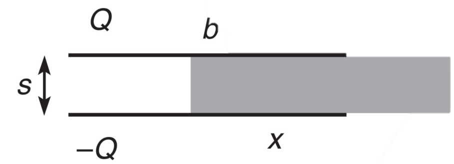
EE 135 Winter 2015
HW #4

Q.1: Consider the setup of Problem 10.2. In terms of the various parameters given there, find the charge densities on the left and right parts (*on metal plates*) of the capacitor. You should find that, as x increases, the charge densities on *both* parts of plates decrease. At first glance this seems a bit absurd, so try to explain intuitively how it is possible.



Q.7: Densities on a capacitor

(You need to work on Problem 10.2 first. Its solution is given at the end of the book.)

The charge density σ_2 on the right part of each plate is κ times the charge density σ_1 on the left part. So

$$\sigma_1(b-x)a + \sigma_2 xa = Q \implies \sigma_1(b-x)a + (\kappa\sigma_1)xa = Q. \tag{639}$$

The two charge densities, σ_1 and $\sigma_2 = \kappa\sigma_1$, are therefore given by

$$\sigma_1 = \frac{Q/a}{b+(\kappa-1)x}, \qquad \sigma_2 = \frac{\kappa Q/a}{b+(\kappa-1)x}. \tag{640}$$

Since $\kappa > 1$, both of these densities decrease as x increases. It is possible for both densities to decrease while the total charge remains at the given value Q, because the charge in the right region increases (while the charge in the left region decreases), but it does so at a slower rate than the area increases; so the density decreases. We *would* have a paradox if the areas stayed the same.

An analogy: 10 people each have the same amount of money. 20 other people each have the same amount, but it is smaller than what the first 10 have. One of these 20 people takes some money from each of the first 10, and also from each of the other 19, so that she now has the same amount as the first 10. The total amount of money held by all 30 people is still the same, but the average amounts in the two groups (now with 11 and 19 people) have both decreased.

Q.2: Materials to be used as insulators or dielectrics in capacitors are rated with respect to *dielectric strength,* defined as the maximum internal electric field the material can support without danger of electrical breakdown. It is customary to express the dielectric strength in kilovolts per mil. (One mil is 0.001 inch, or 0.00254 cm.) For example, Mylar (a Dupont polyester film) is rated as having a dielectric strength of 14 kilovolts/mil when it is used in a thin sheet – as it would be in a typical capacitor. The dielectric constant κ of Mylar is 3.25. Its density is 1.40 g/cm^3.

(a) Calculate the maximum amount of energy that can be stored in a Mylar-filled capacitor, and express it in joules/kg of Mylar.

(b) Assuming the electrodes and case account for 25 percent of the capacitor's weight, how high could the capacitor be lifted by the energy stored in it? Compare the capacitor as an energy storage device with the battery in Exercise 4.41.

TIPS:

(a) Assume it is a parallel plate capacitor and calculate the maximum energy that could be stored in unit volume. You can covert this to energy stored per unit mass.
(b) You can use symbol m for the mass of the whole capacitor. Masses should cancel each other out when you are calculating the height.

The maximum field is 14 kilovolts/mil, which in volts/meter equals

$$E_{\max} = \frac{1.4 \cdot 10^4 \text{ V}}{2.54 \cdot 10^{-5} \text{ m}} = 5.5 \cdot 10^8 \text{ V/m}. \tag{642}$$

The capacitance of the Mylar-filled capacitor is $\kappa\epsilon_0 A/s$. The energy stored is still $C\phi^2/2$, so the maximum possible energy density is

$$\begin{aligned} \frac{\text{energy}}{\text{volume}} &= \frac{1}{2}C\phi^2\frac{1}{V} = \frac{1}{2}\frac{\kappa\epsilon_0 A}{s}(Es)^2\frac{1}{As} = \frac{1}{2}\kappa\epsilon_0 E^2 \\ &= \frac{1}{2}(3.25)\left(8.85 \cdot 10^{-12}\ \frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)(5.5 \cdot 10^8 \text{ V/m})^2 = 4.4 \cdot 10^6 \text{ J/m}^3. \end{aligned} \tag{643}$$

