Note on Excursion Effects and Action Centering

1 Recap of Problem Setup

Please refer to slides "Monday_causal_excursion_effects" for more detailed problem setup.

- Model we assume for excursion effect

$$\mathbb{E}\left[Y_{t+1}(\bar{A}_{t-1}, 1) - Y_{t+1}(\bar{A}_{t-1}, 0) | H_{t-1}, X_t\right] = f(X_t)^\top \theta$$

- Working model for average reward

$$\mathbb{E}\left[Y_{t+1}(\bar{A}_{t-1}, A_t) | H_{t-1}, X_t\right] = g(H_{t-1}, X_t)^\top \eta$$

- **Case 1:** Least squares loss function when π_t only depends on X_t (pre-specified policy):

$$\ell(H_T; \eta, \theta) \triangleq \sum_{t=1}^{T} \left(Y_{t+1} - g(H_{t-1}, X_t)^\top \eta - (A_t - \pi_t) f(X_t)^\top \theta\right)^2$$

- **Case 2:** Least squares loss function when π_t may depend on X_t, H_{t-1} (e.g., when an RL algorithm is used):

$$\ell(H_T; \eta, \theta) \triangleq \sum_{t=1}^{T} W_t \left(Y_{t+1} - g(H_{t-1}, X_t)^\top \eta - (A_t - p_t) f(X_t)^\top \theta\right)^2$$

  where $p_t \triangleq p(X_t)$ for some pre-specified policy p and $W_t = \left(\frac{p_t}{\pi_t(H_{t-1}, X_t)}\right)^{A_t} \left(\frac{1-p_t}{1-\pi_t(H_{t-1}, X_t)}\right)^{1-A_t}$. Note that W_t is a Radon-Nikodym derivative.

- Forming Estimators

$$(\hat{\eta}, \hat{\theta}) = \text{argmin}_{\eta, \theta} \frac{1}{n} \sum_{i=1}^{n} \ell(H_T; \eta, \theta)$$

  Equivalently, $(\eta, \theta) = (\hat{\eta}, \hat{\theta})$ solves

$$0 = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial(\eta, \theta)} \ell(H_T; \eta, \theta)\bigg|_{(\eta, \theta)}$$

2 Case in which π_t only depends on X_t

$$0 = \mathbb{E}\left[\frac{\partial}{\partial\theta}\ell(H_T;\eta,\theta)\right]$$

$$= \sum_{t=1}^{T}\mathbb{E}\left[\left\{Y_{t+1} - g(H_{t-1},X_t)^\top\eta - (A_t - p_t)f(X_t)^\top\theta\right\}(A_t - p_t)f(X_t)\right]$$

Note that $\mathbb{E}\left[g(H_{t-1},X_t)^\top\eta\,(A_t-p_t)\,f(X_t)\middle|H_{t-1},X_t\right]$
$= g(H_{t-1},X_t)^\top\eta\mathbb{E}\left[A_t-\pi_t\middle|H_{t-1},X_t\right]f(X_t) = 0$, by law of iterated expectations,

$$= \sum_{t=1}^{T}\mathbb{E}\Bigg[\left\{Y_{t+1} - (A_t-\pi_t)f(X_t)^\top\theta\right\}(A_t-\pi_t)f(X_t)\Bigg]$$

Let $X_{i,t} \triangleq Y_{i,t+1} - (A_t-\pi_t)f(X_t)^\top\theta$.

$$= \sum_{t=1}^{T}\mathbb{E}\Bigg[\mathbb{E}\left[X_{i,t}(A_t-\pi_t)f(X_t)\middle|H_{t-1},X_t\right]\Bigg]$$

$$= \sum_{t=1}^{T}\mathbb{E}\Bigg[\pi_t(1-\pi_t)f(X_t)\mathbb{E}\left[X_{i,t}\middle|H_{t-1},X_t,A_t=1\right] + (1-\pi_t)(0-\pi_t)f(X_t)\mathbb{E}\left[X_{i,t}\middle|H_{t-1},X_t,A_t=0\right]\Bigg]$$

$$= \sum_{t=1}^{T}\mathbb{E}\Bigg[\pi_t(1-\pi_t)f(X_t)\mathbb{E}\left[Y_{i,t+1}-(1-\pi_t)f(X_t)^\top\theta\middle|H_{t-1},X_t,A_t=1\right] - (1-\pi_t)\pi_t f(X_t)\mathbb{E}\left[Y_{i,t+1}-(0-\pi_t)f(X_t)^\top\theta\middle|H_{t-1},X_t,A_t=0\right]\Bigg]$$

$$= \sum_{t=1}^{T}\mathbb{E}\Bigg[\pi_t(1-\pi_t)f(X_t)\Bigg\{\mathbb{E}\left[Y_{i,t+1}\middle|H_{t-1},X_t,A_t=1\right]-\mathbb{E}\left[Y_{i,t+1}\middle|H_{t-1},X_t,A_t=0\right]-f(X_t)^\top\theta\Bigg\}\Bigg] \tag{2.1}$$

By law of iterated expectations, display (3.1) equals,

$$\sum_{t=1}^{T}\mathbb{E}\Bigg[\mathbb{E}\Bigg[\pi_t(1-\pi_t)f(X_t)\Bigg\{\mathbb{E}\left[Y_{i,t+1}\middle|H_{t-1},X_t,A_t=1\right]-\mathbb{E}\left[Y_{i,t+1}\middle|H_{t-1},X_t,A_t=0\right]-f(X_t)^\top\theta\Bigg\}\Bigg|X_t\Bigg]\Bigg]$$

