MONDAY WEEK 2 NOTES

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1. Sequences and Strings

In this section, we will study sequences, which you typically encounter in a Calculus BC course, and strings, which is a special type of functions.

1.1. **Sequences.**

Definition. A *sequence* is a function $f : D \to S$, where $D \subset \mathbb{Z}$ is a set of consecutive numbers and S is any set (typically $\mathbb{R}$).

Example.

(1) Let $D = \mathbb{N}$ and $S = \mathbb{R}$, and let $f : D \to S$ be the function $f(n) = \frac{1}{n}$. This is the sequence
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$
We sometimes write $f(n) = a_n = \frac{1}{n}$, and the terms of the sequence are $a_1, a_2, \ldots$ (since $n \in \mathbb{N}$).

(2) Let $D = \{1, 2, 3, 4\}$ and $S = \mathbb{N}$. Let $f : D \to S$ be $f(1) = 1$, $f(2) = 3$, $f(3) = 3$, and $f(4) = 2$. This is the sequence
$$1, 3, 3, 2.$$

Definition. A sequence is called *finite* if D is finite.

Since you study sequences in depth in calculus, we will not go over them in much detail here.

1.2. **Strings.**

Definition. Let X be a set. A *string over* X is a finite sequence $f : D \to X$ (meaning D is finite), where we take $D = \{1, 2, 3, \ldots, n\}$.

Example.

(1) $X = \{a, b, c\}$, $D = \{1, 2, 3, 4, 5\}$, and $f : D \to X$ given by the sequence a, a, b, c, b.

(2) $X = \{x, y, z, w\}$, $D = \{1, 2, 3, 4, 5, 6\}$, $g : D \to X$ given by x, y, y, z, w, x.

We typically write strings as sequences of characters without any commas. So the first sequence above would be $aabcb = a^2bcb$ (here the a^2 means 2 consecutive a's). The second string can be written as $xyyzwx = xy^2zwx$.

If $D = \emptyset$, then the unique sequence $D \to X$ is denoted λ (the null string).

Notation. The set of all strings over X is denoted X^*, and the set of all non-null strings over X is denoted X^+.

One more piece of terminology:

Definition. A *bit string* is a string on the set $X = \{0, 1\}$.

1.3. **Operations on Strings.**

Like all objects we have encountered so far, there are operations we can perform.

1.3.1. *Length.*

Definition. The *length* of a string $f : D \to X$ over X is $|D|$, and it is denoted $|f|$.

This should be pretty self-explanatory. We list a few examples:

Example.

(1) If $\alpha = aabab = a^2bab$ is a string over $X = \{a, b, c\}$, then $|\alpha| = 5$.

(2) If $\beta = a^3b^4a^{32}$ over the same set X, then $|\beta| = 39 = 3 + 4 + 32$.

1.3.2. *Concatenation.*

Definition. Let α and β be two strings over X. The *concatenation* is denoted $\alpha\beta$, and it is the string α followed by the string β. Formally, if $\alpha : \{1, 2, \ldots, n\} \to X$ and $\beta = \{1, 2, 3, \ldots, m\} \to X$, then $\alpha\beta : \{1, 2, \ldots, m + n\} \to X$ is given by

$$(\alpha\beta)(i) = \alpha(i) \quad \text{for } i = 1, 2, \ldots, n$$

and

$$(\alpha\beta)(j) = \beta(j - n) \quad \text{for } j = n + 1 \ldots, n + m.$$

Remark. It is clear from the definition that the length of $\alpha\beta$ is $|\alpha\beta| = |\alpha| + |\beta|$.

Example. Let $X = \{a, b\}$. If $\alpha = aba^2$ and $\beta = babab^2$, then

$$\alpha\beta = aba^2babab^2.$$

Notice

$$\beta\alpha = babab^2aba^2 \neq \alpha\beta,$$

so concatenation is not commutative.

Remark. Observe that for any string α over X, we have

$$\lambda\alpha = \alpha\lambda = \alpha.$$

This means that λ is the "identity element" of strings.

Problem. Let X be any nonempty set. Let $f : X^* \times X^* \to X^*$ be the map $f(\alpha, \beta) = \alpha\beta$. Is f injective? Surjective? Bijective?

Solution. The map f being injective means $f(\alpha, \beta) = f(\alpha', \beta')$ implies $\alpha = \alpha'$ and $\beta = \beta'$. But this is not true. If $x \in X$, then the string $xxx = xx \cdot x = x \cdot xx$ can be decomposed in two distinct ways. Therefore f is not injective.

However, f is surjective. Take $\alpha \in X^*$. Then $\alpha = f(\alpha, \lambda)$, so f is surjective. The map f is not bijective as it is not injective.

