Data Assimilation in Dynamical Systems
Lecture 7

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The general problem of the stability of motion

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Preface

In this work some methods are expounded for the resolution of questions concerning the properties of motion and, in particular, of equilibrium, which are known by the terms *stability* and *instability*.

The ordinary questions of this kind, those to which this work is devoted, lead to the study of differential equations of the form

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \ldots, \quad \frac{dx_n}{dt} = X_n,$$

of which the right-hand sides, depending on time t and on unknown functions $x_1, x_2, \ldots, x_n$ of t, may be developed, provided the x_r are sufficiently small in absolute value, in series of positive integer powers of the x_s, and vanish when all these variables are equal to zero.

The problem reduces to finding if it is possible to choose the initial values of the functions x_s so small that, for all time following the initial instant, these functions remain in absolute value less than limits given in advance, which may be as small as one wishes.

Characteristic Exponents

For complex valued function $f(t)$ defined on $[t_0, +\infty)$, the number (or $\pm\infty$) defined as

$$\chi(f) = \limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|$$

is called the characteristic exponent of f.

Examples

$$\chi(c \neq 0) = 0; \quad \chi(0) = -\infty; \quad \chi(t^m) = 0;$$

$$\chi(e^{\alpha t}) = \alpha; \quad \chi(e^{t \sin(\ln t)}) = 1.$$

Properties

- $\chi(f+g) \leq \max\{\chi(f), \chi(g)\}$,
- $\chi(f+g) = \chi(f) \quad \text{if} \quad \chi(f) > \chi(g)$,
- $\chi(f \cdot g) \leq \chi(f) + \chi(g)$,
- $\chi(f)$ is sharp if finite limit $\lim_{t\to\infty} \frac{1}{t} \ln |f(t)|$ exists,
- $\chi(f)$ sharp implies $\chi(f \cdot g) = \chi(f) + \chi(g)$,
- If $F(t) = \int_a^t f(s)ds$, then $\chi(F) \leq \max\{\chi(f), 0\}$,
- For a vector x, $\chi(x) = \chi(||x||)$.

Lyapunov Characteristic Exponents

Consider a FMS $X(t)$ for $x' = A(t)x$ and the quantities

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \log(\|X(t)e_i\|)$$

When sum $\sum_{i=1}^{m} \lambda_i$ is minimized over all FMSs, the FMS $X(t)$ is called normal and the λ_i are called the Lyapunov exponents .
The Lyapunov exponents satisfy

$$\sum_{i=1}^{m} \lambda_i \geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{trace}(A(s))ds$$

A system is regular (Lyapunov) if the time average of the trace has a finite limit and equality holds.

How to Determine Lyapunov Exponents

- Consider $x' = A(t)x$ and the transformation $x = L(t)y$ to obtain $y' = B(t)y$ where

$$B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)L'(t).$$

- If L, L^{-1}, L' are bounded, then L is a Lyapunov transformation and preserves the Lyapunov Exponents.
- Orthogonal Change of Variables
  For $A(t)$ bounded and continuous Perron (1930) and Diliberto (1950) showed that an orthogonal change of variables $Q(t)$ exists such that $B(t)$ is upper triangular.
- For regular, triangular system $y' = B(t)y$,

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t B_{ii}(s)ds$$

LEs of Regular Systems

- Difficulty: Lyapunov exponents of regular systems are not necessarily continuous with respect to perturbations.

$$\begin{aligned} x_1' &= (1 + \tfrac{\pi}{2}\sin(\pi\sqrt{t}))x_1 \\ x_2' &= 0\,, \end{aligned} \tag{1}$$

$\lambda_1 = 1$ and $\lambda_2 = 0$. Since for any $n \in I\!N$

$$\int_{(2n-1)^2}^{(2n)^2} (1 + \frac{\pi}{2}\sin(\pi\sqrt{\tau}))d\tau = 0\ , \tag{2}$$

not integrally separated: [$a > 0$, $d \geq 0$ with $\int_s^t (1 + \frac{\pi}{2}\sin(\pi\sqrt{\tau}))d\tau \geq a(t-s) - d$, $t \geq s$] and hence LEs are not robust.

