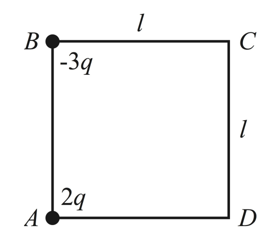
EE 135 Winter 2018
HW #2

Q.1: Designate the corners of a square, *l* on a side, in clockwise order, A, B, C, D. Put charges *2q* at A and −*3q* at B. Determine the value of the line integral of E, from point C to point D. (No actual integration needed!) What is the numerical answer if *q = 10^{-9} C* and *l= 5 cm*.



Q.1: Line integral the easy way

The charges are shown in Fig. 32. The line integral of **E** equals the negative of the change in potential. The potentials at C and D are

$$\phi_C = \frac{1}{4\pi\epsilon_0}\left(-\frac{3q}{\ell}+\frac{2q}{\sqrt{2}\ell}\right) = \frac{q}{4\pi\epsilon_0\ell}(-1.586),$$
$$\phi_D = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{\ell}-\frac{3q}{\sqrt{2}\ell}\right) = \frac{q}{4\pi\epsilon_0\ell}(-0.121). \quad (102)$$

So the line integral from C to D equals

$$\int_C^D \mathbf{E}\cdot d\mathbf{s} = -(\phi_D-\phi_C) = \phi_C - \phi_D = \frac{q}{4\pi\epsilon_0\ell}(-1.464). \quad (103)$$

With the given values of q and ℓ, this becomes

$$\frac{1}{4\pi\epsilon_0}\left(\frac{10^{-9}\mathrm{C}}{0.05\,\mathrm{m}}\right)(-1.464) = -264\ \mathrm{V}. \quad (104)$$

The negative result makes sense, because the field between C and D points at least partly upward, while the $d\mathbf{s}$ in the line integral from C to D points downward.

If you actually want to calculate the line integral, the y component of the field, as a function of y, is (taking the origin to be at the lower left corner, and ignoring the $4\pi\epsilon_0$'s):

$$E_y = \frac{2q}{y^2+\ell^2}\cdot\frac{y}{\sqrt{y^2+\ell^2}} + \frac{3q}{(\ell-y)^2+\ell^2}\cdot\frac{\ell-y}{\sqrt{(\ell-y)^2+\ell^2}}. \quad (105)$$

The integral $\int_C^D \mathbf{E}\cdot d\mathbf{s} = \int_\ell^0 E_y\,dy$ is readily calculated, and you can show that the result is $\phi_C - \phi_D$, where these ϕ's are given in Eq. (102).

Q.2: An interstellar dust grain, roughly spherical with a radius of $3 \cdot 10^{-7}$ m, has acquired a negative charge such that its potential is −0.15 volt. How many extra electrons has it picked up? What is the strength of the electric field at its surface?

The potential is

$$\frac{Q}{4\pi\epsilon_0 R} = -0.15\ \mathrm{V} \implies Q = (-0.15\ \mathrm{V})4\pi\left(8.85\cdot10^{-12}\,\frac{\mathrm{s^2\,C^2}}{\mathrm{kg\,m^3}}\right)(3\cdot10^{-7}\,\mathrm{m})$$
$$= -5\cdot10^{-18}\ \mathrm{C}. \quad (112)$$

The charge of an electron is $-1.6\cdot10^{-19}$ C, so Q corresponds to $n = (5\cdot10^{-18}\ \mathrm{C})/(1.6\cdot10^{-19}\ \mathrm{C}) = 31$ electrons. The field at the surface is $Q/4\pi\epsilon_0R^2$. We could plug in the value of Q we just found, or we could just realize that

$$E = \frac{(Q/4\pi\epsilon_0 R)}{R} = \frac{\phi}{R} = \frac{-0.15\ \mathrm{V}}{3\cdot10^{-7}\,\mathrm{m}} = -5\cdot10^5\,\frac{\mathrm{V}}{\mathrm{m}}, \quad (113)$$

which is a rather large field. Basically, the small R matters more in E than in ϕ, because it is squared in E.

Interestingly, note that the relation $\phi = ER$ says that it takes the same amount of work to drag a test charge out to infinity from the surface of a sphere, as it takes to drag the charge a distance R at full field strength (the value at the surface).

