The orignal ODE of Eisenmann and Weatlauffer is given by:

$$\frac{dE}{dt} = [1 - \alpha(E, \alpha_m)]F_s(t) - F_0(t) + F_{co_2} - F_T(t)\frac{E}{c_{ml}H_{ml}} + F_B \tag{1}$$

$$\text{where} \quad \alpha(E, \alpha_m) = \frac{\alpha_{ml} + \alpha_m}{2} + \frac{\alpha_{ml} - \alpha_m}{2} \tanh\left(\frac{E}{L_i h_c}\right) \tag{2}$$

In equation (1), $F_s(t)$ is the incoming solar radiation, $F_0(t)$ is the amount of longwave radiation(heat) that escapes to space F_{co_2} is the amount of longwave radiation reflected back from clouds and co_2 , $F_T(t)$ has to do with heat exchange with lower latitudes, F_B is the heat input from the ocean below the ice. While these terms are written to be time dependent we can take time averaged values, especially if we wish to consider just a month during the melt season. Monthly averaged values for these quantities can be found in the Apendix that goes with the paper, using these we would have an autonomous ODE, which is usually nice. Equation (2) represents the albedo (the percent of incoming solar radiation the ice reflects) and depends on energy of the ice as a whole. In this equation α_{ml} is the albedo of the ocean mixed layer, L_i the latent heat of fusion for ice, and h_c a chosen characteristic ice thickness which is used to control the smoothness of the parameterization. As E goes up α goes down, this is how they take into account the effect of meltponds. In their model α_m is the maximum attainable albedo of the ice which they take to be 0.6. In reality it can be as high as 0.8 with snow on top of it in cold conditions. This is the part we will modify to create two regimes for our DA problem. The basic argument will be the following, In very cold conditions the maximum attainable albedo of the surface should *tend* toward 0.8 with snow fall and other processes keeping the albedo high. When the ice is in warmer conditions, like melting, the maximum attainable albedo should tend to something lower and there should be a competition of values. Initially as the ice begins to pond the albedo drives down pretty quickly, however as the ice temperature increases the meltponds drain out and the albedo of the ice recovers very quickly. What we will want here is for energies away from zero, but warming conditions, for the albedo to tend to something like say 0.2, but closer to $E = 0$, when the ice is permeable and the ponds can drain exposing the ice surface, the maximum attainable albedo should tend toward that of bare ice 0.6.

In order to accomplish this we will model the rates of change of the maximum attainable albedo with logistic models setting the carrying capacity to be 0.8 (eq. 4) in “cold” conditions and using competing models with carrying capacities of 0.2 and 0.6 (eq. 5) in “warm” conditions. We also let the growth or decay rate in these equations depend on the energy E and some scaling factor K. In cold conditions we want the maximum attainable albedo α_m to approach 0.8 rapidly when the energy is largely negative. For this reason the growth rate is taken with the energy in the numerator (eq. 4). In warm conditions we wish the rate that α_m approaches 0.2 to be faster when the energy is away from zero but to approach 0.6 faster when the energy is near 0. As a result, we take the energy to be in the denominator with the addition of 1 to avoid singularities at $E = 0$

(eq 5) for the logistic model with a carrying capacity of 0.6 and the energy in the numerator for the logistic model with a carrying capacity of 0.2 . To define the discontinuity boundary we need a smooth function $H : \mathbb{R}^2 \to \mathbb{R}$ for which $H = 0$ on the boundary between dynamics of region $S_1 \in \mathbb{R}^2$ and separate region $S_2 \in \mathbb{R}^2$. One possible choice for H can simply be where the albedo of the system $\alpha(E, \alpha_m)$ crosses the $\alpha = 0.6$ threshold. In this case we would define H as,

