Chapter 3

Linear Systems of Differential Equations

Introduction

This chapter connects the linear algebra developed in Chapter 2 with Differential Equations. We define the matrix exponential in §1 and show how it produces the solution to first order systems of differential equations with constant coefficients. We show how the use of eigenvectors and generalized eigenvectors helps to compute matrix exponentials. In §2 we look again at connections between exponential and trigonometric functions, complementing results of Chapter 1, §1.

In §3 we discuss how to reduce a higher order differential equation to a first order system, and show how the "companion matrix" of a polynomial arises in doing this. We show in §4 how the matrix exponential allows us to write down an integral formula (Duhamel's formula) for the solution to a non-homogeneous first order system, and illustrate how this in concert with the reduction process just mentioned, allows us to write down the solution to a non-homogeneous second order differential equation.

Section 5 discusses how to derive first order systems describing the behavior of simple circuits, consisting of resistors, inductors, and capacitors. Here we treat a more general class of circuits than done in Chapter 1, §13.

Section 6 deals with second order systems. While it is the case that second order $n \times n$ systems can always be converted into first order $(2n) \times (2n)$ systems, many such systems have special structure, worthy of separate study. Material on self adjoint transformations from Chapter 2 plays an important role in this section.

In §7 we discuss the Frenet-Serret equations, for a curve in three-dimensional Euclidean space. These equations involve the curvature and torsion of a curve, and also a frame field along the curve, called the Frenet frame, which forms an orthonormal basis of $\mathbb{R}^3$ at each point on the curve. Regarding these equations as a system of differential equations, we discuss the problem of finding a curve with given curvature and torsion. Doing this brings in a number of topics from the previous sections, and from Chapter 2, such as the use of properties of orthogonal matrices.

Having introduced equations with variable coefficients in §7, we concentrate on their treatment in subsequent sections. In §8 we study the solution operator $S(t, s)$ to a homogeneous system, show how it extends the notion of matrix exponential, and extend Duhamel's formula to the variable coefficient setting. In §9 we show how the method of variation of parameters, introduced in Chapter 1, ties in with and becomes a special case of Duhamel's formula.

Section 10 treats power series expansions for a first order linear system with analytic coefficients, and §11 extends the study to equations with regular singular points. These sections provide a systematic treatment of material touched on in Chapter 1, §15. In these sections we use elementary power series techniques. Additional insight can be gleaned from the theory of functions of a complex variable. Readers who have seen some complex variable theory can consult [**Ahl**], pp. 299–312, [**Hart**], pp. 70–83, or [**T**], Vol. 1, pp. 30–31, for material on this.

Appendix A treats logarithms of matrices, a construction inverse to the matrix exponential introduced in §1, establishing results that are of use in §§8 and 11.

1. The matrix exponential

Here we discuss a key concept in matrix analysis, the matrix exponential. Given $A \in M(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we define e^A by the same power series used in Chapter 1 to define e^A for $A \in \mathbb{R}$:

$$e^A = \sum_{k=1}^{\infty} \frac{1}{k!} A^k. \tag{1.1}$$

Note that A can be a real or complex $n \times n$ matrix. In either case, recall from §10 of Chapter 2 that $\|A^k\| \leq \|A\|^k$. Hence the standard ratio test implies (1.1) is absolutely convergent for each $A \in M(n, \mathbb{F})$. Hence

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \tag{1.2}$$

is a convergent power series in t, for all $t \in \mathbb{R}$ (indeed for $t \in \mathbb{C}$). As for all such convergent power series, we can differentiate term by term. We have

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \sum_{k=1}^{\infty} k \frac{t^{k-1}}{k!} A^k \\ &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1} A. \end{aligned} \tag{1.3}$$

We can factor A out on either the left or the right, obtaining

$$\frac{d}{dt} e^{tA} = e^{tA} A = A e^{tA}. \tag{1.4}$$

Hence $x(t) = e^{tA} x_0$ solves the first-order system

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0. \tag{1.5}$$

This is the *unique* solution to (1.5). To see this, let $x(t)$ be any solution to (1.5), and consider

$$u(t) = e^{-tA} x(t). \tag{1.6}$$

Then $u(0) = x(0) = x_0$ and

$$\frac{d}{dt} u(t) = -e^{-tA} A x(t) + e^{-tA} x'(t) = 0, \tag{1.7}$$

so $u(t) \equiv u(0) = x_0$. The same argument yields

$$\frac{d}{dt}\Big(e^{tA} e^{-tA}\Big) = 0, \quad \text{hence} \quad e^{tA} e^{-tA} \equiv I. \tag{1.8}$$

Hence $x(t) = e^{tA} x_0$, as asserted.

Using a variant of the computation (1.7), we show that the matrix exponential has the following property, which generalizes the identity $e^{s+t} = e^s e^t$ for real s, t, established in Chapter 1.

Proposition 1.1. *Given $A \in M(n, \mathbb{C})$, $s, t \in \mathbb{R}$,*

$$e^{(s+t)A} = e^{sA} e^{tA}. \tag{1.9}$$

Proof. Using the Leibniz formula for the derivative of a product, plus (1.4), we have

$$\frac{d}{dt}\Big(e^{(s+t)A}e^{-tA}\Big) = e^{(s+t)A}Ae^{-tA} - e^{(s+t)A}Ae^{-tA} = 0. \tag{1.10}$$

Hence $e^{(s+t)A}e^{-tA}$ is independent of t, so

$$e^{(s+t)A}e^{-tA} = e^{sA}, \quad \forall\ s,t \in \mathbb{R}. \tag{1.11}$$

Taking $s = 0$ yields $e^{tA}e^{-tA} = I$ (as we have already seen in (1.8)) or $e^{-tA} = (e^{tA})^{-1}$, so we can multiply both sides of (1.11) on the right by e^{tA} and obtain (1.9).

Now, generally, for $A, B \in M(n, \mathbb{F})$,

$$e^A e^B \neq e^{A+B}. \tag{1.12}$$

However, we do have the following.

Proposition 1.2. *Given $A, B \in M(n, \mathbb{C})$,*

$$AB = BA \Longrightarrow e^{A+B} = e^A e^B. \tag{1.13}$$

Proof. We compute

$$\begin{aligned} &\frac{d}{dt}\Big(e^{t(A+B)}e^{-tB}e^{-tA}\Big) \\ &\quad = e^{t(A+B)}(A+B)e^{-tB}e^{-tA} - e^{t(A+B)}Be^{-tB}e^{-tA} - e^{t(A+B)}e^{-tB}Ae^{-tA}. \end{aligned} \tag{1.14}$$

Now $AB = BA \Rightarrow AB^k = B^kA$, hence

$$e^{-tB}A = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}B^kA = Ae^{-tB}, \tag{1.15}$$

so (1.14) vanishes. Hence $e^{t(A+B)}e^{-tB}e^{-tA}$ is independent of t, so

$$e^{t(A+B)}e^{-tB}e^{-tA} = I, \tag{1.16}$$

the value at $t = 0$. Multiplying through on the right by e^{tA} and then by e^{tB} gives

$$e^{t(A+B)} = e^{tA}e^{tB}. \tag{1.17}$$

Setting $t = 1$ gives (1.13).

We now look at examples of matrix exponentials. We start with some computations via the infinite series (1.2). Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.18}$$

Then

$$A^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix}, \quad B^2 = B^3 = \cdots = 0, \tag{1.19}$$

so

$$e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}, \quad e^{tB} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{1.20}$$

Note that A and B do not commute, and neither do e^{tA} and e^{tB}, for general $t \neq 0$. On the other hand, if we take

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = I + B, \tag{1.21}$$

since I and B commute, we have without further effort that

$$e^{tC} = e^{tI} e^{tB} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}. \tag{1.22}$$

We turn to constructions of matrix exponentials via use of eigenvalues and eigenvectors. Suppose v_j is an eigenvector of A with eigenvalue λ_j,

$$Av_j = \lambda_j v_j. \tag{1.23}$$

Then $A^k v_j = \lambda_j^k v_j$, and hence

$$e^{tA} v_j = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k v_j = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_j^k v_j = e^{t\lambda_j} v_j. \tag{1.24}$$

This enables us to construct $e^{tA}v$ for each $v \in \mathbb{C}^n$ if $A \in M(n, \mathbb{C})$ and $\mathbb{C}^n$ has a basis of eigenvectors, $\{v_j : 1 \le j \le n\}$. In such a case, write v as a linear combination of the eigenvectors,

$$v = c_1 v_1 + \cdots + c_n v_n, \tag{1.25}$$

and then

$$\begin{aligned} e^{tA} v &= c_1 e^{tA} v_1 + \cdots + c_n e^{tA} v_n \\ &= c_1 e^{t\lambda_1} v_1 + \cdots + c_n e^{t\lambda_n} v_n. \end{aligned} \tag{1.26}$$

We illustrate this process with some examples.

Example 1. Take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.27}$$

One has $\det(\lambda I - A) = \lambda^2 - 1$, hence eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = -1, \tag{1.28}$$

with corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.29}$$

Hence

$$e^{tA} v_1 = e^t v_1, \quad e^{tA} v_2 = e^{-t} v_2. \tag{1.30}$$

To write out e^{tA} as a 2×2 matrix, note that the first and second columns of this matrix are given respectively by

$$e^{tA} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e^{tA} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.31}$$

To compute this, we write $(1,0)^t$ and $(0,1)^t$ as linear combinations of the eigenvectors. We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.32}$$

Hence

$$\begin{aligned} e^{tA} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{2} e^{tA} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{tA} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{e^t}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{e^{-t}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(e^t + e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) \end{pmatrix}, \end{aligned} \tag{1.33}$$

and similarly

$$e^{tA} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t + e^{-t}) \end{pmatrix}. \tag{1.34}$$

Recalling that

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}, \tag{1.35}$$

we have

$$e^{tA} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \tag{1.36}$$

Example 2. Take

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}. \tag{1.37}$$

One has $\det(\lambda I - A) = \lambda^2 - 2\lambda - 2$, hence eigenvalues

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i, \tag{1.38}$$

with corresponding eigenvectors

$$v_1 = \begin{pmatrix} -2 \\ 1+i \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1-i \end{pmatrix}. \tag{1.39}$$

We have

$$e^{tA} v_1 = e^{(1+i)t} v_1, \quad e^{tA} v_2 = e^{(1-i)t} v_2. \tag{1.40}$$

We can write

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= -\frac{i+1}{4} \begin{pmatrix} -2 \\ 1+i \end{pmatrix} + \frac{i-1}{4} \begin{pmatrix} -2 \\ 1-i \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= -\frac{i}{2} \begin{pmatrix} -2 \\ 1+i \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -2 \\ 1-i \end{pmatrix}, \end{aligned} \tag{1.41}$$

to obtain

$$\begin{aligned} e^{tA} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= -\frac{i+1}{4} e^{(1+i)t} \begin{pmatrix} -2 \\ 1+i \end{pmatrix} + \frac{i-1}{4} e^{(1-i)t} \begin{pmatrix} -2 \\ 1-i \end{pmatrix} \\ &= \frac{e^t}{4} \begin{pmatrix} (2i+2)e^{it} + (2-2i)e^{-it} \\ -2ie^{it} + 2ie^{-it} \end{pmatrix}, \end{aligned} \tag{1.42}$$

and

$$\begin{aligned} e^{tA} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= -\frac{i}{2} e^{(1+i)t} \begin{pmatrix} -2 \\ 1+i \end{pmatrix} + \frac{i}{2} e^{(1-i)t} \begin{pmatrix} -2 \\ 1-i \end{pmatrix} \\ &= \frac{e^t}{2} \begin{pmatrix} 2ie^{it} - 2ie^{-it} \\ (1-i)e^{it} + (1+i)e^{-it} \end{pmatrix}. \end{aligned} \tag{1.43}$$

We can write these in terms of trigonometric functions, using the fundamental Euler identities

$$e^{it} = \cot t + i \sin t, \quad e^{-it} = \cos t - i \sin t, \tag{1.44}$$

established in §1 of Chapter 1. (See §2 of this chapter for more on this.) These yield

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}, \tag{1.45}$$

and an inspection of the formulas above gives

$$e^{tA}\begin{pmatrix}1\\0\end{pmatrix} = e^t\begin{pmatrix}\cos t - \sin t\\ \sin t\end{pmatrix}, \quad e^{tA}\begin{pmatrix}0\\1\end{pmatrix} = e^t\begin{pmatrix}-2\sin t\\ \cos t + \sin t\end{pmatrix}, \tag{1.46}$$

hence

$$e^{tA} = e^t\begin{pmatrix}\cos t - \sin t & -2\sin t\\ \sin t & \cos t + \sin t\end{pmatrix}. \tag{1.47}$$

As was shown in Chapter 2, §6, if $A \in M(n, \mathbb{C})$ has n distinct eigenvalues, then $\mathbb{C}^n$ has a basis of eigenvectors. If A has multiple eigenvalues, $\mathbb{C}^n$ might or might not have a basis of eigenvectors, though as shown in §7 of Chapter 2, there will be a basis of generalized eigenvectors. If v is a generalized eienvector of A, say

$$(A - \lambda I)^m v = 0, \tag{1.48}$$

then

$$e^{t(A-\lambda I)}v = \sum_{k<m} \frac{t^k}{k!}(A - \lambda I)^k v, \tag{1.49}$$

so

$$e^{tA}v = e^{t\lambda}\sum_{k<m} \frac{t^k}{k!}(A - \lambda I)^k v. \tag{1.50}$$

Example 3. Consider the 3×3 matrix A used in (7.26) of Chapter 2:

$$A = \begin{pmatrix}2 & 3 & 3\\ 0 & 2 & 3\\ 0 & 0 & 1\end{pmatrix}. \tag{1.51}$$

Here 2 is a double eigenvalue and 1 a simple eigenvalue. Calculations done in Chapter 2, §7, yield

(1.52)
$$(A-2I)\begin{pmatrix}1\\0\\0\end{pmatrix}=0,\quad (A-2I)\begin{pmatrix}0\\1\\0\end{pmatrix}=3\begin{pmatrix}1\\0\\0\end{pmatrix},\quad (A-I)\begin{pmatrix}6\\-3\\1\end{pmatrix}=0.$$

Hence

$$e^{tA}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}e^{2t}\\0\\0\end{pmatrix},\tag{1.53}$$

$$\begin{aligned}e^{tA}\begin{pmatrix}0\\1\\0\end{pmatrix}&=e^{2t}\sum_{k=0}^{1}\frac{t^k}{k!}(A-2I)^k\begin{pmatrix}0\\1\\0\end{pmatrix}\\&=e^{2t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}3t\\0\\0\end{pmatrix}\right],\end{aligned}\tag{1.54}$$

and

$$e^{tA}\begin{pmatrix}6\\-3\\1\end{pmatrix}=e^{t}\begin{pmatrix}6\\-3\\1\end{pmatrix}.\tag{1.55}$$

Note that

$$e^{tA}\begin{pmatrix}0\\0\\1\end{pmatrix}=e^{tA}\begin{pmatrix}6\\-3\\1\end{pmatrix}-6e^{tA}\begin{pmatrix}1\\0\\0\end{pmatrix}+3e^{tA}\begin{pmatrix}0\\1\\0\end{pmatrix}.\tag{1.56}$$

Putting these calculations together yields

$$e^{tA}=\begin{pmatrix}e^{2t}&3te^{2t}&6e^t-6e^{2t}+9te^{2t}\\0&e^{2t}&-3e^t+3e^{2t}\\0&0&e^t\end{pmatrix}.\tag{1.57}$$

Example 4. Consider the 3×3 matrix

$$A=\begin{pmatrix}1&2&0\\3&1&3\\0&-2&1\end{pmatrix}.\tag{1.58}$$

A computation gives $\det(\lambda I - A) = (\lambda - 1)^3$. Hence for $N = A - I$ we have $\operatorname{Spec}(N) = \{0\}$, so we know N is nilpotent (by Proposition 8.1 of Chapter 2). In fact, a calculation gives

$$N = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 3 \\ 0 & -2 & 0 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ -6 & 0 & -6 \end{pmatrix}, \quad N^3 = 0. \tag{1.59}$$

Hence

$$\begin{aligned} e^{tA} &= e^t \Big[I + tN + \frac{t^2}{2} N^2 \Big] \\ &= e^t \begin{pmatrix} 1+3t^2 & 2t & 1+3t^2 \\ 3t & 1 & 3t \\ 1-3t^2 & -2t & 1-3t^2 \end{pmatrix}. \end{aligned} \tag{1.60}$$

Exercises

1. Use the method of eigenvalues and eigenvectors given in (1.23)–(1.26) to compute e^{tA} for each of the following:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

2. Use the method given in (1.48)–(1.50) and illustrated in (1.51)–(1.60) to compute e^{tA} for each of the following:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

3. Show that

$$e^{t(A+B)} = e^{tA} e^{tB}, \quad \forall t \Longrightarrow AB = BA.$$

Hint. Set $X(t) = e^{t(A+B)}$, $Y(t) = e^{tA}e^{tB}$. Show that

$$X \equiv Y \Longrightarrow X'(t) - Y'(t) = Be^{t(A+B)} - e^{tA}Be^{tB} \equiv 0,$$

and hence that

$$X \equiv Y \Longrightarrow Be^{tA} = e^{tA}B, \quad \forall\, t.$$

4. Given $A \in M(n, \mathbb{C})$, suppose $\Phi(t)$ is an $n \times n$ matrix valued solution to

$$\frac{d}{dt}\Phi(t) = A\Phi(t).$$

Show that

$$\Phi(t) = e^{tA}B,$$

where $B = \Phi(0)$. Deduce that $\Phi(t)$ is invertible for all $t \in \mathbb{R}$ if and only if $\Phi(0)$ is invertible, and that in such a case

$$e^{(t-s)A} = \Phi(t)\Phi(s)^{-1}.$$

(For a generalization, see (8.13).)

5. Let $A, B \in M(n, \mathbb{C})$ and assume B is invertible. Show that

$$(B^{-1}AB)^k = B^{-1}A^kB,$$

and use this to show that

$$e^{tB^{-1}AB} = B^{-1}e^{tA}B.$$

6. Show that if A is diagonal, i.e.,

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix},$$

then

$$e^{tA} = \begin{pmatrix} e^{ta_{11}} & & \\ & \ddots & \\ & & e^{ta_{nn}} \end{pmatrix}.$$

Exercises 7–10 bear on the identity

$$\det e^{tA} = e^{t \operatorname{Tr} A}, \tag{1.61}$$

given $A \in M(n, \mathbb{C})$.