Q.3: As a distribution of electric charge, the gold nucleus can be described as a sphere of radius $6 \cdot 10^{-15}$ m with a charge Q = *79e* distributed fairly uniformly through its interior. What is the potential φ_0 at the center of the nucleus, expressed in megavolts? (TIP: First derive a general formula for φ_0 for a sphere of charge Q and radius a. Do this by using Gauss's law to find the internal and external electric field and then integrating to find the potential. You should redo this here, even though it was done in an example in the textbook and class.)

Since volume is proportional to r^3, the amount of charge inside radius r is $Q(r^3/a^3)$. The field at radius r is effectively due to the charge inside r, so for $r \le a$ the field is

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qr^3/a^3}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{a^3}. \tag{117}$$

Outside the sphere the field is simply $Q/4\pi\epsilon_0 r^2$. The potential at the surface of the sphere relative to infinity is

$$\phi(a) - \phi(\infty) = -\int_\infty^a E\,dr = -\int_\infty^a \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{a}, \tag{118}$$

and the potential at the center of the sphere relative to the surface is

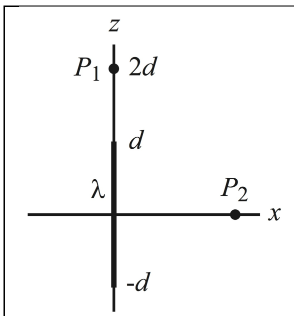
$$\phi(0) - \phi(a) = -\int_a^0 E\,dr = -\int_a^0 \frac{1}{4\pi\epsilon_0}\frac{Qr\,dr}{a^3} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}. \tag{119}$$

Adding the two preceding equations gives $\phi(0) - \phi(\infty) = (1/4\pi\epsilon_0)(3Q/2a)$. For the problem at hand, this yields (assuming $\phi(\infty) = 0$ as usual)

$$\phi(0) = \frac{1}{4\pi\epsilon_0}\frac{3Q}{2a} = \left(9 \cdot 10^9\,\frac{\text{kg m}^3}{\text{s}^2\,\text{C}^2}\right)\frac{3(79 \cdot 1.6 \cdot 10^{-19}\,\text{C})}{2(6 \cdot 10^{-15}\,\text{m})} = 2.84 \cdot 10^7\ \text{V}, \tag{120}$$

or 28.4 megavolts.

Q.4: A thin rod extends along the z axis from $z = -d$ to $z = d$. The rod carries a charge uniformly distributed along its length with linear charge density λ. By integrating over this charge distribution, calculate the potential at a point P_1 on the z axis with coordinates *(0, 0, 2d)*. By another integration find the potential at a point P_2 on the x axis and locate this point to make the potential equal to the potential at P_1. (Do not use Gauss's Law. Symmetry is broken since it is a finite length wire.)



Q.4: Potential from a rod

At point P_1 in Fig. 36 we have

$$\phi_1 = \frac{1}{4\pi\epsilon_0}\int_{-d}^{d}\frac{\lambda\,dz}{2d - z} = -\frac{\lambda}{4\pi\epsilon_0}\ln(2d - z)\Big|_{-d}^{d} = -\frac{\lambda}{4\pi\epsilon_0}\ln\frac{d}{3d} = \frac{\lambda}{4\pi\epsilon_0}\ln 3. \quad (12$$

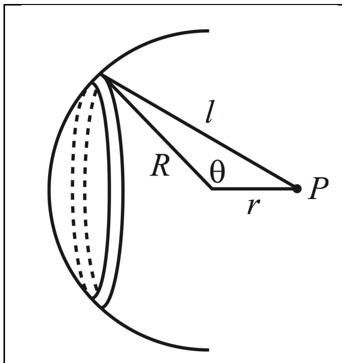
At point P_2 with a general x value, we have (using the integral table in Appendix I

$$\phi_2 = \frac{1}{4\pi\epsilon_0}\int_{-d}^{d}\frac{\lambda\,dz}{\sqrt{x^2 + z^2}} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\sqrt{x^2 + z^2} + z\right)\Big|_{-d}^{d} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{x^2 + d^2} + d}{\sqrt{x^2 + d^2} - d}\right) \quad (12$$

These two potentials are equal when

$$\frac{\sqrt{x^2 + d^2} + d}{\sqrt{x^2 + d^2} - d} = 3 \implies 4d = 2\sqrt{x^2 + d^2} \implies x = \sqrt{3}\,d. \quad (12$$

Q.5: Following the strategy in Problem 2.7 *(solution is available at the end of the textbook)*, find the electric field at the center of a hemispherical shell with radius R and uniform surface charge density σ. Note that this time, you will find φ as a **function of r** and then take the derivative *(see figure on the right for the definition or **r**)*. You might find it easier to Taylor-expand φ before differentiating (*Please note that sqrt(1+x)≈ 1+x/2 if x is much smaller than 1*).



