠

EXERCISE 17.13 **(b)**

Let $h(x) = x \ \forall x \in \mathbb{Q}$ and $h(x) = 0 \ \forall x \in \mathbb{R} \setminus \mathbb{Q}$.

Claim:

h is continuous at $x = 0$ and no other point.

***Proof*:**

For any $\epsilon > 0$, if $|x - 0| < \epsilon$, then $|h(x) - h(0)|$ is either 0 (if x is $\in \mathbb{R} \setminus \mathbb{Q}$) or $|x|$ (if $x \in \mathbb{Q}$, and thus in both cases $< \epsilon$). Thus h is continuous at $x = 0$. $\forall$ other x, consider two sequences with limit x, one $(r_n) \in \mathbb{Q}$, and another, $(x_n) \in \mathbb{R} \setminus \mathbb{Q}$. Then $\lim h(x_n) = 0$ and $\lim h(r_n) = x \neq 0$. Hence, $\exists$ disconituity for h at $x \neq 0$, as desired. ♠

EXERCISE 17.14

Claim:

f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.

***Proof*:**

For $x = \frac{p}{q} \in \mathbb{Q}$, define a sequence $x_n \in \mathbb{R} \setminus \mathbb{Q} : \lim x_n = x \Longrightarrow \lim f(x_n) = 0$, but $f(x) = \frac{1}{q} \neq 0 \Longrightarrow f$ is discontinuous at x. For an irrational x and any $\epsilon > 0$, let $\delta > 0$ be defined as the distance from x to the closest irreducible fraction $\frac{p}{q}$ with denominator $q \leq \frac{1}{\epsilon}$. Then for any $x' : |x' - x| < \delta \Longrightarrow |f(x') - f(x)| < 0 \Longrightarrow f$ continuous at $x \in \mathbb{R} \setminus \mathbb{Q}$, as desired. ♠

EXERCISE 18.2

The limit x_0 (or y_0) of the subsequence $x_{n_k} \in (a, b)$ (or $y_{n_k} \in (a, b)$) may be an endpoint a or b of the interval and thus lie outside of the domain of the function.)

EXERCISE 18.4

Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S that converges to a number $x_0 \notin S$.

Claim:

$\exists$ an unbounded continuous finction on S.

***Proof*:**

Let $x_0 \notin S$. We are given a sequence $(x_n) \in S$ which converges to x_0. Then $|x_n - x_0| :=$ the distance to x_0 is continuous and strictly positive on S. Define $f := \frac{1}{|x_n - x_0|} \implies f$ is well-defined and continuous on $S \implies \lim_{n\to\infty} f = \infty$, as desired. ♠

EXERCISE 18.6

Claim:

$x = \cos x$ for some $x \in \left(0, \frac{\pi}{2}\right)$.

***Proof*:**

We know $f(x) = x - \cos x$ is continuous $\in \left[0, \frac{\pi}{2}\right]$, < 0 at $x = 0$ and > 0 at $x = \frac{\pi}{2}$. Implement the Intermediate Value Theorem. Hence, $\exists$ an $x \in \left(0, \frac{\pi}{2}\right) : f(x) = 0$, as desired. ♠

EXERCISE 18.8

Suppose that f is a real-valued function continuous on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$.

Before commencing, we will state the properties that

$$0 \cdot a = 0 \ \forall a \in \mathbb{Z}$$

and

$$a \cdot b < 0 \iff a < 0, b > 0 \text{ or } a > 0, b < 0 \ \forall a, b \in \mathbb{Z}$$

Claim:

$\exists$ an x between a and $b : f(x) = 0$.

***Proof*:**

Given $f(a)f(b) < 0$, either

Case 1: $f(a) < 0 \implies f(b) > 0 \implies f(a) < 0 < f(b)$, or

Case 2: $f(a) > 0 \implies f(b) < 0 \implies f(b) < 0 < f(a)$.

In either case, the Intermediate Value Theorem tells us that $\exists\, x \in (a, b) : f(x) = 0$, as desired. ♠

EXERCISE 18.10

Suppose that f is continuous on $[0,2]$ and $f(0)=2$.

Claim:

$x, y \in [0,2] : |x-y| = 1$ and $f(x) = f(y)$.

***Proof*:**

Let $g(x) = f(x+1) - f(x) \implies g$ is continuous on $[0,1]$ and $g(0) = f(1) - f(0) = f(1) - f(2) = -g(1)$. Implement the Intermediate Value Theorem. $\exists x \in [0,1] : g(x) = 0$, as desired. ♠

EXERCISE 19.2

(a) *Claim*:

$$f(x) = 3x + 1 \text{ is uniformly continuous on } \mathbb{R}.$$

***Proof*:**

Let $\epsilon > 0$ be given. We want $|f(x) - f(y)| = |(3x+1) - (3y+1)| < \epsilon$ for $|x-y| < \delta$ with $x, y \in \mathbb{R}$. We know $|(3x+1)-(3y+1)| = |3x-3y| = 3|x-y|$. Take $\delta := \frac{\epsilon}{3}$. Then

$$|x-y| < \delta \implies |x-y| < \frac{\epsilon}{3} \implies 3|x-y| < \epsilon \implies |3x-3y| < \epsilon \implies |3x-3y+1-1| < \epsilon \implies$$

$$|3x+1-3y-1| < \epsilon \implies |(3x+1)-(3y+1)| < \epsilon \implies |f(x)-f(y)| < \epsilon,$$

as desired. ♠

(b) Claim:

$$f(x) = x^2 \text{ is uniformly continuous on } [0,3].$$

***Proof*:**

Let $\epsilon > 0$ be given. We want $|f(x) - f(y)| = |x^2 - y^2| < \epsilon$ for $|x-y| < \delta$ with $x, y \in [0,3]$. We know $|x^2 - y^2| = |(x-y)(x+y)| = |x-y| \cdot |x+y|$. With $x, y \in [0,3]$, let $|x+y| \leq |3+3| = 6$. Define $\delta := \frac{\epsilon}{6}$. Then for any $x, y \in [0,3]$ with $|x-y| < \delta$,

$$|f(x)-f(y)| = |x^2-y^2| = |(x-y)(x+y)| = |x-y| \cdot |x+y| < |x-y| \cdot 6 = \epsilon,$$

as desired. ♠

(c) Claim:

$$f(x) = \frac{1}{x} \text{ is uniformly continuous on } \left[\frac{1}{2}, \infty\right).$$

***Proof*:**
Let $\epsilon > 0$ be given. We want $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| < \epsilon$ for $|x - y| < \delta$, with $x, y \in \left[\frac{1}{2}, \infty\right)$. We know $|\frac{1}{x} - \frac{1}{y}| = |\frac{y-x}{xy}| = |(\frac{1}{xy})(y - x)| = |x - y| \cdot |\frac{1}{xy}|$. With $x, y \in \left[\frac{1}{2}, \infty\right)$, let $\frac{1}{xy} \leq \frac{1}{\frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{\frac{1}{4}} = 4$. Define $\delta := \frac{\epsilon}{4}$. Then for any $x, y \in \left[\frac{1}{2}, \infty\right)$ with $|x - y| < \delta$,

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = |x - y| \cdot \left|\frac{1}{xy}\right| < |x - y| \cdot 4 = \epsilon,$$

as desired. ♠.

EXERCISE 19.4

(a) *Claim*:

If f is uniformly continuous on a bounded set S, then f is a bounded function on S.

***Proof*:**
By contradiction. Assume that f is uniformly continuous on a bounded set S *and* an unbounded function on S. Then $\exists$ a sequence, (x_n) in the domain of f such that $|f(x_n)| \geq n$. By the Bolzano-Weierstrass Theorem, the sequence contains a convergent subsequence (x_{n_k}), since the domain is bounded. A convergent subsequence is Cauchy (by definition) and hence the sequence of values $f(x_{n_k})$ is Cauchy by the property of uniformly continuous functions. But the sequence $|f(x_{n_k})| \geq n_k$ is unbounded, contradicting our assumption in the outset of this proof. Hence, if f is uniformly continuous on a bounded set S, then f is a bounded function on S, as desired. ♠

(b) *Claim*:

$$\frac{1}{x^2} \text{ is not uniformly continuous on } (0, 1).$$

***Proof*:**
We want to show that if $\frac{1}{x^2}$ is not a bounded function on $(0, 1)$, then $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$[21]. So it will suffice to show that

$$f(x) = \frac{1}{x^2} \text{ is not a bounded function on } (0, 1).$$

Let $M > 1$ be arbitrary $\Longrightarrow 0 < \frac{1}{M^2} < 1$. We want to show that $\frac{1}{x^2} > M$ for some $x \in (0, 1) \Longrightarrow x < \frac{1}{\sqrt{M}}$. Let $x = \frac{1}{\sqrt{M+1}}$, since $0 < \frac{1}{\sqrt{M+1}} < \frac{1}{\sqrt{M}} < 1$. But then for $f(x) = \frac{1}{x^2}$, we see that

$$\frac{1}{x^2} = \frac{1}{\left(\frac{1}{\sqrt{M+1}}\right)^2} = M + 1 > M,$$

[21] This is the *contrapositive* of what was proved in part **(a)**.

which shows that $f(x) = \frac{1}{x^2}$ is not a bounded function on $(0,1)$, as desired. ♠

EXERCISE 19.6

(a) *Claim*:

$f(x) = \sqrt{x}$ is uniformly continuous on $(0,1]$, although f' is unbounded.

