Investigation of Ventricular Pressure and Volume

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Introduction

In this essay, we will investigate how the ventricular pressure and the volume of the left ventricle vary during a nomal heartbeat. We will use piecewise linear functions to interpolate the pressure and volume data and solve some questions accordingly.

Statement of the Problem

During a normal heartbeat the left ventricle undergoes 4 phases: I) isovolumetric contraction; II) ejection; III) isovolumetric relaxation; IV) filling. During phase I), the ventricular pressure increases. When the ventricular pressure exceeds the aortic valve pressure, we enter in phase II) where the ventricle reducing its volume. As blood is ejected, the ventricular pressure starts to decrease until the aortic valve closes again. At this point, we are in phase III) in which the ventricle is keeping the blood volume constant,and the pressure decreases. When the ventricular pressure becomes smaller the the atrial pressure, we enter in phase IV) where the ventricle again increases its volume. As the blood fills the ventricle again pressure increases and we return back to phase I). The following data for pressure and volume was recorded during the four phases together with the time (starting from a initial time $t_0 = 0$)

Phase I						Phase II					
t_I[ms]	0.	11	22	34	-	t_{II}[ms]	68	119	170	221	255
p_I[mmHg]	8.1	32.2	56.2	79.9	-	p_{II}[mmHg]	95.3	115.0	125.1	126.3	112.9
v_I[ml]	120.0	119.8	119.6	119.4	-	v_{II}[ml]	115.1	105.3	85.4	70.7	55.8
Phase III						Phase IV					
t_{III}[ms]	272	280	289	297	-	t_{IV}[ms]	331	425	510	578	646
p_{III}[mmHg]	90.6	70.0	50.1	30.6	-	p_{IV}[mmHg]	13.5	8.4	6.2	5.1	7.5
v_{III}[ml]	50.7	49.7	49.5	49.2	-	v_{IV}[ml]	50	64.3	79.7	94.1	109.0

Table 1: Pressure and Volume at Different Times during the Four Phases

We furthur investigates the following questions:

1. What are the values of the pressure when the ventricular volume is 90ml?
2. What are the values of the volume when the pressure is 110 mmHg?
3. We also want to assess the ventricular function using the stroke work, which refers to the work done by the ventricle to eject a volume of blood. Stroke work is best depicted by the use of a pressure-volume diagrams, in which stroke work is the area within the pressure-volume loop. What is the stroke work for this set of data?

Numerical Methods

We will use interpolation to solve the problems. To begin with, we notice that the values of ventricular volume keep decreasing at Phase I, II and III, and keep increasing at Phase IV. Therefore, we can separate the data points into two parts, Phase I, II and III, and Phase III, and regard p, the pressure as a function of v, the volume, for the two parts, respectively. Also, in order to form a complete cycle, we add the last point of Phase III, and the first point of Phase I, to part two. We reorganize the data from Table 1 accordingly and show them below in Table 2 and 3, and Figure 1.

p_1[mmHg]	30.6	50.1	70.0	90.6	112.9	126.3	125.1	115.0	95.3	79.9	56.2	32.2	8.1
v_1[ml]	49.2	49.5	49.7	50.7	55.8	70.7	85.4	105.3	115.1	119.4	119.6	119.8	120.0

Table 2: Pressure vs. Volume for Part I

p_2[mmHg]	30.6	13.5	8.4	6.2	5.1	7.5	8.1
v_2[ml]	49.2	50	64.3	79.7	94.1	109.0	120.0

