EE 135 Winter 2015

Midterm Exam

SOLUTIONS

NAME ________________________

You are allowed to use a calculator, the textbook, previous homework, and class notes. Please write answers in spaces provided. Do not write in the spaces below, they are for the grading of the exam. Thank you for following these directions.

1) 20

2) 30

3) 30

4) 20

5*) 25

Total * <u>100+25=125</u>

Q.1-Q.4 is required.

* Q.5 is a bonus question.

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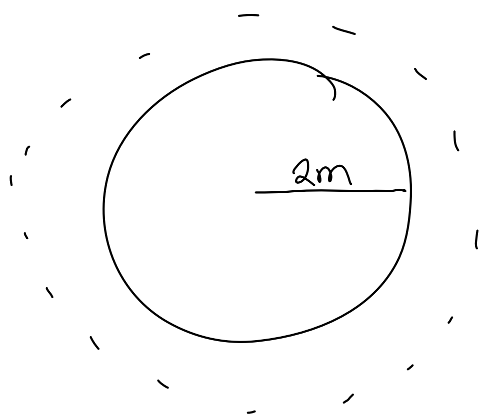
1. If the electric field strength in air exceeds 3.0×10^6 N/C, the air becomes a conductor. Using this fact, determine the maximum amount of charge that can be carried by a metal sphere 2.0 m in radius.



using G-LAW

$$E = 3\times10^{6}\ N/C$$

$$E = \frac{q}{4\pi R^2 \varepsilon_0} \quad \left(\oiint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} \right)$$

$$R = 2m \Rightarrow q = \frac{(2^2)(3\times10^6)}{8.99\times10^9}$$

$$\boxed{q = 1.3\times10^{-3} C}$$

2. You come across a spherically symmetric electric field with the following form:

$$
\begin{aligned}
\vec{E}(r) &= E_0\left(\frac{r}{R}\right)^2 \hat{r} && 0 \le r \le R \\
&= 0 && R < r < 2R \\
&= E_0\left(\frac{r}{R} - 2\right)^2 \hat{r} && 2R \le r \le 3R \\
&= E_0\left(\frac{3R}{r}\right)^2 \hat{r} && 3R \le r \le 4R \\
&= 0 && r > R
\end{aligned}
$$

$\hat{r}$ is the radial unit vector in spherical coordinates.

(a) For all r, what is the charge $Q(r)$ contained within a radius r?

(b) Calculate the charge density $\rho(r)$ everywhere.

(c) Are there any surface charges in this charge distribution? If so, identify their location and give the magnitude of the surface charge density $\boldsymbol{\sigma}$ at each such location.

(a)

$\vec{E} \cdot A = E\, 4\pi r^2 = \frac{Q_{in}}{\varepsilon_0}$

$$
Q_{in} = \begin{cases}
E_0\left(\frac{r}{R}\right)^2 4\pi r^2 \varepsilon_0 & 0 < r < R \\
0 & R < r < 2R \\
E_0\left(\frac{r}{R}-2\right)^2 \pi r^2 \varepsilon_0 & 2R \le r \le 3R \\
E_0\left(\frac{3R}{r}\right)^2 4\pi r^2 \varepsilon_0 & 3R \le r \le 4R \\
0 & r > 4R
\end{cases}
$$

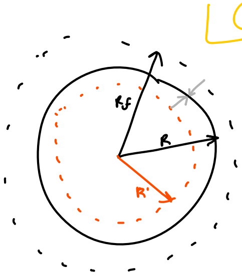
discontinuity

b) $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$$
\rho = \begin{cases}
\varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 E_0 \left(\frac{r}{R}\right)^2\right] = \varepsilon_0 \frac{1}{r^2} \cdot \frac{4r^3 E_0}{R^2} = \frac{4r\varepsilon_0 E}{R^2} & 0 \le r \le R \\
0 & R < r < 2R \\
\varepsilon_0 \frac{1}{r} \frac{\partial}{\partial r}\left[r^2 E_0 \left(\frac{r}{R}-2\right)^2\right] = \frac{\varepsilon_0}{r} E_0 \left(\frac{4r^3}{R^3} - \frac{Rr^2}{R} + 8r\right) = \varepsilon_0 E_0 \left(\frac{4r}{R^2} - \frac{12}{R} + \frac{8}{r}\right) & 2R \le r \le 3R \\
\varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \cdot E_0 \left(\frac{3R}{r}\right)^2\right] = \varepsilon_0 \cdot \frac{1}{r^2} \cdot 0 = 0 & 3R \le r \le 4R \\
0 & r > 4R
\end{cases}
$$

