Investigation of Blood Pressure in a Capillary Network

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Introduction

This essay investigates how the blood pressure is varying in the capillaries. We will model the capillary bed with a network and we will set up a linear system based on the relation between the blood pressure at each node and the blood flow in each vessel, and solve the system using Gauss-Seidel method.

Statement of the Problem

Capillaries are tiny blood vessels, the smallest units of the blood circulatory system. They group together giving rise to networks called capillary beds (Figure 1) featuring a variable number of elements, say from 10 to 100, depending upon the kind of organ and the specific biological tissue. The oxygenated blood reaches capillary beds from the arterioles, and from capillary beds it is released to the surrounding tissue passing through the membrane of red blood cells. A capillary bed can be described by a network, similar to a hydraulic network; in this model, every capillary is assimilated to a pipeline whose endpoints are called nodes. In the schematic illustration of Figure 2, nodes are represented by empty little circles. From a functional viewpoint, the arteriole feeding the capillary bed can be regarded as a reservoir at uniform pressure (about 50 mmHg). In this model we will assume that at the exiting nodes (those indicated by small black circles in Figure 1 right) the pressure features a constant value, that of the venous pressure, that we can normalize to zero. Blood flows from arterioles to the exiting nodes because of the pressure gap between one node and the following ones (those standing at a hierarchically lower level). Still referring to Figure 2, we denote by $p_j, j=1,2,...,15$ (measured in mmHg) the pressure at the j-th node and by $Q_m, m=1,2,...,31$ (measured in mm^3/s) the flow inside the m–th capillary vessel. For any m, denoting by i and j the end-points of the m–th capillary, we adopt the following constitutive relation:

$$Q_m = \frac{1}{L_m R_m}(p_i - p_j), \tag{1}$$

where R_m denotes the hydraulic resistance per unit length (in (mmHg s)/mm^4) and L_m the capillary length (in mm). Obviously, in considering the node number 1, we will take into account $p_0 = 50$; similarly, in considering the nodes from n. 8 to n. 15, we will set null pressure at outflow nodes (from n. 16 to n. 31). Finally, at any node of the network we will impose a balance equation between inflow and outflow, so that:

$$\begin{aligned} Q_1 &= Q_2 + Q_3 \\ Q_2 &= Q_4 + Q_5 \\ Q_3 &= Q_6 + Q_7 \\ &\dots \end{aligned} \tag{2}$$

and so on for all the capillaries. We consider a constant hydraulic resistance $R_m = 2$. The length of the capillary is take such that $L_1 = 20$, and the length is halved at every bifurcation, such that $L_2 = L_3 = 10$, $L_4 = L_5 = L_6 = L_7 = 5$ and so on.

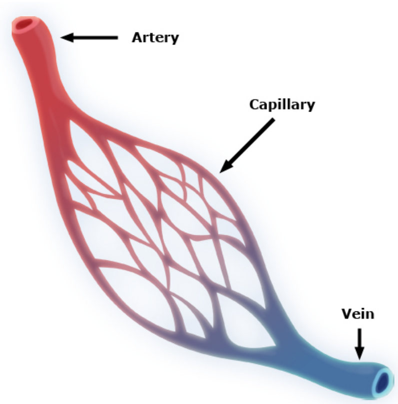

Figure 1. A simplified illustration of a capillary bed

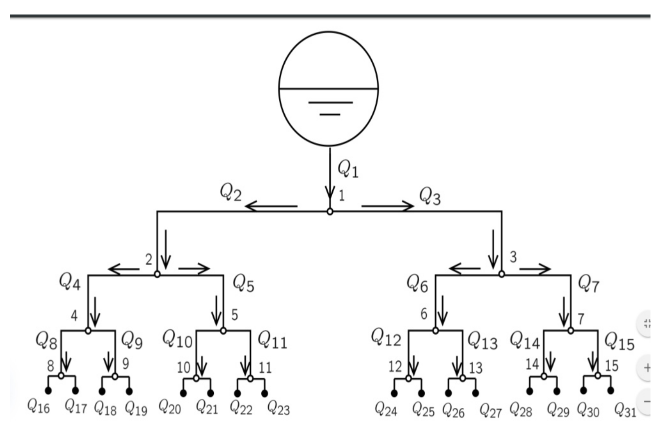

Figure 2. Schematization of a capillary bed

Numerical Methods

We then investigate how the pressure is varying in the capillaries by setting up the linear system and solving it using an iterative method.

