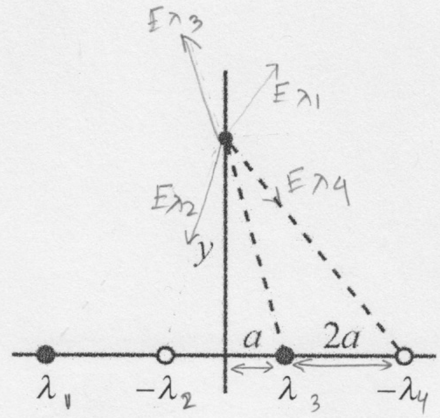
1. Four wires, with **linear charge densities** λ, −λ, λ, and −λ, are located at equally spaced intervals on the x axis. Their x values are −3a, −a, a, and 3a, respectively. Find the point on the y axis for which the electric field is zero? If so, find the y value.



y, x

$\lambda = |\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4|$

$(\hat{a}_x)$ unit vector

due to symmetry $\Sigma E_y = 0$ [E_y for λ_1 and $-\lambda_4$ are equal and opposit, E_y for $-\lambda_2$ and λ_3 are equal and opposit]

now ΣE_x at location $(0, y)$ is (x position)

$$= \underbrace{\underbrace{\frac{\lambda_1}{2\pi\epsilon\sqrt{y^2+9a^2}}}_{|E\lambda_1| \text{ for line charge}} \cdot \underbrace{\frac{3a}{\sqrt{y^2+9a^2}}}_{\text{projection along x-axis}}}_{E\lambda_{1x}} (\hat{a}_x)$$

$$- \frac{\lambda_2}{2\pi\epsilon\sqrt{y^2+a^2}} \cdot \frac{a}{\sqrt{y^2+a^2}} (\hat{a}_x)$$

$$- \frac{\lambda_3}{2\pi\epsilon\sqrt{y^2+a^2}} \cdot \frac{a}{\sqrt{y^2+a^2}} (\hat{a}_x)$$

$$+ \frac{\lambda_4}{2\pi\epsilon\sqrt{y^2+9a^2}} \cdot \frac{3a}{\sqrt{y^2+9a^2}} (\hat{a}_x)$$

we want $\Sigma E_x = 0$.

so,

$$\frac{\lambda_1}{2\pi\epsilon\sqrt{y^2+9a^2}} \cdot \frac{3a}{\sqrt{y^2+9a^2}}(\hat{a}_x) - \frac{\lambda_2}{2\pi\epsilon\sqrt{y^2+a^2}} \cdot \frac{a}{\sqrt{y^2+a^2}}(\hat{a}_x)$$

$$- \frac{\lambda_3}{2\pi\epsilon\sqrt{y^2+a^2}} \cdot \frac{a}{\sqrt{y^2+a^2}}(\hat{a}_x) + \frac{\lambda_4}{2\pi\epsilon\sqrt{y^2+9a^2}} \cdot \frac{3a}{\sqrt{y^2+9a^2}}(\hat{a}_x) = 0$$

$$\Rightarrow \frac{3}{y^2+9a^2} - \frac{1}{y^2+a^2} - \frac{1}{y^2+a^2} + \frac{3}{y^2+9a^2} = 0$$

$$\Rightarrow \frac{6}{y^2+9a^2} = \frac{2}{y^2+a^2}$$

$$\Rightarrow 3y^2+3a^2 = y^2+9a^2$$

$$\Rightarrow 2y^2 = 6a^2$$

$$\Rightarrow \boxed{y = \pm\sqrt{3}\,a}$$ [there are $\Sigma \vec{E} = 0$ point on both side]

2. A spherically symmetric electric field

$$\vec{E}(r) = E_0\left(\frac{r}{r_0}\right)\exp\left(-r^2/r_0^2\right)\hat{r}$$

fills all of space. The parameter E_0 has the units of electric field; r_0 has the units of length.

(a) Find an expression for the charge density $\rho(r)$, valid everywhere.

(b) Sketch this charge density versus radius.

(c) Find the radius r at which the charge density is zero *(other than at infinity)*.

(d) Find an expression for the electric potential π everywhere.

(e) Find the total energy stored in the electric field.

The integral

$$\int_0^\infty r^4 \exp\left(-r^2/a^2\right)dr = \frac{3a^5}{8}\sqrt{\pi}$$

may prove very useful to you.

