Investigation of Robustness of a Mathematical Model

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Introduction

Statement of the Problem

During a normal heartbeat the left ventricle undergoes 4 phases: I) isovolumetric contraction; II) ejection; III) isovolumetric relaxation; IV) filling. During phase I), the valves are closed and no blood can get in or out of the ventricle. During this phase the ventricular muscle starts to contract rapidly increasing the ventricular pressure. When the ventricular pressure exceeds the aortic valve pressure, we enter in phase II) where the valve opens and the blood flows out of the ventricle reducing its volume. As blood is ejected, the ventricular pressure starts to decrease until the aortic valve closes again. At this point, we are in phase III) in which the ventricle is relaxing with both valves closed once again keeping the blood volume constant. As the ventricle relaxes, the pressure decreases. When the ventricular pressure becomes smaller the the atrial pressure, we enter in phase IV) where the mitral valve opens and blood fills the ventricle again increasing its volume. As the blood fills the ventricle again pressure increases and we return back to phase I). The following data for pressure and volume was recorded during the four phases together with the time (starting from a initial time $t_0 = 0$)

Phase I						Phase II					
t_I[ms]	0.	11	22	34	-	t_{II}[ms]	68	119	170	221	255
p_I[mmHg]	8.1	32.2	56.2	79.9	-	p_{II}[mmHg]	95.3	115.0	125.1	126.3	112.9
v_I[ml]	120.0	119.8	119.6	119.4	-	v_{II}[ml]	115.1	105.3	85.4	70.7	55.8
Phase III						Phase IV					
t_{III}[ms]	272	280	289	297	-	t_{IV}[ms]	331	425	510	578	646
p_{III}[mmHg]	90.6	70.0	50.1	30.6	-	p_{IV}[mmHg]	13.5	8.4	6.2	5.1	7.5
v_{III}[ml]	50.7	49.7	49.5	49.2	-	v_{IV}[ml]	50	64.3	79.7	94.1	109.0

Table 1: Pressure and Volume at Different Times during the Four Phases

We furthur investigates the following questions:

1. What are the values of the pressure when the ventricular volume is 90ml?

2. What are the values of the volume when the pressure is 110 mmHg?

3. We also want to assess the ventricular function using the stroke work. The stroke work refers to the work done by the ventricle to eject a volume of blood. The force that is applied to the volume of blood is the intraventricular pressure. Stroke work is best depicted by the use of a pressure-volume diagrams, in which stroke work is the area within the pressure-volume loop. What is the stroke work for this set of data?

Numerical Methods

We will use interpolation to solve the problems. To begin with, we notice that, the values of ventricular volume keep decreasing at Phase I, II and III, and keep increasing at Phase IV. Therefore, we can separate the data points into two parts, Phase I, II and III, and Phase III, and regard p, the pressure as a function of v, the volume, for the two parts, respectively. Also, in order to form a complete cycle, we add the last point of Phase III, and the first point of Phase I, to part two. We reorganize the data from Table 1 accordingly and show them below in Table 2 and 3, and Figure 1.

p_1[mmHg]	30.6	50.1	70.0	90.6	112.9	126.3	125.1	115.0	95.3	79.9	56.2	32.2	8.1
v_1[ml]	49.2	49.5	49.7	50.7	55.8	70.7	85.4	105.3	115.1	119.4	119.6	119.8	120.0

Table 2: Pressure vs. Volume for Part I

p_2[mmHg]	30.6	13.5	8.4	6.2	5.1	7.5	8.1
v_2[ml]	49.2	50	64.3	79.7	94.1	109.0	120.0

