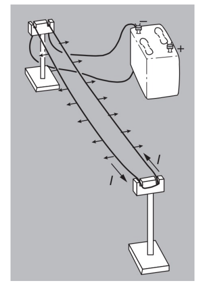
# EE 135 Winter 2018
# HW #5 Solutions

**Q.1:** Suppose the current I that flows in the circuit in Figure is 20 amperes. The distance between the wires is 5 cm. How large is the force, per meter of length, that pushes horizontally on one of the wires?



**Q.1: Force between wires**

The magnetic field due to one of the wires in Fig. 5.1(b), at the location of the other, is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(20\,\text{A})}{2\pi(0.05\,\text{m})} = 8 \cdot 10^{-5}\,\text{T}. \tag{427}$$

The force per unit length on each wire is then $IB = (20\,\text{A})(8 \cdot 10^{-5}\,\text{T}) = 1.6 \cdot 10^{-3}\,\text{N/m}$, and it is repulsive.

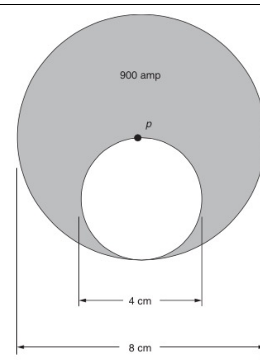
**Q.2:** A current of 8000 amperes flows through an aluminum rod 4 cm in diameter. Assuming the current density is uniform through the cross section, find the strength of the magnetic field at 1 cm, at 2 cm, and at 3 cm from the axis of the rod.

The radius is 2 cm, so 1/4 of the cross-sectional area, and hence current (so 2000 A), is enclosed within $r = 1$ cm. The current enclosed in both the $r = 2$ cm and $r = 3$ cm cases is 8000 A. So we have

$$\begin{aligned}
B_1 &= \frac{\mu_0 I_1}{2\pi r_1} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(2000\,\text{A})}{2\pi(0.01\,\text{m})} = 0.04\,\text{T}, \\
B_2 &= \frac{\mu_0 I_2}{2\pi r_2} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(8000\,\text{A})}{2\pi(0.02\,\text{m})} = 0.08\,\text{T}, \\
B_3 &= \frac{\mu_0 I_3}{2\pi r_3} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(8000\,\text{A})}{2\pi(0.03\,\text{m})} = 0.0533\,\text{T}.
\end{aligned} \tag{435}$$

These fields are 400, 800, and 533 gauss, respectively.

**Q.3:** A long copper rod 8 cm in diameter has an off-center cylindrical hole, as shown in Fig. Q3, down its full length. This conductor carries a current of 900 amps flowing in the direction "into the paper." What is the direction, and strength in tesla, of the magnetic field at the point P that lies on the axis of the outer cylinder




**Q.3: Off-center hole**

The given setup is equivalent to the superposition of a complete solid rod with current flowing into the page plus a smaller rod (where the hole is) with current flowing out of the page. If the two current densities are equal and opposite, then there will be zero current in the hole, in agreement with the given setup. Given the ratio of the areas of the two circular cross sections, currents of 1200 A into the page and 300 A out of the page will yield the given 900 A into the page. The large rod produces zero field on its axis, so the desired field is due entirely to the smaller rod with 300 A coming out of the page. The magnitude of the field is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(300\,\text{A})}{2\pi(0.02\,\text{m})} = 0.003\ \text{T}, \tag{436}$$

or 30 gauss, and it points to the left. A more remarkable fact (see Exercise 6.38) is that the field is 30 gauss pointing to the left not only at $P$ but everywhere inside the cylindrical hole.

**Q.4:** How should the current density inside a thick cylindrical wire depend on r so that the magnetic field has constant magnitude inside the wire?

If $I_r$ is the current inside radius $r$, then Ampere's law gives

$$B \cdot 2\pi r = \mu_0 I_r \implies B = \frac{\mu_0 I_r}{2\pi r}. \tag{438}$$

If we want $B$ to be independent of $r$, then we need $I_r$ to be proportional to $r$. $I_r$ is found by integrating the current density $J(r)$:

$$I_r = \int J\, da = \int_0^r J(r') \cdot (2\pi r'\, dr'). \tag{439}$$

It is easiest to guess and check the form of $J(r')$. If $J(r')$ is proportional to $1/r'$, then it takes the form of $J(r') = \alpha/r'$, so

$$I_r = \int_0^r (\alpha/r')(2\pi r'\, dr') = 2\pi\alpha r, \tag{440}$$

as desired. The field is then

$$B = \frac{\mu_0 I_r}{2\pi r} = \frac{\mu_0(2\pi\alpha r)}{2\pi r} = \mu_0\alpha. \tag{441}$$

The above "$1/r$" result for the current density is the same result that holds for the charge density in the case of the electric field due to a charged cylinder or sphere. In both of these cases the electric field is independent of $r$ if the density $\rho$ is proportional to $1/r$.