Q.5: Field from a hemisphere

(Recall that we already found this electric field in a simpler manner in the example in Section 1.8. The present method is more involved because we need to do more than calculate φ at just one point; we need to know φ as a function of r so that we can take its derivative.)

As in Problem 2.7, our strategy will be to find the potential at radius r, and then take the derivative to find the field. The calculation is the same as in Problem 2.7, except that the limits of integration are modified. If we define θ in the same way is in Fig. 12.28, it now runs from $\pi/2$ to π. Following the steps in the solution to Problem2.7, the potential at point P in Fig. 41 is (we'll keep things in terms of the density σ)

$$\begin{aligned} \phi(r) &= \int_{\pi/2}^{\pi} \frac{2\pi R^2 \sigma \sin\theta \, d\theta}{4\pi\epsilon_0 \sqrt{R^2 + r^2 - 2rR\cos\theta}} \\ &= \frac{\sigma R}{2\epsilon_0 r} \sqrt{R^2 + r^2 - 2rR\cos\theta} \bigg|_{\pi/2}^{\pi} \\ &= \frac{\sigma R}{2\epsilon_0 r} \Big((R + r) - \sqrt{R^2 + r^2} \Big). \end{aligned} \tag{138}$$

We are concerned with small r, because we want to know the field at the center. For small r we can write $\sqrt{R^2 + r^2} = R\sqrt{1 + r^2/R^2} \approx R(1 + r^2/2R^2)$. So the potential near the center is

$$\phi(r) = \frac{\sigma R}{2\epsilon_0 r} \Big(r - r^2/2R \Big) = \frac{\sigma R}{2\epsilon_0} \Big(1 - r/2R \Big) \tag{139}$$

The field at the center is then

$$E(r) = -\frac{d\phi}{dr} = \frac{\sigma}{4\epsilon_0} \, . \tag{140}$$

You can check that you arrive at the same result if you take P to be on the left side of the center. You will need to be careful about the limits of integration and various signs.

Q.6: A hollow spherical shell has radius R and charge Q uniformly distributed on it. Problem 1.32 *(solution is available at the end of the textbook)* presented two methods for calculating the potential energy of this system. Calculate the energy in a third way, by using Eq. (2.32).

The relevant volume in the integral in Eq. (2.32) is all located right on the surface of the shell where the potential ϕ takes on the uniform value of $Q/4\pi\epsilon_0 R$. We can therefore take ϕ outside the integral, yielding

$$U = \frac{1}{2}\phi \int \rho \, dv = \frac{1}{2}\phi Q = \frac{1}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right) Q = \frac{Q^2}{8\pi\epsilon_0 R} \, . \tag{171}$$

Q.7: Use the result stated in Eq. (2.30) to show that the energy stored in the electric field of the charged disk described in Section 2.6 equals *$(2/3\pi^2\varepsilon_0)(Q^2/a)$*. *Hint:* Consider the work done in building the disk of charge out from zero radius to radius *a* by adding successive rings of width *dr*. Express the total energy in terms of the radius *a* and the total charge $Q = \pi a^2\sigma$.

From Eq. (2.30) the potential at a point on the rim of a disk with radius r is $\phi_r = \sigma r/\pi\epsilon_0$. Adding on a ring with charge $dq = \sigma 2\pi r\, dr$ requires an energy of $dU = \phi_r\, dq = 2\sigma^2 r^2\, dr/\epsilon_0$. The total amount of energy required to assemble the disk of charge is therefore

$$U = \int_0^a dU = \frac{2\sigma^2}{\epsilon_0}\int_0^a r^2\, dr = \frac{2\sigma^2 a^3}{3\epsilon_0}. \tag{161}$$

But $Q = \pi a^2\sigma \Longrightarrow \sigma = Q/\pi a^2$, so we can write

$$U = \frac{2}{3\pi^2\epsilon_0}\frac{Q^2}{a} \approx \frac{0.0675}{\epsilon_0}\frac{Q^2}{a}. \tag{162}$$

From Problem 1.32 or Exercise 2.58, the energy required to build up a hollow spherical shell with radius a and charge Q is

$$U = \frac{1}{8\pi\epsilon_0}\frac{Q^2}{a} \approx \frac{0.0398}{\epsilon_0}\frac{Q^2}{a}. \tag{163}$$

As expected, this is smaller than the result for the disk, because the charges are generally closer to each other in the case of the disk.

