STOR 435 Homework 20

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1. No.

   Suppose they are independent, then by definition of independence, we would have

   $$p_{X,Y}(0,1) = p_X(0)p_Y(1) \neq 0$$

   However, we also have

   $$p_{X,Y}(0,1) = \mathbb{P}(\text{Ruoyu is asked 0 questions and answers 1 question adequately}),$$

   which is apparently 0. Therefore we have a contradiction, and therefore they are not independent.

2.

   a) $p_{X_1,X_2,X_3}(x_1,x_2,x_3) = p_{X_1}(x_1)p_{X_2}(x_2)p_{X_3}(x_3) = 0.0002e^{-0.1x_1-0.05x_2-0.04x_3}$

   b) $p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2) = 0.005e^{-0.1x_1-0.05x_2}$

   c) $\mathbb{P}(X_1 < X_2) = \int_0^\infty \int_0^{x_2} 0.005e^{-0.1x_1-0.05x_2}dx_1dx_2 = \dfrac{2}{3}$

   d) $\mathbb{P}(X_1 \leqslant 10, X_2 \leqslant 10, X_3 > 10) = F_{X_1}(10)F_{X_2}(10)(1 - F_{X_3}(10))$
   $= (1 - e^{-0.1\times10})(1 - e^{-0.05\times10})(e^{-0.04\times10}) \approx 0.1667$

3.

   a) $p_{U_1,U_2,U_3}(u_1,u_2,u_3) = p_{U_1}(u_1)p_{U_2}(u_2)p_{U_3}(u_3) = 1$

   b) $\mathbb{P}(\max(U_1,U_2,U_3) \leqslant 0.5) = \mathbb{P}(U_1 \leqslant 0.5, U_2 \leqslant 0.5, U_3 \leqslant 0.5) = F_{U_1}(0.5)F_{U_2}(0.5)F_{U_2}(0.5)$
   $= \dfrac{0.5-0}{1-0} \times \dfrac{0.5-0}{1-0} \times \dfrac{0.5-0}{1-0} = \dfrac{1}{8}$

4. $\mathbb{E}(X^2(Y+1)^2) = \mathbb{E}(X^2)\mathbb{E}((Y+1)^2) = \mathbb{E}(X^2)(\mathbb{E}(Y^2) + 2\mathbb{E}(Y) + 1)$
   $= (\mathrm{Var}(X) + \mathbb{E}(X)^2)(\mathrm{Var}(Y) + \mathbb{E}(Y)^2 + 2 + 1) = 1 \times 5 = 5$

5. $\mathbb{P}(W \leqslant 0.5) = \mathbb{P}(Y < 0.5) + \mathbb{P}\left(Y \geqslant 0.5 \wedge X \leqslant \sqrt{\dfrac{0.5}{Y}}\right)$
   $= \int_0^{0.5} 2y\,dy + \int_{0.5}^1 \int_0^{\sqrt{\frac{0.5}{y}}} 2y6x(1-x)\,dx\,dy = \dfrac{1}{4} + \int_{\frac{1}{2}}^1 3 - \sqrt{2}y^{-\frac{1}{2}}\,dy = \frac{15}{4} - 2\sqrt{2} \approx 0.9216$

6. $M_Y(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t) = e^{3(e^t-1)}e^{0.5(e^t-1)}e^{2(e^t-1)} = e^{5.5(e^t-1)}$
   $p_Y(y) = e^{-5.5}\dfrac{5.5^y}{y!}$, $\mathbb{P}(Y=6) = p_Y(6) \approx 0.1571$