(a) $\vec{E}(r) = E_0 \left(\frac{r}{r_0}\right) \exp\left(-r^2/r_0^2\right)\hat{a}_r$; $\vec{E}(r) = E_r\,\hat{a}_r + \cancel{E_\theta}\,\hat{a}_\theta + \cancel{E_\varphi}\,\hat{a}_\varphi$ (with $E_\theta = 0$, $E_\varphi = 0$)

$$\rho(r) = \nabla\cdot\vec{D} = \nabla\cdot\epsilon\vec{E} = \epsilon\,\nabla\cdot\vec{E}$$

$$= \epsilon\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}\right]$$

$$= \epsilon\left[\frac{1}{r^2}\frac{\partial}{\partial r}\, r^2 E_0\left(\frac{r}{r_0}\right)\exp\left(-r^2/r_0^2\right)\right]$$

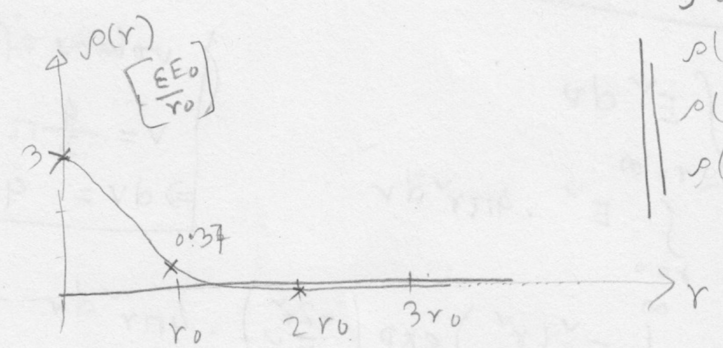
$$= \epsilon\frac{1}{r^2}\left(\frac{E_0}{r_0}\right)\frac{\partial}{\partial r}\left[r^3\exp\left(-r^2/r_0^2\right)\right]$$

$$= \epsilon\frac{1}{r^2}\left(\frac{E_0}{r_0}\right)\left[3r^2\exp\left(-r^2/r_0^2\right) + r^3\left(-\frac{1}{r_0^2}\right)\exp\left(\frac{-r^2}{r_0^2}\right)\cdot(2r)\right]$$

$$= \epsilon\frac{1}{r^2}\left(\frac{E_0}{r_0}\right)\left[3r^2\exp\left(-r^2/r_0^2\right) - 2r^4\cdot\frac{1}{r_0^2}\cdot\exp\left(\frac{-r^2}{r_0^2}\right)\right]$$

$$= \epsilon\frac{E_0}{r_0}\exp\left(-r^2/r_0^2\right)\left[3 - \frac{2r^2}{r_0^2}\right]$$

(b)



$\rho(r) = \rho(-r)$

$\rho(0) = 3\frac{\epsilon E_0}{r_0}$

$\rho(r_0) = \frac{\epsilon E_0}{r_0}\exp(-1) = 0.36\frac{\epsilon E_0}{r_0}$

$\rho(2r_0) = \frac{\epsilon E_0}{r_0}\exp(-4)(-5)$

$= -0.0915\frac{\epsilon E_0}{r_0}$

$\rho(3r_0) = \frac{\epsilon E_0}{r_0}\exp(-9)(-15)$

$= -0.0019\frac{\epsilon E_0}{r_0}$

(c) $\rho(r) = 0 \Rightarrow \epsilon\frac{E_0}{r_0}\exp\left(-\frac{r^2}{r_0^2}\right)\left[3 - \frac{2r^2}{r_0^2}\right] = 0$

$\Rightarrow \exp\left(-\frac{r^2}{r_0^2}\right) = 0$ or $\left[3 - \frac{2r^2}{r_0^2}\right] = 0$

$\Rightarrow \boxed{r \to \infty \quad \text{or} \quad r = \sqrt{\tfrac{3}{2}}\, r_0}$

(d) potential, $\Omega(r) = -\int_{r=\infty}^{r} \bar{E} \cdot d\bar{r}$

$$= -\int_{r=\infty}^{r} E_0 \left(\frac{r}{r_0}\right) \exp\left(-r^2/r_0^2\right) \, dr$$

$$= -\frac{E_0}{r_0} \int_{r=\infty}^{r} r \exp\left(-r^2/r_0^2\right) dr$$

$$= -\frac{E_0}{r_0} \int_{p=\infty}^{p=r^2/r_0^2} \frac{r_0^2}{2} \cdot e^{-p} \, dp$$

