On counterfactual inference with factors models and nearest neighbors

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To think about

- Limitations of the model and how to possibly relax it
- Limitations of the analysis and how to possible sharpen it

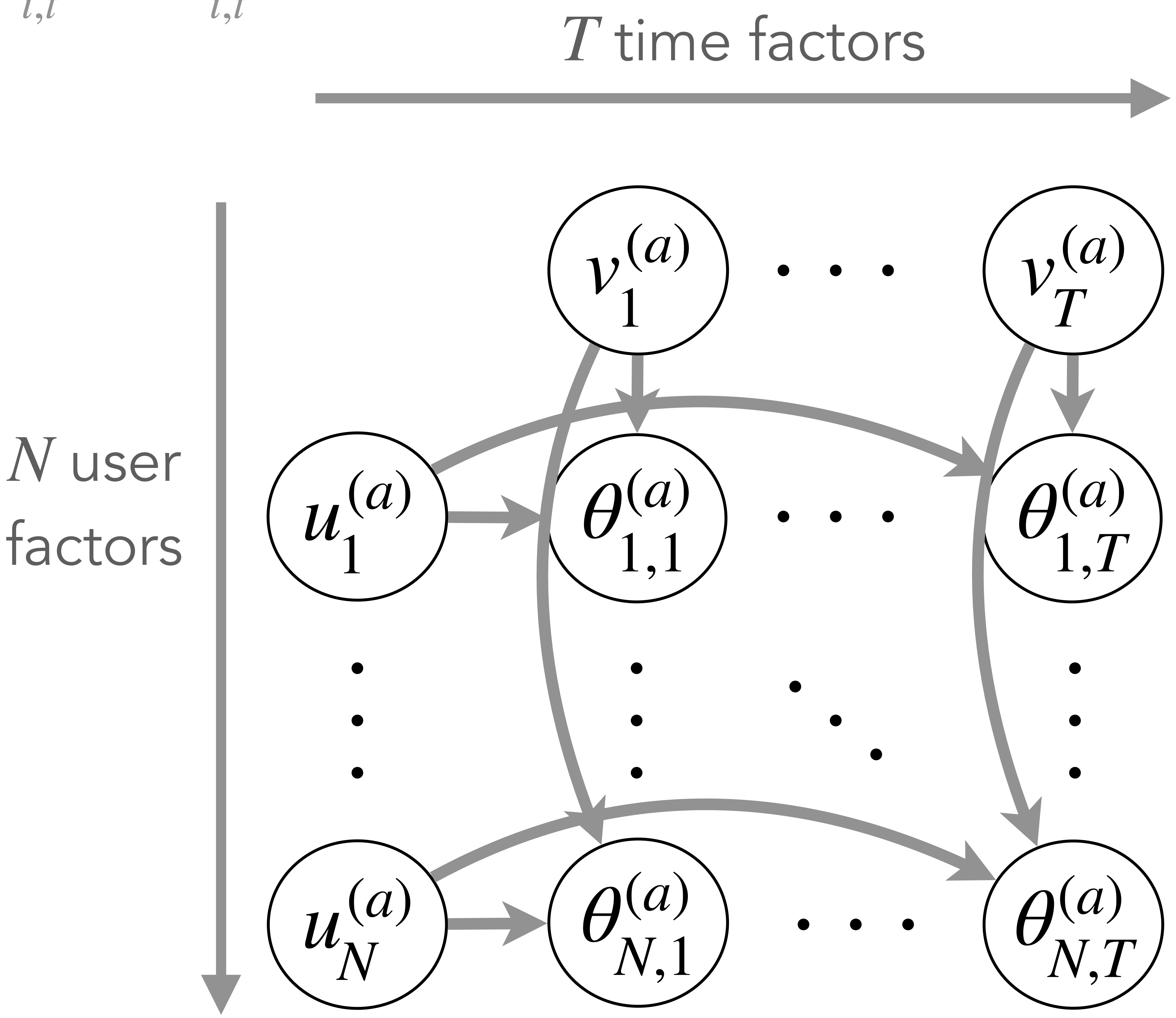
Recap: Latent factor model

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

$$\theta_{i,t}^{(a)} \triangleq \langle u_i^{(a)}, v_t^{(a)} \rangle$$

user factor (e.g., personal traits)

time factor (e.g., societal, weather changes)



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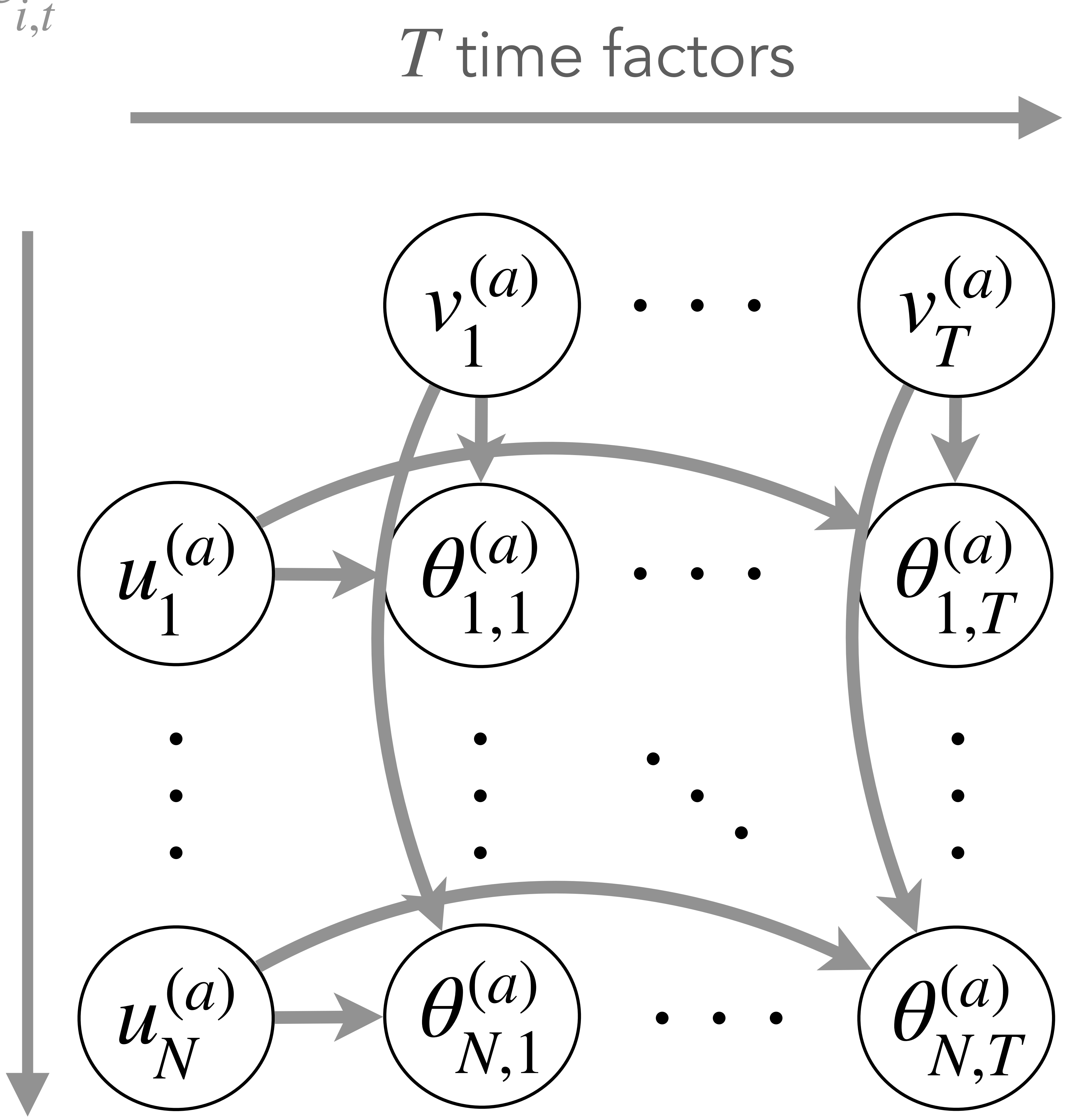
$$\theta_{i,t}^{(a)} \triangleq \langle u_i^{(a)}, v_t^{(a)} \rangle$$

- A generalization of mixed effects model

- No parametric assumptions on the **unknown** distributions of latent factors or noise

- Paper also considers $\theta_{i,t}^{(a)} \triangleq f(u_i^{(a)}, v_t^{(a)})$ for **unknown** Lipschitz f



User nearest neighbors estimator for treatment a

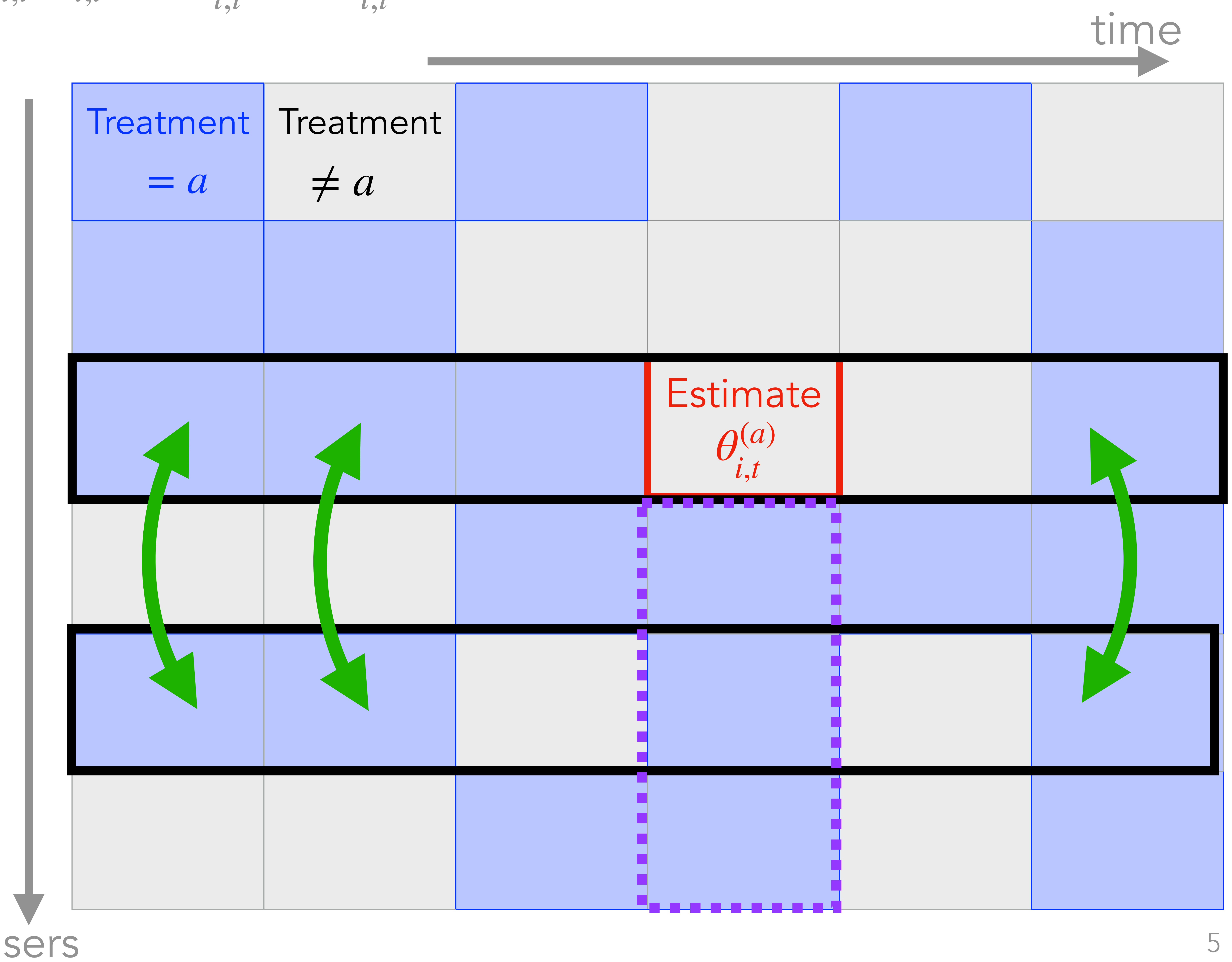
$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

1. Compute distance between user pairs i, j under treatment a **using all data**

$$\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)} + \frac{\hat{\sigma}_\rho}{\sqrt{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}}$$

