

SpinDoctor User Guide

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The user is advised to download the latest version of the User Guide from <https://github.com/jingrebeccali/SpinDoctor>.

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1 Introduction

The MATLAB Toolbox SpinDoctor [1] is a simulation pipeline that

1. allows the user to define a geometrical configuration;
2. solves the Bloch-Torrey equation in that geometrical configuration;
3. fits the apparent diffusion coefficient from the simulated signal.

It includes two other methods for calculating the apparent diffusion coefficient:

1. The first is a homogenized apparent diffusion coefficient mathematical model, which was obtained recently using homogenization techniques on the Bloch-Torrey equation. In the homogenized model, the apparent diffusion coefficient of a geometrical configuration can be computed after solving a diffusion equation subject to a time-dependent Neumann boundary condition, under the assumption of negligible water exchange between compartments.
2. The second module computes the short time approximation formula for the apparent diffusion coefficient. The short time approximation implemented in SpinDoctor includes a recent generalization of this formula to account for finite pulse duration in the pulsed gradient spin echo.

Both of these two apparent diffusion coefficient calculations are sensitive to the diffusion-encoding gradient direction, unlike many previous works where the anisotropy is neglected in analytical model development.

In nutshell, SpinDoctor

1. solves the Bloch-Torrey equation in three dimensions to obtain the diffusion magnetic resonance imaging signal;
2. robustly fits the diffusion magnetic resonance imaging signal to obtain the apparent diffusion coefficient;
3. solves the homogenized apparent diffusion coefficient model in three dimensions to obtain the apparent diffusion coefficient;
4. computes the short-time approximation of the apparent diffusion coefficient;
5. computes useful geometrical quantities such as the compartment volumes and surface areas;
6. allows permeable membranes for the Bloch-Torrey equation (the homogenized apparent diffusion coefficient model assumes negligible permeability);
7. displays the gradient-direction dependent signal or apparent diffusion coefficient in three dimensions.

SpinDoctor provides the following built-in functionalities:

1. the placement of non-overlapping spherical cells (with an optional nucleus) of different radii close to each other;
2. the placement of non-overlapping cylindrical cells (with an optional myelin layer) of different radii close to each other in a canonical configuration where they are parallel to the z -axis;
3. the inclusion of an extra-cellular space that is enclosed either
 - (a) in a tight wrapping around the cells; or
 - (b) in a rectangular box.
4. the deformation of the canonical configuration by bending and twisting.

Built-in diffusion-encoding pulse sequences include

1. the Pulsed Gradient Spin Echo;
2. the Oscillating Gradient Spin Echo (cos- and sin- type gradients);
3. the Double Pulsed Gradient Spin Echo.

There is also support for custom diffusion-encoding pulse sequences.

SpinDoctor uses the following methods:

1. it generates a good quality surface triangulation of the user specified geometrical configuration by calling built-in MATLAB computational geometry functions;
2. it creates a good quality tetrahedra finite elements mesh from the above surface triangulation by calling Tetgen [2], an external package (executable files are included in the Toolbox package);
3. it constructs finite element matrices for linear finite elements on tetrahedra (P1 finite elements) using routines from [3];
4. it adds additional degrees of freedom on the compartment interfaces to allow permeability conditions for the Bloch-Torrey equation using the formalism in [4];
5. it solves the semi-discretized finite elements method equations by calling built-in MATLAB routines for solving ordinary differential equations.

Abbreviations

MRI - magnetic resonance imaging

dMRI - diffusion magnetic resonance imaging

ADC - apparent diffusion coefficient

HADC - homogenized ADC

PGSE - pulsed gradient spin echo

OGSE - oscillating gradient

ECS - extra-cellular space

BTPDE - Bloch-Torrey partial differential equation

PDE - partial differential equation

ODE - ordinary differential equation

HARDI - high angular resolution diffusion imaging

STA - short time approximation

FE - finite elements

FEM - finite element method

MF – matrix formalism

2 Installation

The SpinDoctor Toolbox has been tested with MATLAB R2019a-R2020b. The user is advised to download the latest version of the User Guide from <https://github.com/jingrebeccali/SpinDoctor>. Examples of drivers that run some typical simulations also can be found there.

The SpinDoctor Toolbox has been developed in the MATLAB R2020b and require no additional MATLAB toolboxes. SpinDoctor has support for parallel computations, if the MATLAB Parallel Computing Toolbox is available.

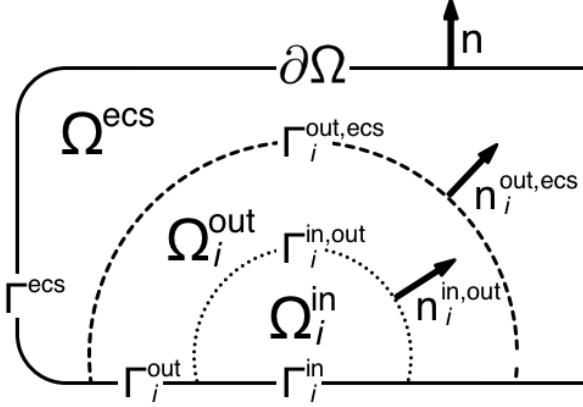


Figure 1: Domain notation, showing the ECS and the inner and outer compartments of the i -th cell, as well as their interfaces and boundaries. Note that this setup is also coherent for $\Omega_i^{\text{in}} = \emptyset$ or $\Omega^{\text{ecs}} = \emptyset$, the latter case requiring $N_{\text{cell}} = 1$.

3 Mathematical background

Suppose the user would like to simulate a geometrical configuration of cells with an optional myelin layer or a nucleus. If spins will be leaving the cells or if the user wants to simulate the extra-cellular space (ECS), then the ECS will enclose the geometrical shapes. Let Ω^{ecs} be the ECS, Ω_i^{in} the nucleus (or the axon) and Ω_i^{out} the cytoplasm (or the myelin layer) of the i th cell, $i = 1, \dots, N_{\text{cell}}$. Let $\Omega = \bigcup_{i=1}^{N_{\text{cell}}} \Omega_i^{\text{in}} \cup \Omega_i^{\text{out}} \cup \Omega^{\text{ecs}}$ denote the domain, and $\partial\Omega$ its outer boundary. Let $\Gamma_i^{\text{in}} = \partial\Omega \cap \Omega_i^{\text{in}}$, $\Gamma_i^{\text{out}} = \partial\Omega \cap \Omega_i^{\text{out}}$, and $\Gamma^{\text{ecs}} = \partial\Omega \cap \Omega^{\text{ecs}}$ denote the outer boundaries of the *in*, *out* and *ECS* compartments respectively. In addition, we denote $\Gamma_i^{\text{in},\text{out}} = \Omega_i^{\text{in}} \cap \Omega_i^{\text{out}}$ the interfaces between the *in* and *out* compartments, and $\Gamma_i^{\text{out},\text{ecs}} = \Omega_i^{\text{out}} \cap \Omega^{\text{ecs}}$ the interfaces between the *out* and *ECS* compartments. Finally we let $\Gamma = \bigcup_{i=1}^{N_{\text{cell}}} \Gamma_i^{\text{in},\text{out}} \cup \Gamma_i^{\text{out},\text{ecs}} \cup \Gamma_i^{\text{in}} \cup \Gamma_i^{\text{out}} \cup \Gamma^{\text{ecs}}$ denote the union of all interfaces and boundaries. These definitions are illustrated in Figure 1.

3.1 Bloch-Torrey PDE

In diffusion MRI, a time-varying magnetic field gradient is applied to the tissue to encode water diffusion. Denoting the effective time profile of the diffusion-encoding magnetic field gradient by $f : [0, T_{\text{echo}}] \rightarrow [-1, 1]$, and letting the vector \mathbf{g} contain the amplitude and direction information of the magnetic field gradient, the complex transverse water proton magnetization in the rotating frame satisfies the Bloch-Torrey PDE:

$$\frac{\partial}{\partial t} M(\mathbf{x}, t) = -i\gamma f(t) \mathbf{g} \cdot \mathbf{x} M(\mathbf{x}, t) + \nabla \cdot (\sigma(\mathbf{x}) \nabla M(\mathbf{x}, t)), \quad (\mathbf{x}, t) \in \Omega \times [0, T_{\text{echo}}], \quad (1)$$

where $\gamma = 2.67513 \times 10^8 \text{ rad s}^{-1} \text{T}^{-1}$ is the gyromagnetic ratio of the water proton, i is the imaginary unit, and σ is a scalar (isotropic) diffusion coefficient. SpinDoctor also supports diffusion tensors. The magnetization is a function of position \mathbf{x} and time t , and depends on the diffusion gradient vector \mathbf{g} and the time profile $f(t)$. The initial spin density is given by

$$M(\mathbf{x}, 0) = \rho(\mathbf{x}). \quad (2)$$

All the spatial-dependent quantities given above are defined compartment-wise:

$$M(\mathbf{x}, t) = \begin{cases} M_i^{\text{in}}(\mathbf{x}, t), & \mathbf{x} \in \Omega_i^{\text{in}} \\ M_i^{\text{out}}(\mathbf{x}, t), & \mathbf{x} \in \Omega_i^{\text{out}}, \\ M^{\text{ecs}}(\mathbf{x}, t), & \mathbf{x} \in \Omega^{\text{ecs}} \end{cases} \quad \rho(\mathbf{x}) = \begin{cases} \rho^{\text{in}}, & \mathbf{x} \in \Omega_i^{\text{in}} \\ \rho^{\text{out}}, & \mathbf{x} \in \Omega_i^{\text{out}}, \\ \rho^{\text{ecs}}, & \mathbf{x} \in \Omega^{\text{ecs}} \end{cases} \quad \sigma(\mathbf{x}) = \begin{cases} \sigma^{\text{in}}, & \mathbf{x} \in \Omega_i^{\text{in}} \\ \sigma^{\text{out}}, & \mathbf{x} \in \Omega_i^{\text{out}} \\ \sigma^{\text{ecs}}, & \mathbf{x} \in \Omega^{\text{ecs}} \end{cases}$$

The Bloch-Torrey PDE is also supplemented by boundary and interface conditions. We recall that Γ_i^{in} , Γ_i^{out} , and Γ_i^{ecs} denote the outer boundaries, while $\Gamma_i^{\text{in},\text{out}}$ and $\Gamma_i^{\text{out},\text{ecs}}$ denote the interfaces between compartments. Denoting $\llbracket a \rrbracket_m^n = a_n - a_m$, we define three types of boundary conditions:

- Flux continuity at interfaces (conservation of spins):

$$\begin{aligned} \llbracket \sigma \nabla M \rrbracket_{i,\text{in}}^{i,\text{out}} \cdot \mathbf{n}_i^{\text{in,out}} &= 0, \quad \text{on } \Gamma_i^{\text{in,out}} \\ \llbracket \sigma \nabla M \rrbracket_{i,\text{out}}^{\text{ecs}} \cdot \mathbf{n}_i^{\text{out,ecs}} &= 0, \quad \text{on } \Gamma_i^{\text{out,ecs}} \end{aligned} \quad (3)$$

- Interface permeability (water exchange between compartments):

$$\begin{aligned} \sigma^{\text{in}} \nabla M_i^{\text{in}} \cdot \mathbf{n}_i^{\text{in,out}} &= \kappa^{\text{in,out}} \llbracket M \rrbracket_{i,\text{in}}^{i,\text{out}}, \quad \text{on } \Gamma_i^{\text{in,out}} \\ \sigma^{\text{out}} \nabla M_i^{\text{out}} \cdot \mathbf{n}_i^{\text{out,ecs}} &= \kappa^{\text{out,ecs}} \llbracket M \rrbracket_{i,\text{out}}^{\text{ecs}}, \quad \text{on } \Gamma_i^{\text{out,ecs}} \end{aligned} \quad (4)$$

- Outer boundary conditions (water protons leaving the domain):

$$\begin{aligned} \sigma^{\text{in}} \nabla M_i^{\text{in}} \cdot \mathbf{n} &= -\kappa^{\text{in}} M_i^{\text{in}}, \quad \text{on } \Gamma_i^{\text{in}} \\ \sigma^{\text{out}} \nabla M_i^{\text{out}} \cdot \mathbf{n} &= -\kappa^{\text{out}} M_i^{\text{out}}, \quad \text{on } \Gamma_i^{\text{out}} \\ \sigma^{\text{ecs}} \nabla M^{\text{ecs}} \cdot \mathbf{n} &= -\kappa^{\text{ecs}} M^{\text{ecs}}, \quad \text{on } \Gamma^{\text{ecs}} \end{aligned} \quad (5)$$

Here, the coefficients $\kappa^{\text{in,out}}$, and $\kappa^{\text{out,ecs}}$ denote the interface permeabilities. In SpinDoctor, these coefficients are assumed to be strictly positive, otherwise there would not be any coupling between compartments. The coefficients κ^{in} , κ^{out} , and κ^{ecs} denote the outer boundary relaxivities (Robin boundary conditions). These coefficients may be set to zero, in which case we obtain reflective (Neumann) boundary conditions, where spins will not be entering or leaving the domain. In the following, we will put all these coefficients into the field κ defined on Γ :

$$\kappa(\mathbf{x}) = \begin{cases} \kappa^{\text{in,out}}, & \mathbf{x} \in \Gamma_i^{\text{in,out}} \\ \kappa^{\text{out,ecs}}, & \mathbf{x} \in \Gamma_i^{\text{out,ecs}} \\ \kappa^{\text{in}}, & \mathbf{x} \in \Gamma_i^{\text{in}} \\ \kappa^{\text{out}}, & \mathbf{x} \in \Gamma_i^{\text{out}} \\ \kappa^{\text{ecs}}, & \mathbf{x} \in \Gamma^{\text{ecs}} \end{cases}. \quad (6)$$

3.2 Diffusion MRI model

The diffusion MRI signal is measured at echo time $t = T_{\text{echo}}$. This signal is computed as the spatial integral of $M(\cdot, T_{\text{echo}})$:

$$S = \int_{\mathbf{x} \in \Omega} M(\mathbf{x}, T_{\text{echo}}) d\mathbf{x}, \quad (7)$$

and represents the output of the SpindDoctor diffusion MRI model. The model input is the gradient sequence, encoded by the amplitude and direction \mathbf{g} and effective time profile f . Some commonly used time profiles (diffusion-encoding sequences) are:

1. The pulsed-gradient spin echo (PGSE) [5] sequence, with two rectangular pulses of duration δ , separated by a time interval $\Delta - \delta$, for which the profile $f(t)$ is

$$f(t) = \begin{cases} 1, & t_1 \leq t \leq t_1 + \delta, \\ -1, & t_1 + \Delta < t \leq t_1 + \Delta + \delta, \\ 0, & \text{otherwise,} \end{cases} \quad (8)$$

where t_1 is the starting time of the first gradient pulse with $t_1 + \Delta > T_{\text{echo}}/2$.

2. The oscillating gradient spin echo (OGSE) sequence [6, 7] was introduced to reach short diffusion times. An OGSE sequence usually consists of two oscillating pulses of duration δ , each containing n periods, hence the frequency is $\omega = n \frac{2\pi}{\delta}$, separated by a time interval $\Delta - \delta$. For a cosine OGSE, the profile $f(t)$ is

$$f(t) = \begin{cases} \cos(n \frac{2\pi}{\delta} t), & t_1 < t \leq t_1 + \delta, \\ -\cos(n \frac{2\pi}{\delta} (t - \Delta)), & t_1 + \Delta < t \leq t_1 + \Delta + \delta, \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

where $\Delta = T_{\text{echo}}/2$.

By default, SpinDoctor assumes $t_1 = 0$ and $T_{\text{echo}} = \Delta + \delta$, meaning that ρ denotes the spin density just before the gradient sequence is applied and S is measured immediately after the end of the pulses. The user can however set up custom sequences taking into account the times before and after the pulses, or model the pulse transitions.

In a dMRI experiment, the pulsed sequence (time profile f) is usually fixed, while \mathbf{g} is varied in amplitude (and possibly also in direction). S is usually plotted against a quantity called the b -value. The b -value depends on \mathbf{g} and f and is defined as

$$b(\mathbf{g}, f) = \gamma^2 \|\mathbf{g}\|^2 \int_0^{T_{\text{echo}}} dt \left(\int_0^t f(s) ds \right)^2.$$

For PGSE, the b-value is [5]:

$$b(\mathbf{g}, \delta, \Delta) = \gamma^2 \|\mathbf{g}\|^2 \delta^2 (\Delta - \delta/3). \quad (10)$$

For the cosine OGSE with *integer* number of periods n in each of the two durations δ , the corresponding b -value is [8]:

$$b(\mathbf{g}, \delta) = \gamma^2 \|\mathbf{g}\|^2 \frac{\delta^3}{4n^2 \pi^2} = \gamma^2 \|\mathbf{g}\|^2 \frac{\delta}{\omega^2}. \quad (11)$$

The reason for these definitions is that in a homogeneous medium, the signal attenuation is $e^{-\sigma b}$, where σ is the intrinsic diffusion coefficient.

3.3 Fitting the ADC from the dMRI signal

An important quantity that can be derived from the dMRI signal is the “Apparent Diffusion Coefficient” (ADC), which gives an indication of the root mean squared distance travelled by water molecules in the gradient direction $\mathbf{g}/\|\mathbf{g}\|$, averaged over all starting positions:

$$\sigma_{\text{ADC}} = -\frac{\partial}{\partial b} \log \frac{S(b)}{S(0)} \Big|_{b=0}. \quad (12)$$

We numerically compute ADC by a polynomial fit of

$$\log \frac{S(b)}{S_{\text{initial}}} = c_0 + c_1 b + \cdots + c_n b^n,$$

increasing n from 1 onwards until we get the value of c_1 to be stable within a numerical tolerance, where $S_{\text{initial}} = \int_{\Omega} \rho(\mathbf{x}) d\mathbf{x}$. The first coefficient c_0 will be equal to $\log(S(0)/S_{\text{initial}})$, which is zero if ρ is constant.

The ADC may be interpreted as a correction to the intrinsic diffusion coefficient, taking into account the deviation from the free diffusion arising from interior interfaces, non-isotropic diffusion, fluid movements etc. For sufficiently small b-values, the signal attenuation is given by

$$e^{-\sigma_{\text{ADC}} b}.$$

3.4 HADC model

In a previous work [9], a PDE model for the time-dependent ADC was obtained starting from the Bloch-Torrey equation, using homogenization techniques. In the case of negligible water exchange between compartments (low permeability), there is no coupling between the compartments, at least to the quadratic order in \mathbf{g} , which is the ADC term. The ADC in compartment Ω is given by

$$\sigma_{\text{HADC}} = \sigma - \frac{\int_0^{T_{\text{echo}}} F(t) h(t) dt}{\int_0^{T_{\text{echo}}} F(t)^2 dt}, \quad (13)$$

where $F(t) = \int_0^t f(s) ds$, and

$$h(t) = \frac{1}{|\Omega|} \int_{\partial\Omega} \omega(\mathbf{x}, t) \mathbf{u}_g \cdot \mathbf{n}(\mathbf{x}) d\Gamma(\mathbf{x}) \quad (14)$$

is a quantity related to the directional gradient of a function ω that is the solution of the homogeneous diffusion equation with Neumann boundary condition and zero initial condition:

$$\begin{aligned}\frac{\partial}{\partial t}\omega(\mathbf{x}, t) &= \nabla \cdot (\sigma \nabla \omega(\mathbf{x}, t)), \quad \mathbf{x} \in \Omega, \\ \sigma \nabla \omega(\mathbf{x}, t) \cdot \mathbf{n} &= \sigma F(t) \mathbf{u}_g \cdot \mathbf{n}, \quad \mathbf{x} \in \partial\Omega, \\ \omega(\mathbf{x}, 0) &= 0, \quad \mathbf{x} \in \Omega,\end{aligned}\tag{15}$$

\mathbf{n} being the outward normal and $t \in [0, T_{\text{echo}}]$ and \mathbf{u}_g the unit gradient direction. The above set of equations, (13)-(15), comprise the homogenized model that we call the HADC model.

