COMMUN. STATIST.—THEORY METH., 29(12), 2693–2715 (2000)

DISTRIBUTIONS OF THE RATIOS OF INDEPENDENT BETA VARIABLES AND APPLICATIONS

T. Pham-Gia

Department of Mathematics and Statistics
Université de Moncton
Moncton, N.B., Canada E1A 3E9

Key Words and Phrases : hypergeometric functions; beta distribution; stress-strength; availability.

ABSTRACT

We establish the exact expressions of X_1/X_2 and of $X_1/(X_1+X_2)$, where X_1 and X_2 are independent beta random variables of the first type, and provide some of their applications, in reliability and availability.

1. INTRODUCTION

Beta variables of the first type are frequently used as statistical models for phenomena known to vary within two finite bounds.

Indeed, due to the rich variety of its density shape, from U-shaped to linear, to uniform to unimodal and to J-shaped and reversed J-shaped, the beta distribution is among the most versatile ones.

Varying within (0,1), the *standard beta* is usually taken as the prior distribution for the proportion p and forms a conjugate family within the beta prior-Bernoulli sampling scheme. On the other hand, ratios of two random variables are often encountered in statistical applications. For example, the ratio of two independent general gammas were studied by both Bowman and Shenton (1998) and Pham-Gia and Turkkan (1999), in relation respectively to average conductivity and to availability. However, due to the complexity of the related distribution, seldom do we see their full use and the precise expression of its characteristics. In this article we will consider the standard beta variable with density :

$$f(\alpha,\beta;x) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha,\beta),\ \alpha,\beta>0 \text{ and } 0\le x\le 1 \quad (1)$$

with $B(\alpha,\beta)$ being the beta function, and compute the precise expressions of the densities of X_1/X_2 and of $X_1/(X_1+X_2)$. We also provide applications of these two densities, in the fields of stress-strength analysis and availability. It is noted that for two independent standard betas, their sum was studied in Pham-Gia and Turkkan (1994) and their difference by Pham-Gia and Turkkan (1993) while Steece (1996) studied their product.

With the result presented in this article, we now have the precise densities associated with all four operations on X_1 and X_2.

Several interesting results on products and ratios of standard betas are presented in the literature (Johnson, Kotz and Balakhrishnan (1995, chapter 25)). However, most of them require that the coefficients α and β be integers, whereas our results here do not require this condition and hence, have a much more general scope.

For the *general beta,* in four parameters, defined on an interval (a,b), with density:

$f(\alpha,\beta;a,b;y) = (y-a)^{\alpha-1}(b-y)^{\beta-1}\Big/\left[B(\alpha,\beta)\times(b-a)^{\alpha+\beta-1}\right]$ where $\alpha,\beta>0$ and $a \le y \le b$, and denoted $\text{gbeta}(\alpha,\beta;a,b;x)$, although it can be reduced to the standard beta by the linear transformation $x=\dfrac{(y-a)}{(b-a)}$, the density of the linear combination of two independent general betas, defined respectively on (a,b) and (c,d), is given by Pham-Gia and Turkkan (1998) and cannot be, in the general case, obtained from the first two previously mentioned results.

2. DENSITIES OF THE RATIOS

DEFINITION: The Gauss hypergeometric function in 3 parameters

a, b, and c, denoted ${}_2F_1$, is defined as follows:

${}_2F_1(a,b;c;x) = \sum_{m=0}^{\infty} \frac{(a,m)\times(b,m)}{(c,m)} \cdot \frac{x^m}{m!}$, $|x| < 1$, with Pochhammer coefficients $(a, m) = a(a+1)K\ (a+m-1) = \Gamma(a+m)/\Gamma(a)$, for $m \geq 1$ and $(a, 0) = 1$.

It has an integral representation of the form:

$$ {}_2F_1(a, b; c; x) = \int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}}{B(a,c-a)} du. \tag{2}$$

Its generalization to two variables, x_1 and x_2, called the Appell First Hypergeometric function, and denoted F_1, (Pham-Gia and Duong (1984)), is defined by $\left(a, b_1, b_2 \in R, 0 < |x_1|, |x_2| < 1\right)$:

$$F_1(a;\ b_1;\ b_2;\ c;\ x_1;\ x_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a,m+n)}{(c,m+n)} \times$$

$$(b_1,\ m) \times (b_2,\ n) \times \frac{x_1^m}{m\ !} \times \frac{x_2^n}{n!} \tag{3}$$

F_1 is used in the expressions of the densities of the sum and difference of two independent beta variables. (Pham-Gia and Turkkan, 1994 and 1993).