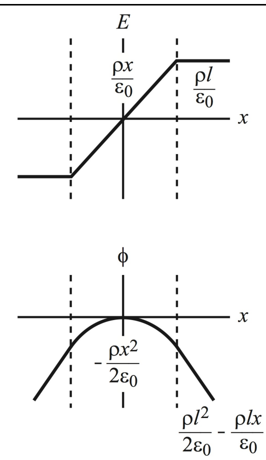
Q.8: A rectangular slab with uniform volume charge density ρ has thickness *$2l$* in the *x* direction and infinite extent in the *y* and *z* directions. Let the *x* coordinate be measured relative to the center plane of the slab. For values of *x* both inside and outside the slab:

(a) find the electric field $E(x)$. (you can do this by considering the amount of charge on either side of *x*, or by using Gauss's law, *TIP: note that the electric field inside and the outside of the slab are diferent.*);

(b) find the potential $\varphi(x)$, with $\varphi=0$ taken to be at $x=0$;

(c) verify that $\rho(x) = \varepsilon_0\nabla \cdot \mathbf{E}$ and $\rho(x) = -\varepsilon_0\nabla^2\varphi$.

(TIP: Electric field and voltage variation is shown in figure right. Please work on your solutions until you find the corresponding curve for your electric field or voltage variation.)



Q.8: Electric field and voltage variation with x

(a) At position x inside the slab, there is a slab with thickness $\ell - x$ to the right of x, which acts effectively like a sheet with surface charge density $\sigma_{\rm R} = (\ell - x)\rho$. Likewise, to the left of x we effectively have a sheet with surface charge density $\sigma_{\rm L} = (\ell + x)\rho$. Since the electric field from a sheet is $\sigma/2\epsilon_0$, the net field at position x inside the slab is

$$E = \frac{(\ell + x)\rho}{2\epsilon_0} - \frac{(\ell - x)\rho}{2\epsilon_0} = \frac{\rho x}{\epsilon_0}, \tag{200}$$

and it is directed away from the center plane (if ρ is positive). You can also quickly obtain this by using a Gaussian surface that extends a distance x on either side of the center plane.

Outside the slab, the slab acts effectively like a sheet with surface charge density $\rho(2\ell)$, so the field has magnitude $(2\rho\ell)/2\epsilon_0 = \rho\ell/\epsilon_0$ and is directed away from the slab. $E(x)$ is continuous at $x = \pm\ell$, as it should be since there are no surface charge densities in the setup.

(b) The potential relative to $x = 0$ is $\phi = -\int_0^x E\,dx$. Inside the slab this gives

$$\phi_{\rm in}(x) = -\int_0^x \frac{\rho x}{\epsilon_0} = -\frac{\rho x^2}{2\epsilon_0}. \tag{201}$$

Outside the slab, we must continue the integral past $x = \pm\ell$. On the right side of the slab, where $x > \ell$, the potential is

$$\begin{aligned} \phi(x) &= -\int_0^\ell E_x\,dx - \int_\ell^x E_x\,dx = -\int_0^\ell \frac{\rho x}{\epsilon_0}\,dx - \int_\ell^x \frac{\rho\ell}{\epsilon_0}\,dx \\ &= -\frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell}{\epsilon_0}(x - \ell) = \frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell x}{\epsilon_0}. \end{aligned} \tag{202}$$

On the left side of the slab, where $x < -\ell$, you can show that the only change in ϕ is that there is a relative "+" sign between the terms (basically, just change ℓ to $-\ell$). So the potential outside the slab equals

$$\phi_{\rm out}(x) = \frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell|x|}{\epsilon_0}. \tag{203}$$

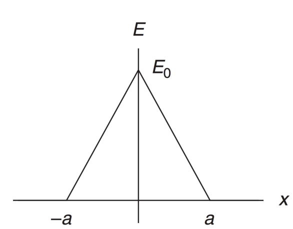
From Eqs. (201) and (203) we see that $\phi(x)$ is continuous at the boundaries at $x = \pm\ell$, as it should be. Plots of $E(x)$ and $\phi(x)$ are shown in Fig. 54.