7. Show that if (1.61) holds for $A = A_1$ and if $A_2 = B^{-1}A_1B$, then (1.61) holds for $A = A_2$.

8. Show that (1.61) holds whenever A is diagonalizable.
Hint. Use Exercises 5–6.

9. Assume $A \in M(n, \mathbb{C})$ is upper triangular:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{pmatrix}. \tag{1.62}$$

Show that e^{tA} is upper triangular, of the form

$$e^{tA} = \begin{pmatrix} e_{11}(t) & \cdots & e_{1n}(t) \\ & \ddots & \vdots \\ & & e_{nn}(t) \end{pmatrix}, \quad e_{jj}(t) = e^{ta_{jj}}.$$

10. Deduce that (1.61) holds when A has the form (1.62). Then deduce that (1.61) holds for all $A \in M(n, \mathbb{C})$.

11. Let $A(t)$ be a smooth function of t with values in $M(n, \mathbb{C})$. Show that

$$A(0) = 0 \Longrightarrow \frac{d}{dt} e^{A(t)}\Big|_{t=0} = A'(0). \tag{1.63}$$

Hint. Take the power series expansion of $e^{A(t)}$, in powers of $A(t)$.

12. Let $A(t)$ be a smooth $M(n, \mathbb{C})$-valued function of $t \in I$ and assume

$$A(s)A(t) = A(t)A(s), \quad \forall\, s, t \in I. \tag{1.64}$$

Show that

$$\frac{d}{dt} e^{A(t)} = A'(t)e^{A(t)} = e^{A(t)}A'(t). \tag{1.65}$$

Hint. Show that if (1.64) holds,

$$\frac{d}{dt} e^{A(t)} = \frac{d}{ds} e^{A(s)-A(t)} e^{A(t)}\Big|_{s=t},$$

and apply Exercise 11.

13. Here is an alternative approach to Proposition 1.2. Assume

$$A, B \in M(n, \mathbb{C}), \quad AB = BA. \tag{1.66}$$

Show that

$$(A+B)^m = \sum_{j=0}^{m} \binom{m}{j} A^j B^{m-j}, \quad \binom{m}{j} = \frac{m!}{j!(m-j)!}. \tag{1.67}$$

From here, show that

$$\begin{aligned} e^{A+B} &= \sum_{m=0}^{\infty} \frac{1}{m!}(A+B)^m \\ &= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{1}{j!(m-j)!} A^j B^{m-j}. \end{aligned} \tag{1.68}$$

Then take $n = m - j$ and show this is

$$\begin{aligned} &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{j!n!} A^j B^n \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{n=0}^{\infty} \frac{1}{n!} B^n \\ &= e^A e^B, \end{aligned} \tag{1.69}$$

so

$$e^{A+B} = e^A e^B. \tag{1.70}$$

14. As an alternative to the proof of (1.4), given in (1.3), which depends on term by term differentiation of power series, verify that, for $A \in M(n, \mathbb{C})$,

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \lim_{h \to 0} \frac{1}{h} \bigl(e^{(t+h)A} - e^{tA}\bigr) \\ &= e^{tA} \lim_{h \to 0} \frac{1}{h} \bigl(e^{hA} - I\bigr) \\ &= e^{tA} A \\ &= A e^{tA}, \end{aligned} \tag{1.71}$$

the second identity in (1.71) by (1.70), the third by the definition (1.2), and the fourth by commutativity.

2. Exponentials and trigonometric functions

In Chapter 1 we have seen how to use complex exponentials to give a self-contained treatment of basic results on the trigonometric functions $\cos t$ and $\sin t$. Here we present a variant, using matrix exponentials. We begin by looking at

$$x(t) = e^{tJ}x_0, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

which solves

$$x'(t) = Jx(t), \quad x(0) = x_0 \in \mathbb{R}^2. \tag{2.2}$$

We first note that the planar curve $x(t)$ moves about on a circle centered about the origin. Indeed,

$$\begin{aligned} \frac{d}{dt}\|x(t)\|^2 = \frac{d}{dt}\big(x(t)\cdot x(t)\big) &= x'(t)\cdot x(t) + x(t)\cdot x'(t) \\ &= Jx(t)\cdot x(t) + x(t)\cdot Jx(t) \\ &= 0, \end{aligned} \tag{2.3}$$

since $J^t = -J$. Thus $\|x(t)\| = \|x_0\|$ is constant. Furthermore the velocity $v(t) = x'(t)$ has constant magnitude; in fact

$$\|v(t)\|^2 = v(t)\cdot v(t) = Jx(t)\cdot Jx(t) = \|x(t)\|^2, \tag{2.4}$$

since $J^tJ = -J^2 = I$.

For example,

$$\begin{pmatrix} c(t) \\ s(t) \end{pmatrix} = e^{tJ}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.5}$$

is a curve, moving on the unit circle $x_1^2 + x_2^2 = 1$, at unit speed, with initial position $x(0) = (1,0)^t$ and initial velocity $v(0) = (0,1)^t$. Now in trigonometry the functions $\cos t$ and $\sin t$ are defined to be the x_1 and x_2 coordinates of such a parametrization of the unit circle, so we have

$$\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = e^{tJ}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.6}$$

The differential equation (2.2) then gives

$$\frac{d}{dt}\cos t = -\sin t, \quad \frac{d}{dt}\sin t = \cos t. \tag{2.7}$$

Using

$$e^{tJ}\begin{pmatrix}0\\1\end{pmatrix}=e^{tJ}J\begin{pmatrix}1\\0\end{pmatrix}=Je^{tJ}\begin{pmatrix}1\\0\end{pmatrix},$$

we have a formula for $e^{tJ}\binom{0}{1}$, which together with (2.6) yields

$$e^{tJ}=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}=(\cos t)I+(\sin t)J. \tag{2.8}$$

Then the identity $e^{(s+t)J}=e^{sJ}e^{tJ}$ yields the following identities, when matrix multiplication is carried out:

$$\begin{aligned}\cos(s+t)&=(\cos s)(\cos t)-(\sin s)(\sin t),\\ \sin(s+t)&=(\cos s)(\sin t)+(\sin s)(\cos t).\end{aligned} \tag{2.9}$$

We now show how the treatment of $\sin t$ and $\cos t$ presented above is really quite close to that given in Chapter 1, §1. To start, we note that if $\mathbb{C}$ is regarded as a real vector space, with basis $e_1=1,\ e_2=i$, and hence identified with $\mathbb{R}^2$, via

$$z=x+iy\leftrightarrow\begin{pmatrix}x\\y\end{pmatrix}, \tag{2.10}$$

then the matrix representation for the linear transformation $z\mapsto iz$ is given by J:

$$iz=-y+ix,\quad J\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}. \tag{2.11}$$

More generally, the linear transformation $z\mapsto(c+is)z$ has matrix representation

$$\begin{pmatrix}c & -s\\ s & c\end{pmatrix}. \tag{2.12}$$

Taking this into account, we see that the identity (2.8) is equivalent to

$$e^{it}=\cos t+i\sin t, \tag{2.13}$$

which is Euler's formula, as in (1.39) of Chapter 1.

Here is another approach to the evaluation of e^{tJ}. We compute the eigenvalues and eigenvectors of J:

$$\lambda_1=i,\quad \lambda_2=-i;\quad v_1=\begin{pmatrix}1\\-i\end{pmatrix},\quad v_2=\begin{pmatrix}1\\i\end{pmatrix}. \tag{2.14}$$

Then, using the fact that $e^{tJ}v_k=e^{t\lambda_k}v_k$, we have

$$e^{tJ}\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{2}e^{it}\begin{pmatrix}1\\-i\end{pmatrix}+\frac{1}{2}e^{-it}\begin{pmatrix}1\\i\end{pmatrix}. \tag{2.15}$$

Comparison with (2.6) gives

$$\cos t=\frac{1}{2}(e^{it}+e^{-it}),\quad \sin t=\frac{1}{2i}(e^{it}-e^{-it}), \tag{2.16}$$

again leading to (2.13).

Exercises

1. Recall Skew(n) and $SO(n)$, defined by (11.7) and (12.4) of Chapter 2. Show that

$$A \in \mathrm{Skew}(n) \Longrightarrow e^{tA} \in SO(n), \quad \forall\, t \in \mathbb{R}. \tag{2.17}$$

Note how this generalizes (2.3).

2. Given an $n \times n$ matrix A, let us set

$$\cos tA = \frac{1}{2}(e^{itA} + e^{-itA}), \quad \sin tA = \frac{1}{2i}(e^{itA} - e^{-itA}). \tag{2.18}$$

Show that

$$\frac{d}{dt}\cos tA = -A \sin tA, \quad \frac{d}{dt}\sin tA = A\cos tA. \tag{2.19}$$

3. In the context of Exercise 2, show that

$$\cos tA = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(tA)^{2k}, \quad \sin tA = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(tA)^{2k+1}. \tag{2.20}$$

4. Show that

$$\begin{aligned} Av = \lambda v \Longrightarrow (\cos tA)v &= (\cos t\lambda)v, \\ (\sin tA)v &= (\sin t\lambda)v. \end{aligned}$$

5. Compute $\cos tA$ and $\sin tA$ in each of the following cases:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

6. Suppose $A \in M(n, \mathbb{C})$ and

$$B = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix} \in M(2n, \mathbb{C}).$$

Show that

$$e^{tB} = \begin{pmatrix} \cos tA & -\sin tA \\ \sin tA & \cos tA \end{pmatrix}.$$

3. First-order systems derived from higher-order equations

There is a standard process to convert an nth order differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = 0 \tag{3.1}$$

to a first-order system. Set

$$x_0(t) = y(t),\ x_1(t) = y'(t), \dots, x_{n-1}(t) = y^{(n-1)}(t). \tag{3.2}$$

Then $x = (x_0, \dots, x_{n-1})^t$ satisfies

$$\begin{aligned} x_0' &= x_1 \\ &\vdots \\ x_{n-2}' &= x_{n-1} \\ x_{n-1}' &= -a_{n-1}x_{n-1} - \cdots - a_0 x_0, \end{aligned} \tag{3.3}$$

or equivalently

$$\frac{dx}{dt} = Ax, \tag{3.4}$$

with

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}. \tag{3.5}$$

The matrix A given by (3.5) is called the *companion matrix* of the polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0. \tag{3.6}$$

Note that a direct search of solutions to (3.1) of the form $e^{\lambda t}$ leads one to solve $p(\lambda) = 0$. Thus the following result is naturally suggested.

Proposition 3.1. *If $p(\lambda)$ is a polynomial of the form (3.6), with companion matrix A, given by (3.5), then*

$$p(\lambda) = \det(\lambda I - A). \tag{3.7}$$

Proof. We look at

$$\lambda I - A = \begin{pmatrix} \lambda & -1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & -1 \\ a_0 & a_1 & \cdots & a_{n-2} & \lambda + a_{n-1} \end{pmatrix}, \tag{3.8}$$

and compute its determinant by expanding by minors down the first column. We see that

$$\det(\lambda I - A) = \lambda \det(\lambda I - \widetilde{A}) + (-1)^{n-1} a_0 \det B, \tag{3.9}$$

where

$$\begin{aligned} &\widetilde{A} \ \text{ is the companion matrix of } \ \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_1, \\ &B \ \text{ is lower triangular, with } \ -1\text{'s on the diagonal.} \end{aligned} \tag{3.10}$$

By induction on n, we have $\det(\lambda I - \widetilde{A}) = \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_1$, while $\det B = (-1)^{n-1}$. Substituting this into (3.9) gives (3.7).

Converse construction

We next show that each solution to a first order $n \times n$ system of the form (3.4) (for general $A \in M(n, \mathbb{F})$) also satisfies an nth order scalar ODE. Indeed, if (3.4) holds, then

$$x^{(k)} = Ax^{(k-1)} = \cdots = A^k x. \tag{3.11}$$

Now if $p(\lambda)$ is given by (3.7), and say

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0, \tag{3.12}$$

then, by the Cayley-Hamilton theorem (cf. (8.10) of Chapter 2),

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0. \tag{3.13}$$

Hence

$$\begin{aligned} x^{(n)} &= A^n x \\ &= -a_{n-1}A^{n-1}x - \cdots - a_1 Ax - a_0 x \\ &= -a_{n-1}x^{(n-1)} - \cdots - a_1 x' - a_0 x, \end{aligned} \tag{3.14}$$

so we have the asserted nth order scalar equation:

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0. \tag{3.15}$$

Remark. If the minimal polynomial $q(\lambda)$ of A has degree m, less than n, we can replace p by q and derive analogues of (3.14)–(3.15), giving a single differential equation of degree m for x.

Exercises

1. Using the method (3.12)–(3.15), convert

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} x$$

into a second order scalar equation.

2. Using the method (3.2)–(3.3), convert

$$y'' - 3y' + 2y = 0$$

into a 2×2 first order system.

In Exercises 3–4, assume that λ_1 is a root of multiplicity $k \geq 2$ for the polynomial $p(\lambda)$ given by (3.6).

3. Verify that $e^{\lambda_1 t}, te^{\lambda_1 t}, \dots, t^{k-1}e^{\lambda_1 t}$ are solutions to (3.1).

4. Deduce that, for each $j = 0, \dots, k-1$, the system (3.3) has a solution of the form

$$x(t) = (t^j + \alpha t^{j-1} + \cdots + \beta)e^{t\lambda_1}v, \tag{3.16}$$

(with v depending on j).

5. For given $A \in M(n, \mathbb{C})$, suppose $x' = Ax$ has a solution of the form (3.16). Show that λ_1 must be a root of multiplicity $\geq j+1$ of the minimal polynomial of A.
Hint. Take into account the remark below (3.15).

6. Using Exercises 3–5, show that the minimal polynomial of the companion matrix A in (3.5) must be the characteristic polynomial $p(\lambda)$.

4. Non-homogeneous equations and Duhamel's formula

In §§1–3 we have focused on homogeneous equations, $x' - Ax = 0$. Here we consider the non-homogeneous equation

$$\frac{dx}{dt} - Ax = f(t), \quad x(0) = x_0 \in \mathbb{C}^n. \tag{4.1}$$

Here $A \in M(n, \mathbb{C})$ and $f(t)$ takes values in $\mathbb{C}^n$. The key to solving this is to recognize that the left side of (4.1) is equal to

$$e^{tA} \frac{d}{dt}\Big(e^{-tA} x(t)\Big), \tag{4.2}$$

as follows from the product formula for the derivative and the defining property of e^{tA}, given in (1.4). Thus (4.1) is equivalent to

$$\frac{d}{dt}\Big(e^{-tA} x(t)\Big) = e^{-tA} f(t), \quad x(0) = x_0, \tag{4.3}$$

and integration yields

$$e^{-tA} x(t) = x_0 + \int_0^t e^{-sA} f(s)\, ds. \tag{4.4}$$

Applying e^{tA} to both sides then gives the solution:

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} f(s)\, ds. \tag{4.5}$$

This is called Duhamel's formula.

Example. We combine methods of this section and §3 (and also §2) to solve

$$y'' + y = f(t), \quad y(0) = y_0,\ y'(0) = y_1. \tag{4.6}$$

As in §3, set $x = (x_0, x_1) = (y, y')$, to obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{4.7}$$

Recognizing the 2×2 matrix above as $-J$, and recalling from §2 that

$$e^{(s-t)J} = \begin{pmatrix} \cos(s-t) & -\sin(s-t) \\ \sin(s-t) & \cos(s-t) \end{pmatrix}, \tag{4.8}$$

we obtain

(4.9)
$$\begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos(s-t) & -\sin(s-t) \\ \sin(s-t) & \cos(s-t) \end{pmatrix}\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds,$$

and hence

$$y(t) = (\cos t) y_0 + (\sin t) y_1 + \int_0^t \sin(t-s)\, f(s)\, ds. \tag{4.10}$$

Use of Duhamel's formula is a good replacement for the method of variation of parameters, discussed in §14 of Chapter 1. See §9 of this chapter for more on this. Here, we discuss a variant of the method of undetermined coefficients, introduced for single second-order equations in §10 of Chapter 1.

We consider the following special case, the first-order $n \times n$ system

$$\frac{dx}{dt} - Ax = (\cos \sigma t)v, \tag{4.11}$$

given $\sigma \in \mathbb{R}$, $v \in \mathbb{R}^n$, and $A \in M(n, \mathbb{R})$ (or we could use complex coefficients). We assume

$$i\sigma,\ -i\sigma \notin \operatorname{Spec} A, \tag{4.12}$$

and look for a solution to (4.11) of the form

$$x_p(t) = (\cos \sigma t)a + (\sin \sigma t)b, \quad a, b \in \mathbb{R}^n. \tag{4.13}$$

Substitution into (4.11) leads to success with

$$\begin{aligned} a &= -A(A^2 + \sigma I)^{-1}v, \\ b &= -\sigma(A^2 + \sigma I)^{-1}v. \end{aligned} \tag{4.14}$$

If (4.12) does not hold, (4.14) fails, and (4.11) might not have a solution of the form (4.13). Of course, (4.5) will work; (4.11) will have a solution of the form

$$x(t) = \int_0^t (\cos \sigma s) e^{(t-s)A} v\, ds. \tag{4.15}$$

When (4.12) holds and (4.13) works, the general solution to (4.11) is

$$x(t) = e^{tA}u_0 + (\cos \sigma t)a + (\sin \sigma t)b, \quad u_0 \in \mathbb{R}^n, \tag{4.16}$$

u_0 related to $x(0)$ by

$$x(0) = u_0 + a. \tag{4.17}$$

If all the eigenvalues of A have negative real part, $e^{tA}u_0$ will decay to 0 as $t \to +\infty$. Then $e^{tA}u_0$ is called the *transient* part of the solution. The other part, $(\cos \sigma t)A + (\sin \sigma t)b$, is called the *steady state* solution.

