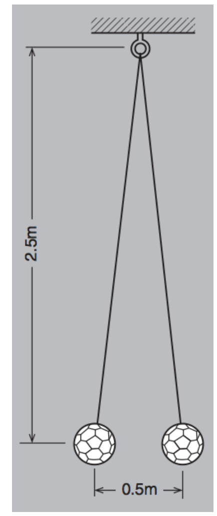
EE 135 Winter 2018
HW #1 SOLUTIONS

Q.1: Two volleyballs, mass *0.3 kg* each, tethered by nylon strings and charged with an electrostatic generator, hang as shown in Fig. Q.1. What is the charge on each, assuming the charges are equal?



Q.1: Repelling Volley Balls

Consider one of the balls. The vertical component of the tension in the string must equal the gravitational force on the ball. And the horizontal component must equal the electric force. The angle that the string makes with the horizontal is given by $\tan\theta = 10$, so we have

$$\frac{T_y}{T_x} = 10 \implies \frac{F_g}{F_e} = 10 \implies \frac{mg}{q^2/4\pi\epsilon_0 r^2} = 10. \tag{4}$$

Therefore,

$$\begin{aligned} q^2 &= \frac{1}{10}(4\pi\epsilon_0)mgr^2 = (0.4)\pi\left(8.85\cdot 10^{-12}\,\frac{\mathrm{s^2\,C^2}}{\mathrm{kg\,m^3}}\right)(0.3\,\mathrm{kg})(9.8\,\mathrm{m/s^2})(0.5\,\mathrm{m})^2 \\ &= 8.17\cdot 10^{-12}\,\mathrm{C}^2 \implies q = 2.9\cdot 10^{-6}\,\mathrm{C}. \end{aligned} \tag{5}$$

Q.2: Calculate the potential energy, per ion, for an infinite 1D ionic crystal with separation *a*; that is, a row of equally spaced charges of magnitude *e* and alternating sign. Hint: The power-series expansion of *ln(1 + x)* may be of use.

Suppose the array has been built inward from the left (that is, from negative infinity) as far as a particular negative ion. To add the next positive ion on the right, the amount of external work required is

$$\frac{1}{4\pi\epsilon_0}\left(-\frac{e^2}{a}+\frac{e^2}{2a}-\frac{e^2}{3a}+\cdots\right) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right). \tag{21}$$

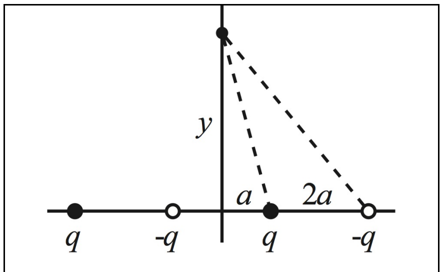
The expansion of $\ln(1+x)$ is $x - x^2/2 + x^3/3 - \cdots$, converging for $-1 < x \le 1$. Evidently the sum in parentheses above is just $\ln 2$, or 0.693. The energy of the infinite chain *per ion* is therefore $-(0.693)e^2/4\pi\epsilon_0 a$. Note that this is an exact result; it does not assume that a is small. After all, it wouldn't make any sense to say that "a is small," because there is no other length scale in the setup that we can compare a with.

The addition of further particles on the right doesn't affect the energy involved in assembling the previous ones, so this result is indeed the energy per ion in the entire infinite (in both directions) chain. The result is negative, which means that it requires energy to move the ions away from each other. This makes sense, because the two nearest neighbors are of the opposite sign.

If the signs of all the ions were the same (instead of alternating), then the sum in Eq. (21) would be $(1+1/2+1/3+1/4+\cdots)$, which diverges. It would take an infinite amount of energy to assemble such a chain.

An alternative solution is to compute the potential energy of a given ion due to the full infinite (in both directions) chain. This is essentially the same calculation as above, except with a factor of 2 due to the ions on each side of the given one. If we then sum over all ions (or a very large number N) to find the total energy of the chain, we have counted each pair twice. So in finding the potential energy per ion, we must divide by 2 (along with N). The factors of 2 and N cancel, and we arrive at the above result.

Q.3: Four charges, q, −q, q, and −q, are located at equally spaced intervals on the x axis. Their x values are −3a, −a, a, and 3a, respectively. Does there exist a point on the y axis for which the electric field is zero? If so, find the y value.



