WEDNESDAY WEEK 2 NOTES

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1. Equivalence Relations

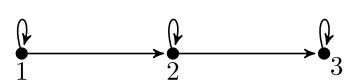
1.1. **Types of Relations.** There are four types of relations that we will focus on, and three of them will be very important in the next section.

1.1.1. *Reflexive.*

Definition. A relation R on a set X is called *reflexive* if $\forall x \in X$, $(x, x) \in R$. That is, every $x \in X$ is related to itself.

Remark. Using what we know about negation, we can say that a relation is NOT reflexive if $\exists x \in X$ such that $(x, x) \notin R$.

Example.
(1) If $X = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 3), (2, 2), (3, 3)\}$, then R is reflexive. To verify this, just go through each element x of X and verify that $(x, x) \in R$. We can also see this graphically. The graph of this relation is:



The relation R being reflexive means every vertex has a loop to itself.
(2) If $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (b, d), (b, b), (c, a), (d, d)\}$, then R is not reflexive as $(c, c) \notin R$.

Problem. Suppose we define a relation on $\mathbb{Z}$ by saying $x \sim y$ if $3|(x - y)$. Is this relation reflexive?

Solution. We need to see if $x \sim x$ for any $x \in \mathbb{Z}$. So let $x \in \mathbb{Z}$. The statement $x \sim x$ is equivalent to asking whether $3|(x - x)$, or $3|0$. This is true as $3 \cdot 0 = 0$. Therefore, $x \sim x$, and since $x \in \mathbb{Z}$ was arbitrary, we have proved this relation is reflexive.

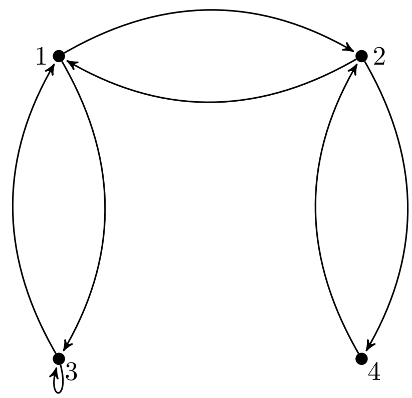
1.1.2. *Symmetric.*

Definition. A relation R on a set X is called *symmetric* if $\forall x \in X\ \forall y \in X\ (x, y) \in R \implies (y, x) \in R$.

Remark. Again, we can negate this statement to find that R is not symmetric if $\exists x \in X\ \exists y \in X$ with $(x, y) \in R \wedge (y, x) \notin R$.

Example.
(1) Let $X = \{1, 2, 3, 4\}$ and define a relation by $x \sim y$ if $x|y$. This is not symmetric, as $(2, 4) \in R$ and $(4, 2) \notin R$.
(2) If $X = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 1), (3, 1), (3, 3), (1, 3), (4, 2), (2, 4)\}$. This is symmetric. To verify this, one could go through and check each pair of elements and verify that we can switch the order and still remain in R. For example, $(1, 2) \in R$ and $(2, 1) \in R$. We also have $(3, 1) \in R$ and

$(1, 3) \in R$, and so on. We could also, again, look at the graph.



One can tell symmetry from the graph if every pair of vertices either has zero edges or two edges (one in each direction) between them. So the fact that 3 and 4 have no edges between them is ok, but since 3 has an edge going to 1, we must verify that 1 also has an edge going to 3, and so on.

Problem. Consider the relation on $\mathbb{Z}$ defined previously: $x \sim y$ if $3|(x-y)$. Prove this relation is symmetric.

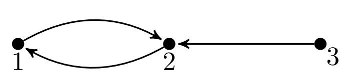
Solution. Suppose $x, y \in \mathbb{Z}$ with $x \sim y$. We want to show $y \sim x$. Since $x \sim y$, we know there exists $k \in \mathbb{Z}$ with $3k = x - y$. But then $y - x = 3(-k)$, meaning $3|(y - x)$. Therefore $y \sim x$, as desired. Therefore, the relation is symmetric.

1.1.3. *Antisymmetric.*

Definition. A relation R on a set X is called *antisymmetric* if $\forall x \in X\ \forall y \in X\ ((x, y) \in R \wedge (y, x) \in R) \implies x = y$. That is, the only way two elements can each be related to the other is if they are the same element.

