

Let  $N := (K, \Sigma, \Delta, s, F)$  be the finite automaton that recognizes  $L$ , i.e.  $L(N) = L$ . Let  $E := \{q \in K \mid \exists f \in F, \text{ s.t. } \exists \text{ path from } q \text{ to } f\}$ ,  $F' := E \cup F$ ,  $N' := (K, \Sigma, \Delta, s, F')$ .

Claim:  $N'$  recognizes  $M$ , i.e.  $L(N') = M$ .

Proof:

We first show  $L(N') \supseteq M$ . Let  $m \in M$  and  $mm' = l \in L$ . Since  $N$  recognizes  $L$ ,  $\exists p \in K, f \in F$  s.t.  $(s, mm') \vdash_N^* (p, m') \vdash_N^* (f, \epsilon)$ . By definition of  $F'$ ,  $p \in F'$ . Further,  $(s, m) \vdash_N^* (p, \epsilon)$ . Note that  $N$  and  $N'$  only differ by the final states, so  $(s, m) \vdash_{N'}^* (p, \epsilon)$ , which means  $m \in L(N')$ .

We then show  $L(N') \subseteq M$ . Let  $m \in L(N')$  and  $(s, m) \vdash_{N'}^* (p, \epsilon), p \in F'$ . By definition of  $F'$ ,  $\exists m', f \in F$  s.t.  $(p, m') \vdash_{N'}^* (f, \epsilon)$ . Therefore,  $(s, mm') \vdash_{N'}^* (f, \epsilon)$ , and since  $N$  and  $N'$  only differ by the final states,  $(s, mm') \vdash_N^* (f, \epsilon)$ , which means  $mm' \in L(N) = L$ , which means  $m \in M$ .

By Claim,  $M$  is regular.