Q.3: Zero Field

The setup is shown in Fig. 5. We know that $E_y = 0$ on the y axis, by symmetry, so we need only worry about E_x. We want the leftward E_x from the two middle charges to cancel the rightward E_x from the two outer charges. This implies that

$$2 \cdot \frac{1}{4\pi\epsilon_0}\frac{q}{y^2+a^2} \cdot \frac{a}{\sqrt{y^2+a^2}} = 2 \cdot \frac{1}{4\pi\epsilon_0}\frac{q}{y^2+(3a)^2} \cdot \frac{3a}{\sqrt{y^2+(3a)^2}}, \tag{25}$$

where the second factor on each side of the equation comes from the act of taking the horizontal component. Simplifying this gives

$$\frac{1}{(y^2+a^2)^{3/2}} = \frac{3}{(y^2+9a^2)^{3/2}} \implies y^2 + 9a^2 = 3^{2/3}(y^2+a^2)$$

$$\implies y = a\sqrt{\frac{9-3^{2/3}}{3^{2/3}-1}} \approx (2.53)a. \tag{26}$$

In retrospect, we know that there must exist a point on the y axis with $E_x = 0$, by a continuity argument. For small y, the field points leftward, because the two middle charges dominate. But for large y, the field points rightward, because the two outer charges dominate. (This is true because for large y, the distances to the four charges are all essentially the same, but the slope of the lines to the outer charges is smaller than the slope of the lines to the middle charges (it is 1/3 as large). So the x component of the field due to the outer charges is 3 times as large, all other things being equal.) Therefore, by continuity, there must exist a point on the y axis where E_x equals zero.

Q.4: (a) What is the electric field at the center of a hollow hemispherical shell with radius *R* and uniform surface charge density *σ* ? (This is a special case of Problem 1.12, but you can solve the present problem much more easily from scratch, without going through all the messy integrals of Problem 1.12.). *(Use spherical coordinates).*
(b) Use your result to show that the electric field at the center of a solid hemisphere with radius R and uniform volume charge density *ρ* equals *ρR/4ε₀*. *(Use spherical coordinates).*

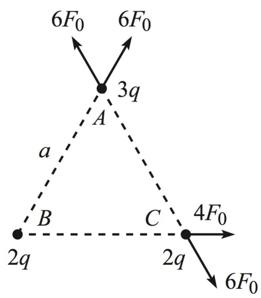
Consider the ring shown in Fig. 9, defined by the angle θ and subtending an angle $d\theta$. Its area is $2\pi(R\cos\theta)(R\,d\theta)$, so its charge is $\sigma(2\pi R^2\cos\theta\,d\theta)$. The horizontal component of the field at the center of the hemisphere is zero, by symmetry. So we need only worry about the vertical component from each piece of the ring, which brings in a factor of $\sin\theta$. Adding up these components from all the pieces of the ring gives the magnitude of the field at the center of the hemisphere, due to the given ring, as

$$dE = \frac{\sigma(2\pi R^2\cos\theta\,d\theta)}{4\pi\epsilon_0 R^2}\sin\theta = \frac{\sigma\sin\theta\cos\theta\,d\theta}{2\epsilon_0}. \tag{35}$$

The field points downward if σ is positive. Integrating over all the rings (θ runs from 0 to $\pi/2$) gives the total field at the center as

$$E = \int_0^{\pi/2}\frac{\sigma\sin\theta\cos\theta\,d\theta}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0}\frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{\sigma}{4\epsilon_0}. \tag{36}$$

Q.5: Three positive charges, A, B, and C, of *3·10⁻⁶*, *2·10⁻⁶*, and *2·10⁻⁶ coulombs*, respectively, are located at the corners of an equilateral triangle of side *0.2 m*.
(a) Find the magnitude in newtons of the force on each charge.
(b) Find the magnitude in newtons/coulomb of the electric field at the center of the triangle.



