Project 4: Zombies Outbreak

Chenxi Huang, Siyang Jing and Yingqiao Zhou

November 27, 2017

Introduction

This report introduces four zombies outbreak models, each of which is a system of first-order, nonlinear differential equations. We will use the Heun method to give numerical approximations of these models and analyze the behaviors of different classes in each model. Specifically, with one given initial condition of a specific class in a model, we will evaluate how the changes of initial conditions of other classes and the changes of parameters influence the decaying rate of that class.

Statement of Problem

The four zombie outbreak: Basic model, Latent infection model, Quarantine model and Treatment model is given as follows.

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S, \\ Z' &= \beta SZ + \zeta R - \alpha SZ, \\ R' &= \delta S + \alpha SZ - \zeta R. \end{aligned}$$

Part I: Basic model

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S, \\ I' &= \beta SZ - \rho I - \delta I \\ Z' &= \rho I + \zeta R - \alpha SZ, \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R. \end{aligned}$$

Part II: Latent infection model

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S, \\ I' &= \beta SZ - \rho I - \delta I - \kappa I \\ Z' &= \rho I + \zeta R - \alpha SZ - \sigma Z, \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R + \gamma Q. \\ Q' &= \kappa I + \sigma Z - \gamma Q \end{aligned}$$

Part III: Quarantine model

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S + cZ, \\ I' &= \beta SZ - \rho I - \delta I \\ Z' &= \rho I + \zeta R - \alpha SZ - cZ, \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R. \end{aligned}$$

Part IV: Treatment model

For all the models above, S, Z and R denote the susceptible, zombie and removed class respectively; I in Part II denote the infected class and Q in Part III denote the quarantine class; Π, β and δ represent the rate of birth, transmission from Q and non-zombie related death rate respectively. $\alpha, \zeta$and ρrepresent the rate of zombies defeated, the rate of resurrection from R to Z and the rate of transmission from I to Z. $\kappa, \sigma$and γrepresent the rates of entering at Q from I , from Z and the rate of I being killed. c represents the rate of zombies being cured. In this project, $\alpha = \rho = 0.005$, $\beta = 0.0095$, $\zeta = 0.0001$, $\Pi = \delta = 0.0001$ and $S(0) = S_0 = 3 \times 10^8$.

We need to evaluate under which initial condition (Z_0, R_0) would human kind (S) survive in Part I, the effect of the latent infection (I) to the model in Part II, the effect of quarantine (Q) to the model in Part III and at which rate(c)to treat zombies so that human kind(S) could survive in Part IV.

Numerical Method

To solve for the four models, we will use Heun's method given by the following formula:

$$u_{n+1} = u_n + \frac{h}{2}[f_n + f(t_{n+1}, u_n + hf_n)] \quad (1)$$

Heun's method is an explicit method, which demands less computational cost than implicit method. An explicit method could be evaluated directly in terms of known quantities at the previous time step, whereas an implicit method generally requires a matrix or iterative solution to compute the new quantities since unknowns are at the both sides of an equation. Besides, the Heun's method is a method of order 2, and thereby gives more accurate approximations than first-order methods. The estimate of truncation error τ_n is derived as follows.

$\tau_n = \frac{u(t_{n+1})-u(t_n)}{h} - \frac{1}{2}(f_n + f(t_{n+1}, u(t_n) + hu'(t_n)))$
$= u'(t_n) + \frac{h}{2!}u^{(2)}(t_n) + \mathcal{O}(h^2) - \frac{1}{2}\left[u'(t_n) + u'(t_n) + hf'(t_n, u_n) + \mathcal{O}(h^2)\right]$
$= u'(t_n) + \frac{h}{2}u^{(2)}(t_n) + \mathcal{O}(h^2) - \frac{1}{2}(2 \cdot u'(t_n) + hu^{(2)}(t_n) + \mathcal{O}(h^2)) = \mathcal{O}(h^2)$

