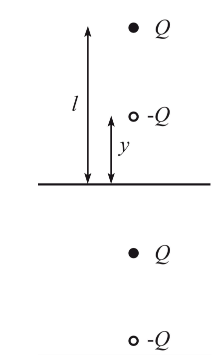
# EE 135 Winter 2016
# HW #3

**Q.1:** A positive point charge $Q$ is fixed a distance $l$ above a horizontal conducting plane. An equal negative charge $-Q$ is to be located somewhere along the perpendicular dropped from $Q$ to the plane. Where can $-Q$ be placed so that the total force on it will be zero?




**Q.1: Two charges and a plane**

First note that such a location must exist, due to a continuity argument: If the $-Q$ charge is placed only slightly below the fixed $Q$ charge, the upward attractive force from the $Q$ charge will dominate. But if the $-Q$ charge is placed only slightly above the conducting plane, the downward attractive force from the $+Q$ image charge will dominate. So somewhere in between, the force on the $-Q$ charge must be zero.

Let $y$ be the distance from the $-Q$ charge to the plane. The field above the plane due to the two given charges along with the induced charge on the plane is identical to the field due to the two given charges along with the two image charges below the plane shown in Fig. 71. The given $-Q$ charge feels the fields due to the other three charges. Taking upward to be positive, the force on the given $-Q$ charge is

$$F = \frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{(\ell-y)^2} - \frac{1}{(2y)^2} + \frac{1}{(\ell+y)^2}\right). \tag{231}$$

Setting this equal to zero yields

$$\frac{1}{4y^2} = \frac{2(\ell^2+y^2)}{(\ell^2-y^2)^2} \implies 7y^4 + 10\ell^2y^2 - \ell^4 = 0. \tag{232}$$

This is a quadratic equation in $y^2$. We are concerned with the positive root (since $y^2$ is positive), which is

$$y^2 = \frac{(-5+4\sqrt{2})\ell^2}{7} \approx (0.0938)\ell^2 \implies y = (0.306)\ell. \tag{233}$$

**Q.2:** By solving the problem of the point charge and the plane conductor, we have, in effect, solved every problem that can be constructed from it by superposition. For instance, suppose we have a straight wire 200 meters long, uniformly charged with $10^{-5}$ C per meter of length, running parallel to the earth at a height of 5 meters. What is the field strength at the surface of the earth, immediately below the wire? (For steady fields the earth behaves like a good conductor.) You may work in the approximation where the length of the wire is much greater than its height. What is the electrical force acting on the wire?

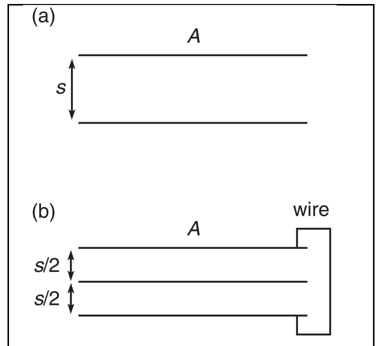
Let $L = 200\,\text{m}$, $h = 5\,\text{m}$, and $\lambda = 10^{-5}\,\text{C/m}$. By superposition, the relevant image charge is an oppositely-charged wire of the same length below the surface of the earth. Because $L$ is much larger than $h$, we can consider (except near the ends of the wire) the wires to be of infinite length, as far as finding the field goes. The field at the surface of the earth is due to both of the wires, so it points downward with magnitude

$$E_{\text{surface}} = 2\cdot\frac{\lambda}{2\pi\epsilon_0 h} = \frac{10^{-5}\,\text{C/m}}{\pi\left(8.85\cdot10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)(5\,\text{m})} = 7.2\cdot10^4\,\frac{\text{V}}{\text{m}}. \tag{234}$$

The electrical force on the given wire is the force due to the field arising from the image-charge wire. The total charge on the given wire is $q = \lambda L = (10^{-5}\,\text{C/m})(200\,\text{m}) = 2\cdot10^{-3}\,\text{C}$. Over nearly the whole length of the wire, the field due to the image-charge wire is essentially $\lambda/2\pi\epsilon_0(2h) = 1.8\cdot10^4\,\text{V/m}$, directed downward. This is a quarter of the field we found above, which involved two wires and half the distance. Neglecting the decrease in field near the ends of the wire, the force on the given wire is $qE = (2\cdot10^{-3}\,\text{C})(1.8\cdot10^4\,\text{V/m}) = 36\,\text{N}$, directed downward. In terms of the various parameters, this force is $qE = (\lambda L)(\lambda/2\pi\epsilon_0(2h)) = (\lambda^2/4\pi\epsilon_0)(L/h)$.

