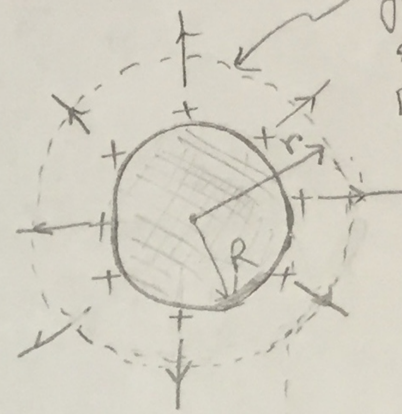
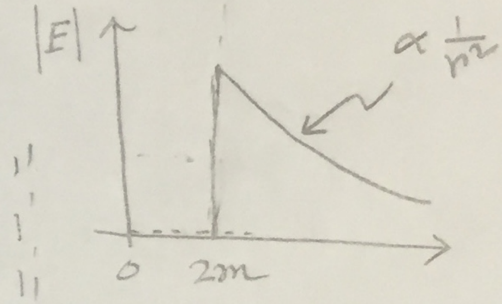
1. If the electric field strength in air exceeds 3.0×10^6 N/C, the air becomes a conductor. Using this fact, determine the maximum amount of charge that can be carried by a metal sphere 2.0 m in radius.

We can calculate the electric field outside the metal sphere considering symmetry of electric field by applying Gauss's law.



Considering spherical gaussian surface of radius "r";



(Electric field maximum at the surface)

electric field is zero inside metal sphere.

all charges are at the surface of metal sphere.

$$Q = \int dQ = \underbrace{\Psi}_{\text{flux}} = \int d\Psi = \int \vec{D} \cdot d\vec{A}$$

$$\Rightarrow Q = \int \vec{D} \cdot d\vec{A}$$

$$\Rightarrow Q = \int \epsilon \vec{E} \cdot d\vec{A}$$

$$\Rightarrow Q = \epsilon \int \vec{E} \cdot d\vec{A} \quad \text{[uniform medium air]}$$

[electric field is in radial direction $(\hat{a}_r)$ and surface $d\vec{A}$ is also in radial direction $(\hat{a}_r)$]

$$\Rightarrow Q = \epsilon \int E \, dA = \epsilon \, E(r) \int dA$$

[electric field at a particular radius is constant, from symmetry argument]

$$\Rightarrow Q = \epsilon \, E(r) \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta \, d\theta \, d\varphi$$

[$dA = r^2 \sin\theta \, d\theta \, d\varphi$ is spherical coordinates.]

$$\Rightarrow Q = \epsilon \, E(r) \cdot 4\pi r^2$$

[you may directly put $\int dA = 4\pi r^2$]

$\Rightarrow Q = \epsilon E(r)\, 4\pi r^2$

$\Rightarrow E(r) = \dfrac{Q}{4\pi\epsilon r^2}$

$E_{max} = \dfrac{Q}{4\pi\epsilon R^2}$; [$r = R$ is the radius of metal sphere]

now, $E_{max} = 3\times10^6$ N/C.

$R = 2m$.

$\Rightarrow Q_{max} = 4\pi\epsilon R^2 E_{max}$

$\Rightarrow \boxed{Q_{max} = 1.335\times10^{-3}\ C}$

[for air $\epsilon = \epsilon_r \epsilon_0$, $\epsilon_r \approx 1$]

2. You come across a spherically symmetric electric field with the following form:

$$\vec{E}(r) = E_0\left(\frac{r}{R}\right)^2 \hat{r} \qquad 0 \le r \le R$$
$$= 0 \qquad R < r < 2R$$
$$= E_0\left(\frac{r}{R} - 2\right)^2 \hat{r} \qquad 2R \le r \le 3R$$
$$= E_0\left(\frac{3R}{r}\right)^2 \hat{r} \qquad 3R \le r \le 4R$$
$$= 0 \qquad r > 4R$$

$\hat{r}$ is the radial unit vector in spherical coordinates.

(a) For all r, what is the charge $Q(r)$ contained within a radius r?

(b) Calculate the charge density $\rho(r)$ everywhere.