$$H(E, \alpha_m) = \alpha(E, \alpha_m) - 0.6. \tag{3}$$

Which will define our two sets of dynamics,

$$\begin{aligned}
\frac{dE}{dt} &= [1 - \alpha(E, \alpha_m)]F_s(t) - F_0(t) + F_{co_2} - F_T(t)\frac{E}{c_{ml}H_{ml}} + F_B \\
\frac{d\alpha_m}{dt} &= \frac{E^2}{K^2}\alpha_m\left(1 - \frac{\alpha_m}{0.8}\right) + \frac{K^2}{1+E^2}\alpha_m\left(1 - \frac{\alpha_m}{0.6}\right) \quad \text{in} \quad S_1 = \{(E, \alpha_m) : H(E, \alpha_m) > 0\}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\frac{dE}{dt} &= [1 - \alpha(E, \alpha_m)]F_s(t) - F_0(t) + F_{co_2} - F_T(t)\frac{E}{c_{ml}H_{ml}} + F_B \\
\frac{d\alpha_m}{dt} &= \frac{K^2}{1+E^2}\alpha_m\left(1 - \frac{\alpha_m}{0.6}\right) + \frac{E^2}{K^2}\alpha_m\left(1 - \frac{\alpha_m}{0.2}\right) \quad \text{in} \quad S_2 = \{(E, \alpha_m) : H(E, \alpha_m) < 0\}
\end{aligned} \tag{5}$$

All of this will generate a third system that we can study, the sliding system. Let $x = (E, \alpha_m)$ and $\vec{f}_1(x)$ be the dynamics in equations (4) and $\vec{f}_2(x)$ that in equations (5). We will define the boundary points with the set,

$$\Sigma = \{x \in \mathbb{R}^n \ : H(x) = 0\}. \tag{6}$$

We introduce the function,

$$\sigma(x) = \left(\nabla H \cdot \vec{f}_1(x)\right)\left(\nabla H \cdot \vec{f}_2(x)\right) \tag{7}$$

and use it to define the sets,

$$\begin{aligned}
&\text{Crossing Points Set:} && \Sigma_c = \{x \in \Sigma : \sigma(x) > 0\} && (8)\\
&\text{Sliding Points Set:} && \Sigma_s = \{x \in \Sigma : \sigma(x) \leq 0\} && (9)\\
&\text{Regular Sliding Points Set:} && \hat{\Sigma}_s = \{x \in \Sigma_s : \nabla H \cdot (\vec{f}_2(x) - \vec{f}_1(x)) \neq 0\} && (10)
\end{aligned}$$

On the set $\hat{\Sigma}_s$ we can define the sliding system,

$$\frac{dx}{dt} = \lambda(x)\vec{f}_1(x) + (1 - \lambda(x))\vec{f}_2(x) \quad x \in \hat{\Sigma}_s \tag{11}$$

$$\lambda(x) = \frac{\nabla H \cdot \vec{f}_2(x)}{\nabla H \cdot \left(\vec{f}_2(x) - \vec{f}_1(x)\right)} \tag{12}$$

For our particular problem we will need to define some kind of "concentration" from our state variables. We will also need to define a non-unique satellite seen concentration.

"Ice concentration": $$C_i = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{E}{200}\right)\right)\frac{\alpha(E,\alpha_m)}{\alpha_m} \tag{13}$$

"Pond concentration": $$C_p = C_i max\left(0, \frac{(0.6 - \alpha(E,\alpha_m))}{0.6}\right) \tag{14}$$

"Satellite Radiances" $$\langle E\alpha_m, \alpha_m - \alpha(E,\alpha_m), \alpha_m\alpha\rangle \tag{15}$$

"Satellite retrieval concentration": $$C_{sat} = C_i - C_p \tag{16}$$

Here we are defining the concentration of the ice C_i to be the current albedo divided by the maximum attainable surface albedo at time t multiplied by a smoother that forces the concentration to go to zero as the maximum attainable albedo approaches 0.2. We take the area fraction of ponds C_p on the surface of any ice to be either 0 or the concentration of ice times relative difference between the albedo of bare ice and the albedo at time t when the albedo is below that of bare ice, 0.6. We then define the satellite observed radiances, the direct observation, to which our machine learned observation operator will map state variables. We will also define a concentration retrieval value, a proxy for the inverted concentration, in this case it will just be the ice concentration minus the concentration of the ponds on the ice surface, since the ponds obscure the ice.

At this point I think what we want to do is generate some data to play with using this system. We should solve things numerically in both S_1 and S_2 as well as on the boundary. Note that when we reach the boundary we must solve the sliding system. I think putting some stochasticity in the α_m term is important.

Ty, I think you probably know the most about how to handle trajectories which approach the discontinuity boundary.