EE 135 Winter 2018
HW #6 Solutions

Q.1: Find a vector potential that will correspond to a uniform field in the *z* direction: $B_x = 0$, $B_y = 0$, $B_z = B_0$.

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we want

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \qquad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, \qquad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0. \tag{445}$$

From inspection, a few choices for $\mathbf{A}$ that satisfy these equations are $\mathbf{A} = (0, B_0 x, 0)$, or $(-B_0 y, 0, 0)$, or $(-B_0 y/2, B_0 x/2, 0)$. In general, any vector of the form $(-ay, bx, 0)$ works if $a + b = B_0$. And even more generally, adding on any vector with zero curl also works.

Q.2: A round wire of radius r_0 carries a current *I* distributed uniformly over the cross section of the wire. Let the axis of the wire be the *z*-axis, $\hat{z}$ with the direction of the current. Show that a vector potential of the form $\vec{A} = A_0\hat{z}\left(x^2 + y^2\right)$ will correctly give the magnetic field $\vec{B}$ of this current at all points inside the wire. What is the value of the constant, A_0?

Since area is proportional to r^2, the current contained within a radius r is $I_r = Ir^2/r_0^2$. The magnitude of the magnetic field at radius r is then

$$B(r) = \frac{\mu_0 I_r}{2\pi r} = \frac{\mu_0 I r}{2\pi r_0^2}, \tag{446}$$

and it points in the positive $\hat{\boldsymbol{\theta}}$ direction. The $\hat{\boldsymbol{\theta}}$ vector equals $(-y/r, x/r, 0)$ because this vector has length 1 and has zero dot product with the radial vector $(x, y, 0)$. So the Cartesian components of $\mathbf{B}$ are

$$B_x = -\frac{y}{r}B = -\frac{\mu_0 I y}{2\pi r_0^2}, \qquad \text{and} \qquad B_y = \frac{x}{r}B = \frac{\mu_0 I x}{2\pi r_0^2}. \tag{447}$$

The magnetic field associated with the potential $\mathbf{A} = A_0\hat{\mathbf{z}}(x^2 + y^2)$ is

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{x}}\frac{\partial A_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial A_z}{\partial x} = 2A_0 y\hat{\mathbf{x}} - 2A_0 x\hat{\mathbf{y}}. \tag{448}$$

This agrees with the $\mathbf{B}$ in Eq. (447) if $A_0 = -\mu_0 I/4\pi r_0^2$.

Alternatively, in cylindrical coordinates we have $\mathbf{A} = A_0\hat{\mathbf{z}}r^2$. From Eq. (F.2) in Appendix F the associated magnetic field is $\mathbf{B} = \nabla \times \mathbf{A} = -(\partial A_z/\partial r)\hat{\boldsymbol{\theta}} = -2A_0 r\hat{\boldsymbol{\theta}}$. Comparing this with the B in Eq. (446), which points in the positive $\hat{\boldsymbol{\theta}}$ direction, we find $A_0 = -\mu_0 I/4\pi r_0^2$, as above.

Since A_0 is negative, $\mathbf{A}$ points in the direction opposite to the current (which points in the positive $\hat{\mathbf{z}}$ direction). You might be wondering how this can be, in view of the fact that Eq. (6.44) seems to say that $\mathbf{A}$ points in the same direction as $\mathbf{J}$. The answer is that we can add an arbitrary constant to the $\mathbf{A}$ in Eq. (6.44), and it will still yield the same value of $\mathbf{B} = \nabla \times \mathbf{A}$. Adding on a sufficiently large vector pointing in the negative $\hat{\mathbf{z}}$ direction will make $\mathbf{A}$ point opposite to $\mathbf{J}$.

Q.3: An electron is moving at a speed *0.01c* on a circular orbit of radius 10^{-10} *m*. What is the strength of the resulting magnetic field at the center of the orbit? (The numbers given are typical, in order of magnitude, for an electron in an atom.)