Table 3: Pressure vs. Volume for Part II

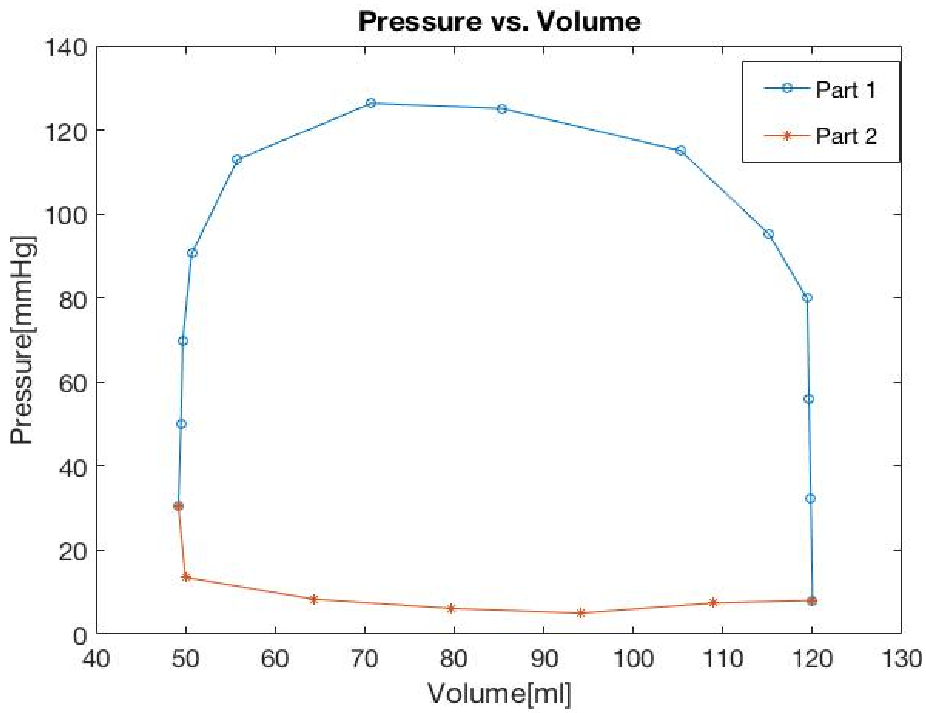

Figure 1. Pressure vs. Volume for Part 1 and 2

We then interpolate the two sets of data points respectively. Results for the several selected interpolating methods, i.e. polynomial interpolation, piecewise quadratic interpolation, and piecewise linear interpolation, are shown below in Figure 2.

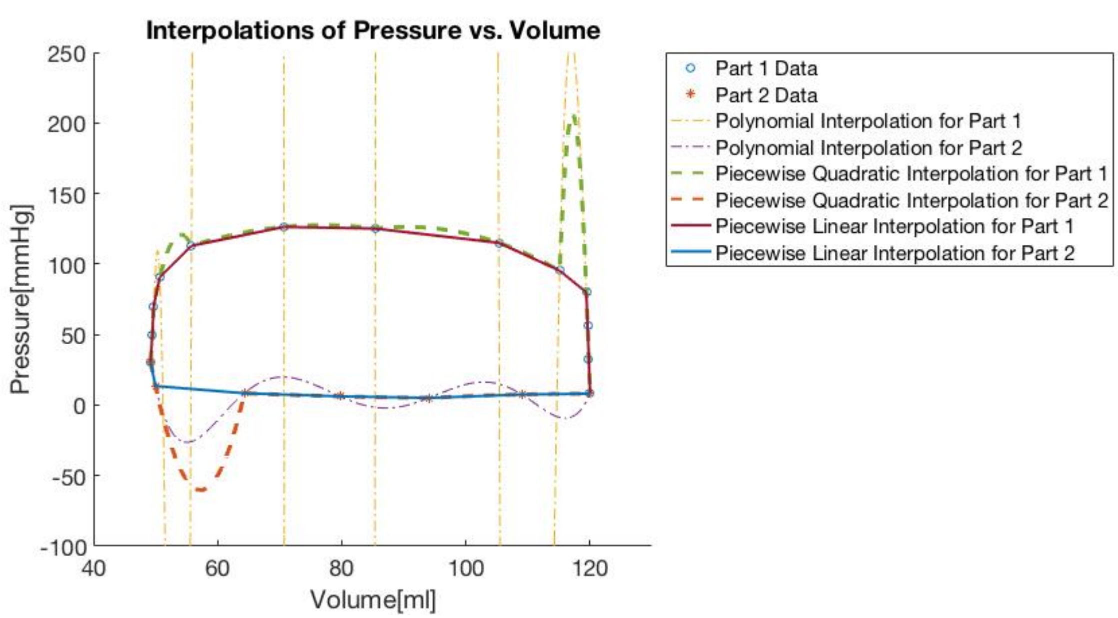

Figure 2. Interpolations of Pressure vs. Volume

From Figure 2, we notice that the interpolating functions for polynomial interpolation and piecewise quadratic interpolation have some unsatisfying behaviors. The interpolating functions of polynomial interpolation for Part 1 and 2 are considerably oscillating, especially the function for Part 1. The interpolating function of piecewise quadratic interpolation for Part 1 changes drastically between $v = 115.1$ and $v = 119.4$, and the interpolating function of piecewise quadratic interpolation for Part 2 also changes unreasonably between $v = 50$ and $v = 64.3$.

Therefore, we will choose piecewise linear interpolation. Though the interpolating functions of piecewise linear interpolation, as seen from Figure 2, are probably not smooth enough at the nodes, they at least do not have erroneous behaviors such as oscillation and drastic change, like other interpolation methods do.

