Ensemble Kalman Filter (EnKF)

This week you have been hearing about some techniques for combining data and model predictions to obtain an optimal estimate of the true state of a physical system. In particular we have been considering the following

Best estimate x^a

$$\textit{True Physics}: \quad x_{k+1} = f(x_k) + w_k \quad x_k, w_k \in R^n$$

unknown physics

$$\textit{Model}: \quad x^f_{k+1} = f\left(x^f_k\right) \quad x^f_k \in R^n$$

$$\textit{Measurements}: \quad y_k = h(x_k) + z_k \quad y_k, z_k \in R^m$$

R

Here $f(x)$ drives the dynamics of our system and $h(x)$ is a function which takes our state space to our measuremt space. Further w_k, z_k are assumed to be uncorelated stationary zero-mean white noise(Gaussian) processes with covariance matricies given by Q_k and R_k respectively.

What is the best estimate?

$e = (x - x^f)$ $\quad$ $x^a = \min_x E[Tr(ee^T)]$ $\quad$ $\vec{x} = [x, y, z]$

$$x^a = \min_x E\left[(x - x^f)^2 + (y - y^f)^2 + (z - z^f)^2\right]$$

You have seen in the videos and worked out in the worksheets that when f and h are linear,

$$f(x_k) = A_k x_k \qquad\qquad h(x_k) = H_k x_k$$

matrix $\qquad$ matrix

That the optimal estimate of the true state and the error at step k is given by:

$$x^a_k = x^f_k + K(y^0_k - Hx^f_k)$$

where

$$\boxed{K_k = B^f_k H^T \left(HB^f_k H^T + R\right)^{-1}}$$

$$B^a_k = B^f_k - K_k H B^f_k$$

Here B^a_k is the error covariance we were minimizing the trace of!

$$B^f_k = E\left[(x_k - x^f_k)(x_n - x^f_k)^T\right]$$

$$B^k_a = E\left[(x_k - x^a_k)(x_k - x^a_k)^T\right]$$

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Prediction Step

$$x_{k+1}^f = A_k x_k^a$$

$$B_{k+1}^f = A_k B_k A_k^T + Q_k \quad \text{(Ricatti Equation)}$$

Update Step

$$K_k = B_k^f H^T \left(H B_k^f H^T + R \right)^{-1}$$

$$x_k^a = x_k^f + K(y_k^0 - H x_k^f)$$

$$B_k^a = B_k^f - K_k H B_k^f$$

$$B_k^f = E[(x_k - x_k^f)(x_k - x_k^f)^T]$$

$$\boxed{B_{xy_k}^f} = E[(x_k - x_k^f)(\underbrace{H_k x_k}_{y_k} - \underbrace{H_k x_k^f}_{y_k^f})^T] = B_k^f H_k^T \rightarrow E[(x_k - x_k^f)(H(x_k - x_k^f))^T]$$

$$= E[(x_k - x_k^f)(x_k - x_k^f)^T H^T] = \underbrace{E[(x_k - x_k^f)(x_k - x_k^f)^T]}_{B_k^f} H^T$$

$$\boxed{B_{yy}^f} = E[(H_k x_k - H_k x_k^f)(H_k x_k - H_k x_k^f)^T] = H_k B_k^f (B_k^f)^T + R_k$$

With these definitions we can rewrite the Kalman Gain Matrix as

$$\boxed{K_k = B_{xyk}^f \left(B_{yyk}^f \right)^{-1}}$$

This is all well and good provided our dynamics and operation operators are linear, what if they are not?

One very common approach is just to linearize them, for example the matrix A_k can just be taken to be the jacobian of f at time step k, however if you have lots of variables, oftern the case for most real applications, this can be prohibitavely computationally expensive. That linearization scheme is often refered to as the Extended Kalman Filter (EKF).

Taylor

$J(f)$

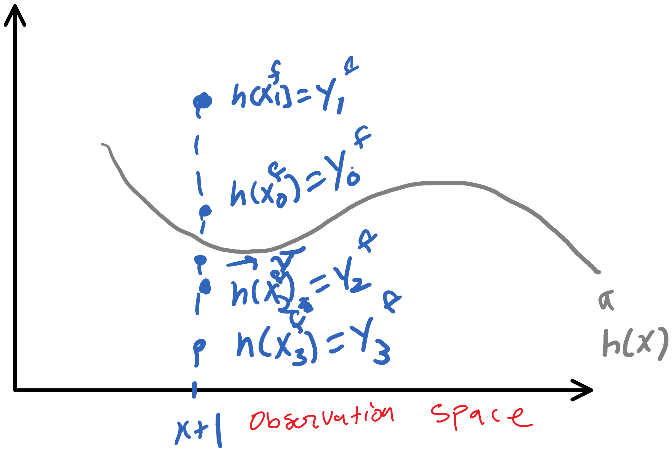
(XKF)

