WEDNESDAY WEEK 3 NOTES

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1. Counting

This section is the start of our next major topic. There will be many subsections here, and since this is an incredibly important chapter, we will be looking at lots of examples.

1.1. Basic Counting Principles.

At the end of the day, most of counting boils down to "choices." Keeping track of what choices you have the make will make counting problems that much easier. Consider the following example:

Problem. Dinner at a restaurant consists of an appetizer, main course, and dessert. There are five choices of appetizer, six main courses, and four possible desserts. How many possible dinners are there?

You may have encountered this sort of problem in school already. If you recall, the answer boils down to multiplication: $5 \cdot 6 \cdot 4 = 120$ possible dinners. This example illustrates the **multiplication principle**.

Principle (Multiplication Principle)**.** Let $A_1, A_2, \ldots, A_k$ be ordered events. Suppose there are n_1 possible outcomes for event A_1, n_2 possible outcomes for event A_2, and so on. Then the total number of possible outcomes for: A_1 then A_2 then A_3, and so on, is $n_1 n_2 n_3 \cdots n_k$.

Intuitively, one can imagine that the number of outcomes is in bijection with $A_1 \times A_2 \times \cdots \times A_k$, which we know has size $|A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|$. Regardless of how you want to think about it, this principle is arguably the most useful counting principle.

Example. Suppose we are in the situation in the earlier problem. Suppose dessert and appetizer are optional. How many dinners are possible? Well now, instead of five choices of appetizer, we actually have six: the five choices, plus the option of getting nothing, which is a sixth choice. Similarly, we now have five choices for dessert (the four plus the option of getting nothing). Thus, there are $6 \cdot 6 \cdot 5$ possible dinners.

Let us look at some examples you may not have seen before.

Problem. How many strings of length 4 can be made from the letters A, B, C, D, E, F if:

(a) Repetition is allowed.
(b) Repetition is not allowed.
(c) Repetition is not allowed, and the first letter must be B.
(d) Repetition is not allowed, and the first letter cannot be a B.

Solution.

(a) If repetition is allowed, then there are 6 choices for each letter in the string, thus there are 6^4 total strings.
(b) We now have to be more careful. There are 6 choices for the first letter. After we choose this letter, there are only 5 choices remaining for the second one. Then there will be only 4 choices for the third, and then 3 for the last one. Thus, there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ total strings.
(c) If B must be the first letter, then there is only one choice for the first letter. After this is chosen, there are 5 choices for the second letter as repetition is not allowed, then 4 for the third letter, and 3 for the last. Therefore, there are $1 \cdot 5 \cdot 4 \cdot 3 = 60$ total strings.
(d) Since the first letter cannot be B, there are only 5 options for that one. For the second letter, it cannot be that first letter, so there are 5 options for this one as well (since we are allowing B now). Then there are 4 for the third letter and 3 for the last. Thus, there are $5 \cdot 5 \cdot 4 \cdot 3 = 300$ total options.

Remark. We could also have approached (d) above in the following way: to count the number of strings that don't start with B, we could take the total number of strings with repetition not allowed and subtract off the number that do start with B. This would give $360 - 60 = 300$, which is exactly the number we found in (d). The general principle is that if $A \subset B$, then $|B \backslash A| = |B| - |A|$.

Problem. Suppose X and Y are sets with $|X| = n$ and $|Y| = m$. How many function $f : X \to Y$ are there?

Solution. To describe a function, we must assign a unique $y \in X$ to each $x \in X$. Thus, we have n choices to make (one for each $x \in X$), and for each choice, we have $m = |Y|$ options to choose from. Therefore, there are n^m total functions.

Problem. Let X be a set with $|X| = n$. Use the multiplication principle to prove $|\mathcal{P}(X)| = 2^n$.

Solution. We need to figure out how to reduce the problem of counting subsets to choices. If we think about it some, we notice that to determine a subset, we just need to decide whether a given element is in the subset or not. If $X = \{x_1, \ldots, x_n\}$, then to determine a subset S, we need to decide whether $x_i \in S$ or $x_i \notin S$, for all i. This means we have n choices, each with two options. Thus, by the multiplication principle, $|\mathcal{P}(X)| = 2^n$.