Exercises

1. Given $A \in M(n, \mathbb{C})$, set

$$E_k(t) = \sum_{j=0}^{k} \frac{t^j}{j!} A^j.$$

Verify that $E_k'(t) = AE_{k-1}(t)$ and that

$$\frac{d}{dt}\big(E_k(t)e^{-tA}\big) = -\frac{t^k}{k!} A^{k+1}\, e^{-tA}.$$

2. Verify that, if A is invertible,

$$\int_0^t s^k e^{-sA}\, ds = -k!\, A^{-(k+1)} \big[E_k(t)e^{-tA} - I\big].$$

3. Solve the initial value problem

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

4. Solve the initial value problem

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

5. Solve the initial value problem

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

6. Produce analogues of (4.8)–(4.10) for

$$y'' - 3y' + 2y = f(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

In Exercises 7–8, take $X, Y \in M(n, \mathbb{C})$ and

$$U(t,s) = e^{t(X+sY)}, \quad U_s(t,s) = \frac{\partial}{\partial s} U(t,s). \tag{4.18}$$

7. Show that U_s satisfies

$$\frac{\partial U_s}{\partial t} = (X + sY)U_s + YU, \quad U_s(0,s) = 0.$$

8. Use Duhamel's formula to show that

$$U_s(t,s) = \int_0^t e^{(t-\tau)(X+sY)} Y e^{\tau(X+sY)}\, d\tau.$$

Deduce that

$$\frac{d}{ds} e^{X+sY}\Big|_{s=0} = e^X \int_0^1 e^{-\tau X} Y e^{\tau X}\, d\tau. \tag{4.19}$$

9. Assume $X(t)$ is a smooth function of $t \in I$ with values in $M(n, \mathbb{C})$. Show that, for $t \in I$,

$$\frac{d}{dt} e^{X(t)} = e^{X(t)} \int_0^1 e^{-\tau X(t)} X'(t) e^{\tau X(t)}\, d\tau. \tag{4.20}$$

10. In the context of Exercise 9, assume

$$t, t' \in I \Longrightarrow X(t)X(t') = X(t')X(t).$$

In such a case, simplify (4.20), and compare the result with that of Exercise 10 in §1.

5. Simple electrical circuits

Here we extend the scope of the treatment of electrical circuits in §13 of Chapter 1. Rules worked out by Kirchhoff and others in the 1800s allow one to write down a system of linear differential equations describing the voltages and currents running along a variety of electrical circuits, containing resistors, capacitors, and inductors.

There are two types of basic laws. The first type consists of two rules known as Kirchhoff's laws:

(A) The sum of the voltage drops around any closed loop is zero.
(B) The sum of the currents at any node is zero.

The second type of law specifies the voltage drop across each circuit element:

$$\text{(a)} \qquad \text{Resistor:} \quad V = IR,$$

$$\text{(b)} \qquad \text{Inductor:} \quad V = L\frac{dI}{dt},$$

$$\text{(c)} \qquad \text{Capacitor:} \quad V = \frac{Q}{C}.$$

In each case, V is the voltage drop (in volts), I is the current (in amps), R is the resistance (in ohms), L is the inductance (in henrys), C is the capacitance (in farads), and Q is the charge (in coulombs). We refer to §13 of Chapter 1 for basic information about these units. The rule (c) is supplemented by the following formula for the current across a capacitor:

$$\text{(c2)} \qquad I = \frac{dQ}{dt}.$$

In (b) and (c2), time is measured in seconds.

Rules (A), (B), and (a) give algebraic relations among the various voltages and currents, while rules (b) and (c)–(c2) give differential equations, namely

$$L\frac{dI}{dt} = V \qquad \text{(Inductor)}, \tag{5.1}$$

$$C\frac{dV}{dt} = I \qquad \text{(Capacitor)}. \tag{5.2}$$

Note that (5.2) results from applying d/dt to (c) and then using (c2). If a circuit has k capacitors and ℓ inductors, we get an $m \times m$ system of first order differential equations, with $m = k + \ell$.

We illustrate the formulation of such differential equations for circuits presented in Figure 5.1 and Figure 5.2. In each case, the circuit elements are numbered. We denote by V_j the voltage drop across element j and by I_j the current across element j.

Figure 5.1

Figure 5.1 depicts a classical RLC circuit, such as treated in §13 of Chapter 1. Rules (A), (B), and (a) give

$$\begin{aligned} V_1 + V_2 + V_3 &= E(t), \\ I_1 = I_2 &= I_3, \\ V_1 &= RI_1. \end{aligned} \tag{5.3}$$

Equations (5.1)–(5.3) yield a system of two ODEs, for I_3 and V_2:

$$L\frac{dI_3}{dt} = V_3, \qquad C\frac{dV_2}{dt} = I_2. \tag{5.4}$$

We need to express V_3 and I_2 in terms of I_3, V_2, and $E(t)$, using (5.3). In fact, we have

$$\begin{aligned} &V_3 = E(t) - V_1 - V_2 = E(t) - RI_1 - V_2 = E(t) - RI_3 - V_2, \\ &I_2 = I_3, \end{aligned} \tag{5.5}$$

so we get the system

$$\begin{aligned} L\frac{dI_3}{dt} &= -RI_3 - V_2 + E(t), \\ C\frac{dV_2}{dt} &= I_3, \end{aligned} \tag{5.6}$$

or, in matrix form,

$$\frac{d}{dt}\begin{pmatrix} I_3 \\ V_2 \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}\begin{pmatrix} I_3 \\ V_2 \end{pmatrix} + \frac{1}{L}\begin{pmatrix} E(t) \\ 0 \end{pmatrix}. \tag{5.7}$$

Figure 5.2

Note that the characteristic polynomial of the matrix

$$A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \tag{5.8}$$

is

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}, \tag{5.9}$$

with roots

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - 4\frac{L}{C}}. \tag{5.10}$$

Let us now look at the slightly more complicated circuit depicted in Figure 5.2. Again we get a 2×2 system of differential equations. Rules (A), (B), and (a) give

$$\begin{gathered} V_1 + V_2 + V_4 = E(t), \quad V_2 = V_3, \\ I_1 = I_2 + I_3 = I_4, \\ V_3 = R_3 I_3, \quad V_4 = R_4 I_4. \end{gathered} \tag{5.11}$$

Equations (5.1)–(5.2) yield differential equations for I_1 and V_2:

$$C\frac{dV_2}{dt} = I_2, \qquad L\frac{dI_1}{dt} = V_1. \tag{5.12}$$

Figure 5.3

We need to express I_2 and V_1 in terms of V_2, I_1, and $E(t)$, using (5.11). In fact, we have

$$\begin{aligned} I_2 &= I_1 - I_3 = I_1 - \frac{1}{R_3}V_3 = I_1 - \frac{1}{R_3}V_2, \\ V_1 &= E(t) - V_2 - V_4 = E(t) - V_2 - R_4I_4 = E(t) - V_2 - R_4I_1, \end{aligned} \tag{5.13}$$

so we get the system

$$\begin{aligned} C\frac{dV_2}{dt} &= -\frac{1}{R_3}V_2 + I_1, \\ L\frac{dI_1}{dt} &= -V_2 - R_4I_1 + E(t), \end{aligned} \tag{5.14}$$

or, in matrix form,

$$\frac{d}{dt}\begin{pmatrix} V_2 \\ I_1 \end{pmatrix} = \begin{pmatrix} -1/R_3C & 1/C \\ -1/L & -R_4/L \end{pmatrix}\begin{pmatrix} V_2 \\ I_1 \end{pmatrix} + \frac{1}{L}\begin{pmatrix} 0 \\ E(t) \end{pmatrix}. \tag{5.15}$$

Exercises

1. Work out the 3×3 system of differential equations describing the behavior of the circuit depicted in Figure 5.3. Assume

$$E(t) = 5\ \sin 12t \quad \text{volts}.$$

2. Using methods developed in §4, solve the 2×2 system (5.7) when

$$R = 5 \;\text{ ohms}, \quad L = 4 \;\text{ henrys}, \quad C = 1 \;\text{ farad},$$

and

$$E(t) = 5 \;\cos 2t \;\text{ volts},$$

with initial data

$$I_3(0) = 0 \;\text{ amps}, \quad V_2(0) = 5 \;\text{ volts}.$$

3. Solve the 2×2 system (5.15) when

$$R_3 = 1 \;\text{ ohm}, \quad R_4 = 4 \text{ ohms}, \quad L = 4 \text{ henrys}, \quad C = 2 \;\text{ farads},$$

and

$$E(t) = 2 \;\cos 2t \;\text{ volts}.$$

4. Use the method of (4.11)–(4.14) to find the steady state solution to (5.7), when

$$E(t) = A \;\cos \sigma t.$$

Take A, σ, R and L fixed and allow C to vary. Show that the amplitude of the steady state solution is maximal (we say resonance is achieved) when

$$LC = \frac{1}{\sigma^2},$$

recovering calculations of (13.7)–(13.13) in Chapter 1.

5. Work out the analogue of Exercise 4 with the system (5.7) replaced by (5.15). Is the condition for resonance the same as in Exercise 4?

6. Draw an electrical circuit that leads to a 4×4 system of differential equations, and write down said system.

6. Second-order systems

Interacting physical systems often give rise to second-order systems of differential equations. Consider for example a system of n objects, of mass $m_1, \dots, m_n$, connected to each other and to two walls by $n+1$ springs, with spring constants $k_1, \dots, k_{n+1}$, as in Fig. 6.1. We assume the masses slide

Figure 6.1

without friction. Denote by x_j the position of the jth mass and by y_j the degree to which the jth spring is stretched. The equations of motion are

$$m_j x_j'' = -k_j y_j + k_{j+1} y_{j+1}, \quad 1 \le j \le n, \tag{6.1}$$

and for certain constants a_j,

$$\begin{aligned} y_j &= x_j - x_{j+1} + a_j, \quad 2 \le j \le n, \\ y_1 &= x_1 + a_1, \quad y_{n+1} = -x_n + a_{n+1}. \end{aligned} \tag{6.2}$$

Substituting (6.2) into (6.1) yields an $n \times n$ system, which we can write in matrix form as

$$Mx'' = -Kx + b, \tag{6.3}$$

where $x = (x_1, \dots, x_n)^t$, $b = (-k_1 a_1 + k_2 a_2, \dots, -k_n a_n + k_{n+1} a_{n+1})^t$,

$$M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}, \tag{6.4}$$

and

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & \ddots & & \\ & & \ddots & & \\ & & \ddots & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}. \tag{6.5}$$

We assume $m_j > 0$ and $k_j > 0$ for each j. Then clearly M is a positive definite matrix and K is a real symmetric matrix.

Proposition 6.1. *If each $k_j > 0$, then K, given by (6.5), is positive definite.*

Proof. We have

$$x \cdot Kx = \sum_{j=1}^{n} (k_j + k_{j+1}) x_j^2 - 2 \sum_{j=2}^{n} k_j x_{j-1} x_j. \tag{6.6}$$

Now

$$2x_{j-1}x_j \le x_{j-1}^2 + x_j^2, \tag{6.7}$$

so

$$\begin{aligned} x \cdot Kx \ge & \sum_{j=1}^{n} k_j x_j^2 + \sum_{j=2}^{n+1} k_j x_{j-1}^2 \\ & - \sum_{j=2}^{n} k_j x_j^2 - \sum_{j=2}^{n} k_j x_{j-1}^2 \\ \ge\ & k_1 x_1^2 + k_{n+1} x_n^2. \end{aligned} \tag{6.8}$$

Furthermore note that the inequality in (6.7) is strict unless $x_{j-1} = x_j$ so the inequality in (6.8) is strict unless $x_{j-1} = x_j$ for *each* $j \in \{2, \dots, n\}$, i.e., unless $x_1 = \cdots = x_n$. This proves that $x \cdot Kx > 0$ whenever $x \in \mathbb{R}^n,\ x \ne 0$.

To be more precise, we can sharpen (6.7) to

$$2x_{j-1}x_j = x_{j-1}^2 + x_j^2 - (x_j - x_{j-1})^2, \tag{6.9}$$

and then (6.8) is sharpened to

$$x \cdot Kx = k_1 x_1^2 + k_{n+1} x_n^2 + \sum_{j=2}^{n} k_j (x_j - x_{j-1})^2. \tag{6.10}$$

If we set

$$\kappa = \min\{k_j : 1 \le j \le n+1\}, \tag{6.11}$$

then (6.10) implies

$$x \cdot Kx \ge \kappa \Big(x_1^2 + x_n^2 + \sum_{j=2}^{n} (x_j - x_{j-1})^2 \Big). \tag{6.12}$$

The system (6.3) is inhomogeneous, but it is readily converted into the homogeneous system

$$Mz'' = -Kz, \quad z = x - K^{-1}b. \tag{6.13}$$

This in turn can be rewritten

$$z'' = -M^{-1}Kz. \tag{6.14}$$

Note that

$$L = M^{-1/2}KM^{-1/2} \Longrightarrow M^{-1}K = M^{-1/2}LM^{1/2}, \tag{6.15}$$

where

$$M^{1/2} = \begin{pmatrix} m_1^{1/2} & & \\ & \ddots & \\ & & m_n^{1/2} \end{pmatrix}. \tag{6.16}$$

Proposition 6.2. *The matrix L is positive definite.*

Proof. $x \cdot Lx = (M^{-1/2}x) \cdot K(M^{-1/2}x) > 0$ whenever $x \neq 0$.

According to (6.15), $M^{-1}K$ and L are similar, so we have:

Corollary 6.3. *For M and K of the form (6.4)–(6.5), with $m_j, k_j > 0$, the matrix $M^{-1}K$ is diagonalizable, and all its eigenvalues are positive.*

It follows that $\mathbb{R}^n$ has a basis $\{v_1, \dots, v_n\}$ satisfying

$$M^{-1}Kv_j = \lambda_j^2 v_j, \quad \lambda_j > 0. \tag{6.17}$$

Then the initial value problem

$$Mz'' = -Kz, \quad z(0) = z_0, \quad z'(0) = z_1 \tag{6.18}$$

has the solution

$$z(t) = \sum_{j=1}^{n} \Big(\alpha_j \cos \lambda_j t + \frac{\beta_j}{\lambda_j} \sin \lambda_j t\Big) v_j, \tag{6.19}$$

where the coefficients α_j and β_j are given by

$$z_0 = \sum \alpha_j v_j, \quad z_1 = \sum \beta_j v_j. \tag{6.20}$$

An alternative approach to the system (6.14) is to set

$$u = M^{1/2} z, \tag{6.21}$$

for which (6.14) becomes

$$u'' = -Lu, \tag{6.22}$$

with L given by (6.15). Then $\mathbb{R}^n$ has an orthonormal basis $\{w_j : 1 \le j \le n\}$, satisfying

$$Lw_j = \lambda_j^2 w_j, \quad \text{namely} \;\; w_j = M^{1/2} v_j, \tag{6.23}$$

with v_j as in (6.17). Note that we can set

$$L = A^2, \quad Aw_j = \lambda_j w_j, \tag{6.24}$$

and (6.22) becomes

$$u'' + A^2 u = 0. \tag{6.25}$$

One way to convert (6.25) to a first order $(2n) \times (2n)$ system is to set

$$v = Au, \quad w = u'. \tag{6.26}$$

Then (6.25) becomes

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = X \begin{pmatrix} v \\ w \end{pmatrix}, \quad X = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}. \tag{6.27}$$

It is useful to note that when X is given by (6.26), then

$$e^{tX} = \begin{pmatrix} \cos tA & \sin tA \\ -\sin tA & \cos tA \end{pmatrix}, \tag{6.28}$$

where $\cos tA$ and $\sin tA$ are given in Exercises 6–7 in §2. One way to see this is to let $\Phi(t)$ denote the right side of (6.28) and use (2.19) to see that

$$\frac{d}{dt}\Phi(t) = X\Phi(t), \quad \Phi(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{6.29}$$

Then Exercise 4 of §1 implies $e^{tX} \equiv \Phi(t)$. These calculations imply that the solution to (6.25), with initial data $u(0) = u_0,\ u'(0) = u_1$, is given by

$$u(t) = (\cos tA)u_0 + A^{-1}(\sin tA)u_1. \tag{6.30}$$

Compare (6.18)–(6.20). This works for each invertible $A \in M(n, \mathbb{C})$.

We move to the inhomogeneous variant of (6.14), which as above we can transform to the following inhomogeneous variant of (6.25):

$$u'' + A^2 u = f(t), \quad u(0) = u_0,\ u'(0) = u_1. \tag{6.31}$$

Using the substitution (6.26), we get

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = X\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} Au_0 \\ u_1 \end{pmatrix}. \tag{6.32}$$

Duhamel's formula applies to give

$$\begin{pmatrix} v(t) \\ w(y) \end{pmatrix} = e^{tX}\begin{pmatrix} Au_0 \\ u_1 \end{pmatrix} + \int_0^t e^{(t-s)X}\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds. \tag{6.33}$$

Using the formula (6.28) for e^{tX}, we see that the resulting formula for $v(t)$ in (6.33) is equivalent to

$$u(t) = (\cos tA)u_0 + A^{-1}(\sin tA)u_1 + \int_0^t A^{-1}\sin(t-s)A\, f(s)\, ds. \tag{6.34}$$

This is the analogue of Duhamel's formula for the solution to (6.31).

We now return to the coupled spring problem and modify (6.1)–(6.2) to allow for friction. Thus we replace (6.1) by

$$m_j x_j'' = -k_j y_j + k_{j+1} y_j - d_j x_j', \tag{6.35}$$

where y_j are as in (6.2) and $d_j > 0$ are friction coefficients, Then (6.3) is replaced by

$$Mx'' = -Kx - Dx' + b, \tag{6.36}$$

with b as in (6.3), M and K as in (6.4)–(6.5), and

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \quad d_j > 0. \tag{6.37}$$

As in (6.13), we can convert (6.36) to the homogeneous system

$$Mz'' = -Kz - Dz', \quad z = x - K^{-1}b. \tag{6.38}$$

If we set $u = M^{1/2}z$, as in (6.21), then, parallel to (6.22)–(6.24), we get

$$u'' + Bu' + A^2 u = 0, \tag{6.39}$$

where A^2 is as in (6.24), with $L = M^{-1/2}KM^{-1/2}$, as in (6.22), and

$$B = M^{-1/2}DM^{-1/2} = \begin{pmatrix} d_1/m_1 & & \\ & \ddots & \\ & & d_n/m_n \end{pmatrix}. \tag{6.40}$$

The substitution (6.26) converts the $n \times n$ second order system (6.39) to the $(2n) \times (2n)$ first order system

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & A \\ -A & -B \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}. \tag{6.41}$$

We can write (6.41) as

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = (X+Y)\begin{pmatrix} v \\ w \end{pmatrix}, \tag{6.42}$$

with X as in (6.27) and

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & -B \end{pmatrix}. \tag{6.43}$$

Note that

$$XY = \begin{pmatrix} 0 & -AB \\ 0 & 0 \end{pmatrix}, \quad YX = \begin{pmatrix} 0 & 0 \\ BA & 0 \end{pmatrix}, \tag{6.44}$$

so these matrices do not commute. Thus $e^{t(X+Y)}$ might be difficult to calculate even when A and B commute. Such commutativity would hold, for example, if $m_1 = \cdots = m_n$ and $d_1 = \cdots = d_n$, in which case B is a scalar multiple of the identity matrix.