Stability of LEs: Integral Separation

- When are Lyapunov exponents (LEs) stable?

  "The LEs are stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\sup_{t \in I\!R^+} ||E(t)|| < \delta$ implies

  $$|\lambda_i - \gamma_i| < \epsilon\ ,\ \ i = 1, \ldots, m,$$

  where the γ_i's are the Lyapunov exponents of the perturbed system $y' = [A(t) + E(t)]y$."
- A fundamental matrix solution is integrally separated if there exist $a > 0$, $1 \geq d > 0$ such that

  $$\frac{||X_i(t)||}{||X_i(s)||} \div \frac{||X_{i+1}(t)||}{||X_{i+1}(s)||} \geq de^{a(t-s)}\ ,\ \ t \geq s.$$

- Integral separation implies distinct LEs.

Stability: integral separation

- [Bylov and Izobov, Millionshchikov] Distinct Lyapunov exponents are stable if and only if a fundamental matrix solution is integrally separated.
- Stability for non-distinct Lyapunov exponents a bit more involved.
- [Millionshchikov, Palmer] "In the Banach space $\mathcal{B}$ of continuous bounded matrix valued functions A, with norm $||A|| = \sup\limits_{t \geq 0} ||A(t)||$, systems with integral separation form an open and dense subset. I.e., integral separation is generic in $\mathcal{B}$."
- Integral separation is preserved by Lyapunov transformations.
- For triangular systems, $i < j$, $a_{ij} > 0$, $d_{ij} \geq 0$, $t \geq s$, $\int_s^t B_{ii}(\tau) - B_{jj}(\tau) d\tau \geq a_{ij}(t-s) - d_{ij}$.

Sacker-Sell Spectrum

- $\Sigma_{\mathrm{ED}} = \{\lambda \in I\!R : x' = [A(t) - \lambda I]x \ \text{ does not have ED}\}$.
- ED=Exponential Dichotomy: A FMS X admits ED if there exist a projection P and constants $\alpha > 0$, and $K \geq 1$, such that

$$\|X(t)PX^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq Ke^{\alpha(t-s)}, \quad t \leq s.$$

- Sacker-Sell spectrum stable/robust/continuous w.r.t. perturbations as a consequence of “Roughness Theorem” for ED.
- $0 \notin \Sigma_{\mathrm{ED}}$ implies existence of bounded solution to inhomogeneous equation.

Bounded Inhomogeneous

- $x'(t) = A(t)x(t) + g(t)$, g bounded, non-zero.
- Variation of Constants formula

$$x(t) = \int_0^t X(t)PX^{-1}(s)g(s)ds - \int_t^\infty X(t)(I-P)X^{-1}(s)g(s)ds$$

  $X(t)$ is a fundamental matrix solution
- $\sup_t \|x(t)\| \le \frac{2K}{\alpha} \sup_t \|g(t)\|$

Nonzero LEs $\not\Rightarrow$ Bdd Inhom

- Stability of zero solution vs. bounded inhomogeneous:
- Consider $x' = a(t)x + g(t)$ where for $0 < \alpha < 1$,

$$a(t) = \sin\left(\frac{t - j(j-1)\pi}{j}\right) - \alpha, \;\; j(j-1)\pi \leq t < j(j+1)\pi.$$

- The system is regular and the Lyapunov exponent $\lambda_1 = -\alpha$.
- Take $g(t) = 1$, then there does not exist a bounded solution even though $\lambda_1 < 0$.
- Note: Sacker-Sell spectrum $\Sigma_{\text{ED}} = [-1 - \alpha, 1 - \alpha]$.
- In general, Lyapunov exponents contained within Sacker-Sell spectrum.

Mini Project 1

- Verify that there does not exist a bounded solution for $g(t) = 1$ in the previous example.
- Verify equation (2) for the regular but not integrally separated example.