First, we set up the linear system by substituting each $Q_m(m=1,2,...15)$ in (2) with corresponding p_i and p_j, using equation (1). We get the following

$$\begin{gathered}
\frac{1}{L_1R_m}(p_0-p_1)=\frac{1}{L_2R_m}(p_1-p_2)+\frac{1}{L_3R_m}(p_1-p_3)\\
\frac{1}{L_2R_m}(p_1-p_2)=\frac{1}{L_4R_m}(p_2-p_4)+\frac{1}{L_5R_m}(p_2-p_5)\\
...\\
\frac{1}{L_{15}R_m}(p_7-p_{15})=\frac{1}{L_{30}R_m}(p_{15}-p_{30})+\frac{1}{L_{31}R_m}(p_{15}-p_{31})
\end{gathered} \tag{3}$$

Second, we substitute $p_i, i>15$ with 0, p_0 with 50, and L_m and R_m with correponding values. Then we move all the p_i's to the right side and constants (in this model, just p_0) to the left side, and we have the following linear system:

$$\begin{pmatrix}
\frac{1}{8} & \frac{-1}{20} & \frac{-1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-1}{20} & \frac{1}{4} & 0 & \frac{-1}{10} & \frac{-1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-1}{20} & 0 & \frac{1}{4} & 0 & 0 & \frac{-1}{10} & \frac{-1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-1}{10} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 & 0\\
0 & 0 & \frac{-1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0\\
0 & 0 & \frac{-1}{10} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5}\\
0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{-1}{5} & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\\ p_9\\ p_{10}\\ p_{11}\\ p_{12}\\ p_{13}\\ p_{14}\\ p_{15} \end{pmatrix}
=
\begin{pmatrix} \frac{5}{4}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}$$

$$(4)$$

Let us denote the system as $\boldsymbol{Ax}=\boldsymbol{b}$. We well use Gauss-Seidel iterative method to solve the system. The reasons for choosing Gauss-Seidel method are listed below:

1. The matrix $\boldsymbol{A}$ is strictly diagonally dominated, which guarantees the convergence of Gauss-Seidel iterations.

2. The matrix $\boldsymbol{A}$ is not ill-conditioned; condition number for the lower triangular part of $\boldsymbol{A}$ is roughly 9. Gauss-Seidel method requires the inversion of the lower triangular part of $\boldsymbol{A}$. The relatively small condition number guarantees that the inversion will not lead to large error.

3. $\boldsymbol{A}$ is not positive definite, so we cannot use descent methods such as conjugate gradient and steepest descent, which are faster than Gauss-Seidel method.

4. The volume of data is considerably small, as the system is only 15-dimensional. Therefore, we do not need to worry much about the speed, or convergence rate, of the method, and thus Gauss-Seidel method, though definitely not among the faster methods, satisfies our need.

We then apply Gauss-Seidel method to system (4), with starting point $\boldsymbol{x}_0 = \boldsymbol{0}$, the zero vector of $\mathbb{R}^{15}$, and we stop the iteration when we reach the maximum number of iterations of 10^8, or when we reach a relative residual tolerance of 10^{-5}.

Results

The iteration stops after 15 iterations and the result $\boldsymbol{x}$ is displayed below, after rounding each entry to five decimal digits:

$$\boldsymbol{x} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{15} \end{pmatrix} = \begin{pmatrix} 12.46334 \\ 3.07918 \\ 3.07918 \\ 0.73314 \\ 0.73314 \\ 0.73314 \\ 0.73314 \\ 0.14663 \\ 0.14663 \\ 0.14663 \\ 0.14663 \\ 0.14663 \\ 0.14663 \\ 0.14663 \\ 0.14663 \end{pmatrix}. \tag{5}$$

We can then calculate Q_m, $m = 1, 2, \ldots, 15$ using equation (1) and verify that they do satisfy the equations in (2), to further confirm the correctness of our result. But that's not of our primary interest. We plot $\boldsymbol{x}$ in Figure 3 and show the calculated $p_m, m = 1, 2, \ldots, 15$ in the schematization of the capillary bed in Figure 4, to furthur illustrate how the blood presure varies in the capillaries.