When the positive, self adjoint operators A and B do commute, we can make the following direct attack on the system (6.39). We know (cf. Exercise 3 in §11, Chapter 2) that $\mathbb{R}^n$ has an orthonormal basis $\{w_1, \dots, w_n\}$ for which

$$Aw_j = \lambda_j w_j, \quad Bw_j = 2\mu_j w_j, \quad \lambda_j, \mu_j > 0. \tag{6.45}$$

Then we can write a solution to (6.39) as

$$u(t) = \sum u_j(t) w_j, \tag{6.46}$$

where the real-valued coefficients $u_j(t)$ satisfy the equations

$$u_j'' + 2\mu_j u_j' + \lambda_j^2 u_j = 0, \tag{6.47}$$

with solutions that are linear combinations:

$$e^{-\mu_j t}\Big(\alpha_j \cos\sqrt{\lambda^2-\mu_j^2}t+\beta_j \sin\sqrt{\lambda_j^2-\mu^2 t}\Big), \quad \lambda_j>\mu_j,$$
$$e^{-\mu_j t}\Big(\alpha_j e^{\sqrt{\mu_j^2-\lambda_j^2}t}+\beta_j e^{-\sqrt{\mu_j^2-\lambda_j^2}t}\Big), \quad \lambda_j<\mu_j, \tag{6.48}$$
$$e^{-\mu_j t}(\alpha_j+\beta_j t), \quad \lambda_j=\mu_j.$$

These three cases correspond to modes that are said to be underdamped, overdamped, and critically damped, respectively.

In cases where A and B do not commute, analysis of (6.39) is less explicit, but we can establish the following decay result.

Proposition 6.4. *If $A, B \in M(n,\mathbb{C})$ are positive definite, then all of the eigenvalues of $Z = \begin{pmatrix} 0 & A \\ -A & -B \end{pmatrix}$ have negative real part.*

Proof. Let's say $(v,w)^t \neq 0$ and $Z(v,w)^t = \lambda(v,w)^t$. Then

$$Aw = \lambda v, \quad Av + Bw = -\lambda w, \tag{6.49}$$

and

$$(Z(v,w)^t,(v,w)^t) = -(Bw,w) + [(Aw,v)-(Av,w)], \tag{6.50}$$

while also

$$(Z(v,w)^t,(v,w)^t) = \lambda(\|v\|^2+\|w\|^2). \tag{6.51}$$

The two terms on the right sie of (6.50) are real and purely imaginary, respectively, so we obtain

$$(\operatorname{Re}\lambda)(\|v\|^2+\|w\|^2) = -(Bw,w). \tag{6.52}$$

If $(v,w)^t \neq 0$, we deduce that either $\operatorname{Re}\lambda < 0$ or $w=0$. If $w=0$, then (6.49) gives $Av=0$, hence $v=0$. Hence $w \neq 0$, and $\operatorname{Re}\lambda<0$, as asserted.

Exercises

1. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. Use the results of Exercise 1 to find the eigenvalues and eigenvectors of K, given by (6.5), in case $n = 3$ and
$$k_1 = k_2 = k_3 = k_4 = k.$$

3. Find the general solution to
$$u'' + Bu' + A^2 u = 0,$$
in case $A^2 = K$, with K as in Exercise 2, and $B = I$.

4. Find the general solution to
$$u'' + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u' + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u = 0.$$

5. Produce a second order system of differential equations for the spring system presented in Fig. 6.2. (Assume there is no friction.)

6. Generalizing the treatment of (6.25), consider
$$u'' + Lu = 0, \quad L \in M(N, \mathbb{C}). \tag{6.53}$$
Assume $\mathbb{C}^N$ has a basis of eigenvectors v_j, such that $Lv_j = \lambda_j^2 v_j$, $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$. Show that the general solution to (6.53) has the form
$$u(t) = \sum_{j=1}^{N} (\alpha_j e^{\lambda_j t} + \beta_j e^{-\lambda_j t}) v_j, \quad \alpha_j, \beta_j \in \mathbb{C}. \tag{6.54}$$
How is this modified if some $\lambda_j = 0$?

7. Find the general solution to
$$u'' + \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} u = 0,$$
and to
$$u'' + \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} u = 0.$$

Figure 6.2

7. Curves in $\mathbb{R}^3$ and the Frenet-Serret equations

Given a curve $c(t) = (x(t), y(t), z(t))$ in 3-space, we define its velocity and acceleration by

$$v(t) = c'(t), \quad a(t) = v'(t) = c''(t). \tag{7.1}$$

We also define its speed $s'(t)$ and arclength by

$$s'(t) = \|v(t)\|, \quad s(t) = \int_{t_0}^{t} s'(\tau)\, d\tau, \tag{7.2}$$

assuming we start at $t = t_0$. We define the unit tangent vector to the curve as

$$T(t) = \frac{v(t)}{\|v(t)\|}. \tag{7.3}$$

Henceforth we assume the curve is parametrized by arclength.

We define the *curvature* $\kappa(s)$ of the curve and the normal $N(s)$ by

$$\kappa(s) = \left\| \frac{dT}{ds} \right\|, \quad \frac{dT}{ds} = \kappa(s) N(s). \tag{7.4}$$

Note that

$$T(s) \cdot T(s) = 1 \Longrightarrow T'(s) \cdot T(s) = 0, \tag{7.5}$$

so indeed $N(s)$ is orthogonal to $T(s)$. We then define the binormal $B(s)$ by

$$B(s) = T(s) \times N(s). \tag{7.6}$$

For each s, the vectors $T(s)$, $N(s)$ and $B(s)$ are mutually orthogonal unit vectors, known as the Frenet frame for the curve $c(s)$. Rules governing the cross product yield

$$T(s) = N(s) \times B(s), \quad N(s) = B(s) \times T(s). \tag{7.7}$$

(For material on the cross product, see the exercises at the end of §12 of Chapter 2.)

The *torsion* of a curve measures the change in the plane generated by $T(s)$ and $N(s)$, or equivalently it measures the rate of change of $B(s)$. Note that, parallel to (7.5),

$$B(s) \cdot B(s) = 1 \Longrightarrow B'(s) \cdot B(s) = 0.$$

Also, differentiating (7.6) and using (7.4), we have

$$B'(s) = T'(s) \times N(s) + T(s) \times N'(s) = T(s) \times N'(s) \Longrightarrow B'(s) \cdot T(s) = 0. \tag{7.8}$$

We deduce that $B'(s)$ is parallel to $N(s)$. We define the torsion by

$$\frac{dB}{ds} = -\tau(s)N(s). \tag{7.9}$$

We complement the formulas (7.4) and (7.9) for dT/ds and dB/ds with one for dN/ds. Since $N(s) = B(s) \times T(s)$, we have

$$\frac{dN}{ds} = \frac{dB}{ds} \times T + B \times \frac{dT}{ds} = \tau N \times T + \kappa B \times N, \tag{7.10}$$

or

$$\frac{dN}{ds} = -\kappa(s)T(s) + \tau(s)B(s). \tag{7.11}$$

Together, (7.4), (7.9) and (7.11) are known as the Frenet-Serret formulas.

Example. Pick $a, b > 0$ and consider the helix

$$c(t) = (a \cos t, a \sin t, bt). \tag{7.12}$$

Then $v(t) = (-a \sin t, a \cos t, b)$ and $\|v(t)\| = \sqrt{a^2 + b^2}$, so we can pick $s = t\sqrt{a^2 + b^2}$ to parametrize by arc length. We have

$$T(s) = \frac{1}{\sqrt{a^2 + b^2}}(-a \sin t, a \cos t, b), \tag{7.13}$$

hence

$$\frac{dT}{ds} = \frac{1}{a^2+b^2}(-a\cos t, -a\sin t, 0). \tag{7.14}$$

By (7.4), this gives

$$\kappa(s) = \frac{a}{a^2+b^2}, \quad N(s) = (-\cos t, -\sin t, 0). \tag{7.15}$$

Hence

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{a^2+b^2}}(b\sin t, -b\cos t, a). \tag{7.16}$$

Then

$$\frac{dB}{ds} = \frac{1}{a^2+b^2}(b\cos t, b\sin t, 0), \tag{7.17}$$

so, by (7.9),

$$\tau(s) = \frac{b}{a^2+b^2}. \tag{7.18}$$

In particular, for the helix (7.12), we see that the curvature and torsion are *constant*.

Let us collect the Frenet-Serret equations

$$\begin{aligned} \frac{dT}{ds} &= \quad \kappa N \\ \frac{dN}{ds} &= -\kappa T \qquad\qquad + \tau B \\ \frac{dB}{ds} &= \quad - \tau N \end{aligned} \tag{7.19}$$

for a smooth curve $c(s)$ in $\mathbb{R}^3$, parametrized by arclength, with unit tangent $T(s)$, normal $N(s)$, and binormal $B(s)$, given by

$$N(s) = \frac{1}{\kappa(s)}T'(s), \quad B(s) = T(s) \times N(s), \tag{7.20}$$

assuming $\kappa(s) = \|T'(s)\| > 0$.

The basic existence and uniqueness theory, which will be presented in Chapter 4, applies to (7.19). If $\kappa(s)$ and $\tau(s)$ are given smooth functions on an interval $I = (a, b)$ and $s_0 \in I$, then, given $T_0, N_0, B_0 \in \mathbb{R}^3$, (7.19) has a unique solution on $s \in I$ satisfying

$$T(s_0) = T_0, \quad N(s_0) = N_0, \quad B(s_0) = B_0. \tag{7.21}$$

In fact, the case when $\kappa(s)$ and $\tau(s)$ are analytic will be subsumed in the material of §10 of this chapter. We now establish the following.

Proposition 7.1. *Assume κ and τ are given smooth functions on I, with $\kappa > 0$ on I. Assume $\{T_0, N_0, B_0\}$ is an orthonormal basis of $\mathbb{R}^3$, such that $B_0 = T_0 \times N_0$. Then there exists a smooth, unit-speed curve $c(s)$, $s \in I$, for which the solution to (7.19) and (7.21) is the Frenet frame.*

To construct the curve, take $T(s), N(s)$, and $B(s)$ to solve (7.19) and (7.21), pick $p \in \mathbb{R}^3$ and set

$$c(s) = p + \int_{s_0}^{s} T(\sigma)\, d\sigma, \tag{7.22}$$

so $T(s) = c'(s)$ is the velocity of this curve. To deduce that $\{T(s), N(s), B(s)\}$ is the Frenet frame for $c(s)$, for all $s \in I$, we need to know:

$$\{T(s), N(s), B(s)\} \text{ orthonormal, with } B(s) = T(s) \times N(s), \quad \forall\, s \in I. \tag{7.23}$$

In order to pursue the analysis further, it is convenient to form the 3×3 matrix-valued function

$$F(s) = (T(s), N(s), B(s)), \tag{7.24}$$

whose *columns* consist respectively of $T(s)$, $N(s)$, and $B(s)$. Then (7.23) is equivalent to

$$F(s) \in SO(3), \quad \forall\, s \in I, \tag{7.25}$$

with $SO(3)$ defined as in (12.4) of Chapter 2. The hypothesis on $\{T_0, N_0, B_0\}$ stated in Proposition 7.1 is equivalent to $F_0 = (T_0, N_0, B_0) \in SO(3)$. Now $F(s)$ satisfies the differential equation

$$F'(s) = F(s)A(s), \quad F(s_0) = F_0, \tag{7.26}$$

where

$$A(s) = \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}. \tag{7.27}$$

Note that

$$\frac{dF^*}{ds} = A(s)^* F(s)^* = -A(s)F(s)^*, \tag{7.28}$$

since $A(s)$ in (7.27) is skew-adjoint. Hence

$$\begin{aligned}\frac{d}{ds}\,F(s)F(s)^* &= \frac{dF}{ds}F(s)^* + F(s)\frac{dF^*}{ds} \\ &= F(s)A(s)F(s)^* - F(s)A(s)F(s)^* \\ &= 0.\end{aligned} \tag{7.29}$$

Thus, whenever (7.26)–(7.27) hold,

$$F_0F_0^* = I \Longrightarrow F(s)F(s)^* \equiv I, \tag{7.30}$$

and we have (7.23).

Let us specialize the system (7.19), or equivalently (7.26), to the case where κ and τ are *constant*, i.e.,

$$F'(s) = F(s)A, \quad A = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \tag{7.31}$$

with solution

$$F(s) = F_0\, e^{(s-s_0)A}. \tag{7.32}$$

We have already seen in that a helix of the form (7.12) has curvature κ and torsion τ, with

$$\kappa = \frac{a}{a^2+b^2}, \quad \tau = \frac{b}{a^2+b^2}, \tag{7.33}$$

and hence

$$a = \frac{\kappa}{\kappa^2+\tau^2}, \quad b = \frac{\tau}{\kappa^2+\tau^2}. \tag{7.34}$$

In (7.12), s and t are related by $t = s\sqrt{\kappa^2+\tau^2}$.

We can also see such a helix arise via a direct calculation of e^{sA}, which we now produce. First, a straightforward calculation gives, for A as in (7.31),

$$\det(\lambda I - A) = \lambda(\lambda^2+\kappa^2+\tau^2), \tag{7.35}$$

hence

$$\operatorname{Spec}(A) = \{0, \pm i\sqrt{\kappa^2+\tau^2}\}. \tag{7.36}$$

An inspection shows that we can take

$$v_1 = \frac{1}{\sqrt{\kappa^2+\tau^2}}\begin{pmatrix}\tau\\0\\\kappa\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad v_3 = \frac{1}{\sqrt{\kappa^2+\tau^2}}\begin{pmatrix}-\kappa\\0\\\tau\end{pmatrix}, \tag{7.37}$$

and then

$$Av_1 = 0, \quad Av_2 = \sqrt{\kappa^2+\tau^2}\,v_3, \quad Av_3 = -\sqrt{\kappa^2+\tau^2}\,v_2. \tag{7.38}$$

In particular, with respect to the basis $\{v_2, v_3\}$ of $V = \operatorname{Span}\{v_2, v_3\}$, $A|_V$ has the matrix representation

$$B = \sqrt{\kappa^2+\tau^2}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}. \tag{7.39}$$

We see that

$$e^{sA}v_1 = v_1, \tag{7.40}$$

while, in light of the calculations giving (2.8),

$$\begin{aligned} e^{sA}v_2 &= (\cos s\sqrt{\kappa^2+\tau^2})v_2 + (\sin s\sqrt{\kappa^2+\tau^2})v_3,\\ e^{sA}v_3 &= -(\sin s\sqrt{\kappa^2+\tau^2})v_2 + (\cos s\sqrt{\kappa^2+\tau^2})v_3. \end{aligned} \tag{7.41}$$

Exercises

1. Consider a curve $c(t)$ in $\mathbb{R}^3$, not necessarily parametrized by arclength. Show that the acceleration $a(t)$ is given by

$$a(t) = \frac{d^2s}{dt^2}T + \kappa\Big(\frac{ds}{dt}\Big)^2 N. \tag{7.42}$$

Hint. Differentiate $v(t) = (ds/dt)T(t)$ and use the chain rule $dT/dt = (ds/dt)(dT/ds)$, plus (7.4).

2. Show that

$$\kappa B = \frac{v \times a}{\|v\|^3}. \tag{7.43}$$

Hint. Take the cross product of both sides of (7.42) with T, and use (7.6).

3. In the setting of Exercises 1–2, show that

$$\kappa^2\tau\|v\|^6 = -a\cdot(v\times a'). \tag{7.44}$$

Deduce from (7.43)–(7.44) that

$$\tau = \frac{(v\times a)\cdot a'}{\|v\times a\|^2}. \tag{7.45}$$

Hint. Proceed from (7.43) to

$$\frac{d}{dt}(\kappa\|v\|^3)\,B + \kappa\|v\|^3\frac{dB}{dt} = \frac{d}{dt}(v\times a) = v\times a',$$

and use $dB/dt = -\tau(ds/dt)N$, as a consequence of (7.9). Then dot with a, and use $a\cdot N = \kappa\|v\|^2$, from (7.42), to get (7.44).

4. Consider the curve $c(t)$ in $\mathbb{R}^3$ given by

$$c(t) = (a\,\cos t, b\,\sin t, t),$$

where a and b are given positive constants. Compute the curvature, torsion, and Frenet frame.
Hint. Use (7.43) to compute κ and B. Then use $N = B\times T$. Use (7.45) to compute τ.

5. Suppose c and $\tilde{c}$ are two curves, both parametrized by arc length over $0\le s\le L$, and both having the same curvature $\kappa(s)>0$ and the same torsion $\tau(s)$. Show that there exit $x_0\in\mathbb{R}^3$ and $A\in O(3)$ such that

$$\tilde{c}(s) = Ac(s) + x_0, \quad \forall\ s\in[0,L].$$

Hint. To begin, show that if their Frenet frames coincide at $s=0$, i.e., $\widetilde{T}(0)=T(0)$, $\widetilde{N}(0)=N(0)$, $\widetilde{B}(0)=B(0)$, then $\widetilde{T}\equiv T$, $\widetilde{N}\equiv N$, $\widetilde{B}\equiv B$.

6. Suppose c is a curve in $\mathbb{R}^3$ with curvature $\kappa>0$. Show that there exists a plane in which $c(t)$ lies for all t if and only if $\tau\equiv 0$.
Hint. When $\tau\equiv 0$, the plane should be parallel to the orthogonal complement of B.

8. Variable coefficient systems

Here we consider a variable coefficient $n\times n$ first order system

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0\in\mathbb{C}^n, \tag{8.1}$$

and its inhomogeneous analogue. The general theory, which will be presented in Chapter 4, implies that if $A(t)$ is a smooth function of $t \in I = (a, b)$ and $t_0 \in I$, then (8.1) has a unique solution $x(t)$ for $t \in I$, depending linearly on x_0, so

$$x(t) = S(t, t_0)x_0, \quad S(t, t_0) \in \mathcal{L}(\mathbb{C}^n). \tag{8.2}$$

See §10 of this chapter for power series methods of constructing $S(t, t_0)$, when $A(t)$ is analytic. As we have seen,

$$A(t) \equiv A \Longrightarrow S(t, t_0) = e^{(t-t_0)A}. \tag{8.3}$$

However, for variable coefficient equations there is not such a simple formula, and the matrix entries of $S(t, s)$ can involve a multiplicity of new special functions, such as Bessel functions, Airy functions, Legendre functions, and many more. We will not dwell on this here, but we will note how $S(t, t_0)$ is related to a "complete set" of solutions to (8.1).