***Proof*:**

$f'(x) = \frac{1}{2\sqrt{x}} \to +\infty$ as $x \to 0 \implies f'(x)$ is unbounded. f continuous on the closed interval $[0,1] \implies f$ uniformly continuous on $[0,1] \implies f$ uniformly continuous on the subset $(0,1]$, as desired. ♠

Let $f(x) = \sqrt{x}$ for $x \geq 1$.

(b) *Claim*:

f is uniformly continuous on $[1,\infty)$.

***Proof*:**

Let $\epsilon > 0$ be given. We want $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| < \epsilon$ for $|x-y| < \delta$ with $x, y \in \mathbb{R}$. We know $|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|}$. Take $\delta := 2\epsilon$. Then

$$|x-y| < \delta \implies |x-y| < 2\epsilon \implies \frac{|x-y|}{2} \leq \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|} = |\sqrt{x} - \sqrt{y}| < \epsilon,$$

as desired. ♠

EXERCISE 19.10

The limit

$$\lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} x \sin x \frac{1}{x} = 0,$$

in other words, g is differentiable at $x = 0$ $(g'(0) = 0)$. At $x \neq 0$,

$$g'(x) = 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = 2x \sin\frac{1}{x} - \cos\frac{1}{x},$$

which is bounded. Since $|\cos y| \neq 1 and\, |\sin y| \leq |y|\, \forall y \implies$ for $y = \frac{1}{x}$:

$$\left|2x \sin\frac{1}{x} - \cos\frac{1}{x}\right| = \left|\frac{2}{y} \sin y - \cos y\right| \leq 2 + 1 = 3.$$

Thus, g' is both bounded and defined on $\mathbb{R} \implies g$ is uniformly continuous.

EXERCISE 20.11 **(b)**

Claim:

$$\lim_{x\to b}\frac{\sqrt{x}-\sqrt{b}}{x-b}=\frac{1}{2\sqrt{b}}$$

***Proof*:**

$$\frac{\sqrt{x}-\sqrt{b}}{x-b}=\frac{(\sqrt{x}-\sqrt{b})\cdot\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}+\sqrt{b}}}{x-b}=\frac{\frac{x-b}{\sqrt{x}+\sqrt{b}}}{x-b}=\frac{1}{\sqrt{x}+\sqrt{b}}$$

Hence,

$$\lim_{x\to b}\frac{1}{\sqrt{x}+\sqrt{b}}\longrightarrow\frac{1}{\sqrt{b}+\sqrt{b}}=\frac{1}{2\sqrt{b}}$$

EXERCISE 20.14

Claim:

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty \text{ and } \lim_{x\to 0^-}\frac{1}{x}=-\infty$$

We will split this up into 2 cases/claims.

Claim 1:

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty$$

***Proof*:**

Let $M>0$ and $\delta:=\frac{1}{2M}$. Then $0<x<0+\delta \implies 0<x<\delta \implies f(x)=\frac{1}{x}>\frac{1}{\delta}=\frac{1}{\frac{1}{2M}}=2M>M$. Hence, $\lim_{x\to 0^+}\frac{1}{x}=+\infty$, as desired. ♠

Claim 2:

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty$$

***Proof*:**

Let $M<0$ and $\delta:=\frac{-1}{2M}$. Then $0-\delta<x<0 \implies -\delta<x<0 \implies f(x)=\frac{1}{x}<\frac{-1}{\delta}=\frac{1}{\frac{1}{2M}}=2M<M$. Hence, $\lim_{x\to 0^-}\frac{1}{x}=-\infty$, as desired. ♠

EXERCISE 20.16

Suppose that the limits $L_1=\lim_{x\to a^+}f_1(x)$ and $L_2=\lim_{x\to a^+}f_1(x)$ exist.

(a) *Claim*:

If $f_1(x)\le f_2(x)\ \forall x$ in some interal (a,b), then $L_1\le L_2$.

***Proof*:**

By contradiction. Let $f_1(x) \leq f_2(x)$ and assume $L_1 > L_2$. Corollary 20.8 says

$$\lim_{x \to a^+} f(x) = L \iff \forall \epsilon > 0 \; \exists \; \delta > 0 : a < x < a + \delta \implies |f(x) - L| < \epsilon$$

Define $\epsilon := \frac{1}{2}(L_1 - L_2) \implies$ for $\epsilon > 0$:

$$\exists \; \delta_1 : x \in (a, a + \delta_1), f_1(x) > L_1 - \epsilon = L_1 - \left[\frac{1}{2}(L_1 - L_2)\right] = \frac{1}{2}(L_1 + L_2)$$

$$\exists \; \delta_2 : x \in (a, a + \delta_2), f_2(x) < L_2 + \epsilon = L_2 + \left[\frac{1}{2}(L_1 - L_2)\right] = \frac{1}{2}(L_1 + L_2).$$

Let $\delta := \min\{\delta_1, \delta_2\} \implies$

$$f_1(x) > \frac{1}{2}(L_1 + L_2) > f_2(x),$$

which is a contradiction. Hence, if $f_1(x) \leq f_2(x)$ $\forall x$ in some interal (a, b), then $L_1 \leq L_2$, as desired. ♠

(b) Suppose $f_1(x) < f_2(x)$ $\forall x$ in some interval (a, b). Can you conclude that $L_1 < L_2$.

Consider $f_1(x) = x$ and $f_2(x) = 2x$, $x \in (a, b) \implies f_1(x) < f_2(x)$ $\forall x \in (a, b)$. Consider $a = 0$ and $b = 1 \implies (a, b) = (0, 1)$. We can plainly see that

$$\lim_{x \to 0} f_1(x) = \lim_{x \to 0} f_2(x) = 0^{22},$$

a contradiction $\implies$ we cannot conclude that $L_1 < L_2$ if $f_1(x) < f_2(x)$ $\forall x$ in some interval (a, b).

EXERCISE 20.18

Claim:

$$\text{For } f(x) = \frac{\sqrt{1 + 3x^2} - 1}{x^2}, \; \lim_{x \to 0} f(x) = \frac{3}{2}.$$

***Proof*:**

Non-formal. Rearranging $f(x)$,

$$\frac{\sqrt{1 + 3x^2} - 1}{x^2} = \frac{\sqrt{1 + 3x^2} - 1}{x^2} \cdot \frac{\sqrt{1 + 3x^2} + 1}{\sqrt{1 + 3x^2} + 1} = \frac{(1 + 3x^2) - 1}{(\sqrt{1 + 3x^2} + 1)x^2} = \frac{3}{\sqrt{1 + 3x^2} + 1},$$

which we see is a composition of continuous functions which happen to behave well near $x = 0$. Hence, $\lim_{x \to 0} f(x) \to f(0) = \frac{3}{2}$, as desired. ♠

[22] Since both f_1 and f_2 extend continuously to 0 where they take on the value 0.

EXERCISE 21.6

Let (S_1, d_1), (S_2, d_2), and (S_3, d_3) be metric spaces.