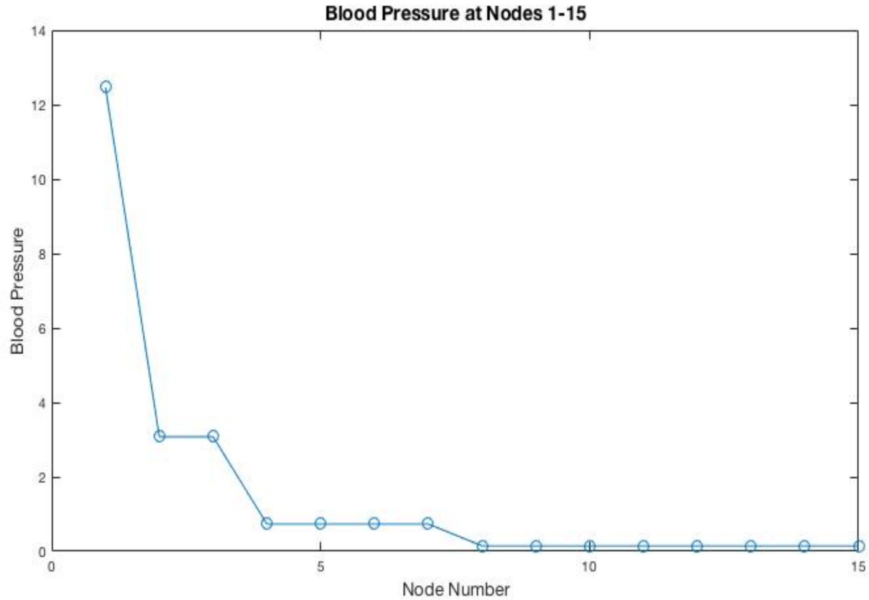

Figure 3. Blood Pressure at Nodes 1-15

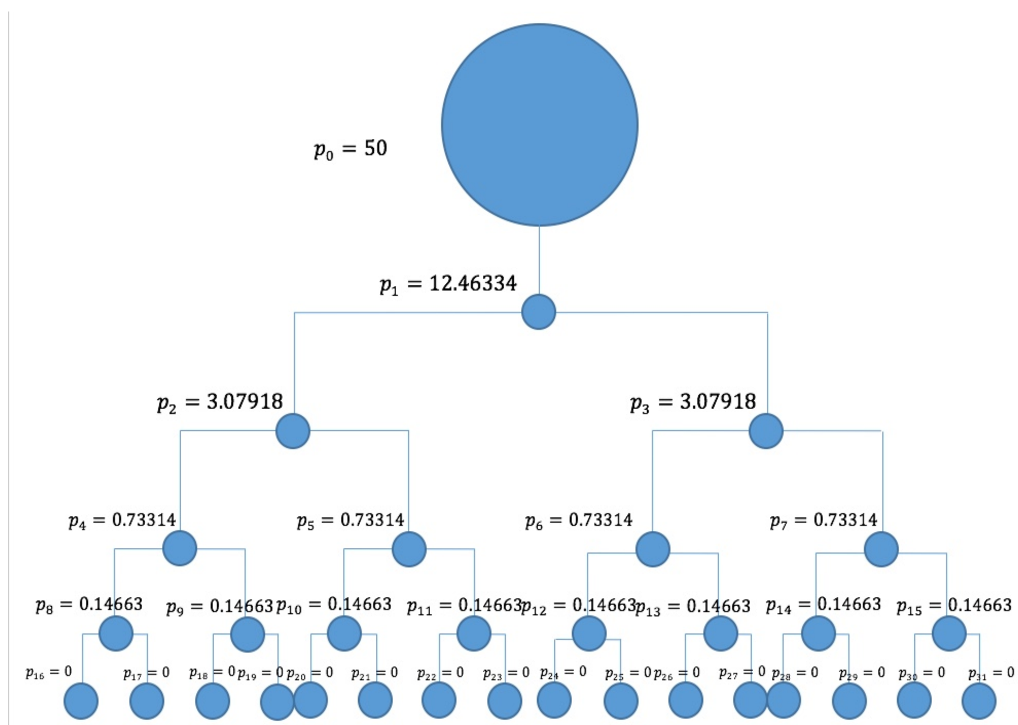

Figure 4. Schematization of the Capillary Bed with Blood Pressure at Nodes

Discussion/Conclusions

We further investigate how the blood pressure varies in the capillaries by inspecting the symmetry of the network. First, we notice in Figure 2 that the network is a perfectly balanced binary tree, i.e. for each bifurcation, or each node p_1 to p_{15}, the two parts of the network separated by the bifurcation are symmetric about the bifurcation. Second, by inspecting the equations (1) and (2), we notice that the equations are also symmetric about each bifurcation, since L_m is equal along the same level. Therefore, we can expect that the blood pressure p_m's at the same level of the network, or equivalently, passing the same number of bifurcations, to be equal, and the same symmetry can be expected for Q_m's.

