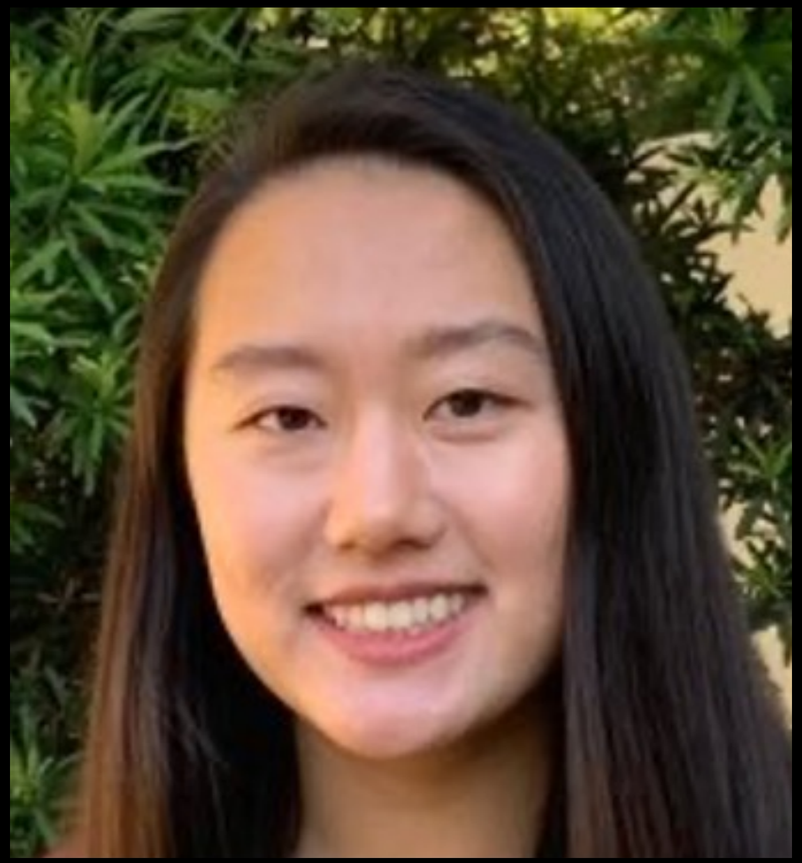
On counterfactual inference with factors models and nearest neighbors

Raaz Dwivedi, CDT Summer School



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Estimate counterfactual means $\{\theta_{i,t}^{(a)}\}$ for $a \in \{0,1\}$, all N users & T times

Challenges:

- ➡ More unknowns than (noisy) observations
- ➡ No parametric model available
- ➡ Intricate dependencies due to
  - ◉ Heterogeneity across users and time
  - ◉ Sequentially adaptive policy
  - ◉ Pooling for policy design

An impossible task without structural assumptions…

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Hope:

- ★ N iid users
- ★ T (dependent) observations per user
- ★ If users are not all too different & multiple observations can help find similarities

A possible task with some structural assumptions…

A factor model for potential outcomes

- The potential outcomes of N units across T decision times are sampled as follows:
  - For each unit $i \in [N]$, sample its ``unit factor'' for each treatment $u_i^{(1)}, u_i^{(0)}$, which remains fixed across time
  - For each decision time $t \in [T]$, sample the corresponding ``time factor'' $v_t^{(1)}, v_t^{(0)}$, that is shared across units
  - The potential outcomes for unit i at time t satisfy

$$\mathbb{E}[R_{i,t}(a) \,|\, u_i^{(a)}, v_t^{(a)}] = \langle u_i^{(a)}, v_t^{(a)} \rangle \triangleq \theta_{i,t}^{(a)}$$

$$\text{and } R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

  - Where $A_{i,t}$ is assigned by bandit algorithm that may be **pooling** data across users

User nearest neighbors estimator for treatment a

1. Compute distance between user pairs i, j under treatment a **using all data**

$$\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)} + \frac{\hat{\sigma}_\rho}{\sqrt{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}}$$

2. Average over user neighbors treated with a at time t

$$\widehat{\theta}_{i,t,\text{user-NN}}^{(a)} = \frac{\sum_{j=1}^{N} R_{j,t} \mathbf{1}(\rho_{i,j}^{(a)} \leq \eta, A_{j,t} = a)}{\sum_{j=1}^{N} \mathbf{1}(\rho_{i,j}^{(a)} \leq \eta, A_{j,t} = a)}$$

User-NN: A non-asymptotic guarantee for each (i, t, a)

Informal theorem: [**Dwivedi**-Tian-Tomkins-Klasnja-Murphy-Shah '22a]
For suitably chosen η & under regularity conditions

- iid latent factors, sub-Gaussian noise
- sequentially adaptive policies with conditionally independent treatments across users that choose a with probability $\geq p^{\dagger}$

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for each user i at each time t, with high probability

$$| \widehat{\theta}^{(a)}_{i,t,\text{user–NN}} - \theta^{(a)}_{i,t} | \lesssim \frac{1}{(p^2 T)^{1/4}} + \frac{1}{(p \cdot \#Neighbors)^{1/2}}$$

($\dagger$Thus p can not decay faster than $\gtrsim T^{-1/2}$)

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$$|\,\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} - \theta^{(a)}_{i,t}\,| \lesssim \frac{1}{(p^2 T)^{1/4}} + \frac{1}{(pN/M)^{1/2}}$$

User factor distribution

↓

(user factors ~ uniform over a finite set of size M)

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(user factors ~ uniform over a finite set of size M)

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{user–NN}} - \theta^{(a)}_{i,t}\,| \lesssim \frac{1}{(p^2T)^{1/4}} + \frac{1}{(pN)^{1/(d+2)}}$$

(user factors ~ Uniform in $[-1,1]^d$)

Rest of the talk will assume p to be constant.