Analysis

1. Basic Model

Let $X_I(t) = \begin{pmatrix} S(t) \\ Z(t) \\ R(t) \end{pmatrix}$, $F_I(X_I) = F_I\begin{pmatrix} S, & Z, & R, \end{pmatrix} = \begin{pmatrix} \Pi - \beta SZ - \delta S \\ \beta SZ + \zeta R - \alpha SZ \\ \delta S + \alpha SZ - \zeta R \end{pmatrix}$, then the differential equation modeling Part I becomes

$$X_I' = F_I(X_I) \quad (2)$$

From equation $S' = \Pi - \beta SZ - \delta S$, both Z and R have negative effect on the growth rate of the susceptible class. Consider an optimal case that minimizes the negative effect with $Z_0 = 0$ and $R_0 = 0$ as they are forced to be nonnegative. We try to numerically solve (2) with such initial condition by applying Heun's method (1) to get the solution. We set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_I(0) = \begin{pmatrix} S_0, & 0, & 0 \end{pmatrix}^T$. Such choice for h and $tspan$ is obtained after several experiments. //////////////We find the solution with $h = 10^{-10}$, $h = 10^{-9}$, and $h = 10^{-8}$ are sufficiently close, and therefore reasonably deduce that $h = 10^{-10}$ is small enough for us to get a reasonably accurate solution. As for time interval, we find that the solution stays considerably stable after $t = 10^{-4}$, and therefore set the time span accordingly.

The process of choosing time step and time interval is similar for numerically solving all the ODE's below, and therefore we will skip the explanation to avoid redundancy. In fact, most of the time steps, or time intervals, chosen below are of the same order of magnitude as this one, respectively.

The solution to the Basic Model with optimal initial condition is given by Figure 1, which implies that the susceptible will eventually die out even for the extreme case in which initially there is no Zombies or the Removed. In that case, we are capable of deducing the conclusion that for any values of Z_0 and R_0, $S(t)$ will eventually decay to zero. Thus, under no initial conditions could human kind survive. Nevertheless, we still want to investigate the behavior of $S(t)$ with different values of Z_0 and R_0.

First, we fix the value of R_0 at 3×10^8. In Figure 2 below, solutions $S(t)$ are plotted with different colors corresponding to the 6 different values of Z_0 ranging from 0 to S_0. We can see that the speed of the decay of population varies, but all decay to sufficiently near zero eventually, again confirming our reasoning that human will extinct regardless of initial condition.

In addition, when $Z_0 = 0$, the behavior of $S(t)$ exhibits significant dependence on R_0. Figure 3 below shows the trend of $S(t)$ with fixed Z_0 at 0, in which different colors correspond to different R_0 values.

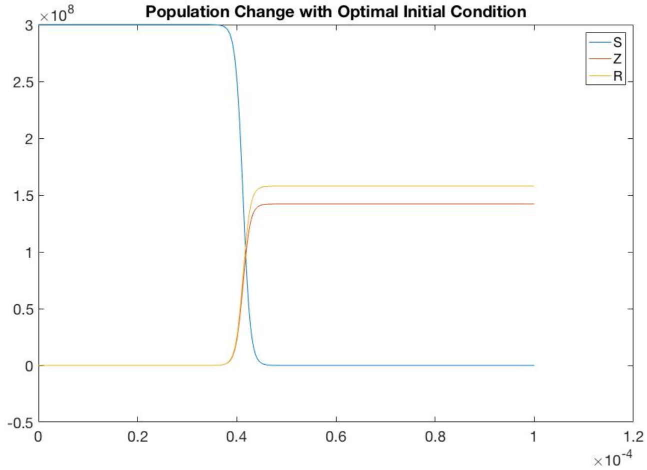
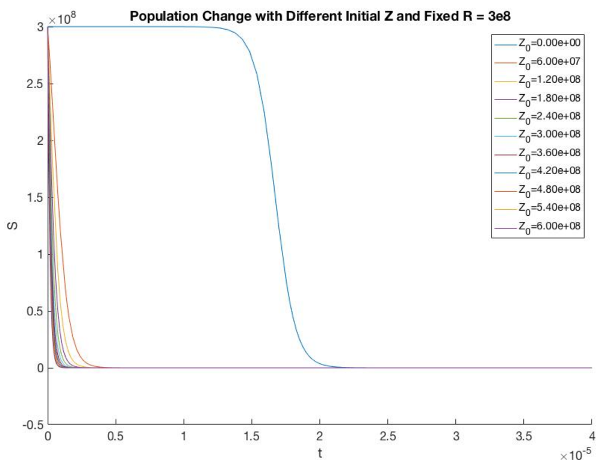