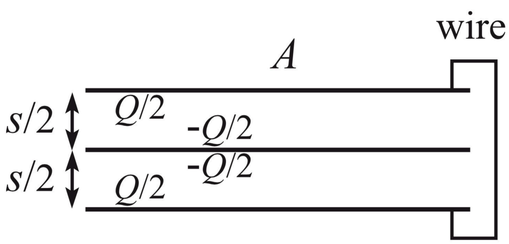
**Q.3:** If the capacitance in Figure Q.3 (a) is $C$, what is the capacitance in Figure Q.3 (b), where a third plate is inserted and the outer plates are connected by a wire?

(Assume that certain charge distributions –in agreement with Gauss's law- and calculate capacitances accordingly. You do not need to use the matrix approach.)



**Q.3: Inserting a plate**

Put charges $Q$ and $-Q$ on the two conductors in each of the two given capacitors. In the bottom capacitor in Fig. 78, one of the conductors consists of the two outer plates, because they are connected by a wire. The charge distributions on the various surfaces are shown. All the factors of $1/2$ arise from symmetry. In the bottom capacitor, the potential difference (which is the difference between either of the outside plates and the inner plate) equals the field times the separation. The field is half of what it is in the top capacitor (because the density $\sigma$ is half), and the separation is also half. So




the potential difference is $(1/2)(1/2) = 1/4$ of what it is in the top capacitor. Since the charge $Q$ on each capacitor is the same, we have

$$Q = C_{\text{top}}\phi, \qquad \text{and} \qquad Q = C_{\text{bottom}}(\phi/4). \tag{261}$$

These quickly give $C_{\text{bottom}} = 4C_{\text{top}}$. So our answer is $4C$.

In the more general case where the middle plate is a fraction $f$ of the distance from one of the outside plates to the other, you can show that the capacitance is $C/[f(1-f)]$. This correctly equals $4C$ when $f = 1/2$. It minimum when $f = 1/2$ and goes to infinity as $f$ goes to 0 or 1.

**Q.4:** A capacitor consists of two coaxial cylinders of length $L$, with outer and inner radii $\boldsymbol{a}$ and $\boldsymbol{b}$. Assume $\boldsymbol{L \gg a-b}$, so that end corrections may be neglected. Show that the capacitance is $\boldsymbol{C = 2\pi\varepsilon_0 L/\ln(a/b)}$. Verify that if the gap between the cylinders, $\boldsymbol{a-b}$, is very small compared with the radius, this result reduces to one that could have been obtained by using the formula for the parallel-plate capacitor.

Neglecting end effects, we can assume that the charge $\pm Q$ is uniformly distributed along each cylinder. The field between the cylinders is that of a line charge with density $\lambda = Q/L$, so $E = \lambda/2\pi\epsilon_0 r = Q/2\pi\epsilon_0 Lr$. The magnitude of the potential difference between the cylinders is then

$$|\Delta\phi| = \int_b^a E\,dr = \int_b^a \frac{Q\,dr}{2\pi\epsilon_0 Lr} = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right). \tag{265}$$

Since $C = Q/|\Delta\phi|$, the capacitance is given by $C = 2\pi\epsilon_0 L/\ln(a/b)$. If $a - b \ll b$, then we can use the Taylor series $\ln(1+\epsilon) \approx \epsilon$ to write

$$\ln\left(\frac{a}{b}\right) = \ln\left(1 + \frac{a-b}{b}\right) \approx \frac{a-b}{b}. \tag{266}$$

So the capacitance becomes $C \approx 2\pi\epsilon_0 bL/(a-b)$. But $2\pi bL$ is the area $A$ of the inner cylinder, and $a - b$ is the separation $s$ between the cylinders. So the capacitance can be written as $C = \epsilon_0 A/s$, which agrees with the standard result for the parallel-plate capacitor.

**Q.5:** Two coaxial aluminum tubes are 30 cm long. The outer diameter of the inner tube is 3 cm, the inner diameter of the outer tube is 4 cm. When these are connected to a 45 volt battery, how much energy is stored in the electric field between the tubes?