User-NN guarantees: **Advantages**

- Asymptotic **confidence intervals** as $N, T \to \infty$:

$$\widehat{\theta}^{(a)}_{i,t,\text{user–NN}} \pm \frac{1.96\,\widehat{\sigma}}{\sqrt{\text{\#neighbors}_{i,t,a}}}$$

$$\downarrow$$

Confidence intervals for treatment effect $\theta^{(1)}_{i,t} - \theta^{(0)}_{i,t}$

Can we improve the slow error rate in T?

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{T^{1/4}} + \frac{1}{\sqrt{N}}\right)$$

$$\big\downarrow$$

$$|\,\text{??} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

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$$|\,??\, - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{time–NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{N^{1/4}} + \frac{1}{\sqrt{T}}\right)^{\star}$$

*- Guarantees under stronger assumptions on policy / modified distance.

Can we improve the <u>slow error rate</u> in T?

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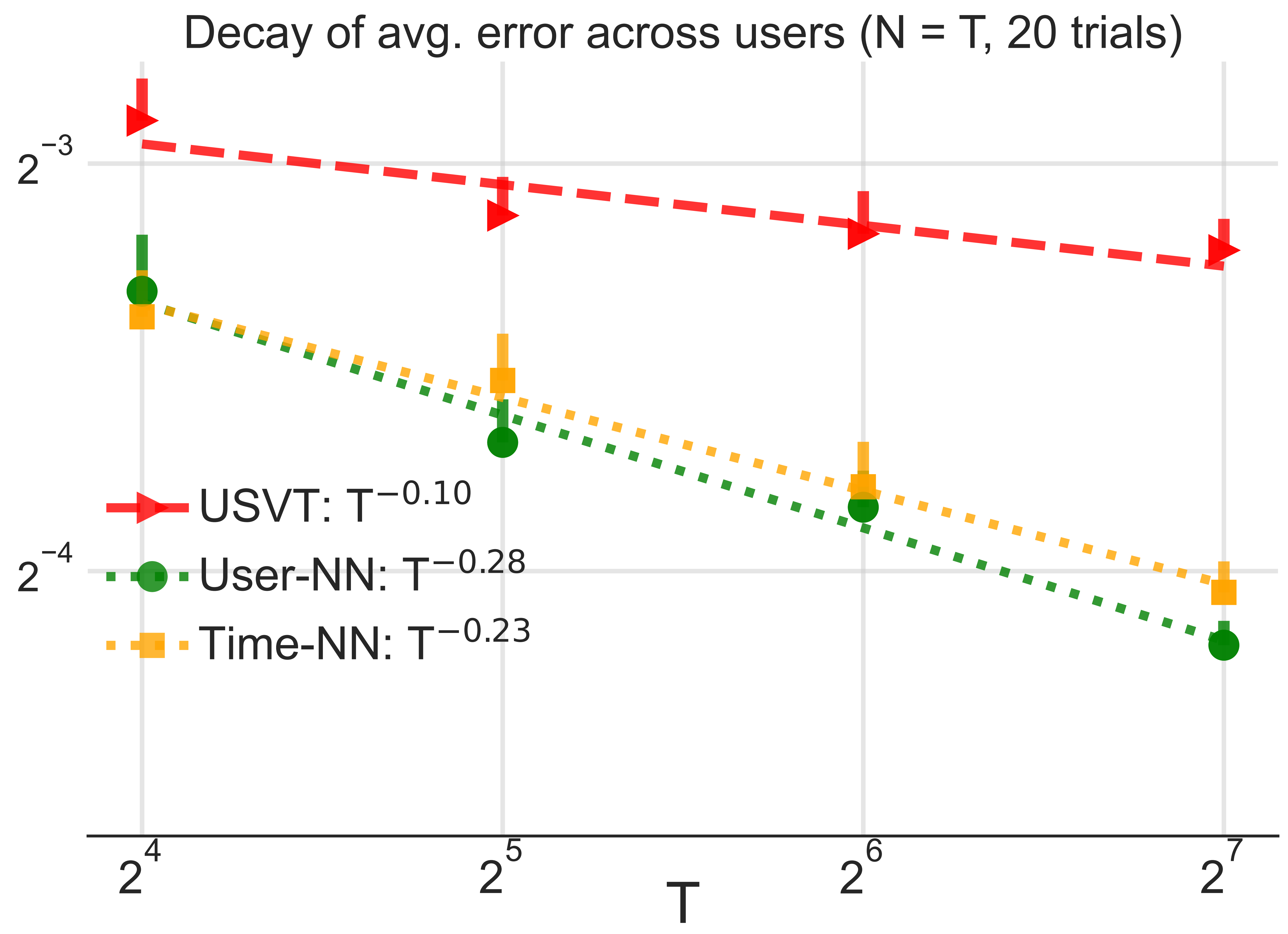
$$|\,??\, - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

$$|\,\widehat{\theta}^{(a)}_{i,t,\text{time-NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{N^{1/4}} + \frac{1}{\sqrt{T}}\right)$$

Uniform factors on $[-0.5, 0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)



USVT: A baseline algorithm from [Chatterjee 2014]