Figure 1: Population Change with $Z_0 = 0$, and $R_0 = 0$



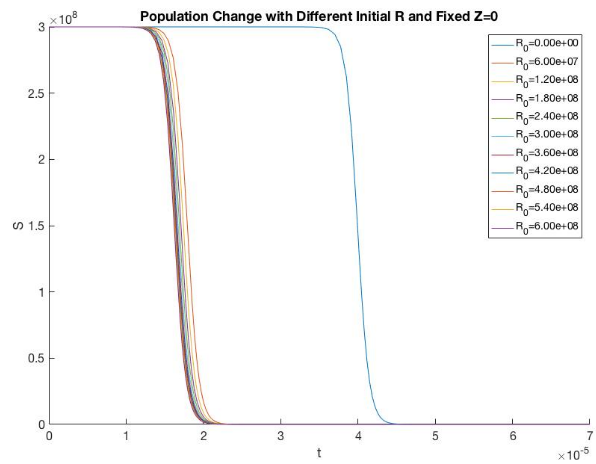


Figure 2: Population Change with Different Z_0 and $R_0 = 3 \times 10^8$

Figure 3: Population Change with Different Initial R and Fixed $Z = 0$

2. Latent infection Model

Let $X_{II}(t) = \begin{pmatrix} S(t) \\ I(t) \\ Z(t) \\ R(t) \end{pmatrix}$, $F_{II}(X_{II}) = F_{II}\begin{pmatrix} S, & I, & Z, & R \end{pmatrix} = \begin{pmatrix} \Pi - \beta SZ - \delta S \\ \beta SZ - \rho I - \delta I \\ \rho I + \zeta R - \alpha SZ \\ \delta S + \delta I + \alpha SZ - \zeta R \end{pmatrix}$, then the differential equation modeling Part II becomes

$$X'_{II} = F_{II}(X_{II}) \tag{3}$$

To check the effect of latent infection, we initialize Z_0, I_0 and R_0 with different values respectively. Assign Z_0 with 6 distinct values ranging from 0 to S_0 with equally-spaced interval; I_0 be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval; R_0 be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval. We apply Heun's method (1) to (3) to get the solution. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_{II}(0) = \begin{pmatrix} S_0, & I_0, & Z_0, & R_0 \end{pmatrix}^T$. Figure 4 below gives solutions with different initial conditions, with different colors corresponding to different Z_0.

Notice that the lines of the same color basically overlap with each other, indicating that changes in I_0 and R_0 does not cause significant change. Thus, the behavior of $S(t)$ is dominated by Z_0. We can also see that for certain values of Z_0, $S(t)$ does not decay to zero, but rather seem to converge to a non-zero number. Such behavior is not observed in Part I. In addition, the final stable human population $S(t_n)$ seems to be dependent on Z_0. Specifically, the more initial zombies there are, the less final human population would be, which is certainly reasonable in this situation.

Therefore, it is reasonable to assume that there exists a threshold value Z_h, such that for Z_0 greater than Z_h, human could not survive, and for Z_0 smaller than Z_h, human could survive.

//TODO

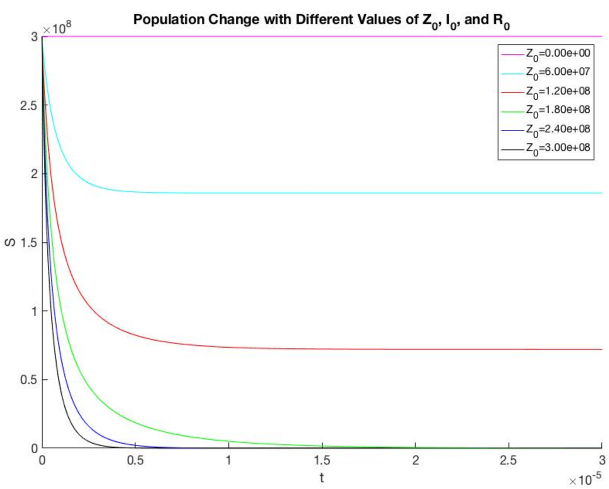
To calculate such Z_h, we first define extinction as

We assume $R_0 = S_0 = 3 \times e^8$ and use the bisection method to find

We find such value to be 1.5946×10^8.

//end TODO

To further evaluate how latent infection affect the human population $S(t)$, fix I_0 and R_o as they are not significantly influencing the behavior of $S(t)$, and assign Z_0 with different values ranging from 1.5×10^8 to 3×10^8 with equally spaced interval. Applying (1) to (2) and(3) with these initial conditions, we get the solutions of $S(t)$ in different parts, as shown in Figure 5, with solid line representing Part II and dashed line representing Part I. Then we can see that latent infection reduced the rate of decay in $S(t)$.