We establish the following

THEOREM :

Let $X_i \sim \text{beta}\left(\alpha_i,\ \beta_i\right)$, $i = 1,\ 2$, be independent standard beta variables.

a) The random variable $W = X_1/X_2$ has density $D_1(\alpha_1, \alpha_2; \beta_1, \beta_2; w)$, given by:

For $0 < w \le 1$,

$$f(w) = B(\alpha_1 + \alpha_2,\ \beta_2)\, w^{\alpha_1 - 1} \cdot$$
$$_2F_1(\alpha_1 + \alpha_2,\ 1 - \beta_1;\ \alpha_1 + \alpha_2 + \beta_2;\ w)/A \text{ and} \tag{4}$$

For $w \ge 1$

$$f(w) = B(\alpha_1 + \alpha_2,\ \beta_1)\, w^{-(1+\alpha_2)} \cdot$$
$$_2F_1\left(\alpha_1 + \alpha_2,\ 1 - \beta_2;\ \alpha_1 + \alpha_2 + \beta_1;\ \frac{1}{w}\right)\Big/A, \quad \text{where}$$

$A = B(\alpha_1,\ \beta_1) \cdot B(\alpha_2,\ \beta_2)$.

b) The density $D_2(\alpha_1, \alpha_2; \beta_1, \beta_2; t)$ of $T = X_1/(X_1 + X_2)$ is given by:

For $0 < t \le 1/2$,

$$g(t) = t^{\alpha_1 - 1}(1 - t)^{\alpha_1 + 1} \cdot B(\alpha_1 + \alpha_2, \beta_2) \cdot$$
$$_2F_1\left(\alpha_1 + \alpha_2, 1 - \beta_1; \alpha_1 + \alpha_2 + \beta_2; \frac{t}{1 - t}\right)\Big/A \text{ and} \tag{5}$$

For $1/2 \le t \le 1$,

$$g(t) = t^{-(\alpha_2 + 1)}(1 - t)^{\alpha_2 - 1} \cdot B(\alpha_1 + \alpha_2, \beta_1) \cdot$$
$$_2F_1(\alpha_1 + \alpha_2, 1 - \beta_2; \alpha_1 + \alpha_2 + \beta_1; \frac{1 - t}{t})/A$$

Proof :

We have $f_i(x_i) = x_i^{\alpha_i - 1}(1 - x_i)^{\beta_i - 1}/B(\alpha_i, \beta_i)$ where $i \in \{1, 2\}$ and

$0 \le x_i \le 1$.

a) Let $W = X_1/X_2$ and $h(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$ be the joint density of X_1 and X_2.

Then, $h(w, x_2) = \dfrac{x_2 (wx_2)^{\alpha_1 - 1}}{A} (1 - wx_2)^{\beta_1 - 1}\, x_2^{\alpha_2 - 1} (1 - x_2)^{\beta_2 - 1}$, where $A_i = B(\alpha_i, \beta_i)$ and $A = A_1 \cdot A_2$, in the transformation $((x_1, x_2,) \to (w, x_2))$.

The marginal density of W is:

$f(w) = \dfrac{w^{\alpha_1 - 1}}{A} \cdot \int_0^\infty x_2^{\alpha_1 + \alpha_2 - 1} (1 - x_2)^{\beta_2} (1 - wx_2)^{\beta_1 - 1}\, dx_2$, with proper integration bounds to be determined.

For $0 \le w \le 1$ we can write

$$f(w) = \int_0^1 h(w, x_2) = \frac{w^{\alpha_1 - 1}}{A} \cdot \int_0^1 x_2^{\alpha_1 + \alpha_2 - 1} (1 - x_2)^{\beta_2} (1 - wx_2)^{\beta_1 - 1}\, dx_2.$$

Using (2), it follows that:

$$f(w) = \frac{w^{\alpha_1 - 1}}{A} \cdot B(\alpha_1 + \alpha_2, \beta_2) \cdot {}_2F_1(\alpha_1 + \alpha_2, 1 - \beta_1; \alpha_1 + \alpha_2 + \beta_2; w).$$

For $w \ge 1$, using $wx_2 = t$, we have:

$$f(w) = \frac{w^{\alpha_1 - 1}}{A} \times \int_0^1 \left(\frac{t}{w}\right)^{\alpha_1 + \alpha_2 - 1} \left(1 - \frac{t}{w}\right)^{\beta_2 - 1} (1 - t)^{\beta_1 - 1}\, \frac{dt}{w} =$$

$$\frac{1}{A \cdot w^{\alpha_2 + 1}} \times \int_0^1 t^{\alpha_1 + \alpha_2 - 1} \left(1 - \frac{t}{w}\right)^{\beta_2 - 1} (1 - t)^{\beta_1 - 1}\, dt.$$