Results

The interpolating functions, $f_1(v)$ and $f_2(v)$, of piecewise linear interpolation for Part 1 and 2, consisting of linear polynomials, are shown below, respectively, after rounding the coefficients to 4 significant figures.

$$f_1(x)\begin{cases} f_{1,1}(x) = 65.00x - 3167, x \in [49.2, 49.5] \\ f_{1,2}(x) = 99.50x - 4875, x \in [49.5, 49.7] \\ f_{1,3}(x) = 20.60\,x - 949.8, x \in [49.7, 50.7] \\ f_{1,4}(x) = 4.373\,x - 131.1, x \in [50.7, 55.8] \\ f_{1,5}(x) = 0.8993x + 62.72, x \in [55.8, 70.7] \\ f_{1,6}(x) = 132.1 - 0.08163x, x \in [70.7, 85.4] \\ f_{1,7}(x) = 168.4 - 0.5075x, x \in [85.4, 105.3] \\ f_{1,8}(x) = 326.7 - 2.010\,x, x \in [105.3, 115.1] \\ f_{1,9}(x) = 507.5 - 3.581x, x \in [115.1, 119.4] \\ f_{1,10}(x) = 14230 - 118.5x, x \in [119.4, 119.6] \\ f_{1,11}(x) = 14410 - 120.0x, x \in [119.6, 119.8] \\ f_{1,12}(x) = 14470 - 120.5\,x, x \in [119.8, 120.0] \end{cases} \quad f_2(x)\begin{cases} f_{2,1}(x) = -21.38x + 1082, x \in [49.2, 50] \\ f_{2,2}(x) = -0.3566\,x + 31.33, x \in [50, 64.3] \\ f_{2,3}(x) = 0.1429x + 17.59, x \in [64.3, 79.7] \\ f_{2,4}(x) = -0.07639x + 12.29, x \in [79.7, 94.1] \\ f_{2,5}(x) = 0.1611\,x - 10.06, x \in [94.1, 109.0] \\ f_{2,6}(x) = 0.05455x + 1.554, x \in [109.0, 120.0] \end{cases} \tag{1}$$

Discussion/Conclusions

We then investigate the error of such interpolation. Let F_1, F_2 denote the real functions describing the relation between pressure and volume for Part 1 and Part 2, respectively. If we assume $F_1, F_2 \in C^2$, then for any volume x between 49.2 and 120.0, assume x belongs to the interval $[v_{1,i}, v_{1,i+1}]$ and $[v_{2,j}, v_{2,j+1}]$, we have

$$\begin{aligned} F_1(x) - f_1(x) &= \tfrac{1}{2}F_1''(\xi_x)(x - v_{1,i})(x - v_{1,i+1}), \\ F_2(x) - f_2(x) &= \tfrac{1}{2}F_2''(\xi_x)(x - v_{2,j})(x - v_{2,j+1}), \end{aligned} \tag{2}$$

where ξ_x is a point between $[v_{1,i}, v_{1,i+1}]$ and $[v_{2,j}, v_{2,j+1}]$, respectively. We can see that if F_1 and F_2 are similar to linear functions, i.e. the second derivatives, $F_1''(\xi_x)$ and $F_2''(\xi_x)$, are bounded by a small number on each interval, then the error would also be small.

For Question 1, we will simply evaluate $f_1(90)$ and $f_2(90)$.

$$\begin{aligned} f_1(90) &= f_{1,7}(90) = 122.8, \\ f_2(90) &= f_{2,4}(90) = 5.413, \end{aligned} \tag{3}$$

Apply (2) to this question, we get

$$\begin{aligned} F_1(90) - f_1(90) &= \tfrac{1}{2}F_1''(\xi_x)(90 - 85.4)(90 - 105.3) = -35.19F_1''(\xi_x), \\ F_2(90) - f_2(90) &= \tfrac{1}{2}F_2''(\xi_x)(90 - 79.7)(90 - 94.1) = -42.23F_2''(\xi_x). \end{aligned} \tag{4}$$

If we assume F_1 is concave and F_2 is convex, i.e. $F_1''(\xi_x) < 0$ and $F_2''(\xi_x) > 0$, which is reasonable since the data seem to form a circle-like shape, as shown in Figure 1, then we can see that $f_1(90)$ is an underestimate of the real value, and $f_2(90)$ and overestimate.

For Question 2, we notice from Table 2 and Figure 1 that $p = 110$mmHg is achieved in Part 1.Then it suffices to find the $x's$ where $f_1(x) = 110$. Since we use piecewise linear interpolation to interpolate the data points, the answers reside in the intervals $[v_{1,i}, v_{1,i+1}]$ where 110 is between $f_{1,i}(v_{1,i}) = p_{1,i}$ and $f_{1,i}(v_{1,i+1}) = p_{1,i+1}$. We find that such intervals are [50.7, 55.8] and [105.3, 115.1], and the corresponding linear functions are $f_{1.4}(x)$, and $f_{1,8}(x)$.We then solve the equations to get the answers x_1 and x_2.

$$\begin{cases} 110 = 4.373\,x_1 - 131.1 \\ 110 = -2.010\,x_2 + 326.7 \end{cases} \Rightarrow \begin{cases} x_1 = 55.14 \\ x_2 = 107.8 \end{cases} \tag{5}$$

For error analysis, the error largely depends the shape of F_1 and F_2 on the intervals. Nonetheless, we do know that if, for each interval, the real values of pressure are bounded by the values on the end points, then the answers do reside in the intervals we find above. Moreover, if the F_1 and F_2 are monotonic on such intervals, then we do know that there is only one answer on each interval.