Comparison

For time dependent $x' = A(t)x$

- Lyapunov exponents
  - Useful for determining stability of zero solution of $x' = A(t)x$,
  - Conditionally stable: Need “integral separation”, etc..

- Sacker-Sell spectrum
  - Useful for determining stability of zero solution of $x' = A(t)x$ and bounded solutions of $x' = A(t)x + g(t)$ for bounded g,
  - Unconditionally stable: “Roughness Theorem”.

For time independent $x' = Ax$, the real parts of the eigenvalues useful for both stability and bounded solution of inhomogenous.

A Constructive Approach

- Consider m-dimensional system (A bounded, piecewise continuous for $t \geq 0$)
$$x' = A(t)x$$

- A Strategy: To extract stability information [e.g. Lyapunov exponents, Sacker-Sell spectrum, etc.] first determine an orthogonal change of variables that brings $A(\cdot)$ to upper triangular.

- $A(t) \mapsto B(t)$, $B(t)$ upper triangular, by $Q(t)$ orthogonal.

- THEOREM: [Dieci,VV'07] Extract stable/robust Lyapunov exponents and Sacker-Sell spectrum from diagonal of $B(\cdot)$ (for a properly chosen $Q(0)$).

Stability Spectra

- For triangular system with stable Lyapunov exponents

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t}\int_0^t B_{ii}(s)ds, \quad \mu_i = \liminf_{t\to\infty} \frac{1}{t}\int_0^t B_{ii}(s)ds.$$

- Sacker-Sell spectrum $\Sigma_{\mathrm{ED}} = \bigcup_{i=1}^{m} [\alpha_i, \beta_i]$,

$$\alpha_i = \liminf_{t\to\infty} \left\{ \inf_{t_0} \frac{1}{t}\int_{t_0}^{t_0+t} B_{ii}(s)ds \right\},$$

$$\beta_i = \limsup_{t\to\infty} \left\{ \sup_{t_0} \frac{1}{t}\int_{t_0}^{t_0+t} B_{ii}(s)ds \right\}.$$

- Finite time computation for problems with recurrent and/or asymptotic behavior.

Stability Spectra

For $j = 1, \ldots$, and $H > 0$, define Steklov (windowed) averages

$$\alpha_j^H = \inf_t \frac{1}{H} \int_t^{t+H} B_{jj}(s) ds, \quad \beta_j^H = \sup_t \frac{1}{H} \int_t^{t+H} B_{jj}(s) ds$$

Then

$$[\alpha_j, \beta_j] \subseteq [\alpha_j^H, \beta_j^H] \,.$$

More generally, the Lyapunov exponent spectrum Σ_L, the Sacker-Sell spectrum Σ_{ED}, the Steklov spectrum $\Sigma_{H>0}$, diagonal spectrum $\Sigma_{H=0}$ related as

$$\Sigma_L \subseteq \Sigma_{ED} \subseteq \Sigma_{H>0} \subseteq \Sigma_{H=0}.$$

Orthogonal Change of Variables

- Difficulty: Equation for Q is nonlinear and nonautonomous:

$$Q' = QS(Q,A), \;\; S(Q,A)_{ij} = \begin{cases} (Q^TAQ)_{ij}, & i > j, \\ 0, & i = j, \\ -(Q^TAQ)_{ji}, & i < j. \end{cases} \tag{3}$$

- Approximate numerically with orthogonal integration scheme so that numerical solution is orthogonal. [Gauss Runge-Kutta methods, "any" method and (modified) Gram-Schmidt, etc..]

- Discrete QR method:
$Q(t_{j+1})R(t_{j+1},t_j) = X(t_{j+1},t_j)Q(t_j), \;\; j = 0, 1, ...$

Discrete QR using Finite Difference Approximation

Consider a mapping (e.g. numerical method for approximation of differential equation):

$$u_{n+1} = F(u_n).$$

For $Q_0 \in I\!R^{m \times p}$ random such that $Q_0^T Q_0 = I$,

$$Q_{n+1} R_n = F'(u_n) Q_n, \;\; n = 0, 1, ...$$

where $Q_{n+1}^T Q_{n+1} = I$ and R_n is upper triangular with positive diagonal elements.