Remark. A relation is not antisymmetric if $\exists x \in X\ \exists y \in X$ with $(x, y) \in R \wedge (y, x) \in R$ but $x \neq y$.

Example.
(1) The relation in example (1) of 6.2.2 is antisymmetric. Indeed, if $x \sim y$ and $y \sim x$, then $x|y$ and $y|x$. But the only way this can happen is if $x = y$.
(2) If $X = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (3, 2)\}$, then R is not antisymmetric: take $x = 1$ and $y = 2$ in the remark above. We could also tell from the graph:



The problem with this graph is that there are two edges between 1 and 2. A relation is antisymmetric if there is at most one edge between any two distinct vertices.

It is important to note that symmetric and antisymmetric are not opposites of one another. Example (2) above is an example of a relation which is neither symmetric nor antisymmetric. The relation on $\mathbb{Z}$ given by $x \sim y$ if $x = y$ is both symmetric and antisymmetric.

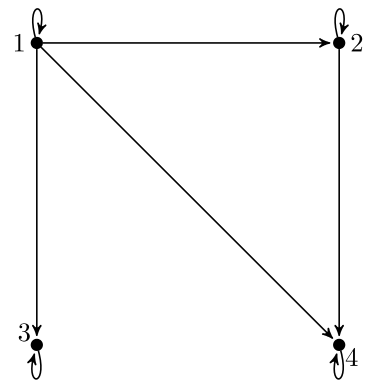
1.1.4. *Transitive.* This is the last property that we will need.

Definition. A relation R on a set X is *transitive* if $\forall x \in X\ \forall y \in X\ \forall z \in X\ ((x, y) \in R \wedge (y, z) \in R) \implies (x, z) \in R$.

Example.
(1) Let $X = \{1, 2, 3, 4\}$ and let R be the relation $x \sim y$ if $x|y$ as before. Observe that

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}.$$

This is transitive. For example, $(1,2) \in R$ and $(2,4) \in R$, and we check that $(1,4) \in R$ as well. The graph of this relation is:



Transitivity translates to: if there is an edge from x to y and an edge from y to z, then there is an edge from x to z.

(2) Again look at the relation on $\mathbb{Z}$ defined by $x \sim y$ if $3|(x-y)$. This relation is transitive. If $x \sim y$ and $y \sim z$, then there exist m, n with $3m = x - y$ and $3n = y - z$. But then

$$x - z = (x-y) + (y-z) = 3m + 3n = 3(m+n),$$

which means $3|(x-z)$, as desired. Therefore, the relation is transitive.

1.2. **Problems.** Let's look at a few problems.

Problem. Define a relation on $\mathbb{R}$ by saying $(x,y) \in R$ if $x \leq y$. Determine whether this relation is reflexive, symmetric, antisymmetric, transitive, or some combination of these.

Solution. This relation is reflexive as $x \leq x$ for any $x \in \mathbb{R}$. It is not symmetric though: $x \leq y$ does not mean $y \leq x$. For example, $4 \leq 5$ but $5 \not\leq 4$. It is antisymmetric: if $x \leq y$ and $y \leq x$, then $x = y$, which is the exact definition of antisymmetry. And finally, it is transitive. If $x \leq y$ and $y \leq z$, then $x \leq z$. Therefore, the relation is reflexive, antisymmetric, and transitive.

Problem. Same problem as above for the relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Q}$.

Solution. This relation is reflexive as $x - x = 0 \in \mathbb{Q}$. It is symmetric as well: take $x, y \in \mathbb{R}$. If $x \sim y$, then $x - y = k \in \mathbb{Q}$. But then $y - x = -k \in \mathbb{Q}$, so $y \sim x$ as well. Therefore the relation is symmetric. It is not antisymmetric, $(0,1) \in R$ and $(1,0) \in R$ but $1 \neq 0$. Lastly, it is transitive. If $x, y, z \in \mathbb{R}$ with $x \sim y$ and $y \sim z$, then $x - y = k \in \mathbb{Q}$ and $y - z = k' \in \mathbb{Q}$. But then $x - z = k + k' \in \mathbb{Q}$. Therefore, $x \sim z$ and the relation is transitive.

Problem. If R and S are relations on X, then we can examine the relation $R \cap S$. If R and S are both transitive, prove $R \cap S$ is transitive.