Using again equation (2), we have:

$$f(w)=\frac{w^{-(\alpha_2+1)}}{A}\times B(\alpha_1+\alpha_2,\beta_1)\cdot {}_2F_1(\alpha_1+\alpha_2,1-\beta_2;\alpha_1+\alpha_2+\beta_1;\frac{1}{w}).$$

b) Let $Y_1=X_1+X_2$ and $T=X_1/(X_1+X_2)$ with $(y_1,t)\in[0,2]\times[0,1]$. The jacobian of the transformation is:

$$J=\left\|\begin{matrix} t & y_1 \\ 1-t & -y_1 \end{matrix}\right\| = y_1 .$$

Since $g(x_1,x_2)=\frac{1}{A}\cdot x_1^{\alpha_1-1}(1-x_1)^{\beta_1-1}x_2^{\alpha_2-1}(1-x_2)^{\beta_2-1}$, changing to (y_1,t) we have:

$$g(y_1,t)=\frac{y_1}{A}\cdot(y_1t)^{\alpha_1-1}(y_1(1-t))^{\alpha_2-1}(1-y_1t)^{\beta_1-1}(1-y_1(1-t))^{\beta_2-1},$$

or $$g(y_1,t)=\frac{y_1^{\alpha_1+\alpha_2-1}\cdot t^{\alpha_1-1}\cdot(1-t)^{\alpha_2-1}(1-y_1t)^{\beta_1-1}(1-(1-t)y_1)^{\beta_2-1}}{A}.$$

The marginal density of variable t is: $g(t)=\int_0^2 g(y_1,t)\,dy_1$, with proper integration bounds to be determined.

Since $\begin{cases} X_1=Y_1\cdot T \\ 0\le X_1\le 1 \end{cases}$ we have $0\le Y_1\le\frac{1}{T}$, while $\begin{cases} X_2=Y_1(1-T) \\ \text{and } 0\le X_2\le 1 \end{cases}$ imply $0\le Y_1\le\frac{1}{1-T}$.

Since $0\le y_1\le 2$, there are two integration regions.

For $0 \leq t \leq \dfrac{1}{2}$ we have:

$$g(t) = \int_0^{1/(1-t)} g(y_1, t)\, dy_1 = \frac{t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}}{A} \cdot$$

$$\int_0^{1/(1-t)} y_1^{\alpha_1+\alpha_2-1}(1-y_1 t)^{\beta_1 - 1}(1-(1-t)y_1)^{\beta_2 - 1} dy_1 .$$

Setting $y_1 = v/(1-t)$ we can write:

$$g(t) = \frac{t^{\alpha_1 - 1}}{A(1-t)^{\alpha_1+1}} \cdot \int_0^1 v^{\alpha_1+\alpha_2-1}(1-\frac{t}{1-t} \cdot v)^{\beta_1 - 1}(1-v)^{\beta_2 - 1} dv .$$

Using expression (2), we obtain:

$$g(t) = \frac{1}{A \cdot (1-t)^2} \cdot \left(\frac{t}{1-t}\right)^{\alpha_1 - 1} \cdot$$

$$B(\alpha_1+\alpha_2, \beta_2) \cdot {}_2F_1(\alpha_1+\alpha_2, 1-\beta_1; \alpha_1+\alpha_2+\beta_2; \frac{t}{1-t}) .$$

For $1/2 \leq t \leq 1$ we have:

$$g(t) = \int_0^{1/t} g(y_1, t)\, dy_1 = \frac{t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}}{A} \cdot$$

$$\int_0^{1/t} y_1^{\alpha_1+\alpha_2-1}(1-y_1 t)^{\beta_1 - 1}(1-(1-t)y_1)^{\beta_2 - 1} dy_1 .$$

Using $u = y_1 t$, $g(t)$ becomes:

$$g(t) = \frac{t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}}{A} \cdot \int_0^1 \left(\frac{u}{t}\right)^{\alpha_1+\alpha_2-1} (1-u)^{\beta_1 - 1}\left[1 - \left(\frac{u}{t} - u\right)\right]^{\beta_2 - 1} dy_1$$

$$\text{or } g(t) = \frac{t^{\alpha_1-1}(1-t)^{\alpha_2-1}}{A} \cdot \int_0^1 \left(\frac{u}{t}\right)^{\alpha_1+\alpha_2-1} (1-u)^{\beta_1-1}\left[1-\left(\frac{1}{t}-1\right)u\right]^{\beta_2-1} \frac{du}{t},$$

which gives:

$$g(t) = \frac{1}{A\ t^2} \cdot \left(\frac{1}{t}-1\right)^{\alpha_2-1} \cdot \int_0^1 u^{\alpha_1+\alpha_2-1}(1-u)^{\beta_1-1}\left[1-\left(\frac{1}{t}-1\right)u\right]^{\beta_2-1} du.$$