Suppose $x_1(t), \dots, x_n(t)$ are n solutions to (8.1) (but with different initial conditions). Fix $t_0 \in I$, and assume

$$x_1(t_0), \dots, x_n(t_0) \text{ are linearly independent in } \mathbb{C}^n, \tag{8.4}$$

or equivalently these vectors form a basis of $\mathbb{C}^n$. Given such solutions $x_j(t)$, we form the $n \times n$ matrix

$$M(t) = \big(x_1(t), \dots, x_n(t)\big), \tag{8.5}$$

whose jth column is $x_j(t)$. This matrix function solves

$$\frac{dM}{dt} = A(t)M(t). \tag{8.6}$$

The condition (8.4) is equivalent to the statement that $M(t_0)$ is invertible. We claim that if M solves (8.6) and $M(t_0)$ is invertible then $M(t)$ is invertible for all $t \in I$. To see this, we use the fact that the invertibility of $M(t)$ is equivalent to the non-vanishing of the quantity

$$W(t) = \det M(t), \tag{8.7}$$

called the *Wronskian* of $\{x_1(t), \dots, x_n(t)\}$. It is also notable that $W(t)$ solves a differential equation. In general we have

$$\frac{d}{dt} \det M(t) = \big(\det M(t)\big) \ \mathrm{Tr}\big(M(t)^{-1}M'(t)\big). \tag{8.8}$$

(See Exercises 1–3 below.) Let $\tilde{I} \subset I$ be the maximal interval containing t_0 on which $M(t)$ is invertible. Then (8.8) holds for $t \in \tilde{I}$. When (8.6) holds, we have $\operatorname{Tr}(M(t)^{-1}M'(t)) = \operatorname{Tr}(M(t)^{-1}A(t)M(t)) = \operatorname{Tr} A(t)$, so the Wronskian solves the differential equation

$$\frac{dW}{dt} = \bigl(\operatorname{Tr} A(t)\bigr)\, W(t). \tag{8.9}$$

Hence

$$W(t) = e^{b(t,s)}W(s), \quad b(t,s) = \int_s^t \operatorname{Tr} A(\tau)\, d\tau. \tag{8.10}$$

This implies $\tilde{I} = I$ and hence gives the asserted invertibility. From here we obtain the following.

Proposition 8.1. *If $M(t)$ solves (8.6) for $t \in I$ and $M(t_0)$ is invertible, then*

$$S(t,t_0) = M(t)M(t_0)^{-1}, \quad \forall\, t \in I. \tag{8.11}$$

Proof. We have seen that $M(t)$ is invertible for all $t \in I$. If $x(t)$ solves (8.1), set

$$y(t) = M(t_0)M(t)^{-1}x(t), \tag{8.12}$$

and apply d/dt. Using the identity

$$\frac{d}{dt}M(t)^{-1} = -M(t)^{-1}M'(t)M(t)^{-1}, \tag{8.13}$$

(see Exercise 4 below), we have

$$\begin{aligned} \frac{dy}{dt} &= -M(t_0)M(t)^{-1}A(t)M(t)M(t)^{-1}x(t) + M(t_0)M(t)^{-1}A(t)x(t) \\ &= 0, \end{aligned} \tag{8.14}$$

hence $y(t) = y(t_0)$ for $t \in I$, i.e.,

$$M(t_0)M(t)^{-1}x(t) \equiv x(t_0). \tag{8.15}$$

Applying $M(t)M(t_0)^{-1}$ to both sides of (8.15) gives (8.11).

Note also that, for $s, t \in I$,

$$S(t,s) = M(t)M(s)^{-1} \tag{8.16}$$

gives $S(t,s)x(s) = x(t)$ for each solution $x(t)$ to (8.1). We also have

$$S(t,t_0) = S(t,s)S(s,t_0), \quad S(t,s) = S(s,t)^{-1}. \tag{8.17}$$

There is a more general version of the Duhamel formula (4.5) for the solution to an inhomogeneous differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_0) = x_0. \tag{8.18}$$

Namely,

$$x(t) = S(t,t_0)x_0 + \int_{t_0}^{t} S(t,s)f(s)\,ds. \tag{8.19}$$

In fact, clearly (8.19) satisfies $x(t_0) = x_0$ and applying d/dt to (8.19) gives the equation (8.18). In view of (8.16), we have the alternative formula

$$x(t) = M(t)M(t_0)^{-1}x_0 + M(t)\int_{t_0}^{t} M(s)^{-1}f(s)\,ds, \tag{8.20}$$

for invertible $M(t)$ as in (8.5)–(8.6).

We note that there is a simple formula for the solution operator $S(t,s)$ to (8.1) in case the following commutativity hypothesis holds:

$$A(t)A(t') = A(t')A(t), \quad \forall\, t,t' \in I. \tag{CA}$$

We claim that if

$$B(t,s) = -\int_{s}^{t} A(\tau)\,d\tau, \tag{8.21}$$

then

$$(CA) \Longrightarrow \frac{d}{dt}\big(e^{B(t,s)}x(t)\big) = e^{B(t,s)}\big(x'(t) - A(t)x(t)\big), \tag{8.22}$$

from which it follows that

$$(CA) \Longrightarrow S(t,s) = e^{\int_s^t A(\tau)\,d\tau}. \tag{8.23}$$

(This identity fails in the absence of the hypothesis (CA).)

To establish (8.22), we note that (CA) implies

$$B(t,s)B(t',s) = B(t',s)B(t,s), \quad \forall\, s,t,t' \in I. \tag{CB}$$

Next,

$$
\begin{aligned}
(CB) \Longrightarrow \lim_{h\to 0} \frac{1}{h}\Big(e^{B(t+h,s)} - e^{B(t,s)}\Big) \\
&= \lim_{h\to 0} \frac{1}{h}\, e^{B(t,s)}\Big(e^{B(t+h,s)-B(t,s)} - I\Big) \\
&= -e^{B(t,s)}A(t) \\
\Longrightarrow \frac{d}{dt}e^{B(t,s)} &= -e^{B(t,s)}A(t),
\end{aligned}
\tag{8.24}
$$

from which (8.22) follows.

Here is an application of (8.23). Let $x(s)$ be a planar curve, on an interval about $s = 0$, parametrized by arc-length, with unit tangent $T(s) = x'(s)$. Then the Frenet-Serret equations (7.1) simplify to $T' = \kappa N$, with $N = JT$, i.e., to

$$
T'(s) = \kappa(s)JT(s), \tag{8.25}
$$

with J as in (2.1). Clearly the commutativity hypothesis (CA) holds for $A(s) = \kappa(s)J$, so we deduce that

$$
T(s) = e^{\lambda(s)J}T(0), \quad \lambda(s) = \int_0^s \kappa(\tau)\,d\tau. \tag{8.26}
$$

Recall that e^{tJ} is given by (2.8), i.e.,

$$
e^{tJ} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{8.27}
$$

Exercises

Exercises 1–3 lead to a proof of the formula (8.8) for the derivative of $\det M(t)$.

1. Let $A \in M(n,\mathbb{C})$. Show that, as $s \to 0$,

$$
\begin{aligned}
\det(I + sA) &= (1 + sa_{11})\cdots(1 + sa_{nn}) + O(s^2) \\
&= 1 + s\ \operatorname{Tr} A + O(s^2),
\end{aligned}
$$

   hence

$$
\frac{d}{ds}\det(I + sA)\big|_{s=0} = \operatorname{Tr} A.
$$

2. Let $B(s)$ be a smooth matrix valued function of s, with $B(0) = I$. Use Exercise 1 to show that
$$\frac{d}{ds} \det B(s)\big|_{s=0} = \operatorname{Tr} B'(0).$$
Hint. Write $B(s) = I + sB'(0) + O(s^2)$.

3. Let $C(s)$ be a smooth matrix valued function, and assume $C(0)$ is invertible. Use Exercise 2 plus
$$\det C(s) = \big(\det C(0)\big) \det B(s), \quad B(s) = C(0)^{-1}C(s)$$
to show that
$$\frac{d}{ds} \det C(s)\big|_{s=0} = \big(\det C(0)\big) \operatorname{Tr} C(0)^{-1}C'(0).$$
Use this to prove (8.8).
Hint. Fix t and set $C(s) = M(t+s)$, so
$$\frac{d}{dt} \det M(t) = \frac{d}{ds} \det C(s)\big|_{s=0}.$$

4. Verify the identity (8.13).
Hint. Set $U(t) = M(t)^{-1}$ and differentiate the identity $U(t)M(t) = I$.

Exercises 5–6 generalize (8.25)–(8.27) from the case of zero torsion (cf. Exercise 6 of §7) to the case

$$\tau(t) = \beta\kappa(t), \quad \beta \text{ constant.} \tag{8.28}$$

5. Assume $x(t)$ is a curve in $\mathbb{R}^3$ for which (8.28) holds. Show that $x(t) = x(0) + \int_0^t T(s)\, ds$, with
$$\big(T(t), N(t), B(t)\big) = e^{\sigma(t)K}\big(T(0), N(0), B(0)\big), \tag{8.29}$$
where
$$K = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\beta \\ 0 & \beta & 0 \end{pmatrix}, \quad \sigma(t) = \int_0^t \kappa(s)\, ds. \tag{8.30}$$

Hint. Use (7.26)–(7.27) and (CA) $\Rightarrow$ (8.21)–(8.23).

6. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$, and let
$$v_1 = (1+\beta^2)^{-1/2}(\beta e_1 + e_3), \quad v_2 = e_2, \quad v_3 = (1+\beta^2)^{-1/2}(e_1 - \beta e_3).$$
Show that v_1, v_2, v_3 form an orthonormal basis of $\mathbb{R}^3$ and, with K as in (8.30),
$$Kv_1 = 0, \quad Kv_2 = -(1+\beta^2)^{1/2} v_3, \quad Kv_3 = (1+\beta^2)^{1/2} v_2.$$
Deduce that
$$\begin{aligned} e^{\sigma K} v_1 &= v_1, \\ e^{\sigma K} v_2 &= (\cos\eta) v_2 - (\sin\eta) v_3, \\ e^{\sigma K} v_3 &= (\sin\eta) v_2 + (\cos\eta) v_3, \end{aligned}$$
where $\eta = (1+\beta^2)^{1/2}\sigma$.

7. Given $B \in M(n, \mathbb{C})$, write down the solution to
$$\frac{dx}{dt} = e^{tB} x, \quad x(0) = x_0.$$
Hint. Use (8.23).

Exercises 8–9 deal with a linear equation with periodic coefficients:
$$\frac{dx}{dt} = A(t)x, \quad A(t+1) = A(t). \tag{8.31}$$
Say $A(t) \in M(n, \mathbb{C})$.

8. Assume $M(t)$ solves (8.6), with $A(t)$ as in (8.31), and $M(0)$ is invertible. Show that
$$M(1) = C \Longrightarrow M(t+1) = M(t)C. \tag{8.32}$$

9. In the setting of Exercise 8, we know $M(t)$ is invertible for all t, so C is invertible. Results of Appendix A yield $X \in M(n, \mathbb{C})$ such that
$$e^X = C. \tag{8.33}$$
Show that
$$P(t) = M(t)e^{-tX} \Longrightarrow P(t+1) = P(t). \tag{8.34}$$
The representation
$$M(t) = P(t)e^{tX} \tag{8.35}$$
is called the *Floquet representation* of $M(t)$.

9. Variation of parameters and Duhamel's formula

An inhomogeneous equation

$$y'' + a(t)y' + b(t)y = f(t) \tag{9.1}$$

can be solved via the method of variation of parameters, if one is given a complete set $u_1(t), u_2(t)$ of solutions to the homogeneous equation

$$u_j'' + a(t)u_j' + b(t)u_j = 0. \tag{9.2}$$

The method (derived already in §12 of Chapter 1 when $a(t)$ and $b(t)$ are constant) consists of seeking a solution to (9.1) in the form

$$y(t) = v_1(t)u_1(t) + v_2(t)u_2(t), \tag{9.3}$$

and finding equations for $v_j(t)$ which can be solved and which work to yield a solution to (9.1). We have

$$y' = v_1u_1' + v_2u_2' + v_1'u_1 + v_2'u_2. \tag{9.4}$$

We impose the condition

$$v_1'u_1 + v_2'u_2 = 0. \tag{9.5}$$

Then $y'' = v_1'u_1' + v_2'u_2' + v_1u_1'' + v_2u_2''$, and plugging in (9.2) gives

$$y'' = v_1'u_1' + v_2'u_2' - (au_1' + bu_1)v_1 - (au_2' + bu_2)v_2, \tag{9.6}$$

hence

$$y'' + ay' + by = v_1'u_1' + v_2'u_2'. \tag{9.7}$$

Thus we have a solution to (9.1) in the form (9.3) provided v_1' and v_2' solve

$$\begin{aligned} v_1'u_1 + v_2'u_2 &= 0, \\ v_1'u_1' + v_2'u_2' &= f. \end{aligned} \tag{9.8}$$

This linear system for v_1', v_2' has the explicit solution

$$v_1' = -\frac{u_2}{W}f, \quad v_2' = \frac{u_1}{W}f, \tag{9.9}$$

where $W(t)$ is the Wronskian:

$$W = u_1u_2' - u_2u_1' = \det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}. \tag{9.10}$$

Then

$$\begin{aligned} v_1(t) &= -\int_{t_0}^t \frac{u_2(s)}{W(s)} f(s)\,ds + C_1, \\ v_2(t) &= \int_{t_0}^t \frac{u_1(s)}{W(s)} f(s)\,ds + C_2. \end{aligned} \tag{9.11}$$

So

$$y(t) = C_1u_1(t) + C_2u_2(t) + \int_{t_0}^t \bigl[u_2(t)u_1(s) - u_1(t)u_2(s)\bigr] \frac{f(s)}{W(s)}\,ds. \tag{9.12}$$

We can connect this formula with that produced in §8 as follows. If $y(t)$ solves (9.1), then $x(t) = (y(t), y'(t))^t$ solves the first order system

$$\frac{dx}{dt} = A(t)x + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \tag{9.13}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix}, \tag{9.14}$$

and a complete set of solutions to the homogeneous version of (9.13) is given by

$$x_j(t) = \begin{pmatrix} u_j(t) \\ u_j'(t) \end{pmatrix}, \quad j = 1, 2. \tag{9.15}$$

Thus we can set

$$M(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix}, \tag{9.16}$$

and as in (8.20) we have

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = M(t)M(t_0)^{-1} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + M(t) \int_{t_0}^t M(s)^{-1} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds, \tag{9.17}$$

solving (9.13) with $y(t_0) = y_0,\ y'(t_0) = y_1$. Note that

$$M(s)^{-1} = \frac{1}{W(s)} \begin{pmatrix} u_2'(s) & -u_2(s) \\ -u_1'(s) & u_1(s) \end{pmatrix}, \tag{9.18}$$

with $W(s)$, the Wronskian, as in (9.10). Thus the last term on the right side of (9.17) is equal to

$$\begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix} \int_{t_0}^t \frac{1}{W(s)} \begin{pmatrix} -u_2(s)f(s) \\ u_1(s)f(s) \end{pmatrix} ds, \tag{9.19}$$

and performing this matrix multiplication yields the integrand in (9.12). Thus we see that Duhamel's formula provides an alternative approach to the method of variation of parameters.

Exercises

1. Use the method of variation of parameters to solve

$$y'' + y = \tan t. \tag{9.20}$$

2. Convert (9.20) to a 2×2 first order system and use Duhamel's formula to solve it. Compare the result with your work on Exercise 1. Compare also with (4.6)–(4.10).

3. Do analogues of Exercises 1–2 for each of the following equations.

$$\text{(a)} \quad y'' + y = e^t,$$

$$\text{(b)} \quad y'' + y = \sin t,$$

$$\text{(c)} \quad y'' + y = t,$$

$$y'' + y = t^2.$$

4. Show that the Wronskian, defined by (9.10), satisfies the equation

$$\frac{dW}{dt} = -a(t)W, \tag{9.21}$$

if u_1 and u_2 solve (9.2).

5. Show that one solution to

$$u'' + 2tu' + 2u = 0 \tag{9.22}$$

is

$$u_1(t) = e^{-t^2}. \tag{9.23}$$

Set up and solve the differential equation for $W(t) = u_1 u_2' - u_2 u_1'$. Then solve the associated first order equation for u_2, to produce a linearly independent solution u_2 to (9.22), in terms of an integral.

6. Do Exercise 5 with (9.22) replaced by

$$u'' + 2u' + u = 0,$$

one of whose solutions is

$$u_1(t) = e^{-t}.$$

10. Power series expansions

Here we produce solutions to initial value problems

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(0) = x_0, \tag{10.1}$$

in terms of a power series expansion,

$$x(t) = x_0 + x_1 t + x_2 t^2 + \cdots = \sum_{k=0}^{\infty} x_k t^k, \tag{10.2}$$

under the hypothesis that the $n \times n$ matrix-valued function $A(t)$ and vector-valued function $f(t)$ are given by power series,

$$A(t) = \sum_{k=0}^{\infty} A_k t^k, \quad f(t) = \sum_{k=0}^{\infty} f_k t^k, \tag{10.3}$$

convergent for $|t| < R_0$. The coefficients x_k in (10.2) will be obtained recursively, as follows. Given $x(t)$ of the form (10.2), we have

$$\frac{dx}{dt} = \sum_{k=1}^{\infty} k x_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) x_{k+1} t^k, \tag{10.4}$$

and

$$\begin{aligned} A(t)x &= \sum_{j=0}^{\infty} A_j t^j \sum_{\ell=0}^{\infty} x_\ell t^\ell \\ &= \sum_{k=0}^{\infty} \Bigl(\sum_{j=0}^{k} A_{k-j} x_j \Bigr) t^k, \end{aligned} \tag{10.5}$$

so the power series on the left and right sides of (10.1) agree if and only if, for each $k \geq 0$,

$$(k+1) x_{k+1} = \sum_{j=0}^{k} A_{k-j} x_j + f_k. \tag{10.6}$$

In particular, the first three recursions are

$$\begin{aligned} x_1 &= A_0 x_0 + f_0, \\ 2x_2 &= A_1 x_0 + A_0 x_1 + f_1, \\ 3x_3 &= A_2 x_0 + A_1 x_1 + A_0 x_2 + f_2. \end{aligned} \tag{10.7}$$

To start the recursion, the initial condition in (10.1) specifies x_0.