For Question 3, we notice, from Figure 1, that the stroke work is the area inside the pressure-volume loop. To compute the area inside the loop, we first integrate the functions f_1 and f_2 to get the areas under their curves, respectively.

$$\begin{aligned} I_1 &= \int_{49.2}^{120.0} f_1(x)\,d\,x \\ I_2 &= \int_{49.2}^{120.0} f_2(x)\,d\,x \end{aligned} \tag{6}$$

Since f_1 and f_2 are piecewise linear functions, we can integrate them using trapezoidal rule piecewise, as shown below in equations (7).

$$\begin{aligned} I_1 &= \sum_{i=1}^{12} \int_{v_{1,i}}^{v_{1.i+1}} f_{1,i}(x)\,d\,x & I_2 &= \sum_{i=1}^{6} \int_{v_{2,i}}^{v_{2.i+1}} f_{2,i}(x)\,d\,x \\ &= \sum_{i=1}^{12} \tfrac{1}{2}(v_{1,i+1} - v_{1,i})(f_{1,i}(v_{1,i}) + f_{1,i}(v_{1,i+1})) & &= \sum_{i=1}^{6} \tfrac{1}{2}(v_{2,i+1} - v_{1,i})(f_{2,i}(v_{2,i}) + f_{1,i}(v_{2,i+1})) \\ &= \sum_{i=1}^{12} \tfrac{1}{2}(v_{1,i+1} - v_{1,i})(p_{1,i} + p_{1,i+1}) & &= \sum_{i=1}^{6} \tfrac{1}{2}(v_{2,i+1} - v_{1,i})(p_{2,i} + p_{2,i+1}) \\ &= 8075.795 & &= 547.675 \end{aligned} \tag{7}$$

Then we subtract the area under the curve for Part 2 from the area under the curve for Part 1 to get the area inside the loop.

$$I = I_1 - I_2 = 7528.12 \tag{8}$$

Denote the real area with J, the real integrations J_1 and J_2. Since such integration is based on functions (1), we have

$$\begin{aligned} J_1 - I_1 &= \int_{49.2}^{120.0} F_1(x) - \int_{49.2}^{120.0} f_1(x) = \sum_{i=1}^{12} \int_{v_{1,i}}^{v_{1,i+1}} \tfrac{1}{2} F_1''(\xi_x)(x - v_{1,i})(x - v_{1,i+1})\,d\,x, \\ J_2 - I_2 &= \int_{49.2}^{120.0} F_2(x) - \int_{49.2}^{120.0} f_2(x) = \sum_{i=1}^{6} \int_{v_{1,i}}^{v_{1,i+1}} \tfrac{1}{2} F_2''(\xi_x)(x - v_{2,i})(x - v_{2,i+1})\,d\,x, \end{aligned} \tag{9}$$

Assume that $|F_1''(x)| < M_1$ and $|F_2''(x)| < M_2$, then, after some computation, we get the following,

$$\begin{aligned} |J_1 - I_1| &\leqslant \tfrac{M_1}{2} \sum_{i=1}^{12} \int_{v_{1,i}}^{v_{1,i+1}} |(x - v_{1,i})(x - v_{1,i+1})|\,d\,x = 1293 M_1, \\ |J_2 - I_2| &\leqslant \tfrac{M_2}{2} \sum_{i=1}^{6} \int_{v_{2,i}}^{v_{2,i+1}} |(x - v_{2,i})(x - v_{2,i+1})|\,d\,x = 1183 M_2, \end{aligned} \tag{10}$$

$$|J - I| = |(J_1 - J_2) - (I_1 - I_2)| \leqslant |J_1 - I_1| + |J_2 - I_2| \leqslant 1293 M_1 + 1183 M_2 \leqslant 1293 * \max(M_1, M_2). \tag{11}$$

In conclusion, we reorganized the data in Table 1 to get data sets Part 1 and Part 2, as shown in Table 2, Table 3 and Figure 1, and generated piecewise polynomial functions (1) to interpolate the the data. We also analyzed the error for such interpolation, as shown in equations (2). For Question 1, we evaluated the functions f_1 and f_2 at the given point $v = 90$ to get the results (3). We also see the error for such estimate in (4). For Question 2, we identified the intervals where the answers reside and solved equations to get the results (5). For Question 3, we evaluated the stroke work by computing the area inside the pressure-volume loop, through integration with piecewise trapezoidal rule, as shown in equation (7), to get the result (8). The error analysis for the integrations are shown in (11).