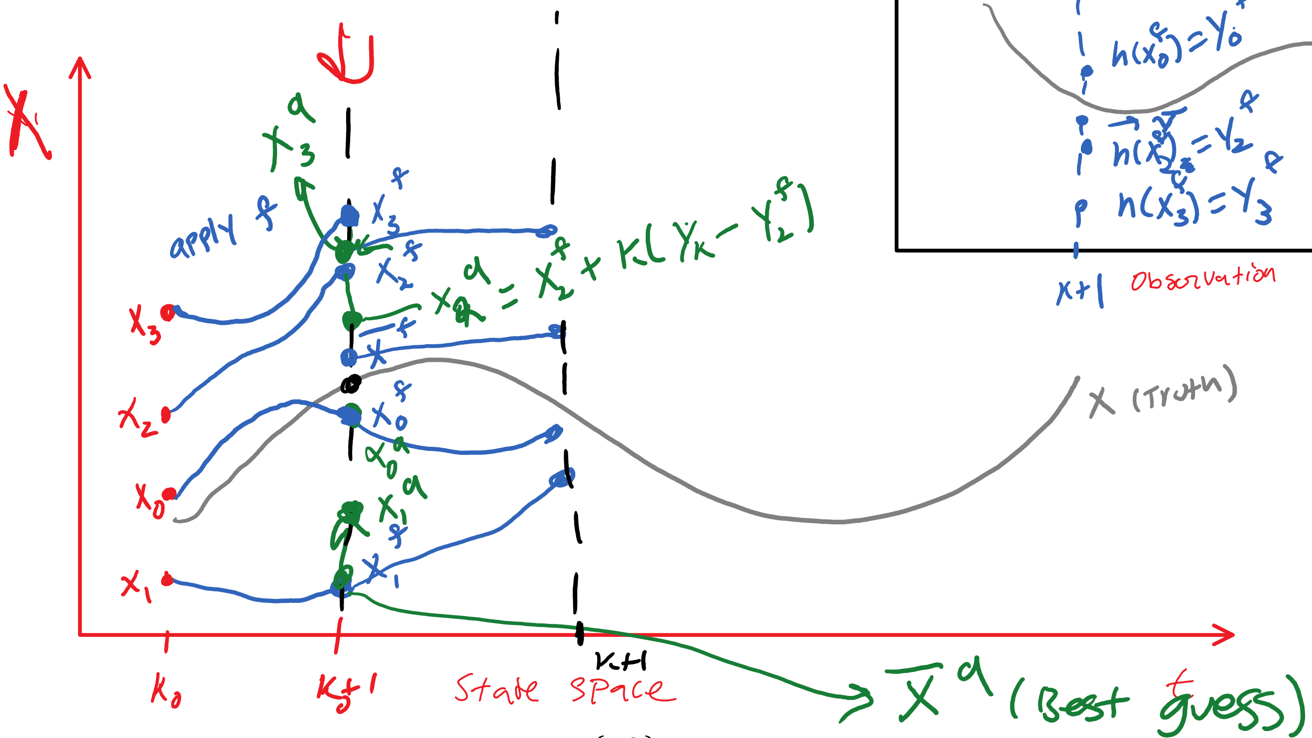
$x_{k+1} = f(x_k)$, $x_{k+1} = A_k x_k$

Another approach to dealing with nonlinear dynamics is through the Ensemble Kalman Filter (EnKF). This approach is centered on approximating $B_k^f, B_{xyk}^f, and\ B_{yyk}^f$ through a clever pertuberation and avreaging scheme. The basic scheme is outlined below.