(a) Let F_0 be the force between two charges of $q = 10^{-6}$ C each, at a distance of $a = 0.2$ m. Then $F_0 = q^2/4\pi\epsilon_0 a^2 = 0.225$ N, as you can verify. The force between B and C has magnitude $(2)(2)F_0 = 4F_0$, and the force between A and either B or C has magnitude $(3)(2)F_0 = 6F_0$. From Fig. 11, the magnitude of the force on A is

$$F_A = 2\cos 30° \cdot 6F_0 = 2.34 \text{ N}. \qquad (39)$$

The magnitude of the force on C is (squaring and adding the horizontal and vertical components)

$$F_C = \left[(4 + 6\cos 60°)^2 + (6\sin 60°)^2\right]^{1/2} F_0 = (8.72)F_0 = 1.96 \text{ N}. \qquad (40)$$

And the force on B has the same magnitude.

(b) Three equal charges of $2\cdot 10^{-6}$ C would yield zero field at the center, by symmetry. So the field at the center is due to the excess charge of $q = 10^{-6}$ C at A. Since A is a distance $a/\sqrt{3}$ from the center, the magnitude of the field at the center of the triangle is

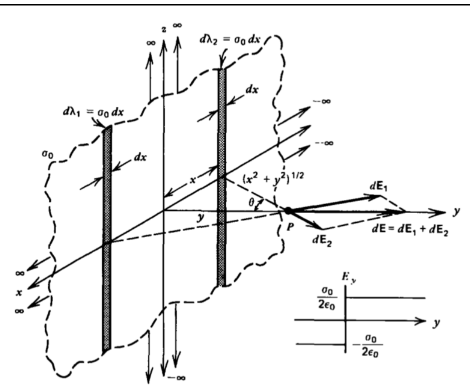
$$E = \frac{q}{4\pi\epsilon_0 (a/\sqrt{3})^2} = \left(9 \cdot 10^9 \frac{\text{kg m}^3}{\text{s}^2\,\text{C}^2}\right) \frac{10^{-6}\text{ C}}{(0.2\text{ m})^2/3} = 6.75 \cdot 10^5 \text{ N/C}. \qquad (41)$$

Q.6: Consider a high-voltage direct current power line that consists of two parallel conductors suspended 3 meters apart. The lines are oppositely charged. If the electric field strength halfway between them is 15 000 N/C, how much excess positive charge resides on a 1 km length of the positive conductor?

The electric field from a single wire is $\lambda/2\pi\epsilon_0 r$. Between the wires the fields from the two wires point in the same direction, so we have

$$\begin{aligned} 15{,}000 \text{ N/C} = 2\frac{\lambda}{2\pi\epsilon_0 r} \quad \Longrightarrow \quad \lambda &= (15{,}000 \text{ N/C})\pi\epsilon_0 r \\ &= (15{,}000 \text{ N/C})(3.14)\left(8.85 \cdot 10^{-12}\frac{\text{s}^2\,\text{C}^2}{\text{kg m}^3}\right)(1.5\text{ m}) \\ &= 6.3 \cdot 10^{-7}\text{ C/m}. \qquad (57) \end{aligned}$$

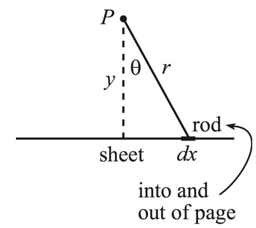
The amount of excess charge on 1 km of the positive wire is then $(1000\text{ m})\lambda = 6.3 \cdot 10^{-4}$ C.



Q.7: Infinite sheet of charge

Q.7: A **uniform infinite sheet of charge** can be thought of as consisting of an **infinite number of adjacent uniformly charged rods**. Using the fact that the field from an infinite rod is $\lambda/2\pi\varepsilon_0 r$, integrate over these rods to show that the field from a sheet with charge density σ is $\sigma/2\varepsilon_0$.

In Fig. 19, the horizontal line represents the sheet, which extends into and out of the page (and also to the left and right). The short segment represents a rod extending into and out of the page, with small width dx. The field at point P due to the rod is $\lambda/2\pi\epsilon_0 r$, where the effective linear charge density of the rod is $\lambda = \sigma\, dx$. This is true because the amount of charge in a length ℓ of the rod can be written as both $\lambda\ell$ (by definition) and $\sigma\ell\, dx$ (because $\ell\, dx$ is the relevant area). The horizontal component of the field cancels with the horizontal component of the field arising from the rod located symmetrically on the left side of P. So (as expected) we care only about the vertical component. This brings in a factor of $\cos\theta$. And since $x = y\tan\theta$, we have $dx = y\, d\theta/\cos^2\theta$. The (vertical) field at P therefore equals