Using again equation (2), we have:

$$g(t) = \frac{1}{A\ t^2} \cdot \left(\frac{1}{t}-1\right)^{\alpha_2-1} B(\alpha_1+\alpha_2, \beta_1) \cdot {}_2F_1(\alpha_1+\alpha_2, 1-\beta_2; \alpha_1+\alpha_2+\beta_1; \frac{1}{t}-1).$$

QED

Figure 1 gives the density of $T = X_1/(X_1+X_2)$ for $X_1 \sim \text{beta}(2.50, 3.75)$ and $X_2 \sim \text{beta}(1.25, 4.06)$. The graph is smooth at the point with abscissa 1/2 in spite of different expressions for the density of T on either side of this point.

REMARKS:

1) Although the above theorem is established for the standard betas, it is immediate that it also applies for general beta variables Y_1 and Y_2, both defined on the same interval $(0, b)$. Through the linear transformation $X_1 = Y_1/b, i = 1, 2$, we have $Y_1/Y_2 = X_1/X_2$ and $Y_1/(Y_1+Y_2) = X_1/(X_1+X_2)$. The more general case where Y_1 and Y_2 have respectively (a, b) and (c, d) as distinct supports, is much

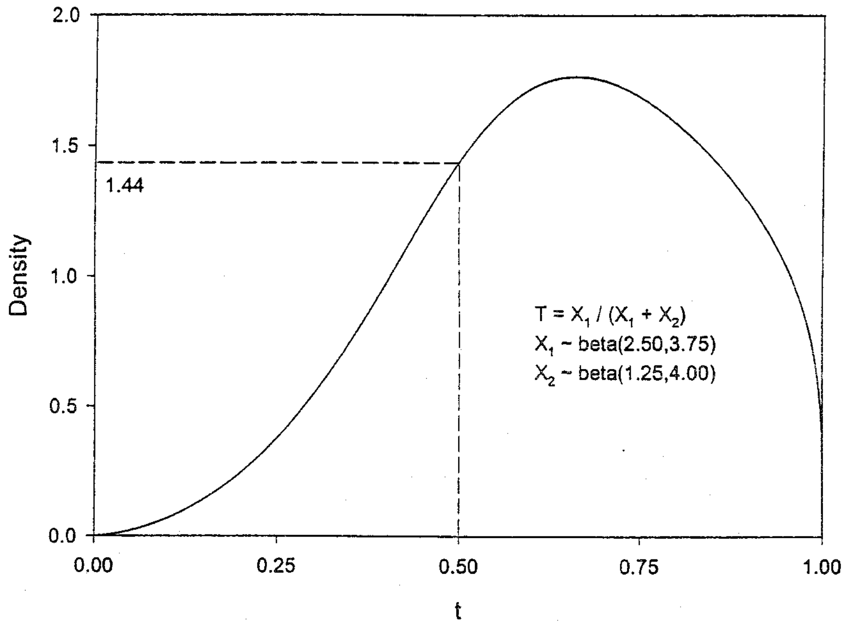

Figure 1: Distribution of $T = X_1/(X_1 + X_2)$, with both X_1 and X_2 standard beta variables.

more complex and will be dealt with in a subsequent paper.

2) Although at first view, moments of order n of W and T seem difficult to compute, they, in fact, reduce to integrals on $(0,1)$ of simple expressions containing ${}_2F_1$. We have:

$$E(W^n) = \Big\{B(\alpha_1 + \alpha_2,\ \beta_2) \int_0^1 w^{n+\alpha_1-1}$$

$${}_2F_1(\alpha_1 + \alpha_2,\ 1-\beta_1;\ \alpha_1 + \alpha_2 + \beta_2;\ w)dw +$$

$$B(\alpha_1+\alpha_2\ ,\ \beta_1)\int_0^1 w^{\alpha_2-(n+1)}$$

$$ {}_2F_1(\alpha_1+\alpha_2,\ 1-\beta_2;\ \alpha_1+\alpha_2+\beta_1;\ w)dw\}/A \tag{6}$$

and

$$E(T^n)=\Big\{B(\alpha_1+\alpha_2,\beta_2)\int_0^1\left[t^{n+\alpha_1-1}/(1+t)^n\right]$$

$$ {}_2F_1(\alpha_1+\alpha_2,1-\beta_1;\alpha_1+\alpha_2+\beta_2;t)\,dt+$$

$$B(\alpha_1+\alpha_2,\beta_1)\int_0^1\left[t^{\alpha_2-1}/(1+t)^n\right]$$

$$ {}_2F_1(\alpha_1+\alpha_2;1-\beta_2;\alpha_1+\alpha_2+\beta_1;t)\,dt\Big\}/A \tag{7}$$

The above expressions can be obtained by performing the appropriate changes of variables but they, themselves, have to be computed numerically.