1. From a series of perturbed forecasts $\{x_k^{f_i}\}$ calculate the mean $\bar{x}_k^f$ create the Error matrix :
$$E_k^f = \left[x_k^{f_1} - \bar{x}_k^f, \cdots, x_k^{f_N} - \bar{x}_k^f \right]$$
2. Using the forecasts and that $y_k^f = h_k x_k^f$ create the observation error matrix
$$E_{y_k}^f = \left[y_k^{f_1} - \bar{y}_k^f, \cdots, y_k^{f_N} - \bar{y}_k^f \right]$$
3. Then estimate the needed covariance matrices as,
$$B_k^f \approx \hat{B}_k^f = \frac{1}{N-1} E_k^f \left(E_k^f \right)^T \qquad B_{yy_k}^f \approx \hat{B}_{y_k}^f = \frac{1}{N-1} E_{y_k}^f \left(E_{y_k}^f \right)^T \qquad B_{xy_k}^f \approx \hat{B}_{xy_k}^f = \frac{1}{N-1} E_k^f \left(E_{y_k}^f \right)^T$$
4. Next preform N parallel data assimilations steps on each member of the forecast enssemble.
$$x_k^{a_i} = x_k^{f_i} + \hat{K}_k \left(y_k^i - h\left(x_k^{f_i} \right) \right) \text{ where } \hat{K}_k = \hat{B}_{xy_k}^f \left(\hat{B}_{yy_k}^f \right)^{-1} \text{ and the observations perturbed as } y_k^i = y_k + z_k^i$$
Real obs
5. Update the approximate error covariance as $\hat{B}_k^a = \frac{1}{N-1} E_k^a (E_k^a)^T$ where $E_k^a = \left[x_k^{a_1} - \bar{x}_k^a, \cdots, x_k^{a_N} - \bar{x}_k^a \right]$
6. Evolve the ensemble of analysis vectors forward perturbed as $x_{k+1}^{f_i} = f(x_k^{a_i}) + w_k^i$ and then repeat!

Ensemble Kalman Filter (EnKF)





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Next preform N parallel data assimilations steps on each member of the forecast enssemble.

$x_k^{a_i} = x_k^{f_i} + \hat{K}_k\left(y_k^i - h\left(x_k^{f_i}\right)\right)$ where $\hat{K}_k = \hat{B}_{xy_k}^f \left(\hat{B}_{yy_k}^f\right)^{-1}$ and the observations perturbed as $y_k^i = y_k + z_k^i$

4. Update the approximate error covariance as,

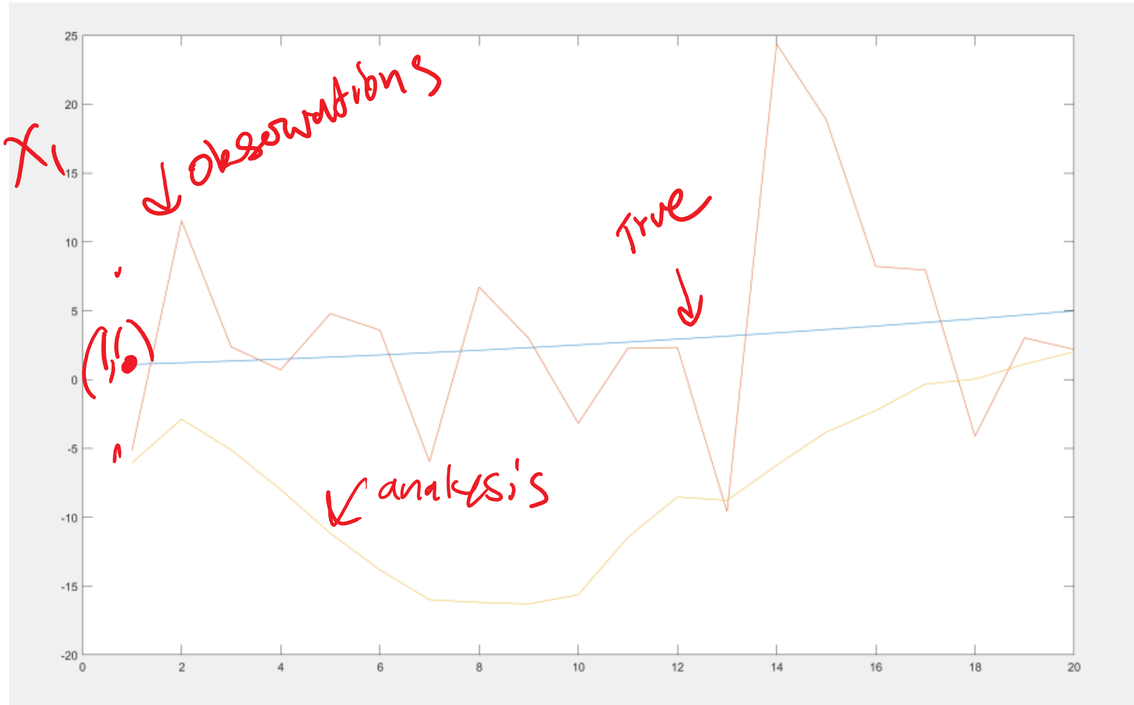
$$\hat{B}_k^a = \frac{1}{N-1} E_k^a \left(E_k^a\right)^T \text{ where } E_k^a = \left[\, x_k^{a_1} - \bar{x}_k^a, \cdots, x_k^{a_N} - \bar{x}_k^a \right]$$

