1 Introduction

1.1 Basics

Invariant sets: $A \subseteq U$, s.t. $A \cdot t = A$

Long time behavior: $\omega(x) = \{y | x \cdot t_n \to y \text{ for some } t_n \to \infty\}$

Stability or attraction:
Fixed point p is stable, if $\forall \epsilon > 0, \exists \delta > 0$, s.t. $B_\delta(p) \cdot t \subset B_\epsilon(p), \forall t \geqslant 0$

Fixed point p is attracting, if (1) stable, (2) $\exists \epsilon > 0, x \in B_\epsilon(p) \implies \omega(x) = \{p\}$

Ex. Stable: surrounded by orbits, not coming in. Attracting: actually goes to the point. Unstable: like saddle points.

Exercise: work out definitions for periodic orbit and invariant sets in general.

1.1.1 Dimension

- in 1, just need to figure out FP's and the signs of the flows.

- in 2, organized by FP and periodic orbits. Poincare-Bendixson theorem: $\omega(x)$ compact $\implies \omega(x)$ contains a FP, or is a periodic orbit.
- Chaotic dynamics: near trajectories separated after certain time

1.2 Linearization

p FP, to understand the dynamics near p

- step1: Linearization of p, i.e. $Df(p)$, linear system: $\dot{y} = Df(p)y$, dynamics of this one tells something of the original.
- Question: to what extent?
- dynamics of linear system: set $A = Df(p)$, spectrum of A, $\sigma(A)$ =eigenvalues of A, we have n eigenvalues counting multiplicity.
- decomposition of $\sigma(A) = \sigma_-(A) \bigcup \sigma_0(A) \bigcup \sigma_+(A)$, decompose $\mathbb{R}^n = E_- \bigoplus E_0 \bigoplus E_+$, each invariant under the corresponding spectrum.
- $\sigma_-(A)$ part: exponentially decay, $\sigma_+(A)$ part: exponentially decay for $t \to -\infty$, i.e. exponential growth, $\sigma_0(A)$ part: undetermined.
- E_- stable subspace, E_0 center subspace, E_+ unstable subspace

1.3 Invariant Manifold

Dynamics near a FP, $p, f(p) = 0, \dot{y} = Df(p)y, A = Df(p), \sigma(A)$
Goes to nonlinearity locally, "curved" versions of $E_{-/0/+}$, $W^{s/c/u}_{loc}$, invariant under the flow relative to a neighborhood of p.
Graph for respective subspace to a complement subspace; each is tangent to their subspace

- W^s_{loc} trajectory decays to p exponentially $t \to \infty$
- W^u_{loc} trajectory decays to p exponentially $t \to -\infty$
- W^c_{loc} trajectory that does neither
- stable/unstable manifolds are determined uniquely on exponential decay condition

Global: $W^s = \bigcup_{t \leq 0} W^s_{loc} \cdot t$, the other two similarly

1.4 Bifurcation Theory

Question: How does dynamics change as parameter varies.
$\dot{x} = f(x, \mu), x \in \mathbb{R}^n, \mu \in \mathbb{R}$

1.4.1 Basic Bifurcations

- n=1
  - saddle-node: stable+unstable $\to$combine to unstable $\to$disappear