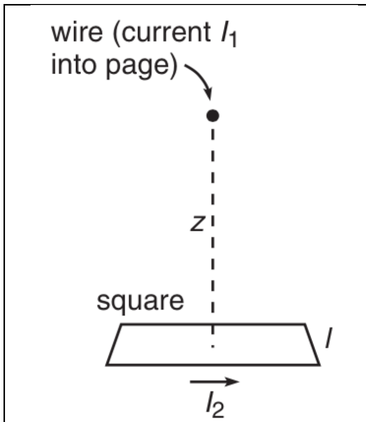
Note that even though the current density diverges at $r = 0$, the actual current does not. There is a finite amount of current in any cross section with radius $r$, and it is given (by construction) by $I_r = 2\pi\alpha r$. Any ring (at any radius) with thickness $dr$ contains the same amount of current, $dI = 2\pi\alpha\, dr$.

We can also solve this exercise by using the differential form of Ampere's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Since $\mathbf{B}$ points tangentially and has a uniform value, it can be written as $\mathbf{B} = B_0\hat{\boldsymbol{\theta}}$. Equation F.2 in Appendix F then gives

$$\nabla \times \mathbf{B} = \frac{1}{r}\frac{\partial(rB_0)}{\partial r}\hat{\mathbf{z}} = \frac{B_0}{r}\hat{\mathbf{z}}. \tag{442}$$

Setting this equal to $\mu_0 J\hat{\mathbf{z}}$ gives $J = B_0/(\mu_0 r)$, consistent with the $1/r$ dependence we found above. The factor of $B_0/\mu_0$ here equals the $\alpha$ from above.

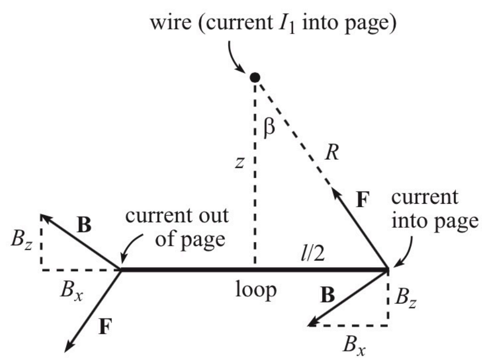
**Q.5:** Figure **Q.5** shows a horizontal infinite straight wire with current $I_1$ pointing into the page, passing a height z above a square horizontal loop with side length l and current $I_2$. Two of the sides of the square are parallel to the wire. As with a circular ring, this square produces a magnetic field that points upward on its axis. The field fans out away from the axis. From the right-hand rule, you can show that the magnetic force on the straight wire points to the right. By Newton's third law, the magnetic force on the square must therefore point to the left.




**Q.5: Force between a wire and a loop**

Your task is to explain qualitatively, by drawing the fields and forces, why the force on the square does indeed point to the left. Then show that the net force equals $\mu_0 I_1 I_2 l^2 / 2\pi R^2$, where $R = \sqrt{z^2 + (l/2)^2}$ is the distance from the wire to the right and left sides of the square.

Consider a little segment in the right-hand side of the square. The current points into the page, and the magnetic field due to the infinite straight wire has magnitude $B_1 = \mu_0 I_1/2\pi R$ and points down to the left, as shown in Fig. 118. From the right-hand rule, the force $q\mathbf{v} \times \mathbf{B}$ on the charges in the current points up to the left, as shown (toward the infinite wire; parallel currents attract). In the left-hand side of the square, the current points out of the page, and the magnetic field due to the infinite wire points up to the left, as shown. The force $q\mathbf{v}\times\mathbf{B}$ on the charges in the current now points down to the left (away from the infinite wire; antiparallel currents repel). The vertical components of the preceding two forces cancel, but the leftward components add. So the net force is leftward, as desired. You can quickly show that the net force on each of the other two sides of the square is zero.




In short, it is the vertical component of $\mathbf{B}$ that matters, because this component changes sign from the right half to the left half of the square. And the direction of the square's current into and out of the page also changes sign. So these two negative signs cancel in $q\mathbf{v} \times \mathbf{B}$, yielding a net leftward force. In contrast, the horizontal component of $\mathbf{B}$ does *not* change sign, so the negation of the current causes a negation of the vertical force. The net vertical force is therefore zero.

Quantitatively, the general form of the force on a wire is $F = IB\ell$. The "$B$" we are concerned with here is the vertical component, which is $B \sin\beta$. The force comes from two sides, so the total horizontal force is

$$F = 2I_2(B_1 \sin\beta)\ell = 2I_2 \left(\frac{\mu_0 I_1}{2\pi R} \cdot \frac{\ell/2}{R}\right) \ell = \frac{\mu_0 I_1 I_2 \ell^2}{2\pi R^2}, \tag{458}$$

where we have used $\sin\beta = (\ell/2)/R$.

The above reasoning shows where the two factors of $R$ in the denominator come from. One comes from the distance to the wire, and the other comes from the fact that the $\mathbf{B}$ field becomes more horizontal (which means that the vertical component decreases) as $R$ gets large.

We weren't concerned with torques in this exercise, but from looking at the vertical forces on the left and right sides (which come from the horizontal component of $\mathbf{B}$), it is clear that there is a torque on the square. It will rotate counterclockwise when viewed from the side. This is consistent with conservation of angular momentum, because the straight wire will gain angular momentum (relative to, say, an origin chosen to be the center of the square) as it moves to the right. This angular momentum will have a clockwise sense, consistent with the fact that the total angular momentum of the system remains constant.