We next address the issue of convergence of the power series thus produced for $x(t)$. We will establish the following

Proposition 10.1. *Under the hypotheses given above, the power series (10.2) converges to the solution $x(t)$ to (10.1), for $|t| < R_0$.*

Proof. The hypotheses on (10.3) imply that for each $R < R_0$, there exist $a, b \in (0, \infty)$ such that

$$\|A_k\| \le aR^{-k}, \quad \|f_k\| \le bR^{-k}, \quad \forall\, k \in \mathbb{Z}^+. \tag{10.8}$$

We will show that, given $r \in (0, R)$, there exists $C \in (0, \infty)$ such that

$$\|x_j\| \le Cr^{-j}, \quad \forall\, j \in \mathbb{Z}^+. \tag{10.9}$$

Such estimates imply that the power series (10.2) converges for $|t| < r$, for each $r < R_0$, hence for $|t| < R_0$.

We will prove (10.9) by induction. The inductive step is to assume it holds for all $j \le k$ and to deduce that it holds for $j = k+1$. This deduction proceeds as follows. We have, by (10.6), (10.8), and (10.9) for $j \le k$,

$$\begin{aligned} (k+1)\|x_{k+1}\| &\le \sum_{j=0}^{k} \|A_{k-j}\| \cdot \|x_j\| + \|f_k\| \\ &\le aC \sum_{j=0}^{k} R^{j-k} r^{-j} + bR^{-k} \\ &= aCr^{-k} \sum_{j=0}^{k} \Bigl(\frac{r}{R}\Bigr)^{k-j} + bR^{-k}. \end{aligned} \tag{10.10}$$

Now, given $0 < r < R$,

$$\sum_{j=0}^{k} \Bigl(\frac{r}{R}\Bigr)^{k-j} < \sum_{j=0}^{\infty} \Bigl(\frac{r}{R}\Bigr)^{j} = \frac{1}{1-\frac{r}{R}} = M(R,r) < \infty. \tag{10.11}$$

Hence

$$(k+1)\|x_{k+1}\| \le aCM(R,r)r^{-k} + br^{-k}. \tag{10.12}$$

We place on C the constraint that

$$C \ge b, \tag{10.13}$$

and obtain

$$\|x_{k+1}\| \le \frac{aM(R,r)+1}{k+1} r \cdot Cr^{-k-1}. \tag{10.14}$$

This gives the desired result

$$\|x_{k+1}\| \le Cr^{-k-1}, \tag{10.15}$$

as long as

$$\frac{aM(R,r)+1}{k+1}r \leq 1. \tag{10.16}$$

Thus, to finish the argument, we pick $K \in \mathbb{N}$ such that

$$K+1 \geq \big[aM(R,r)+1\big]r. \tag{10.17}$$

(Recall that we have a, R, r, and $M(R,r)$.) Then we pick $C \in (0,\infty)$ large enough that (10.9) holds for all $j \in \{0,1,\dots,K\}$, i.e., we take (in addition to (10.13))

$$C \geq \max_{0\leq j\leq K} r^j \|x_j\|. \tag{10.18}$$

Then for all $k \geq K$, the inductive step yielding (10.15) from the validity of (10.9) for all $j \leq k$ holds, and the inductive proof of (10.9) is complete.

For notational simplicity, we have discussed power series expansions about $t = 0$ so far, but the same considerations apply to power series about a more general point t_0. Thus we could replace (10.1) by

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_0) = x_0, \tag{10.19}$$

with $A(t)$ and $f(t)$ given by power series

$$A(t) = \sum_{k=0}^{\infty} A_k (t-t_0)^k, \quad f(t) = \sum_{k=0}^{\infty} f_k (t-t_0)^k, \tag{10.20}$$

for $|t - t_0| < R_0$, and find $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} x_k (t-t_0)^k. \tag{10.21}$$

The recursive formula for the coefficients x_k is again given by (10.6), and (10.8)–(10.18) apply without further change.

It is worth noting that, in (10.20)–(10.21),

$$A_k = \frac{1}{k!}A^{(k)}(t_0), \quad f_k = \frac{1}{k!}f^{(k)}(t_0), \quad x_k = \frac{1}{k!}x^{(k)}(t_0). \tag{10.22}$$

These formulas, say for f_k, arise as follows. Setting $t = t_0$ in (10.20) gives $f_0 = f(t_0)$. Generally, if the power series for $f(t)$ converges for $|t-t_0| < R_0$, so does the power series

$$f'(t) = \sum_{k=1}^{\infty} k f_k (t-t_0)^{k-1}, \tag{10.23}$$

and more generally,

$$f^{(n)}(t) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) f_k (t-t_0)^{k-n}, \tag{10.24}$$

and setting $t = t_0$ in (10.24) gives $f^{(n)}(t_0) = n! f_n$.

As an aside, we mention that a convenient way to prove (10.23) is to define $g(t)$ to be the power series on the right side of (10.23) and show that

$$\int_0^t g(s)\, ds = \sum_{k=1}^{\infty} f_k (t-t_0)^k = f(t) - f(t_0). \tag{10.25}$$

Compare the discussion regarding (1.45)–(1.50) in Chapter 1.

We next establish the following important fact about functions given by convergent power series.

Proposition 10.2. *If $f(t)$ is given by a power series as in (10.20), convergent for $|t-t_0| < R_0$, then f can also be expanded in a power series in $t-t_1$, for each $t_1 \in (t_0 - R_0, t_0 + R_0)$, with radius of convergence $R_0 - |t_0 - t_1|$.*

Proof. As in (10.22), such a power series would necessarily have the form

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_1)}{n!} (t-t_1)^n. \tag{10.26}$$

We claim this converges when $|t - t_1| < R_0 - |t_1 - t_0|$. To see this, apply (10.24) with $t = t_1$ to estimate $|f^{(n)}(t_1)|$. We have

$$\frac{f^{(n)}(t_1)}{n!} = \sum_{k=n}^{\infty} \binom{k}{n} f_k (t_1 - t_0)^{k-n}. \tag{10.27}$$

The assertion that the power series (10.20) for $f(t)$ converges for $|t-t_0| < R_0$ implies that, for each $r < R_0$, there exists $C < \infty$ such that $\|f_k\| \le C r^{-k}$. Hence (10.27) gives

$$\begin{aligned} \frac{\|f^{(n)}(t_1)\|}{n!} &\le C \sum_{k=n}^{\infty} \binom{k}{n} r^{-k} |t_1 - t_0|^{k-n} \\ &= C r^{-n} \sum_{k=n}^{\infty} \binom{k}{n} \Big(\frac{|t_1 - t_0|}{r}\Big)^{k-n}. \end{aligned} \tag{10.28}$$

To evaluate this last quantity, note that

$$\begin{aligned} \varphi(s) = (1-s)^{-(n+1)} &\Rightarrow \varphi^{(\ell)}(0) = (n+1)\cdots(n+\ell) \\ &\Rightarrow \varphi(s) = \sum_{\ell=0}^{\infty} \binom{n+\ell}{n} s^{\ell} = \sum_{k=n}^{\infty} \binom{k}{n} s^{k-n}, \end{aligned} \tag{10.29}$$

for $|s| < 1$, so (10.28) gives

$$\frac{\|f^{(n)}(t_1)\|}{n!} \le Cr^{-n}\Bigl(1 - \frac{|t_1 - t_0|}{r}\Bigr)^{-(n+1)} = C\bigl(r - |t_1 - t_0|\bigr)^{-n}\Bigl(1 - \frac{|t_1 - t_0|}{r}\Bigr)^{-1}. \tag{10.30}$$

This estimate implies the series (10.26) converges for $|t - t_1| < r - |t_1 - t_0|$, for each $r < R_0$, hence for $|t - t_1| < R_0 - |t_1 - t_0|$. The fact that the sum of the series is equal to $f(t)$ is a consequence of the formula (10.32) given below for the remainder $R_k(t, t_1)$ in the expansion

$$f(t) = \sum_{n=0}^{k} \frac{f^{(n)}(t_1)}{n!}(t - t_1)^n + R_k(t, t_1), \tag{10.31}$$

namely,

$$R_k(t, t_1) = \frac{1}{k!} \int_{t_1}^{t} (t - s)^k f^{(k+1)}(s)\, ds. \tag{10.32}$$

See Exercise 1 below for an approach to proving (10.32). The asserted convergence in (10.26) is equivalent to the statement that

$$\|R_k(t, t_1)\| \longrightarrow 0, \quad \text{as} \quad k \to \infty, \tag{10.33}$$

for $|t - t_1| < R_0 - |t_1 - t_0|$, which can be obtained from (10.30), or rather the analogous estimates on $\|f^{(n)}(s)\|/n!$, when s is between t_1 and t. See Exercise 3 below.

A (vector-valued) function f defined on an interval $I = (a, b)$ is said to be an *analytic function* on I if and only if for each $t_1 \in I$, there is an $r_1 > 0$ such that for $|t - t_1| < r_1$, $f(t)$ is given by a convergent power series in $t - t_1$. Again, such a power series is necessarily of the form (10.26). What has just been demonstrated implies that if f is given by a comvergent power series in $t - t_0$ for $|t - t_0| < R_0$, then f is analytic on the interval $(t_0 - R_0, t_0 + R_0)$. The following is a useful fact about analytic functions.

Lemma 10.3. *If f is analytic on (a, b) and $0 < \varepsilon < |b - a|/2$, then there exists $\delta > 0$ such that for each $t_1 \in [a + \varepsilon, b - \varepsilon]$, the power series (10.26) converges whenever $|t - t_1| < \delta$.*

We sketch an approach to the proof of Lemma 10.3. By hypothesis, each point $p \in [a + \varepsilon, b - \varepsilon] = J_\varepsilon$ is the center of an open interval I_p on which f is given by a convergent power series about p. Using results about *compactness* of J_ε presented in Appendix A of Chapter 4, it can be shown that there is a *finite* set $\{p_1, \dots, p_K\} \subset J_\varepsilon$ such that the intervals I_{p_j}, $1 \le j \le K$ cover J_ε. Given this, one can deduce the existence of $\delta > 0$ such that for each $t_1 \in J_\varepsilon$, the interval $(t_1 - \delta, t_1 + \delta)$ is contained in one of the intervals I_{p_j}, $1 \le j \le K$, and then the convergence of the power series of f about t_1 on $(t_1 - \delta, t_1 + \delta)$ follows from Proposition 10.2.

With these tools in hand, we have the following result.

Proposition 10.4. *Assume $A(t)$ and $f(t)$ are analytic on an interval (a, b) and $t_0 \in (a, b)$. Then the initial value problem (10.19) has a unique solution $x(t)$, analytic on (a, b).*

Proof. We know that $A(t)$ and $f(t)$ have power series (10.20), convergent for $|t - t_0| < R_0$, for some $R_0 > 0$, and the calculations from the first part of this section construct the solution $x(t)$ to (10.19) for $t \in (t_0 - R_0, t_0 + R_0)$. If this interval is not all of (a, b), we can extend $x(t)$ as follows.

Suppose $t_0 + R_0 \le b - \varepsilon$, and for such $\varepsilon > 0$ take $\delta > 0$ as in Lemma 10.1. Take $t_1 = t_0 + R_0 - \delta/2$. Then the previous argument constructs $\tilde{x}(t)$ as an analytic function on $(t_1 - \delta, t_1 + \delta)$, satisfying (10.19) with the initial condition given at t_1: $\tilde{x}(t_1) = x(t_1)$. It is readily verified that $\tilde{x} = x$ on $(t_1 - \delta, t_0 + R_0)$, so the solution $x(t)$ is extended past $t_0 + R_0$. Iterating this argument gives the conclusions of the proposition.

To conclude this section, we mention a connection with the study of functions of a complex variable, which the reader could pursue further, consulting texts on complex analysis, such as [**Ahl**]. Here is the general set-up. Let $\Omega \subset \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$. We say f is *complex differentiable* at $z_0 \in \Omega$ provided

$$\lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w} \tag{10.34}$$

exists. Here, $w \to 0$ in $\mathbb{C}$. If this limit exists, we call it

$$f'(z_0). \tag{10.35}$$

We say f is complex differentiable on Ω if is complex differentiable at each $z_0 \in \Omega$.

The relevance of this concept to the material of this section is the following. If $f(t)$ is given by the power series (10.20), absolutely convergent for real $t \in (t_0 - R_0, t_0 + R_0)$, then

$$f(z) = \sum_{k=0}^{\infty} f_k (z - t_0)^k \tag{10.36}$$

is absolutely convergent for $z \in \mathbb{C}$ satisfying $|z - t_0| < R_0$, i.e., on the disk

$$D_{R_0}(t_0) = \{z \in \mathbb{C} : |z - t_0| < R_0\}, \tag{10.37}$$

and it is complex differentiable on this disk. Furthermore, f' is complex differentiable on this disk, etc., including the kth order derivative $f^{(k)}$, and

$$f_k = \frac{f^{(k)}(t_0)}{k!}. \tag{10.38}$$

More is true, namely the following converse.

Theorem 10.5. *Assume f is complex differentiable on the open set $\Omega \subset \mathbb{C}$. Let $t_0 \in \Omega$ and assume $D_{R_0}(t_0) \subset \Omega$. Then f is given by a power series, of the form (10.36), absolutely convergent on $D_{R_0}(t_0)$.*

This is one of the central basic results of complex analysis. A proof can be found in Chapter 5 of [**Ahl**].

Example. Consider

$$f(z) = \frac{1}{z^2 + 1}. \tag{10.39}$$

This is well defined except at $\pm i$, where the denominator vanishes, and one can readily verify that f is complex differentiable on $\mathbb{C} \setminus \{i, -i\}$. It follows from Theorem 10.5 that if $t_0 \in \mathbb{C} \setminus \{i, -i\}$, this function is given by a power series expansion about t_0, absolutely convergent on $D_R(t_0)$, where

$$R = \min\{|t_0 - i|, |t_0 + i|\}. \tag{10.40}$$

In particular,

$$t_0 \in \mathbb{R} \Longrightarrow R = \sqrt{t_0^2 + 1} \tag{10.41}$$

gives the radius of convergence of the power series expansion of $1/(z^2 + 1)$ about t_0. This is easy to see directly for $t_0 = 0$:

$$\frac{1}{z^2 + 1} = \sum_{k=0}^{\infty} (-1)^k z^{2k}. \tag{10.42}$$

However, for other $t_0 \in \mathbb{R}$, it is not so easy to see directly that this function has a power series expansion about t_0 with radius of convergence given by (10.41). The reader might give this a try.

To interface this example with Proposition 10.4, we note that, by this proposition, plus the results just derived on $1/(z^2+1)$, the equation

$$\frac{dx}{dt} = \begin{pmatrix} 1 & (t^2+1)^{-1} \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{10.43}$$

has a solution that is analytic on $(-\infty, \infty)$. The power series expansion for the solution $x(t)$ about t_0 converges for $|t| < 1$ if $t_0 = 0$ (this is an easy consequence of Proposition 10.4 and (10.42)), and for other $t_0 \in \mathbb{R}$, it converges for $|t - t_0| < \sqrt{t_0^2 + 1}$ (as a consequence of Proposition 10.4 and Theorem 10.5).

Exercises

1. Assume f is a smooth (vector valued) function of $t \in (a, b)$. As in (10.31), define $R_k(t, y)$ for $t, y \in (a, b)$, by

$$f(t) = \sum_{n=0}^{k} \frac{f^{(n)}(y)}{n!}(t-y)^n + R_k(t, y). \tag{10.44}$$

Apply $\partial/\partial y$ to both sides of (10.47). Observe that the left side becomes 0 and there is considerable cancellation on the right side. Show that

$$\frac{\partial}{\partial y} R_k(t, y) = -\frac{f^{(k+1)}(y)}{k!}(t-y)^k, \quad R_k(t, t) = 0. \tag{10.45}$$

Integrate this to establish the remainder formula

$$R_k(t, y) = \frac{1}{k!} \int_y^t (t-s)^k f^{(k+1)}(s)\, ds. \tag{10.46}$$

2. Given g, continuous on $[y, t]$, show that there exists $s_1 \in [y, t]$ such that

$$\int_y^t g(s)\, ds = (t-y) g(s_1). \tag{10.47}$$

3. To show that (10.13) holds when the estimate (10.30) holds, and complete the proof of Proposition 10.4, argue as follows. By (10.30),

(10.48)
$$\begin{aligned}\|R_k(t,t_1)\| &\le C(k+1)\max_{s\in[t_1,t]}\Bigl(1-\frac{|s-t_0|}{r}\Bigr)^{-1}\int_{t_1}^{t}(t-s)^k(r-|s-t_0|)^{-k-1}\,ds\\ &\le C'(k+1)\int_{t_1}^{t}(t-s)^k(r-|t_0-s|)^{-k-1}\,ds,\end{aligned}$$

provided $|t_1-t_0|<r$. By (10.47), we have, for some $s_1\in[t_1,t]$,

(10.49)
$$\begin{aligned}\|R_k(t,t_1)\| &\le C'(k+1)|t_1-t|\cdot|t-s_1|^k(r-|s_1-t_0|)^{-k-1}\\ &\le C'(k+1)\frac{|t_1-t|}{r-|s_1-t_0|}\Bigl(\frac{|t-s_1|}{r-|s_1-t_0|}\Bigr)^k.\end{aligned}$$

Show that, provided $|t-t_1|<r-|t_1-t_0|$ and $s_1\in[t_1,t]$, this last quantity tends to 0 as $k\to\infty$.

