Investigation of Robustness of a Mathematical Model

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Introduction

This essay investigates the robustness of a mathematical model used to characterize the mechanical properties of a sample of soft tissue extracted from a pig heart. We will consider a pure extraction experiment and use bisection method to calculate the stretch of the tissue from the model and compare with the actual values in the experiment to test the model.

Statement of the Problem

A sample of soft tissue is extracted from a pig heart muscle in order to characterize its mechanical properties. After treatments the sample is found to behave as an incompressible isotropic nonlinear elastic material characterized by the following stress tensor

$$\boldsymbol{P} = 2W_1\boldsymbol{F} - p\boldsymbol{F}^{-T}, \tag{1}$$

where $\boldsymbol{F}$ is a tensor (in this case a 3×3 matrix) that defines the deformation, p is the pressure in the tissue, $W_1 = \frac{a}{2}(e^{b(I_1-3)})$, $a = 0.5$kPa and $b = 10$ are parameters and $I_1 = \mathrm{trace}(\boldsymbol{F}^T\boldsymbol{F})$.

We want to check how good the mathematical model (1) is, with respect to the experiments. We consider a pure traction experiment, where the sample is assumed to have a cylindrical shape and it is pulled from one of the flat faces. In this case, the pressure field can be eliminated and the resulting force balance for model (1) can be written as

$$\tfrac{a}{2}(e^{b(\lambda^2+\lambda/2-3)})\big(\lambda - \tfrac{1}{\lambda^2}\big) = f \tag{2}$$

where f is the force (per unit area) applied on the tissue, and λ is the amount of stretch in the direction of the pull. The experimental results are shown in Tables 1 and 2.

f[kPa]	-1.5	-0.5	-0.22	-0.08	0	0.08	0.18	0.37	0.73

Table 1. Forces applied to the tissue sample in the experiment

λ	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2

Table 2. Computed values of the stretch in the experiment

Numerical Methods

We evaluate the robustness of the model by calculating the theoretical value of λ, the stretch, given by model (2) using the values of f, the force applied to the sample, shown in table 1, and comparing them with the corresponding actual values of λ shown in table 2.

We calculate the theoretical values of λ by numerically finding the root of function (3) derived from model (2) with different values of f.

$$g(\lambda) = \tfrac{a}{2}(e^{b(\lambda^2+\lambda/2-3)})\big(\lambda - \tfrac{1}{\lambda^2}\big) - f \tag{3}$$

We choose bisection as the numerical method for finding the roots. The reasons are listed below.

First, we can see from Figure 1, the plot of model (2) and different values of f, that for each of the f values, there is only one root in the selected interval, which is required for bisection method to find the correct root we are looking for.