3.5 Short diffusion time approximation of the ADC

A well-known formula for the ADC in the short diffusion time regime is the following short time approximation (STA) [10, 11]:

$$\sigma_{\text{STA}} = \left(1 - \frac{4\sqrt{\sigma}}{3\sqrt{\pi}} \sqrt{\Delta} \frac{A}{dV}\right) \sigma,$$

where $\frac{A}{V}$ is the surface to volume ratio, d is the spatial dimension and σ is the intrinsic diffusion coefficient. In the above formula the pulse duration δ is assumed to be very small compared to Δ . A recent correction to the above formula [9], taking into account the finite pulse duration δ and the gradient direction \mathbf{u}_g , is the following:

$$\sigma_{\text{STA}} = \left(1 - \frac{4\sqrt{\sigma}}{3\sqrt{\pi}} C_{\delta, \Delta} \frac{A_{\mathbf{u}_g}}{V}\right) \sigma, \tag{16}$$

where

$$A_{\mathbf{u}_g} = \int_{\partial\Omega} (\mathbf{u}_g \cdot \mathbf{n}(\mathbf{x}))^2 d\Gamma(\mathbf{x}) = \mathbf{u}_g^T \mathbf{N} \mathbf{u}_g$$

with $\mathbf{N} = \int_{\partial\Omega} \mathbf{n}(\mathbf{x}) \mathbf{n}^T(\mathbf{x}) d\Gamma(\mathbf{x})$ and

$$C_{\delta, \Delta} = \frac{4}{35} \frac{(\Delta + \delta)^{7/2} + (\Delta - \delta)^{7/2} - 2(\delta^{7/2} + \Delta^{7/2})}{\delta^2(\Delta - \delta/3)} = \sqrt{\Delta} \left(1 + \frac{1}{3} \frac{\delta}{\Delta} - \frac{8}{35} \left(\frac{\delta}{\Delta}\right)^{3/2} + \dots\right).$$

When $\delta \ll \Delta$, the value $C_{\delta, \Delta}$ is approximately $\sqrt{\Delta}$.

3.6 Matrix Formalism signal representation

The Matrix Formalism [12, 13] representation of the solution to the Bloch-Torrey PDE is based on the Laplace eigenfunctions of the given geometrical configuration. These are the eigenfunctions of the Laplace operator $-\nabla \cdot \sigma \nabla$, which does not depend on the gradient sequence. The Matrix Formalism truncates the number of Laplace eigenmodes at an acceptable level, by setting a minimum length scale for the eigenfunctions. The gradient sequence dependent Bloch-Torrey operator can then be expressed in the truncated Laplace eigenfunction basis. This allows for a simple analytical expression which approximates the solution to the BTPDE down to an arbitrary length scale.

3.6.1 Eigenvalue decomposition of the Laplace operator

Let $(\phi_n, \lambda_n)_{n \in \mathbb{N}}$ be the L^2 -normalized eigenfunctions and eigenvalues associated to the Laplace operator in the whole domain $\Omega = \bigcup_{i=1}^{N_{\text{cell}}} \Omega_i^{\text{in}} \cup \Omega_i^{\text{out}} \cup \Omega^{\text{ecs}}$ satisfying

$$-\nabla \cdot (\sigma(\mathbf{x}) \nabla \phi(\mathbf{x})) = \lambda \phi(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{17}$$

where

$$\phi(\mathbf{x}) = \begin{cases} \phi^{i, \text{in}}(\mathbf{x}), & \mathbf{x} \in \Omega_i^{\text{in}} \\ \phi^{i, \text{out}}(\mathbf{x}), & \mathbf{x} \in \Omega_i^{\text{out}} \\ \phi^{\text{ecs}}(\mathbf{x}), & \mathbf{x} \in \Omega^{\text{ecs}} \end{cases}. \tag{18}$$

The boundary and interface conditions are the same as for the BTPDE:

$$[\![\sigma \nabla \phi]\!]_{i,\text{in}}^{i,\text{out}} \cdot \mathbf{n}_i^{\text{in,out}} = 0, \quad \text{on } \Gamma_i^{\text{in,out}} \quad (19)$$

$$[\![\sigma \nabla \phi]\!]_{i,\text{out}}^{\text{ecs}} \cdot \mathbf{n}_i^{\text{out,ecs}} = 0, \quad \text{on } \Gamma_i^{\text{out,ecs}} \quad (20)$$

$$\sigma^{\text{in}} \nabla \phi_i^{\text{in}} \cdot \mathbf{n}_i^{\text{in,out}} = \kappa^{\text{in,out}} [\![\phi]\!]_{i,\text{in}}^{i,\text{out}}, \quad \text{on } \Gamma_i^{\text{in,out}} \quad (21)$$

$$\sigma^{\text{out}} \nabla \phi_i^{\text{out}} \cdot \mathbf{n}_i^{\text{out,ecs}} = \kappa^{\text{out,ecs}} [\![\phi]\!]_{i,\text{out}}^{\text{ecs}}, \quad \text{on } \Gamma_i^{\text{out,ecs}} \quad (22)$$

$$\sigma \nabla \phi \cdot \mathbf{n} = -\kappa \phi, \quad \text{on } \partial \Omega. \quad (23)$$

We assume the non-negative real-valued eigenvalues are ordered in non-decreasing order:

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

so that $\lambda_1 = 0$ (this means the first Laplace eigenfunction is the constant function). There is only one constant eigenfunction because for simplicity, we assume that the domain Ω is connected and also coupled through the cell membranes ($\kappa^{\text{in,out}} > 0$ and $\kappa^{\text{out,ecs}} > 0$). If there is no coupling between the compartments, the problem may be reformulated for each compartment separately.

Let \mathbf{L} be the diagonal matrix containing the first N_{eig} Laplace eigenvalues:

$$\mathbf{L} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N_{\text{eig}}}) \in \mathbb{R}^{N_{\text{eig}} \times N_{\text{eig}}}. \quad (24)$$

The matrix \mathbf{L} represents the Laplace operator $-\nabla \cdot \sigma \nabla$ in the truncated Laplace eigenfunction basis.

3.6.2 Bloch-Torrey operator in the Laplace eigenfunction basis

Let $\mathbf{A}(\mathbf{g})$ be the $N_{\text{eig}} \times N_{\text{eig}}$ matrix defined by:

$$\mathbf{A}(\mathbf{g}) = g_x \mathbf{A}^x + g_y \mathbf{A}^y + g_z \mathbf{A}^z, \quad (25)$$

where $\mathbf{g} = (g_x, g_y, g_z)^\top$ is the gradient vector and \mathbf{A}^x , \mathbf{A}^y , and \mathbf{A}^z are three symmetric $N_{\text{eig}} \times N_{\text{eig}}$ matrices whose entries are the first order moments in the coordinate directions of the product of pairs of eigenfunctions:

$$\begin{aligned} A_{mn}^x &= \int_{\Omega} x \phi_m(\mathbf{x}) \phi_n(\mathbf{x}) d\mathbf{x}, \\ A_{mn}^y &= \int_{\Omega} y \phi_m(\mathbf{x}) \phi_n(\mathbf{x}) d\mathbf{x}, \quad 1 \leq m, n \leq N_{\text{eig}}. \\ A_{mn}^z &= \int_{\Omega} z \phi_m(\mathbf{x}) \phi_n(\mathbf{x}) d\mathbf{x}, \end{aligned} \quad (26)$$

Then the general time dependent Bloch-Torrey operator

$$-\nabla \cdot \sigma \nabla + i\gamma f(t) \mathbf{g} \cdot \mathbf{x}, \quad t \in [0, T_{\text{echo}}]$$

in the truncated Laplace eigenfunction basis $(\phi_j)_{1 \leq j \leq N_{\text{eig}}}$ is given by the complex-valued matrix

$$\mathbf{K}(\mathbf{g}, f)(t) = \mathbf{L} + i\gamma f(t) \mathbf{A}(\mathbf{g}), \quad t \in [0, T_{\text{echo}}]. \quad (27)$$

For a constant time profile (PGSE, double-PGSE), the Bloch-Torrey operator

$$-\nabla \cdot \sigma \nabla + i\gamma \mathbf{g} \cdot \mathbf{x}$$

is constant. In the truncated Laplace eigenfunction basis, it is given by the complex-valued matrix

$$\mathbf{K}(\mathbf{g}) = \mathbf{L} + i\gamma \mathbf{A}(\mathbf{g}). \quad (28)$$

3.6.3 Matrix Formalism approximation

Denoting $\phi = (\phi_1, \dots, \phi_{N_{\text{eig}}})^T$ the vector of the first N_{eig} Laplace eigenfunctions, we may project the solution to the Bloch-Torrey PDE (1) onto the truncated Laplace eigenfunction basis:

$$M^{\text{MF}}(\mathbf{x}, t) = \phi^T(\mathbf{x}) \boldsymbol{\nu}(t), \quad (29)$$

where the vector of coefficients $\boldsymbol{\nu} = (\nu_1, \dots, \nu_{N_{\text{eig}}})^T \in \mathbb{C}^{N_{\text{eig}}}$ is the solution to

$$\frac{\partial \boldsymbol{\nu}}{\partial t} = -\mathbf{K}(\mathbf{g}, f)(t) \boldsymbol{\nu}(t), \quad t \in [0, T_{\text{echo}}]. \quad (30)$$

The initial coefficients are given by

$$\boldsymbol{\nu}(0) = \int_{\Omega} \rho(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}. \quad (31)$$

By using a piece-wise constant approximation [13] of the time profile f , we obtain

$$\boldsymbol{\nu}(T_{\text{echo}}) \approx \left(\prod_{i=1}^{N_{\text{int}}} e^{-\delta_i \mathbf{K}_i} \right) \boldsymbol{\nu}(0) = e^{-\delta_{N_{\text{int}}} \mathbf{K}_{N_{\text{int}}}} \dots e^{-\delta_2 \mathbf{K}_2} e^{-\delta_1 \mathbf{K}_1} \boldsymbol{\nu}(0), \quad (32)$$

where $\{\mathcal{I}_i\}_{i=1, \dots, N_{\text{int}}}$ are intervals such that $[0, T_{\text{echo}}] = \bigcup_{i=1}^{N_{\text{int}}} \mathcal{I}_i$, $f(t) = f_i$ for $t \in \mathcal{I}_i$, $\delta_i = |\mathcal{I}_i|$, and $\mathbf{K}_i(\mathbf{g}) = \mathbf{L} + i\gamma f_i \mathbf{A}(\mathbf{g})$. The constants may be computed through quadrature:

$$f_i = \frac{1}{\delta_i} \int_{\mathcal{I}_i} f(t) dt \approx \frac{1}{2} (f(\min \mathcal{I}_i) + f(\max \mathcal{I}_i)). \quad (33)$$

For the PGSE and double-PGSE sequences, the coefficients of the final magnetization are given by

$$\boldsymbol{\nu}(T_{\text{echo}}) = e^{-\delta \mathbf{K}^*} e^{-(\Delta-\delta) \mathbf{L}} e^{-\delta \mathbf{K}} \boldsymbol{\nu}(0) \quad (34)$$

and

$$\boldsymbol{\nu}(T_{\text{echo}}) = e^{-\delta \mathbf{K}^*} e^{-(\Delta-\delta) \mathbf{L}} e^{-\delta \mathbf{K}} e^{-\delta \mathbf{K}^*} e^{-(\Delta-\delta) \mathbf{L}} e^{-\delta \mathbf{K}} \boldsymbol{\nu}(0) \quad (35)$$

respectively. These are the exact solutions to Eq. (30), although there may still be truncation errors from N_{eig} . We note that Eq. (34) and Eq. (35) are particular cases of Eq. (32), with ($N_{\text{int}} = 3$, $\mathcal{I}_1 = [0, \delta]$, $\mathcal{I}_2 = [\delta, \Delta]$, $\mathcal{I}_3 = [\Delta, \Delta + \delta]$) and ($N_{\text{int}} = 6$, $\mathcal{I}_1 = [0, \delta]$, $\mathcal{I}_2 = [\delta, \Delta]$, $\mathcal{I}_3 = [\Delta, \Delta + \delta]$, $\mathcal{I}_4 = [\Delta + \delta, \Delta + 2\delta]$, $\mathcal{I}_5 = [\Delta + 2\delta, 2\Delta + \delta]$, $\mathcal{I}_6 = [2\Delta + \delta, 2\Delta + 2\delta]$) respectively.

We may also note that the matrix \mathbf{L} in the exponent is diagonal, and in this case, the matrix exponential is also diagonal. The notation $*$ denotes the matrix complex conjugate transpose. In order to calculate the non-diagonal matrix exponential $e^{-\delta \mathbf{K}}$ and its conjugate $e^{-\delta \mathbf{K}^*} = (e^{-\delta \mathbf{K}})^*$, the scaling and squaring method [14, 15] is used (the built-in matrix exponential function in MATLAB, `expm`, is called). However, for the purpose of theoretical analysis, we may explicitly diagonalize the matrix $\mathbf{K}(\mathbf{g})$:

$$\mathbf{K}(\mathbf{g}) = \mathbf{V} \mathbf{B} \mathbf{V}^{-1}, \quad (36)$$

where \mathbf{V} has the eigenvectors in the columns and \mathbf{B} has the eigenvalues on the diagonal. Then $e^{-\delta \mathbf{K}} = \mathbf{V} e^{-\delta \mathbf{B}} \mathbf{V}^{-1}$ and $e^{-\delta \mathbf{K}^*} = (\mathbf{V}^{-1})^* e^{-\delta \mathbf{B}^*} \mathbf{V}^*$.

The signal is given by

$$S^{\text{MF}}(f, \mathbf{g}) = \sqrt{|\Omega|} \nu_1(T_{\text{echo}}) \quad (37)$$

where ν_1 is the first entry of $\boldsymbol{\nu}$, as $\int_{\Omega} \phi(\mathbf{x}) d\mathbf{x} = (\sqrt{|\Omega|}, 0, \dots, 0)^T$. Only the first Laplace component remains after integrating, since the first eigenfunction is constant, and the other eigenfunctions are orthogonal to ϕ_1 .

3.6.4 Apparent diffusion coefficient

From the Matrix Formalism signal, the analytical expression of its ADC for a gradient direction given by a unit vector \mathbf{u}_g and a sequence f is the following:

$$\sigma_{\text{ADC}}^{\text{MF}}(\mathbf{u}_g, f) = \sigma \sum_{n=1}^{N_{\text{eig}}} \frac{(\mathbf{u}_g \cdot \mathbf{a}_{1n})^2 \lambda_n \int_0^{T_{\text{echo}}} F(t) \left(\int_0^t e^{-\sigma \lambda_n (t-s)} f(s) ds \right) dt}{\int_0^{T_{\text{echo}}} F^2(t) dt},$$

where the coefficients $\mathbf{a}_{1n} = [A_{1n}^x, A_{1n}^y, A_{1n}^z]^\top$ are given by

$$\begin{aligned} A_{1n}^x &= \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} x \phi_n(\mathbf{x}) d\mathbf{x}, \\ A_{1n}^y &= \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} y \phi_n(\mathbf{x}) d\mathbf{x}, \\ A_{1n}^z &= \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} z \phi_n(\mathbf{x}) d\mathbf{x}. \end{aligned} \quad (38)$$

We remind the reader that the first Laplace eigenfunction is the constant function.

To clarify the relationship between the ADC and the diffusion encoding direction \mathbf{u}_g , we rewrite the Matrix Formalism ADC as

$$\sigma_{\text{ADC}}^{\text{MF}}(\mathbf{u}_g, f) = \mathbf{u}_g^\top \mathbf{D}^{\text{MF}}(f) \mathbf{u}_g, \quad (39)$$

where the Matrix Formalism effective diffusion tensor is given by

$$\mathbf{D}^{\text{MF}}(f) = \sigma \sum_{n=1}^{N_{\text{eig}}} J(\lambda_n, f) \mathbf{a}_{1n} \mathbf{a}_{1n}^\top, \quad (40)$$

with $J(\lambda_n, f)$ depending on λ_n and f :

$$J(\lambda_n, f) = \frac{\lambda_n \int_0^{T_{\text{echo}}} F(t) \left(\int_0^t e^{-\sigma \lambda_n (t-s)} f(s) ds \right) dt}{\int_0^{T_{\text{echo}}} F^2(t) dt}. \quad (41)$$

In the case of a constant initial spin density ρ , we also allow for computing the Matrix Formalism Gaussian Approximation (MFGA) signal, given as

$$S^{\text{MFGA}}(\mathbf{g}, f) = \rho |\Omega| \exp(-\mathbf{u}_g^\top \mathbf{D}^{\text{MF}}(f) \mathbf{u}_g b(\|\mathbf{g}\|, f)), \quad \mathbf{u}_g = \frac{\mathbf{g}}{\|\mathbf{g}\|}. \quad (42)$$

3.6.5 Eigenfunction length scale and orientation

The analytical Laplace eigenvalues $\lambda_{\{1,2,\dots\}}$ of a line segment of length L with diffusivity σ are

$$\lambda_n = \left(\frac{\pi(n-1)}{L} \right)^2 \sigma, \quad n = 1, 2, \dots \quad (43)$$

To make the link between the computed eigenvalue and the spatial scale of the eigenmode, we will convert the computed λ_n into a length scale:

$$\ell(\lambda) = \pi \sqrt{\frac{\bar{\sigma}}{\lambda}}, \quad (44)$$

and characterize the computed eigenmode by $\ell(\lambda_n)$ instead of λ_n . The reference diffusivity $\bar{\sigma}$ is taken as a volume weighted mean of σ^{in} , σ^{out} , and σ^{ecs} :

$$\bar{\sigma} = \frac{1}{|\Omega|} \int_{\Omega} \sigma(\mathbf{x}) d\mathbf{x} \quad (45)$$

To characterize the directional contribution of the eigenmode we use the fact that its contribution to the ADC in the direction \mathbf{u}_g is $J(\lambda_n, f)(\mathbf{u}_g \cdot \mathbf{a}_{1n})^2$. We thus call $\mathbf{a}_{1n} = (A_{1n}^x, A_{1n}^y, A_{1n}^z)^\top$ the “diffusion direction” of the n th eigenmode. We remind that the three components of \mathbf{a}_{1n} are the first moments in the 3 canonical basis directions of the associated eigenfunction.

3.7 Finite element discretization

In SpinDoctor, the finite element space is the space of continuous piecewise linear functions on tetrahedral elements in three dimensions (known as P_1). The tetrahedral elements are defined by

1. a list of N_{node} nodes in three dimensions: $(\mathbf{q}_1, \dots, \mathbf{q}_{N_{\text{node}}}) = (\mathbf{q}^x, \mathbf{q}^y, \mathbf{q}^z)^T \in \mathbb{R}^{3 \times N_{\text{node}}}$;
2. a list of N_{element} tetrahedral elements ($4 \times N_{\text{element}}$ indices referencing the nodes).

The list of nodes includes double nodes that are placed at the interfaces between compartments connected by permeable membranes [16]. From the list of elements it is possible to extract a list of facets, including double facets at the interfaces between connected compartments.