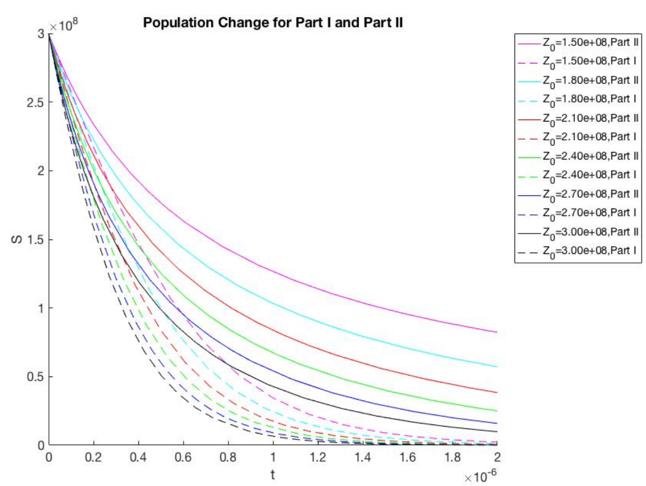


Figure 4: Population Change with Different Values of Z_0, I_0, and R_0

Figure 5: Population Change for Part I and Part II

3. Quarantine Model

Let $X_{III}(t) = \begin{pmatrix} S(t) \\ I(t) \\ Z(t) \\ R(t) \\ Q(t) \end{pmatrix}$, $F_{III}(X_{III}) = F_{III}\begin{pmatrix} S, & I, & Z, & R, & Q \end{pmatrix} = \begin{pmatrix} \Pi - \beta SZ - \delta S \\ \beta SZ - \rho I - \delta I - \kappa I \\ \rho I + \zeta R - \alpha SZ - \sigma Z \\ \delta S + \delta I + \alpha SZ - \zeta R \\ \kappa I + \sigma Z - \gamma Q \end{pmatrix}$, then the differential equation modeling Part III becomes

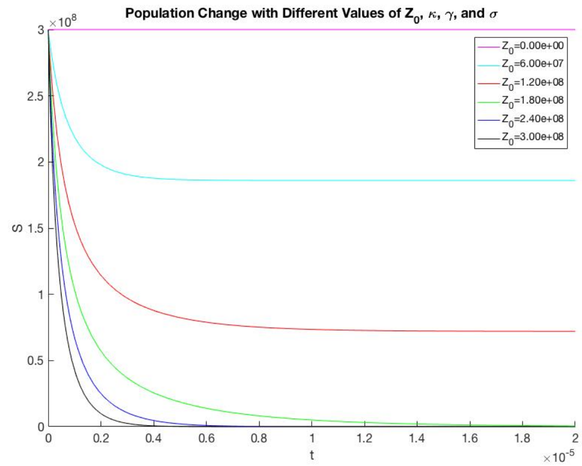
$$X'_{III} = F_{III}(X_{III}) \tag{4}$$

We need to investigate the dependence of S on initial conditions Z_0, I_0, R_0, Q_0, and parametersκ, γ, σ.

First, we check dependence on the initial conditions by initializing Z_0, I_0, R_0 and Q_0 with different values respectively and fixing the parameters κ, γ, and σ. Assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval; I_0, R_0, and Q_0, respectively, be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval. Fix κ, γ, and σ, respectively, to be 0.1. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $x_{III}(0) = \begin{pmatrix} S_0, & I_0, & Z_0, & R_0, & Q_0 \end{pmatrix}^T$. Apply Heun's method (1) to (4) to get the solutions with different initial conditions, which is shown in Figure 6, with different colors corresponding to different Z_0. Notice that the lines of the same color basically overlap with each other, indicating that changes in I_0, R_0, and Q_0 do not cause significant change. Thus, the behavior of $S(t)$ is dominated by Z_0.