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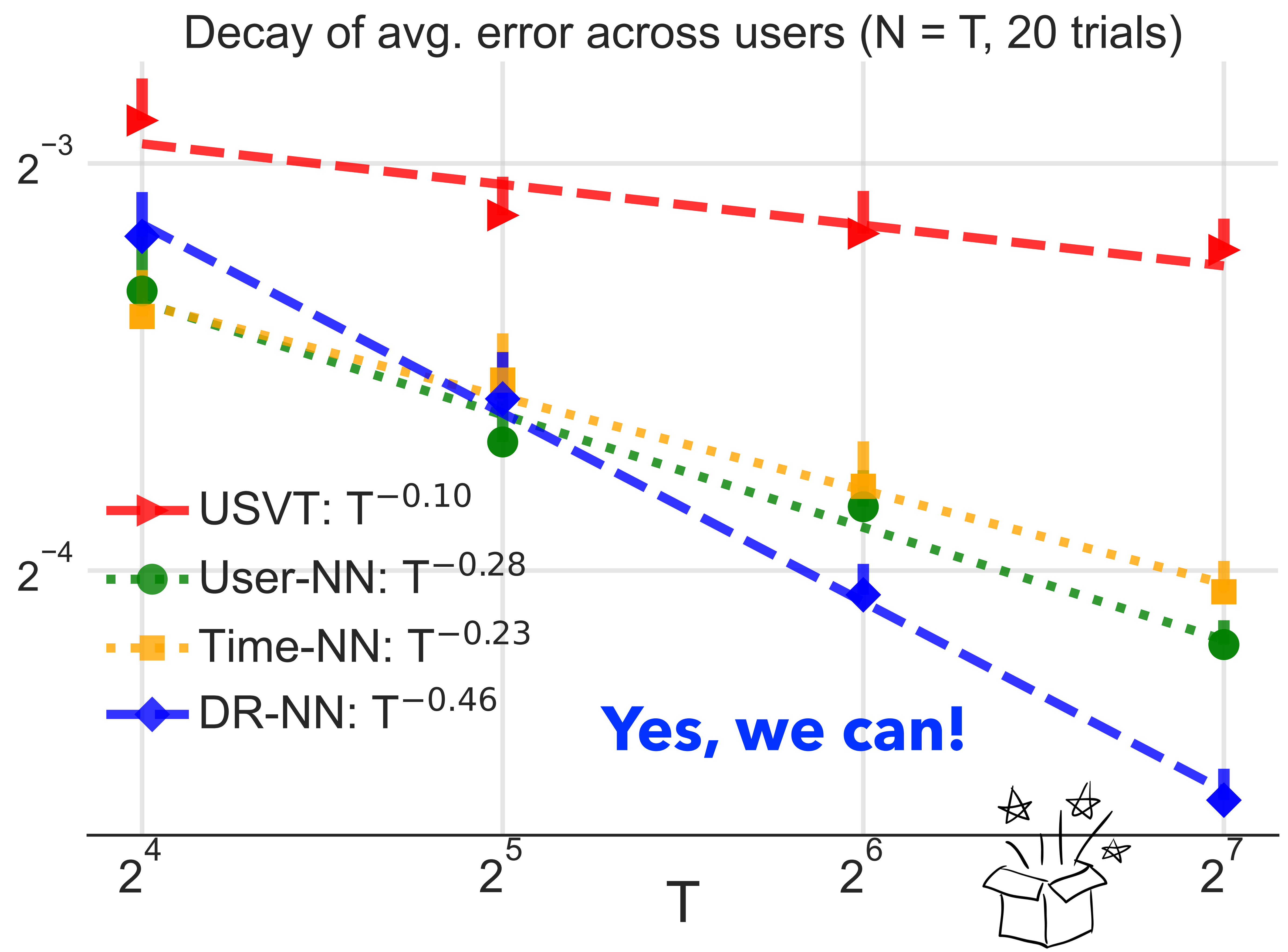


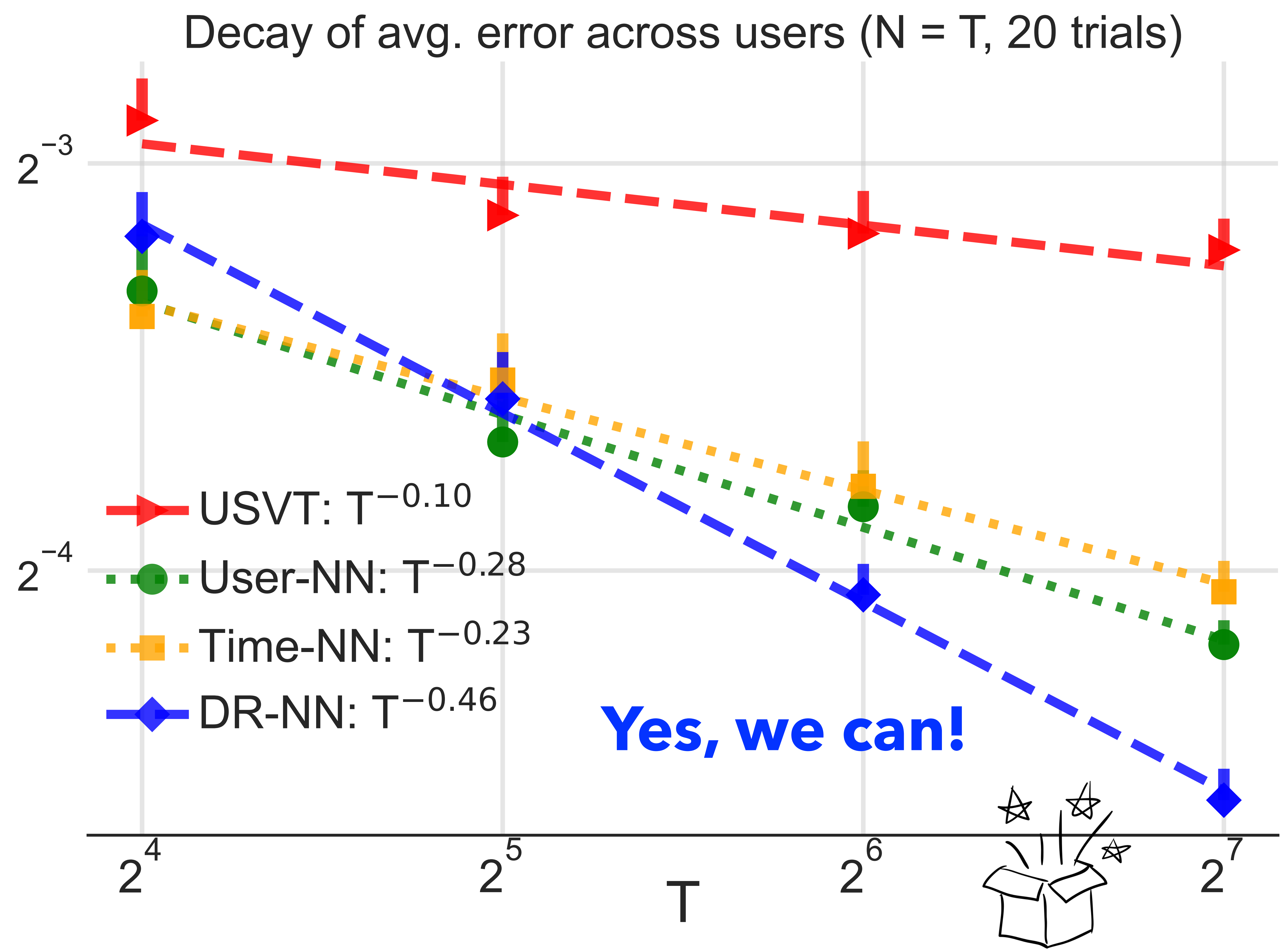
Can we improve the slow error rate in T?