The maximum energy per kilogram of Mylar is therefore

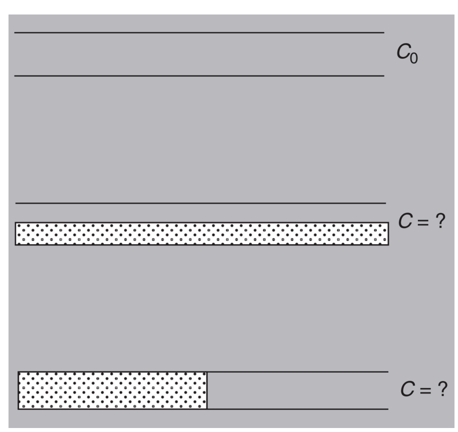
$$\frac{4.4 \cdot 10^6 \text{ J/m}^3}{1400\,\text{kg/m}^3} = 3100 \text{ J/kg}. \tag{644}$$

To determine how high the capacitor could lift itself, let the entire mass of the capacitor be m. Then $3m/4$ of this is Mylar, so conservation of energy gives

$$E = mgh \implies (3100\,\text{J/kg})(3m/4) = mgh \implies h = \frac{(3/4)((3100\,\text{J/kg})}{9.8\,\text{m/s}^2} = 240 \text{ m}. \tag{645}$$

The D cell in Exercise 4.41 had an energy storage of $1.8 \cdot 10^5$ J/kg, which is about 60 times as much as the Mylar capacitor. However, the capacitor can deliver all the stored energy in less than a microsecond!

Q.3: Figure Q.3 shows three capacitors of the same area and plate separation. Call the capacitance of the vacuum capacitor C_0. Each of the others is half-filled with a dielectric, with the same dielectric constant κ, but differently disposed, as shown. Find the capacitance of each of these two capacitors. (Neglect edge effects.)



Q.3: Partially filled capacitors

The second capacitor in the figure consists of two capacitors in series; you can imagine the boundary between them to be two plates with charge Q and $-Q$ superposed. Both of these capacitors have plate separation $s/2$ and area A, so the capacitances are (with the two halves labeled by "v" for vacuum and "d" for dielectric) $C_\text{v} = \epsilon_0 A/(s/2)$ and $C_\text{d} = \kappa\epsilon_0 A/(s/2)$. Since $C_0 = \epsilon_0 A/s$, we have $C_\text{v} = 2C_0$ and $C_\text{d} = 2\kappa C_0$. Problem 3.18 gives the rule for adding capacitors in series, so the desired capacitance is (with "S" for series)

$$\frac{1}{C_\text{S}} = \frac{1}{C_\text{v}} + \frac{1}{C_\text{d}} = \frac{1}{2C_0} + \frac{1}{2\kappa C_0} \quad\Longrightarrow\quad C_\text{S} = \frac{2\kappa}{\kappa+1} C_0. \tag{646}$$

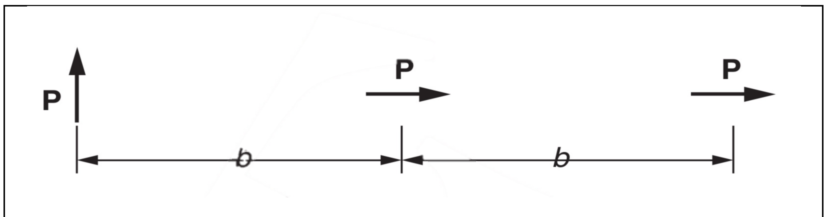
The third capacitor in the figure consists of two capacitors in parallel. They both have plate separation s and area $A/2$, so the capacitances are $C_\text{v} = \epsilon_0(A/2)/s$ and $C_\text{d} = \kappa\epsilon_0(A/2)/s$. These can be written as $C_\text{v} = C_0/2$ and $C_\text{d} = \kappa C_0/2$. Problem 3.18 gives the rule for adding capacitors in parallel, so the desired capacitance is (with "P" for parallel)

$$C_\text{P} = C_\text{v} + C_\text{d} = \frac{C_0}{2} + \frac{\kappa C_0}{2} = \frac{1+\kappa}{2} C_0. \tag{647}$$

If $\kappa = 1$, then both C_S and C_P are equal to C_0, as they should be. For any other value of κ (greater than 1, of course), both C_S and C_P are larger than C_0. This makes sense because the effect of a dielectric is to partially cancel the existing charge, which means that more charge must be added if the same potential is to be maintained.