The finite element space has a set of basis functions whose number is exactly the number of finite element nodes (including ghost nodes):

$$\varphi_k(\mathbf{x}), \quad k = 1, \dots, N_{\text{node}}.$$

The basis function $\varphi_k(\mathbf{x})$ is a piece-wise linear function, non-zero on the tetrahedra that touch the node \mathbf{q}_k , and zero on all other tetrahedra. On a tetrahedron that touches \mathbf{q}_k , $\varphi_k(\mathbf{x})$ is equal to 1 on \mathbf{q}_k and it is equal to 0 on the other 3 vertices of the tetrahedron. This completely describes the piece-wise linear function. Any function in the finite element space can be written as a linear combination of the above basis functions:

$$\sum_{k=1}^{N_{\text{node}}} c_k \varphi_k(\mathbf{x}).$$

To discretize the Bloch-Torrey and Laplace operators, we construct the following finite element matrices: $\mathbf{M}, \mathbf{S}, \mathbf{Q} \in \mathbb{R}^{N_{\text{node}} \times N_{\text{node}}}$, known in the FEM literature as the mass, stiffness, and flux matrices, respectively. The three finite element matrices we need are defined as follows, for $1 \leq j, k \leq N_{\text{node}}$:

$$M_{jk} = \int_{\Omega} \varphi_j(\mathbf{x}) \varphi_k(\mathbf{x}) d\mathbf{x}, \quad (46)$$

$$S_{jk} = \int_{\Omega} \sigma(\mathbf{x}) \nabla \varphi_j(\mathbf{x}) \cdot \nabla \varphi_k(\mathbf{x}) d\mathbf{x}, \quad (47)$$

$$Q_{jk} = w_{jk} \int_{\Gamma} \kappa(\mathbf{x}) \varphi_j(\mathbf{x}) \varphi_k(\mathbf{x}) d\Gamma(\mathbf{x}), \quad w_{jk} = \begin{cases} 1 & \text{if } j, k \text{ in same compartment,} \\ -1 & \text{if } j, k \text{ in different compartments.} \end{cases} \quad (48)$$

We remind the reader that $\Gamma = \bigcup_{i=1}^{N_{\text{cell}}} \Gamma_i^{\text{in,out}} \cup \Gamma_i^{\text{out,ecs}} \cup \Gamma_i^{\text{in}} \cup \Gamma_i^{\text{out}} \cup \Gamma^{\text{ecs}}$ is the union of all interfaces and boundaries.

In SpinDoctor, these matrices are assembled from local element matrices and the assembly process is based on vectorized routines of [3], which replace expensive loops over elements by operations with 3-dimensional arrays. All local element matrices in the assembly of \mathbf{S} , \mathbf{M} and \mathbf{Q} are evaluated at the same time and stored in a full matrix of size $4 \times 4 \times N_{\text{element}}$, where N_{element} denotes the number of tetrahedral elements.

The weak form of the Bloch-Torrey PDE also requires computing the first order moments of the product of pairs of finite element basis functions. We denote these three matrices \mathbf{J}^x , \mathbf{J}^y and \mathbf{J}^z , and they are given by:

$$J_{jk}^u = \int_{\Omega} u(\mathbf{x}) \varphi_j(\mathbf{x}) \varphi_k(\mathbf{x}) d\mathbf{x}, \quad u = x, y, z. \quad (49)$$

For a given gradient vector $\mathbf{g} = (g_x, g_y, g_z)^T$, we denote

$$\mathbf{J}(\mathbf{g}) = g_x \mathbf{J}^x + g_y \mathbf{J}^y + g_z \mathbf{J}^z. \quad (50)$$

The matrices \mathbf{J}^x , \mathbf{J}^y , and \mathbf{J}^z are assembled as coordinate weighted mass matrices, using the same assembly procedure as for \mathbf{M} . The three coordinate functions $\mathbf{x} \mapsto x, y, z$ are replaced by constant (P_0) finite element functions on each tetrahedron. The weights given to the assembly assembly function are given by $\mathbf{q}^x, \mathbf{q}^y, \mathbf{q}^z$; the vectors of x, y , and z coordinates of the finite element nodes. The matrix $\mathbf{J}(\mathbf{g})$ can thus quickly be obtained for different gradient vectors \mathbf{g} , without having to reassemble the entire matrix.

3.7.1 Finite element solution to the Bloch-Torrey PDE

The solution to the Bloch-Torrey partial differential equation (1) may be projected onto the finite element nodal basis $(\varphi_j)_{1 \leq j \leq N_{\text{node}}}$, in which case it is given by

$$M(\mathbf{x}, t) = \sum_{j=1}^{N_{\text{node}}} \xi_j(t) \varphi_j(\mathbf{x}) = \boldsymbol{\xi}^T(t) \boldsymbol{\varphi}(\mathbf{x}), \quad (51)$$

where $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{N_{\text{node}}})^T \in \mathbb{R}^{N_{\text{node}}}$ is the vector of finite element basis functions and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{N_{\text{node}}})^T \in \mathbb{C}^{N_{\text{node}}}$ is the solution to the following system of *ordinary* differential equations (ODE):

$$\mathbf{M} \frac{\partial \boldsymbol{\xi}}{\partial t} = -(\mathrm{i}\gamma f(t)\mathbf{J} + \mathbf{S} + \mathbf{Q}) \boldsymbol{\xi}(t). \quad (52)$$

We note that the notation ^T in Eq. (51) only denotes the *transpose*, as opposed to the complex conjugate transpose *. The complex transverse water proton magnetization M and thus its coefficients $\boldsymbol{\xi}$ may contain complex values. This is not the case for the finite element basis functions $\boldsymbol{\varphi}$, which are real-valued.

3.7.2 Finite element solution to the HADC model

Similarly, the solution to the HADC model (15), ω , may be obtained by solving the equation

$$\mathbf{M} \frac{\partial \boldsymbol{\zeta}}{\partial t} = -\mathbf{S} \boldsymbol{\zeta}(t) + \sigma F(t) \mathbf{G} \mathbf{u}_g, \quad (53)$$

where $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{N_{\text{node}}})$ is the unknown vector of coefficients of the solution $\omega(\mathbf{x}, t) = \boldsymbol{\zeta}^T(t) \boldsymbol{\varphi}(\mathbf{x})$, $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{N_{\text{node}}})$ is the vector of finite element basis functions, and $\mathbf{G} = (\mathbf{G}^x, \mathbf{G}^y, \mathbf{G}^z) \in \mathbb{R}^{N_{\text{node}} \times 3}$. The column vectors are given by

$$\mathbf{G}^u = \sum_{k=1}^{N_{\text{node}}} \int_{\Omega} n_u(\mathbf{x}) \boldsymbol{\varphi}(\mathbf{x}) \varphi_k(\mathbf{x}) d\Gamma(\mathbf{x}), \quad u = x, y, z, \quad (54)$$

where $\mathbf{n} = (n_x, n_y, n_z)^T$ is the outwards unit normal. These column vectors represent the components of the boundary integral of the quantity $\sigma F(t) \mathbf{u}_g \cdot \mathbf{n}$, where the constant diffusivity and gradient sequence dependent part $\sigma F(t) \mathbf{u}_g$ has been factored out. They can thus be assembled independently of the gradient sequence, and be reused when solving for multiple sequences or directions. They are assembled using the same routine as for \mathbf{Q} .

The boundary integral h from Eq. (14) is then given by

$$h(t) = \frac{1}{|\Omega|} \boldsymbol{\zeta}^T(t) \mathbf{G} \mathbf{u}_g, \quad (55)$$

where $|\Omega| = \sum_{j,k=1}^{N_{\text{node}}} M_{jk}$ is computed using the mass matrix.

3.7.3 Finite element Matrix Formalism signal representation

Both of the above ODEs are of dimension N_{node} , the number of finite element nodes. The finite element discretized Matrix Formalism representation uses a different approach, limiting the problem size to N_{eig} , the number of Laplace eigenfunctions (of which the choice is further explored in section 3.7.4). But first we need to solve an eigenvalue problem involving matrices of size $N_{\text{node}} \times N_{\text{node}}$. The finite element discretization changes the continuous Laplace operator eigenvalue problem (17) into the following discrete, generalized *matrix* eigenvalue problem:

$$\begin{aligned} & \text{find } (\lambda_n, \mathbf{p}_n), \quad 1 \leq n \leq N_{\text{eig}}, \text{ such that} \\ & \lambda_n \mathbf{M} \mathbf{p}_n = (\mathbf{S} + \mathbf{Q}) \mathbf{p}_n, \quad \mathbf{p}_n \in \mathbb{R}^{N_{\text{node}}}, \end{aligned} \quad (56)$$

where \mathbf{p}_n is the eigenvector (of N_{node} entries) associated to the eigenvalue λ_n . Moving back to the space of functions (the function space P_1), the eigenfunction $\phi_n(\mathbf{x})$ associated to the eigenvalue λ_n is then

$$\phi_n(\mathbf{x}) = \sum_{j=1}^{N_{\text{node}}} p_n^j \varphi_j(\mathbf{x}), \quad 1 \leq n \leq N_{\text{eig}},$$

where the entries of the eigenvector $\mathbf{p}_n = (p_n^1, \dots, p_n^{N_{\text{node}}})^\top$ are the coefficients of the eigenfunction ϕ_n in the finite element basis $(\varphi_j)_{1 \leq j \leq N_{\text{node}}}$. Using matrix notation, this conversion can also be written

$$\boldsymbol{\phi} = \mathbf{P}^\top \boldsymbol{\varphi}, \quad (57)$$

where $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{N_{\text{node}}})^\top$, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_{N_{\text{eig}}})^\top$ and $\mathbf{P} = (\mathbf{p}_1, \dots, \mathbf{p}_{N_{\text{eig}}}) \in \mathbb{R}^{N_{\text{node}} \times N_{\text{eig}}}$.

Then it is clear that the first order moments of the product of pairs of Laplace eigenfunctions can be written as:

$$\begin{aligned} A_{mn}^x &= \sum_{j=1}^{N_{\text{node}}} \sum_{k=1}^{N_{\text{node}}} p_m^j p_n^k \left(\int_{\Omega} x \varphi_j(\mathbf{x}) \varphi_k(\mathbf{x}) d\mathbf{x} \right) = \mathbf{p}_m^\top \mathbf{J}^x \mathbf{p}_n, \\ A_{mn}^y &= \sum_{j=1}^{N_{\text{node}}} \sum_{k=1}^{N_{\text{node}}} p_m^j p_n^k \left(\int_{\Omega} y \varphi_j(\mathbf{x}) \varphi_k(\mathbf{x}) d\mathbf{x} \right) = \mathbf{p}_m^\top \mathbf{J}^y \mathbf{p}_n, \quad 1 \leq m, n \leq N_{\text{eig}}. \\ A_{mn}^z &= \sum_{j=1}^{N_{\text{node}}} \sum_{k=1}^{N_{\text{node}}} p_m^j p_n^k \left(\int_{\Omega} z \varphi_j(\mathbf{x}) \varphi_k(\mathbf{x}) d\mathbf{x} \right) = \mathbf{p}_m^\top \mathbf{J}^z \mathbf{p}_n, \end{aligned} \quad (58)$$

Using matrix notation, we have

$$\mathbf{A}^u = \int_{\Omega} u(\mathbf{x}) \boldsymbol{\phi}(\mathbf{x}) \boldsymbol{\phi}^\top(\mathbf{x}) d\mathbf{x} = \mathbf{P}^\top \mathbf{J}^u \mathbf{P}, \quad u = x, y, z. \quad (59)$$

These three matrices are computed using a total of six matrix-matrix multiplications.

The generalized eigenvalue problem in (56) is solved using the built-in MATLAB command `eigs`, which can exploit the symmetry of \mathbf{M} and compute a subset of all the eigenvalues, meaning N_{eig} can be much smaller than N_{node} . In earlier versions of SpinDoctor, we depended on the Partial Differential Equation Toolbox of MATLAB to solve the generalized eigenvalue problem, but this is no longer needed in our current implementation.

3.7.4 Choice of eigen-decomposition parameters

We do not want to use the entire set of eigenvalues and eigenvectors $\{\lambda_n, \mathbf{p}_n\}$, $1 \leq n \leq N_{\text{node}}$ of the matrix eigenvalue problem in (56), because the size of \mathbf{M} , \mathbf{S} and \mathbf{Q} is determined by the finite element discretization (it is equal to N_{node} , the number of finite element nodes). This means that most of the rapidly oscillating eigenmodes in the matrix eigenvalue problem are linked to the finite element discretization, and not to the physics of the problem. To link with the physics of the diffusion in the cell geometry, we set a restricted interval in which to keep the computed eigenvalues. We set the interval to be $[0, (\pi/\ell_{\min})^2 \sigma]$, where ℓ_{\min} is the shortest length scale of interest in the cell geometry. In this way, the number of computed eigenmodes, N_{eig} , will be much smaller than N_{node} .

The MATLAB command `eigs` can identify the N_{eig} smallest eigenvalues of the problem in (56). If we choose this number to be large enough such that $\ell(\lambda_{N_{\text{eig}}}) \leq \ell_{\min}$, we can be sure to have found all the modes of interest. Ideally, we thus take

$$N_{\text{eig}} = \min\{n \in \mathbb{N} \mid \ell(\lambda_n) \leq \ell_{\min}\}.$$

In order to choose the number N_{eig} , we can either make a conservative estimate (of the order of N_{node}), and then remove the largest eigenvalues, or we can start out with a smaller first guess for N_{eig} and increment it if the smallest length scale obtained is too large. In particular, having identified the correct number of eigenmodes for a given set of parameters, we can expect this number to be of the same order of magnitude if we change some of the parameters (e.g. permeability κ), and scale linearly with the diffusivity σ .

If $N_{\text{eig}} \geq N_{\text{node}}/2$, the `eig` command is called instead, and a full decomposition is performed. In both cases, we only retain the eigenvalues whose length scales are larger than ℓ_{\min} to compute the Matrix Formalism signal.

Parameter	Variables	Unit
Length	$\mathbf{x}, \mathbf{q}, \ell$	μm
Time	$t, T_{\text{echo}}, \delta, \Delta, T_2$	μs
Diffusivity	$\sigma, \text{ADC}, \mathbf{D}^{\text{MF}}$	$\mu\text{m}^2/\mu\text{s} = \text{mm}^2/\text{s}$
Permeability	κ	$\mu\text{m}/\mu\text{s} = \text{m/s}$
Magnetic field gradient	\mathbf{g}	T/m
Q-value	$q = \gamma \ \mathbf{g}\ $	$(\mu\text{s}\mu\text{m})^{-1}$
B-value	b	$\mu\text{s}/\mu\text{m}^2 = \text{s/mm}^2$
Magnetization	M, ρ	
Signal	S	

Table 1: Physical units of common quantities in SpinDoctor.

3.8 Adding T2-relaxation terms

The equations of the previous sections may be extended to take into account T_2 -relaxation terms without changing all the equations. Denoting T_2^{in} , T_2^{out} and T_2^{ecs} the relaxation times of the compartments Ω_i^{in} , Ω_i^{out} , and Ω^{ecs} respectively, the modified BTPDE is given by

$$\frac{\partial M}{\partial t} = -\text{i}\gamma f(t) \mathbf{g} \cdot \mathbf{x} M(\mathbf{x}, t) + \nabla \cdot (\sigma(\mathbf{x}) \nabla M(\mathbf{x}, t)) - \frac{1}{T_2(\mathbf{x})} M(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad (60)$$

where $T_2(\mathbf{x}) = T_2^{\text{in}} \mathbb{1}_{\bigcup_i \Omega_i^{\text{in}}}(\mathbf{x}) + T_2^{\text{out}} \mathbb{1}_{\bigcup_i \Omega_i^{\text{out}}}(\mathbf{x}) + T_2^{\text{ecs}} \mathbb{1}_{\Omega^{\text{ecs}}}(\mathbf{x})$ is a compartment-wise constant function.

The new terms in the equations, M/T_2 , may be considered as reaction terms that are *added* to the Laplace operator $-\nabla \cdot \sigma \nabla$, and contribute to an immediate attenuation of the solution to the BTPDE. The relaxation times T_2 correspond to the time it takes for the signal to decrease by a factor of $1/e$, which is approximately 63%. If all the relaxation times are equal, the T_2 -modified solution is $M_{T_2}(\mathbf{x}, t) = e^{-t/T_2} M(\mathbf{x}, t)$, where M is the solution to the BTPDE without relaxation.

For this reason, the T_2 -modified signal may be obtained by adding block-wise scaled mass matrices to the original stiffness matrix $\mathbf{S} = \text{diag}(\mathbf{S}_1, \dots, \mathbf{S}_{N_{\text{cmpt}}})$, where N_{cmpt} is the number of compartments:

$$\mathbf{S}_i^{T_2} = \mathbf{S}_i + \frac{1}{T_2^i} \mathbf{M}_i, \quad 1 \leq i \leq N_{\text{cmpt}} \quad (61)$$

Here, the mass matrix is given by $\mathbf{M} = \text{diag}(\mathbf{M}_1, \dots, \mathbf{M}_{N_{\text{cmpt}}})$. Since the coupling between the compartments are handled in the flux matrix \mathbf{Q} , the mass and stiffness matrices are block-diagonal, where one block corresponds to the interactions between the finite element basis functions within the corresponding compartment. When solving the equations, the T_2 -modified solution may be obtained by replacing the original stiffness matrix \mathbf{S} by \mathbf{S}^{T_2} .

Although this is not configured for the STA or HADC models yet, the same logic can however be applied to the Matrix Formalism. Instead of seeking the eigenvalues of the pure Laplace operator $-\nabla \cdot \sigma \nabla$, we seek the eigenvalues of the T_2 modified Laplace operator $-\nabla \cdot \sigma \nabla + 1/T_2$ (where σ and T_2 are constant in each compartment). The only modification needed in the consecutive equations are the same as for the BTPDE; replacing the pure stiffness matrix \mathbf{S} by \mathbf{S}^{T_2} . This does however change the eigenvalues – a positive value will be added to each of the original eigenvalues, meaning the first eigenvalue will no longer be zero, and the decay dynamics will be faster.

3.9 Physical units

The physical units of the input and output quantities for SpinDoctor are shown in Table 1.

4 Software components and algorithms

In this section we discuss the various components of SpinDoctor in more detail.