Uniform factors on $[-0.5,0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)

$$|\,\widehat{\theta}^{(a)}_{i,t,\mathrm{DR\text{-}NN}} - \theta^{(a)}_{i,t}\,| = \tilde{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{N}}\right)$$

With a **suitable variant** of nearest neighbors

[**Dwivedi**-Tian-Tomkins-Klasnja-Murphy-Shah '22b]



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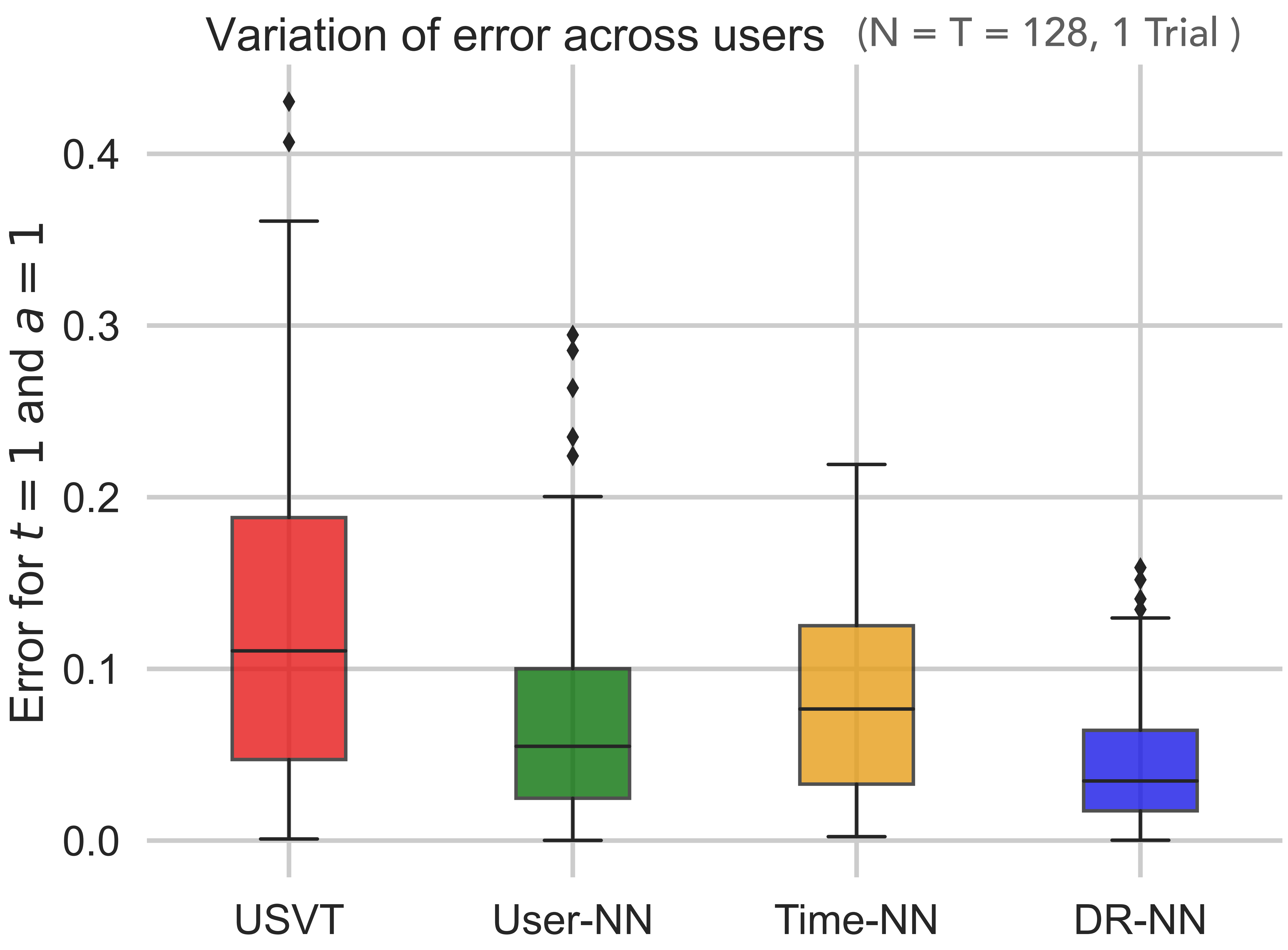
Simulation results