(c) Are there any surface charges in this charge distribution? If so, identify their location and give the magnitude of the surface charge density σ at each such location.

(a) Using gauss's law, we can calculate total amount of flux or flux density over a gausisian surface that is produced by charges lying within the gaussian surface. So, we can write:

$$Q(r) = \int \vec{D} \cdot d\vec{A} = \int \epsilon \vec{E} \cdot d\vec{A} = \epsilon \int \vec{E} \cdot d\vec{A}$$ [considering spherical gaussian surface of radius "r"]

$$= \epsilon \int E\, dA$$ [$\vec{E}$ and $d\vec{A}$ are both in radial direction]

$$= \epsilon \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} E\, r^2 \sin\theta \, d\theta \, d\varphi$$ [integral does not depend on 'dr' at a particular radius]

$$= \epsilon \cdot E \cdot 4\pi r^2$$

$$\Rightarrow \boxed{Q(r) = 4\pi\epsilon E_0 \begin{cases} \frac{r^4}{R^2} & ; 0 \le r \le R \\ 0 & ; R < r \le 2R \\ r^2\left(\frac{r}{R} - 2\right)^2 & ; 2R \le r \le 3R \\ 9R^2 & ; 3R \le r \le 4R \\ 0 & ; r > 4R \end{cases}}$$

(b) from differential form of Gauss's law.

$$\rho(r) = \nabla\cdot\bar{D} = \nabla\cdot\epsilon\vec{E} = \epsilon\,\nabla\cdot\vec{E}$$

$\vec{E}(r)$ is spherically symmetric and from defination of divergance is spherical co-ordinates.

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\left(\frac{\partial E_\phi}{\partial\varphi}\right)$$

polar component of vector

azimuthal component of vector

$$\Rightarrow \nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + 0 + 0$$

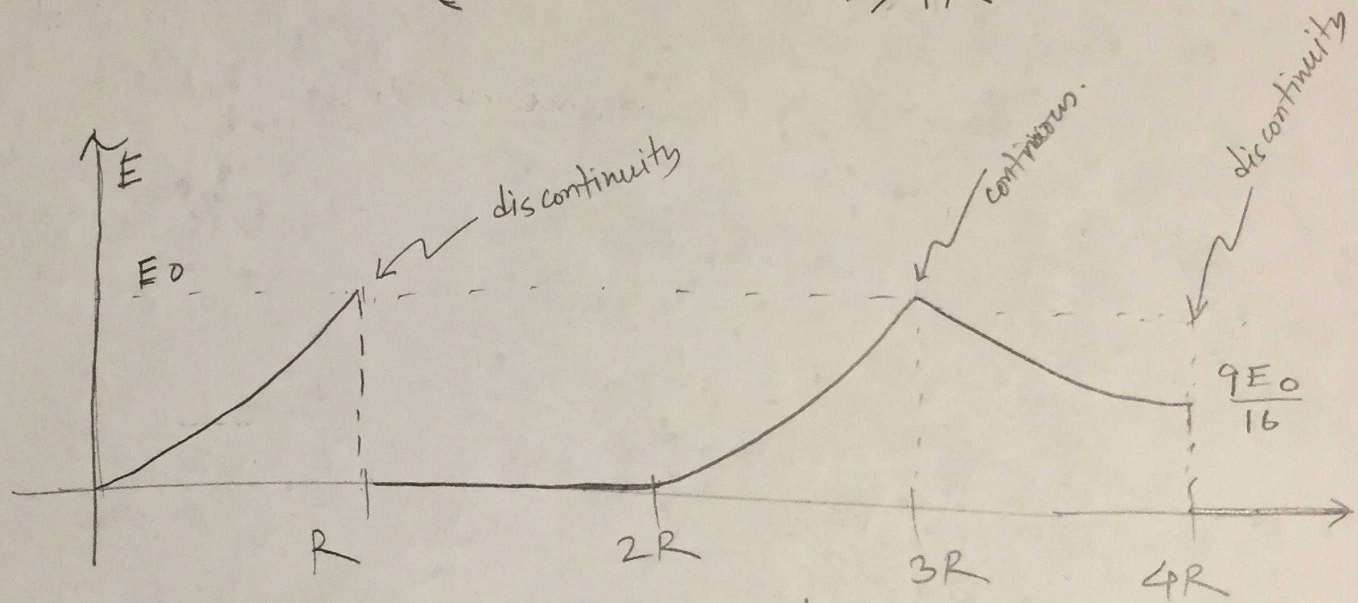
no polar or azimuthal dependence of given field.

