STOR 435 Homework 24

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1.

a) We first notice that $Y|X=x \sim B(x, 0.75)$

$p_{X,Y}(x,y) = p_{Y|X}(y|x)p_X(x) = \frac{1}{6}\binom{x}{y}(0.75)^y(1-0.75)^{x-y} = \frac{\binom{x}{y}4^{-x}3^y}{6}$,

for $0 \leqslant y$, $\max(y,1) \leqslant x \leqslant 6$

$p_{X,Y}(x,y) = 0$, for other values.

b) $p_Y(y) = \sum_{i=\max(y,1)}^{i=6} p_{X,Y}(i,y) = \sum_{i=\max(y,1)}^{i=6} \frac{\binom{i}{y}4^{-i}3^y}{6}$, for $0 \leqslant y \leqslant 6$

// TODO, can simplify?

$p_{X|Y}(x|3) = \frac{p_{X,Y}(x,3)}{p_Y(3)}$

// TODO, closed form?

x	1	2	3	4	5	6
$p_{X\|Y}(x\|3)$	0	0	0.3404	0.3404	0.2128	0.1064

c) 4.0851

// TODO, close form?

2. $\mathrm{Cov}(X,Y) = \rho\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = -0.3\sqrt{4^2 \times 17.6^2} = -21.12$

$\mathrm{Var}(R) = 0.4^2\mathrm{Var}(X) + 0.6^2\mathrm{Var}(Y) + 2 \times 0.4 \times 0.6\mathrm{Cov}(X,Y) = 103.936$

$\sigma_R = \sqrt{\mathrm{Var}(R)} \approx 10.1949$

3.

a) No. Suppose X and Y are independent, then we have $p_{X,Y}(x,y) = p_X(x)p_Y(y)$

$$\left.\begin{array}{l} p_{X,Y}(1,2) \neq 0 \rightarrow p_X(1) \neq 0 \\ p_{X,Y}(2,3) \neq 0 \rightarrow p_Y(3) \neq 0 \end{array}\right\} \rightarrow p_{X,Y}(1,3) = p_X(1)p_Y(3) \neq 0, \text{ contradiction.}$$

Therefore, X and Y are not independent.

Note: This means if independent, 0 has to appear along at least one dimension.

b) 0.5150

c) 0.6314

4.

a) $\mathrm{Cov}(Y_n, Y_{n+j}) = \mathrm{Cov}(X_n + 2X_{n+1}, X_{n+j} + 2X_{n+1+j})$

$= \mathrm{Cov}(X_n, X_{n+j}) + 2\mathrm{Cov}(X_n, X_{n+1+j}) + 2\mathrm{Cov}(X_{n+1}, X_{n+j}) + 4\mathrm{Cov}(X_{n+1}, X_{n+1+j})$

$=0$

Therefore, $\rho_{Y_n,Y_{n+j}} = \dfrac{\text{Cov}(Y_n, Y_{n+j})}{\sqrt{\text{Var}(Y_n)\text{Var}(Y_{n+j})}} = 0$

b) $\text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + 2X_{n+1}, X_{n+1} + 2X_{n+2})$

$=\text{Cov}(X_n, X_{n+1}) + 2\text{Cov}(X_n, X_{n+2}) + 2\text{Cov}(X_{n+1}, X_{n+1}) + 4\text{Cov}(X_{n+1}, X_{n+2})$

$=2\text{Var}(X_{n+1}) = 2\sigma^2$

$\text{Var}(Y_n) = \text{Var}(X_n + 2X_{n+1}) = \text{Var}(X_n) + 4\text{Var}(X_{n+1}) = 5\sigma^2$

$\rho_{Y_n,Y_{n+1}} = \dfrac{\text{Cov}(Y_n, Y_{n+1})}{\sqrt{\text{Var}(Y_n)\text{Var}(Y_{n+1})}} = \dfrac{2\sigma^2}{5\sigma^2} = \dfrac{2}{5}$