ME 639
Introduction to Robotics

Mini Project

2R – Elbow Manipulator

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Robot Description

As shown in figure 1, in the present 2R elbow manipulator, two rods are joined using revolute joints that restrict movement to the xy plane. Rod or link 1 of mass m_1 and length l_1, is attached to ground at the origin via revolute joint J_1. Link 2 of mass m_2 and length l_2 is connected to the other end of link 1 via revolute joint J_2. Thus, the system has 2 degrees of freedom. The end effector (free end of link 2) is denoted by point $E(x, y)$. The links 1 and 2 form angles q_1 and q_2 respectively with the horizontal x axis. Motors present at joints J_1 and J_2 can provide torques τ_1 and τ_2 at the respective joints.

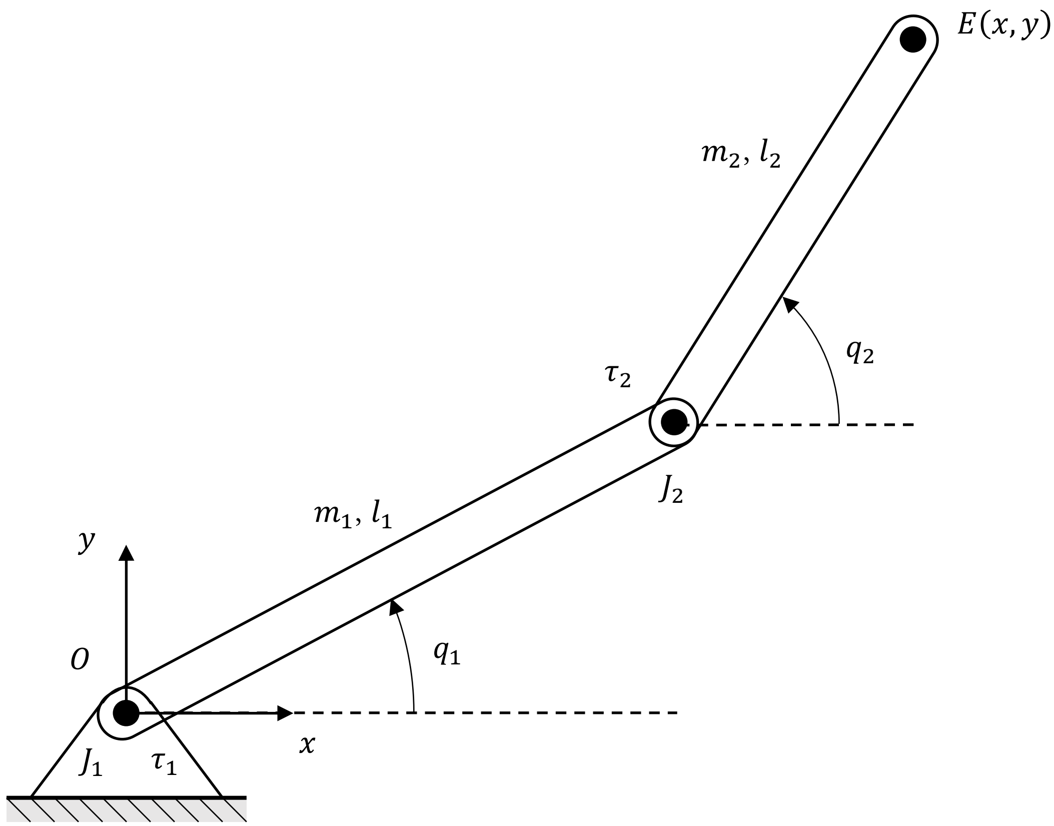

Figure 1 2D Schematic of 2R Elbow Manipulator

Forward Kinematics

From vector algebra, the coordinates of $E(x, y)$ can be found as

$$x = l_1 \cos q_1 + l_2 \cos q_2 \tag{1}$$

$$y = l_1 \sin q_1 + l_2 \sin q_2 \tag{2}$$

Taking the first derivative with respect to time, the x and y components of velocity are found as

$$\dot{x} = -l_1 \sin q_1 \cdot \dot{q}_1 - l_2 \sin q_2 \cdot \dot{q}_2 \tag{3}$$

$$\dot{y} = l_1 \cos q_1 \cdot \dot{q}_1 + l_2 \cos q_2 \cdot \dot{q}_2 \tag{4}$$