so,
$$\rho(r) = \epsilon\,\nabla\cdot\vec{E}$$
$$= \epsilon\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right)$$
$$= \epsilon\,\frac{1}{r^2}\frac{\partial}{\partial r}\begin{cases} r^2 E_0\left(\frac{r}{R}\right)^2 & ; 0 \le r \le R \\ 0 & ; R < r \le 2R \\ r^2 E_0\left(\frac{r}{R}-2\right)^2 & ; 2R \le r \le 3R \\ r^2 E_0\left(\frac{3R}{r}\right)^2 & ; 3R \le r \le 4R \\ 0 & ; r > 4R \end{cases}$$

$$\boxed{\rho(r) = \frac{\epsilon}{r^2}E_0\begin{cases} \frac{4r^3}{R^2} & ; 0 \le r \le R \\ 0 & ; R < r \le 2R \\ \frac{4r^3}{R^2} + 8r - \frac{12r^2}{R} & ; 2R \le r \le 3R \\ 0 & ; 3R < r \le 4R \\ 0 & ; r > 4R \end{cases}}$$

$$\rho(r) = \begin{cases} \dfrac{4\epsilon E_0}{R^2} & ; 0 \le r \le R \\ 0 & ; R < r \le 2R \\ \epsilon E_0\left(\dfrac{4r}{R^2} + \dfrac{8}{r} - \dfrac{12}{R}\right) & ; 2R \le r \le 3R \\ 0 & ; 3R \le r \le 4R \\ 0 & ; r > 4R \end{cases}$$

(c)



from boundary conditions of electrostatics, we know that the tangential field components (E_T) are equal ($E_{T_1} = E_{T_2}$; medium 1 and 2) and the difference of normal components of flux densities (D_N) is equal to surface charge density

$$|D_{N2} - D_{N1}| = \rho_s \text{ (surface charge density)}$$

the problem at hand has a spherical symmetry and assuming uniform medium. we can write.

at $r = R$; $\quad |\epsilon E(R^+) - \epsilon E(R^-)| = \rho_s(R)$ [electric field is in radial direction and normal to spherical surface]

$$\Rightarrow |\epsilon[0 - E_0]| = \rho_s(R)$$

$$\Rightarrow \boxed{|-\epsilon E_0| = \rho_s(R)}$$

at $r = 4R$,

$$\left| D(4R^+) - D(4R^-) \right| = \rho_s(4R)$$

$$\Rightarrow \epsilon \left| 0 - \frac{9E_0}{16} \right| = \rho_s(4R)$$

$$\Rightarrow \boxed{\left| - \epsilon \frac{9E_0}{16} \right| = \rho_s(4R)}$$

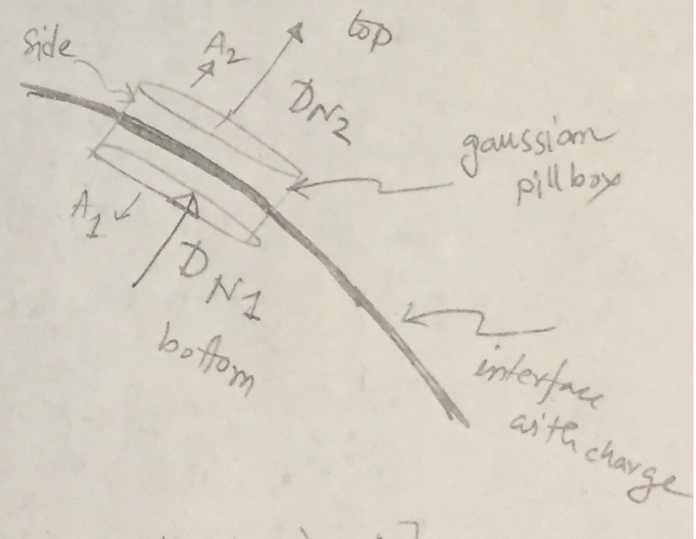
* when there is a change in normal component of electric flux density ($\bar{D}$) at an interface, that change is caused by presence of abrupt charge at the interface surface.