An immediate application of the above theorem concerns the distributions of ratios of individual variates in a Dirichlet distribution. Let $(X_1, X_2, \ldots, X_n)$ have a Dirichlet distribution, with density:

$$f(\theta_0,\theta_1,\ldots,\theta_n;x_1,x_2,\ldots,x_n)=\frac{\Gamma(\sum_{j=0}^{n}\theta_j)}{\prod_{j=0}^{n}\Gamma(\theta_j)}\times\prod_{j=1}^{n}(x_j^{\theta_j-1})\times(1-\sum_{j=1}^{n}x_j)^{\theta_0-1}.$$

defined on the simplex $\sum_{i=1}^{n}x_i\le 1$.

Then for $i\ne j$, we have $X_i/X_j\sim D_1(\theta_i,\theta_j;\psi-\theta_i;\psi-\theta_j;w)$

and $X_i/(X_i+X_j) \sim D_2(\theta_i, \theta_j; \psi-\theta_i, \psi-\theta_j; t)$ since each X_j has a marginal standard beta distribution beta $(\theta_j, \psi-\theta_j)$, where $\psi = \sum_{j=0}^{n} \theta_j$.

Furthermore, let $Y_1 = X_1$, $Y_2 = X_2 / \left[\sum_{k=2}^{n} X_k\right]$ and $Y_j = X_j / \left(\sum_{k=j}^{n} X_k\right)$, $j = 3, K, n$. We know that Y_j : beta$\left(\theta_j, \sum_{k=j+1}^{n} \theta_k\right)$ and are independent of each other (Johnston and Kotz (1972, p. 234)).

Then, by the above theorem, $Y_i/Y_j \sim D_1\left(\theta_i, \theta_j; \sum_{k=i+1}^{n} \theta_k, \sum_{k=j+1}^{n} \theta_k; w\right)$, and $Y_i/(Y_i+Y_j)$: $D_2\left(\alpha_i, \alpha_j; \sum_{k=i+1}^{n} \theta_k, \sum_{k=j+1}^{n} \theta_k; t\right)$ for $i \neq j$.

3. APPLICATIONS IN RELIABILITY

3.1 STRESS-STRENGTH MODEL:

Classical static stress-strength models consider a system with strength S, a random variable with distribution F, subject to a stress

V, with distribution G. The reliability of the system is hence $\gamma = P(S - V > 0)$. For this difference model, γ can be found in closed form when F and G follow some common distributions (see Kapur and Lamberson (1977)). Due to the more complex computation associated with the division of random variables, rarely is the ratio model $P(S/V > 1)$ considered, although this model clearly presents an additional advantage, in the sense that it permits the immediate comparison of values of strength and stress, independently of the unit adopted, while the difference model depends on this unit. For example, if we require that $P(S/V > 1.15)$ be larger than 0.90, we are pretty confident that the system strength is 15 % higher than the system stress at least ninety percent of the times.

On the other hand, although the general beta model is clearly the one to be used for either stress or strength because of the finite limits within which each of them usually varies, other models like the Normal, Gamma and Weibull ones are used instead. This happening could largely be due to the intractability of $P(S > V)$ when one, or both of S and V, are betas. By presenting the explicit expression of the density $S - V$, when both of them are betas, Pham-Gia and Turkkan (1994) have removed most of these difficulties.

As a numerical example, let's consider a system whose strength and stress can be represented by a *general beta* distribution on the

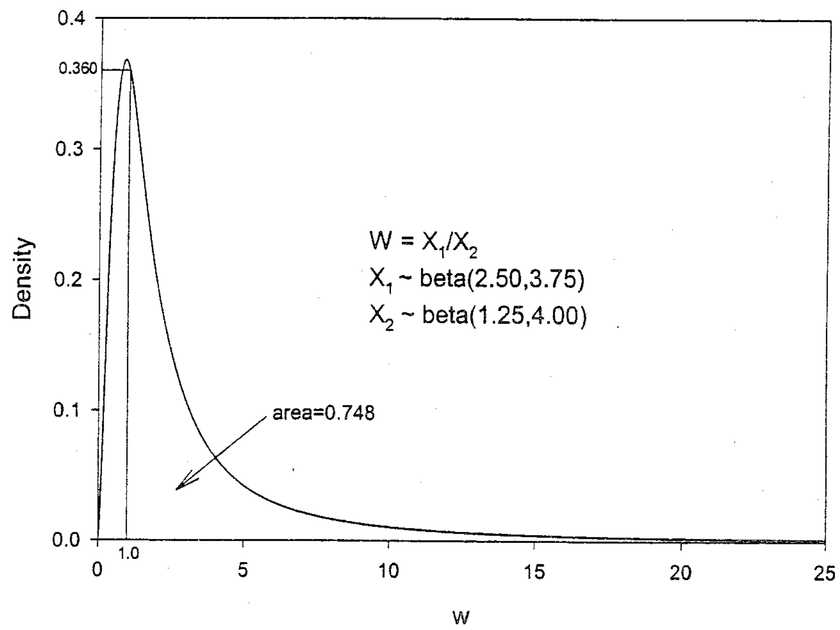

Figure 2: Distribution of X_1 / X_2, with both X_1 and X_2 standard beta variables.