2. Average over user neighbors treated with a at time t

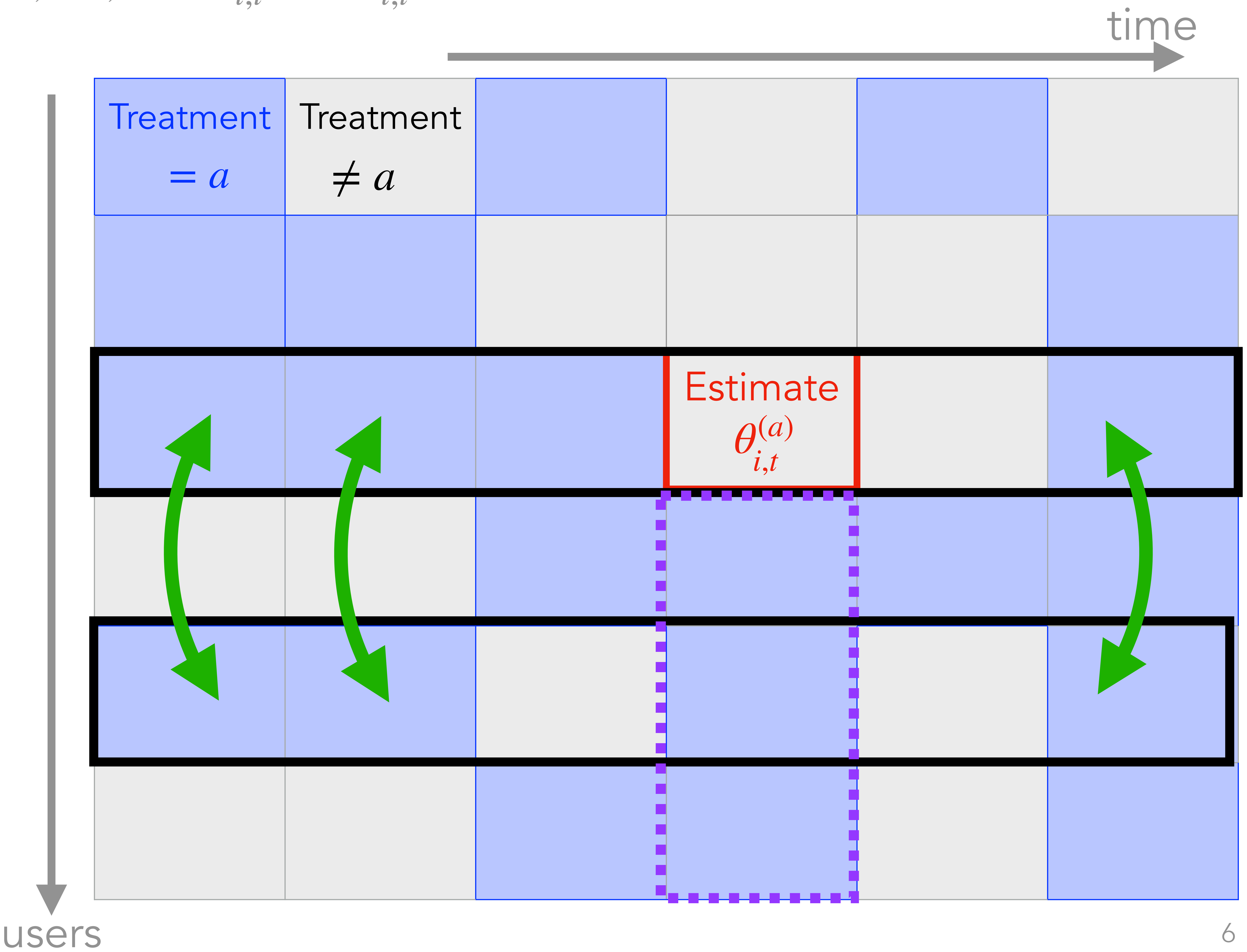
$$\hat{\theta}_{i,t,\text{user-NN}}^{(a)} = \frac{\sum_{j=1}^{N} R_{j,t} \mathbf{1}(\rho_{i,j}^{(a)} \le \eta, A_{j,t} = a)}{\sum_{j=1}^{N} \mathbf{1}(\rho_{i,j}^{(a)} \le \eta, A_{j,t} = a)}$$



User nearest neighbors estimator for treatment a

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

Note that the estimator is agnostic to sampling policy or the generative model for the latent factors



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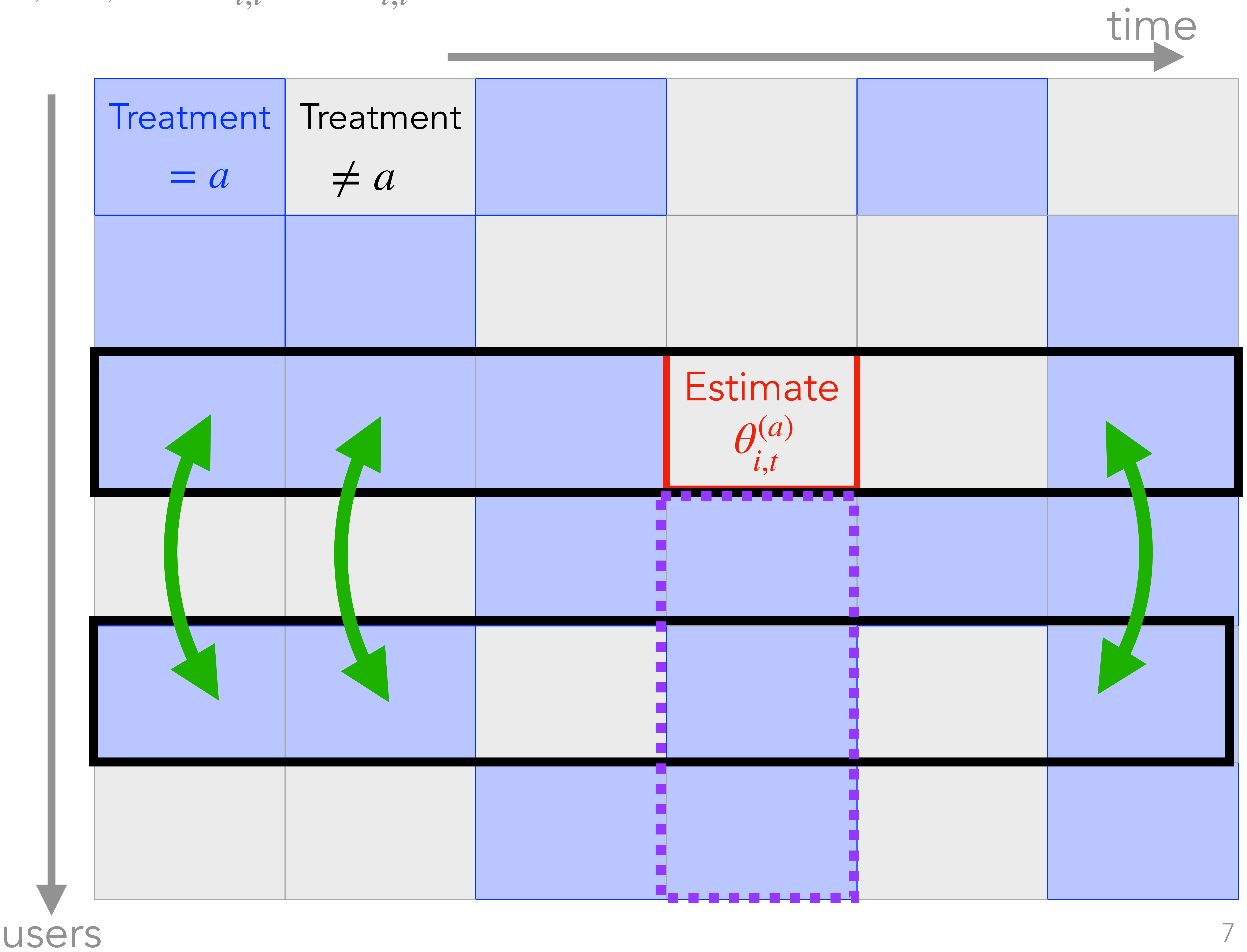

Something should bother you about this estimator when treatments are sampled sequentially !!!

User-NN: A non-asymptotic guarantee for each (i, t, a)

Informal theorem: [**Dwivedi**-Tian-Tomkins-Klasnja-Murphy-Shah '22a]
For suitably chosen η & **under regularity conditions on latent factors**

- iid latent factors, sub-Gaussian noise
- sequentially adaptive policies with conditionally independent treatments across users that choose a with probability $\geq p^{\dagger}$

User-NN: A non-asymptotic guarantee for each (i, t, a)

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For suitably chosen η & **under regularity conditions on latent factors**

- iid latent factors, sub-Gaussian noise
- sequentially adaptive policies with conditionally independent treatments across users that choose a with probability $\geq p^{\dagger}$

for each user i at each time t, with high probability

$$(\widehat{\theta}^{(a)}_{i,t,\text{user–NN}} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\sqrt{p^2 T}} + \frac{1}{p \cdot \#Neighbors}$$

($\dagger$Thus p can not decay faster than $\gtrsim T^{-1/2}$)

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$$(\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\sqrt{p^2 T}} + \frac{1}{pN/M}$$

User factor distribution

$\downarrow$

(user factors ~ uniform over a finite set of size M)

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$$(\widehat{\theta}^{(a)}_{i,t,\text{user–NN}} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\sqrt{p^2 T}} + \frac{1}{(pN)^{2/(d+2)}}$$

User factor distribution

↓

(user factors ~ uniform over a finite set of size M)

(user factors ~ Uniform in $[-1,1]^d$)

User-NN guarantees

- Asymptotic **confidence intervals** as $N, T \to \infty$:

$$\widehat{\theta}^{(a)}_{i,t,\text{user–NN}} \pm \frac{1.96\,\widehat{\sigma}}{\sqrt{\#\text{neighbors}_{i,t,a}}}$$

$$\downarrow$$

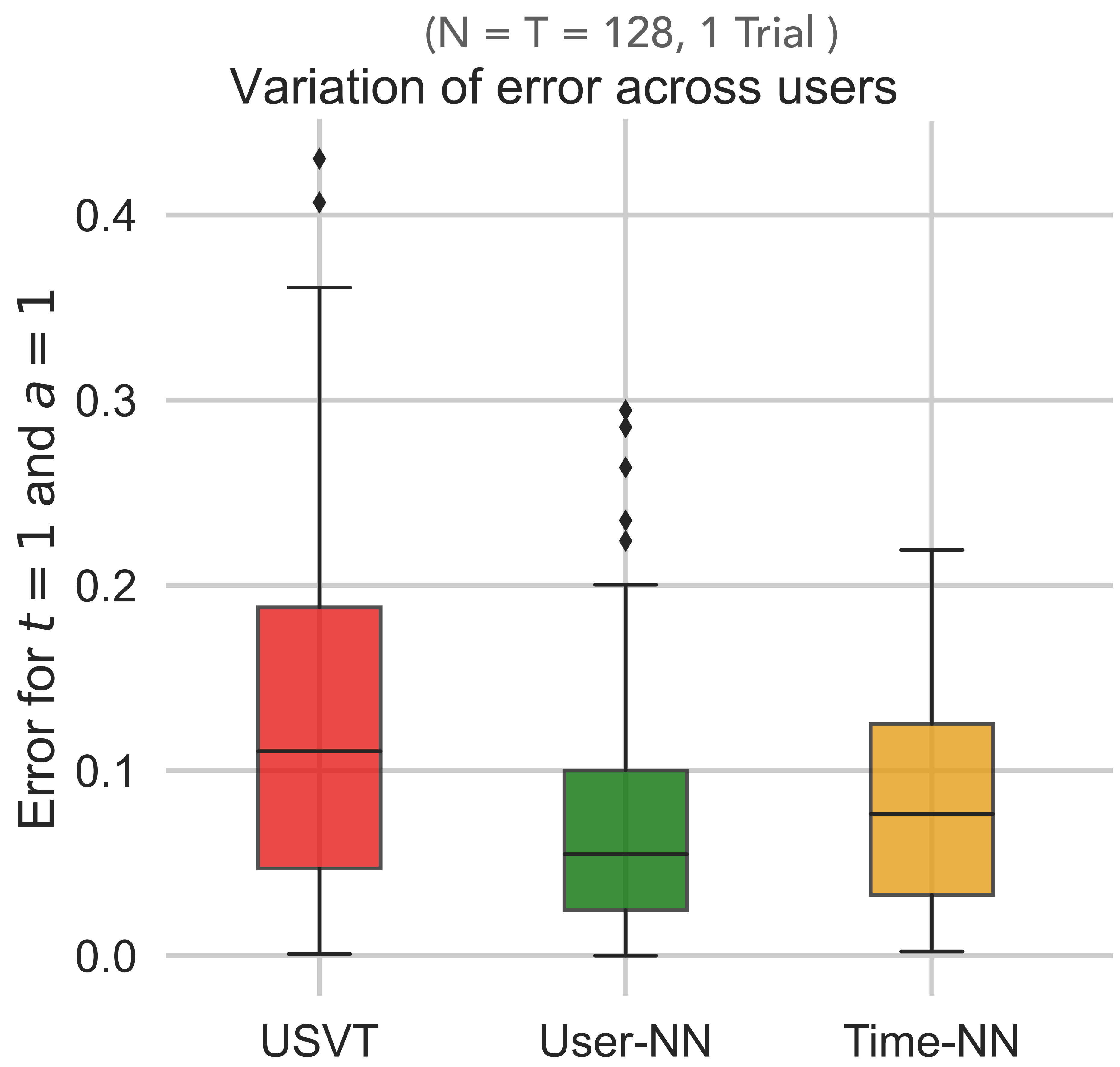
Confidence intervals for treatment effect $\theta^{(1)}_{i,t} - \theta^{(0)}_{i,t}$

$$| \widehat{\theta}^{(a)}_{i,t,\text{user–NN}} - \theta^{(a)}_{i,t} | = \tilde{O}\left(\frac{1}{T^{1/4}} + \frac{1}{\sqrt{N}} \right)$$