Claim:

$$f : S_1 \to S_2 \text{ and } g : S_2 \to S_3 \text{ continuous} \Longrightarrow g \circ f \text{ continuous from } S_1 \text{ into } S_3$$

***Proof*:**

From both the claim and Theorem 21.3, $f^{-1}(U)$ is an open subset of S_1 for every open subset U of S_2, i.e., $f^{-1}(U) = \{s \in S_1 : f(s) \in U\}$ and $g^{-1}(V)$ is an open subset of S_3 for every open subset V of S_3, i.e., $g^{-1}(V) = \{s \in S_3 : f(s) \in V\}$. Define a new mapping, $h(S_1) = g \circ f : S_1 \to S_3$. Then $h^{-1}(V)$ is an open subset of S_1 for every open subset V of S_3, i.e. $h^{-1}(V) = \{s \in S_1 : h(S) \in V\}$, by the continuity of composite functions outlined in Theorem 17.5 and the fact that g and f are both coninuous throughout S. Thus, by Theorems 17.5 and 21.3, $h(S_1) := (g \circ f)(S_1) : S_1 \to S_3$ is continuous, as desired. ♠

EXERCISE 21.10 **(b)** and 21.11 **(b)**

For **21.10 (b)**, in order to show $\exists$ continuous functions mapping $(0, 1) \to \mathbb{R}$, consider

$$f(x) = \frac{\log(2x)}{1 - x}$$

As $x \to 0$, $f(x) \to -\infty$ and as $x \to 1$, $f(x) \to +\infty$. We see $f(x)$ is the composition of continuous functions and the denominator $\neq 0$ through f's domain and is hence continuous, as desired.

For **21.11 (b)**, we can use informal contradiction:

$f : [0, 1] \to \mathbb{R} \Longrightarrow \mathbb{R}$ is compact, since [0, 1] is compact, by Theorem 21.4 (i). But $\mathbb{R}$ is unbounded and thus cannot be compact, which is a contradiction. Hence, there do not exist continuous functions mapping [0, 1] onto $\mathbb{R}$.

EXERCISE 23.1 **(b)**, **(d)**, **(f)** and **(h)**

(b) $\sum \left(\frac{x}{n}\right)^n = \sum \frac{x^n}{n^n} = \sum \left(\frac{1}{n}\right)^n x^n$. If $a_n = \left(\frac{1}{n}\right)^n$, then $\limsup |a_n|^{\frac{1}{n}} = 0$. Therefore, $\beta = 0$, $R = +\infty$ and this series has a radius of convergence $+\infty$ and hence an interval of convergence of $(-\infty, +\infty)$.

(d) $\sum \left(\frac{n3}{3^n}\right) x^n$. If $a_n = \frac{n3}{3^n}$, then $\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$, so $\lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$. Therefore, $\beta = \frac{1}{3}$ and $R = 3$. This series diverges for both $x = 3$ and $x = -3$, hence the radius of convergence is 3 and the interval of convergence is $(-3, 3)$.

(f) $\sum \left(\frac{1}{(n+1)^2 2^n}\right) x^n$. If $a_n = \frac{1}{(n+1)^2 2^n}$, then $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^n}{(n+2)^2 2^{n+1}}$, so $\lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}$. Therefore, $\beta = \frac{1}{2}$ and $R = 2$. This series converges at both $x = 2$ and

$x = -2$, hence the radius of convergence is 2 and the interval of convergence is $[-2, 2]$.

(h) $\sum \left(\frac{(-1)^n}{n^2 \cdot 4^n}\right) x^n$. If $a_n = \frac{(-1)^n}{n^2 \cdot 4^n}$, then $\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(n+1)^2 \cdot 4^{n+1}} \cdot \frac{n^2 \cdot 4^n}{(-1)^n}$, so $\lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$. Therefore, $\beta = \frac{1}{4}$ and $R = 4$. This series converges at both $x = 4$ and $x = -4$, hence the radius of convergence is 4 and the interval of convergence is $[-4, 4]$.

EXERCISE 23.2

(a) $\sum \sqrt{n} x^n$. If $a_n = \sqrt{n}$, then $\frac{a_{n+1}}{a_n} = \frac{\sqrt{n+1}}{\sqrt{n}}$, so $\lim \left|\frac{\sqrt{n+1}}{\sqrt{n}}\right| = 1$. Therefore, $\beta = 1$ and $R = 1$. This series diverges at $x = 1$ and $x = -1$, hence the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

(b) $\sum \frac{1}{n\sqrt{n}} x^n$. If $a_n = \frac{1}{n\sqrt{n}}$, then $|a_n|^{\frac{1}{n}} = \left|\frac{1}{n^{\frac{1}{\sqrt{n}}}}\right|$ and $\limsup \left|\frac{1}{n^{\frac{1}{\sqrt{n}}}}\right| = 1$. At $x = 1$, the series converges with comparison to $\sum \frac{1}{n^p}$ with $p > 1$ and for $x = -1$, the series diverges by the alternating series theorem since the limit of $|a_n|$ approaches 1 and not 0. Hence, the radius on convergence is 1 and the interval of convergence is $(-1, 1]$.

(c) $\sum x^{n!}$. When $|x| \geq 1$, the series diverges since $\lim x^{n!}$ does not tend to 0 as $n \to \infty$. Now for $|x| < 1$, the series converges absolutely by comparison with $\sum |x|^m$[23]. Therefore, $\beta = 1$ and $R = 1$. This series diverges at $x = 1$ and $x = -1$, hence the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

(d) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$. For this series, the radius of convergence is:

$$R = \frac{1}{\lim \left(\frac{3^n}{\sqrt{n}}\right)^{\frac{1}{2n+1}}} = \frac{1}{\sqrt{3}} \lim (3n)^{\frac{1}{4n+2}} = \frac{1}{\sqrt{3}}.$$

When $x = \pm\frac{1}{\sqrt{3}}$, the series explodes in both directions[24]. Hence, the radius of convergence is $\frac{1}{\sqrt{3}}$ and the interval of convergence is $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

EXERCISE 23.6 **(b)**

An example of such a series is

$$\sum_{n>0} \frac{(-x)^n}{n}.$$

[23] Note that $\sum x^{n!} = \sum a_m x^m$ with $a_m = 1$ when $m = n!$ and $a_m = 0$ when $m \neq n!$.

[24] The series turns into $\sum \pm \frac{1}{\sqrt{3n}}$.

This series converges to $-\ln(1+x)$ when $|x|<1$, diverges at $x=-1$ and converges at $x=+1$.

EXERCISE 23.8

$f_n(x) \to 0$ since $\left|n^{-1}\sin nx\right| \neq \frac{1}{n} \to 0$ as $n\to\infty$. But $f_n'(x) = \cos nx \to (-1)^n$ when $x=\pi$, which has no limit.

EXERCISE 24.2

For $x \in [0,\infty)$, let $f_n(x) = \frac{x}{n}$.

(a) For $f_n(x) = \frac{x}{n}$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0$. Hence, $f(x) = 0$.

(b) YES.

Claim: $f_n \to f$ uniformly on $[0,1]$.

***Proof*:**

Let $\epsilon > 0$ be given. Let $N := \frac{1}{\epsilon}$. Then, for $n > N$, $|f_n(x) - 0| = \left|\frac{x}{n}\right| \leq \frac{\frac{1}{\epsilon}}{n} = \frac{1}{n\epsilon} < \epsilon$, as desired. ♠

(c) NO.

Claim: f_n does not converge uniformly to f on $[0,\infty)$.

***Proof*:**

By contradiction. Using the above, let $\epsilon := 1 \implies \exists$ an $N : \left|\frac{x}{n}\right| < 1\ \forall n > N \implies |x| < n\ \forall n > N \implies |x| < n + \xi$ for some $\xi > 0$. But since $x \in [0,\infty)$, x is unbounded, contradicting $|x| < n+\xi$ for some $\xi > 0$. Hence, f_n does not converge uniformly to f on $[0,\infty)$, as desired. ♠

EXERCISE 24.6

Let $f_n(x) = \left(x - \frac{1}{n}\right)^2$ for $x \in [0,1]$.

(a) YES.

Claim:

$f_n(x) = \left(x - \frac{1}{n}\right)^2$ for $x \in [0,1]$ converges pointwise on the set $[0,1]$.

***Proof*:**

$f_n(x) = \left(x-\frac{1}{n}\right)^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2}$ With $x \in [0,1]$, let n grow arbitrarily large $\implies f_n(x) \longrightarrow x^2$, since with n large, $\frac{2x}{n} \longrightarrow 0$ and $\frac{1}{n^2} \longrightarrow 0 \implies f_n(x) \longrightarrow f(x) := x^2$. Hence, given x arbitrary, $f_n(x)$ converges pointwise for $x \in [0,1]$, as desired. ♠

The limit function will thus be $f(x) = x^2$.

(b) YES.

Claim:

$f_n(x) = \left(x - \frac{1}{n}\right)^2$ for $x \in [0,1]$ converges uniformly on the set $[0,1]$.

***Proof*:**

Let $\epsilon > 0$ be given and $N := \frac{1}{\epsilon^2}$. Then for $x \in [0,1]$, we have:

$$\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \left|\frac{1-2xn}{n^2}\right| \leq \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \epsilon,$$

as desired. ♠.