* normal component electric field ($\bar{E}$) can be discontinuous accross a surface due mismatch of dielectric constant.

** voltage is always continuous.

---

proof :



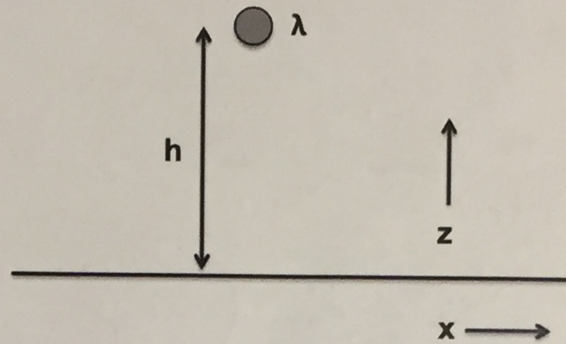
$$\oint \bar{D} \cdot d\bar{A} = Q \text{ (interface charge)}$$

$$\Rightarrow \int_{top} D_{N2} \cdot A_2 - \int_{bottom} D_{N1} A_1 + \underbrace{\cancel{\int_{side}}}_{0} = Q$$

$$\Rightarrow D_{N2} - D_{N1} = \frac{Q}{A} \quad [|A_1| = |A_2| = A]$$

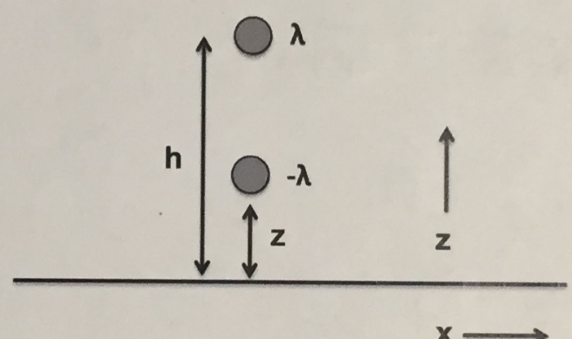
$$\Rightarrow D_{N2} - D_{N1} = \rho_s \quad \text{[surface charge density)}$$

3. An **infinitely long charged wire** (λ linear charge density) is placed parallel to the *y-axis* at a height *h* above a grounded conducting plane.



(a) Calculate the force *(per unit length)* acting on the wire.

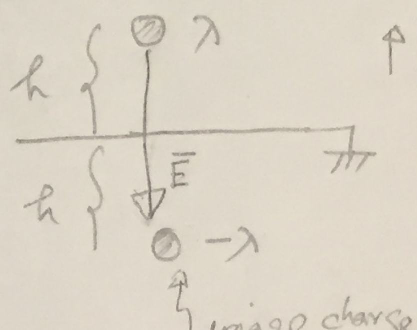
(b) Another wire with a linear charge density $-\lambda$ is to placed somewhere along the perpendicular dropped from the first wire to the plane (Figure below). Where can $-\lambda$ be placed so that the total force on it will be zero? **(z=?)**



(a) considering image charged wire,
electric field at the location of
λ line charge is:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon(2h)}\left(-\hat{a}_z\right)$$

[field due to line charge]



force $\vec{F} = Q\vec{E}$

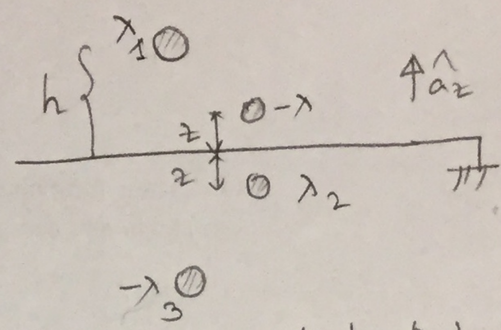
$= \lambda L \vec{E}$

force per unit length,

$$\boxed{\vec{F}_L = \lambda\vec{E} = \frac{\lambda^2}{4\pi\epsilon h}\left(-\hat{a}_z\right)}$$

