

# Investigation of Blood Pressure in a Capillary Network

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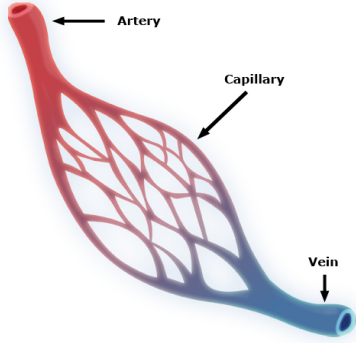
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## Introduction

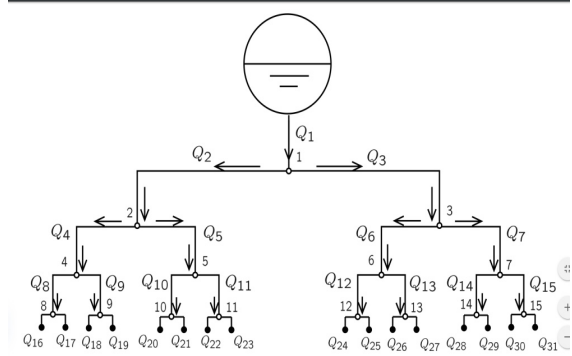
This essay investigates how the blood pressure is varying in the capillaries. We will model the capillary bed with a network and we will set up a linear system based on the relation between the blood pressure at each node and the blood flow in each vessel, and solve the system using Gauss-Seidel method.

## Statement of the Problem

Capillaries group together giving rise to networks called capillary beds (Figure 1). A capillary bed can be described by a network; in this model, every capillary is assimilated to a pipeline whose endpoints are called nodes. In the schematic illustration of Figure 2, nodes are represented by empty little circles. From a functional viewpoint, the arteriole feeding the capillary bed can be regarded as a reservoir at uniform pressure (about 50 mmHg). In this model we will assume that at the exiting nodes (those indicated by small black circles in Figure 2) the pressure features a constant value, that of the venous pressure, that we can normalize to zero. Blood flows from arterioles to the exiting nodes because of the pressure gap between one node and the following ones (those standing at a hierarchically lower level).



**Figure 1.** Illustration of a capillary bed



**Figure 2.** Schematization of a capillary bed

Still referring to Figure 2, we denote by  $p_j, j = 1, 2, \dots, 15$  (measured in mmHg) the pressure at the  $j$ -th node and by  $Q_m, m = 1, 2, \dots, 31$  (measured in  $\text{mm}^3/\text{s}$ ) the flow inside the  $m$ -th capillary vessel. For any  $m$ , denoting by  $i$  and  $j$  the end-points of the  $m$ -th capillary, we adopt the following constitutive relation:

$$Q_m = \frac{1}{L_m R_m} (p_i - p_j), \quad (1)$$

where  $R_m$  denotes the hydraulic resistance per unit length (in  $(\text{mmHg s})/\text{mm}^4$ ) and  $L_m$  the capillary length (in mm). Obviously, in considering the node number 1, we will take into account  $p_0 = 50$ ; similarly, in considering the nodes from n. 8 to n. 15, we will set null pressure at outflow nodes (from n. 16 to n. 31). Finally, at any node of the network we will impose a balance equation between inflow and outflow, so that:

$$Q_1 = Q_2 + Q_3; \quad Q_2 = Q_4 + Q_5; \quad Q_3 = Q_6 + Q_7 \dots \quad (2)$$

and so on for all the capillaries. We consider a constant hydraulic resistance  $R_m = 2$ . The length of the capillary is take such that  $L_1 = 20$ , and the length is halved at every bifurcation, such that  $L_2 = L_3 = 10$ ,  $L_4 = L_5 = L_6 = L_7 = 5$  and so on.