$$\boxed{\rho = \varepsilon_0 \nabla \vec{E} = \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 E\right)}$$

c) finding charge density (inside & outside)



$\boxed{\text{1 At } r = R}$

$E(R^-) = E_0\left(\frac{R}{R}\right)^2 = E_0$

$E(R^+) = 0$

$\Rightarrow E(R^-)A + E(R^+)(-A) = \frac{\sigma A}{\varepsilon_0}$

$\Rightarrow \sigma = \varepsilon_0 \left[E(R^-) - E(R^+)\right] = \varepsilon_0 E$

outer R

$\boxed{\text{2 } r = 2R}$

$E(2R^-) = 0$

$E(2R^+) = E_0\left(\frac{2R}{R} - 2\right)^2 = 0$

$\Rightarrow \sigma = 0$

$\boxed{\text{3 } r = 3R}$

$E(R^-) = E_0\left(\frac{3R}{R} - 2\right)^2 = E_0$

$E(3R^+) = E_0\left(\frac{3R}{3R}\right)^2 = E_0$

$\Rightarrow \sigma = 0$

$\boxed{\text{4 } r = 4R}$

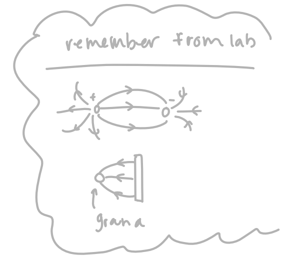
$E(4R^-) = E_0\left(\frac{3R}{4R}\right)^2 = \frac{9}{16} E_0$

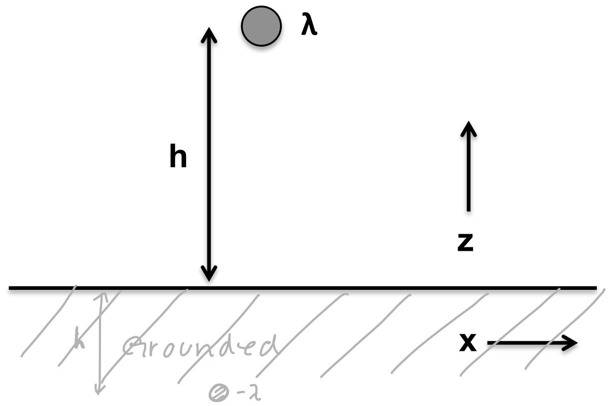
$E(4R^+) = 0$

$\Rightarrow \sigma = \varepsilon_0 \left(\frac{9}{16} E_0 - 0\right)$

$\Rightarrow \sigma = \frac{9}{16} \varepsilon_0 E_0$

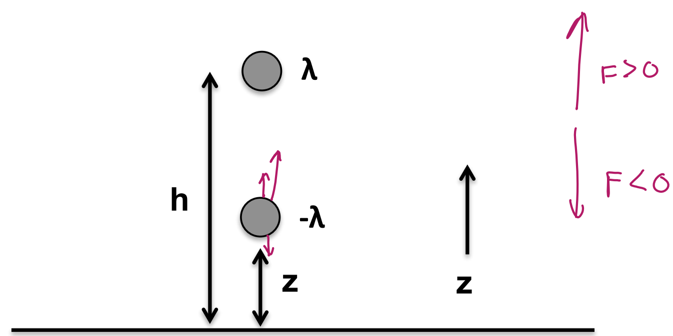
3. An **infinitely long charged wire** (λ linear charge density) is placed parallel to the *y-axis* at a height *h* above a grounded conducting plane.