For high dimensional models or when the tangent linear model is not explicitly known, finite difference approximation

$$F'(u_n) Q_n \approx \frac{1}{\epsilon} [F(u_n + \epsilon Q_n) - F(u_n)], \;\; \epsilon \approx \sqrt{\epsilon_M} \cdot \|F(u_n)\|$$

Example: Lorenz '96 Model $(N = 40, F = 8)$

$$\dot{u}_k = (u_{k+1} - u_{k-2})u_{k-1} - u_k + F \ , \ k = 0, 1, \dots, N-1 \ , \ (\text{mod } N)$$

- Can write in vector form: $\dot{u} = -Iu + N(u)$,
- 13 positive Lyapunov exponents,
- Lyapunov dimension of the attractor $D_L \approx 28$ where for ordered Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$,

$$D_L = k + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{|\lambda_{k+1}|}$$

where k is the maximum value of i such that $\lambda_1 + \lambda_2 + \cdots + \lambda_i > 0$.

Mini Project 2

Download

`people.ku.edu/~erikvv/SAMSIL7.zip`

and use the `matlab` files to determine the number of positive Lyapunov exponents for Lorenz '96 with $F = 4$ and $F = 12$.

Time Dependent Stability

Computational techinques for determining time dependent stablity focused on smooth matrix decompositions of FMS of

$$\dot{x} = A(t)x, \;\; A(\cdot) \text{ continuous and bounded.}$$

- Smooth SVD: $X(t) = U(t)\Sigma(t)V^T(t)$, U, V orthogonal, Σ diagonal;
- Smooth QR : $X(t) = Q(t)R(t)$, Q orthogonal, R upper triangular.

Time Dependent Stability

Equations for Smooth SVD

$$\dot{\Sigma} = diag(C)\Sigma, \quad \dot{U} = UH, \quad \dot{V} = VK, \quad C = U^T AU$$

H, K skew-symmetric matrix functions, $i > j$,

$$H_{ij} = \frac{C_{ij}\Sigma_{jj}^2 - C_{ji}\Sigma_{ii}^2}{\Sigma_{jj}^2 - \Sigma_{ii}^2}, \quad K_{ij} = \frac{C_{ij}C_{ji}\Sigma_{ii}\Sigma_{jj}}{\Sigma_{jj}^2 - \Sigma_{ii}^2}$$

Equations for Smooth QR

Focus on QR and more general case of p linearly independent columns of $X(t)$ so that $Q(t)$ is $m \times p$ and $R(t)$ is $p \times p$, $p \leq m$.

Change of variables: $x(t) = Q(t)y(t)$, $\dot{y}(t) = B(t)y(t)$, $B(t)$ upper triangular.

Continuous QR

Let $t_0 = 0$, $X_0 = Q_0 R_0$. Want DEs for Q ($Q^T(t)Q(t) = I \, \forall t$).

- Differentiate $X = QR \Rightarrow \dot{Q}R + Q\dot{R} = \dot{X} = A(t)QR$. Multiply by Q^T on the left and R^{-1} on the right:

$$Q^T\dot{Q} + \dot{R}R^{-1} = Q^T A(t) Q.$$

- Since $Q^T Q = I$, $Q^T\dot{Q} = -\dot{Q}^T Q$ so that $S := Q^T\dot{Q}$ is skew-symmetric and using $\dot{R}R^{-1}$ upper triangular,

$$S_{ij} = \begin{cases} (Q^T(t)A(t)Q(t))_{ij}, & i > j, \\ 0, & i = j, \\ -S_{ji}, & i < j. \end{cases}$$