Second, we check dependence on the parameters κ, γ, and $\sigma$by initializing Z_0, κ, γ, and $\sigma$with different values, respectively, and fixing the parameters I_0, R_0, and Q_0. Assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval. For each Z_0, assign κ, γ, and σ, respectively, be 6 distinct values ranging from 0 to 1, with equally-spaced interval. Fix I_0, R_0 and Q_0, respectively, to be 3×10^8. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $x_{III}(0) = \begin{pmatrix} S_0, & S_0, & Z_0, & S_0, & S_0 \end{pmatrix}^T$. Apply Heun's method (1) to (4) to get the solutions with different parameters, which is shown in Figure 7, with different colors corresponding to different Z_0. Notice that the lines of the same color basically overlap with each other, indicating that changes in κ, γ, and σ do not cause significant change. Thus, the behavior of $S(t)$ is independent of parameters κ, γ, and σ.

To further evaluate how quarantine affect the human population $S(t)$, fix I_0, R_0 and Q_0, respectively, to be 3×10^8; κ, γ, and σ, respectively, be 0.2 as they are not significantly influencing the behavior of $S(t)$, and assign $Z_0$with 6 distinct values ranging from 0 to S_0, with equally-spaced interval. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $x_I(0) = \begin{pmatrix} S_0, & Z_0, & S_0 \end{pmatrix}^T$, $x_{II}(0) = \begin{pmatrix} S_0, & S_0, & Z_0, & S_0 \end{pmatrix}^T$, $x_{III}(0) = \begin{pmatrix} S_0, & S_0, & Z_0, & S_0, & S_0 \end{pmatrix}^T$. Apply Heun's method (1) to (2), (3), and (4), respectively, to get the solution to each part, as shown in Figure 8, with solid line representing Part III, dashed line representing Part II, and dotted line representing Part I. From the figure we can see the dashed line and solid line basically overlap with each other, thus quarantine does not have a huge influence.