Table 3: Pressure vs. Volume for Part II

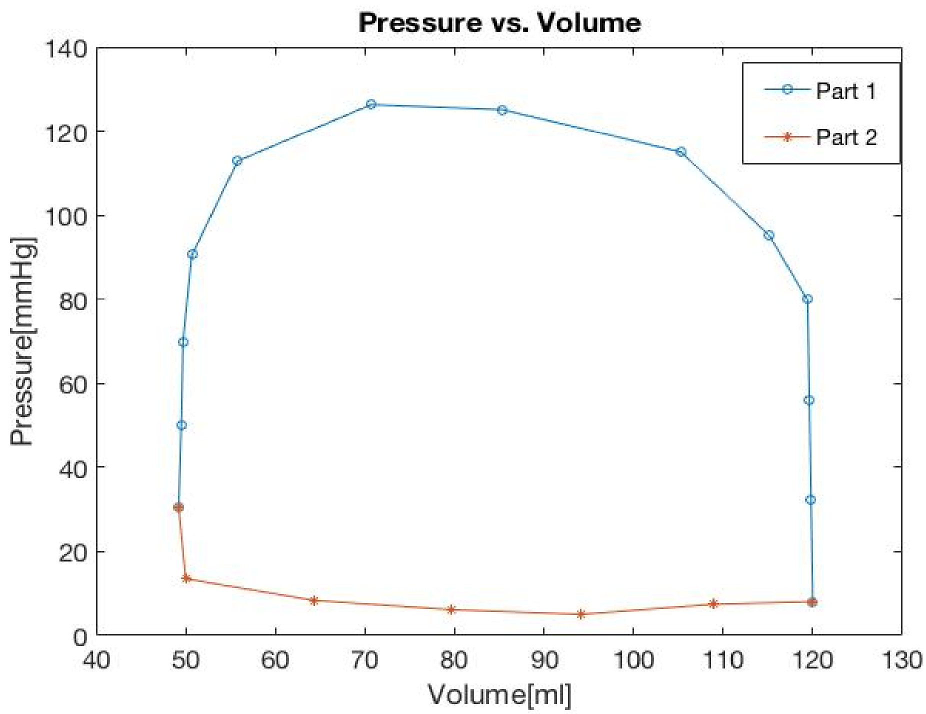

Figure 1. Pressure vs. Volume for Part 1 and 2

We then interpolate the two sets of data points respectively. Results for the several selected interpolating methods, i.e. polynomial interpolation, piecewise quadratic interpolation, and piecewise linear interpolation, are shown below in Figure 2.