Continuous QR

- For $\dot{R} = B(t)R$, $R(0) = R_0$ invertible ($B = \dot{R}R^{-1}$),

$$B(t) := Q^T A(t) Q - S.$$

- Multiply $\dot{Q}R + Q\dot{R} = A(t)QR$ by R^{-1} on the right:

$$\dot{Q} + Q^T B(t) = A(t)Q$$

- Substitute $B(t) = Q^T A(t) Q^T - S$ to obtain

$$\dot{Q} = (I - QQ^T)A(t)Q + QS\,, \quad Q(0) = Q_0\,.$$

- Upper LEs from

$$\begin{aligned} \lambda_i &= \limsup_{t\to\infty} \frac{1}{t} \int_0^t (Q^T(s)A(s)Q(s))_{ii} ds \\ &= \limsup_{t\to\infty} \frac{1}{t} \int_0^t B_{ii}(s) ds\,, \quad i = 1, \ldots, p\,. \end{aligned} \tag{4}$$

Discrete QR

Let $X_n := X(t_{n+1})X^{-1}(t_n)$ for times $0 = t_0 < t_1 < t_2 \cdots$ where $X(t)$ is a fundamental matrix solution of $\dot{x} = A(t)x$.

The discrete QR method (given Q_0)

$$Q_{n+1}R_n = X_n Q_n, \ \ n = 0, 1, ...$$

so that for $k = 0, 1, ...,$

$$Q_{k+1}R_k \cdots R_0 = X(t_{k+1})X^{-1}(t_0)Q_0.$$

This is equivalent to what is obtained for Continuous QR:

$$Q(t_{k+1})R(t_{k+1}) = X(t_{k+1})X^{-1}(t_0)Q_0.$$

Mini Project 3

Derive the differential equations for smooth SVD by differentiating $X(t) = U(t)\Sigma(t)V^T(t)$, using $X'(t) = A(t)X(t)$, and the orthogonality of $U(t)$ and $V(t)$ and that $\Sigma(t)$ is diagonal.

Orthogonal Integration

Differential Equation for $Q(t)$:

$$\dot{Q}(t) = Q(t)S(Q(t), A(t)), \quad S(Q(t), A(t)) := \texttt{skew}(Q^T(t)A(t)Q(t)),$$

In [Dieci, Russell, VV, '94] investigated two classes of numerical methods to preserve orthogonality of $Q(t)$

- Projected Orthogonal Integrators: A standard numerical method plus projection, e.g., via (modified) Gram-Schmidt.
- Automatic Orthogonal Integrators: Showed among "all" one-step, linear multistep methods, Gauss-RK only automatic.

In [Dieci, VV, '95] we developed projected methods for non-square Q

$$\dot{Q} \; = \; (I - QQ^T)AQ + QS(Q, A)\,, \quad Q(0) = Q_0\,. \tag{5}$$

Orthogonal Integration

Preserving orthogonality numerically ensures that

$$\texttt{Trace}(A(t_k)) = \texttt{Trace}(B(t_k))$$

so the sum of the growth rates (Lyapunov exponents, Sacker Sell spectrum, etc.) is preserved.

For square $Q(t)$ consider 3 matrix DEs of increasing difficulty:

$$Q' = SQ, \;\; Q' = S(t)Q, \;\; Q' = QS(Q, A).$$

where $S, S(t)$, and $S(Q, A)$ are real skew-symmetric ($S^T = -S$).

Solution for $Q' = SQ$ is $Q(t) = \exp(S \cdot t)Q(0)$ so that

$$\exp(S^T \cdot t) \exp(S \cdot t) = \exp(-St + St) = exp(0) = I.$$

Orthogonal Integration

If $Q^T(0)Q(0) = I$, then for $t > 0$ using skew-symmetry,

$$\frac{d}{dt}Q^T(t)Q(t) = 0$$

Next consider numerical schemes (applied to $x' = f(x,t)$) that are good candidates to preserve orthogonality:

- $x_{k+1} = x_k + \frac{\Delta t}{2}(f(x_k, t_k) + f(x_{k+1}, t_{k+1}))$ [trapezoid]
- Trapezoid applied to $Q' = SQ$ gives

$$Q_{k+1} = (I - \frac{\Delta t}{2}S)^{-1}(I + \frac{\Delta t}{2}S)Q_k,$$

  Cayley transform orthogonal and orthogonality preserved.
- Trapezoid applied to $Q' = S(t)Q$ gives

$$Q_{k+1} = (I - \frac{\Delta t}{2}S(t_{k+1}))^{-1}(I + \frac{\Delta t}{2}S(t_k))Q_k.$$