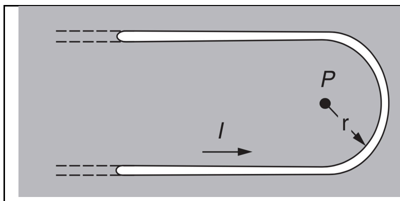
HINT: An electron rotating around a circular orbit is similar to a current carrying ring.

The time for one revolution is $t = 2\pi r/v$, so the average current is $I = e/t = ev/2\pi r$. From Eq. (6.54) the field at the center of the orbit is therefore

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 ev}{4\pi r^2} = \frac{(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2})(1.6 \cdot 10^{-19}\,\text{C})(0.01 \cdot 3 \cdot 10^8\,\text{m/s})}{4\pi(10^{-10}\,\text{m})^2} = 4.8\ \text{T}. \quad (451)$$

Q.4: A long wire is bent into the hairpin-like shape shown in **Fig. Q.4**. Find an exact expression for the magnetic field at the point *P* that lies at the center of the half-circle.

HINT: Decompose hairpin into simple geometries that we have studied in class. There is no need to take any integrals.



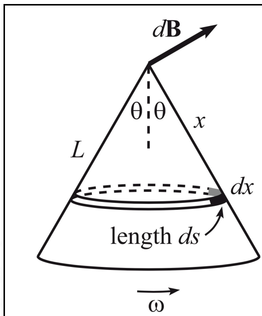
Q.4: Hairpin field.

Each of the two straight segments contributes half the field of an infinite wire. (The contributions do indeed add and not cancel.) The semicircle contributes half the field of an entire ring at the center, which is given by Eq. (6.54). The desired field therefore points out of the page and has magnitude

$$B = 2 \cdot \frac{1}{2}\frac{\mu_0 I}{2\pi r} + \frac{1}{2}\frac{\mu_0 I}{2r} = \left(\frac{1}{2\pi} + \frac{1}{4}\right)\frac{\mu_0 I}{r} = (0.409)\frac{\mu_0 I}{r}. \quad (452)$$

Q.5: A hollow cone (like a party hat) has vertex angle *2θ*, slant height *L*, and surface charge density *σ*. It spins around its symmetry axis with angular frequency *ω*. What is the magnetic field at the tip?

HINT: Use Biot-Savart's law.



Q.5: Field at the tip of a cone.

Consider a circular strip with width dx, a slant-distance x from the tip. The velocity of any point in this strip is $v = \omega(x \sin\theta)$. The amount of charge in the strip that passes a given point during time dt is $dq = \sigma(dx)(v\,dt) = \sigma(dx)(\omega x \sin\theta)\,dt$. The current in the strip is therefore $I = dq/dt = \sigma\omega x \sin\theta\,dx$. (Equivalently, you can use the general result $I = \lambda v$, where $\lambda = \sigma\,dx$ is the effective linear charge density of the ring.)

From the Biot-Savart law, a small piece of the strip with length ds at the location shown in Fig. 120 produces a $d\mathbf{B}$ field at the tip that points up to the right, with magnitude $(\mu_0/4\pi)I\,ds/x^2$. When we integrate over the whole strip, the horizontal components of the $d\mathbf{B}$'s cancel, and we are left with only a vertical component. This brings in a factor of $\sin\theta$.

For a given strip, the ds in the Biot-Savart law integrates up to the length of the strip, which is $s = 2\pi(x \sin\theta)$. The contribution to the (vertical) field from a given strip at slant-distance x, with width dx, is therefore

$$\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{Is}{x^2}\sin\theta = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{(\sigma\omega x \sin\theta\,dx)(2\pi x \sin\theta)}{x^2}\sin\theta = \hat{\mathbf{z}}\frac{1}{2}\mu_0\sigma\omega\sin^3\theta\,dx. \quad (461)$$

Integrating from $x = 0$ to $x = L$ simply gives a factor of L, so the field at the tip is

Integrating from $x = 0$ to $x = L$ simply gives a factor of L, so the field at the tip is

$$\mathbf{B} = \hat{\mathbf{z}}\frac{1}{2}\mu_0\sigma\omega L\sin^3\theta. \tag{462}$$

If $\theta = 0$ we correctly obtain zero field. If $\theta = \pi/2$ we obtain $\mathbf{B} = \hat{\mathbf{z}}\mu_0\sigma\omega L/2$. In this case we just have a flat disk with radius L, and this is indeed the field at the center; see Exercise 6.49.

