There are some issues that you should be able to easily address:

1) The matrix A is not STRICTLY diagonally dominant. Check for example row 4. Is Gauss-Seidel still convergent? Why?

I made a miscalculation to wrongly conclude that A is strictly dominant. It's still convergent. As I wrote in the edited report, I checked the spectral radius of $I - L^{-1}A$ is less than 1 (L is the lower triangular part of A), which is the necessary and sufficient condition for convergence of Gauss-Seidel method.

2) In order to say if A is not ill-conditioned you need to check the condition number of A. If you only take the lower triangular part, then you say the the lower triangular part is not ill-conditioned.

I wrongly assumed that if A is not ill-conditioned and symmetric, then the lower triangular part of A is also not ill-conditioned. I corrected this by directly stating the condition number of L is relatively small.

3) How can you say the matrix is not positive definite? Did you check?

I made huge mistake on this. I made some typos in matlab code. It's actually positive definite, after checking with matlab chol function.

4) Are you sure you could not use the conjugate gradient method? We did not cover this, but since you mention it, how can you support your claim.

Again, I made a huge mistake on this. If A were not positive definite, then conjugate gradient and steepest descent could not be used. Conjugate gradient method and steepest descent are indeed usable. With a 1e-4 residual tolerance, conjugate gradient method converges in 4 iterations, and steepest descent converges in 56 iterations.

5)You claim: "Therefore, we do not need to worry much about the speed, or convergence rate, of the method, and thus Gauss-Seidel method, though definitely not among the faster methods, satisfies our need". What is the order of convergence of Gauss-Seidel? Can other methods be used to solve this problem? Are they faster? In which sense?

Jacobi method is not good, since the spectral radius of the iteration matrix is greater than 1. JOR is usable with proper choice of ω, as stated by theorem 4 of chapter 4.2 of Quateroni. SOR is convergent. And conjugate gradient and steepest descent methods are also convergent.

Actually, conjugate gradient method is probably a better choice to solve the problem. Since it's guaranteed to converge in 15 (dimension of the system) iterations, which is very satisfying. The convergence of Gauss-Seidel method, as demonstrated in many exercises, is highly dependent on the initial guess and the matrix A. Typically, conjugate gradient requires less iterations than Gauss-Seidel, JOR, SOR, and steepest descent.

Nonetheless, in this question Gauss-Seidel is certainly also usable. As stated in the report, we reach 1e-5 residual tolerance after 15 iterations, which is a considerably small and satisfying number. And to keep consistency with my original report, I still use Gauss-Seidel in the final submission.

6) Format your plot such that the labels, the title, and the numbers are clearly readable.

I reformatted some of the plots.

7) You show that at every level the pressure decreases. The different levels are separated by a distance Lm. Can you take this into account?

I'm not sure what you mean by this. I stated that $q_j = q_i - L_m R_m Q_m$, demonstrating that q_j is less than q_i. And in fact, we already take Lm into account when constructing the linear system.

8) Your arguments on the symmetry of the systems are pertinent but not enough to guarantee the same pressure for all nodes at every level. What happens if p31 = 1?

If p31=1, then the binary tree is no longer balanced, and the system is not symmetric. Nonetheless, I elaborated on what I mean by the symmetry of the system.

Investigation of Blood Pressure in a Capillary Network

By Siyang Jing

PID 730042022

Introduction

This essay investigates how the blood pressure is varying in the capillaries. We will model the capillary bed with a network and we will set up a linear system based on the relation between the blood pressure at each node and the blood flow in each vessel, and solve the system using Gauss-Seidel method.

Statement of the Problem

Capillaries group together giving rise to networks called capillary beds (Figure 1). A capillary bed can be described by a network; in this model, every capillary is assimilated to a pipeline whose endpoints are called nodes. In the schematic illustration of Figure 2, nodes are represented by empty little circles. From a functional viewpoint, the arteriole feeding the capillary bed can be regarded as a reservoir at uniform pressure (about 50 mmHg). In this model we will assume that at the exiting nodes (those indicated by small black circles in Figure 2) the pressure features a constant value, that of the venous pressure, that we can normalize to zero. Blood flows from arterioles to the exiting nodes because of the pressure gap between one node and the following ones (those standing at a hierarchically lower level).