5. Evolve the ensemble of analysis vectors forward perturbed as

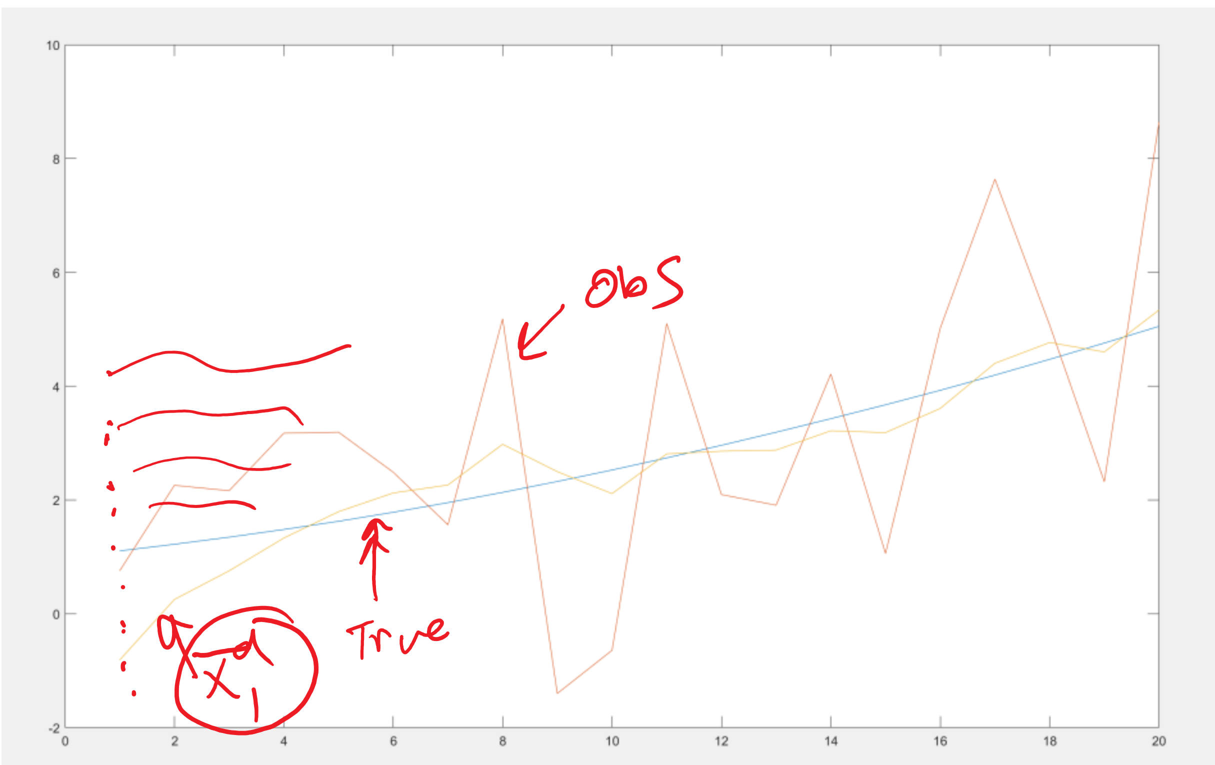
$$x_{k+1}^{f_i} = f\left(x_k^{a_i}\right) + w_k^i$$

and then repeat!

Ensemble Kalman Filter (EnKF)