Expressing the same in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{pmatrix} \cdot \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \tag{5}$$

Taking the second derivative with respect to time, the x and y components of acceleration are obtained

$$\ddot{x} = -l_1 \sin q_1 \cdot \ddot{q}_1 - l_1 \cos q_1 \cdot \dot{q}_1^2 - l_2 \sin q_2 \cdot \ddot{q}_2 - l_2 \cos q_2 \cdot \dot{q}_2^2 \tag{6}$$

$$\ddot{y} = l_1 \cos q_1 \cdot \ddot{q}_1 - l_1 \sin q_1 \cdot \dot{q}_1^2 + l_2 \cos q_2 \cdot \ddot{q}_2 - l_2 \sin q_2 \cdot \dot{q}_2^2 \tag{7}$$

Again, expressing the same in matrix form

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{pmatrix} \cdot \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} - \begin{pmatrix} l_1 \cos q_1 & l_2 \cos q_2 \\ l_1 \sin q_1 & l_2 \sin q_2 \end{pmatrix} \cdot \begin{pmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{pmatrix} \tag{8}$$

Inverse Kinematics

In figure 2, two right angle triangles are constructed to define angles θ, φ and ϕ. θ is the angle between links 1 and 2. Using parallelogram law for vector addition, θ can be found as

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos \theta \tag{9}$$

$$\Rightarrow \theta = \pm \cos^{-1} \left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2} \right) \tag{10}$$

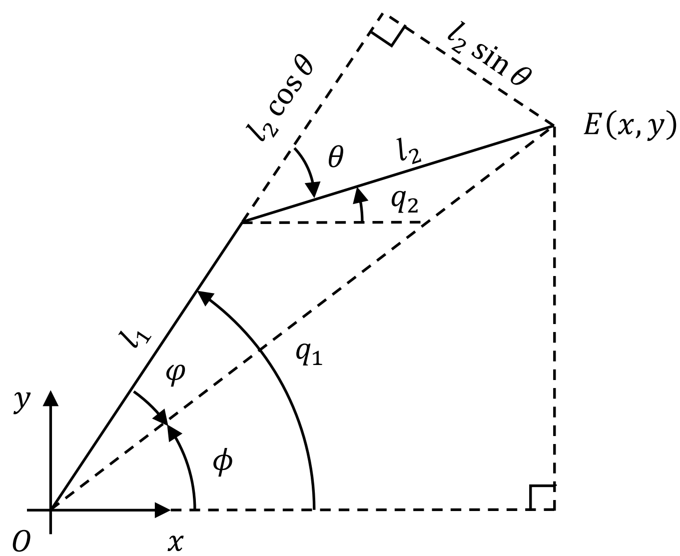

Figure 2 Geometric representation of the robot links on xy plane

The $\pm$ values of θ correspond to elbow-up and elbow-down configuration of the manipulator. To remain consistent and maximize the range of the manipulator, elbow-up configuration is chosen. Therefore,

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right), & x \geq 0 \\ -\cos^{-1}\left(\dfrac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right), & x < 0 \end{cases} \tag{11}$$

From the constructed right triangles, the values of ϕ and φ can be deduced as

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{12}$$

$$\varphi = \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right) \tag{13}$$