Problem. Let X be a set with $|X| = n$. How many ordered pairs of subsets (A, B) are there which satisfy $A \subseteq B \subseteq X$?

Solution. This problem seems like it might be difficult. Let's look at an example first: if $X = \{1\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, and there are three such pairs: $(\emptyset, \emptyset)$, $(\emptyset, \{1\})$, $(\{1\}, \{1\})$. Let's try and reason as we did in the previous problem. For a good ordered pair (A, B) and a given $x \in X$, there are three options: either $x \in A$, $x \in B \backslash A$, or $x \in X \backslash B$. So for each $x \in X$, we have three options. As there are n elements of X, we find the total number of ordered pairs is 3^n.

Looking at these problems, there is a strong relationship between the multiplication principle and conjunctions. Indeed, the principle says that if we have choice 1 AND choice 2, then the number of total options is the product of the number for each choice. But what is the corresponding principle for disjunctions? This is the addition principle.

Principle. Let $X_1, X_2, \ldots, X_k$ be sets with $|X_i| = n_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$. Then

$$|X_1 \cup X_2 \cup \cdots \cup X_k| = n_1 + n_2 + \ldots + n_k.$$

Again, this should be intuitively clear. If we have a bunch of disjoint sets (i.e. with no overlap), then the number of elements in the union should be just the sum of the number of elements of each individual set.

Problem. Suppose we have five computer science books, three math books, and two art books. How many ways are there of selecting two books, each from different subjects?

Solution. Since the books must come from different subjects, there are three possibilities: we choose a math and computer science book, a math and art book, or a computer science and art book. And each of these sets are disjoint. How many ways are there of selecting a math and computer science book. Here we see conjunction and choices, so we will need the multiplication principle. There are 3 choices of math book and 5 of computer science, so there are $3 \cdot 5 = 15$ ways of selecting a math book and a computer science book. Similarly, there are 6 ways of choosing a math and art book, and 10 ways of selecting a computer science and art book. By the addition principle, there are $15 + 10 + 6 = 31$ ways of selecting two books, each from different subjects.

Problem. How many strings of length 4 can be made from the letters A, B, C, D, E, F if repetition is not allowed, and the first letter must be B or a C.

Solution. Let X_1 be the set of strings which start with B and X_2 the set of strings which start with C. They are disjoint (you can't start with both B and C), so we can use the addition principle to find $|X_1 \cup X_2|$. We computed $|X_1|$ in the first problem of the multiplication principle, and we got 60. Similarly, there are 60 strings which start with C. Thus, by the addition principle, there are $60 + 60 = 120$ total such strings.

What happens if the sets are not disjoint? Then we have the method of inclusion/exclusion:

Theorem 1.1 (Inclusion/Exclusion)**.** If X and Y are sets, then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

Proof. We get this from the addition principle. We are going to break up X, Y, and $X \cup Y$ into disjoint sets and use the addition principle for each.

Claim. $X = (X - Y) \cup (X \cap Y)$, and these sets are disjoint: The inclusion $(X - Y) \cup (X \cap Y) \subseteq X$ should be clear. For the reverse inclusion, take $x \in X$. If $x \in Y$, then $x \in X \cap Y$. If $x \notin Y$, then $x \in X \backslash Y$. In either case, $x \in (X\backslash Y) \cup (X \cap Y)$, and we have an equality of sets. They are disjoint because if $x \in (X\backslash Y) \cap (X \cap Y)$, then $x \in Y$ and $x \notin Y$, a contradiction.

By the addition principle, $|X| = |X\backslash Y| + |X \cap Y|$.

Claim. $Y = (Y - X) \cup (X \cap Y)$, and these sets are disjoint: It is the same proof as the first claim.

Thus, $|Y| = |Y\backslash X| + |X \cap Y|$. You showed on your homework that $X \cup Y = (X\backslash Y) \cup (Y\backslash X) \cup (X \cap Y)$. It is easy to prove these sets are disjoint as well. Thus,

$$|X \cup Y| = |X\backslash Y| + |Y\backslash X| + |X \cap Y|.$$

Using our first two equalities, we get

$$|X \cup Y| = |X| - |X \cap Y| + |Y| - |Y \cap X| + |X \cap Y| = |X| + |Y| - |X \cap Y|.$$

□

Let's look at how to apply this.