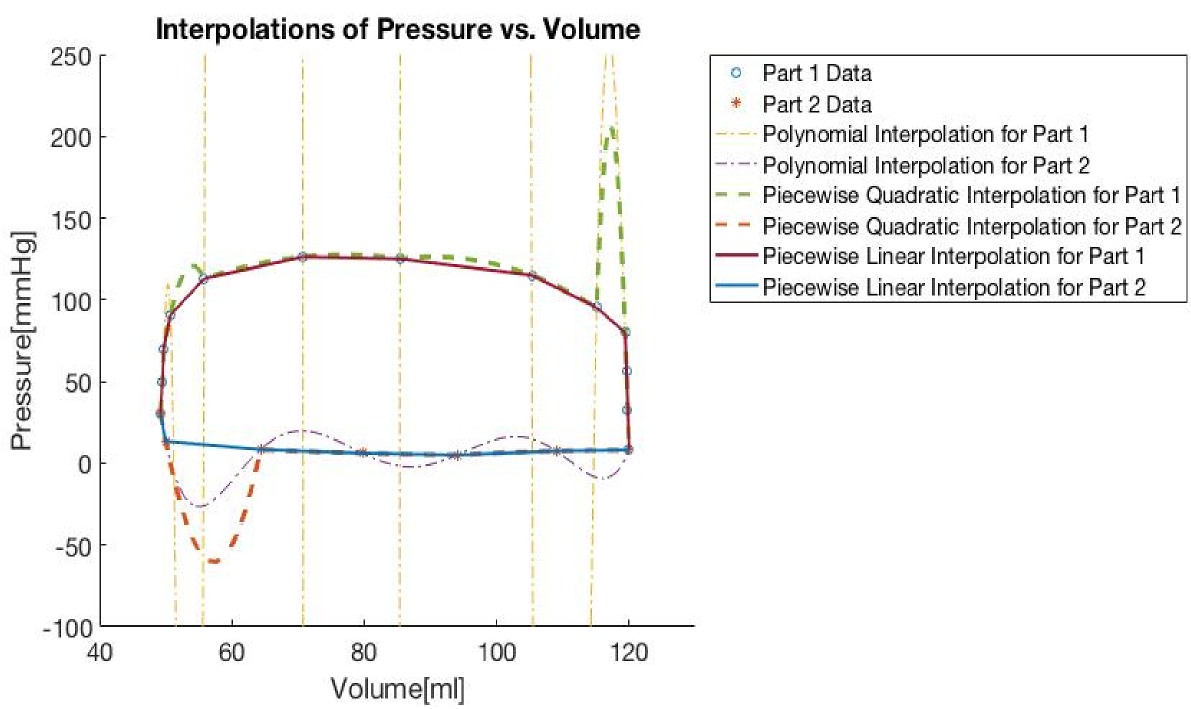

Figure 2. Interpolations of Pressure vs. Volume

From Figure 2, we notice that the interpolating functions for polynomial interpolation and piecewise quadratic interpolation have some unsatisfying behaviors. The interpolating functions of polynomial interpolation for Part 1 and 2 are considerably oscillating, especially the function for Part 1. The interpolating function of piecewise quadratic interpolation for Part 1 changes drastically between $v = 115.1$ and $v = 119.4$, and the interpolating function of piecewise quadratic interpolation for Part 2 also changes unreasonably between $v = 50$ and $v = 64.3$.

Therefore, we will choose piecewise linear interpolation. Although the interpolating functions of piecewise linear interpolation, as seen from Figure 2, are probably not smooth enough at the nodes, they at least do not have erroneous behaviors such as oscillation and drastic change, like other interpolation methods do.

Results

The interpolating functions, $f_1(v)$ and $f_2(v)$, of piecewise linear interpolation for Part 1 and 2, consisting of linear polynomials, are shown below, respectively, after rounding the coefficients to 4 significant figures.

$$f_1(x)\begin{cases} f_{1,1}(x) = 65.00x - 3167, x \in [49.2, 49.5] \\ f_{1,2}(x) = 99.50x - 4875, x \in [49.5, 49.7] \\ f_{1,3}(x) = 20.60\,x - 949.8, x \in [49.7, 50.7] \\ f_{1,4}(x) = 4.373\,x - 131.1, x \in [50.7, 55.8] \\ f_{1,5}(x) = 0.8993x + 62.72, x \in [55.8, 70.7] \\ f_{1,6}(x) = -0.08163x + 132.1, x \in [70.7, 85.4] \\ f_{1,7}(x) = -0.5075x + 168.4, x \in [85.4, 105.3] \\ f_{1,8}(x) = -2.010\,x + 326.7, x \in [105.3, 115.1] \\ f_{1,9}(x) = -3.581x + 507.5, x \in [115.1, 119.4] \\ f_{1,10}(x) = -118.5x + 14230, x \in [119.4, 119.6] \\ f_{1,11}(x) = -120.0x + 14410, x \in [119.6, 119.8] \\ f_{1,12}(x) = -120.5\,x + 14470, x \in [119.8, 120.0] \end{cases} \qquad f_2(x)\begin{cases} f_{2,1}(x) = -21.38x + 1082, x \in [49.2, 50] \\ f_{2,2}(x) = -0.3566\,x + 31.33, x \in [50, 64.3] \\ f_{2,3}(x) = 0.1429x + 17.59, x \in [64.3, 79.7] \\ f_{2,4}(x) = -0.07639x + 12.29, x \in [79.7, 94.1] \\ f_{2,5}(x) = 0.1611\,x - 10.06, x \in [94.1, 109.0] \\ f_{2,6}(x) = 0.05455x + 1.5545, x \in [109.0, 120.0] \end{cases}$$

Discussion/Conclusions

For Question 1, we will simply evaluate $f_1(90)$ and $f_2(90)$.

$$f_1(90) = f_{1,7}(90) = 122.8,$$

$$f_2(90) = f_{2,4}(90) = 5.413,$$

For Question 2, we notice from Table 2 and Figure 1 that $p = 110$mmHg is achieved in Part 1.Then it suffices to find the $x's$ where $f_1(x) = 110$. Since we use piecewise linear interpolation to interpolate the data points, the answers reside in the intervals $[v_{1,i}, v_{1,i+1}]$ where 110 is between $f_{1,i}(v_{1,i}) = p_{1,i}$ and $f_{1,i}(v_{1,i+1}) = p_{1,i+1}$.