$$= \frac{-E_0}{r_0} \cdot \frac{r_0^2}{2} \int_{p=\infty}^{p=\frac{r^2}{r_0^2}} e^{-p} \, dp$$

$$= -\frac{E_0}{r_0} \frac{r_0^2}{2} \left[-e^{-p}\right]_{\infty}^{r^2/r_0^2}$$

$$= \frac{E_0}{r_0} \cdot \frac{r_0^2}{2} \cdot \left[e^{-r^2/r_0^2} - e^{-\infty}\right]$$

$$\boxed{\Omega(r) = \frac{E_0 r_0}{2} e^{-r^2/r_0^2}}$$

let,

$$\frac{r^2}{r_0^2} = p$$

$$\Rightarrow \frac{dp}{dr} = \frac{2r}{r_0^2}$$

$$\Rightarrow dp = \frac{2r}{r_0^2} dr$$

(e) $U = \frac{\epsilon}{2} \int E^2 \, dv$

$$= \frac{\epsilon}{2} \int_{r=0}^{r=\infty} E^2 \cdot 4\pi r^2 \, dr$$

$$= \frac{\epsilon}{2} \int_{r=0}^{\infty} E_0^2 \left(\frac{r^2}{r_0^2}\right) \exp\left(-\frac{2r^2}{r_0^2}\right) \cdot 4\pi r^2 \, dr$$

$$= \frac{2\pi \epsilon E_0^2}{r_0^2} \int_{r=0}^{\infty} r^4 \exp\left(-\frac{2r^2}{r_0^2}\right)$$

$$= \frac{2\pi \epsilon E_0^2}{r_0^2} \cdot 3 \cdot \frac{(r_0/\sqrt{2})^5}{8} \sqrt{\pi}$$

$$\Rightarrow \boxed{U = \frac{3\epsilon}{4} \pi^{3/2} r_0^3 \cdot \frac{1}{4\sqrt{2}} \cdot E_0^2 = \frac{3}{16\sqrt{2}} \pi^{3/2} r_0^3 E_0^2}$$

volume of a sphere,

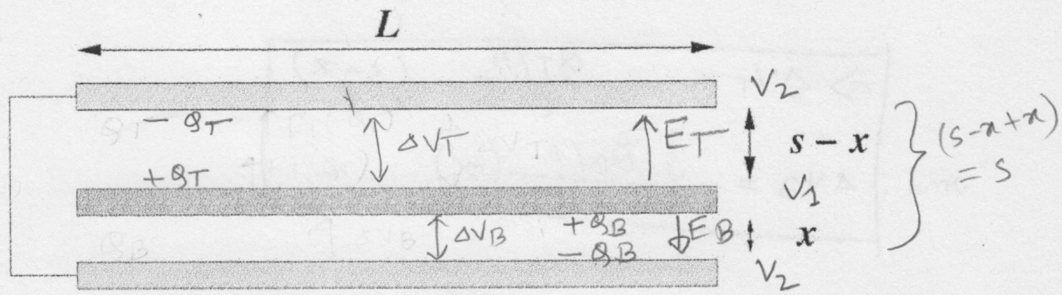
$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow dV = 4\pi r^2 \, dr$$

$$a = \frac{r_0}{\sqrt{2}}$$

3. Three large, square conducting plates are placed parallel to one another as shown. A conducting wire connects the outer plates. A total charge Q is placed on the inner plate; this divides in an amount Q_T on the top of the inner plate and Q_B on the bottom of the inner plate.

The side length of the plates L is far larger than the separations (L >> s, L >> x), so you may ignore the effect of any edge effects in this problem.



(a) Write down expressions for the potential difference between the inner plate and the top plate, and between the inner plate and the bottom plate.

(b) Calculate Q_T and Q_B in terms of Q, x, and s.

(c) Calculate the total energy stored in the electric field of this configuration as a function of x.

(d) Calculate the value of x for which the stored energy in this system is maximum. (Hint: you can do this with a straightforward but moderately tedious calculation.)