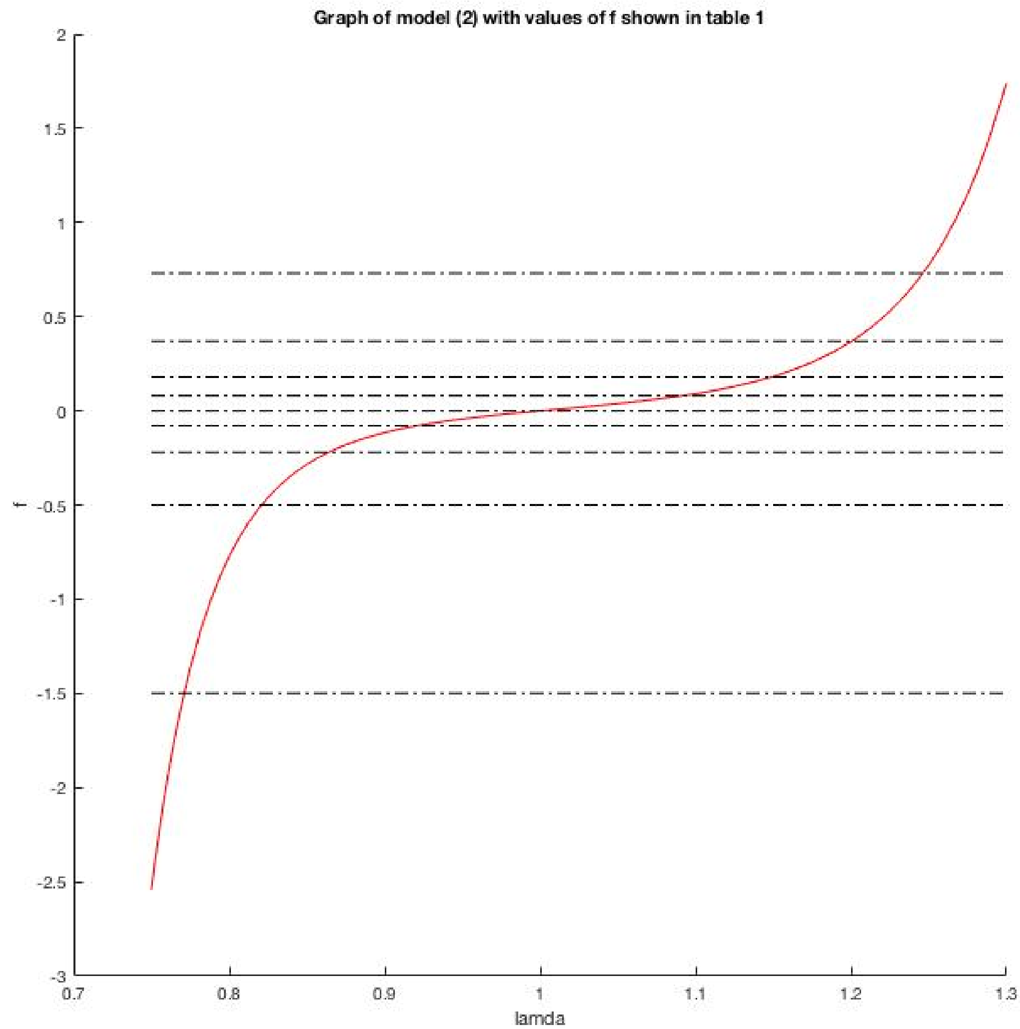

Figure 1. Model (2), f vs. λ

Second, the data obtained from the experiment have only one or two significant digits, and the calculated results need only as much accuracy as the experiment data. Also taking into account the fact that the volume of data is considerably small, we do not care much about the time needed for the computation, and thus the order or speed of convergence of the numerical method, which makes the guaranteed linear convergence of bisection method completely satisfy our need.

Third, some other popular root-finding numerical methods, such as Newton's method, secant method and fixed-point iteration methods, require addtional properties of the function and starting point to guarantee the convergence and give the desired result. We do not need to waste time on verify those properties and as stated in the previous paragraph, bisection method completely satisfies our need.

We then apply the bisection method to function (3) with starting interval $(0.75, 1.3)$, and the tolerance 0.0001. The tolerance is set to 0.0001 to guarantee the second decimal place, i.e. the last digit of experiment data, is not lost through rounding, so that we can make reasonable comparisons between the values.

Results

The calculated results are shown below in table 3, after rounding to the fourth decimal place.

λ	0.7710	0.8208	0.8639	0.9200	1.0000	1.0916	1.1490	1.2005	1.2462

Table 3. Computed values of the stretch from the model

Since the calculated results cannot have more significant digits than the original data, we round the values of λ to the same significant digit of corresponding values of f. The results are shown below in table 4.

λ	0.77	0.8	0.86	0.9	1	1	1.1	1.2	1.2

Table 4. Computed values of the stretch from the model after rounding

Discussion/Conclusions

We compare the calculated and actual values of λ by ploting the data from table 2 and table 4 below in figure 2.