4. Consider the function $g:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ given by

(10.50)
$$g(z)=e^{-1/z^2}.$$

Show that g is complex differentiable on $\mathbb{C}\setminus\{0\}$. Use Theorem 10.5 to deduce that $h:\mathbb{R}\to\mathbb{R}$, given by

(10.51)
$$\begin{aligned}h(t)=e^{-1/t^2},&\quad t\neq 0,\\ 0,&\quad t=0,\end{aligned}$$

is analytic on $\mathbb{R}\setminus\{0\}$. Show that h is not analytic on any interval containing 0. Compute
$$h^{(k)}(0).$$

5. Consider the Airy equation

(10.52)
$$y''=ty,\quad y(0)=y_0,\ y'(0)=y_1,$$

Introduced in (15.9) of Chapter 1. Show that this yields the first order system

(10.53)
$$\frac{dx}{dt}=(A_0+A_1t)x,\quad x(0)=x_0,$$

with

(10.54)
$$A_0=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad A_1=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad x_0=\begin{pmatrix}y_0\\y_1\end{pmatrix}.$$

Note that

$$A_0^2 = A_1^2 = 0, \tag{10.55}$$

and

$$A_0 A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10.56}$$

6. For a system of the form (10.53), whose solution has a power series of the form (10.2), the recursion (10.6) becomes

$$(k+1)x_{k+1} = A_0 x_k + A_1 x_{k-1}, \tag{10.57}$$

with the convention that $x_{-1} = 0$. Assume (10.55) holds. Show that

$$x_{k+3} = \frac{1}{k+3}\Bigl(\frac{1}{k+2} A_0 A_1 + \frac{1}{k+1} A_1 A_0\Bigr) x_k. \tag{10.58}$$

Note that when A_0 and A_1 are given by (10.54), this becomes

$$x_{k+3} = \frac{1}{k+3}\begin{pmatrix} \frac{1}{k+2} & \\ & \frac{1}{k+1} \end{pmatrix} x_k. \tag{10.59}$$

Establish directly from (10.58) that the series $\sum x_k t^k$ is absolutely convergent for all t.

11. Regular singular points

Here we consider equations of the form

$$t\frac{dx}{dt} = A(t)x, \tag{11.1}$$

where x takes values in $\mathbb{C}^n$, and $A(t)$, with values in $M(n, \mathbb{C})$, has a power series convergent for t in some interval $(-T_0, T_0)$,

$$A(t) = A_0 + A_1 t + A_2 t^2 + \cdots. \tag{11.2}$$

The system (1.1) is said to have a regular singular point at $t = 0$. One source of such systems is the following class of second order equations:

$$t^2 u''(t) + t b(t) u'(t) + c(t) u(t) = 0, \tag{11.3}$$

where $b(t)$ and $c(t)$ have convergent power series for $|t| < T_0$. In such a case, one can set

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad v(t) = tu'(t), \tag{11.4}$$

obtaining (11.1) with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -c(t) & 1 - b(t) \end{pmatrix}. \tag{11.5}$$

A paradigm example, studied in Exercises 6–13 of §15, Chapter 1, is the Bessel equation

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} + \Bigl(1 - \frac{\nu^2}{t^2}\Bigr)u = 0, \tag{11.6}$$

which via (11.4) takes the form (11.1), with

$$A(t) = A_0 + A_2 t^2, \quad A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{11.7}$$

It follows from Proposition 10.4 that, given $t_0 \in (0, T_0)$, the equation (11.1), with initial condition $x(t_0) = v$, has a unique solution analytic on $(0, T_0)$. Our goal here is to analyze the behavior of $x(t)$ as $t \searrow 0$.

A starting point for the analysis of (11.1) is the case $A(t) \equiv A_0$, i.e.,

$$t\frac{dx}{dt} = A_0 x. \tag{11.8}$$

The change of variable $z(s) = x(e^s)$ yields

$$\frac{dz}{ds} = A_0 z(s), \tag{11.9}$$

with solution

$$z(s) = e^{sA_0}v, \quad v = z(0), \tag{11.10}$$

hence

$$x(t) = e^{(\log t)A_0} = t^{A_0}v, \quad t > 0, \tag{11.11}$$

the latter identity defining t^{A_0}, for $t > 0$. Compare results on the "Euler equations" in Exercises 1–3, §15, Chapter 1.

Note that if $v \in \mathcal{E}(A_0, \lambda)$, then $t^{A_0} v = t^{\lambda} v$, which either blows up or vanishes as $t \searrow 0$, if $\operatorname{Re} \lambda < 0$ or $\operatorname{Re} \lambda > 0$, respectively, or oscillates rapidly as $t \searrow 0$, if λ is purely imaginary but not zero. On the other hand,

$$v \in \mathcal{N}(A_0) \Longrightarrow t^{A_0} v \equiv v. \tag{11.12}$$

It is useful to have the following extension of this result to the setting of (11.1).

Lemma 11.1. *If $v \in \mathcal{N}(A_0)$, then (11.1) has a solution given by a convergent power series on some interval about the origin,*

$$x(t) = x_0 + x_1 t + x_2 t^2 + \cdots, \quad x_0 = v, \tag{11.13}$$

as long as the eigenvalues of A_0 satisfy a mild condition, given in (11.18) below.

Proof. We produce a recursive formula for the coefficients x_k in (11.13), in the spirit of the calculations of §10. We have

$$t \frac{dx}{dt} = \sum_{k \geq 1} k x_k t^k, \tag{11.14}$$

and

$$\begin{aligned} A(t)x &= \sum_{j \geq 0} A_j t^j \sum_{\ell \geq 0} x_\ell t^\ell \\ &= A_0 x_0 + \sum_{k \geq 1} \sum_{\ell = 0}^{k} A_{k-\ell} x_\ell t^k. \end{aligned} \tag{11.15}$$

Equating the power series in (11.14) and (11.15) would be impossible without our hypothesis that $A_0 x_0 = 0$, but having that, we obtain the recursive formulas, for $k \geq 1$,

$$k x_k = A_0 x_k + \sum_{\ell=0}^{k-1} A_{k-\ell} x_\ell, \tag{11.16}$$

i.e.,

$$(kI - A_0) x_k = \sum_{\ell=0}^{k-1} A_{k-\ell} x_\ell. \tag{11.17}$$

Clearly we can solve uniquely for x_k provided

$$\forall\, k \in \mathbb{N} = \{1, 2, 3, \dots\}, \quad k \notin \operatorname{Spec} A_0. \tag{11.18}$$

This is the condition on $\operatorname{Spec} A_0$ mentioned in the lemma. As long as this holds, we can solve for the coefficients x_k for all $k \in \mathbb{N}$, obtaining (11.13). Estimates on these coefficients implying that (11.13) has a positive radius of convergence are quite similar to those made in §10, and will not be repeated here.

Our next goal is to extend this analysis to solutions to (11.1), for general $A(t)$, of the form (11.2), without such an hypothesis of membership in $\mathcal{N}(A_0)$ as in Lemma 11.1. We will seek a matrix-valued power series

$$U(t) = I + U_1 t + U_2 t^2 + \cdots \tag{11.19}$$

such that under the change of variable

$$x(t) = U(t)y(t), \tag{11.20}$$

(11.1) becomes

$$t\frac{dy}{dt} = A_0 y. \tag{11.21}$$

This will work as long as A_0 does not have two eigenvalues that differ by a nonzero integer, in which case a more elaborate construction will be needed.

To implement (11.20) and achieve (11.21), we have from (11.20) and (11.1) that

$$A(t)U(t)y = t\frac{dx}{dt} = tU(t)\frac{dy}{dt} + tU'(t)y, \tag{11.22}$$

which gives (11.21) provided $U(t)$ satisfies

$$t\frac{dU}{dt} = A(t)U(t) - U(t)A_0. \tag{11.23}$$

Now (11.23) has the same form as (11.1), i.e.,

$$t\frac{dU}{dt} = \mathcal{A}(t)U(t), \tag{11.24}$$

where U takes values in $M(n, \mathbb{C})$ and $\mathcal{A}(t)$ takes values in $\mathcal{L}(M(n, \mathbb{C}))$;

$$\begin{aligned} \mathcal{A}(t)U &= A(t)U(t) - U(t)A_0 \\ &= (\mathcal{A}_0 + \mathcal{A}_1 t + \mathcal{A}_2 t^2 + \cdots)U. \end{aligned} \tag{11.25}$$

In particular,

$$\mathcal{A}_0 U = A_0 U - U A_0 = [A_0, U] = C_{A_0} U, \tag{11.26}$$

the last identity defining $C_{A_0} \in \mathcal{L}(M(n,\mathbb{C}))$. Note that

$$U(0) = I \in \mathcal{N}(C_{A_0}), \tag{11.27}$$

so Lemma 11.1 applies to (11.24), i.e., to (11.23). In this setting, the recursion for U_k, $k \geq 1$, analogous to (11.16)–(11.17), takes the form

$$kU_k = [A_0, U_k] + \sum_{j=0}^{k-1} A_{k-j} U_j, \tag{11.28}$$

i.e.,

$$(kI - C_{A_0})U_k = \sum_{j=0}^{k-1} A_{k-j} U_j. \tag{11.29}$$

Recall $U_0 = I$. The condition for solvability of (11.29) for all $k \in \mathbb{N} = \{1, 2, 3, \dots\}$ is that no positive integer belong to $\operatorname{Spec} C_{A_0}$. Results of Chapter 2, §7 (cf. Exercise 9) yield the following:

$$\operatorname{Spec} A_0 = \{\lambda_j\} \Longrightarrow \operatorname{Spec} C_{A_0} = \{\lambda_j - \lambda_k\}. \tag{11.30}$$

Thus the condition that $\operatorname{Spec} C_{A_0}$ contain no positive integer is equivalent to the condition that A_0 have no two eigenvalues that differ by a nonzero integer. We thus have the following result.

Proposition 11.2. *Assume A_0 has no two eigenvalues that differ by a nonzero integer. Then there exists $T_0 > 0$ and $U(t)$ as in (11.19) with power series convergent for $|t| < T_0$, such that the general solution to (11.1) for $t \in (0, T_0)$ has the form*

$$x(t) = U(t) t^{A_0} v, \quad v \in \mathbb{C}^n. \tag{11.31}$$

Let us see how Proposition 11.2 applies to the Bessel equation (11.6), which we have recast in the form (11.1) with $A(t) = A_0 + A_2 t^2$, as in (11.7). Note that

$$A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix} \Longrightarrow \operatorname{Spec} A_0 = \{\nu, -\nu\}. \tag{11.32}$$

Thus $\operatorname{Spec} C_{A_0} = \{2\nu, 0, -2\nu\}$, and Proposition 11.2 applies whenever ν is not an integer or half-integer. Now as indicated in the exercises in §15 of Chapter 1, there is not an obstruction to series expansions consistent with (11.31) when ν is a half-integer. This is due to the special structure of (11.7), and suggests a more general result, of the following sort. Suppose only even powers of t appear in the series for $A(t)$:

$$A(t) = A_0 + A_2 t^2 + A_4 t^4 + \cdots . \tag{11.33}$$

Then we look for $U(t)$, solving (11.23), in the form

$$U(t) = I + U_2 t^2 + U_4 t^4 + \cdots . \tag{11.34}$$

In such a case, only even powers of t occur in the power series for (11.23), and in place of (11.28)–(11.29), one gets the following recursion formulas for $U_{2k},\ k \geq 1$:

$$2kU_{2k} = [A_0, U_{2k}] + \sum_{j=0}^{k-1} A_{2k-2j} U_{2j}, \tag{11.35}$$

i.e.,

$$(2kI - C_{A_0}) U_{2k} = \sum_{j=0}^{k-1} A_{2k-2j} U_{2j}. \tag{11.36}$$

This is solvable for U_{2k} as long as $2k \notin \operatorname{Spec} C_{A_0}$, and we have the following.

Proposition 11.3. *Assume $A(t)$ satisfies (11.33), and A_0 has no two eigenvalues that differ by a nonzero even integer. Then there exists $T_0 > 0$ and $U(t)$ as in (11.34), with power series convergent for $|t| < T_0$, such that the general solution to (11.1) for $t \in (0, T_0)$ has the form (11.31).*

We return to the Bessel equation (11.6) and consider the case $\nu = 0$. That is, we consider (11.1) with $A(t)$ as in (11.7), and

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{11.37}$$

Proposition 11.3 applies to this case (so does Proposition 11.2), and the general solution to (11.6) is the first entry in (11.31), where $U(t)$ has the form (11.34). Note that in case (11.37), $A_0^2 = 0$, so for $t > 0$,

$$t^{A_0} = \begin{pmatrix} 1 & \log t \\ 0 & 1 \end{pmatrix}. \tag{11.28}$$

Thus there are two linearly independent solutions to

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} + u = 0, \tag{11.39}$$

for $t > 0$, one having the form

$$\sum_{k\ge 0} a_k t^{2k}, \tag{11.40}$$

with coefficients a_k given recursively, and another having the form

$$\sum_{k\ge 0} (b_k + c_k \log t) t^{2k}, \tag{11.41}$$

again with coefficients b_k and c_k given recursively. The solution of the form (11.40) is as in (15.12)–(15.15) of Chapter 1 (with $\nu = 0$), while the solution of the form (11.41) is of a different sort than those constructed in that exercise set.

Proceeding beyond the purview of Propositions 11.2 and 11.3, we now treat the case when A_0 satisfies the following conditions. First,

$$\operatorname{Spec} C_{A_0} \ \text{ contains exactly one positive integer}, \ell, \tag{11.42}$$

and second

$$A_0 \ \text{ is diagonalizable}, \tag{11.43}$$

which implies

$$C_{A_0} \ \text{ is diagonalizable}; \tag{11.44}$$

cf. Chapter 2, §7, Exercise 8. Later we discuss weakening these conditions.

As in Proposition 11.2, we use a transformation of the form (11.20), i.e., $x(t) = U(t)y(t)$, with $U(t)$ as in (11.19), but this time our goal is to obtain for y, not the equation (11.21), but rather one of the form

$$t\frac{dy}{dt} = (A_0 + B_\ell t^\ell) y, \tag{11.45}$$

with the additional property of a special structure on the commutator $[A_0, B_\ell]$, given in (11.55) below. To get (11.45), we use (11.22) to obtain for $U(t)$ the equation

$$t\frac{dU}{dt} = A(t)U(t) - U(t)(A_0 + B_\ell t^\ell), \tag{11.46}$$

in place of (11.23). Taking $A(t)$ as in (11.2) and $U(t)$ as in (11.19), we have

$$t\frac{dU}{dt} = \sum_{k\ge 1} kU_k t^k, \tag{11.47}$$

$$A(t)U(t) - U(t)A_0 = \sum_{k\ge 1}[A_0, U_k]t^k + \sum_{k\ge 1}\Bigl(\sum_{j=0}^{k-1} A_{k-j}U_j\Bigr)t^k, \tag{11.48}$$

and hence solving (11.46) requires for $U_k,\ k \ge 1$, that

$$kU_k = [A_0, U_k] + \sum_{j=0}^{k-1} A_{k-j}U_j - \Gamma_k, \tag{11.49}$$

where

$$\begin{aligned} \Gamma_k = 0, &\qquad k < \ell, \\ B_\ell, &\qquad k = \ell, \\ U_{k-\ell}B_\ell, &\qquad k > \ell. \end{aligned} \tag{11.50}$$

Equivalently,

$$(kI - C_{A_0})U_k = \sum_{j=0}^{k-1} A_{k-j}U_j - \Gamma_k. \tag{11.51}$$

As before, (11.51) has a unique solution for each $k < \ell$, since $C_{A_0} - kI$ is invertible on $M(n, \mathbb{C})$. For $k = \ell$, the equation is

$$(\ell I - C_{A_0})U_\ell = \sum_{j=0}^{\ell-1} A_{\ell-j}U_j - B_\ell. \tag{11.52}$$

This time $C_{A_0} - \ell I$ is not invertible. However, if (11.44) holds,

$$M(n, \mathbb{C}) = \mathcal{N}(C_{A_0} - \ell I) \oplus \mathcal{R}(C_{A_0} - \ell I). \tag{11.53}$$

Consequently, given $\sum_{j=0}^{\ell-1} A_{\ell-j}U_j \in M(n, \mathbb{C})$, we can take

$$B_\ell \in \mathcal{N}(C_{A_0} - \ell I) \tag{11.54}$$

so that the right side of (11.52) belongs to $\mathcal{R}(C_{A_0} - \ell I)$, and then we can find a solution U_ℓ. We can uniquely specify U_ℓ by requiring $U_\ell \in \mathcal{R}(C_{A_0} - \ell I)$, though that is of no great consequence. Having such B_ℓ and U_ℓ, we can proceed to solve (11.51) for each $k > \ell$. Estimates on the coefficients U_k guaranteeing a positive radius of convergence for the power series (11.19) again follow by techniques of §10. We have reduced the problem of representing the general solution to (11.1) for $t \in (0, T_0)$ to that of representing the general solution to (11.45), given that (11.54) holds. The following result accomplishes this latter task. Note that (11.54) is equivalent to

$$[A_0, B_\ell] = \ell B_\ell, \quad \text{i.e.,}\quad A_0B_\ell = B_\ell(A_0 + \ell I). \tag{11.55}$$

Lemma 11.4. *Given A_0, $B_\ell \in M(n, \mathbb{C})$ satisfying (11.55), the general solution to (11.45) on $t > 0$ is given by*

$$y(t) = t^{A_0} t^{B_\ell} v, \quad v \in \mathbb{C}^n. \tag{11.56}$$

Proof. As mentioned earlier in this section, results of §10 imply that for each $v \in \mathbb{C}^n$, there is a unique solution to (11.45) on $t > 0$ satisfying $y(1) = v$. It remains to show that the right side of (11.56) satisfies (11.45). Indeed, if $y(t)$ is given by (11.56), then, for $t > 0$,

$$t\frac{dy}{dt} = A_0 t^{A_0} t^{B_\ell} v + t^{A_0} B_\ell t^{B_\ell} v. \tag{11.57}$$

Now (11.55) implies, for each $m \in \mathbb{N}$,

$$\begin{aligned} A_0^m B_\ell &= A_0^{m-1} B_\ell (A_0 + \ell I) = \cdots \\ &= B_\ell (A_0 + \ell I)^m, \end{aligned} \tag{11.58}$$

which in turn implies

$$e^{sA_0} B_\ell = B_\ell e^{s(A_0 + \ell I)} = B_\ell e^{s\ell} e^{sA_0}, \tag{11.59}$$

hence

$$t^{A_0} B_\ell = B_\ell t^\ell t^{A_0}, \tag{11.60}$$

so (11.57) gives

$$t\frac{dy}{dt} = (A_0 + B_\ell t^\ell) t^{A_0} t^{B_\ell} v, \tag{11.61}$$

as desired.

The construction involving (11.45)–(11.55) plus Lemma 11.4 yields the following.