(b) considering the image

the force on the $(-\lambda)$ linear charge is



$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda|$

$$\vec{F} = \underbrace{\frac{\lambda_1(-\hat{a}_z)}{2\pi\epsilon(h-z)}}_{\text{field}} \underbrace{(-\lambda L)}_{\text{charge}}$$

$$+ \frac{\lambda_2(\hat{a}_z)}{2\pi\epsilon(2z)}(-\lambda L)$$

$$+ \frac{\lambda_3(-\hat{a}_z)}{2\pi\epsilon(h+z)}(-\lambda L)$$

$$\Rightarrow \vec{F} = \left[\frac{\lambda^2 L}{2\pi\epsilon(h-z)} - \frac{\lambda^2 L}{2\pi\epsilon(2z)} + \frac{\lambda^2 L}{2\pi\epsilon(h+z)}\right]\hat{a}_z$$

$$\Rightarrow 0 = \frac{1}{h-z} + \frac{1}{h+z} - \frac{1}{2z}$$ [we want the net force to be zero]

$$\Rightarrow \frac{2h}{h^2-z^2} = \frac{1}{2z}$$

$$\Rightarrow z^2 + 4hz - h^2 = 0$$

$$\Rightarrow z = \frac{-4h \pm \sqrt{(4h)^2 - 4.1.(-h^2)}}{2.1}$$

$$\Rightarrow z = \frac{-4h \pm \sqrt{4 \times 5h^2}}{2}$$

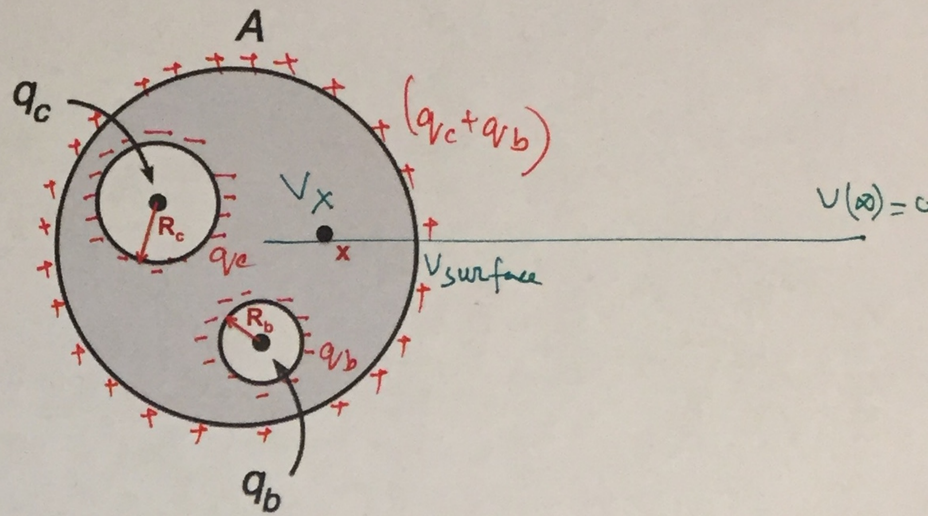
$$= -2h \pm \sqrt{5}\,h$$

$$= h(\pm\sqrt{5} - 2)$$

$$\Rightarrow \boxed{z = h(\sqrt{5} - 2)}$$

[$h(-\sqrt{5} - 2)$ is a -ve value, not acceptable as height]

4. A conducting metal sphere ***(with a radius R_a)*** has two cavities inside with radii R_b and R_c (figure below). Two point charges q_b and q_c are placed inside these cavities.



(a) What are the induced charges on the cavity surfaces?

(b) What are the induced surface charge densities on the cavity surfaces?

(c) What is the surface charge density on the outside surface of metallic sphere?

(d) What is the potential at location ***x*** (assume at potential is zero at infinity) ?