If $\kappa = \infty$, then $C_\text{S} = 2C_0$ and $C_\text{P} = \infty$. The former makes sense because the dielectric is actually a conductor in this case, so we effectively have a vacuum capacitor with separation $s/2$. The latter makes sense because the capacitance of a conductor is infinite, since any charge you dump on it will be neutralized by the shifting of charges within the conductor. So the left half of the third capacitor has infinite capacitance.

Q.4: What are the magnitude and direction of the force on the central dipole caused by the field of the other two dipoles in Fig. Q.4?



Q.4: Forces on a dipole.

Let the middle dipole be located at the origin. Intuitively, the downward field at the origin arising from the left dipole is slightly stronger at the (negative) left end of the middle dipole than at its (positive) right end. The left end therefore feels a larger force upward than the right end feels downward. So there is a net upward force due to the field of the left dipole. Similar reasoning shows that there is a net rightward force due to the field of the right dipole. So the total force on the middle dipole is upward and rightward.

Let's be quantitative. From Eq. (10.26) the x component of the force on a dipole is $F_x = \mathbf{p}\cdot\nabla E_x$, and likewise for the other components. Let's first look at the y force due to the field from the left dipole. This field is $E_y = -p/4\pi\epsilon_0(b+x)^3$ at points on the x axis near the origin. The gradient of this has only an x component, and it is

$$\left.\frac{\partial E_y}{\partial x}\right|_{x=0} = \frac{3p}{4\pi\epsilon_0 b^4}. \tag{667}$$

Therefore,

$$F_y = \mathbf{p}\cdot\nabla E_y = p_x(\nabla E_y)_x = p\frac{3p}{4\pi\epsilon_0 b^4} = \frac{3p^2}{4\pi\epsilon_0 b^4}. \tag{668}$$

Now consider the x force due to the field from the right dipole. This field is $E_x = 2p/4\pi\epsilon_0(b-x)^3$ at points on the x axis near the origin. The gradient of this has only an x component, and it is

$$\left.\frac{\partial E_x}{\partial x}\right|_{x=0} = \frac{6p}{4\pi\epsilon_0 b^4}\,. \tag{669}$$

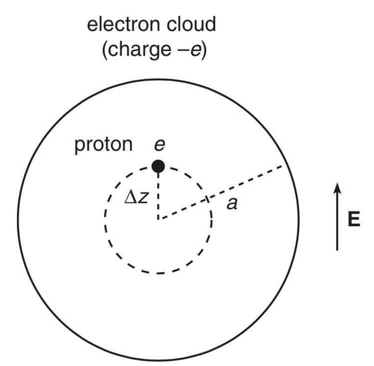
Therefore,

$$F_x = \mathbf{p}\cdot\nabla E_x = p_x(\nabla E_x)_x = p\frac{6p}{4\pi\epsilon_0 b^4} = \frac{6p^2}{4\pi\epsilon_0 b^4}\,. \tag{670}$$

Since $F_y/F_x = 1/2$, the field points up to the right at an angle $\tan\theta = 1/2 \implies \theta = 26.6^\circ$ with respect to the x axis. The magnitude is $F = 3\sqrt{5}p^2/4\pi\epsilon_0 b^4 = (6.71)p^2/4\pi\epsilon_0 b^4$.

Alternatively, you could work out the force from scratch, by letting the dipole consist of two charges $\pm q$ at positions $x = \pm\ell/2$, with $q\ell = p$. If you explicitly calculate the forces on the two charges due to the left and right dipoles, to leading order in ℓ, you will end up with the above values of F_x and F_y. The differences in the forces on the two charges will effectively give the above gradients of the E's.