Uniform latent factors on $[-0.5{,}0.5]^4$, Gaussian noise, pooled ε-greedy policy ($\varepsilon = 0.5$)



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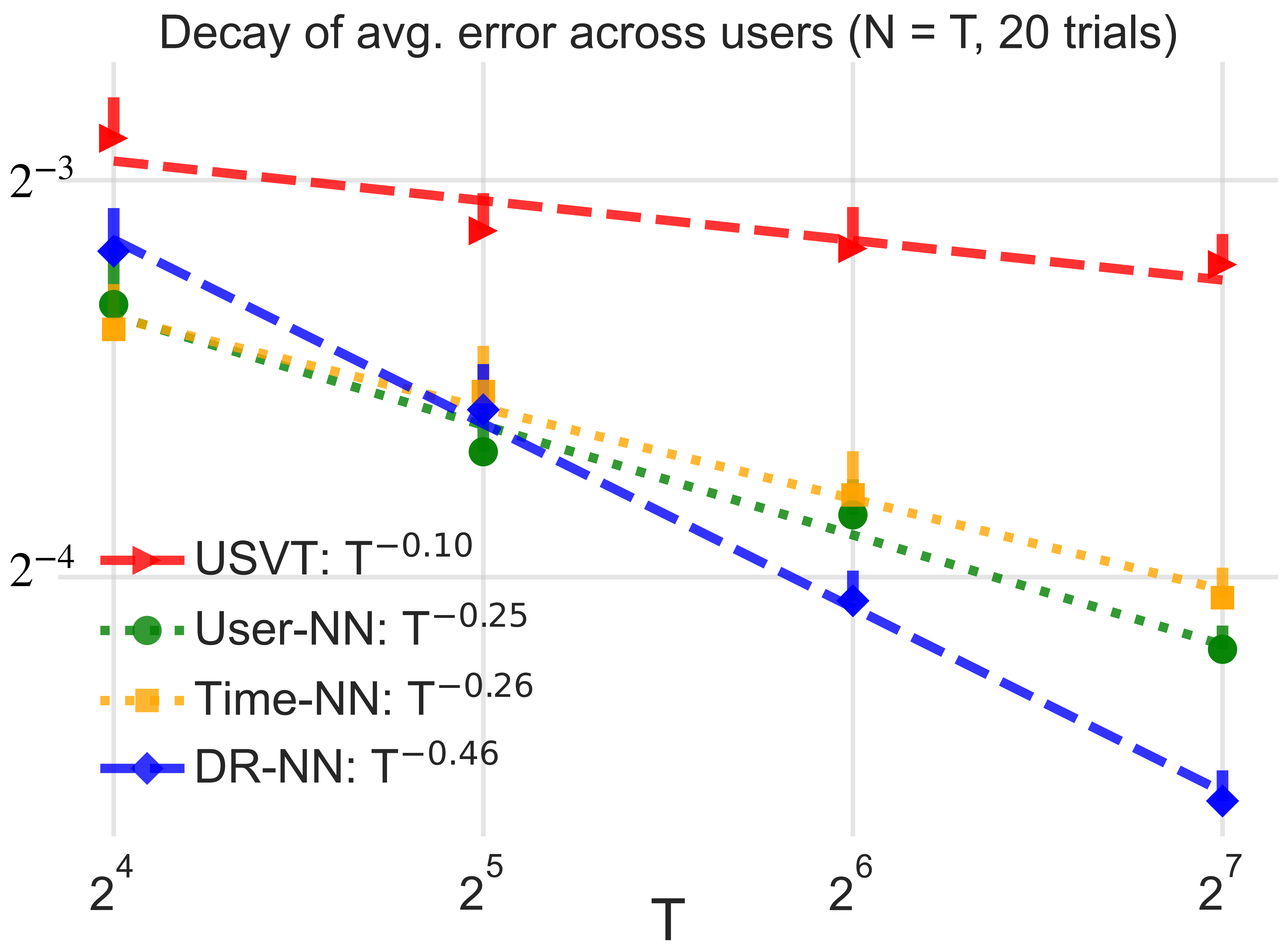
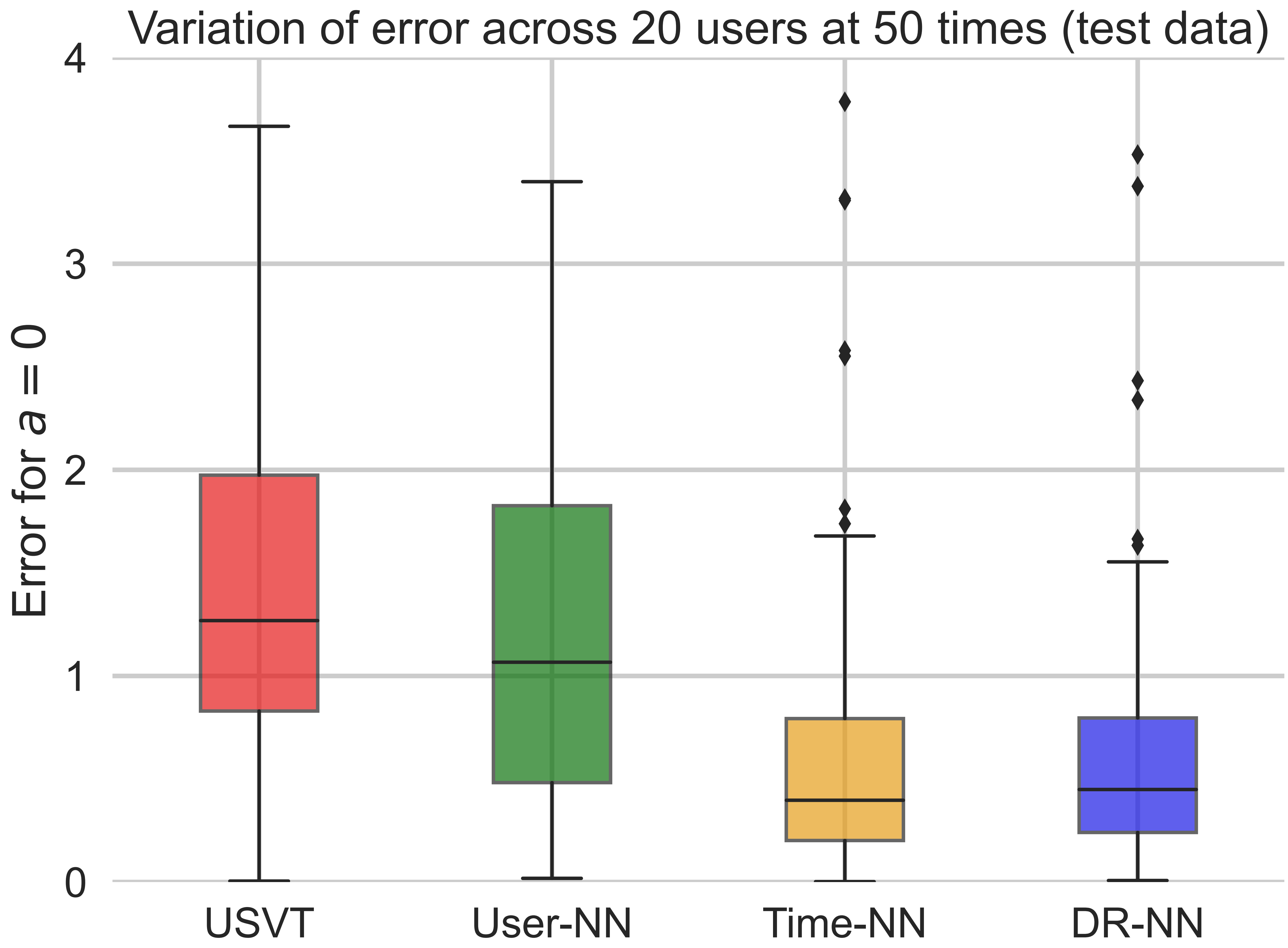
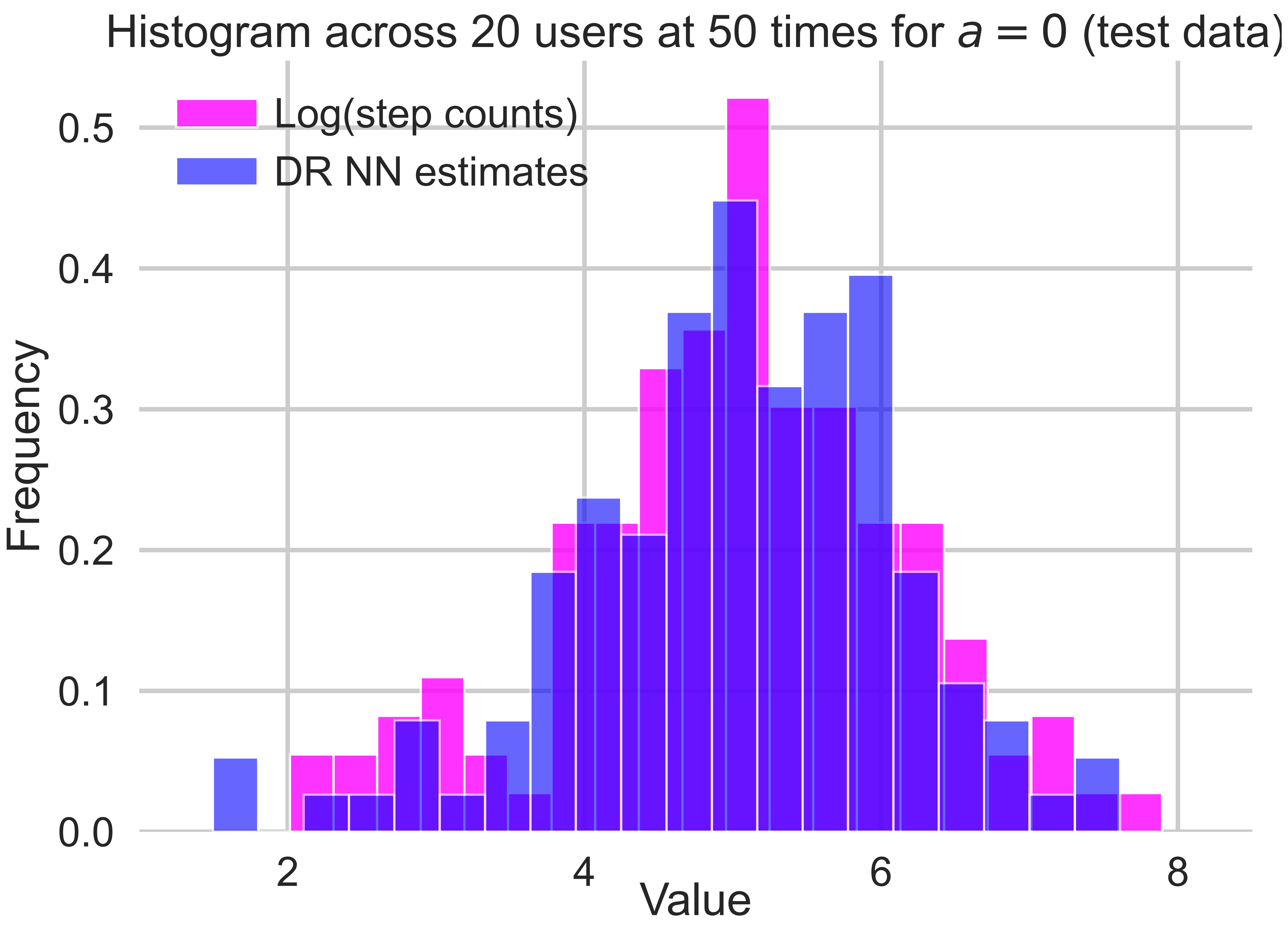