(a) $-q_c$ and $-q_b$ amount of charges

[q_b/q_c can be +ve or −ve charge. Opposit polarity of charge will accumulate on cavity surface]

(b) $\sigma_c = \dfrac{-q_c}{4\pi R_c^2}$; $\sigma_b = \dfrac{-q_b}{4\pi R_b^2}$

(c) (q_c+q_b) amount of charge will accumulate on outside surface of metallic sphere.

$$\sigma = \frac{q_c+q_b}{4\pi R_a^2}$$

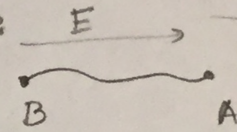
(d) electric field inside metal is zero (0). So, voltage drop inside metal $\left(-\int E\cdot dl\right)$ is zero and thus

$V_x = V_{surface}$

Note: potential difference V_{AB} is the work done by the external forces in moving a unit test charge from point B to point A.

$$V_{AB} = V_A - V_B = \underbrace{-\int_B^A \bar{E}\cdot\bar{dL}}_{\text{work done}}$$

work done by field:



$$= \int_B^A \bar{F}\cdot\bar{dL}$$
$$= \int_B^A q\bar{E}\cdot\bar{dL}$$
$$= \int_B^A \bar{E}\cdot\bar{dL} \quad [q = 1 \text{ unit test charge}]$$

now, $V_x = V_{surface} = -\int_{\infty}^{r=R_a} \vec{E}\cdot\bar{dl}$

$$= -\int_{\infty}^{r=R_a} \frac{q_c+q_b}{4\pi\epsilon r^2}\,\hat{a}_r \cdot dr\,\hat{a}_r$$

$$= -\frac{q_c+q_b}{4\pi\epsilon}\left[-\frac{1}{r}\right]_{\infty}^{R_a}$$

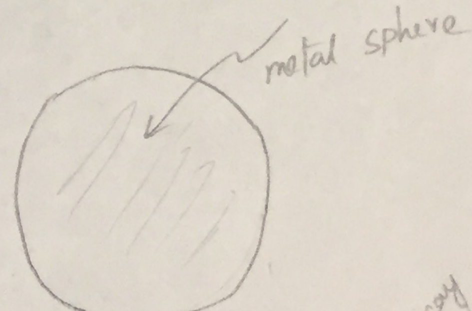
$$\boxed{V_x = \frac{q_c+q_b}{4\pi\epsilon R_a}}$$

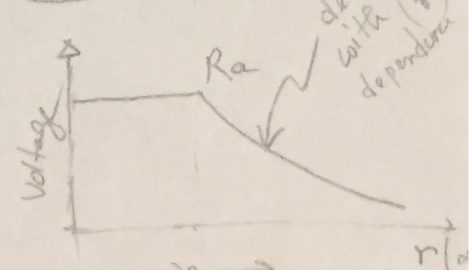
alterantively:

(0 inside metal)

$$V_x = -\int_{\infty}^{x} \bar{E}\cdot\bar{dl} = -\int_{\infty}^{r=R_a} \bar{E}\cdot\bar{dl} - \cancel{\int_{r=R_a}^{x} \bar{E}\cdot\bar{dl}}$$

$$\Rightarrow V_x = \frac{q_c+q_b}{4\pi\epsilon R_a}$$

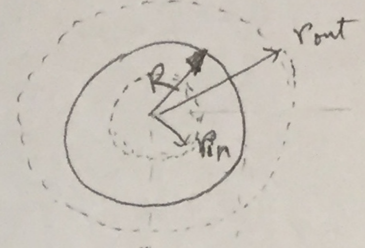




5. A non-conducting sphere of radius R has a charge density **ρ = ar.** where *a* is a constant. Let *r* be the distance from the center of the sphere.

(a) Find the electric field everywhere, both inside and outside the sphere.

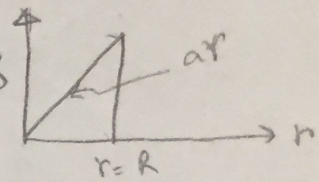
(b) Find the electric potential everywhere, both inside and outside the sphere. Assume that potential is zero at infinity.