We find that such intervals are $[50.7, 55.8]$ and $[105.3, 115.1]$, and the corresponding linear functions are $f_{1.4}(x) =$, and $f_{1,8}(x) =$

We then solve the equations to get the answers x_1 and x_2.

$$\begin{cases} 110 = 4.373\,x_1 - 131.1 \\ 110 = -2.010\,x_2 + 326.7 \end{cases} \Rightarrow \begin{cases} x_1 = 55.14 \\ x_2 = 107.8 \end{cases}$$

For Question 3, we notice from Figure 1, that the stroke work is the area inside the pressure-volume loop. To compute the area inside the loop, we first integrate the functions f_1 and f_2 to get the areas under their curves, respectively.

$$I_1 = \int_{49.2}^{120.0} f_1(x)\,d\,x$$

$$I_2 = \int_{49.2}^{120.0} f_2(x)\,d\,x$$

Since f_1 and f_2 are piecewise linear functions, we can integrate them using trapezoidal rule piecewise, as shown below in Figure 3.

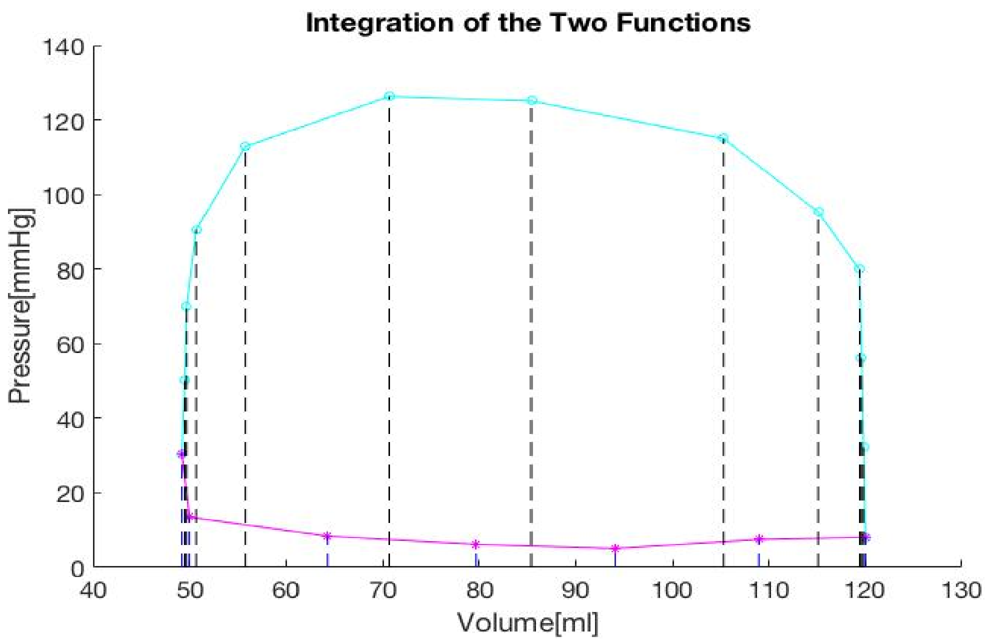

Figure 3. Integration Using Piecewise Trapezoidal Rule

$$\begin{aligned} I_1 &= \textstyle\sum_{i=1}^{12} \int_{v_{1,i}}^{v_{1.i+1}} f_{1,i}(x)\,d\,x \\ &= \textstyle\sum_{i=1}^{12} \frac{1}{2}(v_{1,i+1} - v_{1,i})(f_{1,i}(v_{1,i}) + f_{1,i}(v_{1,i+1})) \\ &= \textstyle\sum_{i=1}^{12} \frac{1}{2}(v_{1,i+1} - v_{1,i})(p_{1,i} + p_{1,i+1}) \\ &= 8075.795 \end{aligned}$$

$$\begin{aligned} I_2 &= \textstyle\sum_{i=1}^{6} \int_{v_{2,i}}^{v_{2.i+1}} f_{2,i}(x)\,d\,x \\ &= \textstyle\sum_{i=1}^{6} \frac{1}{2}(v_{2,i+1} - v_{1,i})(f_{2,i}(v_{2,i}) + f_{1,i}(v_{2,i+1})) \\ &= \textstyle\sum_{i=1}^{6} \frac{1}{2}(v_{2,i+1} - v_{1,i})(p_{2,i} + p_{2,i+1}) \\ &= 547.675 \end{aligned}$$

Then we subtract the area under the curve for Part 2 from the area under the curve for Part 1 to the area inside the loop.

$$I = I_1 - I_2 = 7528.12$$