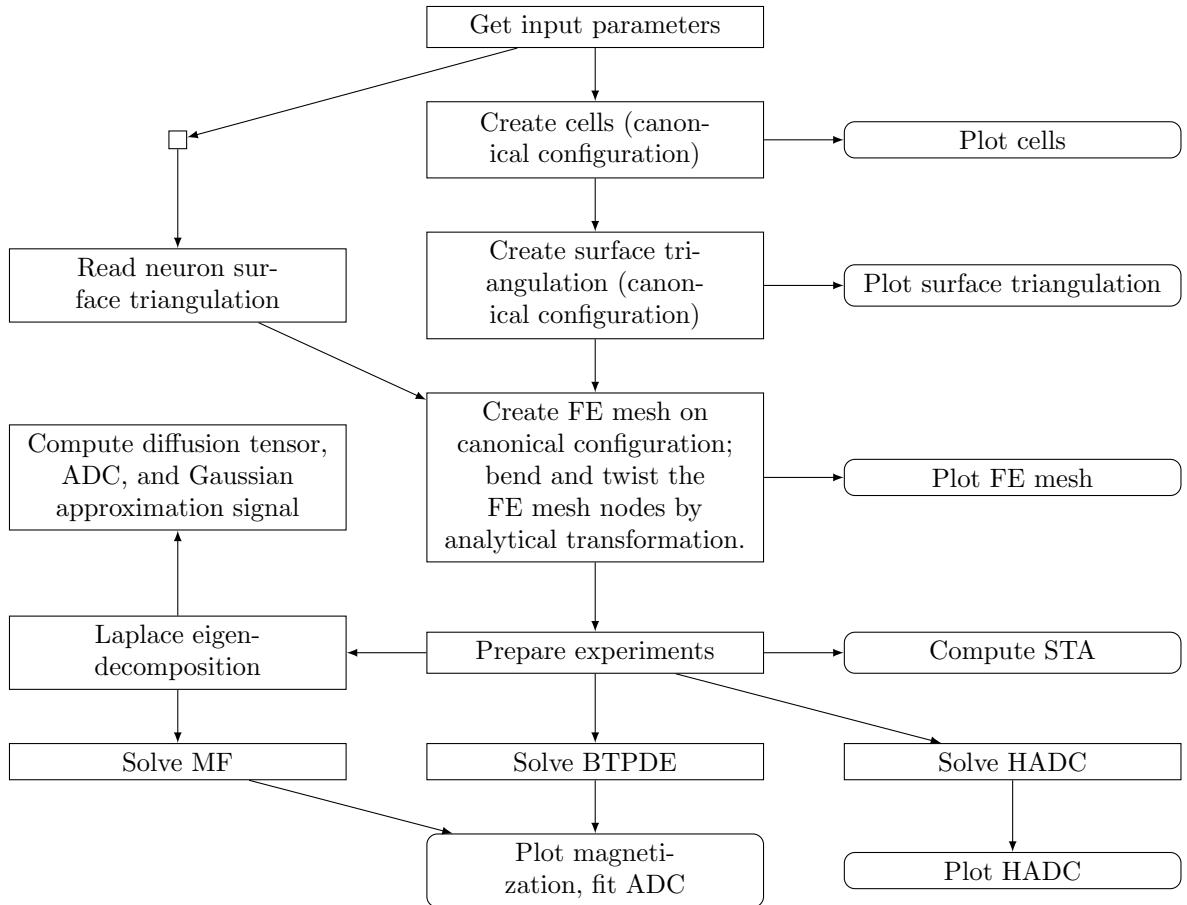


Figure 2: Flow chart describing the work flow of SpinDoctor

4.1 Work flow

Figure 2 is a chart showing the work flow of SpinDoctor. A session starts with creating or loading the geometry. Plotting functions allow for inspecting the resulting geometry. At the end of this process, a finite element mesh is returned, where the nodes are sorted into their respective compartments.

The user also has to prepare the experiments. The function `prepare_pde` creates a compartment numbering system and assigns the material proper to the compartments. The function `prepare_experiments` checks the consistency of the experiments, and prepares some important quantities related to the gradient sequences (q-values, b-values, directions).

SpinDoctor provides different solvers, some of which only work under certain assumptions. The function `solve_btpde` computes the magnetization and signal at a precision only limited by the mesh refinement and ODE-solver tolerances. The ADC may be fitted from the resulting signal. The function `solve_mf` produces the same result as `solve_btpde`, but at a much lower computational cost. This requires having previously computed the Laplace eigenvalues and eigenfunctions, from which a diffusion tensor and Gaussian approximation of the signal may be computed. The functions `solve_hadc` computes the ADC using a homogenized model, that assumes negligible permeability between compartments.

4.2 User provided input parameters

SpinDoctor expects the user to define three structures. In the folder `input_files`, there are several example scripts defining these structures. They are given by:

1. `params_cells`: structure containing *cells parameters*. Format is given in Table 2. If the cells are spheres or cylinders, the parameters for the geometrical configuration are provided in this structure. This structure also defines the file name for saving or loading the geometry.

Field name	Example	Explanation
<code>filename</code>	"myfile"	File name to store cell description or to get neuron mesh data
<code>shape</code>	"cylinder"	"sphere", "cylinder" or "neuron"
<code>ncell</code>	10	Number of cells (1 for neurons)
<code>rmin</code>	1.5	Minimum cell radius
<code>rmax</code>	2.5	Maximum cell radius
<code>dmin</code>	1.5	Minimum (%) distance between cells $\frac{d_{\min}(r_{\min}+r_{\max})}{2}$
<code>dmax</code>	2.5	Maximum (%) distance between cells $\frac{d_{\max}(r_{\min}+r_{\max})}{2}$
<code>height</code>	20	Cylinder height (ignored if not cylinder)
<code>deformation</code>	[0.05 0.05]	$[\alpha \ \beta]$; α defines the amount of bend; β defines the amount of twist (in radians)
<code>include_in</code>	true	Indicator for including in-compartment
<code>in_ratio</code>	0.6	$\frac{r_{\text{in}}}{r_{\text{out}}} \in]0, 1[$
<code>ecs_shape</code>	"no_ecs"	Do not include ECS: "no_ecs"; Box wrap ECS: "box"; Convex hull: "convex_hull"; Tight wrap: "tight_wrap";
<code>ecs_ratio</code>	0.3	ECS thickness as percentage of mean radius
<code>refinement</code>	0.5	Requested TetGen maximum tetrahedron volume (comment line to use default)

Table 2: Structure containing cells parameters.

2. `params_domain`: structure containing simulation *domain parameters*. Format is given in Table 3. This structure provides the material properties for each of the compartments `in`, `out`, `ecs`, as well as their interfaces. If the `in` or `ecs` compartments are not included, the corresponding parameters are ignored.
3. `experiment`: structure containing simulation *experiment parameters*. Format is given in Table 4. This structure defines the model input – that is, the gradient sequences. These sequences are parametrized by their amplitudes, time profiles (sequences), and directions. In addition, this structure decides which solvers to run, and with what parameters. Available ones are `btpde`, `hadc`, `mf`, and `multilayer`.

The sequences (time profiles) are instances of the MATLAB class `Sequence`, which can be of type `PGSE`, `DoublePGSE`, `CosOGSE`, `SinOGSE`, or `CustomSequence`. The latter requires providing a function handle representing the time profile. All the sequences can be called at time t through the `call` method, whose integrals and b-values are obtained by the `integral` and `bvalue_no_q` methods. `echotime` provides the echo time T_{echo} of the sequence. Each sequence comes with utility functions for string representation and information about their characteristic intervals. By default, a `Sequence` only requires a `call` method, while the other quantities can be computed numerically. However, for specific sequences, analytical expressions may be provided for the integral quantities.

4.3 Important output quantities

In Table 5 we list some useful quantities that are the outputs of SpinDoctor. The braces in the “Size” column denote MATLAB cell array structure and the brackets denote MATLAB matrix data structure.

4.4 Important functions

In Table 6 we list some important functions of SpinDoctor. For detailed information about them, including argument lists, please read the online documentation.

In Table 7 we list important functions of the Matrix Formalism Module. For detailed information about them, including argument lists, please read the online documentation.

Field name	Example	Explanation
<code>initial_density_in</code>	1	Initial spin density ρ
<code>initial_density_out</code>	1	
<code>initial_density_ecs</code>	1	
<code>diffusivity_in</code>	0.002	Diffusion coefficient σ . Can also be a 3x3-tensor
<code>diffusivity_out</code>	0.002	
<code>diffusivity_ecs</code>	0.002	
<code>relaxation_in</code>	Inf	T_2 -relaxation times (for no relaxation: Inf)
<code>relaxation_out</code>	5000	
<code>relaxation_ecs</code>	5000	
<code>permeability_in_out</code>	1e-3	Permeability coefficient κ
<code>permeability_out_ecs</code>	1e-4	
<code>permeability_in</code>	0	
<code>permeability_out</code>	0	
<code>permeability_ecs</code>	0	

Table 3: Structure containing simulation domain parameters.

4.5 Create cells (canonical configuration)

SpinDoctor supports the placement of a group of non-overlapping cells in close vicinity to each other. There are two proposed configurations, one composed of spheres, the other composed of cylinders. The algorithm is described in Algorithm 1.

Algorithm 1: Placing N_{cell} non-overlapping cells.

Generate a large number of possible cell centers.

Compute the minimum distance, d , between the current center and previously accepted cells.

Find the intersection of $[d - d_{\max}r_{\text{mean}}, d - d_{\min}r_{\text{mean}}]$ and $[r_{\min}, r_{\max}]$, where $r_{\text{mean}} = \frac{r_{\min} + r_{\max}}{2}$. If the intersection is not empty, then take the middle of the intersection as the new radius and accept the new center. Otherwise, reject the center.

Loop through the possible centers until get N_{cell} accepted cells.

SpinDoctor provides a routine to plot the cells to see if the configuration is acceptable (see Fig. 3).

4.6 Create surface triangulation

Finite element mesh generation software requires a good surface triangulation. This means the surface triangulation needs to be water-tight and does not self-intersect. How closely these requirements are met in floating point arithmetic has a direct impact on the quality of the finite element mesh generated.

It is often difficult to produce a good surface triangulation for arbitrary geometries. Thus, we restrict the allowed shapes to cylinders and spheres. Below in Algorithms 2 and 3 we describe how to obtain a surface triangulation for spherical cells with nucleus, cylindrical cells with myelin layer, and the ECS (box or tightly wrapped). We describe a canonical configuration where the cylinders are placed parallel to the z -axis. More general shapes are obtained from the canonical configuration by coordinate transformation in a later step.

SpinDoctor provides a routine to plot the surface triangulation (see Fig. 4).

4.7 Finite element mesh generation

SpinDoctor calls Tetgen [2], an external package (executable files are included in the toolbox package), to create a tetrahedra finite elements mesh from the surface triangulation generated by Algorithms 2 and 3. The FE mesh is generated on the canonical configuration. The numbering of the compartments and boundaries used by SpinDoctor are given in Tables 8 and 9. The labels are related to the values of the intrinsic diffusion coefficient, the initial spin density, and the permeability requested by the user. Then

Field name	Example	Explanation
<code>ndirection</code>	20	Number of gradient directions
<code>flat_dirs</code>	<code>false</code>	Uniformly distribute directions in 3D or 2D (if more than one direction)
<code>remove_opposite</code>	<code>false</code>	Only solve for half of the directions, copy results to opposite directions
<code>direction</code>	[1.0; 0.0; 0.0]	Gradient direction (if $n_{dir} = 1$), no need to normalize
<code>values</code>	[10 100 400 1000]	$\ \mathbf{g}\ $ -values, q-values or b-values to consider
<code>values_type</code>	"b"	"g", "q" or "b"
<code>sequences</code>	{PGSE(100, 5000)}	Gradient sequences; PGSE(delta, Delta), DoublePGSE(delta, Delta), CosOGSE(delta, Delta, nperiod), SinOGSE(delta, Delta, nperiod) or CustomSequence(delta, Delta, @timeprofile)
<code>btpde</code>	<code>struct</code>	Substructure for solving the BTPDE (comment out to skip)
<code>btpde.ode_solver</code>	@ode23t	Function handle for ODE solver (comment to use default)
<code>btpde.reltol</code>	1e-4	Relative tolerance for ODE solver
<code>btpde.abstol</code>	1e-6	Absolute tolerance for ODE solver
<code>hadc</code>	<code>struct</code>	Substructure for solving the HADC model (comment out to skip)
<code>hadc.ode_solver</code>	@ode15s	Function handle for ODE solver (comment to use default)
<code>hadc.reltol</code>	1e-4	Relative tolerance for ODE solver
<code>hadc.abstol</code>	1e-4	Absolute tolerance for ODE solver
<code>mf</code>	<code>struct</code>	Substructure for solving for the Matrix Formalism (comment out to skip)
<code>mf.length_scale</code>	3	Minimum length scale of Laplace eigenfunctions
<code>mf.neig_max</code>	100	Maximum number of eigenvalues. To compute all: Inf
<code>mf.ninterval</code>	50	Number of intervals to discretize time profile (if not PGSE)
<code>multilayer</code>	<code>struct</code>	Substructure for solving for the multi-layer model (comment out to skip)
<code>multilayer.length_scale</code>	3	Minimum length scale of radial Laplace eigenfunctions
<code>multilayer.eigstep</code>	1e-6	Minimium distance between two radial eigenvalues

Table 4: Structure containing simulation experiment parameters.

Variable name	Size	Explanation
<code>magnetization</code>	$\{\text{ncmpt} \times \text{namp} \times \text{nseq} \times \text{ndir}\}[\text{nnode}]$	Solution of the BTPDE for each compartment
<code>signal</code>	$[\text{ncmpt} \times \text{namp} \times \text{nseq} \times \text{ndir}]$	Signal (integral of magnetization) at T_{echo} in each compartment.
<code>signal_allcmpts</code>	$[\text{namp} \times \text{nseq} \times \text{ndir}]$	Signal (integral of magnetization) at T_{echo} summed over all compartments.
<code>adc</code>	$[\text{ncmpt} \times \text{nseq} \times \text{ndir}]$	ADC in each compartment.
<code>adc_allcmpts</code>	$[\text{nseq} \times \text{ndir}]$	ADC accounting for all compartments.
<code>lap_eig.values</code>	$[\text{neig} \times 1]$	Laplace eigenvalues
<code>lap_eig.funcs</code>	$[\text{nnode} \times \text{neig}]$	Laplace eigenfunctions
<code>lap_eig.moments</code>	$[\text{neig} \times \text{neig} \times 3]$	First order moments of the product pairs of Laplace eigenfunctions
<code>diffusion_tensor</code>	$[3 \times 3 \times \text{nseq}]$	Effective diffusion tensors

Table 5: Some important SpinDoctor output quantities.

Function name	Purpose
<code>prepare_experiments</code>	Prepare magnetic gradient sequences, directions, and BTPDE, HADC and MF experiments
<code>prepare_pde</code>	Set up the PDE model in the geometrical compartments
<code>create_cells</code>	Create the geometrical configuration.
<code>create_femesh</code>	Create or load a finite element mesh
<code>solve_btpde</code>	Compute the BTPDE magnetization and signal
<code>solve_hadc</code>	Compute the ADC from the HADC model
<code>fit_signal</code>	Fit the ADC and signal for zero b-weighting from the BTPDE signal.
<code>compute_adc_sta</code>	Compute the short time approximation in one diffusion-encoding direction.
<code>compute_free_diffusion</code>	Compute the free diffusion ADC and signal.
<code>plot_cells</code>	
<code>plot_surface_triangulation</code>	
<code>plot_femesh</code>	Display the finite elements mesh.
<code>plot_geometry_info</code>	
<code>plot_signal</code>	Display the simulated signal in one diffusion-encoding direction.
<code>plot_timing</code>	
<code>plot_adc</code>	Display the simulated ADC in one diffusion-encoding direction.
<code>plot_field</code>	Display the PDE solution on the finite elements mesh. If the finite elements mesh is too large, the magnetization should not be outputted from the solution of the BTPDE, and this function cannot be used.
<code>plot_hardi</code>	Display simulation results in multiple diffusion-encoding directions.

Table 6: Some important functions in SpinDoctor.

Function name	Purpose
<code>compute_laplace_eig</code>	Compute the Laplace eigenvalues, eigenfunctions and the first order moments of the products of the eigenfunctions.
<code>eig2length, length2eig</code>	Convert the computed eigenvalues into a length scale, and vice-versa
<code>mf_jn</code>	Compute the quantity $J(\lambda_n, f)$
<code>compute_mf_diffusion_tensor</code>	Computes the effective diffusion tensor
<code>plot_diffusion_tensor</code>	Display the effective diffusion tensor
<code>solve_mf</code>	Compute the Matrix Formalism magnetization and signal
<code>compute_mf_signal</code>	Compute the Matrix Formalism total signal only
<code>compute_mfga_signal</code>	Computes the Matrix Formalism Gaussian Approximation total signal

Table 7: Some important functions in the Matrix Formalism Module.

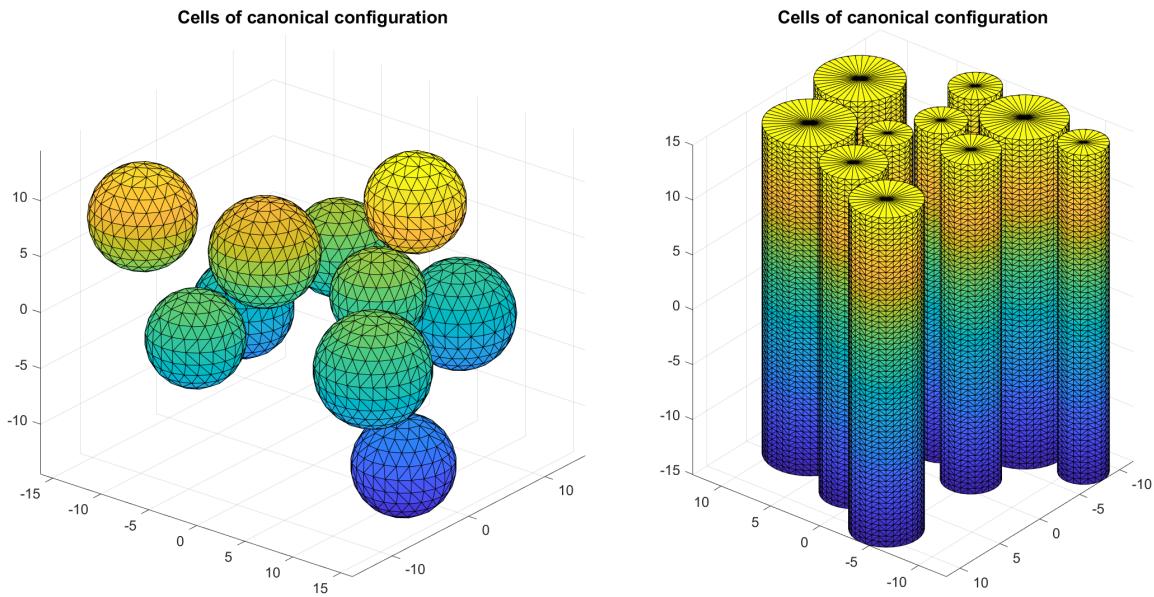


Figure 3: SpinDoctor plots cells in the canonical configuration.

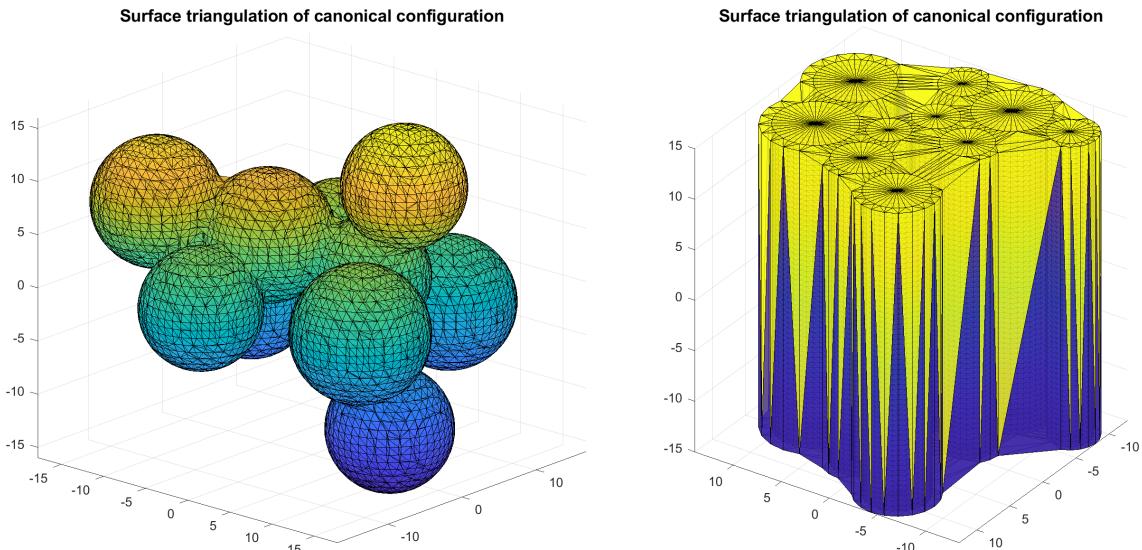


Figure 4: SpinDoctor plots the surface triangulation of the canonical configuration. Left: spherical cells with ECS; Right: cylindrical cells with ECS.