(a) $-Q_T$ amount of charge is going to induce in the inner part of top plate and $-Q_B$ amount of charge is going to induce in the inner part of bottom plate

[if the top and bottom conductors are grounden, there will be no charge accumulation at the outer surface of top and bottom conductor. Otherwise, $+Q_T$ and $+Q_B$ charge is going to accumulate on outer surface of top and bottom plate.]

field between top and inner plate,

["A" is the area of the conducting plates]

$$E_T = E_T(Q_T) + E_T(-Q_T)$$

$$= \frac{Q_T/A}{2\epsilon} + \frac{Q_T/A}{2\epsilon}$$

$$= \frac{Q_T/A}{\epsilon}$$

similarly, $E_B = \frac{Q_B/A}{\epsilon}$

now, $\Delta V_T = V_2 - V_1 = -\int_1^2 E \cdot dl = -\int \frac{Q_T/A}{\epsilon} \cdot dl$

$$\boxed{\begin{aligned} \Rightarrow \Delta V_T &= \frac{Q_T/A}{\epsilon} \cdot (s-x) \\ \text{and, } \Delta V_B &= \frac{Q_B/A}{\epsilon}(x) \end{aligned}}$$

(b) now $\Delta V_T = \Delta V_B$

$$\Rightarrow \frac{Q_T/A}{\epsilon}(s-x) = \frac{Q_B/A}{\epsilon}(x)$$

$$\Rightarrow \frac{Q_T}{Q_B} = \frac{x}{s-x}$$

$$\Rightarrow \frac{Q_T}{Q_T+Q_B} = \frac{x}{s}$$

[$Q_T + Q_B = Q$, total charge in the inner plate]

$$\Rightarrow \boxed{Q_T = \frac{x}{s} \cdot Q}$$

again, $\frac{Q_B}{Q_T} = \frac{s-x}{x}$

$$\Rightarrow \frac{Q_B}{Q_T+Q_B} = \frac{s-x}{s}$$

$$\Rightarrow \boxed{Q_B = \frac{s-x}{s} \cdot Q}$$

(c) $U = \frac{1}{2}\epsilon \int E^2 dV = \frac{1}{2}\epsilon \int E_T^2 dV + \frac{1}{2}\epsilon \int E_B^2 dV$

$$= \frac{1}{2}\epsilon \cdot \frac{Q_T^2/A^2}{\epsilon^2} \cdot A(s-x) + \frac{1}{2}\epsilon \frac{Q_B^2/A^2}{\epsilon^2} \cdot A(x)$$

$$= \frac{1}{2} \cdot \frac{1}{A\epsilon} \cdot \left[Q_T^2(s-x) + Q_B^2(x) \right]$$

$$\Rightarrow U = \frac{1}{2} \frac{1}{AE} \cdot \left[Q_T^2 (s-x) + Q_B^2 (x) \right]$$

$$\Rightarrow U = \frac{1}{2} \frac{1}{AE} \left[\frac{x^2}{s^2} \cdot Q^2 \cdot (s-x) + \frac{(s-x)^2}{s^2} \cdot Q^2 \cdot x \right]$$

$$\Rightarrow U = \frac{1}{2} \frac{1}{AE} \frac{Q^2}{s^2} \left[(s-x) x^2 + (s-x)^2 x \right]$$

$$\Rightarrow U = \frac{1}{2} \frac{1}{AE} \frac{Q^2}{s^2} \left[s x^2 - \cancel{x^3} + s^2 x + \cancel{x^3} - 2 s x^2 \right]$$

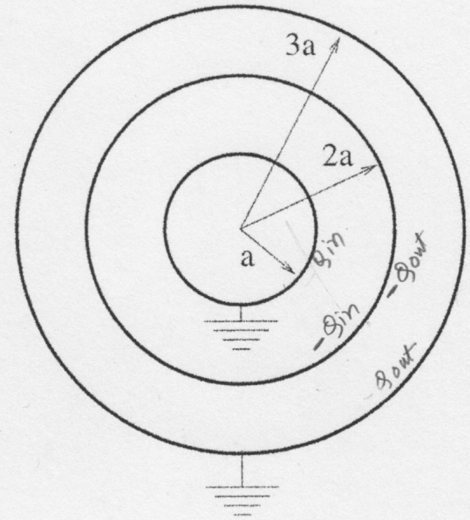
$$\Rightarrow \boxed{U = \frac{1}{2} \frac{1}{AE} \frac{Q^2}{s^2} \left[s^2 x - s x^2 \right]}$$

(d) $$\frac{dU}{\partial x} = \frac{1}{2} \frac{1}{AE} \frac{Q^2}{s^2} \left[s^2 - s(2x) \right]$$

for U_{max}, $\frac{\partial U}{\partial x} = 0 \Rightarrow s^2 - s(2x) = 0$

$$\Rightarrow \boxed{x = s/2}$$

4. A capacitor is made of three conducting concentric spherical shells of radii a, 2a and 3a as shown in the figure below. In what follows we will assume that the shells are thick enough that we may distinguish the inner and outer surfaces, but thin enough that we do not actually need to know what their thickness is.



even if the inner and outer shells were not grounded, there would be no charge accumulated on the inner surface of inner shell and outer surface of outer shell.