Orthogonal Integration

- $x_{k+1} = x_k + \Delta t f(\frac{1}{2}(x_k + x_{k+1}), t_{k+1/2})$ [implicit midpoint]
- Midpoint applied to $Q' = QS(Q, A)$ gives

$$Q_{k+1} = Q_k + \frac{\Delta t}{2}(Q_k + Q_{k+1})S(\frac{1}{2}(Q_k + Q_{k+1}), A(t_{k+1/2}))$$

  which can be shown to be orthogonal. (Use that implicit midpoint is a Symplectic Integrator).

- Midpoint fails to preserve orthogonality when applied to

$$\dot{Q} \; = \; (I - QQ^T)AQ + QS(Q, A)\,, \quad Q(0) = Q_0, Q_0^T Q_0 = I.$$

Projected Integrators: Any stable, consistent one step method applied to equation for Q, then project back onto orthogonal (projection error of the order of the local error).

Mini Project 3

Consider the implicit midpoint method applied to $Q' = QS(Q, A)$:

$$Q_{k+1} = Q_k + \frac{\Delta t}{2}(Q_k + Q_{k+1})S(\frac{1}{2}(Q_k + Q_{k+1}), A(t_{k+1/2})).$$

By directly forming $Q_{k+1}^T Q_{k+1}$ and assuming that $Q_k^T Q_k = I$, show that orthogonality is preserved.

Hint: The Δt term is zero using skew-symmetry, while the Δt^2 squared term is zero using that the implicit midpoint is a Symplectic Integrator.

Integral Separation Structure

For all $i < j$, either

(i) B_{ii} and B_{jj} are integrally separated, i.e. there exist $a > 0$, $d \geq 0$ such that

$$\int_s^t \big(B_{ii}(\tau) - B_{jj}(\tau)\big) d\tau \geq a(t-s) - d, \qquad t \geq s \geq 0\,, \tag{6}$$

or

(ii) B_{ii} and B_{jj} are not integrally separated, but $\forall \epsilon > 0$ there exists $M_{ij}(\epsilon) > 0$ such that

$$\left| \int_s^t \big(B_{ii}(\tau) - B_{jj}(\tau)\big) d\tau \right| \leq M_{ij}(\epsilon) + \epsilon(t-s), \qquad t \geq s \geq 0\,. \tag{7}$$

Condition (6) is equivalent to distinct and robust Lyapunov exponents, while (7) implies that the Lyapunov exponents of the system are robust but not necessarily distinct.

Lyapunov Exponents from Diagonal of $B(t)$

Under the standing assumption of Integral Separation Structure

- there exists a Lyapunov transformation (preserves LEs), that transforms $B(t)$ to
  - $\texttt{diag}(B(t))$ when system is fully integrally separated,
  - $\texttt{blockdiag}(B(t))$ when not fully integrally separated but with integral separation structure.
- Theorem [Dieci-EVV] Given a triangular system $\dot{R} = B(t)R$, with bounded, continuous B, stability spectra (Lyapunov exponents, Sacker-Sell spectrum, etc.) may be obtained from the diagonal subsystem.

Two Dimensional Example

Let

$$B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix}, \quad D(t) = \begin{pmatrix} B_{11}(t) & 0 \\ 0 & B_{22}(t) \end{pmatrix}.$$

Want to define a Lyapunov transformation $L(t)$ such that

$$D(t) = L^{-1}B(t)L - L^{-1}L'$$

Take

$$L(t) = \begin{pmatrix} 1 & w(t) \\ 0 & 1 \end{pmatrix},$$

and substitute to obtain $w' = (B_{11} - B_{22})w + B_{12}$.

If B_{11} and B_{22} are integrally separated, then $w(t)$ is bounded when employing the boundary condition $\lim_{t\to\infty} w(t) = 0$.