## Numerical Methods

We then investigate how the pressure is varying in the capillaries by setting up the linear system and solving it using an iterative method. First, we set up the linear system by substituting each  $Q_m(m = 1, 2, \dots, 15)$  in (2) with corresponding  $p_i$  and  $p_j$ , using equation (1). We get the following

$$\begin{aligned} \frac{1}{L_1 R_m}(p_0 - p_1) &= \frac{1}{L_2 R_m}(p_1 - p_2) + \frac{1}{L_3 R_m}(p_1 - p_3) \\ \frac{1}{L_2 R_m}(p_1 - p_2) &= \frac{1}{L_4 R_m}(p_2 - p_4) + \frac{1}{L_5 R_m}(p_2 - p_5) \\ &\dots \\ \frac{1}{L_{15} R_m}(p_7 - p_{15}) &= \frac{1}{L_{30} R_m}(p_{15} - p_{30}) + \frac{1}{L_{31} R_m}(p_{15} - p_{31}) \end{aligned} \quad (3)$$

Second, we substitute  $p_i, i > 15$  with 0,  $p_0$  with 50, and  $L_m$  and  $R_m$  with corresponding values. Then we move all the  $p_i$ 's to the right side and constants (in this model, just  $p_0$ ) to the left side, and we have the following linear system:

$$\begin{pmatrix} \frac{1}{8} & \frac{-1}{20} & \frac{-1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{20} & \frac{1}{4} & 0 & \frac{-1}{10} & \frac{-1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{20} & 0 & \frac{1}{4} & 0 & 0 & \frac{-1}{10} & \frac{-1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{10} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{10} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{15} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4)$$

Let us denote the system as  $\mathbf{Ax} = \mathbf{b}$ . We will use Gauss-Seidel iterative method to solve the system; the iteration routine for Gauss-Seidel method is  $\mathbf{x}^{(k+1)} = (\mathbf{I} - \mathbf{L}^{-1}\mathbf{A})\mathbf{x}^{(k)} + \mathbf{L}^{-1}\mathbf{b}$ , where  $\mathbf{L}$  is the lower triangular part of  $\mathbf{A}$ , including the diagonal. The reasons for choosing Gauss-Seidel method are listed below:

1. The condition number for the lower triangular part of  $\mathbf{A}$  is roughly 9. Gauss-Seidel method requires the inversion of the lower triangular part of  $\mathbf{A}$ . The relatively small condition number guarantees that the inversion will not lead to large error.
2. The spectral radius of  $(\mathbf{I} - \mathbf{L}^{-1}\mathbf{A})$  is roughly 0.419, which is less than 1;  $\rho(\mathbf{I} - \mathbf{L}^{-1}\mathbf{A}) < 1$  is the necessary and sufficient condition for Gauss-Seidel method to converge.
3.  $\mathbf{A}$  is not positive definite, so we cannot use descent methods such as conjugate gradient and steepest descent, which are faster than Gauss-Seidel method.

We then apply Gauss-Seidel method to system (4), with starting point  $\mathbf{x}^{(0)} = \mathbf{0}$ , the zero vector of  $\mathbb{R}^{15}$ , and we stop the iteration when we reach the maximum number of iterations of  $10^8$ , or when we reach a relative residual tolerance of  $10^{-5}$ .

## Results

The iteration stops after 15 iterations and the result  $\mathbf{x}$  is displayed below, after rounding each entry to four decimal digits:

$$\begin{aligned} \mathbf{x} = & (p_1 \ p_2 \ p_3 \ p_4 \ p_5 \ p_6 \ p_7 \ p_8 \ p_9 \ p_{10} \ p_{11} \ p_{12} \ p_{13} \ p_{14} \ p_{15})^T = \\ & (12.46334 \ 3.07918 \ 3.07918 \ 0.73314 \ 0.73314 \ 0.73314 \ 0.73314 \\ & 0.14663 \ 0.14663 \ 0.14663 \ 0.14663 \ 0.14663 \ 0.14663 \ 0.14663 \ 0.14663)^T. \end{aligned} \quad (5)$$

We can then calculate  $Q_m, m = 1, 2, \dots, 15$  using equation (1) and verify that they do satisfy the equations in (2), to further confirm the correctness of our result. But that's not of our primary interest. We plot  $\mathbf{x}$  in Figure 3, to further illustrate how the blood pressure varies in the capillaries.