The inner and outer shells are **grounded**: their potentials are fixed to be zero ($\varphi = 0$). The middle shell carries some net charge Q. This charge induces a charge Q_{in} on the outer edge of the inner shell and a charge Q_{out} on the inner edge of the outer shell. Note that these charges are taken **from ground**, so the inner and outer shells are **not** electrically neutral.

(a) What is the electric field in the region $r < a$?

(b) What is the electric field in the region $a < r < 2a$? Your answer should be expressed in terms of Q_{in}.

(c) What is the electric field in the region $2a < r < 3a$? Your answer should be expressed in terms of Q_{in} and Q.

(d) What is the electric field in the region $r > 3a$? Using this result, find Q_{out} in terms of Q_{in} and Q.

(e) Find the potential of the middle shell with respect to the **ground** first by going from the inner shell to the middle shell and then by going from the outer shell to the middle shell. Leave your answers in terms of Q_{in} and Q.

(f) Using the answer to part (e), find Q_{in} in terms of Q.

(g) What is the capacitance of this system?

(a, b, c, d)

(a) From Gauss's law:

$$\int \epsilon \bar{E}(r) \cdot d\bar{A} = \text{enclosed charge}$$

$$\Rightarrow \epsilon E(r) \cdot 4\pi r^2 = \begin{cases} 0, & r < a \\ -Q_{in}; & a < r < 2a \\ -Q_{out}; & 2a < r < 3a \\ 0; & r > 3a \end{cases}$$

considering spherical gaussian surface of radius "r"

$$\Rightarrow E(r) = \begin{cases} 0, & r < a \\ \frac{Q_{in}}{4\pi\epsilon r^2}, & a < r < 2a \\ \frac{-Q_{out}}{4\pi\epsilon r^2}; & 2a < r < 3a \\ 0; & r > 3a \end{cases}$$

$$\Rightarrow \boxed{E(r) = \begin{cases} 0, & r < a \\ \frac{Q_{in}}{4\pi\epsilon r^2}, & a < r < 2a \\ \frac{Q + Q_{in}}{4\pi\epsilon r^2}, & 2a < r < 3a \\ 0, & r > 3a \end{cases}}$$

$-Q_{out} + (-Q_{in}) = Q$, net charge in the inner shell

↳ radially outward

(e) $$\varphi(2a^-) - \varphi(a) = -\int_{r=a}^{r=2a} \bar{E} \cdot d\bar{l}$$

$$\Rightarrow \varphi(2a^-) - 0 = -\int_{r=a}^{2a} \frac{-Q_{in}}{4\pi\epsilon r^2} \cdot dr$$

$$= \frac{Q_{in}}{4\pi\epsilon}\left(\frac{1}{2a} - \frac{1}{a}\right)$$

$$\Rightarrow \varphi(2a^-) = -\frac{Q_{in}}{4\pi\epsilon} \cdot \frac{1}{2a}$$

b) $\varphi(2a^+) - \varphi(3a) = -\int_{r=3a}^{2a} \bar{E} \cdot d\bar{r}$

$\Rightarrow \varphi(2a^+) - 0 = -\int_{r=3a}^{2a} \frac{Q+Q_{in}}{4\pi\epsilon r^2}\, dr$

$= \frac{Q+Q_{in}}{4\pi\epsilon}\left(\frac{1}{2a} - \frac{1}{3a}\right)$

$\Rightarrow \varphi(2a^+) = \frac{Q+Q_{in}}{4\pi\epsilon} \cdot \frac{1}{6a}$

(f) now, $\varphi(2a^-) = \varphi(2a^+)$ [voltage on same conducting shell]

$\Rightarrow \frac{-Q_{in}}{4\pi\epsilon}\frac{1}{2a} = \frac{Q+Q_{in}}{4\pi\epsilon}\frac{1}{6a}$

$\Rightarrow -3Q_{in} = Q + Q_{in}$

$\Rightarrow \boxed{Q_{in} = -Q/4}$

(g) $Q = C\varphi$

$\Rightarrow C = \frac{Q}{\varphi} = \frac{Q}{-Q_{in}/8\pi\epsilon a} = \frac{Q}{Q/32\pi\epsilon a} = 32\pi\epsilon a$

$\Rightarrow \boxed{C = 32\pi\epsilon a.}$