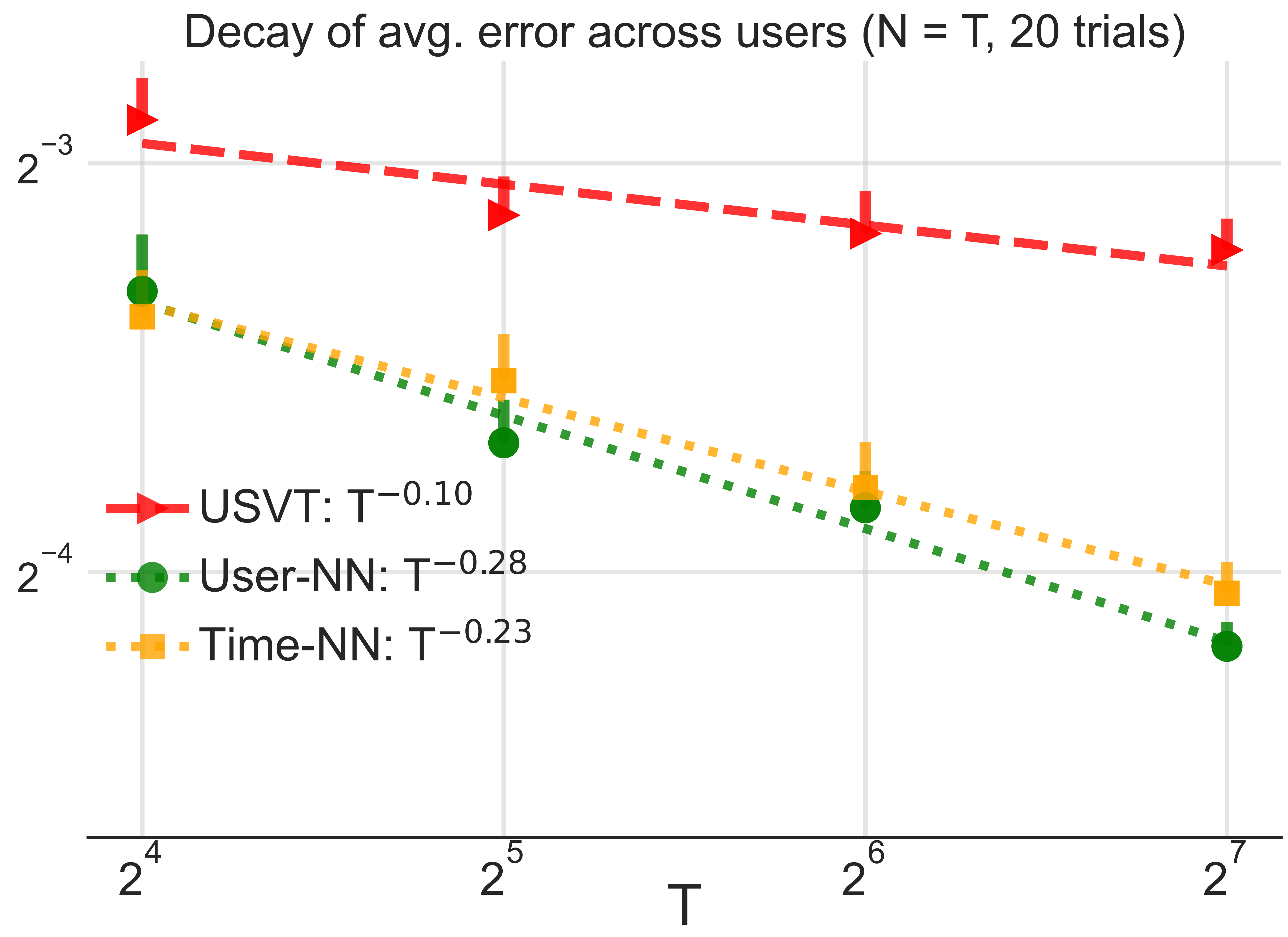
$$| \widehat{\theta}^{(a)}_{i,t,\text{time–NN}} - \theta^{(a)}_{i,t} | = \tilde{O}\left(\frac{1}{N^{1/4}} + \frac{1}{\sqrt{T}} \right)$$

Simulation results

Uniform latent factors on $[-0.5, 0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)



A baseline algorithm from [Chatterjee 2014]



USVT: A baseline algorithm from [Chatterjee 2014]

We prove a general error bound for user NN (with actions sampled by learning policies)

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η'

e_T

NN bias due to threshold

Error in NN distance

NN noise variance

NN bias inflation due to **learning** policy

$$\lambda_\star \triangleq \lambda_{\min}(\mathbf{\Sigma}_v) \quad \text{where } \mathbf{\Sigma}_v = \mathbb{E}[v_{t'} v_{t'}^\top]$$

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \mathbf{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Steps towards deriving the general bound

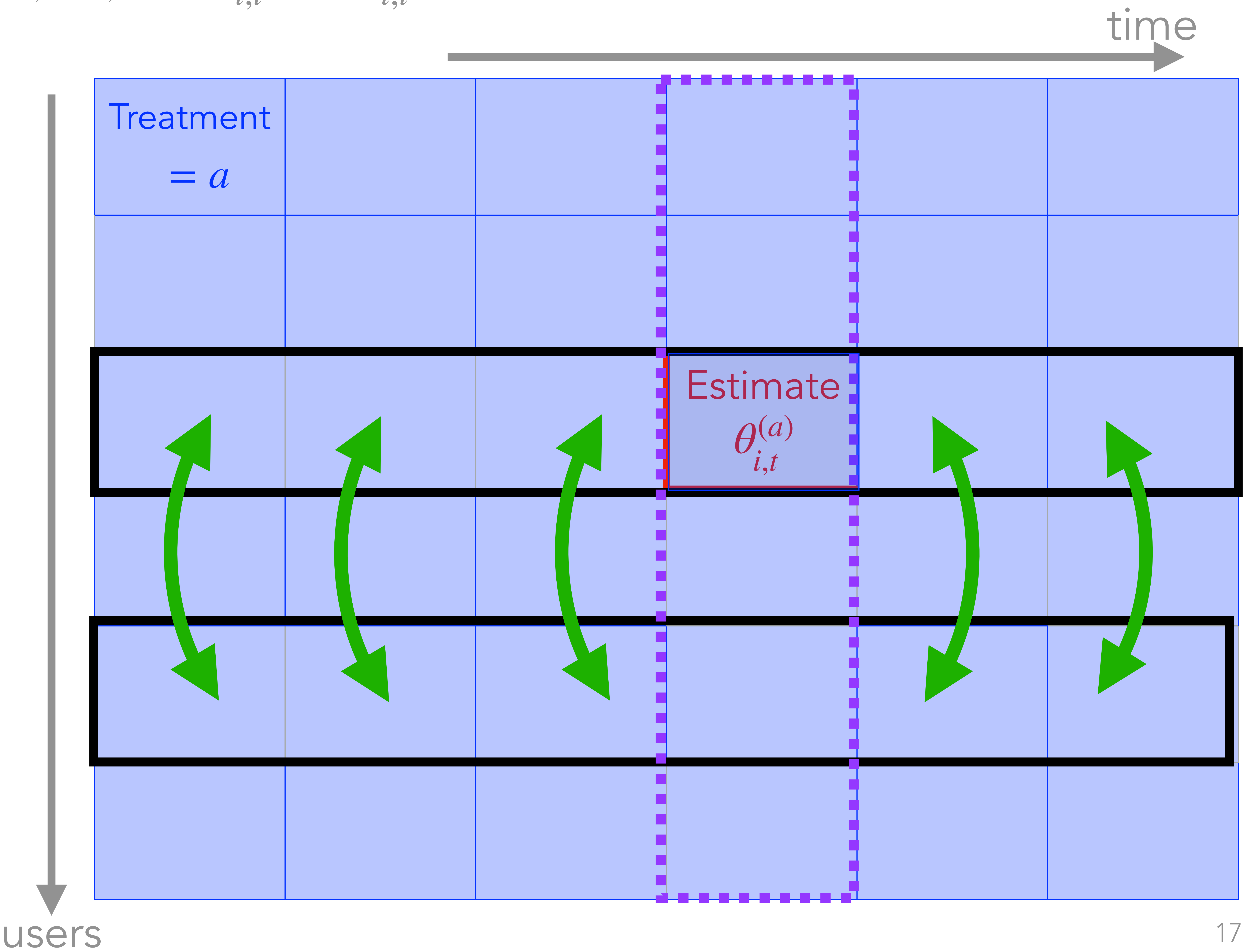
User-NN with data split (no missingness)

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

$$\rho_{i,j}^{(a)} = \frac{\sum_{t' \neq t} (R_{i,t'} - R_{j,t'})^2}{T-1}$$

$$\widehat{\theta}_{i,t,\text{user-NN}}^{(a)} = \frac{\sum_{j=1}^{N} R_{j,t} \mathbf{1}(\rho_{i,j}^{(a)} \leq \eta)}{\sum_{j=1}^{N} \mathbf{1}(\rho_{i,j}^{(a)} \leq \eta)}$$

Why may we want to do a data split?



Proof outline for user-NN (no missingness)

- Simple case: Always assign $A_{j,t} = a$ and $\theta_{i,t}^{(a)} \triangleq u_i v_t$

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

Proof outline for user-NN (no missingness)

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- $\widehat{\theta}_{i,t,\text{user-NN}}^{(a)} = \dfrac{\sum_{j \in user\ nbrs} R_{j,t}}{|user\ nbrs|} = \dfrac{\sum_{j \in user\ nbrs} \theta_{j,t}^{(a)} + \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$

$$= \frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|} v_t + \frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$$

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

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$$= \underbrace{\frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|}}_{\hat{u}_i} v_t + \underbrace{\frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}}_{\bar{\varepsilon}_t}$$

user-nbrs = $\{j : \rho_{i,j}^{(a)} \leq \eta\}$

Proof outline for user-NN (no missingness)

- Simple case: Always assign $A_{j,t} = a$ and $\theta^{(a)}_{i,t} \triangleq u_i v_t$

- $$\widehat{\theta}^{(a)}_{i,t,\text{user–NN}} = \frac{\sum_{j \in user\ nbrs} R_{j,t}}{|user\ nbrs|} = \frac{\sum_{j \in user\ nbrs} \theta^{(a)}_{j,t} + \varepsilon^{(a)}_{j,t}}{|user\ nbrs|}$$

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- $|\theta^{(a)}_{i,t} - \widehat{\theta}^{(a)}_{i,t,\text{user–NN}}| = |u_i v_t - \widehat{\theta}^{(a)}_{i,t,\text{user–NN}}| \leq |u_i v_t - \hat{u}_i v_t| + |\bar{\varepsilon}_t|$

Re-expressing the distance without data-split

- Simple case: Always assign $A_{j,t} = a$ and $\theta^{(a)}_{i,t} \triangleq u_i v_t$

- $\rho^{(a)}_{i,j} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2}{T} = \frac{1}{T} \sum_{t'=1}^{T} \left[(u_i v_{t'} - u_j v_{t'})^2 + (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 + 2(u_i v_{t'} - u_j v_{t'})(\varepsilon_{i,t'} - \varepsilon_{j,t'}) \right]$

$$= (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$$

Re-expressing the distance without data-split

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

- $\rho_{i,j}^{(a)}$ depends on noise at time t
  $\implies$ user-nbrs = $\{j : \rho_{i,j}^{(a)} \leq \eta\}$ are correlated with noise at time t

- $\bar{\varepsilon}_t = \frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$

  —- need not behave as mean of iid noise and need not decay to 0

Re-expressing the distance without data-split

- $\rho^{(a)}_{i,j} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$
- $\rho^{(a)}_{i,j}$ depends on noise at time t
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Data splitting, or, excluding time t data in defining the distance $\rho^{(a)}_{i,j}$ breaks this dependence and the noise at time t is not correlated with user-nbrs.

–- need not behave as mean of iid noise and need not decay to 0

What do these three terms concentrate on?

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t' \neq t} v_{t'}^2}{T-1} + \frac{\sum_{t' \neq t} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T-1} + 2(u_i - u_j) \frac{\sum_{t' \neq t} v_t' (\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T-1}$

Inverting the distance to get a control on $|u_i - u_j|$

- Assume v_t, ε are bounded and $\mathbb{E}[v_t^2] = v_\star^2$ then
    - $|\rho_{i,j}^{(a)} - (u_i - u_j)^2 v_\star^2 - 2\sigma^2| \leq \frac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$ with probability $1 - \delta$
    - Treat δ as a constant
    - Rearranging terms $|u_i - u_j|^2 \leq \frac{1}{v_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \frac{C}{\sqrt{T}}\right)$

Univariate factors: A general error bound for user NN when $A_{j,t} \equiv a$ and we use data split, ignore time t data while computing distance)

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{T-1}}\right) \quad + \quad \frac{\sigma^2}{N_{i,\eta'-e_T}}$$

η' $\quad$ e_T

NN bias due to threshold

Error in NN distance

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$$v_\star^2 = \mathbb{E}[v_{t'}^2]$$