Solution. Suppose $x, y, z \in X$ with $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$. We want to show $(x,z) \in R \cap S$. Since $(x,y) \in R \cap S$, we know $(x,y) \in R$ and $(x,y) \in S$, and similarly for (y,z). But as R is transitive, this implies $(x,z) \in R$. Similarly, as S is transitive, we know $(x,z) \in S$. Therefore $(x,z) \in R \cap S$, and so $R \cap S$ is transitive.

Remark. There is nothing special about transitivity in the previous example. If R and S both have property P (where P is one of the four properties examined above), then $R \cap S$ will also have property P.

As an illustration of this intersection, let R be the relation on $\mathbb{Z}$ defined by $(x,y) \in R$ if $2|(x-y)$. Let S be the relation on $\mathbb{Z}$ given by $(x,y) \in S$ if $3|(x-y)$. What is $R \cap S$? If $(x,y) \in R \cap S$, then $2|(x-y)$ and $3|(x-y)$, so $6|(x-y)$. The claim is that $R \cap S$ is precisely this set, namely,

$$R \cap S = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 6|(x-y)\}.$$

We have proved the inclusion $R \cap S \subseteq \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 6|(x-y)\}$. For the reverse, assume $(a,b) \in \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 6|(x-y)\}$. We want $(a,b) \in R \cap S$. Since $6|(b-a)$, we know $2|(b-a)$, so $(a,b) \in R$. We also know $3|(b-a)$, so $(a,b) \in S$. Therefore, $(a,b) \in R \cap S$, as desired.

2. Equivalence Relations

2.1. Definition and Examples. We now turn our attention to a very specific type of relation, known as an equivalence relation. This concept comes up repeatedly in math classes in every area of math, so we devote an entire section to it.

Definition. A relation R on a set X is called an *equivalence relation* if R is reflexive, symmetric, and transitive.

Since equivalence relations are important, we will look at several examples.

Example. Let R be the relation on $\mathbb{R}$ defined by $x \sim y$ if $x - y \in \mathbb{Z}$. To check this is an equivalence relation, we need to check the three properties:

(1) *Reflexive*: Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$, meaning $x \sim x$, which is what we wanted. Therefore, R is reflexive.
(2) *Symmetric*: Let $x, y \in \mathbb{R}$ with $x \sim y$. We want to show $y \sim x$. Since $x \sim y$, we know $x - y = k \in \mathbb{Z}$. But then $y - x = -k \in \mathbb{Z}$. Therefore, $y \sim x$, and R is symmetric.
(3) *Transitive*: Let $x, y, z \in \mathbb{R}$ with $x \sim y$ and $y \sim z$. We want to show $x \sim z$. Since $x \sim y$, we know $x - y = k \in \mathbb{Z}$, and since $y \sim z$, we know $y - z = l \in \mathbb{Z}$. But then
$$x - z = (x - y) + (y - z) = k + l \in \mathbb{Z}.$$
Thus, $x \sim z$, and R is transitive.

Therefore, R is an equivalence relation.

Example. Fix a positive integer n. We define a relation R on $\mathbb{Z}$ by saying $a \sim b$ if $n|(a - b)$. We saw in the previous section (with $n = 3$) that this relation is reflexive, symmetric, and transitive. There was nothing special about 3 in those examples. Replace the 3 by an n and the same work shows this is an equivalence relation for any $n \in \mathbb{Z}$.

Example. Let S and T be sets, and let $f : S \to T$ be a function. Define a relation R on S by saying $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. This relation is clearly reflexive and symmetric. If $x_1 \sim x_2$ and $x_2 \sim x_3$, then $f(x_1) = f(x_2)$, and $f(x_2) = f(x_3)$, and therefore $f(x_1) = f(x_3)$, meaning $x_1 \sim x_3$. Thus, R is transitive, making it an equivalence relation.

2.2. Equivalence Classes.

Definition. Let R be an equivalence relation on a set X. For $a \in X$, define the *equivalence class* of a to be the set of all elements equivalent to a. It is denoted $[a]$. In other words,
$$[a] = \{x \in X : x \sim a\}.$$

Example. In the first example above, we have
$$[0] = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \mathbb{Z},$$
$$[2] = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \mathbb{Z},$$
$$[1.1] = \{\ldots, -3.9, -2.9, -1.9, -.9, .1, 1.1, 2.1, \ldots\}.$$
How do we get these? So see what $[0]$ is, we need to see which elements are equivalent to zero. We know $x \sim 0$ if and only if $x - 0 \in \mathbb{Z}$, but this just means $x \in \mathbb{Z}$. This is why $[0] = \mathbb{Z}$.