DR-NN error $\ll$ min { user-NN error, time-NN error }

Personalized HeartSteps results

Treatments assigned with Thompson sampling independently for 91 users for 90 days, 5 times a day





DR-NN error ≈ min { user-NN error, time-NN error }

Proof outline for user-NN

- Simple case: Always assign $A_{j,t} = a$ and $\theta_{i,t}^{(a)} \triangleq u_i v_t$

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

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$$= \frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|} v_t + \frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$$

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$$= \underbrace{\frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|}}_{\hat{u}_i} v_t + \underbrace{\frac{\sum_{j \in user\ nbrs} \varepsilon^{(a)}_{j,t}}{|user\ nbrs|}}_{\bar{\varepsilon}_t}$$

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta^{(A_{i,t})}_{i,t} + \varepsilon^{(A_{i,t})}_{i,t}$$

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$$= \underbrace{\frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|}}_{\hat{u}_i} v_t + \underbrace{\frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}}_{\bar{\varepsilon}_t}$$

- $|\theta_{i,t}^{(a)} - \widehat{\theta}_{i,t,\text{user-NN}}^{(a)}| = |u_i v_t - \widehat{\theta}_{i,t,\text{user-NN}}^{(a)}| \leq |u_i v_t - \hat{u}_i v_t| + |\bar{\varepsilon}_t|$

$$R_{i,t} = R_{i,t}(A_{i,t}) \triangleq \theta_{i,t}^{(A_{i,t})} + \varepsilon_{i,t}^{(A_{i,t})}$$

Proof outline for user-NN

- $|u_i v_t - \hat{u}_i v_t| \leq \max_{j \in user\ nbrs} |u_i - u_j| |v_t| \lesssim \sqrt{\eta - 2\sigma^2} + \frac{1}{T^{1/4}}$
- $\bar{\varepsilon}_t = \frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|} \lesssim \frac{\sigma}{\sqrt{|user\ nbrs|}} = \frac{\sigma}{\sqrt{N_\eta}}$

Summary of the unit or time nearest neighbors

- $|u_i v_t - \widehat{\theta}^{(a)}_{i,t,\text{user-NN}}| \leq |u_i v_t - \hat{u}_i v_t| + |\bar{\varepsilon}_t| = O(|u_i - \hat{u}_i|)$
- $|u_i v_t - \widehat{\theta}^{(a)}_{i,t,\text{time-NN}}| \leq |u_i v_t - u_i \hat{v}_t| + |\bar{\varepsilon}_i| = O(|v_t - \hat{v}_t|)$
- Can we combine both to improve the error rate?

In the search of improved estimator…

- Let's ignore the noise term and consider one nearest neighbor. "j" is a user neighbor so that $\hat{u}_i = u_j$ and "t'" is time neighbor so that $\hat{v}_t = v_{t'}$

- $\widehat{\theta}^{(a)}_{i,t,\text{user-NN}} = R_{j,t} = u_j v_t$ and $\widehat{\theta}^{(a)}_{i,t,\text{time-NN}} = R_{i,t'} = u_i v_{t'}$

- Can we combine to improve?

    - Average the two estimates: $\dfrac{u_j v_t + u_i v_{t'}}{2} = \dfrac{R_{j,t} + R_{i,t'}}{2}$

    - Use both neighbors: Outcome of user j at time t': $u_j v_{t'} = R_{j,t'}$

Discussion questions

- What are the limitations of the factor model and the assumptions made for stating the non-asymptotic guarantee?
    - Can you try to weaken these assumptions, to include states, delayed effects?
- **Hard**: Would the “averaged/merged” combination strategy significantly improve the performance?
    - Can you think of other ways to improve the NN estimator for the current model or more generally?

What do we desire?