Problem. A committee of three people (president, vice president, and treasurer) is to be made up from Alice, Bob, Cindy, Dave, Ed, Fred. How many different committees are possible if either Alice or Dave (or both) needs to be on the committee.

Solution. Let X_A be the set of committees with Alice as a member, and let X_D be the set of committees with Dave as a member. We want $|X_A \cup X_D|$. Here, $X_A \cap X_D \neq \emptyset$, so we need to use inclusion/exclusion. For $|X_A|$, we can use the multiplication principle. There are actually three choices to make. First, we need to decide which position Alice will fill. There are three choices for this. Next, we have to choose people for the other two positions. There are 5 choices for the first and 4 for the last. Thus, $|X_A| = 3 \cdot 5 \cdot 4 = 60$. Similarly, $|X_D| = 60$.

For $X_A \cap X_D$, we have three choices. First, we choose the position Alice will fill. Next, we choose the position Dave will fill. Lastly, we choose the person for the final position. Thus, there are $3 \cdot 2 \cdot 4 = 24$ ways of doing this. Thus,

$$|X_A \cup X_D| = 60 + 60 - 24 = 96.$$

Problem. How many numbers between 1 and 100 are divisible by 2 or 5 (or both)?

Problem. Let X_2 be the set of numbers divisible by 2 and X_5 the set of numbers divisible by 5. We want $|X_2 \cup X_5|$. Well $|X_2| = 100/2 = 50$ and $|X_5| = 100/5 = 20$. Lastly, $X_2 \cap X_5$ is the set of numbers divisible by both 2 and 5, which are just multiples of 10. There are 10 such numbers. Thus,

$$|X_2 \cup X_5| = 50 + 20 - 10 = 60.$$

We'll end with one more problem.

Problem. How many 8 bit strings neither start with 11 nor end with 000?

Solution. Let A be the set of 8 bit strings which start with 11, and B the set of 8 bit strings which end with 000. We want the set of 8 bit strings x such that $x \notin A$ and $x \notin B$. Letting U denote all 8 bit strings, this is the same as saying we want

$$x \in U\backslash(A \cup B).$$

To count the number of such strings, we can use inclusion/exclusion:

$$|U\backslash(A \cup B)| = |U| - |A \cup B| = |U| - (|A| + |B| - |A \cap B|).$$

We just need to compute the terms. First, $|U| = 2^8$ since there are 8 choices, each with 2 options. For $|A|$, we only have 6 choices since the first two bits are determined for us already. Thus, $|A| = 2^6$. Similarly, in computing $|B|$, we notice we only have 5 choices since the last three bits must be 0. Thus, $|B| = 2^5$. Lastly, we need $|A \cap B|$. But to be in A and B means that we need to start with 11 and end with 000, so the only choices are the three digits between them. Thus, $|A \cap B| = 2^3$. Putting this all together gives:

$$|U\backslash(A \cup B)| = 2^8 - (2^6 + 2^5 - 2^3) = 256 - (64 + 32 - 8) = 168.$$

2. Permutations and Combinations

2.1. **Permutations.** We will now look at permutations and combinations. Permutations are a very specific application of the multiplication principle. To give the general setup, let X be a set with $|X| = n$. Suppose we want to pick and order r of the n elements of X. For example, if $X = \{x_1, \ldots, x_n\}$, then if we want to pick and order 3 of the elements, we could have $\{x_1, x_2, x_3\}$, or $\{x_2, x_1, x_3\}$, or $\{x_3, x_2, x_1\}$, and so on. Let's make this a definition.

Definition. Given n distinct elements $x_1, \ldots, x_n$, an *r-permutation* of these elements is an ordering of an r-element subset of $\{x_1, \ldots, x_n\}$. The number of r-permutations is denoted $P(n, r)$.

Remark. We only care about $r \leq n$, since it doesn't make sense to choose more than n distinct elements from a set of n elements.