For the second one, we have that $x \sim 2$ precisely when $x - 2 \in \mathbb{Z}$. But if $x - 2 \in \mathbb{Z}$, then $x \in \mathbb{Z}$ as well, which explains the second class.

Lastly, $x \sim 1.1$ if $x - 1.1 \in \mathbb{Z}$. This means $x = 1.1 + k$ for $k \in \mathbb{Z}$. This is how we get the last set.

Example. If $n = 3$ in the second example, how can we describe $[2]$? Well,
$$[2] = \{\ldots, -1, 2, 5, 8, 11, \ldots\}.$$
Taking a closer look, we see that $[2]$ is just the set of integers which have a remainder of 2 when divided by 3. More generally, $[a]$ will be the set of integers which have the same remainder as a when divided by 3.
If n is just an arbitrary integer, then $[a]$ will be the set of integers which share the same remainder as a when divided by n.

Taking an even closer look at the relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Z}$, observe $[0] = [2]$. Why is that? It happened because $0 \sim 2$ under R, so that anything equivalent to 2 is equivalent to 0 (by transitivity), and vice versa. This is a more general fact.

Lemma. If R is an equivalence relation on a set S, then for any $a, b \in S$, $[a] = [b]$ if and only if $a \sim b$.

Proof. Let $a, b \in S$. If $[a] = [b]$, then as $a \in [a]$, we know $a \in [b]$, and thus $a \sim b$.

Conversely, let $a \sim b$. Then $[a] \subseteq [b]$ because if $c \in [a]$, then $c \sim a$, and since $a \sim b$, we know by transitivity that $c \sim b$. Thus $c \in [b]$. The reverse inclusion is similar. Thus, $[a] = [b]$, as desired. □

Consider the relation on $\mathbb{Z}$ given by $a \sim b$ if $5|(b - a)$, an equivalence relation as above. Then we know $[2] = [7]$ since $2 \sim 7$, but $[2] \neq [1]$ as $2 \not\sim 1$.

Definition. Let R be an equivalence relation on a set S. If we consider the equivalence class $[a]$, then a is called a *representative* for the equivalence class.

Remark. All the work above shows that there could be many representatives for the same class. For example, if R on $\mathbb{R}$ is the relation $x \sim y$ if $x - y \in \mathbb{Z}$, then every equivalence class has infinitely many representatives. Take the class $[0]$. We know $[0] = [1]$, so 1 is a representative for the same class. In fact, any integer k can be a representative for the class, as $[0] = [k]$ for any $k \in \mathbb{Z}$. In this example, there are infinitely many possible representatives.

2.3. **Quotients.**

Definition. Let X be a set and R an equivalence relation on X. Then the set of equivalence classes under R is denoted X/R or $X/\sim$. We call this set X *modulo* R or the *quotient of* X *by* R.

Example. If R is the relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Z}$, then what is $\mathbb{R}/\sim$? We know that two real numbers define the same equivalence class if they differ by an integer. Notice that every class has a representative which lies in the interval $[0, 1)$. For example, the class $[5.1234]$ is the same as the class $[.1234]$. Moreover, if $a, b \in [0, 1)$, then $[a] \neq [b]$, since there is no way a and b can differ by an integer (as they are less than distance 1 from one another). Thus,

$$\mathbb{R}/R = \{[c] : c \in [0, 1) \subset \mathbb{R}\}.$$

Example. Fix a positive integer n. Let R be the relation on $\mathbb{Z}$ given by $x \sim y$ if $n|(x - y)$. We typically denote $\mathbb{Z}/R$ by $\mathbb{Z}/n$ to emphasize that it depends on n. What is $\mathbb{Z}/n$? The claim is that

$$\mathbb{Z}/n = \{[0], [1], [2], \ldots, [n - 1]\}.$$

To see this, notice that every equivalence class has some representative between 0 and $n - 1$ (just take the remainder when divided by n). On the other hand, all these classes are inequivalent since the difference between any two of the representatives is less than n, and hence cannot be divisible by n.

We'll investigate equivalence classes more next time.