Our calculation confirms this symmetry. We notice that the blood pressure at the same level of the schematization in Figure 4 is the same, for example, $p_2 = p_3$, $p_4 = p_5 = p_6 = p_7$. We plot the blood pressure in the nodes versus the bifurcations the blood flow has passed in Figure 5.

From Figure 5, we notice that the blood pressure drops significantly after passing a bifurcation. It drops roughly to 1/4 after each bifurcation, and finally drops to zero.

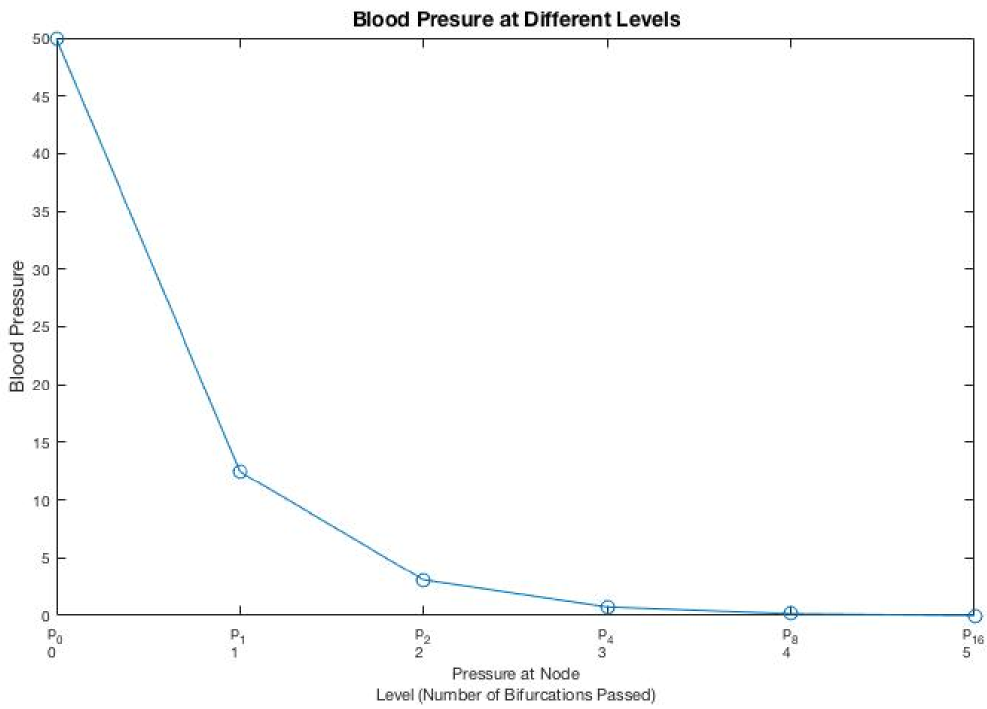

Figure 5. Blood Pressure at Different Levels

To investigate how the blood pressure varies in the capillaries, we modeled the capillary bed with a network, as schematized in Figure 2, and constructed the linear model (4) based on equations (3), which are derived from equations (1) and (2). We then solved the linear system with Gauss-Seidel iterative method and get the results as shown in (5) and Figure 3 and Figure 4. We concluded that the blood pressure decreases significantly after each bifurcation, and that the blood pressure in different nodes that pass the same number of bifurcations is equal, as demonstrated in Figure 5.