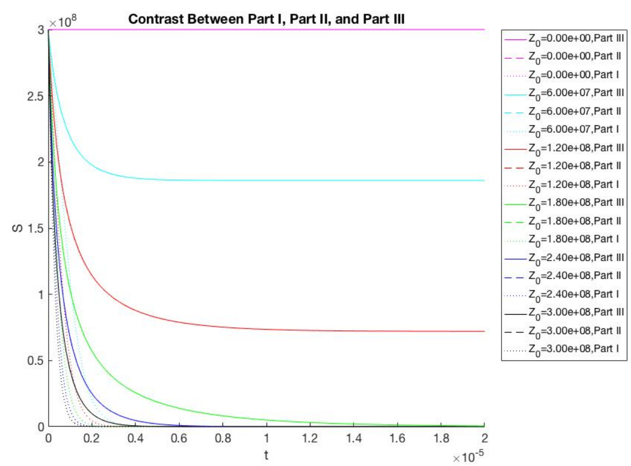


Figure 6: Population Change with Different Values of Z_0, I_0, R_0, and Q_0

Figure 7: Population Change with Different Values of Z_0, κ, γ, and σ

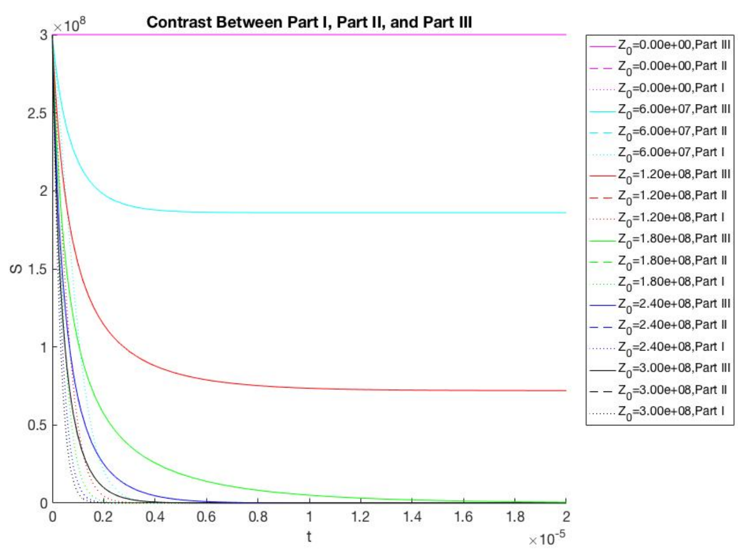


Figure 8: Population Change for Part I, Part II, and Part III

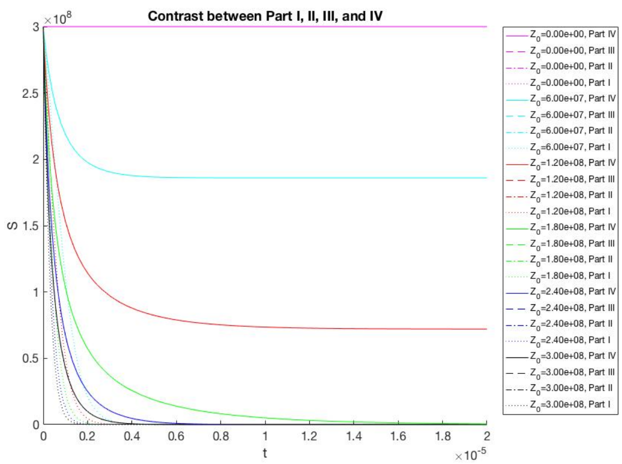
4. Treatment Model

Let $X_{IV}(t) = \begin{pmatrix} S(t) \\ I(t) \\ Z(t) \\ R(t) \end{pmatrix}$, $F_{IV}(X_{IV}) = F_{IV}\begin{pmatrix} S, & I, & Z, & R \end{pmatrix} = \begin{pmatrix} \Pi - \beta SZ - \delta S + cZ \\ \beta SZ - \rho I - \delta I \\ \rho I + \zeta R - \alpha SZ - cZ \\ \delta S + \delta I + \alpha SZ - \zeta R \end{pmatrix}$, then the differential equation modeling Part II becomes

$$X'_{IV} = F_{IV}(X_{IV}) \tag{5}$$

Again, we need to investigate the dependence of $S(t)$ on the parameters Z_0, I_0, R_0, and $c$by initializing Z_0, I_0, R_0, and c with different values respectively. Assign Z_0 with 6 distinct values ranging from 0 to S_0, with equally-spaced interval; I_0 and R_0, respectively, be 11 distinct values ranging from 0 to $2 \cdot S_0$ with equally-spaced interval; c be 11 distinct values ranging from 0 to 1, with equally spaced interval. Set the time step to $h = 10^{-10}$, the time interval to $tspan = [0, 10^{-4}]$, and the initial value to $X_{IV}(0) = \begin{pmatrix} S_0, & I_0, & Z_0, & R_0 \end{pmatrix}^T$. Apply Heun's method (1) to (5) to get the solutions of different initial conditions, with different colors corresponding to different Z_0. Notice that the lines of the same color basically overlap with each other, indicating that changes in I_0, R_0, and $c$do not cause significant change. Thus, the behavior of $S(t)$ is dominated by Z_0 and independent of the parameter c..

To further evaluate how treatment affect the human population $S(t)$, we plot the human population $S(t)$ in different parts on the same figure. Fix $I_0 = S_0$, $R_0 = S_0$, $Q_0 = S_0$, $\kappa = 0.1$, $\gamma = 0.1$, $\sigma = 0.1$, and $c = 0.1$, as they are not significantly influencing the behavior of $S(t)$, and assign Z_0 with different values. The solution to each part is shown in Figure 9, with solid line representing Part IV, dashed line representing Part II, dash-dot line representing Part IV, and dotted line representing Part I. From the figure we can see the dashed line, dash-dot line, and solid line basically overlap with each other, indicating treatment does not have a huge influence on $S(t)$.



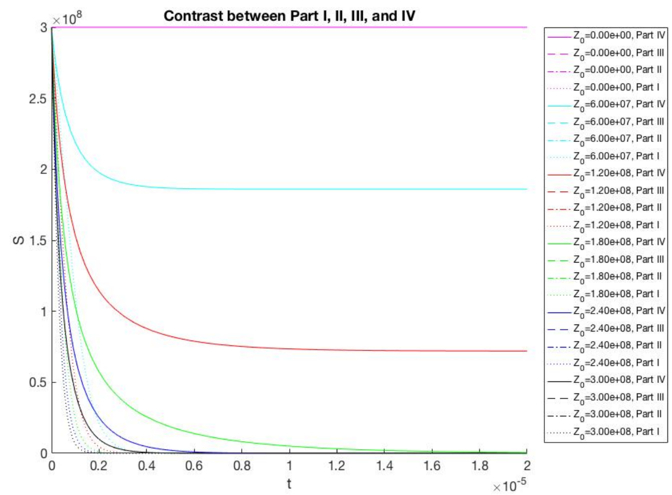


Figure 9: Population Change with Different Values of Z_0, I_0, R_0, and c

Figure 10: Population Change for Part I, II, III, and IV