EXERCISE 24.14

Let $f_n(x) = \frac{nx}{1+n^2x^2}$

(a) *Claim*:

$f_n \longrightarrow 0$ pointwise on $\mathbb{R}$.

***Proof*:**

Let $x \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrary. Let $N := \max\left\{x^2, \frac{\sqrt{x}}{\sqrt{\epsilon}}\right\}$. Then for $n > N$,

$$|f_n - 0| = \left|\frac{nx}{1+n^2x^2}\right| = \left|\frac{x}{\frac{1}{n} + nx^2}\right| \leq \left|\frac{x}{\frac{1}{n} + n^2}\right| \leq \left|\frac{x}{n^2}\right| \leq \frac{x}{N^2} < \epsilon.$$

Hence, $f_n(x)$ converges uniformly to zero, as desired. ♠

(b) *Claim*:

$f_n(x)$ does not converge to 0 uniformly on $[0,1]$.

***Proof*:**

Implement **Remark 24.4**. Hence,

$$f_n'(x) = \frac{n}{1+n^2x^2} - \frac{nxn^2 2x}{(1+n^2x^2)^2} = \frac{n + n^3x^2 - 2n^3x^2}{(1+n^2x^2)^2} = \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

Assume, for contradiction, $\frac{n-n^3x^2}{(1+n^2x^2)^2} \to 0$, which implies $x^2 = \frac{1}{n^2}$ which implies $x = \pm\frac{1}{n}$. But we see that $f_n\left(\frac{1}{n}\right) = \frac{1}{2} \neq 0$, a contradiction, since $x \in [0,1]$ satisfies $x = \pm\frac{1}{n}$. Hence, $f_n(x)$ does not converge to 0 uniformly on $[0,1]$ as desired. ♠

(c) *Claim*:

$f_n(x)$ converges to 0 uniformly on $[1, \infty)$.

***Proof*:**

Implement **Remark 24.4**. With $x \in [1, \infty)$, x is unable to always satisfy $\pm\frac{1}{n}$. Using methods from calculus we can see that $f_n(x) = \frac{nx}{1+n^2x^2}$ assumes its maximum at $x = 1$, which is $\in [1, \infty)$. Since $f_n(1) = \frac{n}{1+n^2} \to 0$ as $n \to \infty$, $f_n(x)$ converges to 0 uniformly on $[1, \infty)$ as desired. ♠

EXERCISE 25.2

Let $f_n(x) = \frac{x^n}{n}$.

Claim: (f_n) is uniformly convergent on $[-1, 1]$.

***Proof*:**

Let $\epsilon > 0$ be given. Let $N := \frac{1}{\epsilon}$. Then for $n > N$ and $\forall x \in [-1, 1]$,

$$|f_n(x) - f(x)| = \left|\frac{x^n}{n} - 0\right| \leq \frac{1}{n}^{25} \leq \frac{1}{N} = \epsilon,$$

as desired. ♠

The limit function is $f_n(x) \to f(x) = 0$ for large n.

EXERCISE 25.6

(a) Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[0, 1]$ to a continuous function.

Implement **Theorem 25.5** and the **Weierstrass M-test**. Since $\sum |a_k| < \infty$ and that $\sum a_k x^k \leq \sum a_k$ because $x \in [0, 1]$, we know $\sum a_k x^k$ converges uniformly on S. Now since the series converges uniformly on S and $a_k x^k$ is continuous, then $\sum a_k x^k$ represents a continuous function on S.

(b) Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1, 1]$?

This is a series which converges at both $x = -1$ (by the alternating series test) and at $x = 1$ (convergent p-series). Now consider the interval $-1 \leq a \leq 1$ and note that $\sum_{n=1}^{\infty} \frac{1}{n^2} a^n$ converges. Since $|n^{-2}x^n| \geq |n^{-2}a^n| = \left(\frac{a^n}{n^2}\right)$ for $x \in [-a, a]$, the Weierstrass M-test shows that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ converges uniformly to a function on $[-a, a]$. Since $|a|$ can be any number ≤ 1, we conclude that f represents a continuous function on $[-1, 1]$[26].

[25] This function takes its max value at $x = 1$.

[26] Note: $\sum_{n=1}^{\infty} \left|\frac{1}{n^2} x^n\right| \leq \frac{\pi^2}{6}$ for $x \in [-1, 1]$.

EXERCISE 25.14

Claim:

If $\sum g_k$ converges uniformly on a set S and if h is a bounded function on S, then $\sum hg_k$ converges uniformly on S.

***Proof*:**

If the series $\sum g_k$ converges uniformly on a set S, then $\sum g_k$ is uniformly Cauchy on S. If h is a bounded function on S, then $\exists$ an $M : |h| \leq M$. Let $\epsilon > 0$ be given and let $N := \frac{\epsilon}{M}$. Then

$$n \geq m > N \Longrightarrow \left|\sum_{k=m}^{n} hg_k\right| \leq \left|\sum_{k=m}^{n} Mg_k\right| = M\left|\sum_{k=m}^{n} g_k\right| \leq M \cdot \frac{\epsilon}{M} = \epsilon,$$

as desired. ♠

EXERCISE 27.2

Show that if f is continuous on $\mathbb{R}$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on each bounded subset of $\mathbb{R}$. *Hint*: Arrange for $|f(x) - p_n(x)| < \frac{1}{n}$ for $|x| \leq n$.

Claim:

If f is continuous on $\mathbb{R}$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on each bounded subset of $\mathbb{R}$.

***Proof*:**

Given $|x| \leq n$, suppose $I = [-n, n]$, a closed and bounded interval; hence $x \in I$ and $f : I \longrightarrow \mathbb{R}$ is a continuous function. Let $g : [0,1] \longrightarrow [-n, n]$ be a bijective map defined by $g(x) = -n + x(2n)$, and hence continuous, i.e., $g(0) = -n$ and $g(1) = n$. Since f is continuous, the composite function, $f \circ g : [0,1] \longrightarrow \mathbb{R}$ is continuous. Hence, for any $\epsilon > 0, \exists N > 0$ such that for any $n \geq N$, the Bernstein Polynomial $B_n(f \circ g)$ satisfies

$$|(f \circ g)(x) - B_n(f \circ g)(x)| < \epsilon \ \forall x \in [0,1]$$

Now g is a continuous injective map and so g has a continuous inverse function defined by

$$g^{-1}(x) = \frac{x+n}{2n} \ x \in [-n, n]$$

Thus, for all $x \in [-n, n]$, $|f(x) - B_N(f \circ g)(g^{-1}(x))| < \epsilon$.

Hence,

$$\left|f(x) - B_N(f \circ g)\left(\frac{x+n}{2n}\right)\right| < \epsilon \ \forall x \in [-n, n]$$

Since $B_N(f \circ g)$ is a polynomial function, $p_\epsilon(x) = B_N(f \circ g)\left(\frac{x+n}{2n}\right)$ is a polynomial function in x and $|f(x) - p_\epsilon(x)| < \epsilon \ \forall x \in I$. If we let $q_n(x) = B_N(f \circ g)\left(\frac{x+n}{2n}\right)$, then

$$q_n(x) = \quad B_N(f \circ g)\left(\frac{x+n}{2n}\right) = \sum_{k=0}^{n} f \circ g\left(\frac{k}{n}\right)\binom{n}{k}\left(\frac{x+n}{2n}\right)^k\left(1 - \left(\frac{x+n}{2n}\right)\right)^{n-k} \tag{36}$$

$$= \qquad \sum_{k=0}^{n} (f \circ g)\left(\frac{k}{n}\right)\binom{n}{k}\left(\frac{x+n}{2n}\right)^k\left(\frac{n-x}{2n}\right)^{n-k} \tag{37}$$

$$= \qquad \sum_{k=0}^{n} f\left(a + \frac{k}{n}(2n)\right)\binom{n}{k}\left(\frac{x+n}{2n}\right)^k\left(\frac{n-x}{2n}\right)^{n-k} \tag{38}$$

It then follows from $|(f \circ g)(x) - B_n(f \circ g)(x)| < \epsilon \forall x \in [0,1]$ that $q_n \longrightarrow f$ uniformly on $[-n, n]$, as desired. ♠

EXERCISE 27.6

Claim:

If $B_n f \to f$ uniformly on $[0,1]$, then f is continuous on $[0,1]$.

***Proof*:**

If $B_n(f) \to f$ uniformly on $[0,1]$, then

$$\forall \epsilon > 0 \ \exists N \ \forall x \in [0,1] \ \forall n > N : |B_n f(x) - f(x)| < \frac{\epsilon}{2}$$

Let $\epsilon > 0$ be given. Let $N := \delta$. Then $x \in [0,1]$ and $|x - x_0| < \delta \Longrightarrow$

$$|f(x) - f(x_0)| = |B_n f(x) - f(x_0) + f(x) - B_n f(x)| \leq |B_n f(x) - f(x_0)| + |B_n f(x) - f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. ♠

EXERCISE 28.2

Use the *definition* of the derivative to calculate the derivatives of the following functions at the indicated points.

