Exercise7

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1 Homework 7: Population Dynamics (20 points)

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In [1]: #Load standard libraries
        import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib inline
```

In this exercise we study the following equation for population dynamics:

$$\frac{dN}{dt} = rN(1 - N/K) - \frac{BN^2}{A^2 + N^2}, \tag{1}$$

where all parameters r, K, A and B are positive. It is a more complex example, in which the growth behaviour depends on whether N is smaller or larger than a critical populations size A.

1.0.1 1. Dimensional analysis: Determine the dimension of the parameters and rewrite the equation in dimensionless form. Note that there are different possibilities. Please formulate a dimensionless time τ that is not defined on the basis of r. Use $n = N/A$ as the dimensionless version of N.

Since the dimension of the left-hand side of the above equation is "number/time", one can notice that r has dimension "1/time", A and K have dimension "number" and B has dimension "number/time".

Dividing the above equation by B and using $n = N/A$ gives us:

$$\frac{dN}{Bdt} = \frac{rnA}{B}(1 - nA/K) - \frac{n^2}{1 + n^2}. \tag{2}$$

Defining the dimensionless time $\tau = Bt/A$, yields

$$\frac{dn}{d\tau} = \frac{rnA}{B}(1 - nA/K) - \frac{n^2}{1 + n^2}. \tag{3}$$

Now, one could also define the dimensionless parameters $a = rA/B$ and $b = K/A$ to obtain the nice and handy equation

$$\frac{dn}{d\tau} = an(1 - n/b) - \frac{n^2}{1 + n^2}. \tag{4}$$

1.0.2 2. Determine the stationary points n^* for $K/A = 7.3$. Note that for $n^* \neq 0$ these values are solutions of a cubic equation; it depends on n and the remaining free parameter. The cubic equation should be derived by yourself analytically; its zero points you can obtain numerically / graphically by using e.g. Mathematica. When do one or three real solutions exist as a function of the remaining free parameter?

(Hint: we do not ask for some analytical formula here! It is enough to vary the free parameter and check using Mathematica which three solutions for the stationary points you get; as stationary points only real solutions are valid. Only one digit after the comma is enough, in other words you vary the free parameter by about 0.05.). Which of the stationary points is stable and unstable?

The stationary points n^* fullfill the condition

$$\frac{dn}{d\tau}(n^*) = 0. \tag{5}$$

Thus, one solution is $n^* = 0$. If $n^* \neq 0$, we get

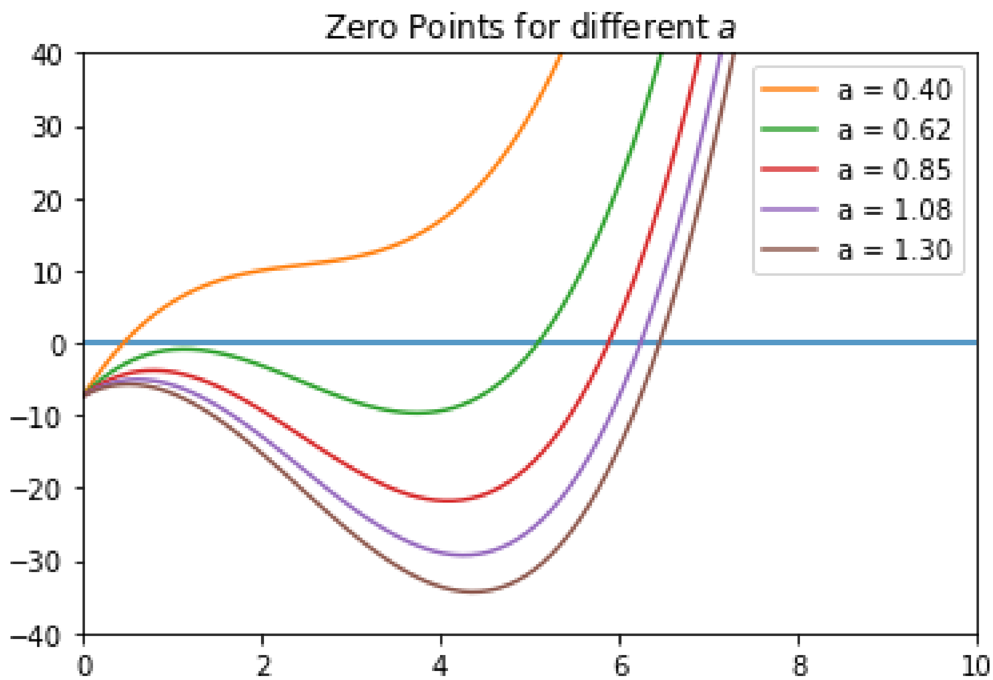
$$0 = a(1 - n/b) - \frac{n}{1 + n^2} \quad \Leftrightarrow \quad 0 = an^3 - abn^2 + (a + b)n - ab. \tag{6}$$

Setting $b = 7.3$, one can obtain solutions of this equation via the zero points of the function $f(n) = n^3 - 7.3n^2 + (1 + 7.3/a)n - 7.3$ (since $a \neq 0$).