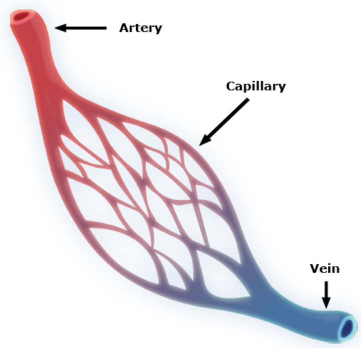

Figure 1. Illustration of a capillary bed

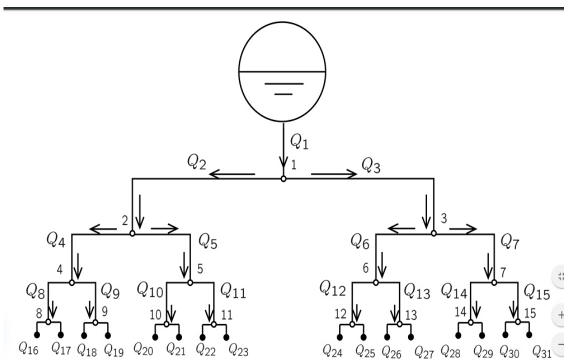

Figure 2. Schematization of a capillary bed

Still referring to Figure 2, we denote by $p_j, j=1,2,\ldots,15$ (measured in mmHg) the pressure at the j-th node and by $Q_m, m=1,2,\ldots,31$ (measured in mm^3/s) the flow inside the m–th capillary vessel. For any m, denoting by i and j the end-points of the m–th capillary, we adopt the following constitutive relation:

$$Q_m = \frac{1}{L_m R_m}(p_i - p_j), \tag{1}$$

where R_m denotes the hydraulic resistance per unit length (in (mmHg s)/mm^4) and L_m the capillary length (in mm). Obviously, in considering the node number 1, we will take into account $p_0 = 50$; similarly, in considering the nodes from n. 8 to n. 15, we will set null pressure at outflow nodes (from n. 16 to n. 31). Finally, at any node of the network we will impose a balance equation between inflow and outflow, so that:

$$Q_1 = Q_2 + Q_3; \qquad Q_2 = Q_4 + Q_5; \qquad Q_3 = Q_6 + Q_7\ldots \tag{2}$$

and so on for all the capillaries. We consider a constant hydraulic resistance $R_m = 2$. The length of the capillary is take such that $L_1 = 20$, and the length is halved at every bifurcation, such that $L_2 = L_3 = 10$, $L_4 = L_5 = L_6 = L_7 = 5$ and so on.

Numerical Methods

We then investigate how the pressure is varying in the capillaries by setting up the linear system and solving it using an iterative method. First, we set up the linear system by substituting each $Q_m (m = 1,\, 2,\, ...15)$ in (2) with corresponding p_i and p_j, using equation (1). We get the following

$$\begin{gathered}
\frac{1}{L_1 R_m}(p_0 - p_1) = \frac{1}{L_2 R_m}(p_1 - p_2) + \frac{1}{L_3 R_m}(p_1 - p_3) \\
\frac{1}{L_2 R_m}(p_1 - p_2) = \frac{1}{L_4 R_m}(p_2 - p_4) + \frac{1}{L_5 R_m}(p_2 - p_5) \\
\cdots \\
\frac{1}{L_{15} R_m}(p_7 - p_{15}) = \frac{1}{L_{30} R_m}(p_{15} - p_{30}) + \frac{1}{L_{31} R_m}(p_{15} - p_{31})
\end{gathered} \tag{3}$$

Second, we substitute p_i, $i > 15$ with 0, p_0 with 50, and L_m and R_m with correponding values. Then we move all the p_i's to the right side and constants (in this model, just p_0) to the left side, and we have the following linear system:

$$\begin{pmatrix}
\frac{1}{8} & \frac{-1}{20} & \frac{-1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{20} & \frac{1}{4} & 0 & \frac{-1}{10} & \frac{-1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{20} & 0 & \frac{1}{4} & 0 & 0 & \frac{-1}{10} & \frac{-1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{10} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} & 0 & 0 \\
0 & 0 & \frac{-1}{10} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-1}{5} \\
0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{5} & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{15} \end{pmatrix}
=
\begin{pmatrix} \frac{5}{4} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

Let us denote the system as $\boldsymbol{Ax} = \boldsymbol{b}$. We will use Gauss-Seidel iterative method to solve the system; the iteration routine for Gauss-Seidel method is $\boldsymbol{x}^{(k+1)} = (\boldsymbol{I} - \boldsymbol{L}^{-1}\boldsymbol{A})\boldsymbol{x}^{(k)} + \boldsymbol{L}^{-1}\boldsymbol{b}$, where $\boldsymbol{L}$ is the lower triangular part of $\boldsymbol{A}$, including the diagonal. The reasons for choosing Gauss-Seidel method are listed below:

1. The condition number for the lower triangular part of $\boldsymbol{A}$ is roughly 9. Gauss-Seidel method requires the inversion of the lower triangular part of $\boldsymbol{A}$. The relatively small condition number guarantees that the inversion will not lead to large error.