(a) $f(x) = x^3$ at $x = 2$.

$$f'(2) = \lim_{x \to 2} \frac{x^3 - 8}{x - 8} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = \lim_{x \to 2} x^2 + 2x + 4 = (2)^2 + 2(2) + 4 = 12$$

(b) $g(x) = x + 2$ at $x = a$.

$$f'(a) = \lim_{x\to a} \frac{(x+2)-(a+2)}{x-a} = \lim_{x\to a} \frac{x+2-a-2}{x-a} = \lim_{x\to a} \frac{x-a}{x-a} = 1$$

(c) $f(x) = x^2 \cos x$ at $x = 0$.

$$f'(0) = \lim_{x\to 0} \frac{x^2\cos x - (0)^2\cos(0)}{x-0} = \lim_{x\to 0} \frac{x^2\cos x - 0}{x-0} = \lim_{x\to 0} x\cos x = 0\cdot 1 = 0$$

(d) $r(x) = \frac{3x+4}{2x-1}$ at $x = 1$

$$r'(1) = \lim_{x\to 1} \frac{\frac{3x+4}{2x-1} - \frac{7}{1}}{x-1} = \lim_{x\to 1} \frac{\frac{-11x+11}{2x-1}}{x-1} = \lim_{x\to 1} \frac{-11x+11}{2x-1}\cdot\frac{1}{x-1} = \lim_{x\to 1} \frac{-11}{2x-1} = -11$$

EXERCISE 28.4

Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Use Theorems 28.3 and 28.4 to show that f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and that $\cos x$ is its derivative.

Using the definition, we see that

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^2 \sin\left(\frac{1}{x}\right) - a^2 \sin\left(\frac{1}{a}\right)}{x-a} \\ &= \lim_{x\to a} x^2 \frac{\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{a}\right)}{x-a} + \sin\left(\frac{1}{a}\right)\frac{x^2-a^2}{x-a}, \end{aligned}$$

where the $\frac{\sin\left(\frac{1}{x}\right)-\sin\left(\frac{1}{a}\right)}{x-a}$ limit represents $\sin'\left(\frac{1}{x}\right)$. We are given that $a \neq 0$, which gives the function $\sin\left(\frac{1}{a}\right)$ some meaning in terms of differentiability and hence, using Theorems 28.3 and 28.4 and the given fact that $\sin' x = \cos x$,

$$\begin{aligned} f'(a) &= a^2 \sin'\left(\frac{1}{a}\right) + \sin\left(\frac{1}{a}\right)(a^2)' = a^2\cos\left(\frac{1}{a}\right)\left(\frac{-1}{a^2}\right) + 2a\sin\left(\frac{1}{a}\right) \\ &= 2a\sin\left(\frac{1}{a}\right) - \cos\left(\frac{1}{a}\right). \end{aligned}$$

(b) Use the definition to show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x} = \lim_{x\to 0} x \sin\frac{1}{x} = 0\cdot\xi,$$

where $|\xi| \leq 1$, and hence the whole expression equals zero, as desired.

(c) Show that f' is not continuous at $x = 0$.

Implement Theorem 28.4, which implies f is differentiable everywhere. Further, $f'(x) = x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ at $x \neq 0$. We know $x\sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0$ and $\cos\left(\frac{1}{x}\right)$ doesn't have a limit at $x = 0$. Hence, the sum $f'(x)$ has no limit at $x \neq 0$ and thus is discontinuous at $x = 0$.

EXERCISE 28.8

Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

(a) *Claim*:

f is continuous $x = 0$.

***Proof*:**

Let $\epsilon > 0$ be given. Let $|x - 0| < \sqrt{\epsilon}$. Then $f(x)$ is either equal to $x^2 \in [0, \epsilon)$ or 0. In either of these 2 cases, $|f(x) - f(0)| < \epsilon \Longrightarrow$ continuity at 0, as desired. ♠

(b) *Claim*:

f is discontinuous at all $x \neq 0$.

***Proof*:**

This will be done in 2 cases.

Case I: $x \neq 0$, $x \in \mathbb{Q}$

Let $\epsilon > 0$ be given. Let $\delta > 0$ be given. Further, let $\epsilon := x^2$. Due to the denseness of the rationals, $\exists$ an irrational number q in $(a - \delta, a + \delta)$. But while $|x - q| < \delta$ and $|f(x) - f(q)| = \epsilon$, δ can be made arbitrarily small (once again, due to the denseness property) $\Longrightarrow f$ is not continuous at x, as desired.♠

Case II: $x \neq 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$

Let $\epsilon > 0$ be given. Let $0 < \delta < \frac{|x|}{2}$ be given. Further, let $\epsilon := \frac{x^2}{10}$. Due to the denseness of the irrationals, $\exists$ a rational number q in $(a - \delta, a + \delta)$. By the triangle inequality, $|q| > \frac{|x|}{2} \Longrightarrow f(q) = \frac{x^2}{4} \Longrightarrow |f(x) - f(q)| = \frac{x^2}{4} > \epsilon$. Since δ can be made arbitrarily small $\Longrightarrow f$ is not continuous at x, as desired.♠

(c) *Claim*:

f is differentiable at $x = 0$.

***Proof*:**

Let $x = 0$ and $a \neq 0$. Then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \frac{f(a)}{a}$, which will will equal a if $a \in \mathbb{Q}$ and 0 otherwise. Both cases show that the limit $\to 0$ as $a \to 0$, and hence f is differentiable at $x = 0$ with derivative equal to 0, as desired. ♠

EXERCISE 28.15

Proof of Leibniz' Rule

Claim:

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$$

***Proof*:**

By Induction on n[27]. Let Leibniz' Rule hold for $n = m$. Then

$$\begin{aligned}(fg)^{(m+1)} = (f'g + fg')^{(m)} &= \sum_{k=0}^{m} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m+1-k)} \\ &= \sum_{k=0}^{m+1} \left[\binom{m}{k-1} + \binom{m}{k} \right] f^{(k)} g^{(m+1-k)} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{(m+1-k)},\end{aligned}$$

where, using Pascal's triangle[28] (from the Binomial Theorem handout), the last equality holds. Hence, Leibniz' Rule holds true, as desired. ♠

EXERCISE 29.4

Let f and g be differentiable functions on an open interval I. Suppose that a, b in I satisfy $a < b$ and $f(a) = f(b) = 0$.

Claim:

$$f'(x) + f(x)g'(x) = 0 \text{ for some } x \in (a, b).$$

***Proof*:**

Consider the function $h(x) = f(x)e^{g(x)}$. Since $f(a) = f(b) = 0 \implies h(a) = h(b) = 0$, implementing Rolle's Theorem, we know $\exists$ some $x \in (a, b) : h'(x) = 0$. Differentiating $h(x)$ yields:

$$\begin{aligned} h'(x) &= f(x)g'(x)e^{g(x)} + f'(x)e^{g(x)} \\ &= e^{g(x)} \left(f(x)g'(x) + f'(x) \right). \end{aligned}$$

Given $h'(x) = 0$ for some $x \in (a, b)$, either $e^{g(x)}$ has to equal 0 for some $x \in (a, b)$ or $f(x)g'(x) + f'(x)$ has to equal 0 for some $x \in (a, b)$. Since we know $e^{g(x)}$ cannot equal 0 for any value $g(x)$, we can safely conclude that $f(x)g'(x) + f'(x) = 0$ for some $x \in (a, b)$, as desired. ♠

[27] For $n = 1$, Leibniz' Rule turns into the product rule, i.e., $(fg)' = f'g + fg'$.

[28] $\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$.

EXERCISE 29.8

Claim:

f is strictly decreasing is $f'(x) < 0 \ \forall x \in (a, b)$.

***Proof*:**

Consider x_1, x_2 where $a < x_1 < x_2 < b$. By the Mean Value Theorem, for some $x \in (x_1, x_2)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) < 0.$$

Since $x_2 - x_1 > 0$ and $f(x_2) - f(x_1) < 0 \Longrightarrow f(x_2) < f(x_1)$, as desired. ♠

Claim:

f is increasing if $f'(x) \geq 0 \ \forall x \in (a, b)$.