interval 0 to 50 Mpa, for example, $S \sim \text{gbeta}(2.5, 3.75; 0, 50; x)$ and $V \sim \text{gbeta}(1.25, 4.00; 0, 50; x)$. To compute the reliability of this system, let's consider the distribution of $W = S/V$, which is the same as the distribution of the ratio of the corresponding standard betas, which is $D_1(2.5, 1.25; 3.75, 4; w)$.

Computations give $\gamma = P(W \geq 1) = 0.9184$, as shown on fig. 2.

If we wish to be more stringent on the safety of this system and require that the strength be at least 25% larger than the stress, let's compute $P(W > 1.25)$. Since this probability is 0.8656, with this requirement, strength is hence superior to stress by 25% for 86% of the times.

Similarly, for any lower bound imposed on W, we can obtain the associated probability.

Note: $D_1(\alpha_1, \alpha_2; \beta_1, \beta_2; w)$ gives the general expression of the density of X_1/X_2 and hence, generalizes the result obtained by Weisberg (1972) who computed $P(X_1/X_2 < c)$ for beta distributions with integer-valued parameters.

TWO-STATE SYSTEM AVAILABILITY:

Let us consider a Markovian system with two states: on and off. The corresponding distributions are exponential, with rates respectively λ and μ. We have $f(\lambda; t) = \lambda \exp(-\lambda t)$ and $f(\mu; t) = \mu \exp(-\mu t), 0 < t$.

Let $A(t) = P$ (system is on at t) be the availability of the system at time t. At time 0, let $A(0) = P_0$.

Classical results give:

$$A(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu - P_0(\lambda+\mu)}{\lambda+\mu} \exp(-(\lambda+\mu)\cdot t) \text{ with } t \geq 0.$$

When $P_0 = 1$, $A(t)$ is the sum of the steady-state availability $A = \frac{\mu}{\lambda + \mu}$ and the time dependent availability $K(t) = (1 - A)\exp(-(\lambda + \mu)t)$, and converges toward A when $t \to \infty$.

Hence, at time t_0 the above result gives the deterministic value of $A(t_0)$ when values of λ and μ are known with precision. In many real-life applications, these values are not known with certainty, and are, usually, given within intervals of variation, which could originate from confidence intervals or previous data analysis, or even, in the case of a Bayesian approach, from expert's opinion elicitation. Furthermore, within these variation intervals, there could be subintervals that are more likely to contain these parameters than others. Under these conditions, general beta distributions on $(0, b)$ are the best suited to be used as distributions for λ and μ, making $A(t_0)$ a random variable.

Let's suppose, without loss of generality, λ: $\text{beta}(\alpha_1, \beta_1)$ and μ: $\text{beta}(\alpha_2, \beta_2)$. Then according to (5), the steady state availability A has a $D_2(\alpha_1, \alpha_2; \beta_1, \beta_2; t)$ distribution, while at time t_0, the time dependent availability, denoted $K(t_0)$, is the product of $(1 - A)$ with

$$L(t_0) = \exp(-(\lambda + \mu)\, t_0). \tag{8}$$

The latter term has as density a transform of the density of the sum of two standard betas, as given by Pham-Gia and Turkkan (1993). Hence, we can compute the density of $K(t_0)$, using standard techniques, although it is quite tedious. However, these two availability terms A and $K(t_0)$ are dependent, and it is rather difficult to obtain the precise density of their sum $A(t_0)$, although we can obtain its mean $\mu(A(t_0))$ and variance $Var(A(t_0))$. We have:

$$\mu(A(t_0)) = \mu(A) + (1 - \mu(A)) \times \mu(L(t_0)) \tag{9}$$

since A and $L(t_0)$ are independent.

On the other hand, we have (see Appendix):

$$Var(A(t_0)) = Var(A) \cdot Var(L(t_0)) +$$

$$Var(A)\left[1 - \mu(L(t_0))\right]^2 + Var(L(t_0))\left[1 - \mu(A)\right]^2 \tag{10}$$

In (9) and (10), $\mu(A)$ and $Var(A)$ can be obtained using (7).