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

User-NN with data split under pure exploration $\mathbb{P}(A_{i,t} = a) = p$

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

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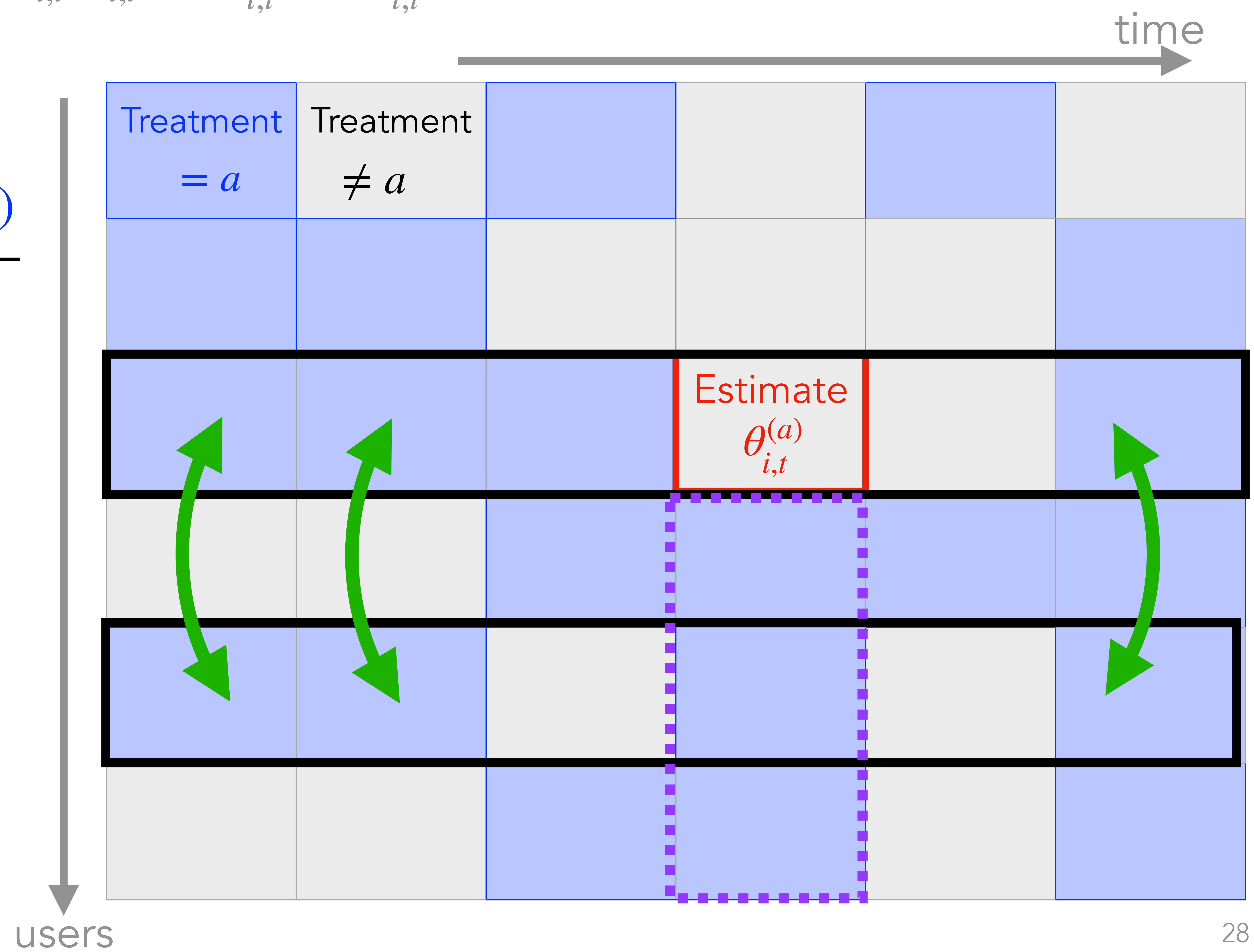
$$\widehat{\theta}_{i,t,\text{user-NN}}^{(a)} = \frac{\sum_{j=1}^{N} R_{j,t} \mathbf{1}(\rho_{i,j}^{(a)} \leq \eta, A_{j,t} = a)}{\sum_{j=1}^{N} \mathbf{1}(\rho_{i,j}^{(a)} \leq \eta, A_{j,t} = a)}$$



Univariate factors + pure exploration policy $\mathbb{P}(A_{i,t} = a) = p$: A general error bound for user NN with data split

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{p^2(T-1)}}\right) \quad + \quad \frac{\sigma^2}{pN_{i,\eta'-e_T}}$$

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User-NN with data split:
Will this trivially work when policy is sequential/learning?

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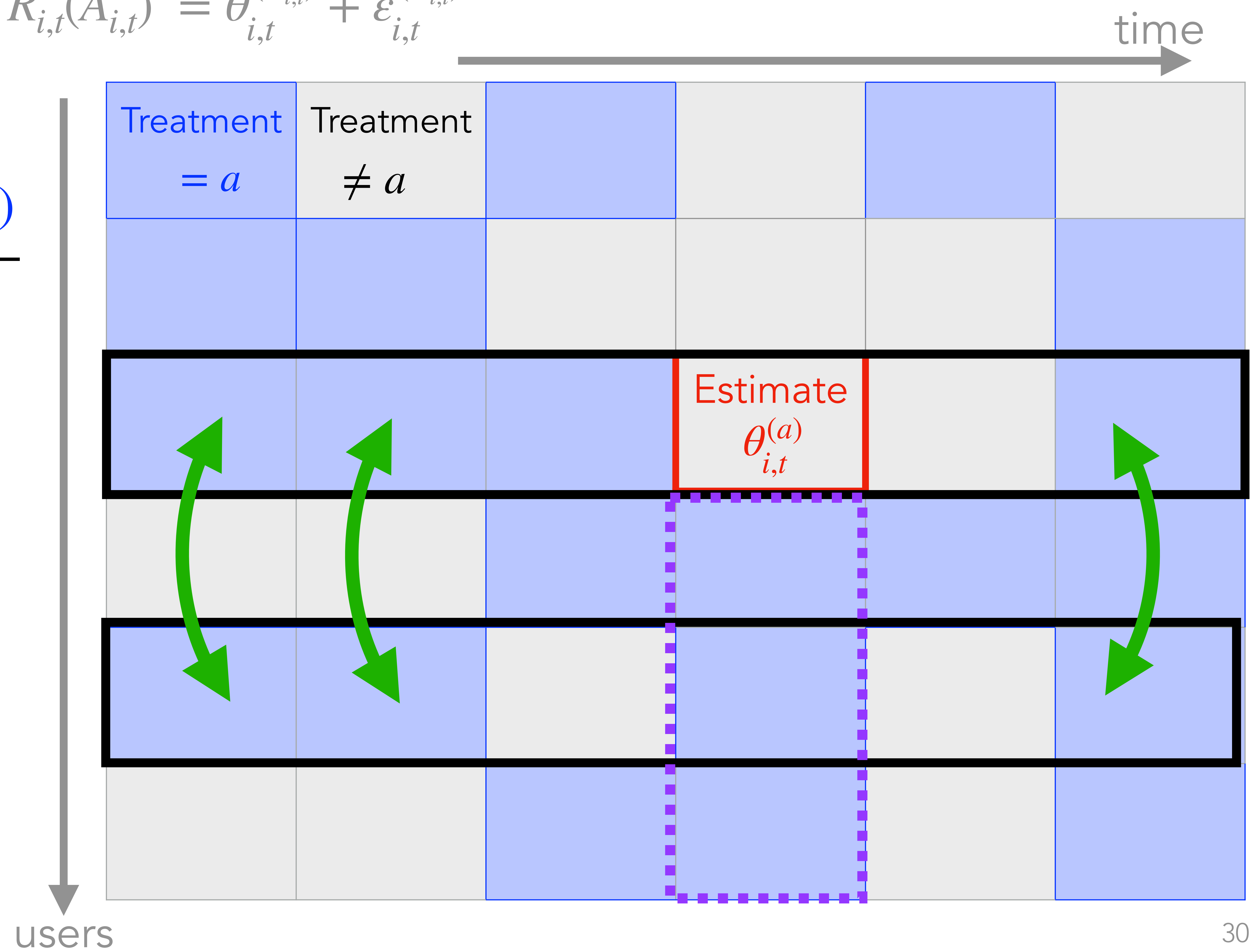

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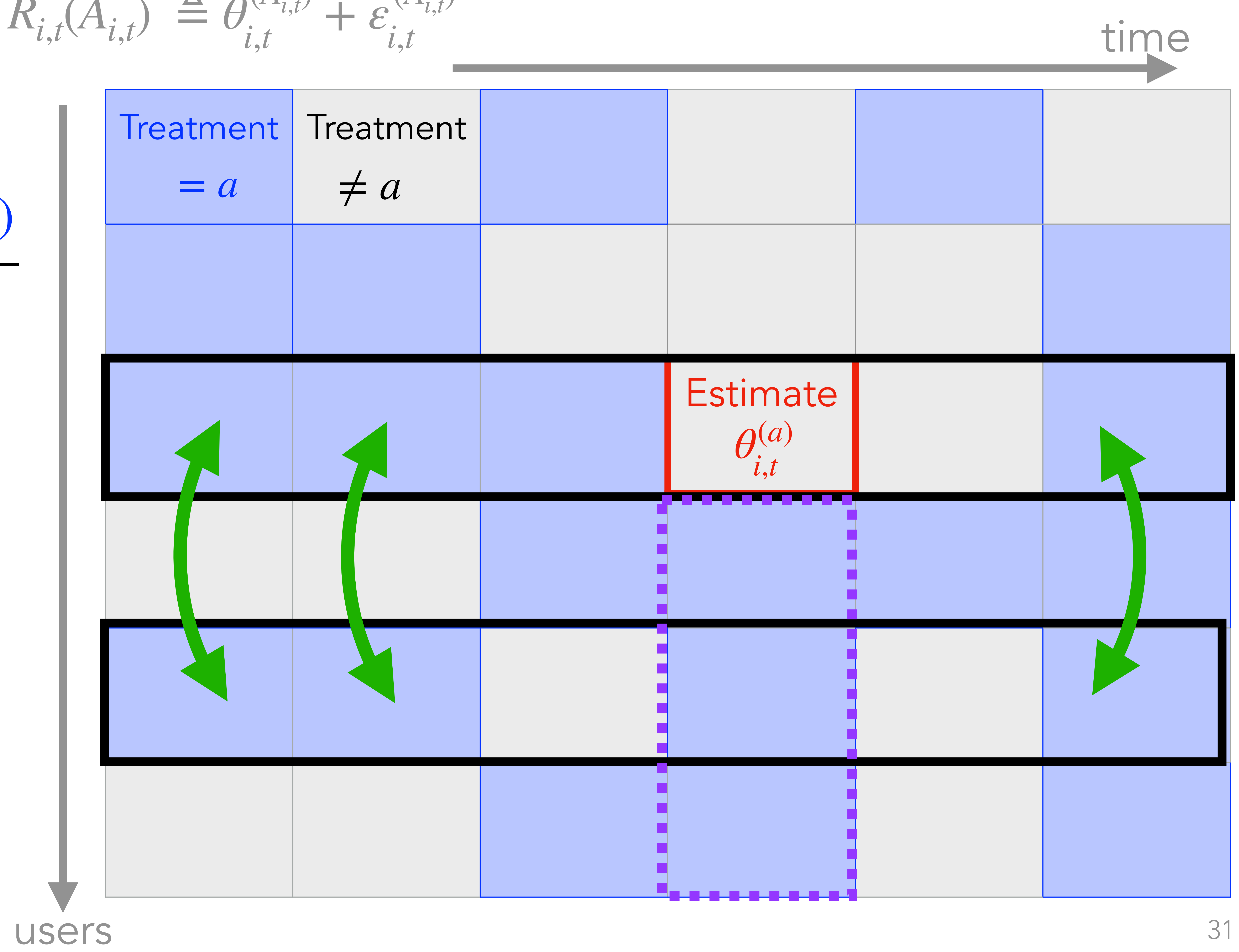
How about this?



Back to our original user-NN: No data-split + learning policy

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$$\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}$$

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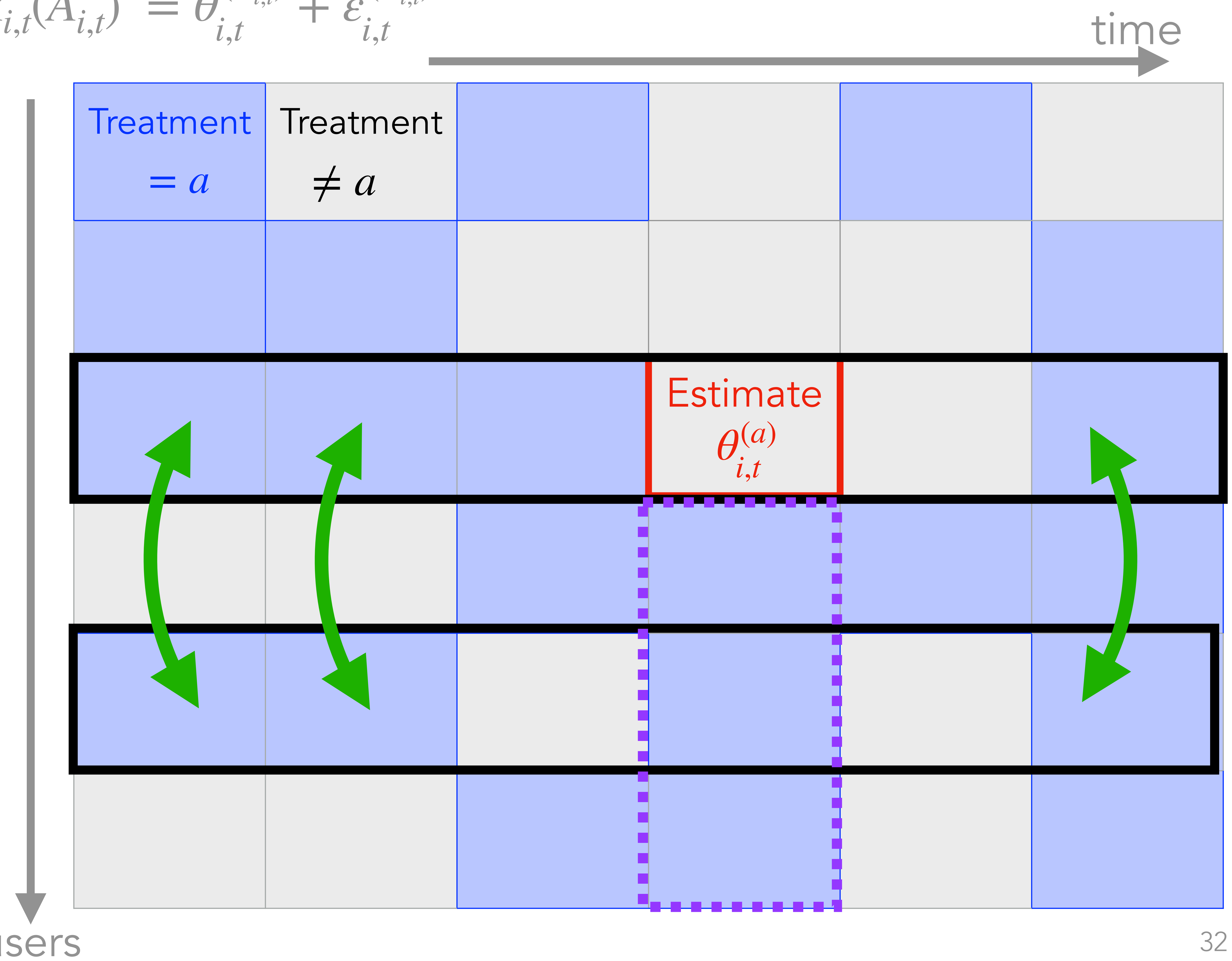