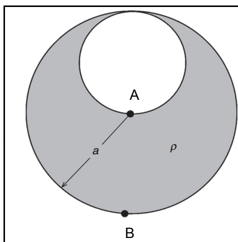
$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} \frac{\sigma\, dx}{2\pi\epsilon_0 r}\cos\theta = \int_{-\pi/2}^{\pi/2} \frac{\sigma(y\, d\theta/\cos^2\theta)}{2\pi\epsilon_0 (y/\cos\theta)}\cos\theta \\
&= \frac{\sigma}{2\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\sigma}{2\epsilon_0}, \qquad (58)
\end{aligned}
$$

as desired. Alternatively, you can write the integral in terms of x. Since $\cos\theta = y/r$ we have

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} \frac{\sigma\, dx}{2\pi\epsilon_0 r}\cdot\frac{y}{r} = \frac{\sigma y}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dx}{x^2+y^2} \\
&= \frac{\sigma y}{2\pi\epsilon_0}\cdot\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\Bigg|_{-\infty}^{\infty} = \frac{\sigma y}{2\pi\epsilon_0}\cdot\frac{\pi}{y} = \frac{\sigma}{2\epsilon_0}. \qquad (59)
\end{aligned}
$$

Q.8: Carved-out sphere:

A sphere of radius a is filled with positive charge with uniform density ρ. Then a smaller sphere of radius *a/2* is carved out, as shown in Q.8, and left empty. What are the direction and magnitude of the electric field at A? At B?



Q.8: Force between two strips

The given setup is equivalent to the superposition of a sphere with radius a and density ρ, plus an off-center sphere with radius $a/2$ and density $-\rho$. The desired fields at A and B are the sums of the fields from these two objects.

The charge in the big sphere is $Q_{\text{b}} = (4/3)\pi a^3\rho$, while the charge in the small sphere is $Q_{\text{s}} = (4/3)\pi(a/2)^3(-\rho) = -Q_{\text{b}}/8$. For convenience, let the field at B due to the big sphere be labeled E_0. Then

$$
E_0 \equiv E_{\text{b},B} = \frac{Q_{\text{b}}}{4\pi\epsilon_0 a^2} = \frac{(4/3)\pi a^3\rho}{4\pi\epsilon_0 a^2} = \frac{a\rho}{3\epsilon_0}, \qquad (67)
$$

and the field is directed downward. The field at A due to the big sphere is $E_{\text{b},A} = 0$. The field at A due to the small (negative) sphere has magnitude

$$
E_{\text{s},A} = \frac{Q_{\text{s}}}{4\pi\epsilon_0 (a/2)^2} = \frac{(4/3)\pi(a/2)^3\rho}{4\pi\epsilon_0(a/2)^2} = \frac{a\rho}{6\epsilon_0} = \frac{E_0}{2}, \qquad (68)
$$

and is directed upward. The field at B due to the small sphere has magnitude

$$
E_{\text{s},B} = \frac{Q_{\text{s}}}{4\pi\epsilon_0 (3a/2)^2} = \frac{(4/3)\pi(a/2)^3\rho}{4\pi\epsilon_0(3a/2)^2} = \frac{a\rho}{54\epsilon_0} = \frac{E_0}{18}, \qquad (69)
$$

and is directed upward.

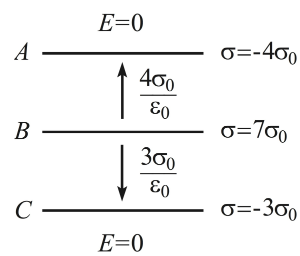
The total field at A is therefore directed upward with magnitude $0 + E_0/2 = a\rho/6\epsilon_0$. And the total field at B is directed downward with magnitude $E_0 - E_0/18 = 17E_0/18 = 17a\rho/54\epsilon_0$.

Q.9: Two infinite plane sheets of surface charge, with densities $3\sigma_0$ and $-2\sigma_0$, are located a distance l apart, parallel to one another. Discuss the electric field of this system.