```
In [27]: def f(n, a):
             return n**3 - 7.3*n**2 + (1+7.3/a)*n - 7.3

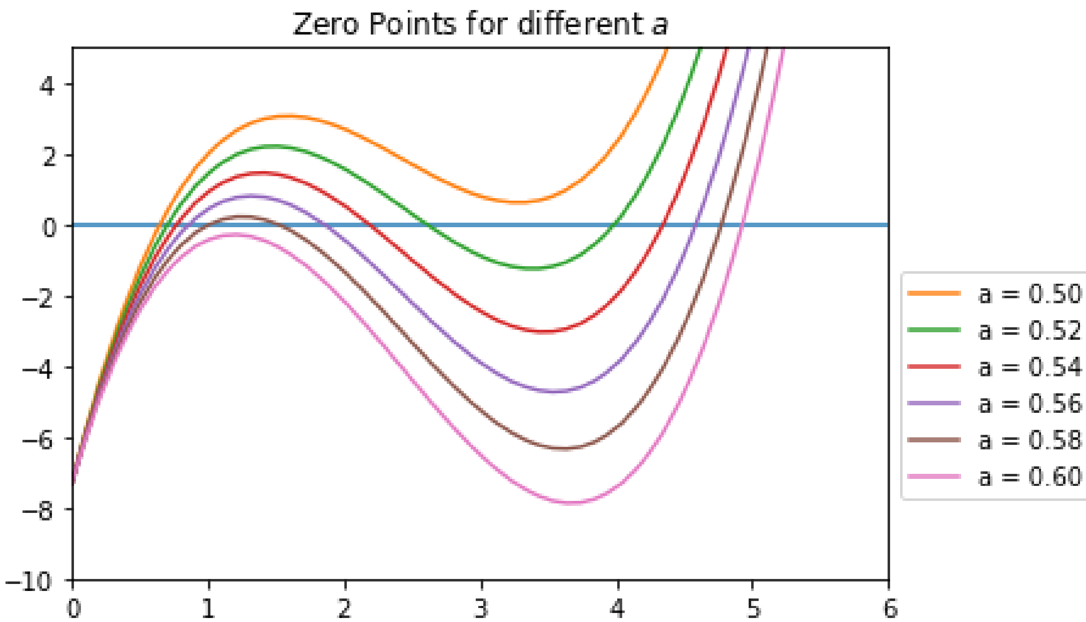
         n_range = np.linspace(0,10,100)

         plt.title(r"Zero Points for different $a$")
         plt.plot(n_range, np.zeros(len(n_range)))
         for a in np.linspace(0.4, 1.3, 5):
             plt.plot(n_range, f(n_range, a), label = "a = {:.2f}".format(a))
         plt.xlim(0, 10)
         plt.ylim(-40, 40)
         plt.legend(bbox_to_anchor=(1,1))
         plt.show()
```



So the relevant interval for a, where we get three solutions is for a between 0.5 and 0.6. For all other values of a, one gets only one solution.

In [39]:
```
plt.title(r"Zero Points for different $a$")
plt.plot(n_range, np.zeros(len(n_range)))
for a in np.linspace(0.5, 0.6, 6):
    plt.plot(n_range, f(n_range, a), label = "a = {:.2f}".format(a))
plt.xlim(0, 6)
plt.ylim(-10, 5)
plt.legend(bbox_to_anchor=(1,0.6))
plt.show()
```



Now, let's choose $a = 0.55$ and determine the solutions with Mathematica:

$$n_1^* = 0.809, \tag{7}$$
$$n_2^* = 2.015, \tag{8}$$
$$n_3^* = 4.475. \tag{9}$$

One can decide, which solution is a stable or unstable stationary point by looking at the derivative of the right-hand side of the differential equation at this stationary point:

$$g(a, b, n) = a\left(1 - \frac{2n}{b}\right) - \frac{2n}{(1+n^2)^2} \tag{10}$$

If this expression is negative/positive for a n^*, we have a stable/unstable stationary point. We see:

$$g(0.55, 7.3, n_1^*) < 0, \tag{11}$$
$$g(0.55, 7.3, n_2^*) > 0, \tag{12}$$
$$g(0.55, 7.3, n_3^*) < 0. \tag{13}$$

```
In [40]: 0.55*(1-2*0.809/7.3) - 2*0.809/(1+0.809**2)**2

Out[40]: -0.16299592930923185

In [41]: 0.55*(1-2*2.015/7.3) - 2*2.015/(1+2.015**2)**2
```

Out[41]: 0.08898411941927517

```
In [42]: 0.55*(1-2*4.475/7.3) - 2*4.475/(1+4.475**2)**2
```

Out[42]: -0.14456041464346583

Thus, n_1^* and n_3^* are stable stationary points and n_2^* is unstable.

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In [ ]:
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