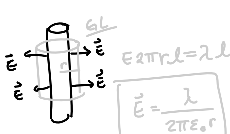


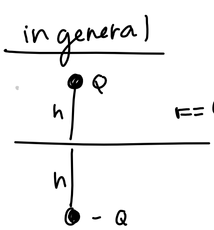
(a) Calculate the force *(per unit length)* acting on the wire.

(b) Another wire with a linear charge density –λ is to placed somewhere along the perpendicular dropped from the first wire to the plane (Figure below). Where can –λ be placed so that the total force on it will be zero? **(z=?)**

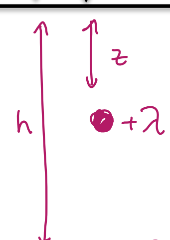


(A)

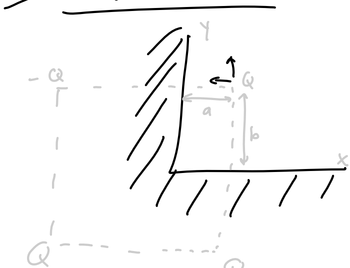


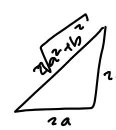


F= E · Q



EXAMPLE (Image math) What If:





(a) $F=\left(\frac{-\lambda}{2\pi\varepsilon_0 2h}\right)\cdot(+\lambda)$

$F=-\frac{\lambda^2}{4\pi h\varepsilon_0}$

- then calculate $\vec{E}$ on wire

(B) Find z to find 0-total force

· using result from Part A

$$F=-\frac{\lambda^2}{2\pi\varepsilon_0(2z)}+\frac{\lambda^2}{2\pi\varepsilon_0(h-z)}+\frac{\lambda^2}{2\pi\varepsilon_0(h+z)}$$

F = 0 ; can now ignore "λ"

↳ $F=\frac{-1}{2z}+\frac{1}{h-z}+\frac{1}{h+z}=0$

$\frac{1}{2z}=\frac{-1}{h+z}+\frac{1}{h-z}$

$h^2-z^2=2z(h-\cancel{z}+h+\cancel{z})$

$h-z^2=4z^2$

➡ $z^2+4hz-h=0$

solve for z

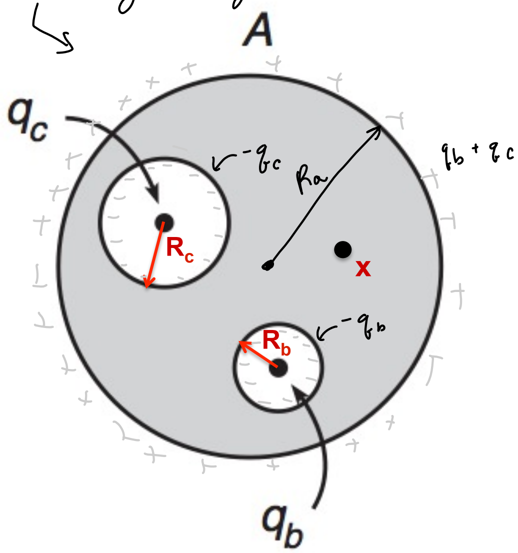
↳ use $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-4h\pm2\sqrt{5}h}{2}\Rightarrow\boxed{z=(\sqrt{5}-2)h}$

$$F=\frac{-Q^2}{4\pi\varepsilon_0(2a)^2}\hat{x}+\frac{Q^2}{4\pi\varepsilon_0(2b)^2}\hat{y}+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{Q^2}{4\pi\varepsilon_0 4(a^2+b^2)}\right)\hat{x}+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{Q^2\hat{y}}{4\pi\varepsilon_0 4(a^2+b^2)}\right)$$

4. A conducting metal sphere ***(with a radius R_a)*** has two cavities inside with radii $\boldsymbol{R_b}$ and $\boldsymbol{R_c}$ (figure below). Two point charges $\boldsymbol{q_b}$ and $\boldsymbol{q_c}$ are placed inside these cavities.

(a) assuming charges are (+)



(a) What are the induced charges on the cavity surfaces?

(b) What are the induced surface charge densities on the cavity surfaces?