1.3.3. *Reverse.*

Definition. If X is a set and α is a string on X, then the *reverse* string, denoted α^R is just the string α backwards. Formally, if $n = |\alpha|$, then

$$\alpha^R(i) = \alpha(n + 1 - i) \quad \text{for } i = 1, 2, \ldots, n.$$

Again, it is clear from the definition that $|\alpha^R| = |\alpha|$ and that $(\alpha^R)^R = \alpha$.

Example. If $\alpha = aba^2$ is a string on $X = \{a, b\}$, then $\alpha^R = aaba = a^2ba$.

Problem. Let X be a nonempty set, and let $f : X^* \to X^*$ be $f(\alpha) = a^R$. Prove f is a bijection.

Solution. To prove f is injective, let $f(\alpha) = f(\beta)$, so $\alpha^R = \beta^R$. Then applying the reverse operation to both sides gives $\alpha = \beta$, meaning f is injective.

To prove f is surjective, simply observe that $f(\alpha^R) = \alpha$. Thus, f is a bijection.

1.3.4. *Substrings.*

Definition. If α is a string over a nonempty set X, then a string β is a *substring* of α if there exist $\gamma, \delta \in X^*$ such that $\alpha = \gamma\beta\delta$.

Informally, a substring is just a subsequence of consecutive characters in the larger string.

Example. If $X = \{a, b, c\}$ and $\alpha = aabcbbac$, then $\beta = bcb$ is a substring with $\gamma = aa$ and $\delta = bac$.

Problem. Let X be a nonempty set, and let $f : X^* \to \mathcal{P}(X^*)$ be the map with $f(\alpha)$ being the set of all substrings of α. Is f injective? Surjective? Bijective?

Solution. Suppose $f(\alpha) = f(\beta)$. We want to see if $\alpha = \beta$. Well α is a substring of itself, and since $f(\alpha) = f(\beta)$, it must be a substring of β. Therefore $\beta = \gamma\alpha\delta$ for some strings γ and δ. By the same reasoning β is a substring of α, so there exists γ' and δ' such that $\alpha = \gamma'\beta\delta'$. Together, they say $|\alpha| = |\beta|$, and the only way this can happen is if $\gamma = \delta = \gamma' = \delta' = \lambda$. Thus, $\alpha = \beta$ and f is injective.

The map f is not surjective. Any set in $\mathcal{P}(X^*)$ which does not include λ cannot be in the image of f. Therefore, f is neither surjective nor bijective.

2. Relations

2.1. Definition and Examples.

Definition. A *binary relation* R from a set X to a set Y is a subset $R \subseteq X \times Y$.

Notation.
(1) If $(x, y) \in R$, then we write xRy or $x \sim y$.
(2) If $X = Y$, then we say R is a relation on X.

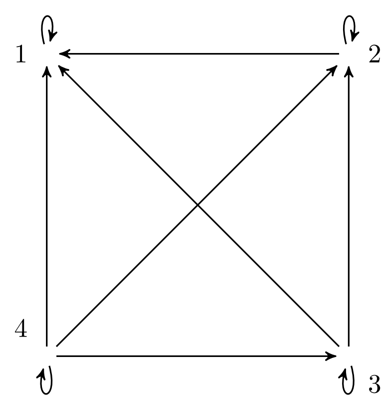
Example. Let $X = \{\text{Bill, Joe, Alice, Dave}\}$ and $Y = \{\text{Math, Physics, Chemistry, History}\}$. Then

$$R = \{(\text{Bill, Math}), (\text{Alice, Physics}), (\text{Bill, History}), (\text{Dave, Chemistry}), (\text{Joe, Physics})\} \subset X \times Y$$

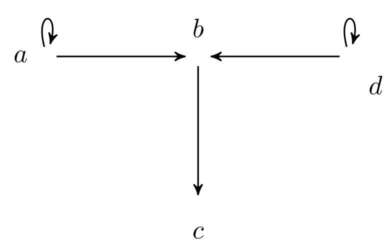
is an example of a relation from X to Y.

One could imagine that this relation encodes classes different students are taking. We could also define relations with rules:

Example.
(1) Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4, 5\}$. Then we can define a relation R by saying $x \sim y$ if $x|y$. We could list out R by writing all the ordered pairs:
$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$
(2) $X = \{1, 2, 3, 4\}$. Define a relation on X by writing $x \sim y$ if $x \geq y$. Thus,
$$R = \{(1, 1), (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2), (3, 3), (4, 3), (4, 4)\}.$$
We could also encode this information in the following diagram:



The vertices in the graph are just the elements of X, and there is an arrow from x to y precisely when $x \sim y$.
(3) Using the previous idea, if $X = \{a, b, c, d\}$, then the following illustrates a relation on X:



This is the relation:
$$R = \{(a, a), (a, b), (b, c), (d, b), (d, d)\}.$$