(a) for $r \leq R$, considering a spherical gaussian surface of radius "r_{in}"

$$\int \bar{D}_{in} \cdot d\bar{A} = \int dq = \int \rho \, dv$$

$$\Rightarrow \epsilon E_{in}(r_{in}) \int dA = \int_{r=0}^{r=r_{in}} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} ar \, (r^2 \sin\theta \, dr \, d\theta \, d\varphi) \; \rho$$



[$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$ is spherical coordinates]

$$\Rightarrow \epsilon E_{in}(r_{in})(4\pi r_{in}^2) = \int_{r=0}^{r_{in}} ar \cdot r^2 4\pi \, dr$$

$V = \frac{4}{3}\pi r^3$
$\Rightarrow dv = 4\pi r^2 dr$

$$\Rightarrow \epsilon E_{in}(r_{in})(4\pi r_{in}^2) = 4\pi a \left[\frac{r^4}{4}\right]_0^{r_{in}}$$

$$\Rightarrow E_{in}(r_{in}) = \frac{a r_{in}^2}{4\epsilon}$$

$$\Rightarrow \boxed{E(r) = \frac{ar^2}{4\epsilon} \; ; \; r \leq R}$$

for $r > R$, considering a spherical gaussian surface of radius "r_{out}"

$$\int \bar{D}_{out} \cdot d\bar{A} = \int dq = \int \rho \, dv$$

$$\Rightarrow \epsilon E_{out}(r_{out}) \int dA = \int_{r=0}^{r=R} \rho \, dv + \int_{r=R}^{r=r_{out}} \rho \, dv \quad 0$$

$$\Rightarrow \epsilon E_{out}(r_{out})(4\pi r_{out}^2) = \frac{4\pi a R^4}{4} + 0$$

$$\Rightarrow E_{out}(r_{out}) = \frac{aR^4}{4\epsilon r^2}$$

$$\Rightarrow \boxed{E(r) = \frac{aR^4}{4\epsilon r^2} \; ; \; r > R}$$

now, voltage for $r \geq R$ (outside the non-conducting sphere)

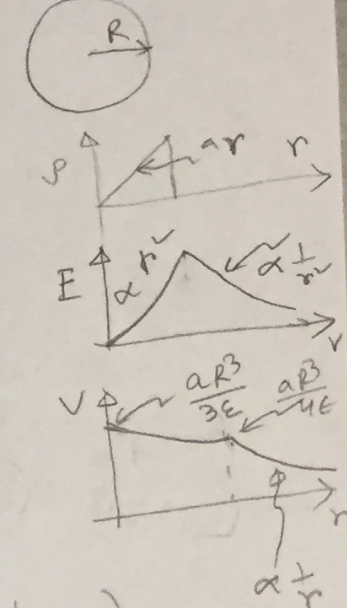
$$V_{out}(r_{out}) = -\int_{r=\infty}^{r=r_{out}} \bar{E} \cdot \bar{dl}$$

$$= -\int_{r=\infty}^{r=r_{out}} \frac{aR^4}{4\epsilon_0 r^2} \cdot dr$$

$$= \frac{aR^4}{4\pi\epsilon} \left[\frac{1}{r}\right]_{\infty}^{r_{out}}$$

$$\Rightarrow V_{out}(r_{out}) = \frac{aR^4}{4\pi\epsilon\, r_{out}}$$

$$\Rightarrow V(r) = \frac{aR^4}{4\pi\epsilon r} \quad ; r \geq R$$



for $r \leq R$ (inside the non-conducting sphere)

$$V_{inside}(r_{in}) = -\int_{r=\infty}^{r_{in}} \vec{E} \cdot \bar{dl}$$

$$= -\int_{r=\infty}^{R} \vec{E} \cdot \bar{dl} - \int_{r=R}^{r=r_{in}} \bar{E} \cdot \bar{dl}$$

$$= \frac{aR^4}{4\pi\epsilon R} - \int_{r=R}^{r=r_{in}} \frac{ar^2}{4\epsilon} \cdot dr$$

$$= \frac{aR^3}{4\pi\epsilon} - \frac{a}{12\epsilon}\left(r_{in}^3 - R^3\right)$$

$$\Rightarrow \boxed{V(r) = \frac{aR^3}{4\pi\epsilon} - \frac{a}{12\epsilon}\left(r^3 - R^3\right) \; ; r \leq R}$$