(c) For a single Cartesian direction, we have $\nabla \cdot \mathbf{E} = \partial E_x/\partial x$ and $\nabla^2\phi = \partial^2\phi/\partial x^2$. The following four relations are indeed all true:

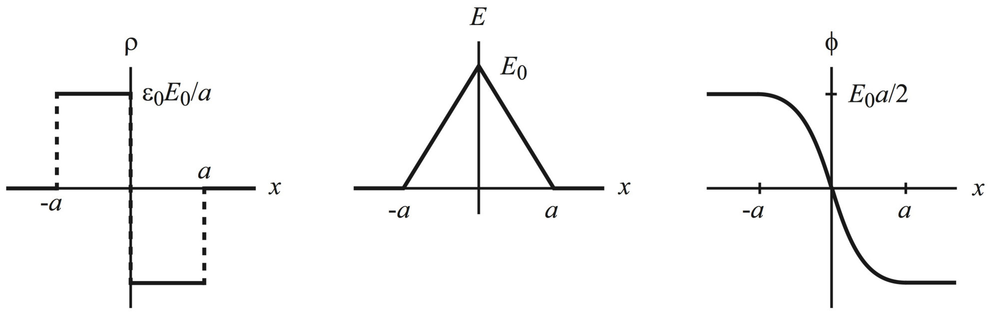
$$\begin{aligned} \text{Inside}: &\quad \rho(x) = \epsilon_0 \nabla \cdot \mathbf{E} \iff \rho = \epsilon_0 \partial(\rho x/\epsilon_0)/\partial x, \\ \text{Outside}: &\quad \rho(x) = \epsilon_0 \nabla \cdot \mathbf{E} \iff 0 = \epsilon_0 \partial(\rho\ell/\epsilon_0)/\partial x, \\ \text{Inside}: &\quad \rho(x) = -\epsilon_0 \nabla^2\phi \iff \rho = -\epsilon_0 \partial^2(-\rho x^2/2\epsilon_0)/\partial x^2, \\ \text{Outside}: &\quad \rho(x) = -\epsilon_0 \nabla^2\phi \iff 0 = -\epsilon_0 \partial^2(\rho\ell^2/2\epsilon_0 \pm \rho\ell x/\epsilon_0)/\partial x^2. \end{aligned} \tag{204}$$

We also have $\mathbf{E} = -\nabla\phi$ both inside and outside, which is true by construction due to the line integrals we calculated in part (b).

Q.9: Find the charge density ρ and potential φ associated with the electric field shown in Figure right. E is independent of y and z. Assume that $\varphi = 0$ at $x = 0$.



Q.9: Electric field



Q.9: Charge, electric field and voltage variation with x for question.9.

The slopes in the triangular part of the plot of E are $\pm E_0/a$, so we quickly find that in the left and right regions near the origin, $E(x)$ takes the form of

$$E_{\mathrm{L}}(x) = (E_0/a)x + E_0 \qquad \text{and} \qquad E_{\mathrm{R}}(x) = -(E_0/a)x + E_0. \tag{205}$$

Gauss's law in differential form is $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$, which in one dimension becomes simply $\rho = \epsilon_0\, \partial E_x/\partial x$. So the charge densities in the left and right regions are

$$\rho_{\mathrm{L}} = \epsilon_0 E_0/a \qquad \text{and} \qquad \rho_{\mathrm{R}} = -\epsilon_0 E_0/a. \tag{206}$$

And $\rho = 0$ outside the $-a \le x \le a$ region. So we have two slabs with opposite charge densities, with the positive slab on the left.