Note, when $x < 0$, the value of ϕ should lie in the range $\left[\frac{\pi}{2}, \pi\right]$. But the principal value range of $\tan^{-1}(x)$ function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. On the other hand, the value of angle φ can lie in any of the 4 quadrants depending on the values of l_1, l_2 and θ. The `arctan2` function in Numpy[1] library of Python implicitly outputs angles in the appropriate quadrant according to the $\pm$ values of the numerator and denominator. Thus, the values of q_1 and q_2 are obtained as

$$q_1 = \phi + \varphi \tag{14}$$

$$q_2 = q_1 - \theta \tag{15}$$

Dynamics

Let a force, $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$, act on the end effector. Let q_1 and q_2 be the independent variables. Therefore, work done by force $\vec{F}$ in moving dq_i

$$\delta W_F = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} dq_i \tag{16}$$

where, $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ is the position vector of the end effector. Work done by motors is given by

$$\delta W_i = \tau_i dq_i \tag{17}$$

The potential energy of the system is obtained as

$$V = m_1 g \left(\frac{l_1}{2}\right) \sin q_1 + m_2 g \left\{ l_1 \sin q_1 + \left(\frac{l_2}{2}\right) \sin q_2 \right\} \tag{18}$$

The kinetic energy of the system is obtained as

$$T = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right) \dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right) \dot{q}_2^2 + \frac{1}{2} m_2 v_2^2 \tag{19}$$

where, v_2 is the velocity of center of mass of link 2. It is given by

$$v_2^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2 \dot{q}_2}{2}\right)^2 + 2(l_1 \dot{q}_1)\left(\frac{l_2 \dot{q}_2}{2}\right) \cos(q_1 - q_2) \tag{20}$$

[1] Documentation available at https://numpy.org/doc/stable/reference/generated/numpy.arctan2.html

Applying Lagrange's equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + \tau_i \tag{21}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + \tau_i \tag{22}$$

Clearly, $\frac{\partial V}{\partial \dot{q}_i} = 0$. Calculating the other partial derivatives,

$$\frac{\partial V}{\partial q_1} = m_1 g \left(\frac{l_1}{2}\right) \cos q_1 + m_2 g l_1 \cos q_1 \tag{23}$$

$$\frac{\partial V}{\partial q_2} = m_2 g \left(\frac{l_2}{2}\right) \cos q_2 \tag{24}$$

$$\frac{\partial T}{\partial q_1} = -m_2 (l_1 \dot{q}_1) \left(\frac{l_2 \dot{q}_2}{2}\right) \sin(q_1 - q_2) \tag{25}$$

$$\frac{\partial T}{\partial q_2} = -m_2 (l_1 \dot{q}_1) \left(\frac{l_2 \dot{q}_2}{2}\right) \sin(q_2 - q_1) \tag{26}$$

$$\frac{\partial T}{\partial \dot{q}_1} = \left(\frac{1}{3} m_1 l_1^2\right) \dot{q}_1 + m_2 l_1^2 \dot{q}_1 + m_2 (l_1) \left(\frac{l_2 \dot{q}_2}{2}\right) \cos(q_1 - q_2) \tag{27}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \left(\frac{1}{3} m_1 l_1^2\right) \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 l_1 \left(\frac{l_2 \ddot{q}_2}{2}\right) \cos(q_1 - q_2) \\ - m_2 l_1 \left(\frac{l_2 \dot{q}_2}{2}\right) \sin(q_1 - q_2)\,(\dot{q}_1 - \dot{q}_2) \end{aligned} \tag{28}$$

$$\frac{\partial T}{\partial \dot{q}_2} = \left(\frac{1}{12} m_2 l_2^2\right) \dot{q}_2 + m_2 \left(\frac{l_2}{2}\right)^2 \dot{q}_2 + m_2 (l_1 \dot{q}_1) \left(\frac{l_2}{2}\right) \cos(q_1 - q_2) \tag{29}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = \left(\frac{1}{12} m_2 l_2^2\right) \ddot{q}_2 + m_2 \left(\frac{l_2}{2}\right)^2 \ddot{q}_2 + m_2 (l_1 \ddot{q}_1) \left(\frac{l_2}{2}\right) \cos(q_1 - q_2) \\ - m_2 (l_1 \dot{q}_1) \left(\frac{l_2}{2}\right) \sin(q_1 - q_2)\,(\dot{q}_1 - \dot{q}_2) \end{aligned} \tag{30}$$