A new sandwich argument for user-nbrs = $\{j : \rho_{i,j}^{(a)} \leq \eta\}$

- First using Azuma-Hoeffding concentration, we show that

$$|\rho_{i,j}^{(a)} - (u_i - u_j)^2 v_\star^2 - 2\sigma^2| \ \leq \frac{C}{\sqrt{p^2 T}}$$

with high probability for a learning policy with exploration p

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with high probability for a learning policy with exploration p

- Define $Q_1 \triangleq \{j : (u_i - u_j)^2 v_\star^2 \leq \left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right)\}$

and $Q_2 \triangleq \{j : (u_i - u_j)^2 v_\star^2 \leq \left(\eta - 2\sigma^2 - \frac{C}{\sqrt{p^2 T}}\right)\}$

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  and $Q_2 \triangleq \{j : (u_i - u_j)^2 v_\star^2 \leq \left(\eta - 2\sigma^2 - \frac{C}{\sqrt{p^2 T}}\right)\}$

- Then we have $Q_2 \subseteq$ User-nbrs $\subseteq Q_1$

Applying the sandwich argument for neighbors

- $\bar{\varepsilon}_t^2 = \dfrac{\left(\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)\right)^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$

Applying the sandwich argument for neighbors

- $\bar{\varepsilon}_t^2 = \dfrac{(\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a))^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$

$$= \frac{\left(\sum_{j \in Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a) + \sum_{j \in user\ nbrs \setminus Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)\right)^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$$

Lower part of the sandwich

Applying the sandwich argument for neighbors

- $\bar{\varepsilon}_t^2 = \dfrac{(\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a))^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$

$$= \frac{\left(\sum_{j \in Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a) + \sum_{j \in user\ nbrs \setminus Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a) \right)^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$$

Lower part of the sandwich

$$\leq 2 \frac{(\sum_{j \in Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a))^2}{(users \in Q_1\ with\ A_{j,t} = a)^2} + 2c_{noise} \frac{(\sum_{j \in Q_2 \setminus Q_1} \mathbf{1}(A_{j,t} = a))^2}{(users \in Q_1\ with\ A_{j,t} = a)^2}$$

Upper part of the sandwich

Applying the sandwich argument for neighbors

- $\bar{\varepsilon}_t^2 = \dfrac{\left(\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)\right)^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$

$$= \frac{\left(\sum_{j \in Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a) + \sum_{j \in user\ nbrs \backslash Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)\right)^2}{(user\ nbrs\ with\ A_{j,t} = a)^2}$$

$$\leq 2 \frac{\left(\sum_{j \in Q_1} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)\right)^2}{(users \in Q_1\ with\ A_{j,t} = a)^2} + 2c_{noise} \frac{\left(\sum_{j \in Q_2 \backslash Q_1} \mathbf{1}(A_{j,t} = a)\right)^2}{(users \in Q_1\ with\ A_{j,t} = a)^2}$$

$$\lesssim \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise} \left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

Draw illustration

Our bound: Obtained by tuning a general error bound for user NN for sequential pooled policies over η

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η' — NN bias due to threshold

e_T — Error in NN distance

NN noise variance

NN bias inflation due to **learning** policy

$$\lambda_\star \triangleq \lambda_{\min}(\boldsymbol{\Sigma}_v) \quad \text{where } \boldsymbol{\Sigma}_v = \mathbb{E}[v_{t'}v_{t'}^\top]$$

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Proof summary for user-NN

- $\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} = \dfrac{\sum_{j \in user\ nbrs} R_{j,t}}{|user\ nbrs|} = \dfrac{\sum_{j \in user\ nbrs} \theta^{(a)}_{j,t} + \varepsilon^{(a)}_{j,t}}{|user\ nbrs|}$

$$= \underbrace{\frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|}}_{\hat{u}_i} v_t + \underbrace{\frac{\sum_{j \in user\ nbrs} \varepsilon^{(a)}_{j,t}}{|user\ nbrs|}}_{\bar{\varepsilon}_t}$$

- $|\theta^{(a)}_{i,t} - \widehat{\theta}^{(a)}_{i,t,\text{user-NN}}| = |u_i v_t - \widehat{\theta}^{(a)}_{i,t,\text{user-NN}}| \leq |u_i v_t - \hat{u}_i v_t| + |\bar{\varepsilon}_t|$

Summary of the proof sketch for unit or time nearest neighbors

- $|u_i v_t - \widehat{\theta}^{(a)}_{i,t,\text{user-NN}}| \leq |u_i v_t - \hat{u}_i v_t| + |\bar{\varepsilon}_t| = O(|u_i - \hat{u}_i|)$
- $|u_i v_t - \widehat{\theta}^{(a)}_{i,t,\text{time-NN}}| \leq |u_i v_t - u_i \hat{v}_t| + |\bar{\varepsilon}_i| = O(|v_t - \hat{v}_t|)$
- Can we combine both to improve the error rate?

Can we make the error rates symmetric in N and T?

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{T^{1/4}} + \frac{1}{\sqrt{N}}\right)$$

$$\downarrow$$

$$|\,\text{??} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

$$\uparrow$$

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{time-NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{N^{1/4}} + \frac{1}{\sqrt{T}}\right)$$

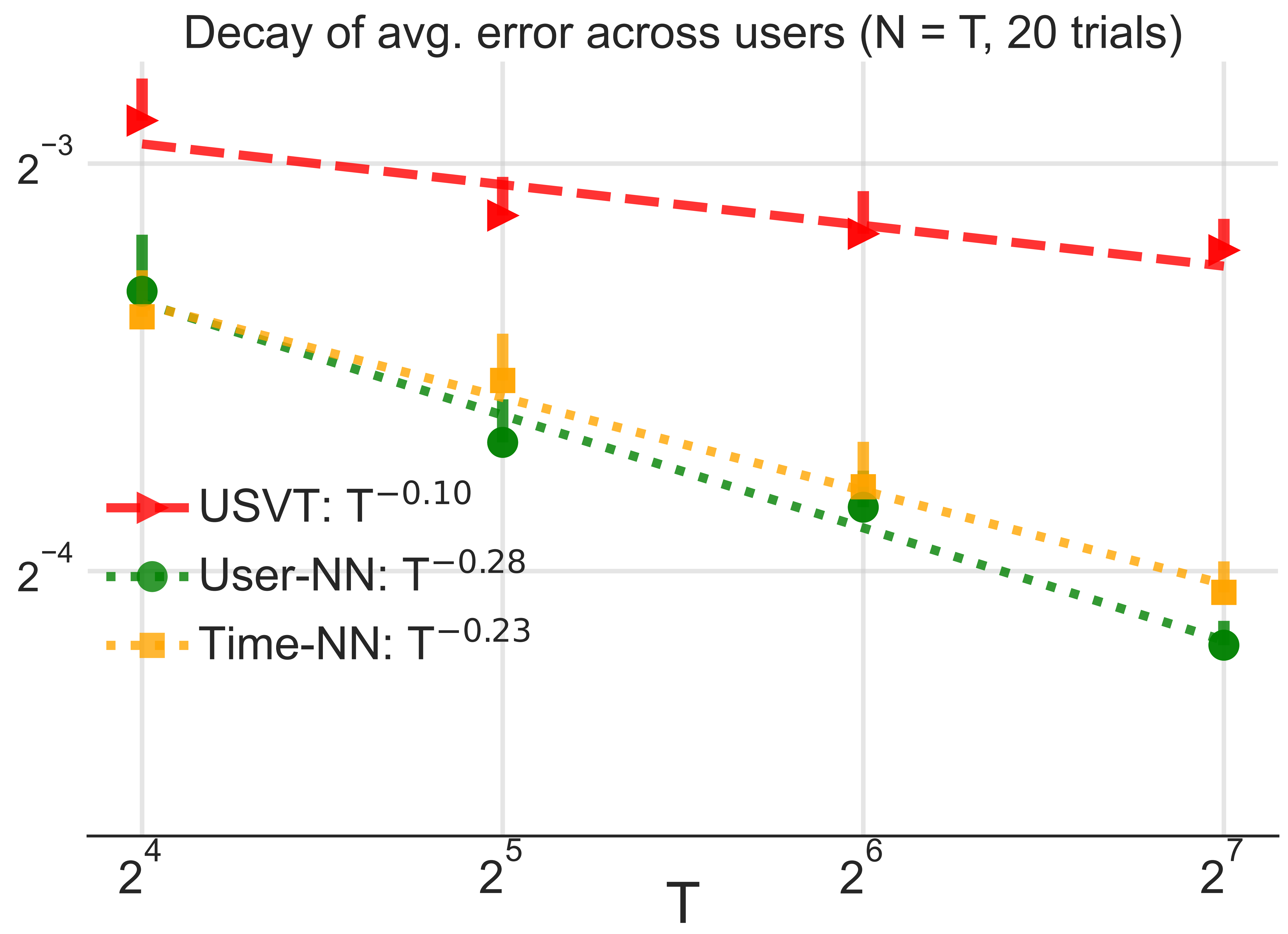
Can we make the error rates symmetric in N and T?

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{T^{1/4}} + \frac{1}{\sqrt{N}}\right)$$

$$|\,\boxed{??} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{time-NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{N^{1/4}} + \frac{1}{\sqrt{T}}\right)$$

Uniform factors on $[-0.5,0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)



USVT: A baseline algorithm from [Chatterjee 2014]

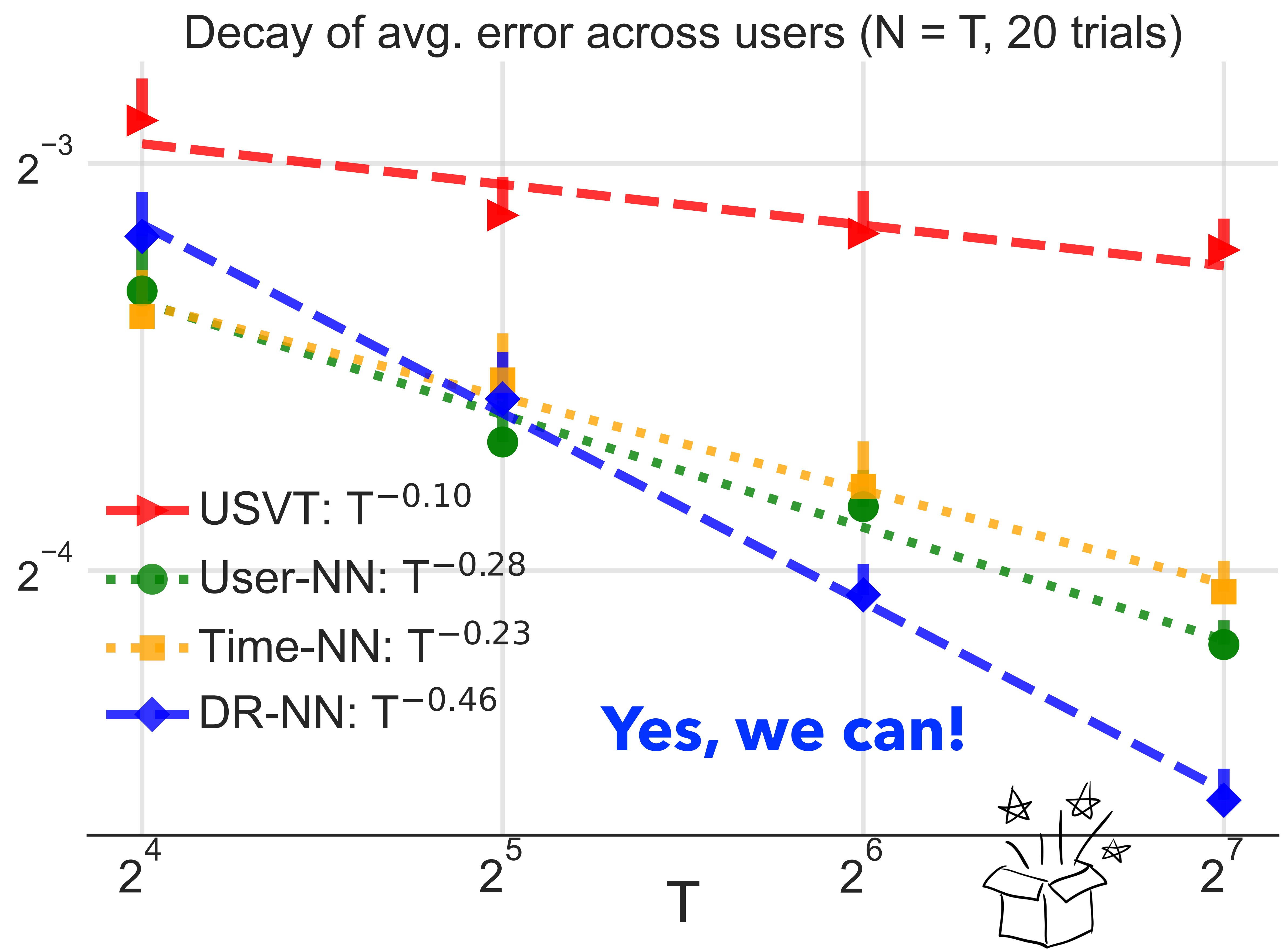
Can we make the error rates symmetric in N and T?

$$|\,\widehat{\theta}^{(a)}_{i,t,\mathrm{DR\text{-}NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

With a **suitable variant** of nearest neighbors

[**Dwivedi**-Tian-Tomkins-Klasnja-Murphy-Shah '22b]

