STOR 435 Homework 22

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1. $Y|X=80 \sim \mathcal{N}\left(\mu=75+0.8\times\frac{16}{10}(80-85), \sigma=\sqrt{1-0.8^2}\times 16\right) \sim \mathcal{N}(\mu=68.6, \sigma=9.6)$

   $P(Y>80|X=80)=1-F_{Y|X}(80|80)=1-\Phi\left(\frac{80-68.6}{9.6}\right)\approx 0.1170$ (table), 0.1175 (exact)

2.

   a) $f_{U,V}(u,v)=f_{X,Y}(uv,v)\left|\det\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}\right|=f_X(uv)f_Y(v)v=\alpha\beta v e^{-\alpha uv-\beta v}$

      $U\in[0,\infty)$, $V\in(0,\infty)$

      // Question: Can $V=0$?

      // If $V=0$, then $U=X/0$ is not really well-defined

      // If $V=0$, then $f_{U,V}(u,v)=0$.

      // From this point of view, V cannot equal 0

      // However, $V=Y$ means V have the same domain as Y, which includes 0

      // Further question: We have assumed g is invertible, but here it's probably not.

      // Consider a simpler case, $W:=1/Y$, what can we say about W?

      // However, for U, $0/0$ could be defined maybe? As a limit?

      // Need a more mathematically rigorous definition for things like X/Y

      // In fact, we haven't seen any mathematically rigorous definition for anything.

   b) $f_V(v)=f_Y(v)=\beta e^{-\beta v}$

3.

   a) $\frac{\partial(x,y)}{\partial(R,\theta)}=\det\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -R\sin(\theta) & R\cos(\theta) \end{pmatrix}=R$

      $f_{R,\theta}(R,\theta)=f_{X,Y}(x,y)R=f_X(x)f_Y(y)R=(2\pi)^{-1}e^{-\frac{1}{2}(x^2+y^2)}R=(2\pi)^{-1}e^{-\frac{1}{2}R^2}R$

   b) $f_\theta(\theta)=\int_0^\infty f_{R,\theta}(R,\theta)dR=(2\pi)^{-1}$

      θ is a uniform distribution on $[0,2\pi]$,

      which confirms the isometric symmetry of Gaussian.

4.

   a) $\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}=\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}=-1/2$

      $f_{Y_1,Y_2}(y_1,y_2)=f_{X_1,X_2}(x_1,x_2)/2=(2\pi)^{-1}e^{-\frac{1}{2}(x_1^2+x_2^2)}/2=\frac{1}{\sqrt{2\pi}\sqrt{2}}e^{-\frac{1}{2}\left(\frac{y_1}{\sqrt{2}}\right)^2}\frac{1}{\sqrt{2\pi}\sqrt{2}}e^{-\frac{1}{2}\left(\frac{y_2}{\sqrt{2}}\right)^2}$

   b) We identify this as the product of two normal pdf's, which means Y_1 and Y_2 are separable, and $Y_1\sim Y_2\sim\mathcal{N}(\mu=0,\sigma^2=2)$