(c) What is the surface charge density on the outside surface of metallic sphere?

(d) What is the potential at location ***x*** (assume at potential is zero at infinity) ?

(b) $\sigma_c = \dfrac{-q_c}{4\pi R_c^2}$

$\sigma_b = \dfrac{-q_b}{4\pi R_b^2}$

(c) $\sigma_{out} = \sigma_c + \sigma_b$

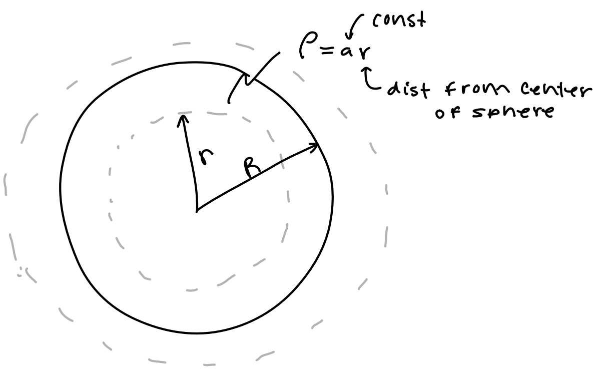
$\sigma_{out} = \dfrac{q_c + q_b}{4\pi R_a^2}$

(d) $V_x = V_A = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q_b + q_c}{R_a}\right)$

5. A non-conducting sphere of radius R has a charge density $\boldsymbol{\rho} = \mathbf{ar.}$ where $\boldsymbol{a}$ is a constant. Let $\boldsymbol{r}$ be the distance from the center of the sphere.

(a) Find the electric field everywhere, both inside and outside the sphere.

(b) Find the electric potential everywhere, both inside and outside the sphere. Assume that potential is zero at infinity.



USE GL

(a) $\oiint_S \vec{E}\cdot d\vec{s} = \dfrac{Q_{in}}{\varepsilon_0}$

because charge density is varying

① r<R: $E(R)\cdot 4\pi r^2 = \dfrac{\int_0^r 4\pi r^2 (ar)\,dr}{\varepsilon_0}$

$\Rightarrow E(r)\, r^2 = \dfrac{a}{\varepsilon_0}\int_0^r r^3\,dr$

$\boxed{E(r) = \dfrac{ar^2}{4\varepsilon_0}}$

② r<R: $E(r)\, 4\pi r^2 = \dfrac{\int_0^R 4\pi r^2 (ar)\,dr}{\varepsilon_0}$

$\Rightarrow E(r) = \dfrac{1}{\varepsilon_0 r^2}\int_0^R ar^3\,dr$

$\boxed{E(r) = \dfrac{aR^4}{4\varepsilon_0 r^2} \quad \left(\int r^3 dr = \frac{1}{4} r^4\right)}$

(b) $V = \int \vec{E}\cdot d\vec{l}$

Potential Outside Sphere

$V_{out} = -\dfrac{R^4 a}{4\varepsilon_0}\int_\infty^R \dfrac{1}{r^2}\,dr = \dfrac{R^4 a}{4\varepsilon_0}\,\dfrac{1}{r}\Big|_\infty^R = \dfrac{aR^3}{4\varepsilon_0}$

$\left(\phi(r) - \phi(\infty) - \int_\infty^r \vec{E}\cdot d\vec{r}\right)$ $\boxed{V_{out} = \dfrac{aR^3}{4\varepsilon_0}}$

Potential Inside Sphere

$V_{in}(r) = V_{out} - \int_R^r \dfrac{ar^2}{4\varepsilon_0}\,dr$

$= \dfrac{aR^3}{4\varepsilon_0} - \dfrac{a}{12\varepsilon_0} r^3\Big|_R^r$

$= \dfrac{aR^3}{4\varepsilon_0} + \dfrac{aR^3}{12\varepsilon_0} - \dfrac{ar^3}{12\varepsilon_0}$

$= \dfrac{aR^3}{4\varepsilon_0} - \dfrac{ar^3}{12\varepsilon_0}$

$\boxed{V_{in}(r) = \dfrac{a}{12\varepsilon_0}\left(4R^3 - r^3\right)}$