2. The spectral radius of $(\boldsymbol{I} - \boldsymbol{L}^{-1}\boldsymbol{A})$ is roughly 0.419, which is less than 1; $\rho(\boldsymbol{I} - \boldsymbol{L}^{-1}\boldsymbol{A}) < 1$ is the necessary and sufficient condition for Gauss-Seidel method to converge.

We then apply Gauss-Seidel method to system (4), with starting point $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, the zero vector of $\mathbb{R}^{15}$, and we stop the iteration when we reach the maximum number of iterations of 10^8, or when we reach a relative residual tolerance of 10^{-5}.

Results

The iteration stops after 15 iterations and the result $\boldsymbol{x}$ is displayed below, after rounding each entry to four decimal digits:

$$\boldsymbol{x} = (\ p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5\ \ p_6\ \ p_7\ \ p_8\ \ p_9\ \ p_{10}\ \ p_{11}\ \ p_{12}\ \ p_{13}\ \ p_{14}\ \ p_{15}\ )^T =$$
$$(\ 12.46334\ \ 3.07918\ \ 3.07918\ \ 0.73314\ \ 0.73314\ \ 0.73314\ \ 0.73314$$
$$0.14663\ \ 0.14663\ \ 0.14663\ \ 0.14663\ \ 0.14663\ \ 0.14663\ \ 0.14663\ \ 0.14663\ )^T. \tag{5}$$

We can then calculate $Q_m, m = 1, 2, ..., 15$ using equation (1) and verify that they do satisfy the equations in (2), to further confirm the correctness of our result. But that's not of our primary interest. We plot $\boldsymbol{x}$ in Figure 3, to furthur illustrate how the blood presure varies in the capillaries.

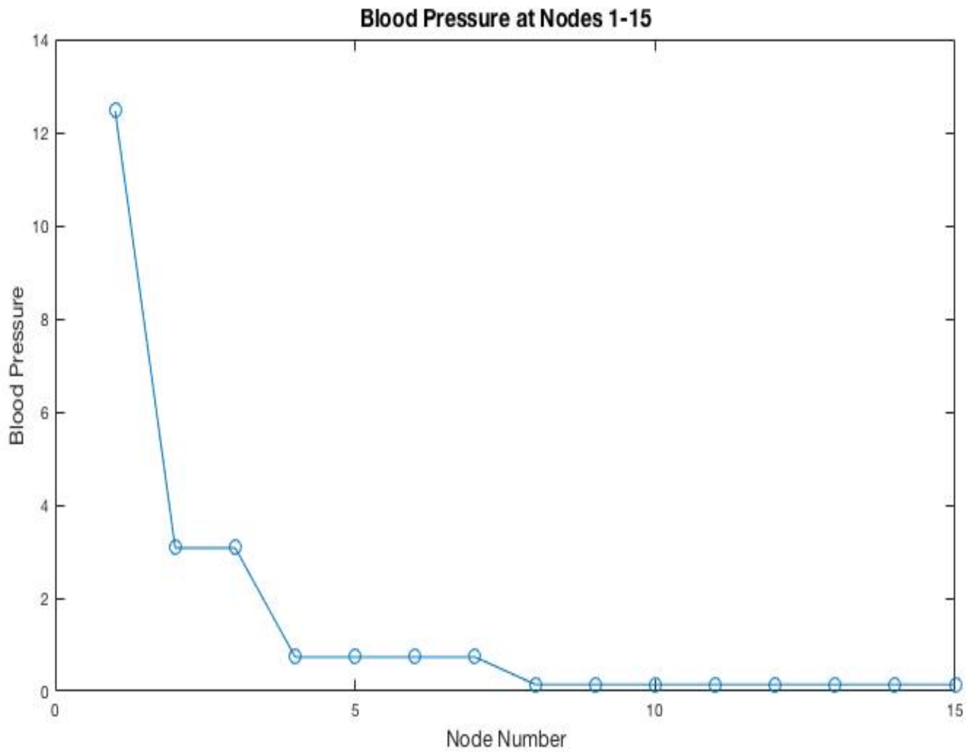

Figure 3. Blood Pressure at Nodes 1-15

Discussion/Coclusions

We further investigate how the blood pressure varies in the capillaries by inspecting the symmetry of the network.