***Proof*:**

Consider x_1, x_2 where $a < x_1 \leq x_2 < b$. By the Mean Value Theorem, for some $x \in (x_1, x_2)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \geq 0.$$

Since $x_2 - x_1 > 0$ and $f(x_2) - f(x_1) \geq 0 \Longrightarrow f(x_2) \geq f(x_1)$, as desired. ♠

Claim:

f is decreasing if $f'(x) \leq 0 \ \forall x \in (a, b)$.

***Proof*:**

Consider x_1, x_2 where $a < x_1 \leq x_2 < b$. By the Mean Value Theorem, for some $x \in (x_1, x_2)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \leq 0.$$

Since $x_2 - x_1 > 0$ and $f(x_2) - f(x_1) \leq 0 \Longrightarrow f(x_2) \leq f(x_1)$, as desired. ♠

EXERCISE 29.10

Let $f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$ at $x \neq 0$, and $f(0) = 0$.

Claim:

$f'(0) > 0$, but f is not increasing on any interval containing 0. Compare this result with Theorem 29.7 (i).

***Proof*:**
We know

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \left[\frac{1}{2} + x \sin \frac{1}{x} \right] = \frac{1}{2} + 0 > 0$$

Conversely, at $x \neq 0$, we get

$$f'(x) = 2x \sin \frac{1}{x} + \frac{1}{2} - \cos \frac{1}{x}.$$

The first term $\to 0$ as $x \to 0$. The last term oscillates between $+1$ and -1 infinitely many times in any neighborhood of $x = 0$. $\frac{1}{2} - 1 < 0 \implies$ in any neighborhod of $x = 0$, f' stay < 0 on some intervals. Implement Theorem 29.7. The function f is therefore decreasing on these intervals $\implies f$ is not increasing in any neighborhod of $x = 0$. If f' were continuous at $x = 0$, then it would remain positive in some neighborhood of $x = 0$. Hence, $\exists$ discontinuity of f' at $x = 0$, as desired. ♠

EXERCISE 29.15

We know that $(x^m)' = mx^{m-1}$ for $m \geq 0$, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ and $\left(x^{\frac{1}{n}}\right)' = x^{\frac{1}{n}-1}$. We can thus calculate $\left(x^{\frac{m}{n}}\right)'$ using the chain rule as follows:

$$\frac{d}{dx}\left(x^{\frac{m}{n}}\right) = \frac{d}{dx}\left(x^m\right)^{\frac{1}{n}} = \frac{y^{\frac{1}{n}-1}}{n}\bigg|_{y=x^m} \cdot mx^{m-1} = \frac{m}{n}x^{m\left(\frac{1}{n}-1\right)}x^{m-1} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

EXERCISE 29.18

Let f be be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0), s_2 = f(s_1)$,...etcetra.

Claim:

(s_n) is a convergent sequence.

***Proof*:**

Implement the Mean Value Theorem. Then, for each $n > 0$,

$$|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| = |f'(y)(s_n - s_{n-1})| \leq a|s_n - s_{n-1}|,$$

which implies

$$|s_{n+1} - s_n| \leq a^n|s_1 - s_0|, \ \forall n > 0,$$

by induction. Further,

$$|s_{m+1} - s_n| \leq \sum_{k=n}^{m} |s_{k+1} - s_k| \leq (s_1 - s_0) \sum_{k=n}^{m} a^k, \ \forall m \geq n > 0.$$

We know the geometric series $\sum a^k$ converges if $a < 1$ and further, its partial sums, $\sum_{k=0}^{n}$ form a Cauchy Sequence. Therefore, by the previous estimate, (s_n) is a Cauchy sequence, and hence converges, as desired. ♠

EXERCISE 30.2

Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{6x}{-\sin x} = -6 \cdot \lim_{x\to 0} \frac{x}{\sin x} = -6 \cdot 1 = -6$.

(b) $\lim_{x\to 0} \frac{-x}{x^3} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{\sec^4 x + 2\sec^2 x \tan^2 x}{3} = \frac{1}{3}$

(c) $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right] = \infty - \infty$, which we can rearrange to suffice for an application of L'Hospitals rule. $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right] = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{1 - \cos x}{x\cos x + \sin x} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0$.

(d) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = 1^\infty$, which when manipulated, can yield an indeterminant form sufficient for the application of L'Hospital's rule.

Let $y = \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$. Then $\ln y = \ln \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} \ln(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{-\tan x}{2x} = \frac{0}{0}$, which is an indeterminant form, so we apply L'Hospital's rule and get $\lim_{x\to 0} \frac{-\sec^2 x}{2} = \frac{-1}{2}$. Thus $\ln y = \frac{-1}{2} \implies e^{\ln y} = e^{\frac{-1}{2}} \implies y = \frac{1}{\sqrt{e}}$. Therefore, the limit as $x \to 0 = \frac{1}{\sqrt{e}}$.

EXERCISE 30.4

Let f be a function defined on some interval $(0, a)$, and define $g(y) = f\left(\frac{1}{y}\right)$ for $y \in (a^{-1}, \infty)$; here we set $a^{-1} = 0$ if $a = \infty$.

Claim:

$\lim_{x\to 0^+} f(x)$ exists if and only if $\lim_{y\to\infty} g(y)$ exists, in which case they are equal.

***Proof*:**

In two parts.

"$\lim_{x\to 0^+} f(x)$ exists $\implies \lim_{y\to\infty} g(y)$ exists".

Assume $\lim_{x \to 0^+} f(x)$ exists and is equal to L. Then for each $\epsilon > 0 \; \exists \; \delta > 0 : 0 < x < \delta \implies |f(x) - L| < \epsilon$. Now, define $y := \frac{1}{x}$. Then $0 < x < \delta \implies \frac{1}{\delta} < y < \infty$. Then for f defined on an interval (c, ∞), for each $\epsilon > 0 \; \exists \; \alpha < \infty : \alpha < \frac{1}{y} \implies \left| f\left(\frac{1}{y}\right) - L \right| < \epsilon \implies \lim_{y \to \infty} g(y)$ exists and is equal to L, as desired.

"$\lim_{y \to \infty} g(y) \implies \lim_{x \to 0^+} f(x)$ exists".
$\lim_{y \to \infty} g(y) = \lim_{y \to \infty} f\left(\frac{1}{y}\right)$, which when implemented with the same transformation, $y := \frac{1}{x} \implies x = \frac{1}{y}$, we see that $\lim_{y \to \infty} f\left(\frac{1}{y}\right) \equiv \lim_{x \to 0^+} f(x)$ and hence $\lim_{x \to 0^+} f(x)$ exists and is finite.

EXERCISE 30.7
For $x \in \mathbb{R}$, let

$$f(x) = x + \cos x \sin x \text{ and } g(x) = e^{\sin x}(x + \cos x \sin x).$$

(a) Since $|\cos x| \leq 1$ and $|\sin x| \leq 1 \implies |\cos x \sin x| \leq 1$, we can conclude

$$\lim_{x \to \infty} f(x) = x + \cos x \sin x \geq \lim_{x \to \infty} x - 1 = +\infty,$$

and hence, $\lim_{x \to \infty} f(x) = x + \cos x \sin x = +\infty$. Since we know $\frac{1}{e} \leq \lim_{x \to \infty} e^{\sin x} \leq e$, this implies

$$\lim_{x \to \infty} \frac{1}{e}(x + \cos x \sin x) \leq \lim_{x \to \infty} e^{\sin x}(x + \cos x \sin x) \leq \lim_{x \to \infty} e(x + \cos x \sin x)$$

which implies

$$\frac{1}{e} \cdot \infty \leq \lim_{x \to \infty} e^{\sin x}(x + \cos x \sin x) \leq e \cdot \infty$$

which implies $\lim_{x \to \infty} e^{\sin x}(x + \cos x \sin x) = +\infty$, by the Squeeze Theorem.
(b) Implement Theorem 28.3 and the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} f(x) &= x + \cos x \sin x \\ f'(x) &= 1 + \cos x(\cos x) + \sin x(-\sin x) \\ &= 1 + \cos^2 x - \sin^2 x \\ &= \cos^2 x + (1 - \sin^2 x) \\ &= \cos^2 x + \cos^2 x \\ &= 2(\cos x)^2 \end{aligned}$$

$$\begin{aligned} g(x) &= e^{\sin x}(x + \cos x \sin x) \\ g'(x) &= e^{\sin x}(2\cos^2 x) + (x + \cos x \sin x) \cos x e^{\sin x} \\ &= e^{\sin x}(2\cos^2 x) + e^{\sin x} \cos x \, [f(x)] \\ &= e^{\sin x} \cos x \, [2 \cos x + f(x)] \end{aligned}$$

$$
\begin{aligned}
\frac{f'(x)}{g'(x)} &= \frac{2(\cos x)^2}{e^{\sin x}\cos x\,[2\cos x + f(x)]} \\
&= \frac{2e^{-\sin x}\cos^2 x}{\cos x\,[2\cos x + f(x)]} \\
&= \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}
\end{aligned}
$$

(d) Using the fact that $|\sin x| \le 1$ and $|\cos x| \le 1$, we see that

$$
\lim_{x\to\infty} \frac{2\cos x}{e^{\sin x}(2\cos x + x + \cos x \sin x)} \le \lim_{x\to\infty} \frac{2}{2e + xe + e} \le \lim_{x\to\infty} \frac{2}{e(3+x)} \to 0 \text{ as } x \to \infty,
$$

so we can conclude that $\frac{f'(x)}{g'(x)}$ tends to 0 for large x. However,

$$
\lim_{x\to\infty} \frac{x + \cos x \sin x}{e^{\sin x}(x + \cos x \sin x)} = \lim_{x\to\infty} \frac{1}{e^{\sin x}},
$$

which will oscillate between e and $\frac{1}{e}$ as $x \to \infty$, which shows that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ doesn't exist.