To check that the units of $\mathbf{B}$ are correct, we can compare it with the B due to a wire, which is $\mu_0 I/2\pi r$. And indeed, $\sigma\omega L$ correctly has the same units as I/r.

Note that the result in Eq. (461) is independent of x, so all rings with the same thickness dx give the same contribution to the field. The reason for this is that the larger a ring is, the larger the current I and length s are, and these effects cancel the effect of the x^2 in the denominator of the Biot-Savart law. Note also where the three factors of $\sin\theta$ come from. For given values of the other parameters, a larger θ means a larger velocity (and hence current), a larger circumference s, and a larger vertical component of each of the $d\mathbf{B}$'s.

Q.6: A disk with radius *R* and surface charge density σ spins with angular frequency *ω*. What is the magnetic field at the center?

Consider a ring with radius r and thickness dr. The effective linear charge density along the ring is $d\lambda = \sigma\,dr$. The speed of all points on the ring is $v = \omega r$, so the current in the ring is $dI = (d\lambda)v = (\sigma\,dr)(\omega r)$. From the Biot-Savart law, a small piece of the ring with length dl produces a $d\mathbf{B}$ field at the center that points perpendicular to the ring and has magnitude $(\mu_0/4\pi)I\,dl/r^2$. Integrating over the whole ring turns the dl into $2\pi r$, so the field at the center due to the ring is $(\mu_0/4\pi)(\sigma\omega r\,dr)(2\pi r)/r^2 = \mu_0\sigma\omega\,dr/2$. Integrating over r (that is, integrating over all the rings in the disk) turns the dr into an R, so the field at the center equals $\mu_0\sigma\omega R/2$. It points perpendicular to the disk, with the direction determined by the righthand rule.

Q.7: What is the maximum electromotive force induced in a coil of 4000 turns, average radius 12 cm, rotating at 30 revolutions per second in the earth's magnetic field where the field intensity is 0.5 gauss? *(1 Gauss = 10^{-4} Tesla)*

The maximum emf equals the maximum value of $d\Phi/dt$. The flux is given by $\Phi = NAB\cos(\omega t + \phi)$, where N is the number of turns and $A = \pi r^2$ is the area. (We'll assume that the coil is oriented optimally, with its axis of rotation lying perpendicular to the field.) The maximum value of $d\Phi/dt = -\omega NAB\sin(\omega t + \phi)$ is then ωNAB, so

$$\mathcal{E}_{\max} = \omega NAB = (2\pi\cdot 30\,\mathrm{s}^{-1})(4000)\big(\pi(0.12\,\mathrm{m})^2\big)(5\cdot 10^{-5}\,\mathrm{T}) = 1.7\ \mathrm{V}. \tag{488}$$

Q.8: In the central region of a solenoid that is connected to a radio frequency power source, the magnetic field oscillates at $2.5\cdot 10^6$ cycles per second with an amplitude of *4 gauss*. What is the amplitude of the oscillating electric field at a point *3 cm* from the axis? (This point lies within the region where the magnetic field is nearly uniform.)