First, we notice in Figure 2 that the network is a perfectly balanced binary tree. For each bifurcation, or each node p_1 to p_{15}, the two parts of the network separated by the bifurcation are symmetric about the bifurcation. In particular, the distances L_m between nodes on the same level are the same, and the pressure at the leaf of the tree, i.e. p_{16} to p_{31} are the same.

Second, by inspecting the equations (3), we notice that the equations are also symmetric about each bifurcation, since L_m is equal along the same level. Therefore, we can expect that the blood pressure p_m's at the same level of the network, or equivalently, passing the same number of bifurcations, to be equal, and the same symmetry can be expected for Q_m's.

Our calculation confirms this symmetry. We notice from Figure 3 that the blood pressure at the same level of the schematization is the same, for example, $p_2 = p_3$, $p_4 = p_5 = p_6 = p_7$. We plot the blood pressure in the nodes versus the bifurcations the blood flow has passed in Figure 4.

From Figure 4, we notice that the blood pressure drops significantly after passing a bifurcation. It drops roughly to $1/4$ after each bifurcation, and finally drops to zero. We can expect this dropping from life experience, and also from equation (1), which is equivalent to $p_j = p_i - L_m R_m Q_m$, showing that with positive L_m, R_m, and Q_m, p_j, which is at lower level than p_i, is smaller than p_i.

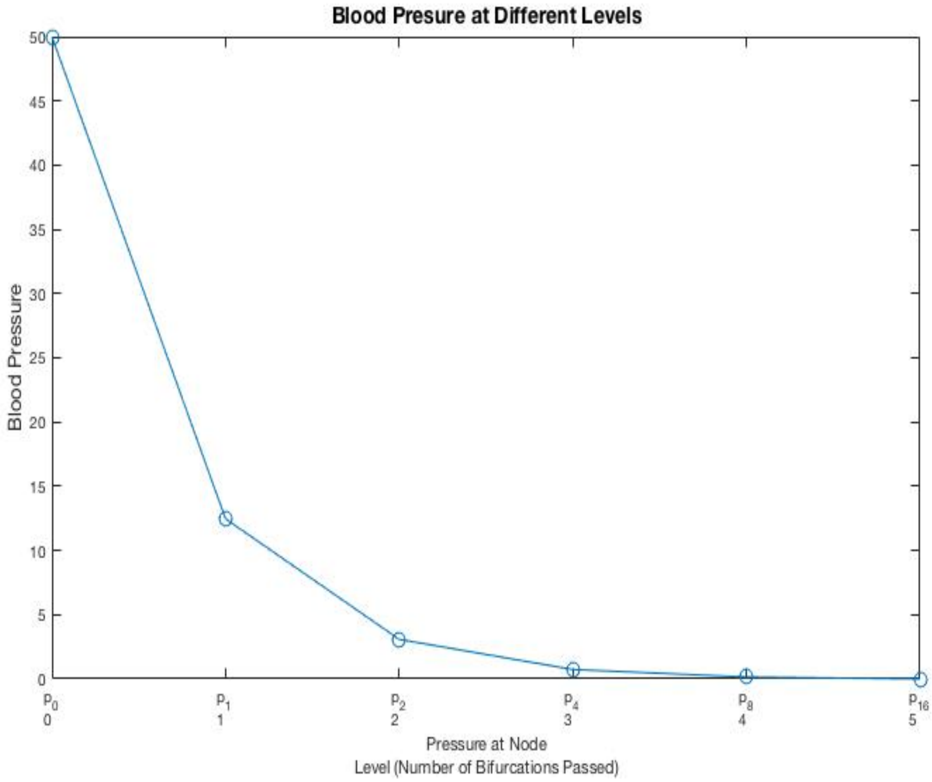

Figure 4. Blood Pressure at Different Levels

To investigate how the blood pressure varies in the capillaries, we modeled the capillary bed with a network, as schematized in Figure 2, and constructed the linear model (4) based on equations (3), which are derived from equations (1) and (2). We then solved the linear system with Gauss-Seidel iterative method and get the results as shown in (5) and Figure 3. We concluded that the blood pressure decreases significantly after each bifurcation, and that the blood pressure in different nodes that pass the same number of bifurcations is equal, as demonstrated in Figure 4.