EXERCISE 31.2

$$
\sinh x = \sum_{n\ge 1} \frac{x^{2n-1}}{(2n-1)!} \text{ and } \cosh x = \sum_{n\ge 0} \frac{x^{2n}}{(2n)!}.
$$

Both of these results follow from the series expansion for $e^x \equiv \sum \frac{x^n}{n!}$ and the fact that $(\sinh x)' = \cosh x$. Convergence for both series can be shown by implementing the Ratio Test.

$$
\begin{aligned}
&\limsup \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \limsup \left| \frac{x^2}{2n(2n+1)} \right| \to 0 < 1, \\
&\limsup \left| \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{x^{2n-1}} \right| = \limsup \left| \frac{x^2}{(2n+1)(2n+2)} \right| \to 0 < 1.
\end{aligned}
$$

EXERCISE 32.2

Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) To calculate the upper Darboux integral for f on the interval $[0, b]$, we need to come up with an upper bound. Consider the partition

$$
P = \{0 = t_0 < t_1 < \cdots < t_n = b\} \text{ where } t_k = b\frac{k}{n} \text{ for each } k.
$$

This implies

$$U(f,P) = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n}^{29} = \left(\frac{b}{n}\right)^2 \sum_{k=1}^{n} k = \left(\frac{b}{n}\right)^2 \frac{n^2+n}{2} = \frac{1}{2}b^2(1+2n^{-1}).$$

$\frac{1}{2}b^2(1+2n^{-1}) \to \frac{1}{2}b^2$ as $n \to \infty \implies U(f) \le \frac{1}{2}b^2$.
We now need to come up with a lower bound. Consider the partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n < b\}$$

which implies

$$U(f,P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1})^{30} \ge \sum_{k=1}^{n} \frac{(t_k + t_{k-1})}{2}(t_k - t_{k-1})^{31} = \frac{1}{2}\sum_{k=1}^{n}(t_k^2 - t_{k-1}^2)^{32} = \frac{1}{2}b^2$$

which implies $U(f) = \frac{1}{2}b^2$.
To calculate the lower Darboux integral for f on the interval $[0,b]$, for each partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n < b\},$$

$\exists$ an irrational number in $[t_{i-1}, t_i]$ for each i, which implies the minimal value of f on $[t_{i-1}, t_i]$ is $0 \implies L(f,P) = 0$ for each $P \implies L(f) = 0$.

(b) Assume $b > 0 \implies U(f) \ne L(f) \implies f$ is not integrable on $[0,b]$, since the Theorems of this chapter won't hold for a degenerate interval.

EXERCISE 32.6

Let f be a bounded function on $[a,b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.

We want to show that $\lim L_n = \lim U_n$. Assume that $\lim L_n < \lim U_n$. Then $\lim(U_n - L_n) > 0$ which is a contradiction. Observe that $L_n \le L(f) \le U(f) \le U_n$. Taking limits of both sides and implementing the Squeeze Theorem yields $\lim L_n \le L(f) \le U(f) \le \lim U_n = \lim L_n$. Thus $L(f) = U(f)$ by the Squeeze Theorem which implies f is integrable by Theorem 32.9.

[29] Each interval has length $\frac{b}{n}$; on the interval $[t_{i-1}, t_i]$, the maximum value of f is attained at $t_i = \frac{i}{n}$.
[30] Each interval has length $\frac{b}{n}$; the supremum of f on $[t_{i-1}, t_i]$ is t_i since you can pick rational numbers in the interval arbitrarily close to t_i.
[31] Since $t_k > t_{k-1}, t_k > \frac{t_k + t_{k-1}}{2}$.
[32] Note that this is a telescoping series, lol.

EXERCISE 33.3 **(a)**

A function f on $[a,b]$ is called a step-function if $\exists$ a partition $P = \{a = u_0 < u_1 < \cdots < u_m = b\}$ of $[a,b]$ such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for x in (u_{j-1}, u_j).

Claim:

f is integrable.

***Proof*:**

Consider a sub-partition of P, called P', where

$$P' = \left\{u_0 < \underline{u}_1 < u_1 < \overline{u}_1 < \cdots < \underline{u}_{n-1} < u_{n-1} < \overline{u}_{n-1} < u_n\right\},$$

where $\overline{u}_i - \underline{u}_i < \epsilon \ \forall i$.

We want to show $|U(f,P) - S| < \epsilon$ and $|S - L(f,P)| < \epsilon$. From P',

$$U(f,P') - L(f,P') = \sum (\overline{u}_i - \underline{u}_i) \cdot |c_i - c_{i-1}| = \sum |c_i - c_{i-1}| \le \epsilon \cdot \max\{c_i\},$$

which will tend to 0 as ϵ tends to 0. Hence,

$$L(f,P') \le \sum_{j=1}^{n} c_j(u_j - u_{j-1}) \le U(f,P'),$$

and since $L(f,P') = U(f,P')$ from above, $\int_a^b f = \sum_{j=1}^{n} c_j(u_j - u_{j-1})$, as desired. ♠

EXERCISE 33.4

Give an example of a function f on $[0,1]$ that is *not* integrable for which $|f|$ is integrable.

Consider the intervale $[0,1]$ and let $f(x) = 1$ for rational $x \in [0,1]$, and let $f(x) = -1$ for irrational $x \in [0,-1]$. For any partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n = 1\},$$

we have

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = 1$$

and

$$L(f,P) = \sum_{k=1}^{n} (-1) \cdot (t_k - t_{k-1}) = -1.$$

It follows that $U(f) \neq L(f)$ which implies that f is not integrable. However, if the absolute value of f is taken, we see that $f(x) = 1 \; \forall x \in \mathbb{R}$, and hence will be integrable since $U(f) = L(f)$.

EXERCISE 33.7

Let f be a bounded function on $[a, b]$, so that $\exists \; B > 0 : |f(x)| \leq B \; \forall x \in [a, b]$.

(a) *Claim*:

$$U(f^2, P) - L(f^2, P) \leq 2B\Big[U(|f|, P) - L(|f|, P)\Big] \; \forall \text{ partitions } P \text{ of } [a, b]$$

***Proof*:**

$$\sum_{k=1}^{n} \left[M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k])\right] \cdot (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \Big([M(|f|, [t_{k-1}, t_k]) + m(|f|, [t_{k-1}, t_k])] \, [M(|f|, [t_{k-1}, t_k]) - m(|f|, [t_{k-1}, t_k])] \Big) \cdot (t_k - t_{k-1})$$

$$\leq 2B \sum_{k=1}^{n} \Big[M(|f|, [t_{k-1}, t_k]) - m(|f|, [t_{k-1}, t_k])\Big] \cdot (t_k - t_{k-1}),$$

as desired[33]. ♠.

(b) *Claim*:

If f is integrable on $[a, b]$, then f^2 is integrable on $[a, b]$.

***Proof*:**

From part (a), we know

$$U(f^2, P) - L(f^2, P) \leq 2B\Big[U(|f|, P) - L(|f|, P)\Big] \; \forall \text{ partitions } P \text{ of } [a, b],$$

and by Theorem 32.5, if f is integrable on $[a, b]$, then for $\epsilon > 0$, $U(f, P) - L(f, P) < \epsilon$.