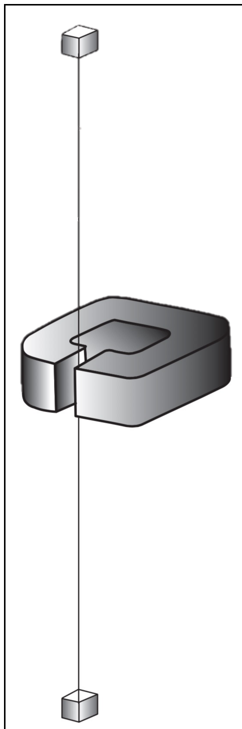
Consider a circle of radius r centered on the axis. The flux through this circle is $\pi r^2 B$. Since B takes the form of $B_0\cos(\omega t+\phi)$, the amplitude of $dB/dt = -\omega B_0\sin(\omega t+\phi)$ is ωB_0. So the amplitude of the emf around the circle is $\mathcal{E}_{\max} = (d\Phi/dt)_{\max} = \pi r^2(\omega B_0)$. The electric field along the circle is related to $\mathcal{E}$ by $2\pi r E = \mathcal{E}$. Therefore, the amplitude of E is

$$E_{\max} = \frac{\mathcal{E}_{\max}}{2\pi r} = \frac{\omega r B_0}{2} = \frac{(2\pi\cdot 2.5\cdot 10^6\,\mathrm{s}^{-1})(0.03\,\mathrm{m})(4\cdot 10^{-4}\,\mathrm{T})}{2} = 94\ \mathrm{V/m}. \tag{489}$$

Q.9: Derive an approximate formula for the mutual inductance of two circular rings of the same radius *a*, arranged like wheels on the same axle with their centers a distance *b* apart. Use an approximation good for $b \gg a$.

From Eq. (6.53), the magnetic field along the axis of a ring of radius a, a distance b from the center, is $B = \mu_0 I a^2/2(a^2+b^2)^{3/2}$. For $b \gg a$ this can be approximated as $B = \mu_0 I a^2/2b^3$. In this limit we can also neglect the variation of B over the interior of the other ring. The flux through the other ring is therefore $\Phi = \pi a^2 B = \mu_0 \pi I a^4/2b^3$. The mutual inductance is then $\Phi/I = \mu_0 \pi a^4/2b^3$.

Q.10: A taut wire passes through the gap of a small magnet (**Fig. Q.10**), where the field strength is 5000 gauss. The length of wire within the gap is 1.8 cm. Calculate the amplitude of the induced alternating voltage when the wire is vibrating at its fundamental frequency of 2000 Hz with an amplitude of 0.03 cm, transverse to the magnetic field.



Q.10 Vibrating wire.

The amplitude of the vibration is $x_0 = 3 \cdot 10^{-4}$ m, the frequency is $\nu = 2000$ Hz, the length of wire within the gap is $\ell = 0.018$ m, and the magnetic field is $B = 0.5$ T.

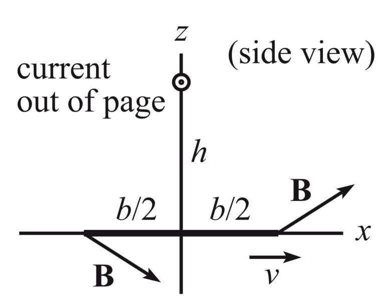
The position of the wire takes the general form of $x_0 \cos(\omega t + \phi)$, so taking the derivative tells us that the maximum speed is $v_{\max} = \omega x_0 = 2\pi\nu x_0$. Imagine connecting the ends of the wire with another wire to form a complete loop, which is in fact what you would be doing if you measured the voltage between the ends by connecting them to a voltmeter. Then the movement of the wire implies that an area is being swept out; the area enclosed by the loop is changing. The maximum rate of swept area equals the maximum speed times ℓ. So the maximum induced voltage is

$$\begin{aligned} \mathcal{E}_{\max} &= \left(\frac{d\Phi}{dt}\right)_{\max} = B\ell v_{\max} = 2\pi B\ell\nu x_0 \\ &= 2\pi(0.5\,\mathrm{T})(0.018\,\mathrm{m})(2000\,\mathrm{s}^{-1})(3\cdot 10^{-4}\,\mathrm{m}) = 0.034\ \mathrm{V}. \end{aligned} \tag{490}$$

This result depends linearly on all four of the given quantities, which makes intuitive sense.