Algorithm 2: Surface triangulation of spherical cells and ECS.

Suppose we have N_{cell} spherical cells with nucleus. Denote a sphere with center c and radius R by $S(c, R)$, we use the built-in functions (convex hull, delaunay triangulation) in MATLAB to get its surface triangulation, $T(c, R)$. Call the radii of the nucleus $r_1, \dots, r_{N_{\text{cell}}}$ and the radii of the cells $R_1, \dots, R_{N_{\text{cell}}}$. Then the boundaries between the cytoplasm and the nucleus are

$$\{\Gamma_i = T(c_i, r_i)\}, \quad i = 1, \dots, N_{\text{cell}};$$

and between the cytoplasm and the ECS

$$\{\Sigma_i = T(c_i, R_i)\}, \quad i = 1, \dots, N_{\text{cell}};$$

For the box ECS, we find the coordinate limits of the set

$$\bigcup_{i=1}^{N_{\text{cell}}} S(c_i, R_i) \subset [x_0, x_f] \times [y_0, y_f] \times [z_0, z_f]$$

and add a gap $k = \text{ecs_gap} \times \max\{x_f - x_0, y_f - y_0, z_f - z_0\}$ to make a box

$$B = [x_0 - k, x_f + k] \times [y_0 - k, y_f + k] \times [z_0 - k, z_f + k].$$

We put 2 triangles on each face of B to make a surface triangulation Ψ with 12 triangles. For the tight-wrap ECS, we increase the cell radius by a gap size and take the union

$$W = \bigcup_{i=1}^{N_{\text{cell}}} S(c_i, R_i + \text{ecs_gap} \times R_{\text{mean}}),$$

where $R_{\text{mean}} = \frac{R_{\min} + R_{\max}}{2}$. We use the alphaShape function in MATLAB to find a surface triangulation Ψ that contains W .

Algorithm 3: Surface triangulation of cylindrical cells and ECS.

Suppose we have N_{cell} cylindrical cells with a myelin layer, all with height H . Denote a disk with center c and radius R by $D(c, R)$, and the circle with the same center and radius by $C(c, R)$. Let the radii of the axons be $r_1, \dots, r_{N_{\text{cell}}}$ and the radii of the cells be $R_1, \dots, R_{N_{\text{cell}}}$, meaning the thickness of the myelin layer is $R_i - r_i$.

The boundary between the axon and the myelin layer is:

$$C(c_i, r_i) \times [-H/2, H/2]$$

We discretize $C(c_i, r_i)$ as a polygon $P(c_i, r_i)$ and place one at $z = -H/2$ and one at $z = H/2$. Then we connect the corresponding vertices of $P(c_i, r_i) \times \{-H/2\}$ and $P(c_i, r_i) \times \{H/2\}$ and add a diagonal on each panel to get a surface triangulation Γ_i .

Between the myelin layer and the ECS we discretize $C(c_i, R_i)$ as a polygon and place one at $z = -H/2$ and one at $z = H/2$ to get a surface triangulation Σ_i .

For the box ECS, we find the coordinate limits of the union of $D(c_i, r_i)$ and add a gap to make a rectangle in two dimensions. Then we place the rectangle at $z = -H/2$ and at $z = H/2$ to get a box. Finally, the box is given a surface triangulation with 12 triangles.

For tight-wrap ECS, we increase the cell radius by a gap size and take the union

$$W = \bigcup_{i=1}^{N_{\text{cell}}} D(c_i, R_i + kR_{\text{mean}}).$$

We use the alphaShape function in MATLAB to find a two dimensional polygon Q that contains W . We place Q at $z = -H/2$ and at $z = H/2$ and connect corresponding vertices, adding a diagonal on each panel. Suppose Q is a polygon with n vertices, then the surface triangulation of the side of the ECS will have $2n$ triangles.

The above procedure produces a surface triangulation for the boundaries that are parallel to z -axis. We now must close the top and bottom. The top and bottom boundaries is just the interior of Q . However, the surface triangulation cannot be done on Q directly. We must cut out $D(c_i, r_i)$, the disk which touches the axon, and $A_i = D(c_i, R_i) - D(c_i, r_i)$, the annulus which touches the myelin. Then we triangulate $Q - \bigcup_{i=1}^{N_{\text{cell}}} D(c_i, R_i)$ using the MATLAB built-in function that triangulates a polygon with holes to get the boundary that touches the ECS. The surface triangulation for A_i and $D(c_i, r_i)$ is straightforward.

Spherical cells without nucleus			
Cmpt		Nucleus	ECS
Label		out	ecs
Number		$[1, \dots, n_{\text{cell}}]$	$n_{\text{cell}} + 1$
Spherical cells with nucleus			
Cmpt	Nucleus	Cytoplasm	ECS
Label	in	out	ecs
Number	$[1, \dots, n_{\text{cell}}]$	$[n_{\text{cell}} + 1, \dots, 2n_{\text{cell}}]$	$2n_{\text{cell}} + 1$
Cylindrical cells without myelin			
Cmpt		Axon	ECS
Label		out	ecs
Number		$[1, \dots, n_{\text{cell}}]$	$n_{\text{cell}} + 1$
Cylindrical cells with myelin			
Cmpt	Axon	Myelin	ECS
Label	in	out	ecs
Number	$[1, \dots, n_{\text{cell}}]$	$[n_{\text{cell}} + 1, \dots, 2n_{\text{cell}}]$	$2n_{\text{cell}} + 1$
Neuron			
Cmpt		Neuron	ECS
Label		out	ecs
Number		1	2

Table 8: The labels and numbers of compartments.

the FE mesh nodes are deformed analytically by a coordinate transformation, described in Algorithm 4.

Algorithm 4: Bending and twisting of the FE mesh of the canonical configuration.

The external package Tetgen [2] generates the finite element mesh that keeps track of the different compartments and the interfaces between them. The mesh is saved in several text files.

The connectivity matrices of the finite elements and facets are not modified by the coordinates transformation described below. The nodes are transformed by bending and twisting as described next.

The set of FE mesh nodes $\{x_i, y_i, z_i\}$ are transformed in the following ways:

Twisting around the z -axis with a user-chosen twisting parameter α_{twist} is defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} \cos(\alpha_{\text{twist}}z) & -\sin(\alpha_{\text{twist}}z) & 0 \\ \sin(\alpha_{\text{twist}}z) & \cos(\alpha_{\text{twist}}z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Bending on the $x - z$ plane with a user-chosen bending parameter α_{bend} is defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x + \alpha_{\text{bend}}z^2 \\ y \\ z \end{bmatrix}.$$

Given $[\alpha_{\text{bend}}, \alpha_{\text{twist}}]$, bending is performed after twisting.

SpinDoctor provides a routine to plot the FE mesh (see Fig. 5 for cylinders and ECS that have been bent and twisted).

4.8 Solve BTPDE

The BTPDE-solver `solve_btpde` is triggered by the presence of the field `experiment.btpde`. The solver first assembles all the finite element matrices, before it loops over gradient amplitudes $\|g\|$ (given by `experiment.values` of type "g", "q" or "b"), pulsed sequences f (given by `experiment.sequences`), and directions \mathbf{u}_g (given by `experiment.directions`). The loop is traversed using linear indices, meaning

Spherical cells without nucleus					
Label		out,ecs			ecs
Number		$[1, \dots, n_{\text{cell}}]$			$n_{\text{cell}} + 1$
Spherical cells with nucleus					
Label	in,out	out,ecs			ecs
Number	$[1, \dots, n_{\text{cell}}]$	$[n_{\text{cell}} + 1, \dots, 2n_{\text{cell}}]$			$2n_{\text{cell}} + 1$
Cylindrical cells without myelin					
Boundary		Cylinder side wall		Cylinder top and bottom	Outer ECS boundary
Label		out,ecs		out	ecs
Number		$[1, \dots, n_{\text{cell}}]$		$[n_{\text{cell}} + 1, \dots, 2n_{\text{cell}}]$	$2n_{\text{cell}} + 1$
Cylindrical cells with myelin					
Boundary	Inner cylinder side wall	Outer cylinder side wall	Inner cylinder top and bottom	Outer cylinder top and bottom	Outer ECS boundary
Label	in,out	out,ecs	in	out	ecs
Number	$[1, \dots, n_{\text{cell}}]$	$[n_{\text{cell}} + 1, \dots, 2n_{\text{cell}}]$	$[2n_{\text{cell}} + 1, \dots, 3n_{\text{cell}}]$	$[3n_{\text{cell}} + 1, \dots, 4n_{\text{cell}}]$	$4n_{\text{cell}} + 1$
Neuron					
Boundary				Neuron	
Label				out	
Number				1	
Neuron with ECS					
Boundary		Neuron			ECS
Label		out,ecs			ecs
Number		1			2

Table 9: The labels and numbers of boundaries.

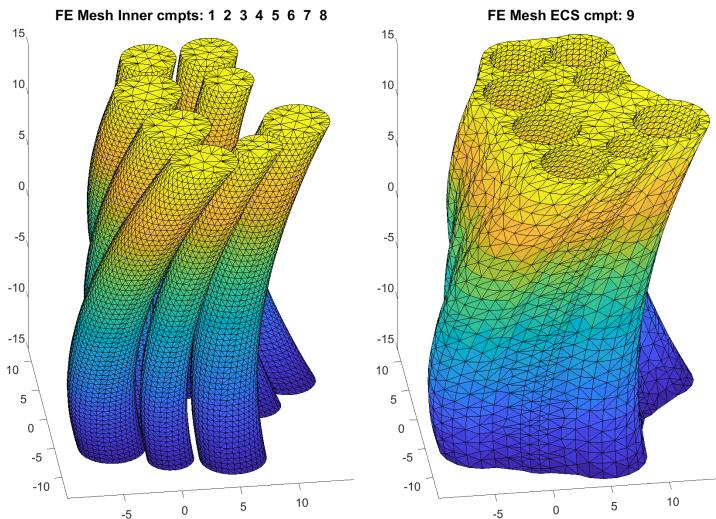


Figure 5: FE mesh of cylinders and ECS after bending and twisting. Compartment number is 1 to 8 for the cylinders and 9 for the ECS.

that the three loops are combined into one. If the Parallel Computing Toolbox is available, the loop is parallelized.

For each iteration, the ODE-solver is set up with the appropriate functions and parameters. The ODE is then solved using one of the built-in ODE routines in MATLAB (provided in `experiment.btpde.ode_solver`), the default being `ode15s`. Only the final magnetization is stored – this assures limited memory usage if multiple solver calls are running in parallel. The signal is computed from the final magnetization.

In addition, the time interval $[0, T_{\text{echo}}]$ is split into multiple subintervals if the sequence f is of type PGSE, DoublePGSE, CosOGSE, or SinOGSE. This is done to take advantage of the sequence being constant on certain intervals, resulting in a simplified ODE-function and a matrix Jacobian (instead of a time dependent Jacobian). The magnetization at the end of one interval is used as initial conditions for the beginning of the next interval. The computational advantage of this scheme is only visible for iterations that would normally take a long time, especially for higher b-values (higher gradient amplitudes or longer pulses). For lower b-values, this procedure may be slower, when the setup of the ODE-solver is larger compared to the actual time-stepping than for higher b-values.

The user may toggle the switch `experiment.remove_opposite`, which skips solving for directions opposite to those already computed. This is based on the symmetry of the BTPDE around the real axis – if the initial spin density is real, flipping the gradient direction only flips the sign of the imaginary part of the complex magnetization. The complex conjugate is thus returned for these opposite directions.

4.9 Solve HADC model

Similarly, the diffusion equation of the HADC model is discretized using P_1 finite elements. The solver is controlled by the field `experiment.hadc`, of which the precease triggers the solver `solve_hadc`. The ODE-solver is given by `experiment.hadc.ode_solver` (default: `ode15s`), with tolerances `experiment.hadc.abstol` and `experiment.hadc.reltol`.

The solver assembles the finite element matrices \mathbf{M} , \mathbf{S} , and \mathbf{G} , before it loops over compartments, sequences and directions using linear indices, possible using parallel iterations if the Parallel Computing Toolbox is available.

The HADC model differs from the BTDPE in that the compartments are assumed to be separated, and we therefore solve for each compartment separately. The HADC does not depend on the gradient amplitudes $\|\mathbf{g}\|$. In addition, the HADC solver has to store all the time steps from the ODE-solver, as time integrals are required.

Each iteration consists of setting up the ODE for the diffusion equation for the given sequence f (with integral F) and direction \mathbf{u}_g . The solution to the diffusion equation $\zeta(t)$ is computed and stored for all time steps. The integral quantity $\int_0^{T_{\text{echo}}} F(t)h(t) dt$ is then computed numerically, using a trapezoidal rule with the time steps $t_1, \dots, t_{N_{\text{step}}}$. The denominator in Eq. (13) is available through the `bvalue_no_q` method of the sequence.

4.10 Solve Matrix Formalism model

The function `compute_laplace_eig` computes the Laplace eigenvalues $(\lambda_n)_{n=1,\dots,N_{\text{eig}}}$, eigenfunction coefficients $\mathbf{P} \in \mathbb{R}^{N_{\text{node}} \times N_{\text{eig}}}$, and first order moments of products of pairs of eigenfunctions \mathbf{A}^x , \mathbf{A}^y , and \mathbf{A}^z , given a length scale ℓ_{\min} and guiding number N_{eig} . The function `solve_mf` can then compute the resulting magnetization and signal, using the same parallel looping scheme as for `solve_btpde` (see Section 4.8). Once the Laplace eigendecomposition has been performed, computing the the magnetization is straightforward, and generally requires only a fraction of the time it takes to call `solve_btpde` directly. It does still however requiring assembling the mass matrix for projecting the initial spin density onto the Laplace eigenbasis and computing the signal.

4.11 Visualization of results

To display results in one diffusion-encoding direction, the user can choose the following options:

1. the signal is plotted against the b-values;

- the ADC is plotted against the geometrical compartment number.

We note that if the ADC from the short time approximation is negative (meaning the short diffusion time assumption is not satisfied), we do not display the negative ADC in the plot.

For results in multiple diffusion-encoding directions uniformly distributed on the three dimensional unit sphere, the user can call the function `plot_hardi`, which plots the signals are plotted as a surface plot. The dots are the end points of the vectors in the diffusion-encoding direction multiplied the signal. The color gives the magnitude of the signal, which can be the signal or the ADC.

The user can also choose to plot the magnetization solution at the surface nodes of the finite elements mesh, using the functions `plot_field` or `plot_field_everywhere`. These functions can also be used to plot eigenfunctions.

5 Finite elements meshes of neurons

The diffusion MRI signal arising from neurons can be numerically simulated by solving the Bloch-Torrey partial differential equation. In order to facilitate the diffusion MRI simulation of realistic neurons by the research community, we constructed finite element meshes for a group of 36 pyramidal neurons and a group of 29 spindle neurons whose morphological descriptions were found in the publicly available neuron repository *NeuroMorpho.Org* [17]. These finite elements meshes range from having 15163 nodes to 622553 nodes. We also broke the neurons into the soma and dendrite branches and created finite elements meshes for these cell components. Through the Neuron Module, these neuron and components finite element meshes can be seamlessly coupled with the functionalities of SpinDoctor to provide the diffusion MRI signal that can be attributed to spins inside neurons.

As this time, we have not implemented the Neuron Module for coupled compartments linked by permeable membranes. Rather, the diffusion MRI signal is computed with zero permeability on the compartment boundaries. The current emphasis of the Neuron Module is to show how the geometrical structure of neurons affect the diffusion MRI signal. Thus, some of the input parameters related to multiple compartment models in SpinDoctor are not applicable in the current version of the Neuron Module. However, we have kept the exactly same input formats as SpinDoctor in anticipation of the future development of the Neuron Module for permeable membranes. In particular, the various compartments in SpinDoctor have designations as IN, OUT, and ECS, and in the Neuron Module, the geometry defined by the finite element mesh is designated as the OUT compartment.

Table 10 lists the names and the finite element mesh sizes of the group of 36 pyramidal neurons and the group of 29 spindle neurons. Table 11 shows the morphological characteristics for the neurons. The neuron models and the measurement data are from [18] and [19].

Neuron ID	Brain region	Average diam. (μm)	Height (μm)	Soma vol. (μm^3)	Total vol. (μm^3)
<i>02a_pyramidal2aFI</i>	fronto-insula	1.27	404.85	19701.05	25639.61
<i>02b_pyramidal1aACC</i>	anterior cingulate	1.58	363.08	9065.56	11579.71
<i>02b_pyramidal1aFI</i>	fronto-insula	1.62	381.56	22475.52	29804.96
<i>03a_pyramidal9aFI</i>	fronto-insula	2.15	532.30	22557.27	30189.28
<i>03a_spindle2aFI</i>	fronto-insula	1.74	387.16	13406.27	17684.23
<i>03a_spindle6aFI</i>	fronto-insula	1.66	501.47	33458.19	37812.72
<i>03b_pyramidal2aACC</i>	anterior cingulate	1.48	189.29	2977.21	4487.44
<i>03b_pyramidal3aACC</i>	anterior cingulate	1.14	188.45	6005.06	6891.04
<i>03b_pyramidal3aFI</i>	fronto-insula	1.84	496.35	32510.62	46154.08
<i>03b_pyramidal4aFI</i>	fronto-insula	1.33	414.70	35253.85	39324.87
<i>03b_pyramidal9aFI</i>	fronto-insula	1.92	430.06	15263.14	20532.57
<i>03b_spindle4aACC</i>	anterior cingulate	1.43	336.33	3098.39	4070.19
<i>03b_spindle5aACC</i>	anterior cingulate	1.49	221.52	11925.78	13242.53
<i>03b_spindle6aACC</i>	anterior cingulate	1.33	398.36	4027.74	6058.67
<i>03b_spindle7aACC</i>	anterior cingulate	1.18	369.51	4982.41	6076.52
<i>04a_pyramidal4aACC</i>	anterior cingulate	1.52	705.96	5684.55	13637.33