Fitting a standard beta (α_3, β_3) to the distribution of $A(t_0)$ by matching its mean $\alpha_3/(\alpha_3 + \beta_3)$ and its variance $(\alpha_3\,\beta_3)/[\alpha_3 + \beta_3]^2\,[\alpha_3 + \beta_3 + 1]$ to $\mu(A(t_0))$ and $Var(A(t_0))$, we obtain the relations:

$$\alpha_3 = \mu(A(t_0))\frac{\left[\mu(A(t_0))(1 - \mu(A(t_0)) - Var(A(t_0))\right]}{Var(A(t_0))} \tag{11}$$

and

$$\beta_3 = \left[1-\mu(A(t_0))\right] \times \frac{\left[\mu(A(t_0))(1-\mu(A(t_0)) - \mathrm{Var}(A(t_0))\right]}{\mathrm{Var}(A(t_0))} \quad (12)$$

We can now obtain an approximate $(1-\theta)$ 100 % credible interval for $A(t_0)$.

The above considerations can easily be extended to series and parallel systems with k repair channels (see Rau (1970)).

For a numerical example, suppose that a Markovian system alternates between functioning and repair, with respective rates λ and μ, and let $P_0 = A(0) = 1$, i.e. the system is on at time origin.

Let's suppose that λ and μ vary within $(0,1)$, eg. λ: beta$(9.75,\ 1.25)$ and μ: beta$(1.15, 10.25)$. We then have $A \sim D_2(1.15, 9.75;\ 10.25, 1.25;\ t)$, according to (5).

Let's consider the time $t_0 = 3.10$. Computing directly the means and variances of A on the one hand, using (7) above, and those of $L(t_0)$ on the other hand (using (8) and the density of $(\lambda+\mu)$ in terms of Appell function F_1, as given by Pham-Gia and Turkkan (1994)), and applying expressions (9) and (10), we obtain :

$\mu(A(t_0)) = 0.88291$ and $\mathrm{Var}(A(t_0)) = 0.001761$.

From relations (11) and (12), the approximating beta distribution for $A(t_0)$ is hence beta$(54.19, 6.78)$. Using the

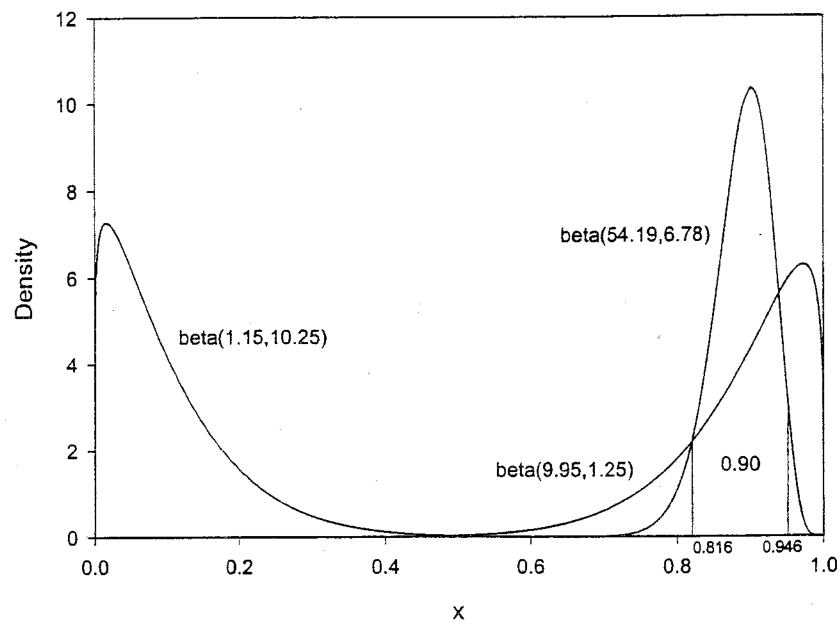

Figure 3 : Approximative beta distribution for $A(3.10)$.

algorithm presented in Turkkan and Pham-Gia (1993), the 90% credible interval for $A(3.10)$, i.e. the credible interval for the availability of the system at time 3.10, is hence $(0.8396, 0.9319)$, as shown in Fig. 3.

3. CONCLUSION

Precise expressions of the densities of the ratios of two beta variables, as obtained above, allow in-depth and analytic studies of several problems in Queueing and Reliability theories, that can only be carried out up to now by numerical or simulation methods. At the same time, the beta distribution can be used much more widely now that all four operations on two independent betas can be exactly carried out, especially in fields where it has been traditionally present, such as order statistics and tolerance regions.