We can also solve this exercise by looking at the qvB magnetic force on the charges in the wire. Multiplying this force by the distance ℓ along the wire over which it acts, and dividing by q to obtain the work per charge, gives a voltage difference of $vB\ell$, as above. This is maximum when v is maximum, since B and ℓ are constants.

Q.11: A long straight stationary wire is parallel to the *y*-axis and passes through the point *z* = *h* on the *z* axis. A current *I* flows in this wire, returning by a remote conductor whose field we may neglect. Lying in the *xy* plane is a square loop with two of its sides, of length *b*, parallel to the long wire. This loop slides with constant speed *v* in the $\hat{x}$ direction. Find the magnitude of the electromotive force induced in the loop at the moment when the center of the loop crosses the *y*-axis.



Q.11 Sliding Loop.

In Fig. 130 the y axis points into the page. We've arbitrarily chosen the current in the wire to flow in the negative y direction (out of the page), but the sign doesn't matter since all we care about is the magnitude of the emf. At the leading edge of the square loop, the magnitude of B is $\mu_0 I/2\pi r$, where $r = \sqrt{h^2 + (b/2)^2}$. Only the z component matters in the flux, and this brings in a factor of $(b/2)/r$. So

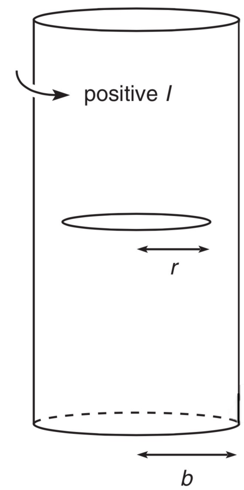
$$B_z = \frac{\mu_0 I}{2\pi r}\frac{b/2}{r} = \frac{\mu_0 I b}{4\pi(h^2 + b^2/4)}. \tag{492}$$

At the trailing edge, B_z has the opposite sign. If the loop moves a small distance $v\,dt$, there is additional positive flux through a thin rectangle with area $b(v\,dt)$ at the leading edge, and also less negative flux through a similar rectangle at the trailing edge. Both of these effects cause the upward flux to increase. Therefore,

$$\mathcal{E} = \frac{d\Phi}{dt} = 2\frac{b(v\,dt)B_z}{dt} = 2bvB_z = \frac{\mu_0 I b^2 v}{2\pi(h^2 + b^2/4)}. \tag{493}$$

The flux is increasing upward. So for our choice of direction of the current in the wire, the induced emf is clockwise when viewed from above, because that creates a downward field inside the loop which opposes the change in flux. For $h = 0$ (or in general for $h \ll b$) $\mathcal{E}$ reduces to $2\mu_0 I v/\pi$. This is independent of b because the field at the leading and trailing edges decreases with b, while the length of the thin rectangles at these edges increases with b.

You can show that our result for $\mathcal{E}$ has the correct units, either by working them out explicitly, or by noting that $\mathcal{E}$ has the units of B (which are the same as $\mu_0 I/2\pi r$) times length squared divided by time, which correctly gives flux per time.



Q.12 Sliding Loop.

Q.12: An infinite solenoid has radius *b* and *n* turns per unit length. The current varies in time according to $I(t) = I_0 \cos\omega t$ (with positive defined as shown in **Fig. Q.12**). A ring with radius $r < b$ and resistance *R* is centered on the solenoid's axis, with its plane perpendicular to the axis.

(a) What is the induced current in the ring?

(b) A given little piece of the ring will feel a magnetic force. For what values of *t* is this force maximum?

(c) What is the effect of the force on the ring? That is, does the force cause the ring to translate, spin, flip over, stretch/shrink, etc.?