2 members
Initial condition STD 10



20 members initial
Condition STD 10

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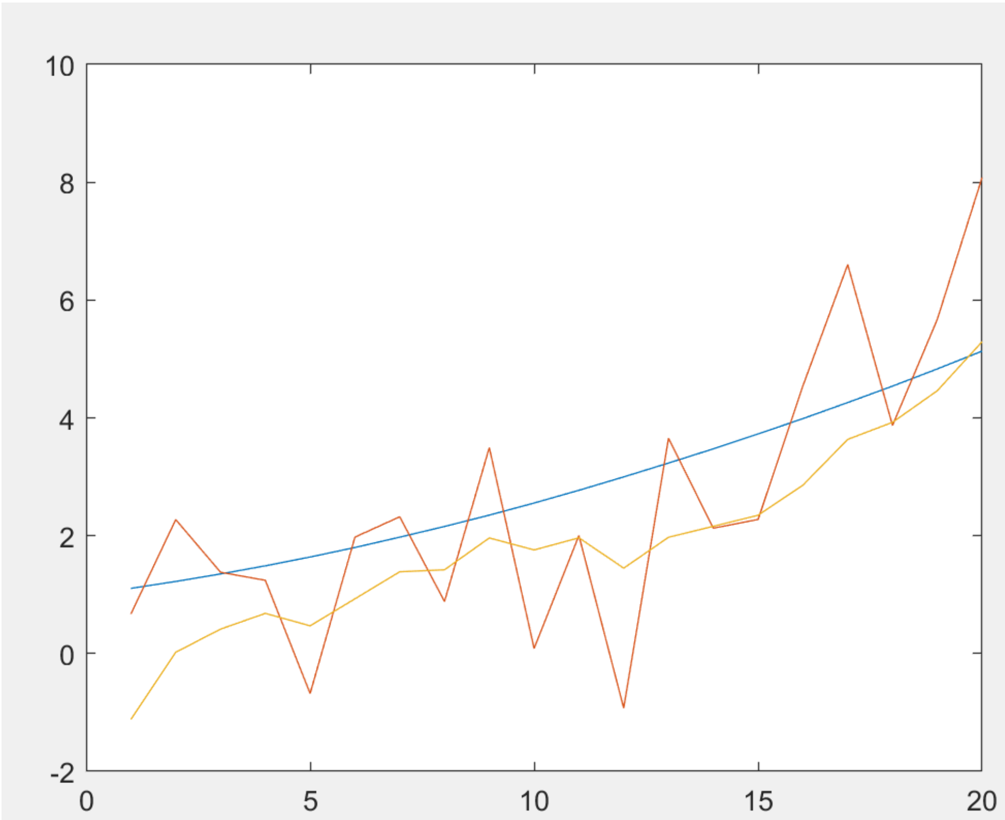
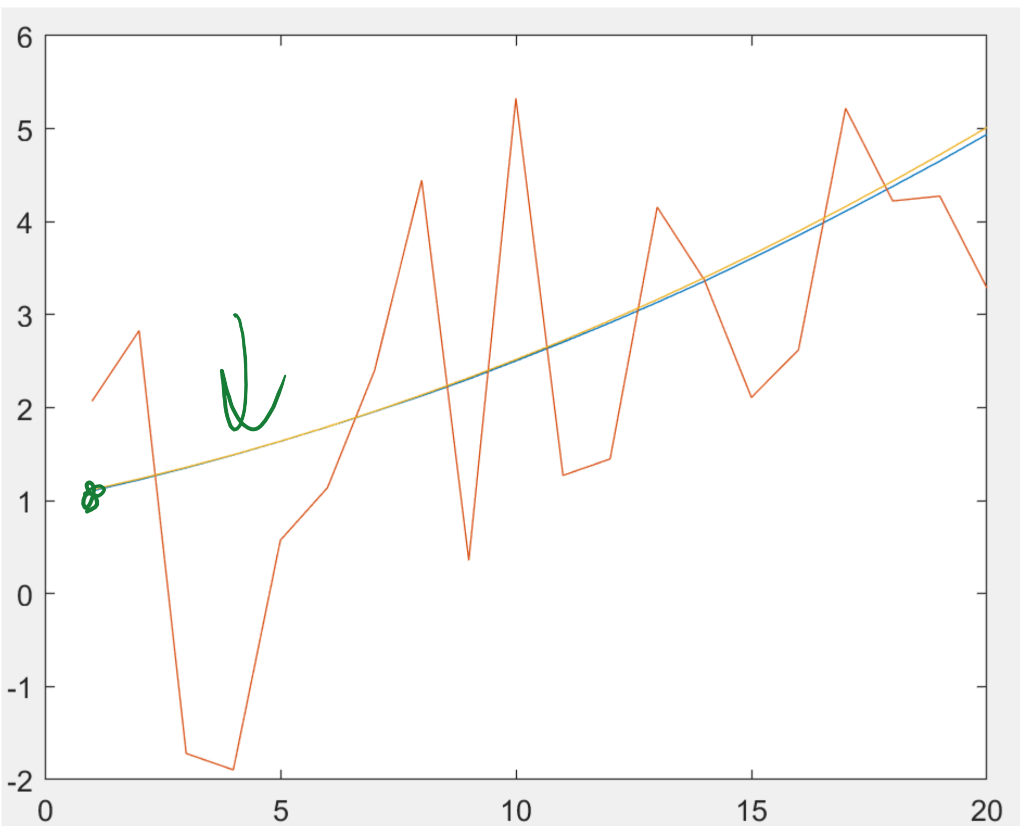


50 members
Initial condition std 10



50 members
Initial condition std 0.1