Since π_t is only a function of X_t,

$$= \sum_{t=1}^{T} \mathbb{E}\Big[\pi_t(1-\pi_t) f(X_t) \Big\{ \mathbb{E}\Big[\mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 1\right] - \mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 0\right] \Big| X_t \Big] - f(X_t)^\top \theta \Big\}\Big]$$

$$= \sum_{t=1}^{T} \mathbb{E}\Big[\pi_t(1-\pi_t) f(X_t) \Big\{ \mathbb{E}\left[Y_{i,t+1}(\bar{A}_{t-1}, 1) - Y_{i,t+1}(\bar{A}_{t-1}, 0) \middle| X_t\right] - f(X_t)^\top \theta \Big\}\Big]$$

The above will equal zero for $\theta = \theta^\star$ by our modelling assumption on the excursion effect.

3 Case in which π_t may depend on H_{t-1}, X_t (e.g. when an RL algorithm is used)

$$0 = \mathbb{E}\left[\frac{\partial}{\partial \theta} \ell(H_T; \eta, \theta)\right]$$

$$= \sum_{t=1}^{T} \mathbb{E}_{\pi_t}\left[W_t \left\{Y_{t+1} - g(H_{t-1}, X_t)^\top \eta - (A_t - p_t) f(X_t)^\top \theta\right\} (A_t - p_t) f(X_t)\right]$$

Since the weights W_t are a Radon-Nikodym derivative (change of measure),

$$= \sum_{t=1}^{T} \mathbb{E}_{p_t}\left[\left\{Y_{t+1} - g(H_{t-1}, X_t)^\top \eta - (A_t - p_t) f(X_t)^\top \theta\right\} (A_t - p_t) f(X_t)\right]$$

Note that $\mathbb{E}_{p_t}\left[g(H_{t-1}, X_t)^\top \eta \left(A_t - p_t\right) f(X_t) \middle| H_{t-1}, X_t\right]$
$= g(H_{t-1}, X_t)^\top \eta \mathbb{E}_{p_t}\left[A_t - p_t \middle| H_{t-1}, X_t\right] f(X_t) = 0$, by law of iterated expectations,

$$= \sum_{t=1}^{T} \mathbb{E}_{p_t}\Big[\left\{Y_{t+1} - (A_t - p_t) f(X_t)^\top \theta\right\} (A_t - p_t) f(X_t)\Big]$$

Let $X_{i,t} \triangleq Y_{i,t+1} - (A_t - p_t) f(X_t)^\top \theta$.

$$= \sum_{t=1}^{T} \mathbb{E}\Big[\mathbb{E}_{p_t}\left[X_{i,t}(A_t - p_t) f(X_t) \middle| H_{t-1}, X_t\right]\Big]$$

$$= \sum_{t=1}^{T} \mathbb{E}\Big[p_t(1-p_t) f(X_t) \mathbb{E}\left[X_{i,t} \middle| H_{t-1}, X_t, A_t = 1\right]$$
$$+ (1-p_t)(0-p_t) f(X_t) \mathbb{E}\left[X_{i,t} \middle| H_{t-1}, X_t, A_t = 0\right]\Big]$$

$$= \sum_{t=1}^{T} \mathbb{E}\Bigg[p_t(1-p_t)f(X_t)\mathbb{E}\left[Y_{i,t+1} - (1-p_t)f(X_t)^\top\theta \middle| H_{t-1}, X_t, A_t = 1\right]$$

$$- (1-p_t)p_t f(X_t)\mathbb{E}\left[Y_{i,t+1} - (0-p_t)f(X_t)^\top\theta \middle| H_{t-1}, X_t, A_t = 0\right]\Bigg]$$

$$= \sum_{t=1}^{T} \mathbb{E}\Bigg[p_t(1-p_t)f(X_t)\Bigg\{\mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 1\right] - \mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 0\right] - f(X_t)^\top\theta\Bigg\}\Bigg] \tag{3.1}$$

By law of iterated expectations, display (3.1) equals,

$$\sum_{t=1}^{T} \mathbb{E}\Bigg[\mathbb{E}\Bigg[p_t(1-p_t)f(X_t)\Bigg\{\mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 1\right] - \mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 0\right] - f(X_t)^\top\theta\Bigg\}\Bigg| X_t\Bigg]\Bigg]$$

Since p_t is only a function of X_t,

$$= \sum_{t=1}^{T} \mathbb{E}\Bigg[p_t(1-p_t)f(X_t)\Bigg\{\mathbb{E}\Bigg[\mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 1\right] - \mathbb{E}\left[Y_{i,t+1} \middle| H_{t-1}, X_t, A_t = 0\right] \Bigg| X_t\Bigg] - f(X_t)^\top\theta\Bigg\}\Bigg]$$

$$= \sum_{t=1}^{T} \mathbb{E}\Bigg[p_t(1-p_t)f(X_t)\Bigg\{\mathbb{E}\left[Y_{i,t+1}(\bar{A}_{t-1}, 1) - Y_{i,t+1}(\bar{A}_{t-1}, 0) \middle| X_t\right] - f(X_t)^\top\theta\Bigg\}\Bigg]$$

The above will equal zero for $\theta = \theta^\star$ by our modelling assumption on the excursion effect.