(a) The magnetic field inside the solenoid is $B(t) = \mu_0 n I(t) = \mu_0 n I_0 \cos\omega t$. Faraday's law applied to the given ring yields

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\pi r^2 \frac{dB}{dt} = \pi r^2 \mu_0 n I_0 \omega \sin\omega t. \tag{497}$$

With the given positive direction of I, the right-hand rule gives the positive direction of B as upward, and then also gives the positive direction of $\mathcal{E}$ as counterclockwise when viewed from above (as for I). The current in the loop is $I_{\text{loop}}(t) = \mathcal{E}/R = (\pi r^2 \mu_0 n I_0 \omega/R)\sin\omega t$.

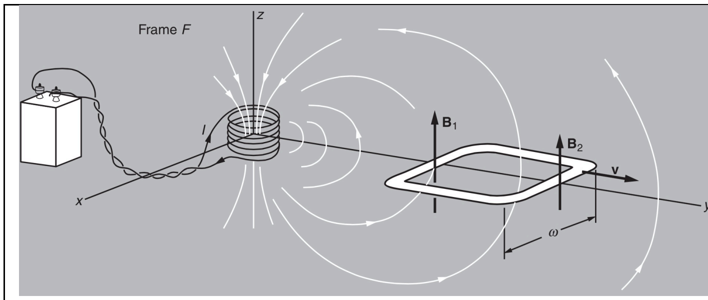
(b) The force on a little piece of the ring is $F(t) = I_{\text{loop}}(t)\, d\boldsymbol{l} \times \mathbf{B}$. With positive I counterclockwise and positive B upward, this force is radial and equals

$$F(t) = \frac{\pi r^2 \mu_0 n I_0 \omega}{R}\sin\omega t \cdot dl \cdot \mu_0 n I_0 \cos\omega t = \frac{\pi r^2 \mu_0^2 n^2 I_0^2 \omega\, dl}{R}\sin\omega t \cos\omega t. \tag{498}$$

The force is radially outward if this quantity is positive, inward if it is negative. Since $\sin\omega t\cos\omega t = (1/2)\sin(2\omega t)$ we see that the force is maximum outward when $\omega t = \pi/4$ (plus multiples of π), and maximum inward when $\omega t = 3\pi/4$ (plus multiples of π).

(c) Since the force lies in the horizontal plane, it serves only to stretch/shrink the ring (negligibly, if the ring is rigid).

Q.13: Suppose the loop in **Fig. Q.13** has a resistance *R*. Show that whoever is pulling the loop along at constant speed does an amount of work during the interval *dt* that agrees precisely with the energy dissipated in the resistance during this interval, assuming that the self-inductance of the loop can be neglected.



Q.13: Work and dissipated energy.

The induced emf is $\mathcal{E} = wv(B_1 - B_2)$, so the current is $I = \mathcal{E}/R = wv(B_1 - B_2)/R$. From Lenz's law it is counterclockwise when viewed from above. From the righthand rule, the forward-directed force that must be applied to the loop to balance the retarding magnetic force and keep the loop moving at constant speed is $F = IB_1w - IB_2w = Iw(B_1 - B_2)$. Using $(B_1 - B_2) = IR/wv$, the rate at which work is done is therefore

$$Fv = Iw(B_1 - B_2)v = Iw\left(\frac{IR}{wv}\right)v = I^2R, \tag{504}$$

which is the power dissipated in the resistance, as we wanted to show.

In Fig. 7.14 the energy that is dissipated in the stationary loop has to be supplied by whatever agency is moving the coil. A force is indeed required to move the coil because of the magnetic field arising from the induced current in the loop. To see how this works out in a simple case, let the coil have the same rectangular shape as the loop, but with N turns. And let the coil have current I_0. Then the difference in the B fields (due to the loop) at the leading and trailing edges of the coil is smaller than the difference in the B fields (due to the coil) at the leading and trailing edges of the loop by a factor of I/I_0 (because these are the currents that produce the B fields) and also by a factor of N. So the $Fv = Iw(B_1 - B_2)v$ relation in Eq. (504) for the rate at which work is done in moving the N-loop coil becomes

$$Fv = N \cdot I_0 w\left((B_1 - B_2)\cdot\frac{I}{I_0}\cdot\frac{1}{N}\right)v = Iw(B_1 - B_2)v, \tag{505}$$

which agrees with the expression in Eq. (504), and hence equals I^2R.