But,

$$U(f^2, P) - L(f^2, P) \leq 2B\Big[U(|f|, P) - L(|f|, P)\Big] < \epsilon \Longrightarrow \frac{U(f^2, P) - L(f^2, P)}{2B} < \epsilon,$$

which implies

$$U(f^2, P) - L(f^2, P) < 2B\epsilon.$$

So define a new ϵ, $\epsilon' := 2B\epsilon > 0$ and we thus have

$$U(f^2, P) - L(f^2, P) < \epsilon',$$

[33] The first equality in this proof is from factoring the perfect squares.

which implies that if f is integrable on $[a, b]$, then f^2 is integrable on $[a, b]$, as desired. ♠

EXERCISE 33.8

Let f and g be integrable functions on $[a, b]$.

(a) *Claim*:

fg is integrable on $[a, b]$.

***Proof*:**

Implement Theorem 33.3. Let f and g be integrable on $[a, b]$. Then, by Theorem 33.3, $f + g$ and $f - g$ are integrable on $[a, b]$. Further, if $f + g$ and $f - g$ are integrable on $[a, b]$, then $(f + g)^2$ and $(f - g)^2$ are both integrable on $[a, b]$, as per the results of exercise 33.7 **(b)**. Since $\frac{1}{4}\left[(f + g)^2 - (f - g)^2\right] = fg$, and Theorem 33.3 tells us a constant times an integrable function is integrable, this shows that if f and g are integrable on $[a, b]$, then fg is integrable on $[a, b]$, as desired. ♠

(b) *Claim*:

$\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$.

***Proof*:**

Implement Theorem(s) 33.3 and 33.5. We know

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$
$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|,$$

which are compositions of functions and constants integrable on $[a, b]$ and hence are integrable by Theorems 33.3 and 33.5, as desired. ♠.

EXERCISE 33.10

Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on $[-1, 1]$.

Let $\epsilon > 0$ be given. Since f is piece-wise continuous, by Definition 33.7, on $[\frac{\epsilon}{4}, 1]$, $\exists$ a partition P_1 of $[\frac{\epsilon}{4}, 1] : U(f_1, P) - L(f_1, P) < \frac{\epsilon}{2}$. Similarly, $\exists$ a partition P_2 of $[-1, -\frac{\epsilon}{4}] : U(f_2, P) - L(f_2, P) < \frac{\epsilon}{2}$. Define $P = P_[1 \Cup P_2$, a partition of $[-1, 1]$. Since

$$\left\{M(f, \left[-\frac{\epsilon}{4}, \frac{\epsilon}{4}\right]) - m(f, \left[-\frac{\epsilon}{4}, \frac{\epsilon}{4}\right])\right\} \cdot \left\{\frac{\epsilon}{4} - \left(-\frac{\epsilon}{4}\right)\right\} < \epsilon,$$

which, when combined with Theorem 32.5, shows f is integrable on $[-1, 1]$.

EXERCISE 33.14

Suppose f and g are continuous functions on $[a,b]$ and that $g(x) \geq 0\ \forall x \in [a,b]$.

(a) *Claim*:

$\exists\ x \in [a,b]$:

$$\int_a^b f(t)g(t)dt = f(x)\int_a^b g(t)dt.$$

***Proof*:**

Given that

$$\int_a^b f(t)g(t)dt = \sum_{i=1}^{\infty} f(t_i)g(t_i)\cdot(t_i - t_{i-1}),$$

where $t_i \in [u_{i-1}, u_i]$ and $f(t_i) = a_i$ and $g(t_i) = b_i$. Further,

$$a_i b_i \geq \min\{a_i\}\cdot b_i \Longrightarrow a_i \geq \min\{a_i\}$$
$$a_i b_i \leq \max\{a_i\}\cdot b_i \Longrightarrow a_i \leq \max\{a_i\}.$$

Then,

$$\int_a^b f(t)g(t)dt \geq f_{\min}\int_a^b g(t)dt$$
$$\int_a^b f(t)g(t)dt \leq f_{\max}\int_a^b g(t)dt.$$

Note: if $b_i = 0$ then the equality holds, trivially. Hence, by the Intermediate Value Theorem, $\exists\ x \in [a,b] : \int_a^b f(t)g(t)dt = f(x)\int_a^b g(t)dt.$, as desired. ♠.

(b) Let $g(x) := \frac{1}{b-a}$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = \int_a^b f(t)g(t)dt = f(x)\cdot\int_a^b g(t)dt = f(x)\cdot 1 = f(x).$$

EXERCISE 34.2

(a)

$$\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt$$

To calculate this integral, let us use the formulation

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)dt \text{ for } x \neq x_0$$

Hence,

$$f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x f(x_0)dt.$$

Thus, by the Fundamental Theorem of Calculus II,

$$\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt = e^{0^2} = 1.$$

(b)

$$\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2}dt$$

Using the same argument as above,

$$\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2}dt = \lim_{h\to 0}\frac{1}{h}\int_0^3 e^{(t+3)^2}dt = e^{3^2} = e^9.$$

EXERCISE 34.6

Let f be a continuous function on $\mathbb{R}$ and define

$$G(x) = \int_0^{\sin x} f(t)dt \text{ for } x \in \mathbb{R}.$$

If f is continuous at $x \in \mathbb{R}$, then $f(\sin x)$ is differentiable on $\mathbb{R}$ as a composition of differentiable functions. Hence, f is differentiable at $\sin x$ and $G' = f(\sin x)\cos x$, per the chain rule.

Let $\epsilon > 0$ and $B > 0 : |f(\sin x)| \leq B\ \forall x \in \mathbb{R}$. Now $\forall x \in \mathbb{R} : |\sin x - \sin 0| \leq |\sin x| < \epsilon$, we have $|F(\sin x) - F(0)| \leq |\int_0^{\sin x} f| \leq |\sin x| < \epsilon$, which shows G is continuous.

EXERCISE 36.1

Show that if f is integrable on [a, b] as in Definition 32.1, then

$$\lim_{d\to b^-}\int_a^d f(x)dx = \int_a^b f(x)dx.$$

It suffices to show that if $|f|$ is bounded by some number B, then

$$\left|\int_a^d f(x)dx - \int_a^b f(x)dx\right| \leq B(b-d).$$

Hence, we want to show that for $|b-d| < \delta \implies |B(b-d)| < \epsilon$. Choose $\delta := \frac{\epsilon}{B}$. Then

$$|B(b-d)| = B|(b-d)| < B\delta = \epsilon.$$

Hence as d approaches b from the left, the difference between the integrals converges to zero, showing they are equivalent.

EXERCISE 36.6

Let f and g be continuous functions on $(a, b) : 0 \leq f(x) \leq g(x)\ \forall x \in (a, b)$; a can be $-\infty$ and b can be $+\infty$.

(a) *Claim*:

$$\int_a^b g(x)dx < \infty \Longrightarrow \int_a^b f(x)dx < \infty$$

***Proof*:**

We know

$$\int_a^b f(x)dx = \int_a^0 f(x)dx + \int_0^b f(x)dx,$$

by Theorem 33.6. Likewise,

$$\int_a^b g(x)dx = \int_a^0 g(x)dx + \int_0^b g(x)dx.$$

From the fact that $0 \leq f(x) \leq g(x)\ \forall x \in (a, b)$, we can assume

$$\left(\int_a^0 f(x)dx + \int_0^b f(x)dx\right) \leq \left(\int_a^0 g(x)dx + \int_0^b g(x)dx\right).$$

Implementing Definition 36.1 and Theorem 19.6 *(Extensions from bounded intervals to unbounded intervals being uniformly continuous and hence integrable)*:

$$\left(\lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to+\infty}\int_0^b f(x)dx\right) \leq \left(\lim_{a\to-\infty}\int_a^0 g(x)dx + \lim_{b\to+\infty}\int_0^b g(x)dx\right).$$

Define a continuous function $h(x) := g(x) - f(x) \Longrightarrow h(x) \geq 0$, since $g(x) \geq f(x)$. Hence,

$$\int h(x)dx := \int \big(g(x) - f(x)\big)dx \Longrightarrow \int g(x)dx = \int \big(h(x) + f(x)\big)dx.$$

We can now see that

$$\int g(x)dx < \infty \Longrightarrow \int \big(h(x)+f(x)\big)dx < \infty \Longrightarrow \int f(x)dx < \infty, \text{ since } \int h(x)dx \geq 0,$$

as desired. ♠

(b) *Claim*:

$$\int_a^b f(x)dx = \infty \Longrightarrow \int_a^b g(x)dx = \infty$$

***Proof*:**
Using the result from part (a),

$$\int f(x)dx = \int g(x)dx - \int h(x)dx,$$

and therefore,

$$\int f(x)dx = \infty \Longrightarrow \int g(x)dx - \int h(x)dx = \infty \Longrightarrow \int g(x)dx = \infty,$$

as desired. ♠.