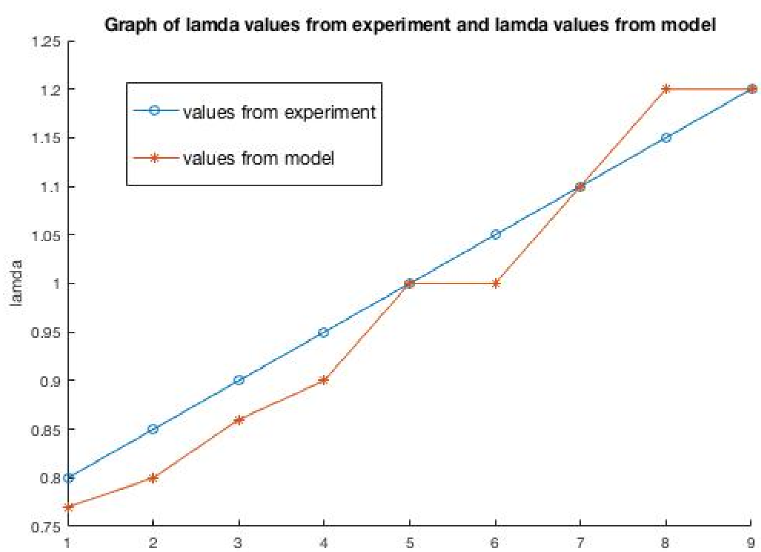

Figure 2. λ values from experiment and λ values from model

From figure 2, we can see that the calculated values from model (2) are close to the corresponding actual values, and some are even the same. Also taking into consideration the relatively low accuracy of the data, we can say the results are resonable approximations to the actual data.

We further investigate the behavior of the model by inspecting the absolute error and relative error of the calculated values of λ. Here, the absolute error is defined as

$$\varepsilon = |\lambda_{\text{actual}} - \lambda_{\text{model}}|, \tag{4}$$

and the relative error is defined as

$$\delta = \left| \frac{\lambda_{\text{actual}} - \lambda_{\text{model}}}{\lambda_{\text{actual}}} \right|, \tag{5}$$

where ε is the absolute error, δ is the relative error, λ_{actual} is the λ value in the experiment, as shown in table 2, and λ_{model} is the λ value obtained from the model, as shown in table 4.

Note that the absolute error and relative error stated here are different from the concepts with the same name frequently used in describing the results of numerical methods. Here the absolute error and relative error are used to describe the model.

The results are listed below in table 5.

ε	0.03	0.05	0.04	0.05	0.00	0.05	0.00	0.05	0.00
δ	0.038	0.059	0.044	0.053	0.000	0.048	0.000	0.043	0.000

Table 5. The absolute and relative errors of λ

We can see that the errors are resonably small; especially, all absolute errors are smaller than 6%, and the average of relative error is 0.032, or 3.2%.

We also see that the behavior of the errors is considerably random with respect to the actual values of λ, as shown in figure 3, which furthur demonstrate the robustness of the model.

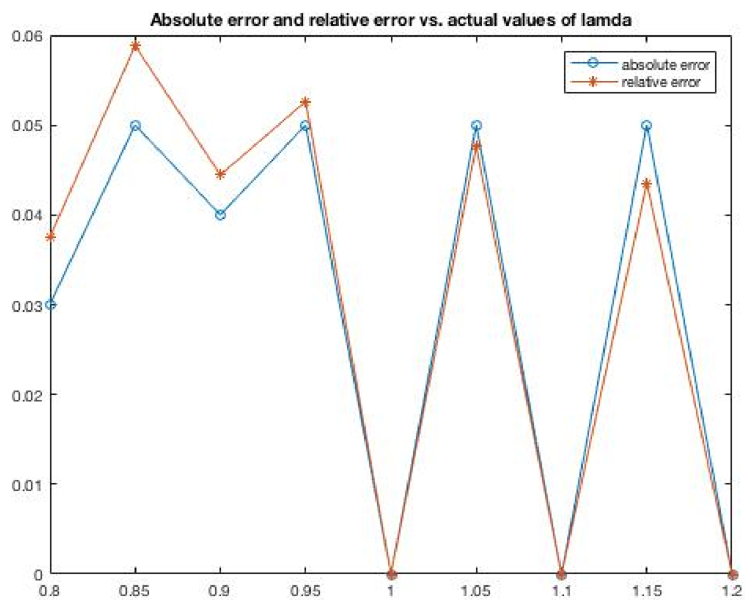

Figure 3. Absolute error and relative error of values of λ from model

We used bisection method to calculate the theoretical values of λ given by model (2) using the values of f shown in table 1, and compared them with the corresponding actual values of λ shown in table 2; the comparison showed that the results are good approximations to the actual values, as demonstrated in figure 2, and the errors are reasonably small and random, as demonstrated in table 5 and figure 3. In conclusion, model (2), or model (1) in this special case, i.e. where the sample is assumed to have a cylindrical shape, and is pulled from one of the flat faces, is a good model to characterize the mechanical properties of the sample of soft tissue extracted from a pig heart muscle.