The field from an infinite sheet with charge density σ has magnitude $\sigma/2\epsilon_0$. It is directed away from the sheet if σ is positive, and toward it if σ is negative. The total field in the given setup equals the superposition of the fields from each sheet; the result is shown in Fig. 23. The field has magnitude $(3\sigma_0 + 2\sigma_0)/2\epsilon_0 = 5\sigma_0/2\epsilon_0$ inside the sheets and $(3\sigma_0 - 2\sigma_0)/2\epsilon_0 = \sigma_0/2\epsilon_0$ outside the sheets. In all regions it is directed away from the $3\sigma_0$ sheet.

If the sheets intersect at right angles, the field is again obtained by superposition, but now the two individual fields are orthogonal. Fig. 24 shows the results in the four regions. The magnitude of the field everywhere is $\sqrt{3^2 + 2^2}(\sigma_0/2\epsilon_0) \approx (1.8)\sigma_0/\epsilon_0$. In all regions it is directed at least partially away from the $3\sigma_0$ sheet and partially toward the $-2\sigma_0$ sheet.

Q.10: Consider three plane charged sheets, *A*, *B*, and *C*. The sheets are parallel with *A* above *B* above *C*. On each sheet there is surface charge of uniform density: $-4\cdot10^{-5}$ C/m^2 on A, $7 \cdot 10^{-5}$ C/m^2 on *B*, and $-3\cdot10^{-5}$ C/m^2 on *C*. (The density given includes charge on both sides of the sheet.) What is the magnitude of the electrical force per unit area on each sheet? Check to see that the total force per unit area on the three sheets is zero.



From Eq. (1.49) the force per unit area on a sheet is $(E_1 + E_2)\sigma/2$, where E_1 and E_2 are the electric fields on either side. The fields in the various regions can be found by the superpositions of the fields from the individual sheets, using the fact that the field due to a given sheet is $\sigma/2\epsilon_0$. With $\sigma_0 \equiv 10^{-5}\,\text{C/m}^2$, the fields in the two middle regions are $4\sigma_0/\epsilon_0$ upward and $3\sigma_0/\epsilon_0$ downward, as shown in Fig. 31. Above and below all three plates the field is zero. The forces per unit area on the three sheets are therefore

$$\begin{aligned}
F_A &= \frac{1}{2}(E_1+E_2)\sigma = \frac{1}{2}\left(0+\frac{4\sigma_0}{\epsilon_0}\right)(-4\sigma_0) = -\frac{8\sigma_0^2}{\epsilon_0} = -90.4\,\frac{\text{N}}{\text{m}^2}\,,\\
F_B &= \frac{1}{2}(E_1+E_2)\sigma = \frac{1}{2}\left(\frac{4\sigma_0}{\epsilon_0}-\frac{3\sigma_0}{\epsilon_0}\right)(7\sigma_0) = \frac{7\sigma_0^2}{2\epsilon_0} = 39.5\,\frac{\text{N}}{\text{m}^2}\,,\\
F_C &= \frac{1}{2}(E_1+E_2)\sigma = \frac{1}{2}\left(0-\frac{3\sigma_0}{\epsilon_0}\right)(-3\sigma_0) = \frac{9\sigma_0^2}{2\epsilon_0} = 50.8\,\frac{\text{N}}{\text{m}^2}\,.
\end{aligned} \tag{89}$$

The sum of these forces per area is $(\sigma_0^2/\epsilon_0)(-8 + 7/2 + 9/2)$. This equals zero as it must, because a system can't exert a net force on itself (otherwise momentum wouldn't be conserved).

The sum of these forces per area is $(\sigma_0^2/\epsilon_0)(-8 + 7/2 + 9/2)$. This equals zero as it must, because a system can't exert a net force on itself (otherwise momentum wouldn't be conserved).

Alternatively, we can find the forces by calculating the field at the location of a given plate due to the *other* two plates. For example, the bottom two plates produce a field of $(7-3)\sigma_0/2\epsilon_0 = 2\sigma_0/\epsilon_0$ at the location of the top plate, which therefore feels a force per unit area equal to $(2\sigma_0/\epsilon_0)(-4\sigma_0) = -8\sigma_0^2/\epsilon_0$, as above. The $(E_1+E_2)/2$ averages in Eq. (89) are simply a way of finding the field at the location of one sheet due to the others.