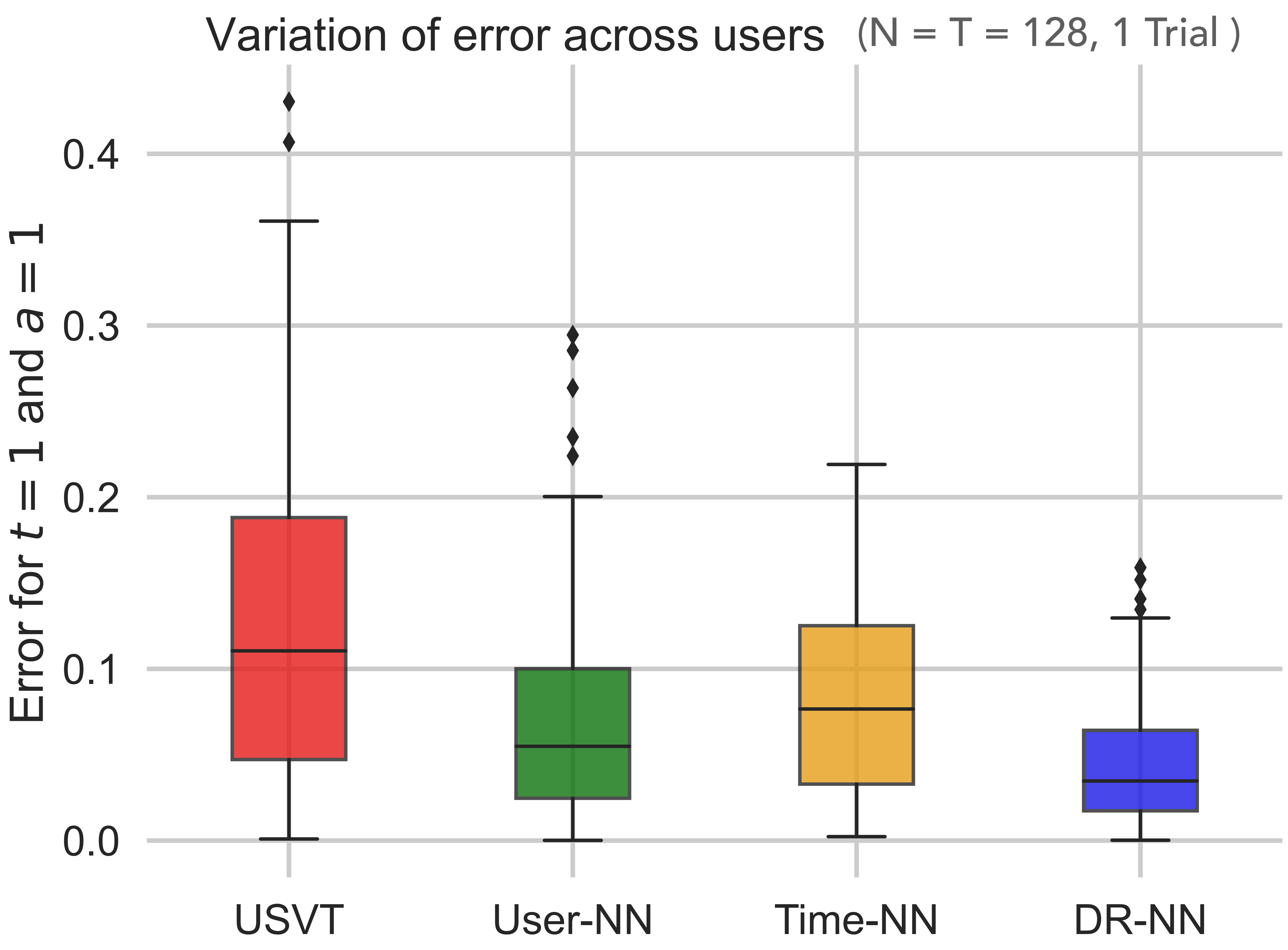
Uniform factors on $[-0.5, 0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)



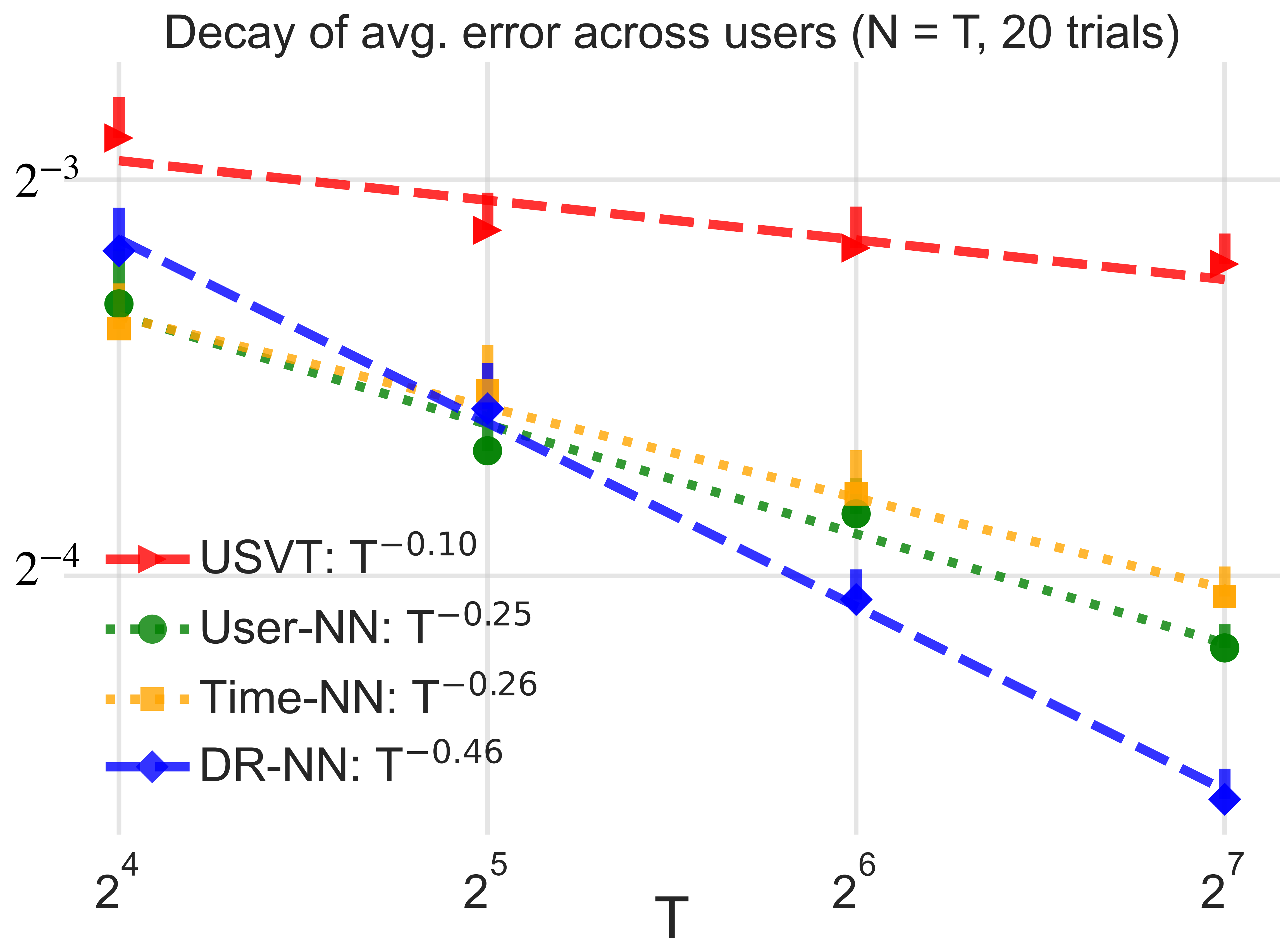
USVT: A baseline algorithm from [Chatterjee 2014]

Simulation results

Uniform latent factors on $[-0.5,0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)



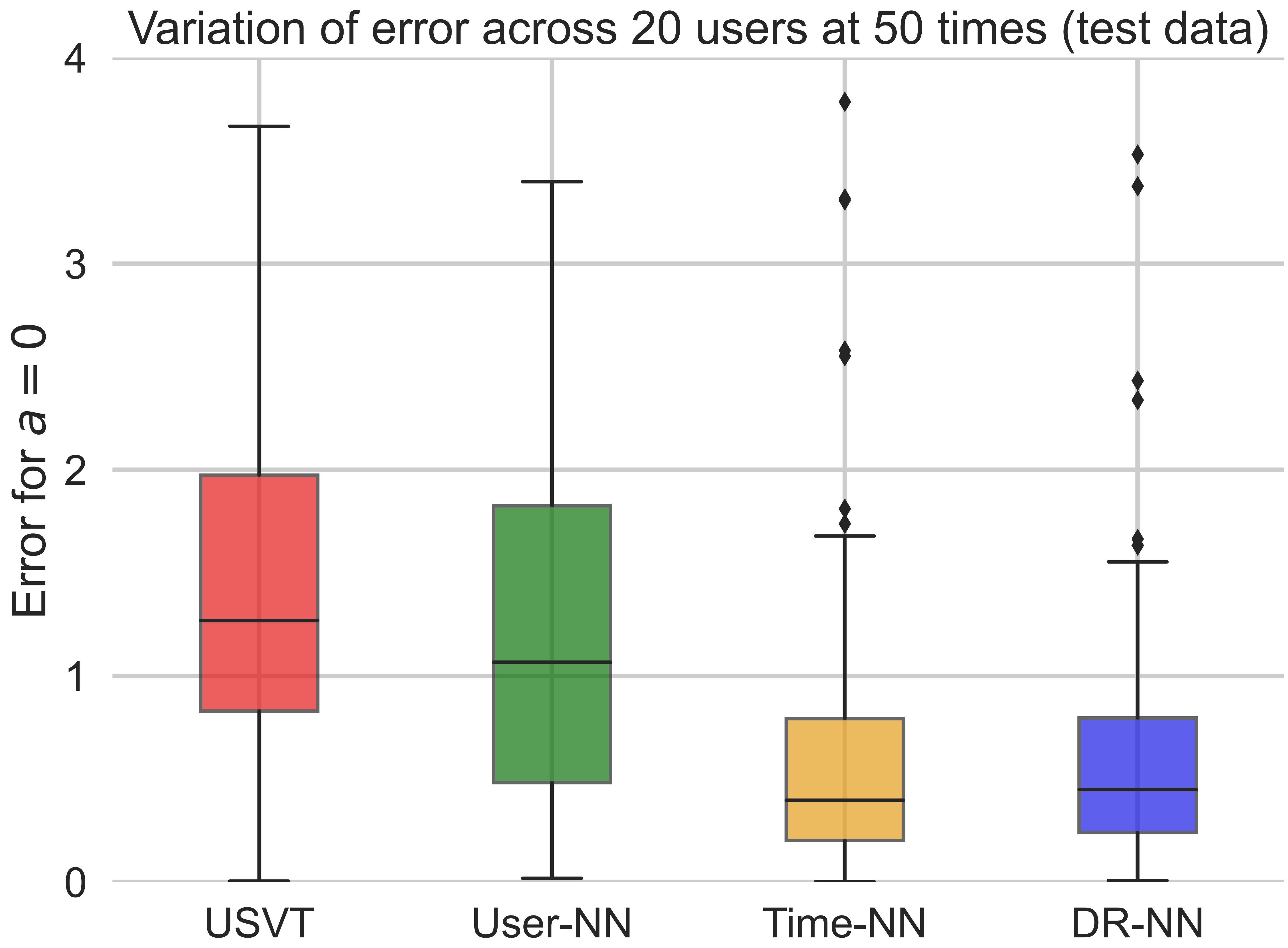
A baseline algorithm from [Chatterjee 2014]

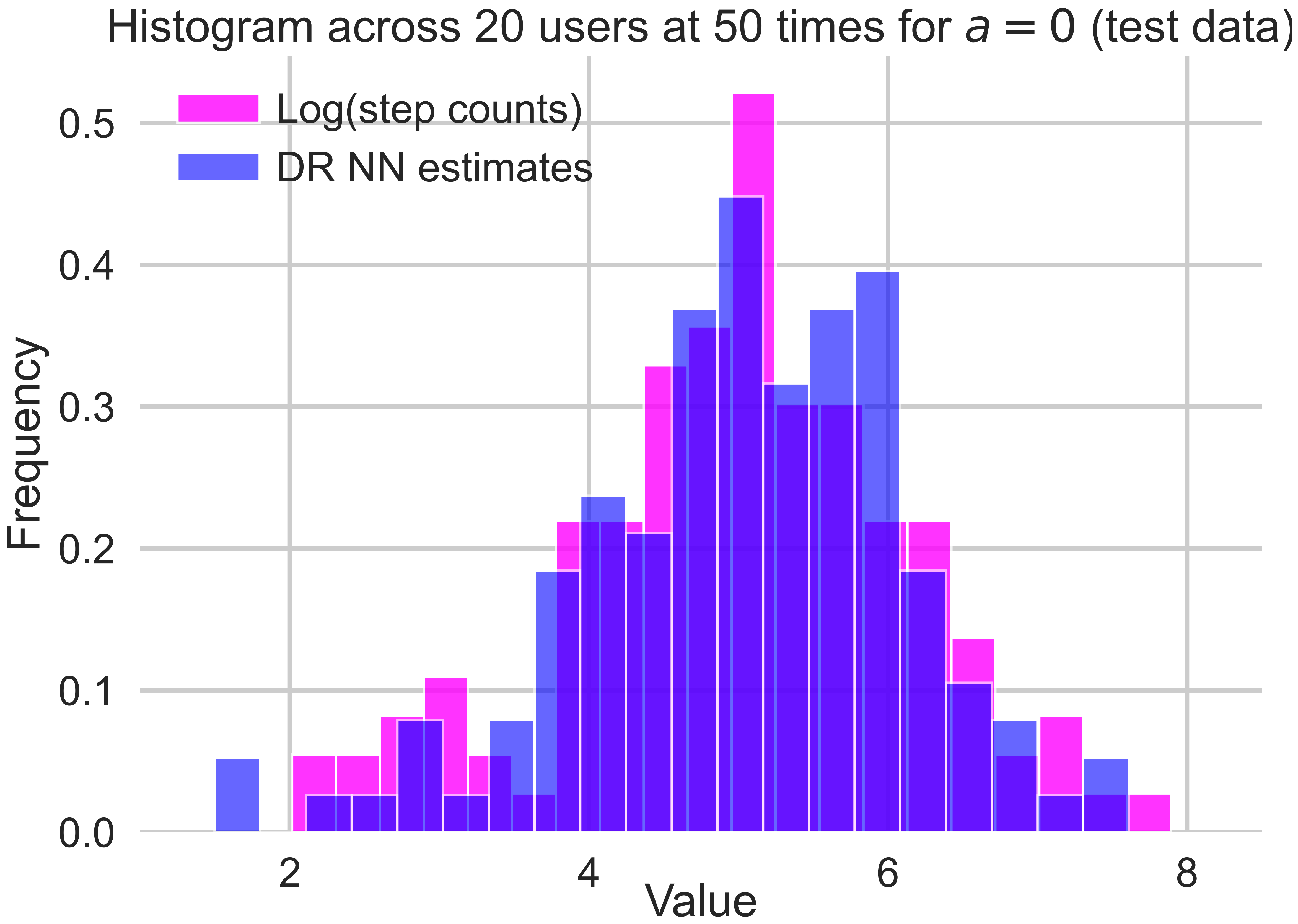


DR-NN error $\ll$ min { user-NN error, time-NN error }

Personalized HeartSteps results

Treatments assigned with Thompson sampling independently for 91 users for 90 days, 5 times a day





DR-NN error ≈ min { user-NN error, time-NN error }

In the search of improved estimator...