Mini Project 4

Show that for two-dimensional integrally separated problems

$$L(t) = \begin{pmatrix} 1 & w(t) \\ 0 & 1 \end{pmatrix},$$

is a Lyapunov transformation that transforms upper triangular $B(t)$ to its diagonal.

Show that $w(t)$ satisfies $w' = (B_{11} - B_{22})w + B_{12}$.

Show that if B_{11} and B_{22} are integrally separated, then $w(t)$ is bounded when employing the boundary condition $\lim_{t\to\infty} w(t) = 0$.

Convergence: QR and SVD

- QR ($X(t) = Q(t)R(t) = Q(t)D(t)Z(t)$):

$$Z = diag(R)^{-1}R, \quad \dot{Z} = EZ,$$

$$E = D^{-1}(B - diag(B))D, \quad D = diag(R)$$

- SVD ($X(t) = U(t)\Sigma(t)V^T(t)$):

$$\dot{\Sigma} = diag(C)\Sigma, \quad \dot{U} = UH, \quad \dot{V} = VK, \quad C = U^TAU$$

H, K skew-symmetric matrix functions, $i > j$,

$$H_{ij} = \frac{C_{ij}\Sigma_{jj}^2 - C_{ji}\Sigma_{ii}^2}{\Sigma_{jj}^2 - \Sigma_{ii}^2}, \quad K_{ij} = \frac{C_{ij}C_{ji}\Sigma_{ii}\Sigma_{jj}}{\Sigma_{jj}^2 - \Sigma_{ii}^2}$$

- Integral separation implies $E \to 0$ and $K \to 0$ exponentially.

Exponential Convergence: SVD

[Dieci, Elia JDE 2006]

- Let $\chi^s(f) = \limsup_{t\to\infty} \frac{1}{t}\log(|f(t)|)$,
- $\alpha_i^j(t) = v_i(t)^T \overline{v}_j$ [$\alpha_i^j \to 0, \alpha_i^i \to 1$],
- For fixed $0 < \tau \leq 1$, $\beta_i^j(t+\tau) = v_i(t+\tau)^T v_j(t)$.
- If the problem is integrally separated, then $\chi^s(\beta_i^j) < 0$ and the Lyapunov exponents λ_i^s are distinct.
- Theorem: If the problem is integrally separated, then for $i \neq j$,

$$\chi^s(\alpha_i^j) \leq A|\lambda_i^s - \lambda_j^s|$$

where $A = \max_{k\neq l} \frac{\chi^s(\beta_k^l)}{|\lambda_k^s - \lambda_l^s|} < 0$ and

$$\chi^s(1-\alpha_j^j) \leq 2\max_{i\neq j} \chi^s(\alpha_i^j).$$

Exponential Convergence: QR

- Under the assumption of “robust Lyapunov exponents” we prove [Dieci, Elia, VV, JDE 2010] exponential convergence of $Z := D(t)^{-1}R(t) \to \overline{R}$, $D = diag(R)$.

- Why is this significant? Write $R(t) = D(t)[D^{-1}(t)R(t)]$. Then $\overline{R}^{-1}$ determines initial conditions that grow/decay based upon growth/decay of diagonal elements of D.

- Why is the exponential convergence important? It reduces the problem of determining exponential dichotomy to finite intervals up to exponentially small perturbations.

Projected Data Assimilation

Consider a nonlinear evolution equation (solution operator of a model)

$$u_{n+1} = F_n(u_n; \boldsymbol{\alpha}), \;\; n = 0, 1, ..., N-1$$

where

- u_n are the state variables at time n,
- $\boldsymbol{\alpha}$ are adjustable model parameters, e.g., global in time.

Write $u_n = u_n^{(0)} + \delta_n$.

If we can decompose the time dependent tangent space into slow variables and fast variables, then we write $\delta_n = P_n\delta_n + (I - P_n)\delta_n$.

Rewrite original nonlinear evolution approximately as two subsystems ...