<i>04a_pyramidal5aACC</i>	anterior cingulate	1.86	410.59	15010.43	24648.35
<i>04b_pyramidal5aFI</i>	fronto-insula	1.78	480.13	10312.87	17184.26
<i>04b_pyramidal6aACC</i>	anterior cingulate	1.41	465.88	3129.97	7497.12
<i>04b_pyramidal6aFI</i>	fronto-insula	1.56	310.21	14718.05	21708.21
<i>04b_pyramidal7aACC</i>	anterior cingulate	1.3	610.42	17060.60	28552.49
<i>04b_spindle3aFI</i>	fronto-insula	2.71	391.14	22569.99	28404.13
<i>05a_pyramidal10aACC</i>	anterior cingulate	1.74	281.20	16604.06	21826.41
<i>05a_pyramidal8aACC</i>	anterior cingulate	1.29	430.37	24709.77	29778.79
<i>05b_pyramidal7aFI</i>	fronto-insula	2.18	281.02	25720.11	32731.05
<i>05b_pyramidal8aFI</i>	fronto-insula	1.66	361.45	32527.06	44679.46
<i>05b_pyramidal9aACC</i>	anterior cingulate	1.56	650.60	23948.05	40014.54
<i>05b_spindle5aFI</i>	fronto-insula	2.35	381.88	15383.08	18190.63
<i>06a_pyramidal11aACC</i>	anterior cingulate	1.46	437.60	17222.02	29995.02
<i>06b_pyramidal10aFI</i>	fronto-insula	1.92	365.18	43127.81	52179.53
<i>06b_pyramidal12aACC</i>	anterior cingulate	1.52	324.94	17181.33	24931.32
<i>06b_spindle8aACC</i>	anterior cingulate	1.92	342.21	18237.49	19462.92
<i>07a_pyramidal13aACC</i>	anterior cingulate	1.37	325.73	6254.53	8738.01
<i>07b_pyramidal14aACC</i>	anterior cingulate	1.67	350.40	16053.07	22772.96
<i>07b_spindle9aACC</i>	anterior cingulate	1.75	437.87	21344.83	27307.48
<i>08a_spindle13aACC</i>	anterior cingulate	1.74	814.45	9911.07	14113.32
<i>08o_pyramidal11aFI</i>	fronto-insula	1.91	421.68	11512.38	24326.94
<i>09o_spindle7aFI</i>	fronto-insula	2.90	472.87	22052.10	27905.89
<i>09o_spindle8aFI</i>	fronto-insula	2.05	376.73	11923.76	15189.32
<i>10a_pyramidal15aACC</i>	anterior cingulate	1.40	341.48	8522.11	10960.84
<i>10a_spindle18aACC</i>	anterior cingulate	1.57	457.90	5895.17	7219.28
<i>11a_pyramidal16aACC</i>	anterior cingulate	1.27	486.31	8807.01	12263.84
<i>11o_pyramidal12aFI</i>	fronto-insula	1.91	369.34	70786.62	79516.92
<i>12a_spindle19aACC</i>	anterior cingulate	2.05	431.22	12178.08	15618.67
<i>12o_spindle9aFI</i>	fronto-insula	3.41	305.31	29983.79	36678.18
<i>13o_spindle10aFI</i>	fronto-insula	2.69	516.92	39866.55	46022.15
<i>15o_spindle12aFI</i>	fronto-insula	3.60	604.57	53192.65	79170.43
<i>16o_spindle13aFI</i>	fronto-insula	2.17	364.66	17467.88	18888.13
<i>17o_pyramidal13aFI</i>	fronto-insula	1.89	340.77	11004.30	21167.19
<i>18o_pyramidal14aFI</i>	fronto-insula	1.74	288.41	69851.56	78999.20
<i>19o_spindle14aFI</i>	fronto-insula	2.18	232.21	10507.15	12905.43
<i>20o_pyramidal15aFI</i>	fronto-insula	1.82	383.18	22344.32	27667.19
<i>21o_spindle15aFI</i>	fronto-insula	2.36	286.33	17567.69	29466.53
<i>22o_pyramidal16aFI</i>	fronto-insula	1.94	585.35	18776.05	29441.43
<i>23o_spindle16aFI</i>	fronto-insula	1.67	420.05	10429.13	13482.93
<i>24o_pyramidal17aFI</i>	fronto-insula	2.04	371.99	40986.40	47377.09
<i>25o_pyramidal18aFI</i>	fronto-insula	1.80	364.05	18587.13	23572.15
<i>25o_spindle17aFI</i>	fronto-insula	1.79	358.70	7897.44	13563.26
<i>26o_spindle18aFI</i>	fronto-insula	2.27	442.65	52911.93	56084.44
<i>27o_spindle19aFI</i>	fronto-insula	1.73	275.08	20640.14	25423.96
<i>28o_spindle20aFI</i>	fronto-insula	3.00	520.69	35442.59	51267.07
<i>28o_spindle21aFI</i>	fronto-insula	2.62	298.57	35579.06	37783.31
<i>29o_spindle22aFI</i>	fronto-insula	3.52	402.84	62928.22	83279.12
<i>31o_pyramidal19aFI</i>	fronto-insula	2.26	303.55	65950.80	86376.72

Table 11: The morphological characteristics of the neurons.

As an illustration, we show the finite elements meshes of 4 spindle neurons in Figure 6, and in Figure 7 we show the finite elements meshes of 4 pyramidal neurons.

Neuron ID	Num of FE mesh nodes	Neuron ID	Num of FE mesh nodes
<i>03a_spindle2aFI</i>	38202	<i>02a_pyramidal2aFI</i>	119156
<i>03a_spindle6aFI</i>	44000	<i>02b_pyramidal1aACC</i>	45216
<i>03b_spindle4aACC</i>	17370	<i>02b_pyramidal1aFI</i>	105384
<i>03b_spindle5aACC</i>	26345	<i>03a_pyramidal9aFI</i>	81530
<i>03b_spindle6aACC</i>	26792	<i>03b_pyramidal2aACC</i>	28183
<i>03b_spindle7aACC</i>	21618	<i>03b_pyramidal3aACC</i>	27607
<i>04b_spindle3aFI</i>	51265	<i>03b_pyramidal3aFI</i>	151362
<i>05b_spindle5aFI</i>	22457	<i>03b_pyramidal4aFI</i>	96177
<i>06b_spindle8aACC</i>	15163	<i>03b_pyramidal9aFI</i>	66162
<i>07b_spindle9aACC</i>	54952	<i>04a_pyramidal4aACC</i>	150897
<i>08a_spindle13aACC</i>	46293	<i>04a_pyramidal5aACC</i>	89256
<i>09o_spindle7aFI</i>	38992	<i>04b_pyramidal5aFI</i>	95784
<i>09o_spindle8aFI</i>	60755	<i>04b_pyramidal6aACC</i>	87195
<i>10a_spindle18aACC</i>	25797	<i>04b_pyramidal6aFI</i>	90482
<i>12a_spindle19aACC</i>	31841	<i>04b_pyramidal7aACC</i>	622553
<i>12o_spindle9aFI</i>	29320	<i>05a_pyramidal10aACC</i>	201506
<i>13o_spindle10aFI</i>	43081	<i>05a_pyramidal8aACC</i>	139975
<i>15o_spindle12aFI</i>	101548	<i>05b_pyramidal7aFI</i>	208203
<i>16o_spindle13aFI</i>	18266	<i>05b_pyramidal8aFI</i>	124350
<i>19o_spindle14aFI</i>	25786	<i>05b_pyramidal9aACC</i>	366659
<i>21o_spindle15aFI</i>	28822	<i>06a_pyramidal11aACC</i>	319574
<i>23o_spindle16aFI</i>	30073	<i>06b_pyramidal10aFI</i>	106808
<i>25o_spindle17aFI</i>	52919	<i>06b_pyramidal12aACC</i>	277718
<i>26o_spindle18aFI</i>	36239	<i>07a_pyramidal13aACC</i>	155854
<i>27o_spindle19aFI</i>	50807	<i>07b_pyramidal14aACC</i>	309789
<i>28o_spindle20aFI</i>	56036	<i>08o_pyramidal11aFI</i>	419651
<i>28o_spindle21aFI</i>	17581	<i>10a_pyramidal15aACC</i>	56184
<i>29o_spindle22aFI</i>	18414	<i>11a_pyramidal16aACC</i>	222732
<i>30o_spindle23aFI</i>	26357	<i>11o_pyramidal12aFI</i>	380293
<i>22o_pyramidal16aFI</i>	389878	<i>17o_pyramidal13aFI</i>	326989
<i>24o_pyramidal17aFI</i>	245058	<i>18o_pyramidal14aFI</i>	338453
<i>25o_pyramidal18aFI</i>	71209	<i>20o_pyramidal15aFI</i>	247116
<i>31o_pyramidal19aFI</i>	619390		

Table 10: Names and sizes of all the neuron finite elements meshes generated by Tetgen with default settings (H_{tetgen} is not defined). The number of FE elements (not shown) is approximately four times the number of FE nodes.

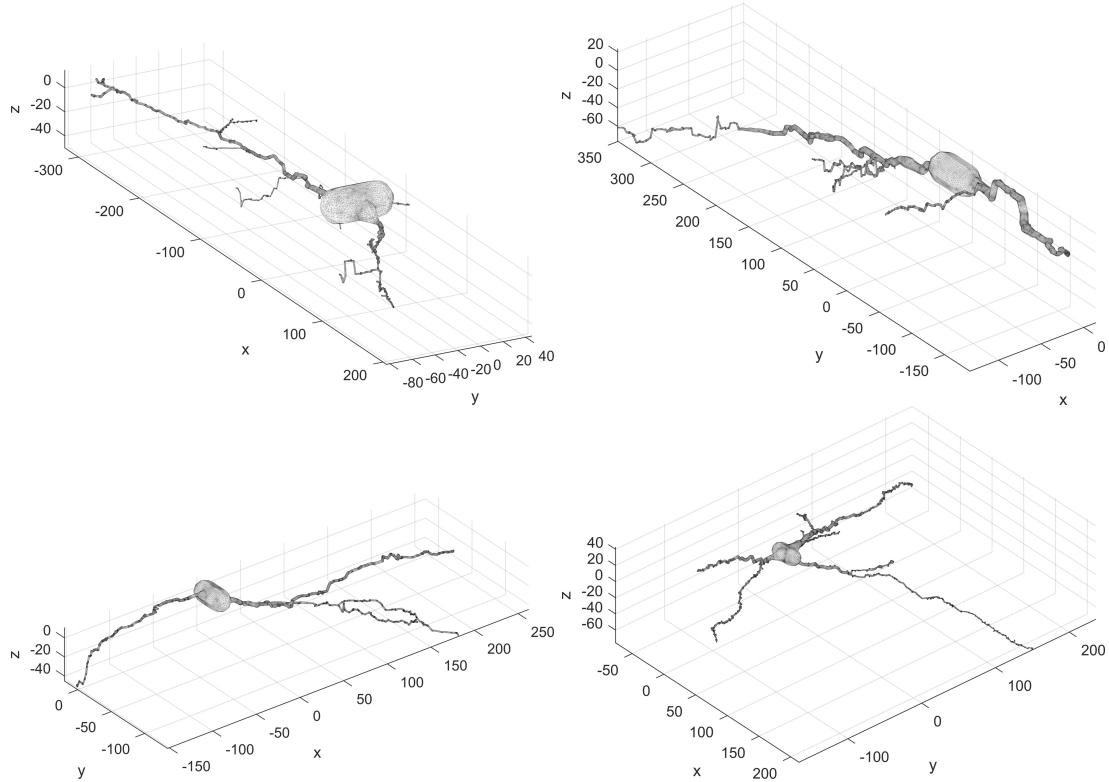


Figure 6: The finite elements meshes of four spindle neurons. The unit is μm . Top left: *03a_spindle6aFI*. Top right: *28o_spindle20aFI*. Bottom left: *09o_spindle8aFI*. Bottom right: *25o_spindle17aFI*.

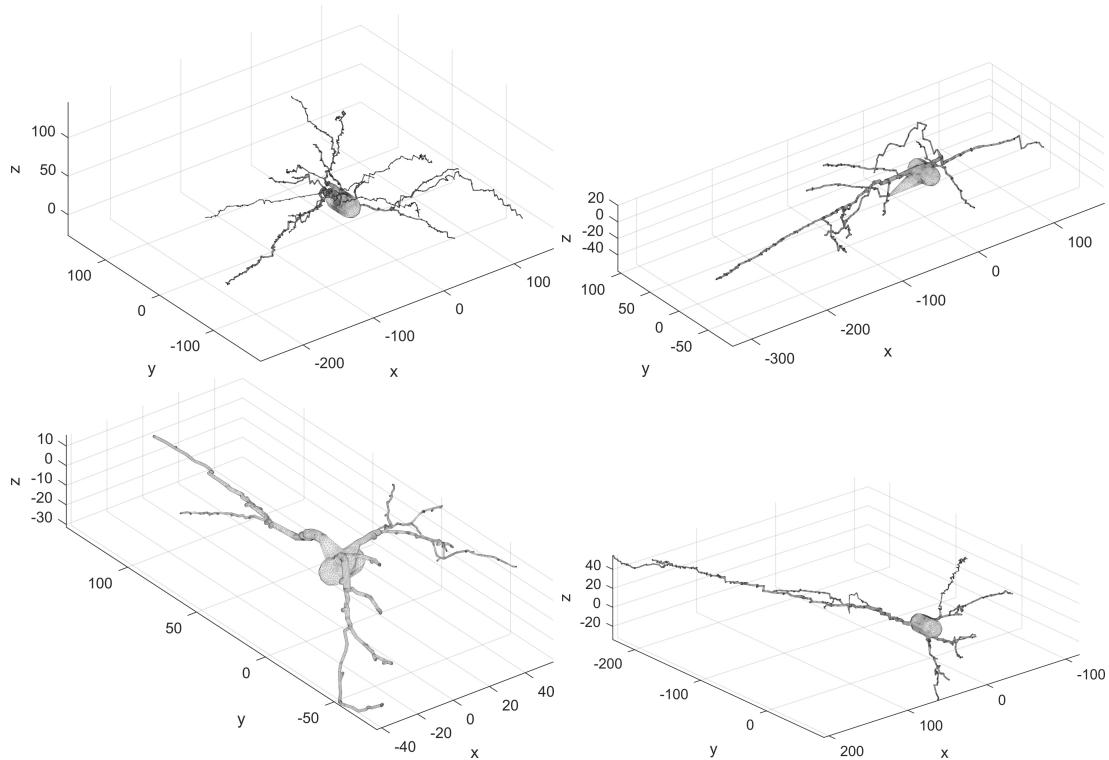


Figure 7: The finite elements meshes of four pyramidal neurons. The unit is μm . Top left: *02a_pyramidal2aFI*. Top right: *03b_pyramidal9aFI*. Bottom left: *03b_pyramidal2aACC*. Bottom right: *10a_pyramidal15aACC*.

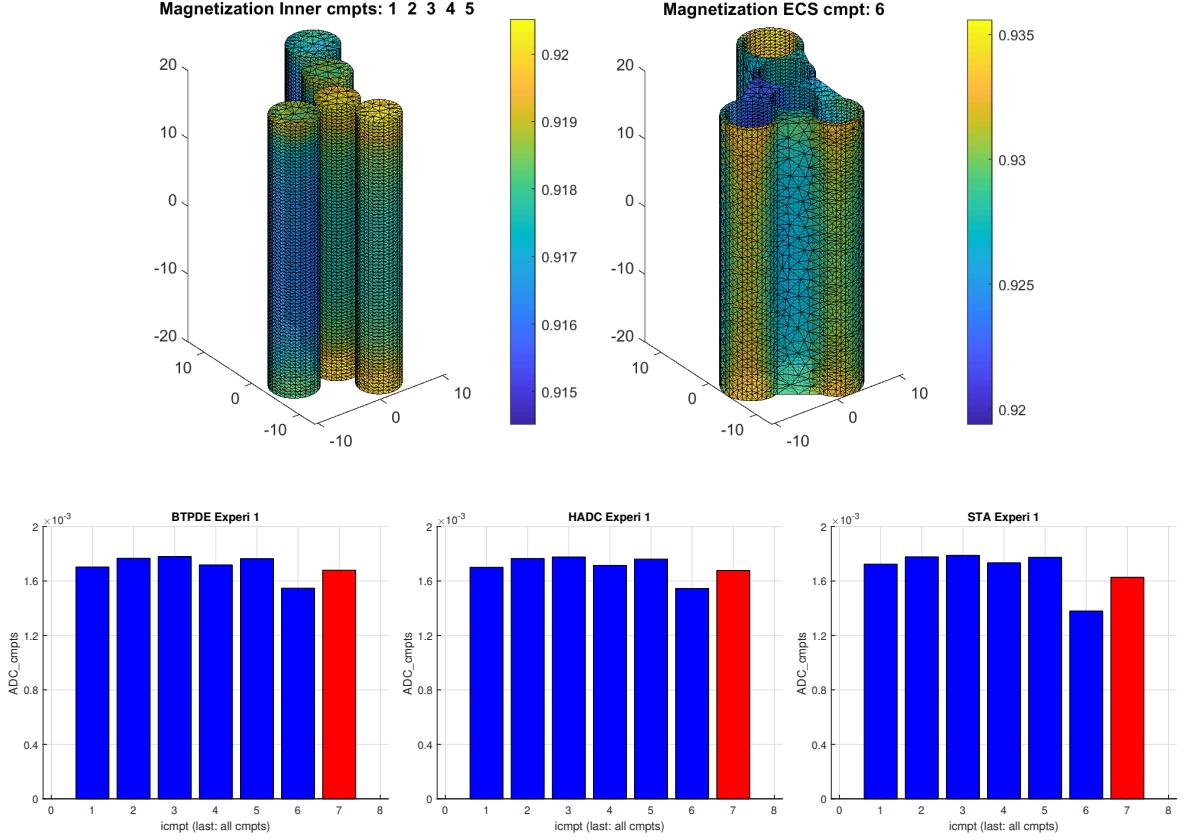


Figure 8: Geometry: 5 cylinders, tight wrap ECS, $\text{ecs_gap} = 0.2$, $\mathbf{u}_g = (1, 1, 1)^\top / \sqrt{3}$, $\sigma^{\text{out}} = \sigma^{\text{ecs}} = 2 \times 10^{-3} \text{mm}^2/\text{s}$, $\kappa = 0 \text{m}/\text{s}$, OGSE cosine ($\delta = 14 \text{ms}$, $\Delta = 14 \text{ms}$, nperiod = 6). The vertical bars indicate the ADC in each compartment. The ADC in the rightmost position is the ADC that takes into account the diffusion in all the compartments.

6 SpinDoctor examples

In this section we show some prototypical examples using the available functionalities of SpinDoctor.

6.1 Comparison of BTPDE and HADC with Short Time Approximation

In Fig. 8 we show that both BTPDE and HADC solutions match the STA values at short diffusion times for cylindrical cells (compartments 1 to 5). We also show that for the ECS (compartment 6), the STA is too low, because it does not account for the fact that spins in the ECS can diffuse around several cylinders. This also shows that when the interfaces are impermeable, the BTPDE ADC and that from the HADC model are identical. The diffusion-encoding sequence here is cosine OGSE with 6 periods.

6.2 Permeable membranes

In Fig. 9 we show the effect of permeability: the Bloch-Torrey PDE model includes permeable membranes ($\kappa = 1 \times 10^{-3} \text{m}/\text{s}$) whereas the HADC has impermeable membranes. We see in the permeable case, the ADC in the spheres are higher than in the impermeable case, whereas the ECS show reduced ADC because the faster diffusing spins in the ECS are allowed to move into the slowly diffusing spherical cells. We note that in the permeable case, the ADC in each compartment is obtained by using the fitting formula involving the logarithm of the dMRI signal, and we defined the “signal” in a compartment as the total magnetization in that compartment at T_{echo} , which is just the integral of the solution of the Bloch-Torrey PDE in that compartment.