- Let's ignore the noise term and consider one nearest neighbor. "j" is a user neighbor so that $\hat{u}_i = u_j$ and "t'" is time neighbor so that $\hat{v}_t = v_{t'}$

- $\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} = R_{j,t} = u_j v_t$ and $\widehat{\theta}^{(a)}_{i,t,\text{time-NN}} = R_{i,t'} = u_i v_{t'}$

- Can we combine to improve?

  - Average the two estimates: $\dfrac{u_j v_t + u_i v_{t'}}{2} = \dfrac{R_{j,t} + R_{i,t'}}{2}$

  - Use both neighbors: Outcome of user j at time t': $u_j v_{t'} = R_{j,t'}$

Discussion questions

- What are the limitations of the factor model and the assumptions made for stating the non-asymptotic guarantee? Can you try to weaken these assumptions, to include states, delayed effects?

- Given all these counterfactual estimates, what kind of quantities could you investigate? How would you use them for between study analyses or to help the design of next study?

- **Hard**: Would the “averaged/merged” combination strategy significantly improve the performance?
    - Can you think of other ways to improve the NN estimator for the current model or more generally?

Error of the "averaged" estimate

$$|u_i v_t - \frac{u_j v_t + u_i v_{t'}}{2}| = \frac{|u_i v_t - u_j v_t + u_i v_t - u_i v_{t'}|}{2} \leq \frac{|u_i v_t - u_j v_t| + |u_i v_t - u_i v_{t'}|}{2}$$

$$= \frac{|u_i - u_j||v_t| + |u_i||v_t - v_t'|}{2}$$

$$= \frac{O(|u_i - u_j|) + O(|v_t - v_t'|)}{2}$$

$$= \frac{1}{2}(\text{User-NN Error} + \text{Time-NN Error})$$

$$\approx \max \{\text{ User-NN Error, Time-NN Error}\}$$

Error of the "merged" estimate

$$|u_i v_t - u_j v_{t'}| = |u_i v_t - u_j v_t + u_j v_t - u_j v_{t'}| \leq |u_i v_t - u_j v_t| + |u_j v_t - u_j v_{t'}|$$

$$= |u_i - u_j||v_t| + |u_j||v_t - v_t'|$$

$$= O(|u_i - u_j|) + O(|v_t - v_t'|)$$

$$= \text{User-NN Error} + \text{Time-NN Error}$$

$$\approx \max\{\text{User-NN Error, Time-NN Error}\}$$

What do we desire?

- **Convert + to ×:** $|u_i v_t - \text{??}| = |u_i - u_j| \times |v_t - v_{t'}|$

  $= \text{User-NN Error} \times \text{Time-NN Error}$

  or **max** to **min:** $\approx \min \{\text{User-NN Error, Time-NN Error}\}$

What should be our estimator? Let's expand the RHS...

$$\cancel{u_i v_t} - ?? = (u_i - u_j) \times (v_t - v_{t'})$$

$$= \cancel{u_i v_t} - u_j v_t - u_i v_{t'} + u_j v_{t'}$$

$$\Rightarrow \qquad ?? = u_j v_t + u_i v_{t'} - u_j v_{t'}$$

$$R_{j,t} \quad + \quad R_{i,t'} \quad - R_{j,t'}$$

This is our improved nearest neighbors estimator!

$$\cancel{u_i v_t} - \text{??} = (u_i - u_j) \times (v_t - v_{t'})$$

$$= \cancel{u_i v_t} - u_j v_t - u_i v_{t'} + u_j v_{t'}$$

$$\Rightarrow \qquad \text{??} = u_j v_t + u_i v_{t'} - u_j v_{t'}$$

$$\widehat{\theta}^{(a)}_{i,t,\mathrm{DR\text{-}NN}} = \frac{\sum_{j,t'} (R_{j,t} + R_{i,t'} - R_{j,t'}) \mathbf{1}_{i,t,j,t'}}{\sum_{j,t'} \mathbf{1}_{i,t,j,t'}}$$

$$\mathbf{1}_{i,t,j,t'} = \mathbf{1}(\rho^{(a)}_{i,j} \leq \eta,\ \ \rho^{(a)}_{t,t'} \leq \eta', A_{j,t} = A_{i,t'} = A_{j,t'} = a)$$

This is our improved nearest neighbors estimator!

$$\cancel{u_i v_t} - ?? = (u_i - u_j) \times (v_t - v_{t'})$$

$$= \cancel{u_i v_t} - u_j v_t - u_i v_{t'} + u_j v_{t'}$$

$$\Rightarrow \quad ?? = u_j v_t + u_i v_{t'} - u_j v_{t'}$$

DR-NN error $\approx$ **user-NN error** $\times$ **time-NN error**

$\lesssim$ **min{user-NN error, time-NN error}**

$\boxed{\textbf{Doubly robust to heterogeneity in user factors \& time factors}}$

Double robustness, double machine learning…

[… Cassel+ ’77, Robinson ’88, Särndal+ ’89, Robins+ ’94, ’95, ’08, ’09, Newey+ ’94, ’18, Bickel+ ’98, van der Laan+ ’03, Lunceford+ ’04, Davidian+ ’05, Li+ ’11, Jiang+ ’15, Chernozhukov+ ’18, Hirshberg+ ’18, Diaz ’19, Arkhangelsky+ ’21, Dorn+ ’21 …]

Simplified view of doubly robust estimator for uv

Estimate	Error
$\hat{u}\hat{v}$	$O(\lvert\hat{u}-u\rvert + \lvert\hat{v}-v\rvert)$
$\downarrow$	$\downarrow$
$\hat{u}v + u\hat{v} - \hat{u}\hat{v}$	$O(\lvert\hat{u}-u\rvert \times \lvert\hat{v}-v\rvert)$

Problem setting	u	v
Factor model (this talk)	user factor	time factor
Observational studies (Causal inference)	propensity function	mean outcome function
Off policy evaluation (Reinforcement learning)	importance ratio	reward function

Double robustness, double machine learning…

[… Cassel+ '77, Robinson '88, Särndal+ '89, Robins+ '94, '95, '08, '09, Newey+ '94, '18, Bickel+ '98, van der Laan+ '03, Lunceford+ '04, Davidian+ '05, Li+ '11, Jiang+ '15, Chernozhukov+ '18, Hirshberg+ '18, Diaz '19, Arkhangelsky+ '21, Dorn+ '21 …]

Appendix [not covered in lecture]: Further details on derivations

Disclaimer

- c, C are universal constants that might take a different value in each appearance

Proof sketch for user-NN

- Simple case: Always assign $A_{j,t} = a$ and $\theta_{i,t}^{(a)} \triangleq u_i v_t$
- $\widehat{\theta}_{i,t,\text{user-NN}}^{(a)} = \dfrac{\sum_{j \in user\ nbrs} R_{j,t}}{|user\ nbrs|} = \dfrac{\sum_{j \in user\ nbrs} \theta_{j,t}^{(a)} + \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$

$$= \frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|} v_t + \frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$$

- $|u_i v_t - \hat{u}_i v_t| \leq \max_{j \in user\ nbrs} |u_i - u_j| |v_t| \lesssim \sqrt{\eta - 2\sigma^2} + \dfrac{1}{T^{1/4}}$
- $\bar{\varepsilon}_t = \dfrac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|} \lesssim \dfrac{\sigma}{\sqrt{|user\ nbrs|}} = \dfrac{\sigma}{\sqrt{N_\eta}}$

Our goal: Control $\max_{j \in user\ nbrs} |u_i - u_j||v_t|$

- $|v_t|$ is bounded so suffices to bound $\max_{j \in user\ nbrs} |u_i - u_j|$

- user neighbours = $\{\rho_{i,j}^{(a)} \leq \eta\}$

    - $$\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)} + \frac{\hat{\sigma}_\rho}{\sqrt{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}}$$

    - In theory, we ignore the second term

Controlling the bias via concentration of distance

- Simple case: Always assign $A_{j,t} = a$ and $\theta_{i,t}^{(a)} \triangleq u_i v_t$

- $\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2}{T}$

Re-expressing the distance

- Simple case: Always assign $A_{j,t} = a$ and $\theta^{(a)}_{i,t} \triangleq u_i v_t$

- $\rho^{(a)}_{i,j} = \dfrac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2}{T}$

$$= \frac{1}{T}\sum_{t'=1}^{T}\left[(u_i v_{t'} - u_j v_{t'})^2 + (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 + 2(u_i v_{t'} - u_j v_{t'})(\varepsilon_{i,t'} - \varepsilon_{j,t'})\right]$$

$$= (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j)\,\frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$$

Re-expressing the distance: Collecting into three terms

- $\rho^{(a)}_{i,j} = (u_i - u_j)^2 \boxed{\frac{\sum_{t'=1}^{T} v_{t'}^2}{T}} + \boxed{\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T}} + 2(u_i - u_j) \boxed{\frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}}$

What do these three terms concentrate on?

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

Recall our assumptions –

v_t are iid

$\varepsilon_{i,t}$ are iid zero mean with variance σ

v_t and $\varepsilon_{i,t}$ are independent of each other

What do these three terms concentrate on?

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$
- $|\frac{\sum_{t'=1}^{T} v_{t'}^2}{T} - ?| \lesssim ?$
- $|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} - ?| \lesssim ?$
- $|\frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T} - ?| \lesssim ?$

Recall our assumptions –
v_t are iid
$\varepsilon_{i,t}$ are iid zero mean with variance σ
v_t and $\varepsilon_{i,t}$ are independent of each other

Tools for concentration

- **Markov's inequality:** Let $X_1, X_2, \ldots, X_T$ be iid random variables with mean μ and variance Var(X), then

$$\mathbb{P}\left[\left|\frac{\sum_{i=1}^{T} X_i}{T} - \mu\right| \leq \sqrt{\frac{\mathrm{Var}(X)}{\delta T}}\right] \geq 1 - \delta$$

Tools for concentration

- **Markov's inequality:** Let $X_1, X_2, \ldots, X_T$ be iid random variables with mean μ and variance Var(X), then

$$\mathbb{P}\left[\left|\frac{\sum_{i=1}^T X_i}{T} - \mu\right| \leq \sqrt{\frac{\text{Var}(X)}{\delta T}}\right] \geq 1 - \delta$$

- **Chernoff-Hoeffding bound:** If X_i have mean μ and are $\gamma-$sub-Gaussian, i.e., $\mathbb{E}[e^{t(X-\mu)}] \leq e^{t^2\gamma^2/2}$ then

$$\mathbb{P}\left[\left|\frac{\sum_{i=1}^T X_i}{T} - \mu\right| \leq \gamma\sqrt{2\log(1/\delta)}\right] \geq 1 - \delta$$

  - Useful fact if $|X_i| \leq c$, then we can use $\gamma = c$

What do these three terms concentrate around?

Their means!

- $\rho^{(a)}_{i,j} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j)\frac{\sum_{t'=1}^{T} v'_t(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

- $\left|\frac{\sum_{t'=1}^{T} v_{t'}^2}{T} - \mathbb{E}[v_{t'}^2]\right| \lesssim \frac{c\sqrt{\text{Var}(v_{t'}^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(v_{t'}^2) \cdot \log(1/\delta)}}{\sqrt{T}}$

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- $\rho^{(a)}_{i,j} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

- $\left| \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} - \mathbb{E}[v_{t'}^2] \right| \lesssim \frac{c\sqrt{\mathrm{Var}(v_{t'}^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(v_{t'}^2) \cdot \log(1/\delta)}}{\sqrt{T}}$

- $\left| \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} - 2\sigma^2 \right| \lesssim \frac{c\sqrt{2\mathrm{Var}(\varepsilon^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(\varepsilon^2) \cdot \log(1/\delta)}}{\sqrt{T}}$

What do these three terms concentrate around?

Their means!