Thus, the equations of motion are

$$\begin{aligned}\left(\frac{1}{3}m_1l_1^2\right)\ddot{q}_1 + m_2l_1^2\ddot{q}_1 + m_2l_1\left(\frac{l_2\ddot{q}_2}{2}\right)\cos(q_1-q_2) \\ - m_2l_1\left(\frac{l_2\dot{q}_2}{2}\right)\sin(q_1-q_2)\,(\dot{q}_1-\dot{q}_2) + m_2(l_1\dot{q}_1)\left(\frac{l_2\dot{q}_2}{2}\right)\sin(q_1-q_2) \\ + m_1g\left(\frac{l_1}{2}\right)\cos q_1 + m_2gl_1\cos q_1 = -F_xl_1\sin q_i + F_yl_1\cos q_1 + \tau_1\end{aligned} \tag{31}$$

$$\begin{aligned}\left(\frac{1}{12}m_2l_2^2\right)\ddot{q}_2 + m_2\left(\frac{l_2}{2}\right)^2\ddot{q}_2 + m_2(l_1\ddot{q}_1)\left(\frac{l_2}{2}\right)\cos(q_1-q_2) \\ - m_2(l_1\dot{q}_1)\left(\frac{l_2}{2}\right)\sin(q_1-q_2)\,(\dot{q}_1-\dot{q}_2) + m_2(l_1\dot{q}_1)\left(\frac{l_2\dot{q}_2}{2}\right)\sin(q_2-q_1) \\ + m_2g\left(\frac{l_2}{2}\right)\cos q_2 = -F_xl_2\sin q_2 + F_yl_2\cos q_2 + \tau_2\end{aligned} \tag{32}$$

Further simplifying,

$$\begin{aligned}\left(\frac{1}{3}m_1l_1^2\right)\ddot{q}_1 + m_2l_1^2\ddot{q}_1 + m_2l_1\left(\frac{l_2\ddot{q}_2}{2}\right)\cos(q_1-q_2) + m_2l_1\left(\frac{l_2\dot{q}_2^2}{2}\right)\sin(q_1-q_2) \\ + m_1g\left(\frac{l_1}{2}\right)\cos q_1 + m_2gl_1\cos q_1 = -F_xl_1\sin q_i + F_yl_1\cos q_1 + \tau_1\end{aligned} \tag{33}$$

$$\begin{aligned}\left(\frac{1}{3}m_2l_2^2\right)\ddot{q}_2 + m_2(l_1\ddot{q}_1)\left(\frac{l_2}{2}\right)\cos(q_1-q_2) - m_2(l_1\dot{q}_1^2)\left(\frac{l_2}{2}\right)\sin(q_1-q_2) \\ + m_2g\left(\frac{l_2}{2}\right)\cos q_2 = -F_xl_2\sin q_2 + F_yl_2\cos q_2 + \tau_2\end{aligned} \tag{34}$$

If mass of the links is ignored, the dynamics is simplified to

$$\tau_1 = F_xl_1\sin q_1 - F_yl_1\cos q_1 \tag{35}$$

$$\tau_2 = F_xl_2\sin q_2 - F_yl_2\cos q_2 \tag{36}$$

Expressing the same in the matrix form

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} l_1 \sin q_1 & -l_1 \cos q_1 \\ l_2 \sin q_2 & -l_2 \cos q_2 \end{pmatrix} \cdot \begin{pmatrix} F_x \\ F_y \end{pmatrix} \tag{37}$$

Statics

For the system to be in equilibrium with the external forces, $\ddot{q}_i = \dot{q}_i = 0$. Thus,

$$\tau_1 = m_1 g \left(\frac{l_1}{2}\right) \cos q_1 + m_2 g l_1 \cos q_1 + F_x l_1 \sin q_i - F_y l_1 \cos q_1 \tag{38}$$

$$\tau_2 = m_2 g \left(\frac{l_2}{2}\right) \cos q_2 + F_x l_2 \sin q_2 - F_y l_2 \cos q_2 \tag{39}$$