Q.14: A coil with resistance of 0.01 ohm and self-inductance 0.50 milli-henry is connected across a large 12 volt battery of negligible internal resistance. How long after the switch is closed will the current reach 90 percent of its final value? At that time, how much energy, in joules, is stored in the magnetic field? How much energy has been withdrawn from the battery up to that time?

From Eq. (7.69) the current is $I(t) = I_0\big(1 - e^{-(R/L)t}\big)$, where $I_0 = \mathcal{E}_0/R$. In the problem at hand,

$$I_0 = \frac{\mathcal{E}_0}{R} = \frac{12\,\mathrm{V}}{0.01\,\Omega} = 1200\,\mathrm{A} \qquad \text{and} \qquad \frac{R}{L} = \frac{0.01\,\Omega}{0.5\cdot 10^{-3}\,\mathrm{H}} = 20\,\mathrm{s}^{-1}. \tag{516}$$

So the time scale is $L/R = 0.05\,\mathrm{s}$. The current reaches a value of $(0.9)I_0$ when

$$e^{-(R/L)t} = 0.1 \implies (20\,\mathrm{s}^{-1})t = \ln 10 \implies t = 0.115\,\mathrm{s}. \tag{517}$$

At this time, the current is $I = (0.9)(1200) = 1080\,\mathrm{A}$, so the energy stored in the magnetic field is

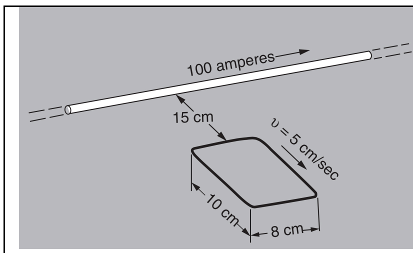
$$\frac{1}{2}LI^2 = \frac{1}{2}(0.5\cdot 10^{-3}\,\mathrm{H})(1080\,\mathrm{A})^2 = 292\,\mathrm{J}. \tag{518}$$

The instantaneous power delivered by the battery is $\mathcal{E}_0 I$, but since I is changing we must perform an integral to find the energy delivered by the battery between $t = 0$ and $t = 0.115\,\text{s}$:

$$\begin{aligned}
\int_0^{t=0.115\,\text{s}} \mathcal{E}_0 I(t')\,dt' &= \mathcal{E}_0 I_0 \int_0^t \left(1 - e^{-(R/L)t'}\right) dt' \\
&= \mathcal{E}_0 I_0 \left(t' + \frac{L}{R} e^{-(R/L)t'}\right)\Bigg|_0^t \\
&= \mathcal{E}_0 I_0 \left(t + \frac{L}{R} e^{-(R/L)t} - \frac{L}{R}\right) \\
&= (12\,\text{V})(1200\,\text{A})\Big(0.115\,\text{s} + (0.05\,\text{s})(0.1) - (0.05\,\text{s})\Big) \\
&= 1008\ \text{J}. \qquad (519)
\end{aligned}$$

From conservation of energy, apparently $1008\,\text{J} - 292\,\text{J} = 716\,\text{J}$ has been dissipated in the resistor. The task of Problem 7.15 is to show that the energy delivered by the battery does indeed equal the energy stored in the magnetic field plus the energy dissipated in the resistor, at any general time t.

Q.15: Calculate the electromotive force in the moving loop in **Fig. Q.15** at the instant when it is in the position shown. Assume the resistance of the loop is so great that the effect of the current in the loop itself is negligible. Estimate very roughly how large a resistance would be safe, in this respect. Indicate the direction in which current would flow in the loop, at the instant shown.



Q.15: Induced emf in a loop.