We'll solve this exercise first by using the energy density in the electric field, and then by using the capacitance. If $\lambda$ is the charge per unit length on the inner cylinder (with $-\lambda$ on the outer cylinder), then the field between the cylinders (ignoring end effects) is $\lambda/2\pi\epsilon_0 r$. The energy stored in the field is therefore (with $\ell = 0.3\,\text{m}$ being the length)

$$U = \frac{\epsilon_0}{2}\int E^2\,dv = \frac{\epsilon_0}{2}\int_{r_1}^{r_2}\left(\frac{\lambda}{2\pi\epsilon_0 r}\right)^2 2\pi r\ell\,dr = \frac{\lambda^2\ell}{4\pi\epsilon_0}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\lambda^2\ell}{4\pi\epsilon_0}\ln\left(\frac{r_2}{r_1}\right). \tag{292}$$

To write this in terms of the (magnitude of the) potential difference $\phi$ between the tubes, instead of in terms of $\lambda$, note that

$$\phi = \int_{r_1}^{r_2} E\,dr = \int_{r_1}^{r_2}\frac{\lambda\,dr}{2\pi\epsilon_0 r} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_2}{r_1}\right). \tag{293}$$

Solving for $\lambda$ and plugging the result into Eq. (292) gives

$$\begin{aligned} U &= \left(\frac{2\pi\epsilon_0\phi}{\ln(r_2/r_1)}\right)^2\frac{\ell}{4\pi\epsilon_0}\ln(r_2/r_1) = \frac{\pi\epsilon_0\ell\phi^2}{\ln(r_2/r_1)} \\ &= \frac{\pi\left(8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)(0.3\,\text{m})(45\,\text{V})^2}{\ln(4/3)} = 5.9\cdot 10^{-8}\ \text{J}. \end{aligned} \tag{294}$$

**Q.6:** Calculate the electrical force that acts on one plate of a parallel-plate *vacuum* capacitor. The potential difference between the plates is 10 volts, and the plates are squares 20 cm on a side with a separation of 3 cm. If the plates are insulated so the charge cannot change, how much external work could be done by letting the plates come together? Does this equal the energy that was initially stored in the electric field?

From Eq. (1.49), the force per unit area on one of the plates is $\sigma$ times the average of the fields on either side of the plate. (Equivalently, it is $\sigma$ times the field from the other plate.) This average field is $E/2$, where $E$ is the field between the plates. But $E$ equals $\sigma/\epsilon_0$, so $\sigma = \epsilon_0 E$ (it will be more useful to write the field in terms of $E$ than $\sigma$). The force per unit area is therefore

$$\frac{F}{A} = \sigma\frac{E}{2} = (\epsilon_0 E)\frac{E}{2} \implies F = A\frac{\epsilon_0 E^2}{2}. \tag{305}$$

Since $E$ is given by $\phi/s$, we can write $F$ in terms of the potential as

$$F = \frac{A\epsilon_0\phi^2}{2s^2} = \frac{(0.2\,\text{m})^2\left(8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)(10\,\text{V})^2}{2(0.03\,\text{m})^2} = 2.0\cdot 10^{-8}\ \text{N}. \tag{306}$$

If the charge is held constant as the plates come together, then the electric field is independent of the separation, so we see from Eq. (305) that the force is also independent of the separation. (Equivalently, $\phi$ is proportional to $s$ in Eq. (306), so $F$ is independent of $s$.) The total work done by the electric force (which could be used to lift an external object, etc.) is then $W = F \cdot s = (2.0 \cdot 10^{-8}\ \text{N})(0.03\,\text{m}) = 6 \cdot 10^{-10}$ J. Note that the work can be written symbolically as

$$W = F \cdot s = \frac{A\epsilon_0 E^2}{2} \cdot s = (As)\frac{\epsilon_0 E^2}{2} = (\text{volume})\frac{\epsilon_0 E^2}{2}. \tag{307}$$

Since $\epsilon_0 E^2/2$ is the energy density, the work does indeed equal the energy initially stored in the field. Alternatively, the work can be written in terms of $\phi$ as (using $C = \epsilon_0 A/s$ for a parallel-plate capacitor)

$$W = F \cdot s = \frac{A\epsilon_0 \phi^2}{2s^2} \cdot s = \frac{1}{2}\frac{\epsilon_0 A}{s}\phi^2 = \frac{1}{2}C\phi^2, \tag{308}$$

which is the energy stored in the capacitor.

What is the work done if the plates remain connected to the 10 volt battery? In this case, since $\phi$ is constant, the force of $A\epsilon_0\phi^2/2s^2$ in Eq. (306) grows like $1/s^2$ as $s$ goes to zero. The integral of this diverges near zero, so the work is theoretically infinite. However, eventually the battery won't be able to supply the necessary charge to the plates, so $\phi$ will inevitably decrease.