Since $\mathbf{E} = -\nabla\phi$ (which in one dimension becomes $\mathbf{E} = -\hat{\mathbf{x}}\, \partial\phi/\partial x$), we simply need to integrate $E(x)$ to obtain $\phi(x)$. We find

$$\phi_{\mathrm{L}}(x) = -E_0 x^2/2a - E_0 x \qquad \text{and} \qquad E_{\mathrm{R}}(x) = E_0 x^2/2a - E_0 x. \tag{207}$$

There is technically a constant of integration in each of these expressions, but the constants are zero if we take $\phi = 0$ at $x = 0$. Since $\mathbf{E} = 0$ outside the $-a \le x \le a$ region, ϕ is constant, taking on the values it has at the boundaries, namely $\pm E_0 a/2$. The plots of ρ, E, and ϕ are shown in Fig. 55.

A double check: At $x = 0$, the two slabs act effectively like sheets with charge densities $\pm\sigma = \pm\rho a$. They each create a field pointing to the right with magnitude $\sigma/2\epsilon_0$, so the total field at $x = 0$ is $2(\rho a)/2\epsilon_0 = \rho a/\epsilon_0$. And since we found above that $\rho = \epsilon_0 E_0/a$, this field equals E_0, in agreement with the given value.

Q.10: Problem 1.24 and Exercise 1.83 presented two methods for calculating the energy per unit length stored in a cylinder with radius *a* and uniform charge density *ρ*. Calculate the energy in a third way, by using Eq. (2.32). If you take the *φ = 0* point to be at infinity, you will obtain an infinite result. So instead take it to be at a given radius R outside the cylinder. You will then be calculating the energy relative to the configuration where the charge is distributed over a cylinder with radius *R*. In terms of the total charge *λ* per unit length in the final cylinder, show that the energy per unit length can be written as $(\lambda^2/4\pi\varepsilon_0)\,(1/4 + \ln(R/a))$.

The electric field outside the cylinder is $\lambda/2\pi\epsilon_0 r$, where $\lambda = \rho\pi a^2$ is the charge per unit length in the cylinder. So the field outside is $E_{\mathrm{out}} = \rho a^2/2\epsilon_0 r$. The field at radius r inside the cylinder is due to the charge within radius r, so the effective charge per unit length is $\lambda_r = \rho\pi r^2$. The field inside is therefore $E_{\mathrm{in}} = \lambda_r/2\pi\epsilon_0 r = \rho r/2\epsilon_0$.

The potential at radius r inside the cylinder, relative to a given radius R outside the cylinder, is the negative integral of the field from R down to r. We must break this integral into two pieces:

$$\phi(r) = -\int_R^a \frac{\rho a^2}{2\epsilon_0 r'}\,dr' - \int_a^r \frac{\rho r'}{2\epsilon_0}\,dr' = \frac{\rho a^2}{2\epsilon_0}\ln\left(\frac{R}{a}\right) + \frac{\rho}{4\epsilon_0}(a^2 - r^2). \tag{172}$$

Equation (2.32) then gives the energy (relative to the configuration where the charge is distributed over a cylinder with radius R) in a length ℓ of the cylinder as

$$
\begin{aligned}
U = \frac{1}{2}\int \rho\phi\, dv &= \frac{1}{2}\int_0^a \rho \left[\frac{\rho a^2}{2\epsilon_0}\ln\left(\frac{R}{a}\right) + \frac{\rho}{4\epsilon_0}(a^2 - r^2)\right] 2\pi r\ell\, dr \\
\Longrightarrow \quad \frac{U}{\ell} &= \frac{\pi\rho^2}{2\epsilon_0}\int_0^a \left[a^2\ln\left(\frac{R}{a}\right) + \frac{1}{2}(a^2 - r^2)\right] r\, dr \\
&= \frac{\pi\rho^2}{2\epsilon_0}\left[\frac{a^4}{2}\ln\left(\frac{R}{a}\right) + \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)\right] \\
&= \frac{\rho^2\pi^2 a^4}{4\pi\epsilon_0}\left[\ln\left(\frac{R}{a}\right) + \frac{1}{4}\right] \\
&= \frac{\lambda^2}{4\pi\epsilon_0}\left[\ln\left(\frac{R}{a}\right) + \frac{1}{4}\right],
\end{aligned}
\tag{173}
$$

as desired. If $R = a$, so that all of the charge is initially distributed over the surface of the cylinder, then it takes an amount of work per unit length equal to $\lambda^2/16\pi\epsilon_0$ to move the charge inward and distribute it uniformly throughout the volume of the cylinder.