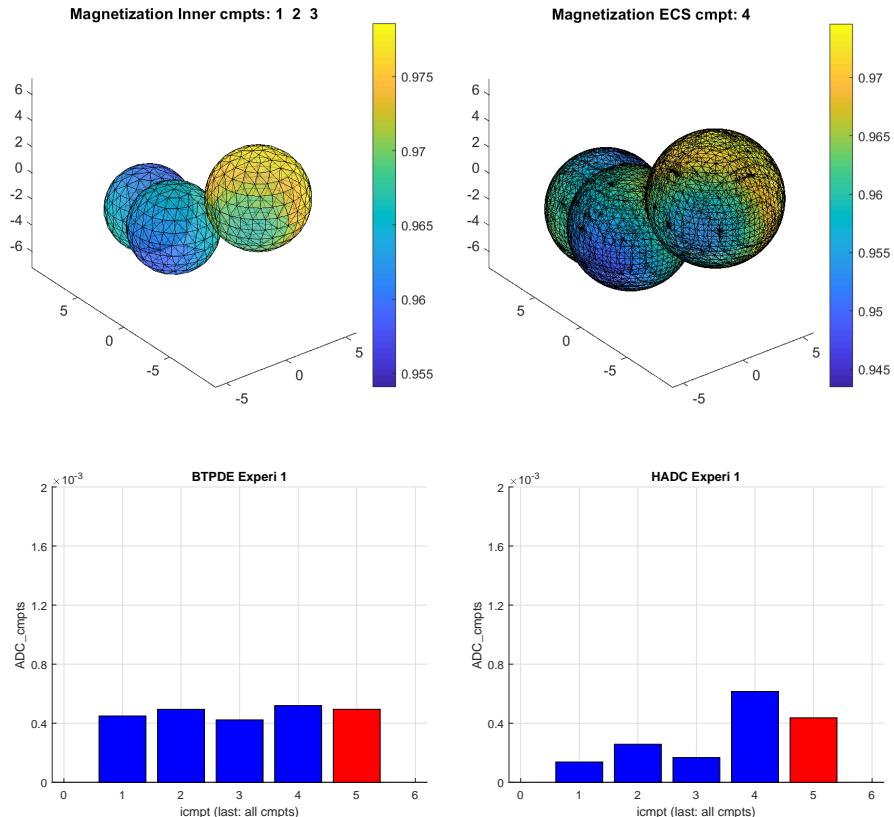


Figure 9: Geometry: 3 spheres, tight wrap ECS, $\text{ecs_gap} = 0.3$, $\mathbf{u}_g = (1, 1, 0)^T / \sqrt{2}$, $\sigma^{\text{in}} = \sigma^{ecs} = 2 \times 10^{-3} \text{ mm}^2/\text{s}$, $\kappa = 1 \times 10^{-3} \text{ m/s}$ (left), $\kappa = 0 \text{ m/s}$ (right). PGSE ($\delta = 5 \text{ ms}$, $\Delta = 5 \text{ ms}$). The vertical bars indicate the ADC in each compartment. The ADC in the rightmost position is the ADC that takes into account the diffusion in all the compartments.

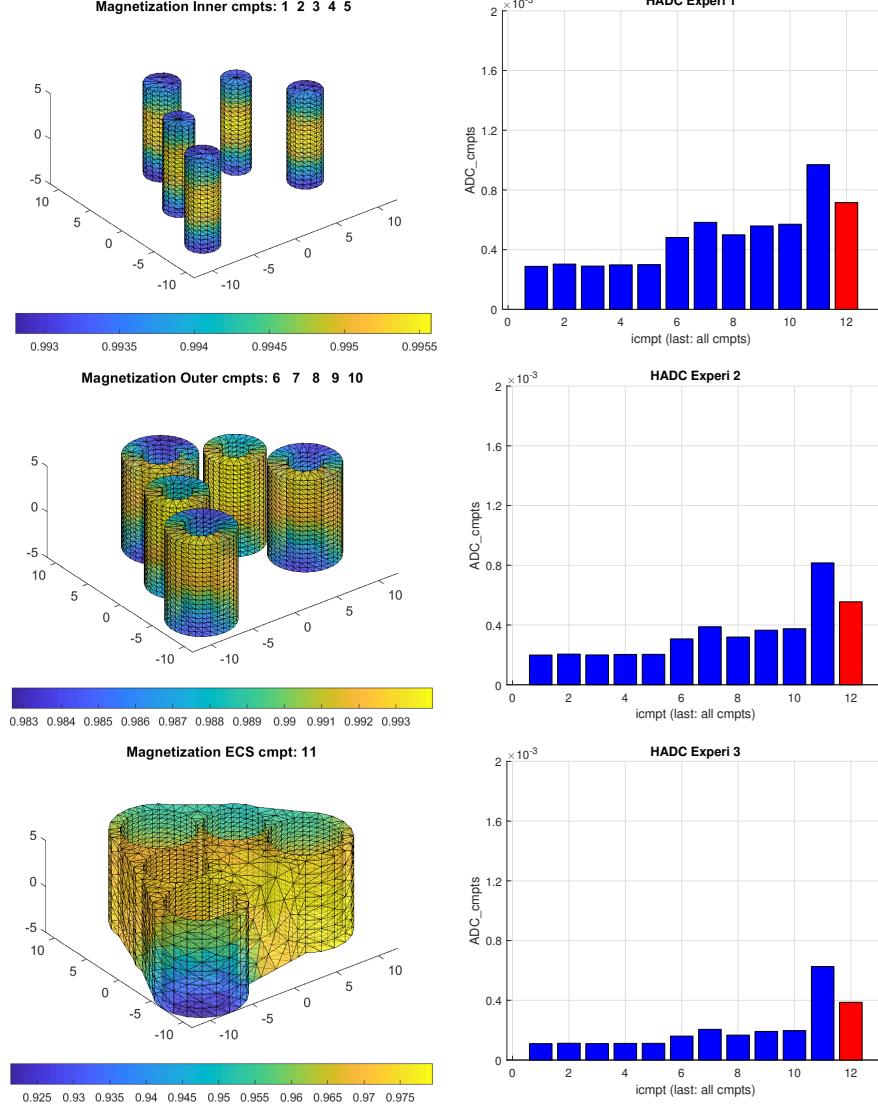


Figure 10: Geometry: 5 cylinders, myelin layer, $r_{in}/r_{out} = 0.5$, tight wrap ECS, $ecs_gap = 0.3$, $\kappa = 0\text{m/s}$, $\mathbf{u}_g = (1, 1, 1)^T/\sqrt{3}$, $\sigma^{in} = \sigma^{out} = \sigma^{ecs} = 2 \times 10^{-3}\text{mm}^2/\text{s}$, 3 experiments: PGSE ($\delta = 5\text{ms}$, $\Delta = 5, 10, 20\text{ms}$). Left: the magnetization at $\Delta = 5\text{ms}$. Right: the ADC values. The vertical bars indicate the ADC in each compartment. The ADC in the rightmost position is the ADC that takes into account the diffusion in all the compartments.

6.3 Myelin layer

In Fig. 10 we show the diffusion in cylindrical cells, the myelin layer, and the ECS. The ADC is higher in the myelin layer than in the cells, because for spins in the myelin layer diffusion occurs in the tangential direction (around the circle). At longer diffusion times, the ADC of both the myelin layer and the cells becomes very low. The ADC is the highest in the ECS, because the diffusion distance can be longer than the diameter of a cell, since the diffusing spins can move around multiple cells.

6.4 Twisting and bending

In Fig. 11 we show the effect of bending and twisting in cylindrical cells in multiple gradient directions. The HADC is obtained in 20 directions uniformly distributed in the sphere. We used spherical harmonics interpolation to interpolate the HADC in the entire sphere. Then we deformed the radius of the unit sphere to be proportional to the interpolated HADC and plotted the 3D shape. The color axis also indicates the value of the interpolated HADC.

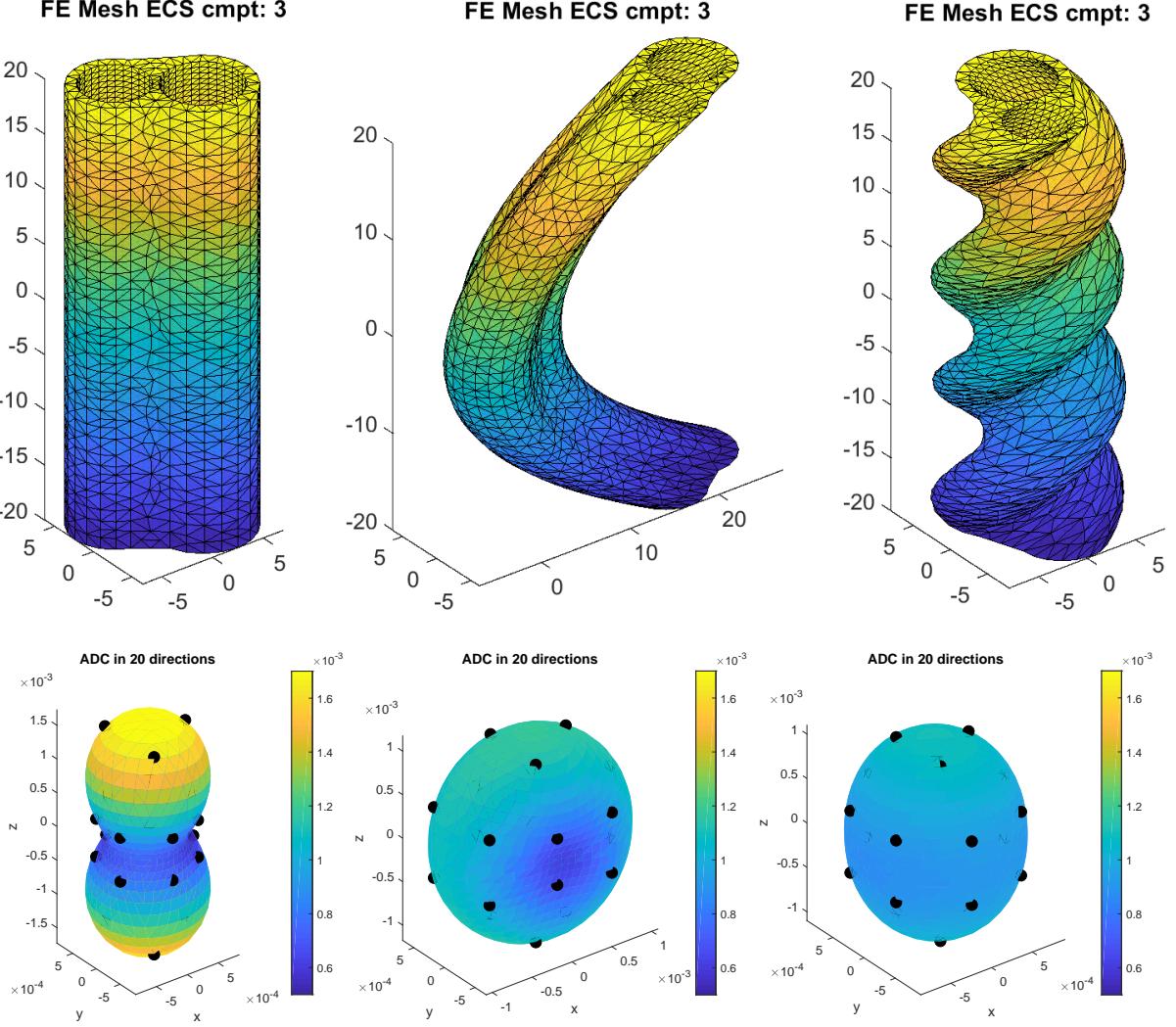


Figure 11: Geometry: 2 cylinders, no myelin layer, tight wrap ECS, $\text{ecs_gap} = 0.3$, $\kappa = 0\text{m/s}$, $\sigma^{\text{out}} = \sigma^{\text{ecs}} = 2 \times 10^{-3}\text{mm}^2/\text{s}$, PGSE ($\delta = 2.5\text{ms}$, $\Delta = 5\text{ms}$). Left: canonical configuration. Middle: bend parameter = 0.05. Right: twist parameter = 0.30. Top: FE mesh of the ECS (the FE mesh of the axon compartments numbered 1 and 2 not shown). Bottom: interpolated values of the HADC on the unit sphere, and then the sphere was distorted to reflect the value of the HADC. The color axis also gives the value of the HADC in the various gradient directions. The black dots indicate the 20 original gradient-directions in which the HADC was simulated. The spherical harmonics interpolation takes the 20 original directions into 900 directions uniformly distributed on the sphere.

6.5 Neuron examples

In Figures 12, 13, and 14 we display figures from the functions `plot_femesh`, `plot_field`, `plot_signal`, `plot_hardi`. The geometrical configuration is the spindle neuron `03b_spindle6aFI`. The intrinsic diffusion coefficient is set to $\mathcal{D}_0 = 2 \times 10^{-3}\text{mm}^2/\text{s}$, with nonpermeable membrane. We simulated 2 diffusion-encoding sequences:

1. f_1 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$);
2. f_2 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 73\text{ms}$).

6.6 Matrix Formalism examples

Below we display some example outputs from the Matrix Formalism Module. The geometrical configuration is one dendrite branch of a spindle neuron, `03b_spindle6aACC_dendrites_2`, shown in Figure 15.

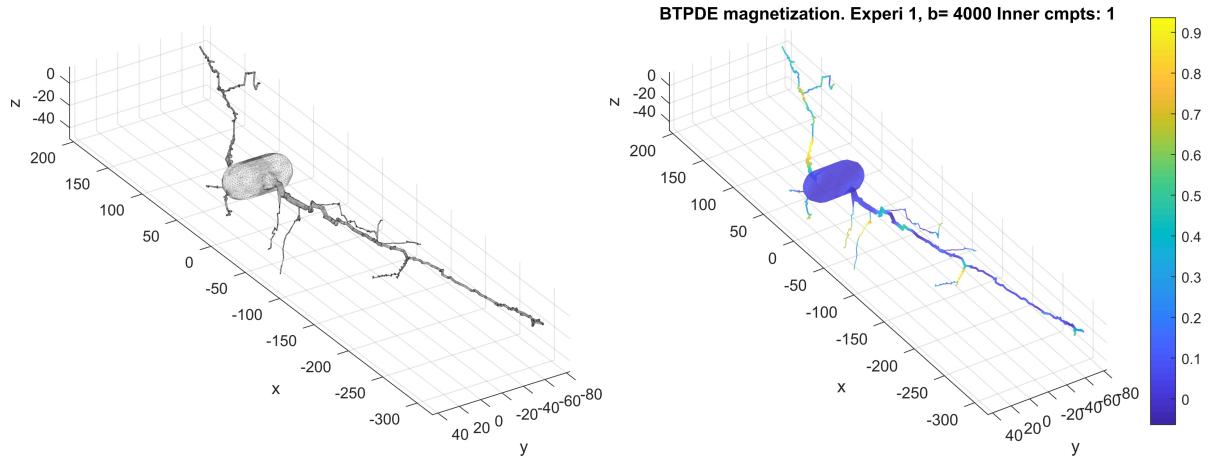


Figure 12: Left: The finite elements mesh of the neuron *03b_spindle6aFI* (using the command `plot_femesh`). The unit is μm . Right: The PDE solution (magnetization) on the neuron *03b_spindle6aFI* in the diffusion-encoding direction $(1, 1, 1)$ (using the command `plot_field`). The diffusion-encoding sequence is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$). The b -value is $b = 4000\text{s/mm}^2$.

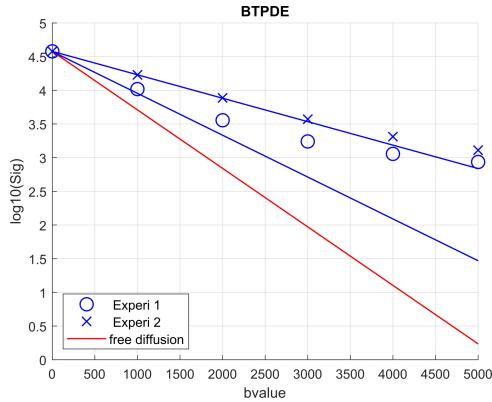


Figure 13: The (non-normalized) simulated signal of the neuron *03b_spindle6aFI* in the diffusion-encoding direction $(1, 1, 1)$ (using the command `plot_signal`). For Experiment 1, the diffusion-encoding sequence is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$). For Experiment 2, the diffusion-encoding sequence is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 73\text{ms}$).

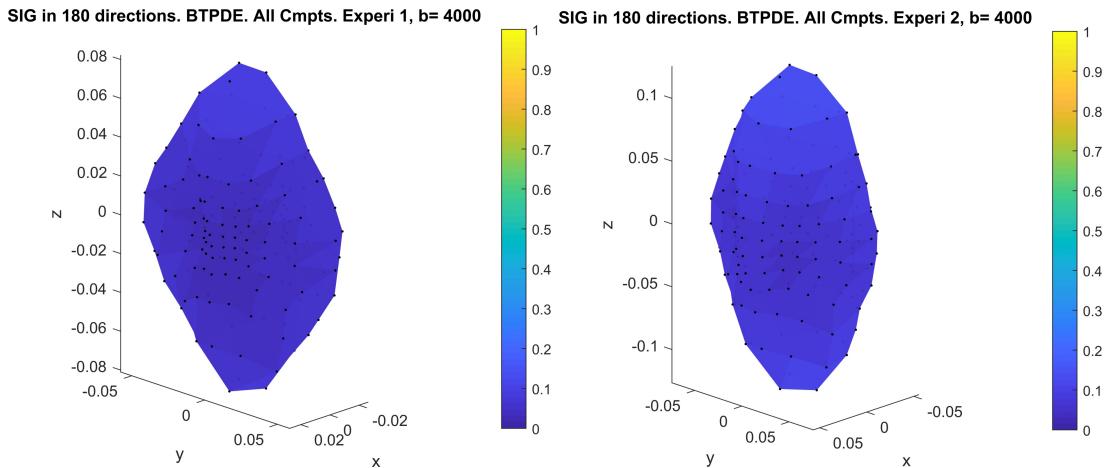


Figure 14: The simulated signal in 180 diffusion-encoding directions for the neuron *03b_spindle6aFI* (using the command `plot_hardi`). Left: The signal in 180 directions with PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$) and $b = 4000\text{s/mm}^2$. Right: The signal in 180 directions with PGSE ($\delta = 10.6\text{ms}$, $\Delta = 73\text{ms}$) and $b = 4000\text{s/mm}^2$.

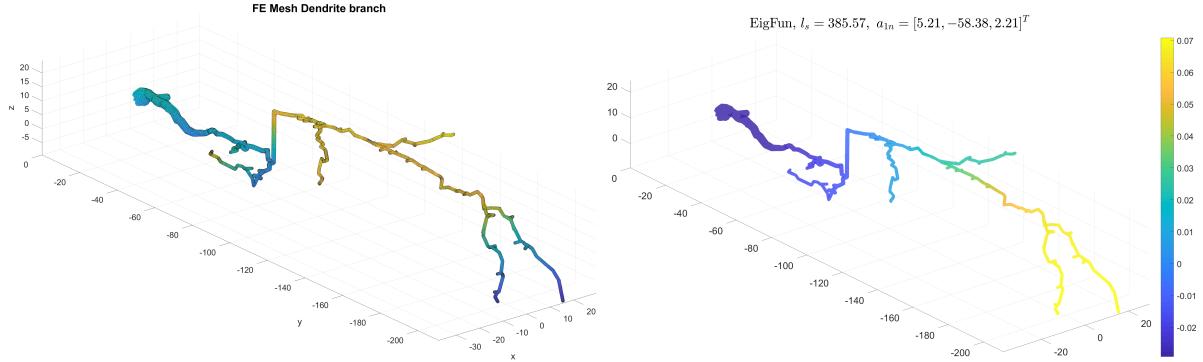


Figure 15: Left: The finite elements mesh of the spindle neuron dendrite branch `03b_spindle6aACC_dendrites_2` (using the command `plot_femesh`). Right: The eigenfunction with the associated length scale $L = 385.57\mu\text{m}$ and the first moments vector $\mathbf{a}_{1n} = [5.21, -58.38, 2.21]^\top$ (using the command `plot_field`).

The intrinsic diffusion coefficient is set to $D_0 = 2 \times 10^{-3}\text{mm}^2/\text{s}$, with an impermeable membrane. We simulated 2 diffusion-encoding sequences:

1. f_1 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$),
2. f_2 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 73\text{ms}$).