The magnetic field due to an infinite current-carrying wire is $\mu_0 I/2\pi r$, so the fields at the near and far sides of the rectangle are

$$B_1 = \frac{\mu_0(100\,\text{A})}{2\pi(0.15\,\text{m})} = 1.33\cdot 10^{-4}\ \text{T}, \qquad \text{and} \qquad B_2 = \frac{\mu_0(100\,\text{A})}{2\pi(0.25\,\text{m})} = 0.8\cdot 10^{-4}\ \text{T}. \quad (499)$$

The induced emf is therefore

$$\mathcal{E} = \frac{d\Phi}{dt} = wv(B_1 - B_2) = (0.08\,\text{m})(5\,\text{m/s})(0.53\cdot 10^{-4}\,\text{T}) = 2.1\cdot 10^{-5}\ \text{V}. \qquad (500)$$

The flux is downward and decreasing, so $\mathcal{E}$ will be in the direction to drive a current that would make more flux downward. The current is therefore clockwise when viewed from above.

Let's now estimate roughly how large the resistance must be to make the effect of the current in the loop negligible. The current in the loop at any instant is $I' = \mathcal{E}/R$. This causes a field B' and a flux Φ' through the loop. Because $\mathcal{E}$ is changing with time as the loop moves away from the wire, Φ' is changing too, resulting in an extra emf $\mathcal{E}'$, which we have so far ignored. The question is, how large must R be so that $\mathcal{E}'$ is negligible compared with $\mathcal{E}$?

As a very rough estimate, we have $B' \approx \mu_0 I'/2\pi d$, where d is a typical dimension of the loop, say, $d = 5$ cm. The flux[1] is then $\Phi' \approx B'A = (\mu_0 I'/2\pi d)w\ell$, where ℓ is the length of the loop (we could set $\ell \approx w \approx d$ here since we're being rough, but we'll keep them separate). The (very rough) time characteristic of the change in Φ' is $\tau = h/v$, where h is the mean distance from the loop to the wire (20 cm), and v is the speed of the loop. So in order of magnitude, we have (using $I' = \mathcal{E}/R$)

$$\mathcal{E}' = \frac{d\Phi'}{dt'} \approx \frac{\Phi'}{\tau} = \frac{\mu_0 I' w\ell}{2\pi d}\cdot\frac{v}{h} = \frac{\mu_0 \mathcal{E} w\ell v}{2\pi R h d}. \qquad (501)$$

Our goal is to have $\mathcal{E}' \ll \mathcal{E}$, which is equivalent to

$$\begin{aligned}
\frac{\mu_0 w \ell v}{2\pi R h d} &\ll 1 \implies R \gg \frac{\mu_0 w \ell v}{2\pi h d} \\
\implies R &\gg \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(0.08\,\text{m})(0.1\,\text{m})(5\,\text{m/s})}{2\pi(0.2\,\text{m})(0.05\,\text{m})} = 8 \cdot 10^{-7}\,\Omega. \qquad (502)
\end{aligned}$$

This is roughly equal to $10^{-6}\,\Omega$, so the condition that the current in the loop is negligible (more precisely, the condition that $\mathcal{E}' \ll \mathcal{E}$) can be written as $R \gg 10^{-6}\,\Omega$. This is a rather small resistance, so this bound is easily satisfied by a typical copper wire.

We can alternatively write the condition in Eq. (502) as

$$\frac{\mu_0 w \ell}{2\pi d} \cdot \frac{1}{R} \ll \frac{h}{v}. \qquad (503)$$

But from the definition of the self-inductance L, the above expression for Φ' yields $L = \mu_0 w \ell / 2\pi d$. So the condition can be written as $L/R \ll \tau$. In other words, the inductive time constant of the loop itself, L/R, should be short compared with the time scale of the change of the externally induced emf.

Note that in the case where all of the above lengths (w, ℓ, h, d) are of the same order of magnitude (which is the case here), the condition reduces to the simple expression (ignoring the 2π): $R \gg \mu_0 v$. We therefore see that the smallness of the above lower bound on R (in SI units) comes from the smallness of μ_0 (in SI units).