What should $P(n, r)$ be? It is an application of the multiplication principle. We want to select and order r things from the n objects. Well, there are n choices for the first element. Then there are $(n-1)$ elements for the second, since we cannot choose the first element again. Then there are $(n-2)$ choices for the third element, and so on. Thus, we have:

Theorem 2.1. $P(n, r) = n(n-1)(n-2)\cdots(n-(r-1)) = \frac{n!}{(n-r)!}$.

Proof. The first equality is the multiplication principle, and the second is straightforward from the definition of factorials. $\square$

Example. How many 2-permutations are there of the elements a, b, c? This should be $P(3, 2) = \frac{3!}{(3-2)!} = 6$. We could get this from the multiplication principle by saying there are 3 choices for the first slot and 2 for the second slot, giving $3 \cdot 2 = 6$ total 2-permutations of the three elements. We can even list them:

$$ab, ba, ac, ca, bc, cb.$$

Problem. A committee of three people (president, vice president, treasurer) is to be chosen from a group of 10 people. How many committees are possible?

Solution. We have ten people, and we want an ordered set of three people (ordered because the positions are different). Thus, the total number of committees is $P(10, 3) = \frac{10!}{7!} = 720$. Notice, again, that this is just the multiplication principle at work.

Definition. A *permutation* of a set $X = \{x_1, \ldots, x_n\}$ is an n-permutation of the elements $x_1, \ldots, x_n$.

How many permutations are there? There $P(n, n) = \frac{n!}{(n-n)!} = n!$ such permutations. We will need this in a second.

2.2. **Combinations.**

Definition. Given a set $X = \{x_1, \ldots, x_n\}$, an *r-combination* (where $r \leq n$) is an unordered selection of r elements of X. The number of r-combinations is denoted $C(n, r)$.

The only difference between permutations and combinations is the importance of order. Permutations are ordered, and combinations are unordered. This helps us figure out what $C(n, r)$ should be.

Example. If $X = \{a, b, c\}$, then there are three 2-combinations: ab, ac, bc. Notice the difference with permutations. Here, ab and ba is the same unordered set of elements, but they would be different elements when doing permutations.

Definition. Let $n \geq 0$ and $0 \leq r \leq n$. We let $\binom{n}{r}$ denote the quantity

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proposition. $C(n, r) = \binom{n}{r}$.

Proof. Let $X = \{x_1, \ldots, x_n\}$. Given an r-combination of X, there are $r!$ rearrangements of these r elements. This produces all r permutations involving these chosen elements. Thus,

$$P(n,r) = C(n,r) \cdot r!,$$

which means

$$C(n,r) = \frac{P(n,r)}{r!} = \binom{n}{r}.$$

□

Corollary. The quantity $\binom{n}{r}$ is an integer.

Problem. How many length 8 bit strings are there which contain exactly 5 zeroes?

Solution. Again, let's look at choices. Before, we went bit by bit and looked at choices (for example, when showing there are 2^8 length 8 bit strings). But here, we need to decide where the 0's are going to be. There are 8 slots, and we have to choose 5 for the zeroes. It doesn't matter which order we choose the slots in. Thus, there are $\binom{8}{5} = \frac{8!}{3!5!}$ length 8 bit strings which contain exactly 5 zeroes.

Problem. How many strings can be formed using all the letters of BOOKKEEPER.

Solution. To use all the letters means it needs to be a string of length 10. There need to be 3 E's, 2 K's, 2 O's, and 1 each of B, P, and R. There are 10 total slots, and we have to choose 3 of them for the E's: there are $\binom{10}{3}$ such choices. Then, out of the 7 remaining slots, we must choose 2 for the K's: there are $\binom{7}{2}$ ways to do this. Then there are $\binom{5}{2}$ ways of choosing the slots for the O's, then $\binom{3}{1}$ for the B, $\binom{2}{1}$ for the P, and $\binom{1}{1}$ for the R. Thus, there are

$$\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{10!}{3!2!2!1!1!1!}$$

rearrangements of the letters at BOOKKEEPER.

Remark. The other way to look at this solution is the following: if all the letters were different, then there would be 10! rearrangements of the letters. But they aren't all different. If we had EEEKKOOBPR, then any rearrangement of the E's would produce the same string, and any rearrangement of the K's would produce the same string, and so on. So we have overcounted. The quantity in the denominator is the amount we've overcounted by: the number of rearrangements of each of the individual characters.