STOR 435 Homework 19

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1. $n = 46$; $p_1 = p_3 = p_5 = p_7 = p_8 = p_{10} = p_{12} = \frac{31}{365}$, $p_4 = p_6 = p_9 = p_{11} = \frac{30}{365}$, $p_2 = \frac{28}{365}$

2.

a) Apparently (X_1, X_2, X_3) obeys a multinomial distribution, with

$n = 150$, $p_1 = p^2$, $p_2 = 2p(1-p)$, $p_3 = (1-p)^2$

Therefore, the pmf for (X_1, X_2, X_3) is:

$$p_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{150!}{x_1!x_2!x_3!} p^{2x_1}(2p(1-p))^{x_2}(1-p)^{2x_3} = \frac{150!}{x_1!x_2!x_3!} 2^{x_2} p^{2x_1+x_2}(1-p)^{2x_3+x_2}$$

b) $(X_1 + X_2, X_3)$ also obeys a multinomial distribution, with

$n = 150$, $p_{1+2} = p^2 + 2p(1-p) = p(2-p)$, $p_3 = (1-p)^2$

Therefore, the pmf for $(X_1 + X_2, X_3)$ is:

$$p_{X_1+X_2,X_3}(a,b) = \frac{150!}{a!b!}(p(2-p))^a(1-p)^{2b}$$

c) $g(p) = p_{X_1,X_2,X_3}(45,80,25) = \frac{150!}{45!80!25!} p^{90}(2p(1-p))^{80}(1-p)^{50} = \frac{150!2^{80}}{45!80!25!} p^{170}(1-p)^{130}$

$=52012508718072112444317858605528438647415157500705058713440669889468678679063333\backslash$
$47481600\,(1-p)^{130}p^{170}$

d) Let $c = \frac{150!2^{80}}{45!80!25!}$, $\frac{\partial}{\partial p} g(p) = c\frac{\partial}{\partial p} p^{170}(1-p)^{130} = c(170 - 300p)p^{169}(1-p)^{129}$.

Let $\hat{p} := \underset{p\in[0,1]}{\operatorname{argmax}}(g(p))$.

We observe that $g(0) = g(1) = 0$, and that when $p \in (0,1)$, $g(p) > 0$. We also observe that $\frac{\partial}{\partial p} g(p)$ has only one zero on $(0,1)$, and therefore, $\frac{\partial}{\partial p} g(\hat{p}) = 0$.

$$\frac{\partial}{\partial p} g(\hat{p}) = 0 \Leftrightarrow c(170 - 300\hat{p})\hat{p}^{169}(1-\hat{p})^{129} = 0 \Leftrightarrow 170 - 300\hat{p} = 0 \Leftrightarrow \hat{p} = \frac{17}{30}$$

3.

a) $\int_0^1 \int_0^y c\,x^2y^2\,dx\,dy = \int_0^1 \frac{c}{3} y^5 dy = \frac{c}{18} = 1 \Leftrightarrow c = 18$

b) $f_X(x) = \begin{cases} \int_x^1 18x^2y^2 dy = 6x^2 - 6x^5 & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$

c) $f_Y(y) = \begin{cases} \int_0^y 18x^2y^2 dx = 6y^5 & \text{if } 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$

d) Generally, $f(x,y) \neq f_X(x) f_Y(y)$, and therefore, they are not independent.

4.

a) $f_X(x) = \begin{cases} \int_0^1 x + y dy = x + \frac{1}{2} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

b) $f_Y(y) = \begin{cases} \int_0^1 x + y dx = y + \frac{1}{2} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

c) Generally, $f(x, y) \neq f_X(x) f_Y(y)$, and therefore, they are not independent.

5.

a) $f(r, s) = f_R(r) f_S(s) = 1$

b) $P(h(x) \text{has real roots}) = P(S^2 - 4R \geqslant 0)$

We first find probability distribution of S^2 and $-4R$:

$F_{S^2}(a) = F_S(\sqrt{a}) = \sqrt{a},\ a \in [0, 1]$

$f_{S^2}(a) = \frac{1}{2} a^{-\frac{1}{2}},\ a \in [0, 1]$

$F_{-4R}(a) = 1 - F_R\left(-\frac{a}{4}\right) = 1 + \frac{a}{4},\ a \in [-4, 0]$

$f_{-4R}(a) = \frac{1}{4},\ a \in [-4, 0]$

$$f_{S^2 - 4R}(a) = \int_{-\infty}^{\infty} f_{S^2}(t) f_{-4R^2}(a - t) dt = \begin{cases} \int_0^{a+4} \frac{1}{2} t^{-\frac{1}{2}} \frac{1}{4} dt = \frac{1}{4}(a + 4)^{\frac{1}{2}} & \text{if } a \in [-4, -3] \\ \int_0^1 \frac{1}{2} t^{-\frac{1}{2}} \frac{1}{4} dt = \frac{1}{4} & \text{if } a \in [-3, 0] \\ \int_a^1 \frac{1}{2} t^{-\frac{1}{2}} \frac{1}{4} dt = \frac{1}{4} - \frac{1}{4} a^{\frac{1}{2}} & \text{if } a \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$P(S^2 - 4R \geqslant 0) = \int_0^1 \frac{1}{4} - \frac{1}{4} a^{\frac{1}{2}} da = \frac{1}{12}$