Dynamic Splitting of State Space Model

For $k = 0, 1, 2, ...$

P System:

$$u_{n+1}^{(k)} + P_{n+1}\delta_{n+1} = P_{n+1}F_n(u_n^{(k)} + P_n\delta_n),$$
$$u_n^{(k+\frac{1}{2})} = u_n^{(k)} + P_n\delta_n, \;\; n = 0, 1, ..., N,$$

I-P System:

$$u_{n+1}^{(k+\frac{1}{2})} + (I - P_{n+1})\delta_{n+1} = (I - P_{n+1})F_n(u_n^{(k+\frac{1}{2})} + (I - P_n)\delta_n),$$
$$u_n^{(k+1)} = u_n^{(k+\frac{1}{2})} + (I - P_n)\delta_n, \;\; n = 0, 1, ..., N.$$

Roughly speaking the first subsystem contains the slow variables (positive, zero, and slightly negative Lyapunov exponents) and the second subsystem contains the fast variables (strongly negative Lyapunov exponents).

Unstable Subspace

Importance of observations rich in unstable subspace

- Assimilation in the Unstable Subspace (AUS) [Trevisan, D'Isidoro, Talagrand '10 Q.J.R. Meteorol. Soc., Palatella, Carrassi, Trevisan '13 J. Phys A, ...]
- Error analysis in DA for hyperbolic system [González-Tokman, Hunt '13 Phys D]
- Adaptive observation operators and unstable subspace [K.J.H. Law, D. Sanz-Alonso, A. Shukla, A.M. Stuart '16 Phys D]
- Convergence of covariances matrices in unstable subspace [Bosquet etal. '17 SIAM UQ]

Forming time dependent projections

Discrete QR algorithm for determining Lyapunov exponents, local in time stability information, etc.:

For $Q_0 \in I\!R^{m \times p}$ random such that $Q_0^T Q_0 = I$,

$$Q_{n+1} R_n = F'(u_n) Q_n, \;\; n = 0, 1, ...$$

where $Q_{n+1}^T Q_{n+1} = I$ and R_n is upper triangular with positive diagonal elements.

Orthogonal Projections:

$$P_n = Q_n Q_n^T, \;\; (I - P_n) = I - Q_n Q_n^T$$

Slow/Fast Splitting and Techniques for P and $I - P$

Using the framework of a slow/fast splitting we

- may employ different DA/parameter estimation techniques in each subsystem,
- obtain an explicit representations for the time dependent unstable subspace.

Interested in systems that are hyperbolic in flavor (finite number of positive Lyapunov exponents, few zero Lyapunov exponents, potentially many negative Lyapunov exponents).

Slow/Fast Splitting and Techniques for P and $I - P$

Techniques for P System:

- Particle filters that are effective for low dimensional problem, a new class of techniques based upon residual correction (Pseudo Orbit DA (PDA) [Du & Smith I & II, '14 J. Atmos. Sci. 2014], shadowing refinement [Grebogi, Hammel, Yorke, and Sauer, Phys Rev Lett (1990)]), ...

Techniques I-P System:

- Techniques such as ETKF, LETKF, 4DVar (essentially a shooting method starting from u_0^b, trying to match observations, basin of attraction shrinks in the presence of positive Lyapunov exponents).

Shadowing

For a discrete time dynamical system

$$u_{n+1} = F_n(u_n), \;\; n = 0, ..., N-1,$$

shadowing involves showing the existence of a solution to the DS near an approximate solution.

For approximate solution $\{u_n^{(0)}\}_{n=0}^N$ satisfying

$$u_{n+1}^{(0)} - F_n(u_n^{(0)}) = \delta_n, \;\; n = 0, ..., N-1,$$

want to show existence of true solution $u_n = u_n^{(0)} + \epsilon_n$.

If $\delta = \sup_n \|\delta_n\|, \; \epsilon = \sup_n \|\epsilon_n\|$, then $\epsilon = C\delta$ where C is a function of the hyperbolicity or exponential dichtomy and ”strength” of nonlinearity.

For example, allows for global error analysis over long time intervals for DS with both positive and negative Lyapunov exponents.