7 Commented drivers for typical simulations

7.1 Solve BTPDE, HADC, MF or multilayer models and compare apparent diffusion coefficients

```

1 %DRIVER_SPINDOCTOR Solve BTPDE, HADC, MF or multilayer.
2 %   Compare different ADC. Plot results in many directions.
3 %
4 %   It is highly recommended to read this driver to understand the
5 %   workflow
6 %   of SpinDoctor.
7 %
8 %   The user is advised to read the latest version
9 %   from \url{https://github.com/jingrebeccali/SpinDoctor}
10
11 clear
12 restoredefaultpath
13
14 % Add SpinDoctor
15 addpath(genpath("src"));
16
17 %% Define inputs
18
19 % Define the directory where the input file is kept
20 addpath input_files
21
22 % Load input parameters using input file script. The following
23 % structures will be added to the workspace:
24 %   params_cells
25 %   params_domain

```

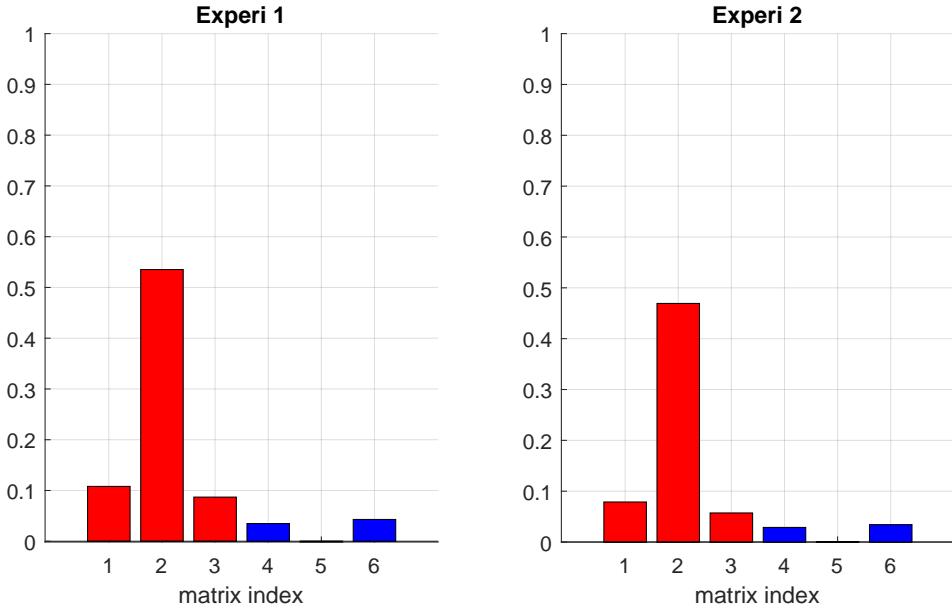


Figure 16: The 6 entries of the normalized diffusion tensor $\mathbf{D}^{\text{MF}}(f)/\mathcal{D}_0$ for the dendrite branch. Indices 1 to 6 are in the order of $\{D_{xx}^{\text{MF}}(f), D_{yy}^{\text{MF}}(f), D_{zz}^{\text{MF}}(f), D_{xy}^{\text{MF}}(f), D_{xz}^{\text{MF}}(f), D_{yz}^{\text{MF}}(f)\}/\mathcal{D}_0$. The diagonal entries of $\mathbf{D}^{\text{MF}}(f)/\mathcal{D}_0$ are shown in red and the off-diagonal entries shown in blue (if non-zero). From left to right: Experiment 1: f_1 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$), Experiment 2: f_2 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 73\text{ms}$). (Using the command `plot_diffusion_tensor`).

```

26 %     experiment
27 % Choose input script:
28
29 % define_params_1axon_multilayer;
30 % define_params_1sphere_multilayer;
31 % define_params_15spheres;
32 % define_params_2axons_deform;
33 % define_params_5axons_myelin_relax;
34 % define_params_neuron;
35 define_params_4axons_flat;
36 % define_params_30axons_flat;
37
38 % Choose to see some of the typical plots or not
39 do_plots = true;
40
41
42 %% Prepare experiments
43
44 % Create geometrical configuration
45 cells = create_cells(params_cells);
46
47 % Set up the PDE model in the geometrical compartments.
48 params_domain = prepare_pde(params_cells, params_domain);
49
50 % Prepare experiments (gradient sequence, bvalues, qvalues)
51 experiment = prepare_experiments(experiment);
52
53 % Create or load finite element mesh
54 [femesh, surfaces] = create_femesh(cells, params_cells, params_domain);
55
56 % Get volume and surface area quantities from mesh

```

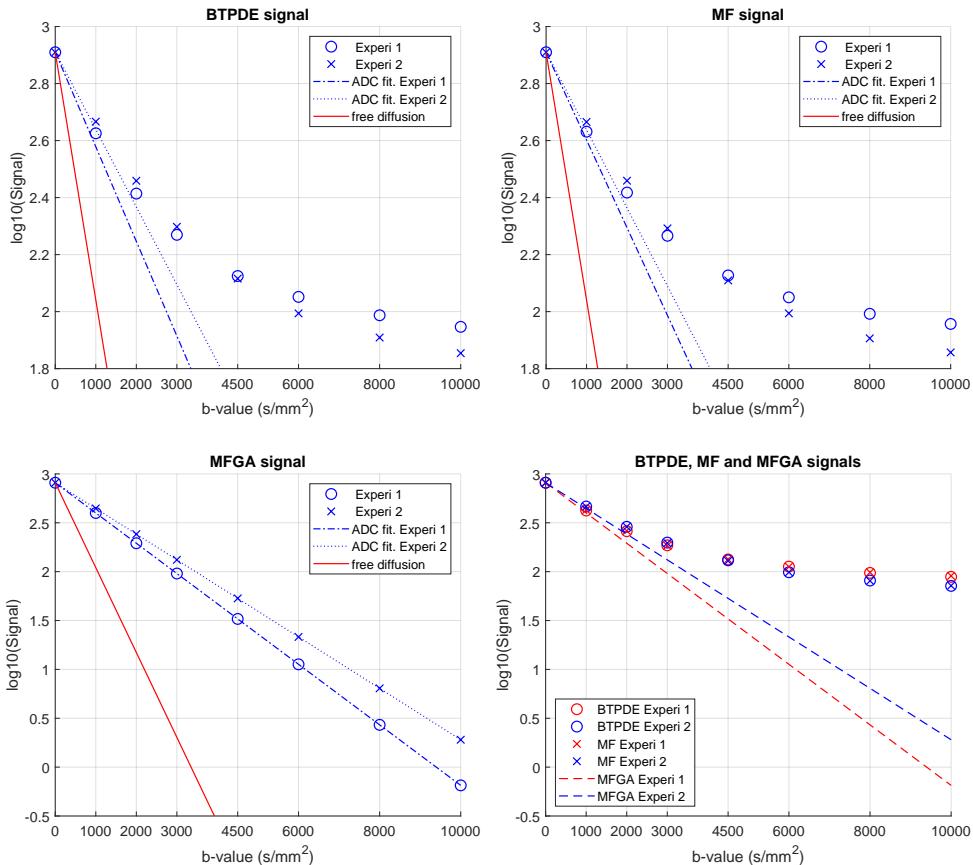


Figure 17: Top left: the BTPDE signals. Top right: the MF signals. Bottom left: the MFGA signals. The markers indicate the values of the simulated signals, the blue lines indicate the ADC fit of those signals, the red line is the signal of free diffusion at the intrinsic diffusion coefficient. Bottom right: the BTPDE, the MF, the MFGA signals plotted together. Experiment 1: f_1 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$), Experiment 2: f_2 is PGSE ($\delta = 10.6\text{ms}$, $\Delta = 73\text{ms}$). Diffusion-encoding direction $\mathbf{u}_g = (1, 1, 0)^T / \sqrt{2}$ (using the command `plot_signal`).

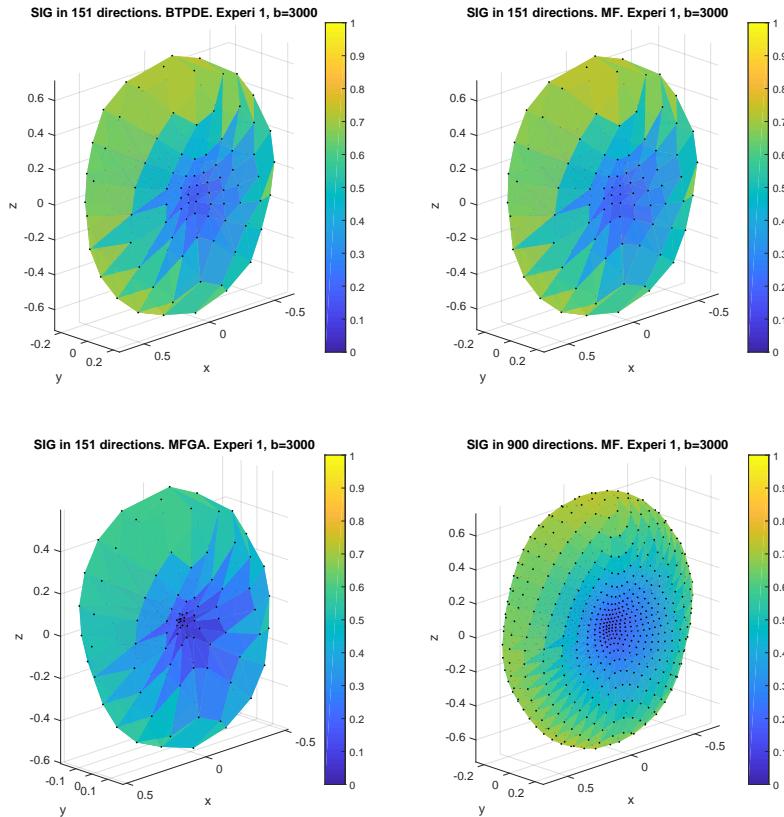


Figure 18: Top left: the BTPDE signals, S^{BTPDE}/S_0 , in 151 diffusion-encoding directions. Top right: the MF signals, S^{MF}/S_0 , in 151 diffusion-encoding directions. Bottom left: the MFGA signals, S^{MFGA}/S_0 , in 151 diffusion-encoding directions. Bottom right: the MF signals, S^{MF}/S_0 , in 900 diffusion-encoding directions. The black points are the magnitude of the signal attenuation multiplied by the diffusion-encoding direction. The color indicates the value of the signal attenuation. $b = 3000\text{s/mm}^2$, PGSE, $\delta = 10.6\text{ms}$, $\Delta = 13\text{ms}$. (Using the command `plot_hardi`).

```

57 [volumes, surface_areas] = get_vol_sa(femesh);
58
59 % Compute volume weighted mean of diffusivities over compartments
60 % Take trace of each diffusion tensor, divide by 3
61 mean_diffusivity = trace(sum(params_domain.diffusivity ...
62 .* shiftdim(volumes, -1), 3)) / (3 * sum(volumes));
63
64 % Initial total signal
65 initial_signal = params_domain.initial_density * volumes';
66
67
68 %% Perform experiments
69
70 % Short time approximation (STA) of the ADC
71 [sta_adc, sta_adc_allcmpts] = compute_adc_sta(femesh, params_domain,
72 experiment);
73
74 % Free diffusion signal
75 free = compute_free_diffusion(experiment.bvalues, params_domain.
76 diffusivity, ...
77 volumes, params_domain.initial_density);
78
79 % Perform BTPDE experiments
80 if isfield(experiment, "btpde")
81     % Solve BTPDE
82     btpde = solve_btpde(femesh, params_domain, experiment);
83
84     % Fit ADC from signal
85     btpde_fit = fit_signal(btpde.signal, btpde.signal_allcmpts,
86     experiment.bvalues);
87
88     % BTPDE direction averaged magnetization
89     btpde.magnetization_avg = average_magnetization(btpde.magnetization
90         );
91 end
92
93 % Perform HADC experiment
94 if isfield(experiment, "hadc")
95     % Solve HADC model
96     hadc = solve_hadc(femesh, params_domain, experiment);
97 end
98
99 % Perform MF experiments
100 if isfield(experiment, "mf")
101     % Perform Laplace eigendecomposition
102     eiglim = length2eig(experiment.mf.length_scale, mean_diffusivity);
103     lap_eig = compute_laplace_eig(femesh, params_domain, eiglim,
104     experiment.mf.neig_max);
105
106     % Compute length scales of eigenvalues
107     lap_eig.length_scales = eig2length(lap_eig.values, mean_diffusivity
108         );
109
110     % Compute the JN value that relates the eigenmodes to their
111     % contribution
112     % to the Matrix Formalism signal for a diffusion-encoding sequence
113     mf_jn = compute_mf_jn(lap_eig.values, mean_diffusivity, experiment)
114         ;

```

```

107
108 % Compute the Matrix Formalism effective diffusion tensor
109 diffusion_tensor = compute_mf_diffusion_tensor(lap_eig, mf_jn,
110     mean_diffusivity);
111
112 % Compute MF magnetization
113 mf = solve_mf(femesh, params_domain, experiment, lap_eig);
114
115 % Fit ADC from MF signal
116 mf_fit = fit_signal(mf.signal, mf.signal_allcmpts, experiment.
117     bvalues);
118
119 % MF direction averaged magnetization
120 mf.magnetization_avg = average_magnetization(mf.magnetization);
121
122 % Compute MFGA signal
123 mfga = compute_mfga_signal(experiment, initial_signal,
124     diffusion_tensor);
125 end
126
127 % Perform multilayer experiment
128 if isfield(experiment, "multilayer")
129     % Solve multilayer analytical model
130     ml_signal = solve_multilayer(params_cells, params_domain,
131         experiment);
132 end
133
134 %% Postprocess
135
136 if ~do_plots
137     return
138 end
139
140 if params_cells.shape == "sphere" || params_cells.shape == "cylinder"
141     % Plot cells in canonical configuration
142     plot_cells(cells, params_cells);
143 end
144
145 % Plot surface triangulation
146 plot_surface_triangulation(surfaces);
147
148 % Plot the finite element mesh
149 plot_femesh(femesh, params_domain.compartments);
150 % plot_femesh_everywhere(femesh, "");
151
152 % Plot information about the geometry
153 plot_geometry_info(params_domain, volumes, surface_areas);
154
155 % Get sizes
156 ncompartment = length(params_domain.compartments);
157 namplitude = length(experiment.values);
158 nsequence = length(experiment.sequences);
159 ndirection = experiment.ndirection;
160
161 % Plot BTPDE magnetization in some directions
162 if isfield(experiment, "btpde")
163     for idir = 1 :ndirection

```

```

161 for iseq = 1 % :nsequence
162     for iamp = 1 % :namplitude
163         b = experiment.bvalues(iamp, iseq);
164         title_str = sprintf(...%
165             "BTPDE magnetization. Sequence %d of %d, b=%.2f",
166             ...
167             iseq, nsequence, b);
168         field = btpde.magnetization(:, iamp, iseq, idir);
169         % plot_field(femesh, field, params_domain.compartments,
170         %             title_str);
171         plot_field_everywhere(femesh, field, title_str);
172         % caxis([0 1]);
173     end
174 end
175 clear field
176
177 % Plot results in one or many directions
178 if ndirection == 1
179     % Plot ADC short time approximation
180     plot_adc(sta_adc, sta_adc_allcmpts, "STA");
181
182     % Plot BTPDE results
183     if isfield(experiment, "btpde")
184         % Plot the BTPDE signal, the S0*exp(-ADC*b) curve, and the
185         % free diffusion curves together.
186         plot_signal(experiment.bvalues, btpde.signal_allcmpts, free.
187             signal_allcmpts, ...
188             btpde_fit.S0_allcmpts, btpde_fit.adc_allcmpts, "BTPDE")
189
190     % Plot ADC fitted from BTPDE
191     plot_adc(btpde_fit.adc, btpde_fit.adc_allcmpts, "BTPDE");
192
193     % Plot computational time
194     plot_timing(btpde.itertimes, femesh, "BTPDE", "B-value");
195 end
196
197 % Plot results from HADC experiment
198 if isfield(experiment, "hadc")
199     % Plot ADC
200     plot_adc(hadc.adc, hadc.adc_allcmpts, "HADC");
201
202     % Plot computational time
203     plot_timing(hadc.itertimes, femesh, "HADC", "Compartment");
204 end
205 else
206     % Plot STA ADC
207     plot_hardi(experiment.directions, sta_adc_allcmpts, "STA ADC all
208         compartments");
209
210     % Plot BTPDE signal in all directions
211     if isfield(experiment, "btpde")
212         % Signal before applying the gradient pulse sequence
213         S0 = sum(params_domain.initial_density .* volumes);
214
215         % Plot normalized signals
216         title_str = "BTPDE total magnetization (normalized)";

```

```

215 plot_hardi(experiment.directions, btpde.signal_allcmpts / S0,
216     title_str);
217
218 % Plot HADC in all directions
219 if isfield(experiment, "hadc")
220     % Plot normalized ADC (ADC/DO)
221     title_str = sprintf("HADC all compartments");
222     plot_hardi(experiment.directions, hadc.adc_allcmpts /
223         mean_diffusivity, title_str);
224 end
225 end % One or many directions
226
227 if isfield(experiment, "mf")
228     % Plot Matrix Formalism effective diffusion tensor
229     plot_diffusion_tensor(diffusion_tensor, mean_diffusivity);
230
231     % Relative error between BTPDE and MF signal
232     signal_allcmpts_relerr = abs(mf.signal_allcmpts - btpde.
233         signal_allcmpts) ...
234         ./ max(abs(btpde.signal_allcmpts), [], 3);
235
236     % Difference between BTPDE and MF signal, normalized by initial
237     % signal
238     signal_allcmpts_abserr_vol = abs(mf.signal_allcmpts - btpde.
239         signal_allcmpts) ./ initial_signal;
240
241 if ndirection == 1
242     % Plot BTPDE, MF and MFGA signals
243     plot_signal(experiment.bvalues, btpde.signal_allcmpts, free.
244         signal_allcmpts, btpde_fit.S0_allcmpts, btpde_fit.adc, "
245             BTPDE signal");
246     plot_signal(experiment.bvalues, mf.signal_allcmpts, free.
247         signal_allcmpts, mf_fit.S0_allcmpts, mf_fit.adc, "MF signal
248             ");
249     % plot_signal(experiment.bvalues, mfga.signal_allcmpts, free.
250         signal_allcmpts, initial_signal, mfga.adc_allcmpts, "MFGA
251         signal");
252     plot_signal_btpde_mf(experiment, btpde.signal_allcmpts, mf.
253         signal_allcmpts, mfga.signal_allcmpts);
254
255     % Display difference
256     disp("Error (normalized by initial signal)");
257     disp(signal_allcmpts_abserr_vol);
258     disp("Relative error:");
259     disp(signal_allcmpts_relerr);
260 else
261     % Plot HARDI signal
262     plot_hardi(experiment.directions, real(btpde.signal_allcmpts) /
263         initial_signal, "BTPDE signal")
264     plot_hardi(experiment.directions, real(mf.signal_allcmpts) /
265         initial_signal, "MF signal")
266
267     % Plot relative difference
268     fig_title = "Rel diff between BTPDE and MF";
269     plot_hardi(experiment.directions, signal_allcmpts_relerr,
270         fig_title);

```

```
258
259 % Plot difference normalized by volume
260 fig_title = "Diff between BTPDE and MF normalized by volume";
261 plot_hardi(experiment.directions, signal_allcmpts_abserr_vol,
262             fig_title);
263 end
```

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