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$
- $|\frac{\sum_{t'=1}^{T} v_{t'}^2}{T} - \mathbb{E}[v_{t'}^2]| \lesssim \frac{c\sqrt{\mathrm{Var}(v_{t'}^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(v_{t'}^2) \cdot \log(1/\delta)}}{\sqrt{T}}$
- $|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} - 2\sigma^2| \lesssim \frac{c\sqrt{2\mathrm{Var}(\varepsilon^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(\varepsilon^2) \cdot \log(1/\delta)}}{\sqrt{T}}$
- $|\frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T} - 0| \lesssim \frac{c\sqrt{2\mathrm{Var}(v_t\varepsilon)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(v_t\varepsilon) \cdot \log(1/\delta)}}{\sqrt{T}}$

Inverting the distance to get a control on $|u_i - u_j|$

- Assume v_t, ε are bounded and $\mathbb{E}[v_t^2] = v_\star^2$ then
  - $|\rho_{i,j}^{(a)} - (u_i - u_j)^2 v_\star^2 - 2\sigma^2| \leq \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$ with probability $1 - \delta$
  - Treat δ as a constant
  - Rearranging terms $|u_i - u_j|^2 \leq \dfrac{1}{v_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \dfrac{C}{\sqrt{T}}\right)$

Inverting the distance to get a control on $|u_i - u_j|$

- Assume v_t, ε are bounded and $\mathbb{E}[v_t^2] = v_\star^2$ then
  - $|\rho_{i,j}^{(a)} - (u_i - u_j)^2 v_\star^2 - 2\sigma^2| \leq \frac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$ with probability $1 - \delta$
  - Treat δ as a constant
  - Rearranging terms $|u_i - u_j|^2 \leq \frac{1}{v_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \frac{C}{\sqrt{T}}\right)$
- So if $\rho_{i,j}^{(a)} \leq \eta \Longrightarrow |u_i - u_j| \lesssim \sqrt{\eta - 2\sigma^2} + \frac{C}{T^{1/4}}$ if $v_\star^2 > 0$.

But how many users would satisfy $\rho_{i,j}^{(a)} \leq \eta$?

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq r\}|$

- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$

But how many users would satisfy $\rho_{i,j}^{(a)} \leq \eta$?

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq r\}|$
- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \frac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}|}{|user\ nbrs|} \lesssim \frac{\sigma}{\sqrt{N_{i,\gamma}}}$

Univariate factors:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{T}}\right) \quad + \quad \frac{\sigma^2}{N_{i,\eta'-e_T}}$$

η' | e_T

NN bias due to threshold | Error in NN distance | NN noise variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

Univariate factors + constant policy $\mathbb{P}(A_{i,t} = a) =$:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{p^2 T}}\right) \quad + \quad \frac{\sigma^2}{pN_{i,\eta'-e_T}}$$

η'

NN bias
due to threshold

e_T

Error in
NN distance

NN noise
variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

Multivariate factors + learning policy with $\mathbb{P}(A_{i,t} = a \mid History_{t-1}) \geq p$: A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η'

NN bias
due to threshold

e_T

Error in
NN distance

NN noise
variance

NN bias
inflation due to
learning policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

Scalings of $N_{i,\gamma}$

- When factors are sampled independently and uniformly from a discrete set $\{\Delta, \ldots, (M-1)\Delta\}$
    - $N_{i,r} \geq cN/M$ for any $r \geq 0$ if $v_\star > 0$.
- When factors are sampled independently and uniformly from a continuous set $[0,1]$
    - $N_{i,r} \geq c\sqrt{r/v_\star}$ for any $r \geq 0$.
- **HW:** You can now tune η to get refined error bounds.

Multivariate factors: Bias analysis

- Assume v_t, ε are bounded and $\mathbb{E}[v_t v_t^\top] = \Sigma_v$ then

- $|\rho_{i,j}^{(a)} - (u_i - u_j)^\top \Sigma_v (u_i - u_j) - 2\sigma^2| \leq \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$ with probability $1 - \delta$

- Treat δ as a constant

- Rearranging terms $\|u_i - u_j\|_2^2 \leq \dfrac{1}{\lambda_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \dfrac{C}{\sqrt{T}}\right)$ where $\lambda_\star = \lambda_{\min}(\Sigma_v)$

- So if $\rho_{i,j}^{(a)} \leq \eta \Longrightarrow \|u_i - u_j\|_2 \lesssim \sqrt{\eta - 2\sigma^2} + \dfrac{C}{T^{1/4}}$ if $\lambda_\star^2 > 0$.

Multivariate factors: Variance analysis

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^\top \Sigma_v (u_i - u_j) \leq r\}|$
- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \frac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}|}{|user\ nbrs|} \lesssim \frac{\sigma}{\sqrt{N_{i,\gamma}}}$

Multivariate factors:
A general error bound for user NN when $A_{i,t}$ is always a

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{T}}\right) + \frac{\sigma^2}{N_{i,\eta'-e_T}}$$

η'

NN bias
due to threshold

e_T

Error in
NN distance

NN noise
variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \mathbf{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Multivariate factors + learning policy with exploration p:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η'

e_T

NN bias
due to threshold

Error in
NN distance

NN noise
variance

NN bias
inflation due to
learning policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Constant policy: Bias analysis

- Assume $A_{j,t}$ are iid Bernoulli random variables p – constant MRT – Like in HeartSteps V1
    - Let $a = 1$, then what is the distribution of $B_{i,j,t'} \triangleq \mathbf{1}(A_{i,t'}{=}A_{j,t'}{=}a)$?
    - $\rho_{i,j}^{(a)} = \dfrac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}}$
      $$= \frac{\sum_{t'=1}^{T} \left[(u_i v_{t'} - u_j v_{t'})^2 + (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 + 2(u_i v_{t'} - u_j v_{t'})(\varepsilon_{i,t'} - \varepsilon_{j,t'}) \right] \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}}$$

Bias analysis: The denominator changes

- $\left|(u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - (u_i - u_j)^2\, \mathbb{E}[v_{t'}^2]\right| \lesssim \frac{c_v^2 \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$

- $\left|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - 2\sigma^2\right| \lesssim \frac{c_\varepsilon \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$

- $|\frac{T_{i,j}}{T} - p^2| \lesssim \frac{c\sqrt{\log(1/\delta)}}{\sqrt{T}}$
    - A better bound available: $T_{i,j} \geq cp^2T$ with probability $\geq 1 - e^{-cp^2T}$.

Bias analysis: The denominator changes

- Assume v_t, ε are bounded and $\mathbb{E}[v_t v_t^\top] = \Sigma_v$ then

- Hence $|\rho_{i,j}^{(a)} - (u_i - u_j)^\top \Sigma_v (u_i - u_j) - 2\sigma^2| \leq \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{p^2 T}}$ with probability $1 - \delta$

- Treat δ as a constant

- Rearranging terms $\|u_i - u_j\|_2^2 \leq \dfrac{1}{\lambda_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \dfrac{C}{\sqrt{p^2 T}}\right)$ where $\lambda_\star = \lambda_{\min}(\Sigma_v)$

- So if $\rho_{i,j}^{(a)} \leq \eta \Longrightarrow \|u_i - u_j\|_2 \lesssim \sqrt{\eta - 2\sigma^2} + \dfrac{C}{p^{1/2}T^{1/4}}$ if $\lambda_\star^2 > 0$.

Constant policy: Variance analysis: denominator changes

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^\top \Sigma_v (u_i - u_j) \leq r\}|$
- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \frac{C\sqrt{\log(c/\delta)}}{\sqrt{p^2 T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)|}{|user\ nbrs\ with\ A_{j,t} = a|} \lesssim \frac{\sigma}{\sqrt{pN_{i,\gamma}}}$

Multivariate factors + constant policy:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{p^2 T}}\right) \quad + \quad \frac{\sigma^2}{pN_{i,\eta'-e_T}}$$

η' $\quad$ e_T

NN bias
due to threshold

Error in
NN distance

NN noise
variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Multivariate factors + learning policy with exploration p:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η'

e_T

NN bias due to threshold

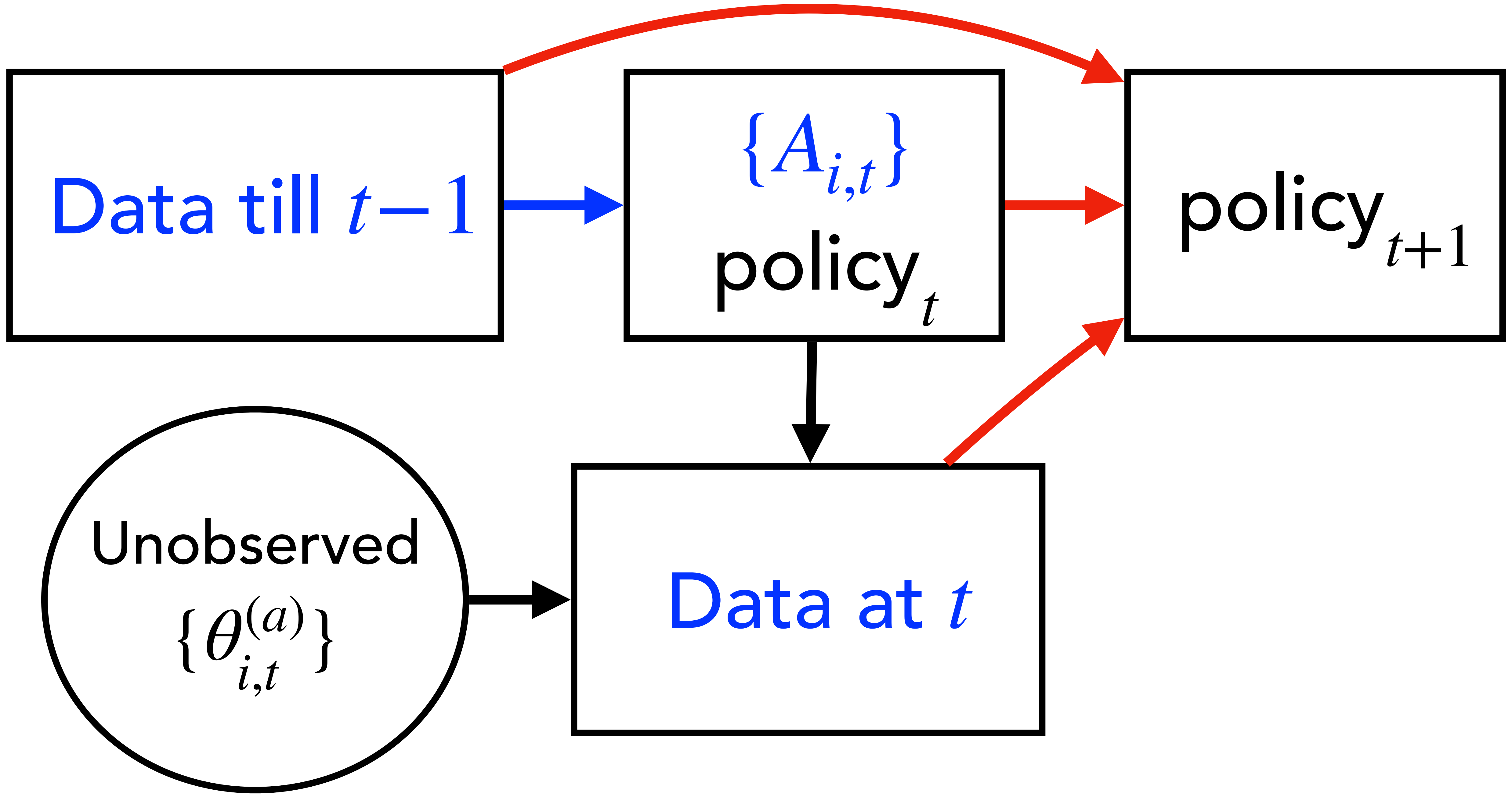
Error in NN distance

NN noise variance

NN bias inflation due to **learning** policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Learning policy:
Sequential dependence between observations



Learning policy: Bias analysis
Similar except now with Martingales

- Still goes through using "Azuma-Hoeffing bounds" and careful Martingale construction

- $\left|(u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - (u_i - u_j)^2\, \mathbb{E}[v_{t'}^2]\right| \lesssim \frac{c_v^2 \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$

- $\left|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - 2\sigma^2\right| \lesssim \frac{c_\varepsilon \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$

- $|\frac{T_{i,j}}{T} - p^2| \lesssim \frac{c\sqrt{\log(1/\delta)}}{\sqrt{T}}$

  - A better bound available: $T_{i,j} \geq cp^2T$ with probability $\geq 1 - e^{-cp^2T}$.

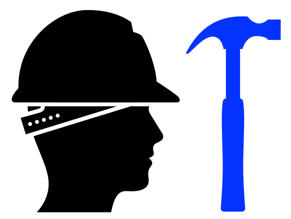
Learning policy: Bias bounds
Essentially same as the MRT bound

- If $\rho_{i,j}^{(a)} \leq \eta \quad \Longrightarrow \quad \|u_i - u_j\|_2 \lesssim \frac{1}{\lambda_\star}(\sqrt{\eta - 2\sigma^2} + \frac{C}{p^{1/2}T^{1/4}})$ if $\lambda_\star^2 > 0.$

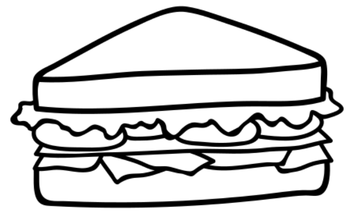
Learning policy: Variance analysis — Non-trivial changes

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^\top \Sigma_v (u_i - u_j) \leq r\}|$
- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{p^2 T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \dfrac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)|}{|user\ nbrs\ with\ A_{j,t} = a|}$

? noise at t **correlated** with user neighbors (learning policy)



Martingale concentration, **new sandwich argument** for NN



Learning policy: Variance bounds — Has a "bias" like term

- $\bar{\varepsilon}_t^2 = \left(\frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)|}{|user\ nbrs\ with\ A_{j,t} = a|}\right)^2$

$$\lesssim \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

Martingale concentration, **new sandwich argument** for NN

Multivariate factors + learning policy with exploration p: A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η'

NN bias
due to threshold

e_T

Error in
NN distance

NN noise
variance

NN bias
inflation due to
learning policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \mathbf{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$