Proposition 11.5. *Assume $A_0 \in M(n, \mathbb{C})$ has the property (11.42) and is diagonalizable. Then there exist $T_0 > 0$, $U(t)$ as in (11.19,) and $B_\ell \in M(n, \mathbb{C})$, satisfying (11.55), such that the general solution to (11.1) on $t \in (0, T_0)$ is*

$$x(t) = U(t) t^{A_0} t^{B_\ell} v, \quad v \in \mathbb{C}^n. \tag{11.62}$$

The following is an important property of B_ℓ.

Proposition 11.6. *In the setting of Proposition 11.5, B_ℓ is nilpotent.*

Proof. This follows readily from (11.55), which implies that for each $\lambda_j \in \operatorname{Spec} A_0$,

$$B_\ell : \mathcal{GE}(A_0, \lambda_j) \longrightarrow \mathcal{GE}(A_0, \lambda_j + \ell). \tag{11.63}$$

Remark. Note that if $B_\ell^{m+1} = 0$, then, for $t > 0$,

$$t^{B_\ell} = \sum_{k=0}^{m} \frac{1}{k!} (\log t)^k B_\ell^k. \tag{11.64}$$

Let us apply these results to the Bessel equation (11.6) in case $\nu = n$ is a positive integer. We are hence looking at (11.1) when

$$A(t) = A_0 + A_2 t^2, \quad A_0 = \begin{pmatrix} 0 & 1 \\ n^2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{11.65}$$

We have

$$\operatorname{Spec} A_0 = \{n, -n\}, \quad \operatorname{Spec} C_{A_0} = \{2n, 0, -2n\}. \tag{11.66}$$

Clearly A_0 is diagonalizable. The recursion (11.51) for U_k takes the form

$$(kI - C_{A_0}) U_k = \Sigma_k + \Gamma_k, \tag{11.67}$$

where

$$\begin{aligned} \Sigma_k = 0, \qquad & k < 2, \\ A_2, \qquad & k = 2, \\ A_2 U_{k-2}, \quad & k > 2, \end{aligned} \tag{11.68}$$

and

$$\begin{aligned} \Gamma_k = 0, \qquad & k < 2n, \\ B_{2n}, \qquad & k = 2n, \\ U_{k-2n} B_{2n}, \quad & k > 2n. \end{aligned} \tag{11.69}$$

In particular, the critical equation (11.52) is

$$(2nI - C_{A_0}) U_{2n} = A_2 U_{2n-2} + B_{2n}, \tag{11.70}$$

and we solve this after picking

$$B_{2n} \in \mathcal{N}(C_{A_0} - 2nI), \tag{11.71}$$

such that the right side of (11.69) belongs to $\mathcal{R}(C_{A_0} - 2nI)$. We have from Chapter 2, §7, Exercise 9, that (since A_0 is diagonalizable)

$$\mathcal{N}(C_{A_0} - 2nI) = \mathrm{Span}\{vw^t : v \in \mathcal{E}(A_0, n),\ w \in \mathcal{E}(A_0^t, -n)\}. \tag{11.72}$$

For A_0 in (11.64), v is a multiple of $(1, n)^t$ and w^t is a multiple of $(-n, 1)$, so

$$B_{2n} = \beta_n B_{2n}^{\#}, \quad \beta_n \in \mathbb{C}, \quad B_{2n}^{\#} = \begin{pmatrix} -n & 1 \\ -n^2 & n \end{pmatrix}. \tag{11.73}$$

Note that $B_{2n}^2 = 0$. Consequently the general solution to (11.1) in this case takes the form

$$x(t) = U(t) t^{A_0} t^{B_{2n}} v, \tag{11.74}$$

with

$$t^{B_{2n}} = I + (\log t) B_{2n}. \tag{11.75}$$

Note that

$$\mathcal{N}(B_{2n}) = \mathrm{Span} \begin{pmatrix} 1 \\ n \end{pmatrix} = \mathcal{E}(A_0, n), \tag{11.76}$$

so

$$U(t) t^{A_0} t^{B_{2n}} \begin{pmatrix} 1 \\ n \end{pmatrix} = U(t) t^{A_0} \begin{pmatrix} 1 \\ n \end{pmatrix} = U(t) t^n \begin{pmatrix} 1 \\ n \end{pmatrix} \tag{11.77}$$

is a regular solution to (11.1). Its first component is, up to a constant multiple, the solution $J_n(t)$ (in case $\nu = n$) studied in Chapter 1, §15, Exercises 6–13. The recursion gives results similar to (15.13)–(15.15) of Chapter 1, and $U(t)$ has infinite radius of convergence. Note also that

$$A_0 \begin{pmatrix} 1 \\ -n \end{pmatrix} = -n \begin{pmatrix} 1 \\ -n \end{pmatrix}, \quad B_{2n} \begin{pmatrix} 1 \\ -n \end{pmatrix} = -2n\beta_n \begin{pmatrix} 1 \\ n \end{pmatrix}, \tag{11.78}$$

which in concert with (11.75)–(11.76) gives

$$\begin{aligned} U(t) t^{A_0} t^{B_{2n}} \begin{pmatrix} 1 \\ -n \end{pmatrix} &= U(t) t^{A_0} \begin{pmatrix} 1 \\ -n \end{pmatrix} - 2n\beta_n (\log t) U(t) t^{A_0} \begin{pmatrix} 1 \\ n \end{pmatrix} \\ &= U(t) t^{-n} \begin{pmatrix} 1 \\ -n \end{pmatrix} - 2n\beta_n (\log t) U(t) t^n \begin{pmatrix} 1 \\ n \end{pmatrix}. \end{aligned} \tag{11.79}$$

The first component gives a solution to (11.6), with $\nu = n$, complementary to $J_n(t)$, for $t > 0$.

REMARK. When (11.1) is a 2×2 system, either Proposition 11.2 or Proposition 11.5 will be applicable. Indeed, if $A_0 \in M(n,\mathbb{C})$ and its two eigenvalues differ by a nonzero integer ℓ, then A_0 is diagonalizable, and

$$\operatorname{Spec} C_{A_0} = \{\ell, 0, -\ell\},$$

so (11.42) holds.

To extend the scope of Proposition 11.5, let us first note that the hypothesis (11.43) that A_0 is diagonalizable was used only to pass to (11.53), so we can replace this hypothesis by

$$\ell \in \mathbb{N} \cap \operatorname{Spec} C_{A_0} \Longrightarrow M(n,\mathbb{C}) = \mathcal{N}(C_{A_0} - \ell I) \oplus \mathcal{R}(C_{A_0} - \ell I). \tag{11.80}$$

We now show that we can drop hypothesis (11.42). In general, if $\operatorname{Spec} C_{A_0} \cap \mathbb{N} \neq \emptyset$, we have a finite set

$$\operatorname{Spec} C_{A_0} \cap \mathbb{N} = \{\ell_j : 1 \le j \le m\}; \tag{11.81}$$

say $\ell_1 < \cdots < \ell_m$. In this more general setting, we use a transformation of the form (11.20), i.e., $x(t) = U(t)y(t)$, with $U(t)$ as in (11.19), to obtain for y an equation of the form

$$t\frac{dy}{dt} = (A_0 + B_{\ell_1} t^{\ell_1} + \cdots + B_{\ell_m} t^{\ell_m})y, \tag{11.82}$$

with commutator properties on $[A_0, B_{\ell_j}]$ analogous to (11.55) (see (11.85) below). In this setting, in place of (11.46), we aim for

$$t\frac{dU}{dt} = A(t)U(t) - U(t)(A_0 + B_{\ell_1} t^{\ell_1} + \cdots + B_{\ell_m} t^{\ell_m}). \tag{11.83}$$

We continue to have (11.47)–(11.51), with a natural replacement for Γ_k, which the reader can supply. The equation (11.51) for $k \notin \{\ell_1, \dots, \ell_m\}$ is uniquely solvable for U_k because $kI - C_{A_0}$ is invertible. For $k = \ell_j$, $1 \le j \le m$, one can pick

$$B_{\ell_j} \in \mathcal{N}(C_{A_0} - \ell_j I), \tag{11.84}$$

and solve the appropriate variant of (11.52), using (11.80). Note that (11.84) is equivalent to

$$[A_0, B_{\ell_j}] = \ell_j B_{\ell_j}, \quad \text{i.e.,} \quad A_0 B_{\ell_j} = B_{\ell_j}(A_0 + \ell_j I). \tag{11.85}$$

Proposition 11.7. *Assume $A_0 \in M(n, \mathbb{C})$ has the property (11.80). For each ℓ_j as in (11.81), take B_{ℓ_j} as indicated above, and set*

$$B = B_{\ell_1} + \cdots + B_{\ell_m}. \tag{11.86}$$

Then there exist $T_0 > 0$ and $U(t)$ as in (11.19) such that the general solution to (11.1) on $t \in (0, T_0)$ is

$$x(t) = U(t) t^{A_0} t^B v, \quad v \in \mathbb{C}^n. \tag{11.87}$$

Proof. It suffices to show that the general solution to (11.82) is

$$y(t) = t^{A_0} t^B v, \quad v \in \mathbb{C}^n, \tag{11.88}$$

given that B_{ℓ_j} satisfy (11.85). In turn, it suffices to show that if $y(t)$ is given by (11.88), then (11.82) holds. To verify this, write

$$t\frac{dy}{dt} = A_0 t^{A_0} t^B v + t^{A_0}(B_{\ell_1} + \cdots + B_{\ell_m}) t^B v. \tag{11.89}$$

Now (11.85) yields

$$A_0^k B_{\ell_j} = B_{\ell_j}(A_0 + \ell_j I)^k, \quad \text{hence} \quad t^{A_0} B_{\ell_j} = B_{\ell_j} t^{\ell_j} t^{A_0}, \tag{11.90}$$

which together with (11.89) yields (11.82), as needed.

Parallel to Proposition 11.6, we have the following.

Proposition 11.8. *In the setting of Proposition 11.7, B is nilpotent.*

Proof. By (11.85), we have, for each $\lambda_j \in \operatorname{Spec} A_0$,

$$B : \mathcal{GE}(A_0, \lambda_j) \longrightarrow \mathcal{GE}(A_0, \lambda_j + \ell_1) \oplus \cdots \oplus \mathcal{GE}(A_0, \lambda_j + \ell_m), \tag{11.91}$$

which readily implies nilpotence.

Exercises

1. In place of (11.3), consider second order equations of the form

$$t u''(t) + b(t) u'(t) + c(t) u(t) = 0, \tag{11.92}$$

where $b(t)$ and $c(t)$ have convergent power series in t for $|t| < T_0$. In such a case, show that setting

$$x(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} \tag{11.93}$$

yields a sysyem of the form (11.1) with

$$A(t) = \begin{pmatrix} 0 & t \\ -c(t) & -b(t) \end{pmatrix}. \tag{11.94}$$

Contrast this with what you would get by multiplying (11.92) by t and using the formula (11.4) for $x(t)$.

2. Make note of how to extend the study of (11.1) to

$$(t - t_0)\frac{dx}{dt} = A(t)x, \tag{11.95}$$

when $A(t) = \sum_{k\geq 0} A_k(t - t_0)^k$ for $|t - t_0| < T_0$. We say t_0 is a regular singular point for (11.95).

3. The following is known as the hypergeometric equation:

$$t(1 - t)u''(t) + [\gamma - (\alpha + \beta + 1)t]u'(t) - \alpha\beta u(t) = 0. \tag{11.96}$$

Show that $t_0 = 0$ and $t_0 = 1$ are regular singular points and construct solutions near these points, given α, β, and γ.

4. The following is known as the confluent hypergeometric equation:

$$tu''(t) + (\gamma - t)u'(t) - \alpha u(t) = 0. \tag{11.97}$$

Show that $t_0 = 0$ is a regular singular point and construct solutions near this point, given α and γ.

5. Let $B(t)$ be analytic in t for $|t| > a$. We say that the equation

$$\frac{dy}{dt} = B(t)y \tag{11.98}$$

has a regular singular point at infinity provided that the change of variable

$$x(t) = y\Big(\frac{1}{t}\Big) \tag{11.99}$$

transforms (11.98) to an equation with a regular singular point at $t = 0$. Specify for which $B(t)$ this happens.

6. Show that the hypergeometric equation (11.96) has a regular singular point at infinity.

7. What can you say about the behavior as $t \searrow 0$ of solutions to (11.1) when

$$A(t) = A_0 + A_1 t, \quad A_0 = \begin{pmatrix} 2 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ & & 1 \\ & & 0 \end{pmatrix}?$$

8. What can you say about the behavior as $t \searrow 0$ of solutions to (11.1) when

$$A(t) = A_0 + A_1 t, \quad A_0 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ & & 1 \\ & & 0 \end{pmatrix}?$$

9. In the context of Lemma 11.4, i.e., with A_0 and B_ℓ satisfying (11.55), show that

$$B_\ell \text{ and } e^{2\pi i A_0} \text{ commute.}$$

More generally, in the context of Proposition 11.7, with A_0 and B satisfying (11.85)–(11.86), show that

$$B \text{ and } e^{2\pi i A_0} \text{ commute.}$$

Deduce that for all $t > 0$

$$t^{A_0} e^{2\pi i A_0} t^B e^{2\pi i B} = t^{A_0} t^B C, \quad C = e^{2\pi i A_0} e^{2\pi i B}. \tag{11.100}$$

10. In the setting of Exercise 9, pick $E \in M(n, \mathbb{C})$ such that

$$e^{2\pi i E} = C. \tag{11.101}$$

(Cf. Appendix A.) Set

$$Q(t) = t^{A_0} t^B t^{-E}, \quad t > 0. \tag{11.102}$$

Show that there exists $m \in \mathbb{Z}^+$ such that $t^m Q(t)$ is a polynomial in t, with coefficients in $M(n, \mathbb{C})$. Deduce that, in the setting of Proposition 11.7, the general solution to (11.1) on $t \in (0, T_0)$ is

$$x(t) = U(t)Q(t)t^E v, \quad c \in \mathbb{C}^n, \tag{11.103}$$

with $U(t)$ as in (11.19), E as in (11.101), and $Q(t)$ as in (11.102), so that $t^m Q(t)$ is a polynomial in t.

A. Logarithms of matrices

Given $C \in M(n, \mathbb{C})$, we say $X \in M(n, \mathbb{C})$ is a logarithm of C provided

$$e^X = C. \tag{A.1}$$

In this appendix, we aim to prove the following:

Proposition A.1. *If $C \in M(n, \mathbb{C})$ is invertible, there exists $X \in M(n, \mathbb{C})$ satisfying (A.1).*

Let us start with the case $n = 1$, i.e., $C \in \mathbb{C}$. In case C is a positive real number, we can take $X = \log C$, defined as in Chapter 1, §1; cf. (1.21)–(1.27). More generally, for $C \in \mathbb{C} \setminus 0$, we can write

$$C = |C|e^{i\theta}, \quad X = \log|C| + i\theta. \tag{A.2}$$

Note that the logarithm X of C is not uniquely defined. If $X \in \mathbb{C}$ solves (A.1), so does $X + 2\pi i k$ for each $k \in \mathbb{Z}$. As is customary, for $C \in \mathbb{C} \setminus 0$, we will denote any such solution by $\log C$.

Let us now take an invertible $C \in M(n, \mathbb{C})$ with $n > 1$, and look for a logarithm, i.e., a solution to (A.1). Such a logarithm is easy to produce if C is diagonalizable, i.e., if for some invertible $B \in M(n, \mathbb{C})$,

$$B^{-1}CB = D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}. \tag{A.3}$$

Then

$$Y = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}, \quad \mu_k = \log \lambda_k \Longrightarrow e^Y = D, \tag{A.4}$$

and so

$$e^{BYB^{-1}} = BDB^{-1} = C. \tag{A.5}$$

Similar arguments, in concert with results of Chapter 2, §§7–8, show that to prove Proposition A.1 it suffices to construct a logarithm of

$$C = \lambda(I+N), \quad \lambda \in \mathbb{C} \setminus 0, \quad N^n = 0. \tag{A.6}$$

In turn, if we can solve for Y the equation

$$e^Y = I + N, \tag{A.7}$$

given N nilpotent, then

$$\mu = \log \lambda \Longrightarrow e^{\mu I + Y} = \lambda(I+N), \tag{A.8}$$

so it suffices to solve (A.7) for $Y \in M(n, \mathbb{C})$.

We will produce a solution Y in the form of a power series in N. To prepare for this, we first strike off on a slight tangent and produce a series solution to

$$e^{X(t)} = I + tA, \quad A \in M(n, \mathbb{C}), \ \|tA\| < 1. \tag{A.9}$$

Taking a cue from the power series for $\log(1+t)$ given in Chapter 1, (1.56), we establish the following.

Proposition A.2. *In case $\|tA\| < 1$, (A.9) is solved by*

$$\begin{aligned} X(t) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k A^k \\ &= tA - \frac{t^2}{2} A^2 + \frac{t^3}{3} A^3 - \cdots . \end{aligned} \tag{A.10}$$

Proof. If $X(t)$ is given by (A.10), we have

$$\begin{aligned} \frac{dX}{dt} &= A - tA^2 + t^2 A^3 - \cdots \\ &= A(I - tA + t^2 A^2 - \cdots) \\ &= A(I+tA)^{-1}. \end{aligned} \tag{A.12}$$

Hence

$$\frac{d}{dt} e^{-X(t)} = -e^{-X(t)} A (I+tA)^{-1}, \tag{A.12}$$

for $|t| < 1/\|A\|$; cf. §1, Exercise 10. It follows that

$$\frac{d}{dt}\Big(e^{-X(t)}(I+tA)\Big) = e^{-X(t)}\Big(-A(I+tA)^{-1}(I+tA) + A\Big) = 0, \tag{A.13}$$

so

$$e^{-X(t)}(I+tA) \equiv e^{-X(0)} = I, \tag{A.14}$$

which implies (A.9).

The task of solving (A.7) and hence completing the proof of Proposition A.1 is accomplished by the following result.

Proposition A.3. *If $N \in M(n, \mathbb{C})$ is nilpotent, then for all $t \in \mathbb{R}$,*

$$e^{Y(t)} = I + tN \tag{A.15}$$

is solved by

$$Y(t) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} t^k N^k. \tag{A.16}$$

Proof. If $Y(t)$ is given by (A.16), we see that $Y(t)$ is nilpotent and that $e^{Y(t)}$ is a polynomial in t. Thus both sides of (A.15) are polynomials in t, and Proposition A.2 implies they are equal for $|t| < 1/\|N\|$, so (A.15) holds for all $t \in \mathbb{R}$.