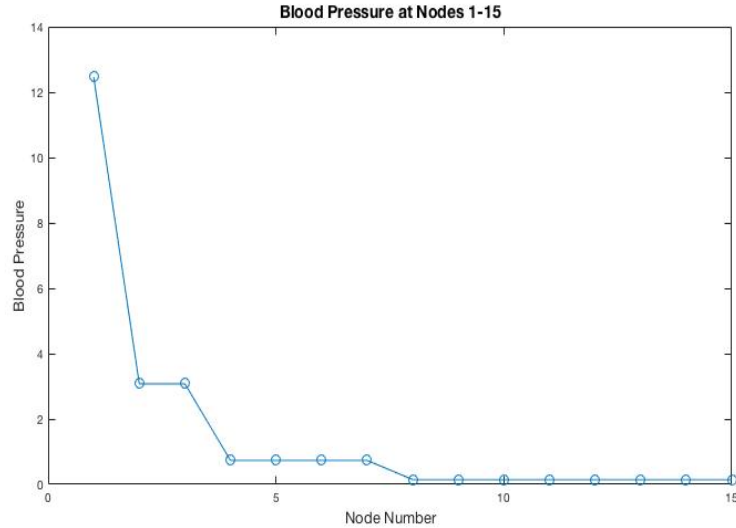


Figure 3. Blood Pressure at Nodes 1-15

## Discussion/Coclusions

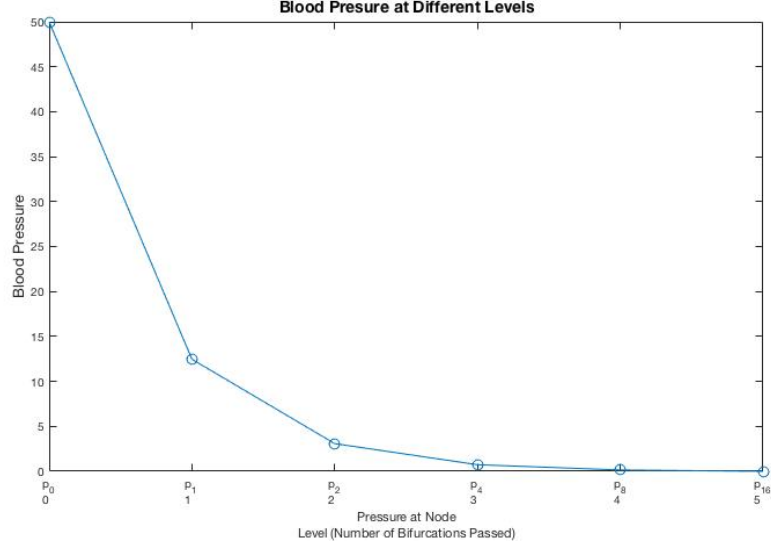
We further investigate how the blood pressure varies in the capillaries by inspecting the symmetry of the network.

First, we notice in Figure 2 that the network is a perfectly balanced binary tree. For each bifurcation, or each node  $p_1$  to  $p_{15}$ , the two parts of the network separated by the bifurcation are symmetric about the bifurcation. In particular, the distances  $L_m$  between nodes on the same level are the same, and the pressure at the leaf of the tree, i.e.  $p_{16}$  to  $p_{31}$  are the same.

Second, by inspecting the equations (3), we notice that the equations are also symmetric about each bifurcation, since  $L_m$  is equal along the same level. Therefore, we can expect that the blood pressure  $p_m$ 's at the same level of the network, or equivalently, passing the same number of bifurcations, to be equal, and the same symmetry can be expected for  $Q_m$ 's.

Our calculation confirms this symmetry. We notice from Figure 3 that the blood pressure at the same level of the schematization is the same, for example,  $p_2 = p_3$ ,  $p_4 = p_5 = p_6 = p_7$ . We plot the blood pressure in the nodes versus the bifurcations the blood flow has passed in Figure 4.

From Figure 4, we notice that the blood pressure drops significantly after passing a bifurcation. It drops roughly to  $1/4$  after each bifurcation, and finally drops to zero. We can expect this dropping from life experience, and also from equation (1), which is equivalent to  $p_j = p_i - L_m R_m Q_m$ , showing that with positive  $L_m$ ,  $R_m$ , and  $Q_m$ ,  $p_j$ , which is at lower level than  $p_i$ , is smaller than  $p_i$ .



**Figure 4.** Blood Pressure at Different Levels

To investigate how the blood pressure varies in the capillaries, we modeled the capillary bed with a network, as schematized in Figure 2, and constructed the linear model (4) based on equations (3), which are derived from equations (1) and (2). We then solved the linear system with Gauss-Seidel iterative method and get the results as shown in (5) and Figure 3. We concluded that the blood pressure decreases significantly after each bifurcation, and that the blood pressure in different nodes that pass the same number of bifurcations is equal, as demonstrated in Figure 4.