APPENDIX

In this appendix we will prove that when X and Y independent r.v, for $W = X + (1-X)Y$, we have $E(W) = \mu_X + \mu_Y - \mu_X\mu_Y$ and $\mathrm{Var}(W) = \mathrm{Var}(X)\cdot\mathrm{Var}(Y) + \mathrm{Var}(X)(1-\mu_Y)^2 + \mathrm{Var}(Y)(1-\mu_X)^2$.

Proof:

Since $W = X + (1-X)Y$, let's set $W_1 = X + Y$ and $W_2 = XY$;

$E(W) = E(W_1 - W_2) = \mu_X + \mu_Y - \mu_X\mu_Y$. On the other hand,

$\mathrm{Var}(W) = \mathrm{Var}(W_1) + \mathrm{Var}(W_2) - 2\,\mathrm{cov}(W_1, W_2)$.

We have $\mathrm{Var}(W_1) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ while

$\mathrm{Var}(W_2) = E\left[(XY - \mu_X\mu_Y)^2\right] = E(X^2)E(Y^2) - \mu_X^2\mu_Y^2$

$= (\mathrm{Var}(X) + \mu_X^2)(\mathrm{Var}(Y) + \mu_Y^2) - \mu_X^2\mu_Y^2$.

Also, $\operatorname{Cov}(W_1, W_2) = E\left[\left\{(X+Y)-(\mu_X+\mu_Y)\right\}\times\left\{XY-\mu_X\mu_Y\right\}\right]$

$$= E\left\{(X^2Y+Y^2X)-\mu_X\mu_Y(\mu_X+\mu_Y)\right\}$$

$$= \mu_Y\left[E(X^2)-\mu_X^2\right]+\mu_X\left[E(Y^2)-\mu_Y^2\right] = \mu_Y \operatorname{Var}(X)+\mu_X \operatorname{Var}(Y).$$

Hence, $\operatorname{Var}(W) = \operatorname{Var}(X)\operatorname{Var}(Y) + \operatorname{Var}(X)(1+\mu_Y^2 - 2\mu_Y)$

$+ \operatorname{var}(Y)(1+\mu_X^2 - 2\mu_X)$

setting $X = A = \dfrac{\mu}{(\lambda+\mu)}$ and $Y = \exp(-(\lambda+\mu)\, t_0)$, we have X and Y independent, $A(t_0) = X + (1-X)Y$, and obtain expressions (9) and (10) for $\mu(A(t_0))$ and $\operatorname{Var}(A(t_0))$.

ACKNOWLEDGEMENT

Research supported by NSERC Grant A9249 (Canada). The author wishes to thank Mr. A. M. Essombé Edimo for carrying out several complex computations required by the paper and an anonymous referee for several helpful suggestions.

REFERENCES

Bowman, K.O. and Shenton, L.R. (1998). Distribution of the Ratio of Gamma Variables, *Commun. Stat.-Simula.*, 27(1), 1-19.

Dennis, S. Y. (1996). On the Distribution of Products of Independent

Beta Variates, *Commun. in Stat.-Theory and Methods*, 23, 1895-1913.

Johnson, N., Kotz, S. and Balakrishnan (1995). *Continuous Univariate Distribution*, 2nd Edition, John Wiley and Sons, NY.

Kapur, K.C. and Lamberson, L.R. (1977). *Reliability Engineering Design*, Wiley, NY.

Pham-Gia, T. and Duong, O.P. (1989). The Generalized Beta and F-Distributions in Statistical Modeling, *Math. Comput. Modeling*, 12, 12, 1613-1625.

Pham-Gia, T. and Turkkan, N. (1993). Bayesian Analysis of the Difference of Two Proportions, *Commu. Stat-Theory Meth.*, 22(6), 1755-1771.

Pham-Gia, T. and Turkkan, N. (1994). Reliability of a Standby System with Beta Component Lifelength, *IEEE Trans. Rel.*, 71-75.

Pham-Gia, T. and Turkkan, N. (1998). Distribution of the Linear

Combination of Two General Betas and Applications, *Commun. Stat-Theory Meth.*, 27(7), 1851-1869.

Pham-Gia, T. and Turkkan, N.(1999). System Availability in a Gamma Alternating Renewal Process, *Naval Research Logistics*, 46, 822-844.

Rau, J. (1970). *Optimization and Probability in System Engineering*, Van Nostrand, NY.

Steece, B.M.P. (1976). On the Exact Distribution of the Product of Independent Beta-Distributed Variables, *Metron*, 187-190.

Turkkan, N. and Pham-Gia, T. (1993). Computation of the Highest Posterior Density Region in Bayesian Analysis, *J. Stat. Comput. Simulat.*, 44, 243-250.

Weisberg, H.I. (1972). Bayesian Comparison of Two Ordered Multinomial Populations, *Biometrics*, 28, 859-867.

Received December, 1999; Revised June, 2000.