STOR 435 Homework 21

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1.

a) $M_Y(t) = \prod_{i=1}^{i=30} M_{X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{30}$

$Y \sim \text{Gamma}\left(\alpha = 30, \lambda = \frac{1}{8}\right),\ f_Y(y) = \frac{1}{8^{30}\Gamma(\alpha)} x^{29} e^{-\frac{x}{8}}$

b) $M_{\bar{Y}}(t) = \mathbb{E}\left(e^{t\frac{Y}{30}}\right) = M_Y\left(\frac{t}{30}\right) = \left(\frac{\lambda}{\lambda - \frac{t}{30}}\right)^{30} = \left(\frac{30\lambda}{30\lambda - t}\right)^{30}$

$\bar{Y} \sim \text{Gamma}\left(\alpha = 30, \lambda = \frac{15}{4}\right),\ f_Y(y) = \frac{15^{30}}{4^{30}\Gamma(\alpha)} x^{29} e^{-\frac{15x}{4}}$

2. We first note that for a normal random variable $X \sim N(\mu, \sigma^2)$, $aX \sim N(a\mu, a^2\sigma^2)$

a) $M_Y(t) = M_{X_1}(2t) M_{X_2}(-3t) M_{X_3}(-5t) = e^{(2\mu_1 - 3\mu_2 - 5\mu_3)t + \frac{4\sigma_1^2 + 9\sigma_2^2 + 25\sigma_3^2}{2} t^2} = e^{-31t + \frac{2077}{2} t^2}$

$Y \sim N(\mu = -31, \sigma^2 = 2077)$

b) $\mathbb{P}(Y > 3) = 1 - F_Y(3) \approx 0.2278$

3. $\mathbb{E}(X^2) = \text{Var}(X) + \mathbb{E}(X)^2 = 8,\ \mathbb{E}(Y^2) = \text{Var}(Y) + \mathbb{E}(Y)^2 = 18,\ \mathbb{E}(Z^2) = \text{Var}(Z) + \mathbb{E}(Z)^2 = 37$

a) $\mathbb{E}(g(X, Y, Z)) = 2 + \mathbb{E}(Y)\mathbb{E}(Z) - 3\mathbb{E}(X^2)\mathbb{E}(Y) + 6\mathbb{E}(X^2)\mathbb{E}(Y^2)\mathbb{E}(Z^2) = 31901$

b) $\mathbb{P}(2X + Y \geqslant Z - Y) = \mathbb{P}(2X + 2Y - Z \geqslant 0)$ $(*)$

$W := 2X + 2Y - Z \sim N(\mu = 1, \sigma^2 = 88)$, therefore $(*) = 1 - F_W(0) \approx 0.5424$

4. $Y \sim N(\mu = 20 \times 10 + 30 \times 11, \sigma^2 = 20^2 \times 1 + 30^2 \times 1.2) \sim N(\mu = 530, \sigma^2 = 1480)$

$\mathbb{P}(Y > 480) = 1 - F_Y(480) = 0.5135$

5.

a) $\int_0^1 \int_0^1 c(x^2 + y^2) dx dy = c\frac{2}{3} = 1 \Rightarrow c = \frac{3}{2}$

$f_Y(y) = \int_0^1 c(x^2 + y^2) dx = \frac{1}{2} + \frac{3}{2} y^2$

b) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{c(x^2 + y^2)}{c\left(\frac{1}{3} + y^2\right)} = \frac{x^2 + y^2}{1/3 + y^2}$

c) $\mathbb{P}(1/4 < X < 1/2 | Y = 3/4) = \int_{1/4}^{1/2} \frac{x^2 + (3/4)^2}{1/3 + (3/4)^2} dx = \frac{17}{86}$