Q.5: A hydrogen atom is placed in an electric field *E*. The proton and the electron cloud are pulled in opposite directions. Assume simplistically (since we are only concerned with a rough result here) that the electron cloud takes the form of a uniform sphere with radius *a*, with the proton a distance *z* from the center, as shown in Fig. 10.40. Find Δ*z*, and show that your result agrees with Eq. (10.27).



Q.5: Polarized hydrogen dipole

Since volume is proportional to r^3, the negative charge inside radius Δz is $q = -e(\Delta z/a)^3$. Gauss's law therefore gives the field due to the inner part of the electron cloud as

$$\int \mathbf{E}_e\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \implies E_e\cdot 4\pi(\Delta z)^2 = \frac{-e(\Delta z)^3}{\epsilon_0 a^3} \implies E_e = \frac{-e\Delta z}{4\pi\epsilon_0 a^3}\,. \tag{671}$$

This field pulls the proton downward. In equilibrium, it must be equal and opposite to the applied field E that pushes the proton upward. Hence Δz is given by

$$\Delta z = \frac{4\pi\epsilon_0 E a^3}{e}\,, \tag{672}$$

which agrees with Eq. (10.27). The hydrogen atom won't actually remain spherically symmetric, but that won't affect the rough size of Δz.

Q.6: If you don't trust the E = $-P/3\varepsilon_0$ result we obtained in Section 10.9 for the field inside a uniformly polarized sphere, you will find it more believable if you check it in a special case. By direct integration of the contributions from the σ = *P* cos θ surface charge density, show that the field at the center is directed downward (assuming P points upward) with magnitude $P/3\varepsilon_0$.

Consider a horizontal ring at an angle θ down from the top of the sphere, with angular span $d\theta$. The area of this ring is $2\pi(R\sin\theta)(R\,d\theta)$. Since the density is $\sigma = P\cos\theta$, the charge in the ring is $q = 2\pi PR^2\sin\theta\cos\theta\,d\theta$. A little bit of charge dq in a ring in the upper hemisphere creates a diagonally downward field of $dq/4\pi\epsilon_0 R^2$ at the center of the sphere. But by symmetry we are concerned only with the vertical component, which brings in a factor of $\cos\theta$. Integrating over all the dq's in a ring simply gives the total charge q in the ring. The net field from the ring therefore points downward with magnitude

$$\frac{2\pi PR^2\sin\theta\cos\theta\,d\theta}{4\pi\epsilon_0 R^2}\cos\theta = \frac{P\sin\theta\cos^2\theta\,d\theta}{2\epsilon_0}. \tag{679}$$

You can verify that this expression is valid for rings in the lower hemisphere too; all contributions to the field point downward. Integrating over θ from 0 to π gives a total magnitude of

$$E = \int_0^\pi \frac{P\sin\theta\cos^2\theta\,d\theta}{2\epsilon_0} = -\frac{P\cos^3\theta}{6\epsilon_0}\bigg|_0^\pi = \frac{P}{3\epsilon_0}, \tag{680}$$

as desired. The direction is downward.

Q.7: By considering how the introduction of a dielectric changes the energy stored in a capacitor, show that the correct expression for the energy density in a dielectric must be $\boldsymbol{\varepsilon E^2/2}$. Then compare the energy stored in the electric field with that stored in the magnetic field in the wave studied in Section 10.15

With a dielectric present, the capacitance of a parallel-plate capacitor is $C = \kappa\epsilon_0 A/s \equiv \epsilon A/s$. The energy stored is still $C\phi^2/2$, because it equals $Q\phi/2$ for all the same reasons as in the vacuum case (imagine a battery doing work in transferring charge from one plate to the other). So the energy density is

$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2}C\phi^2\frac{1}{V} = \frac{1}{2}\frac{\epsilon A}{s}(Es)^2\frac{1}{As} = \frac{\epsilon E^2}{2}, \tag{690}$$