- **Convert $+$ to $\times$:** $\quad |u_i v_t - \text{??}| = |u_i - u_j| \times |v_t - v_{t'}|$

$$= \text{User-NN Error} \times \text{Time-NN Error}$$

or **max** to **min:**

$$\approx \min \{\text{User-NN Error, Time-NN Error}\}$$

What should be our estimator? Let's expand the RHS...

$$\cancel{u_i v_t} - ?? = (u_i - u_j) \times (v_t - v_{t'})$$

$$= \cancel{u_i v_t} - u_j v_t - u_i v_{t'} + u_j v_{t'}$$

$$\Rightarrow \qquad ?? = u_j v_t + u_i v_{t'} - u_j v_{t'}$$

$$R_{j,t} \quad + \quad R_{i,t'} \quad - R_{j,t'}$$

This is our improved nearest neighbors estimator!

$$\cancel{u_i v_t} - ?? = (u_i - u_j) \times (v_t - v_{t'})$$

$$= \cancel{u_i v_t} - u_j v_t - u_i v_{t'} + u_j v_{t'}$$

$$\Rightarrow \qquad ?? = u_j v_t + u_i v_{t'} - u_j v_{t'}$$

$$\widehat{\theta}^{(a)}_{i,t,\mathrm{DR\text{-}NN}} = \frac{\sum_{j,t'} (R_{j,t} + R_{i,t'} - R_{j,t'}) \mathbf{1}_{i,t,j,t'}}{\sum_{j,t'} \mathbf{1}_{i,t,j,t'}}$$

$$\mathbf{1}_{i,t,j,t'} = \mathbf{1}(\rho^{(a)}_{i,j} \leq \eta, \;\; \rho^{(a)}_{t,t'} \leq \eta', A_{j,t} = A_{i,t'} = A_{j,t'} = a)$$

This is our improved nearest neighbors estimator!

$$\cancel{u_i v_t} - \text{??} = (u_i - u_j) \times (v_t - v_{t'})$$

$$= \cancel{u_i v_t} - u_j v_t - u_i v_{t'} + u_j v_{t'}$$

$$\Rightarrow \qquad \text{??} = u_j v_t + u_i v_{t'} - u_j v_{t'}$$

DR-NN error $\approx$ user-NN error $\times$ time-NN error

$\lesssim$ min{user-NN error, time-NN error}

Doubly robust to heterogeneity in user factors & time factors

Double robustness, double machine learning…

[… Cassel+ '77, Robinson '88, Särndal+ '89, Robins+ '94, '95, '08, '09, Newey+ '94, '18, Bickel+ '98, van der Laan+ '03, Lunceford+ '04, Davidian+ '05, Li+ '11, Jiang+ '15, Chernozhukov+ '18, Hirshberg+ '18, Diaz '19, Arkhangelsky+ '21, Dorn+ '21 …]

Disclaimer

- c, C are universal constants that might take a different value in each appearance

Proof sketch for user-NN

- Simple case: Always assign $A_{j,t} = a$ and $\theta_{i,t}^{(a)} \triangleq u_i v_t$
- $\hat{\theta}_{i,t,\text{user-NN}}^{(a)} = \dfrac{\sum_{j \in user\ nbrs} R_{j,t}}{|user\ nbrs|} = \dfrac{\sum_{j \in user\ nbrs} \theta_{j,t}^{(a)} + \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$

$$= \frac{\sum_{j \in user\ nbrs} u_j}{|user\ nbrs|} v_t + \frac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|}$$

- $|u_i v_t - \hat{u}_i v_t| \leq \max_{j \in user\ nbrs} |u_i - u_j| |v_t| \lesssim \sqrt{\eta - 2\sigma^2} + \dfrac{1}{T^{1/4}}$
- $\bar{\varepsilon}_t = \dfrac{\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}}{|user\ nbrs|} \lesssim \dfrac{\sigma}{\sqrt{|user\ nbrs|}} = \dfrac{\sigma}{\sqrt{N_\eta}}$

Our goal: Control $\max_{j \in user\ nbrs} |u_i - u_j||v_t|$

- $|v_t|$ is bounded so suffices to bound $\max_{j \in user\ nbrs} |u_i - u_j|$
- user neighbours = $\{\rho_{i,j}^{(a)} \leq \eta\}$
  - $\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)} + \frac{\hat{\sigma}_\rho}{\sqrt{\sum_{t'=1}^{T} \mathbf{1}(A_{i,t'} = A_{j,t'} = a)}}$
  - In theory, we ignore the second term

Controlling the bias via concentration of distance

- Simple case: Always assign $A_{j,t} = a$ and $\theta^{(a)}_{i,t} \triangleq u_i v_t$

- $\rho^{(a)}_{i,j} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2}{T}$

Re-expressing the distance

- Simple case: Always assign $A_{j,t} = a$ and $\theta_{i,t}^{(a)} \triangleq u_i v_t$

- $\rho_{i,j}^{(a)} = \frac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2}{T}$

$$= \frac{1}{T} \sum_{t'=1}^{T} \left[(u_i v_{t'} - u_j v_{t'})^2 + (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 + 2(u_i v_{t'} - u_j v_{t'})(\varepsilon_{i,t'} - \varepsilon_{j,t'}) \right]$$

$$= (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$$

Re-expressing the distance: Collecting into three terms

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v'_t(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

What do these three terms concentrate on?

- $\rho^{(a)}_{i,j} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

Recall our assumptions –

v_t are iid

$\varepsilon_{i,t}$ are iid zero mean with variance σ

v_t and $\varepsilon_{i,t}$ are independent of each other

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- $|\frac{\sum_{t'=1}^{T} v_{t'}^2}{T} - ?| \lesssim ?$
- $|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} - ?| \lesssim ?$
- $|\frac{\sum_{t'=1}^{T} v'_t(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T} - ?| \lesssim ?$

Recall our assumptions –

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$\varepsilon_{i,t}$ are iid zero mean with variance σ

v_t and $\varepsilon_{i,t}$ are independent of each other

Tools for concentration

- **Markov's inequality:** Let $X_1, X_2, \ldots, X_T$ be iid random variables with mean μ and variance Var(X), then

$$\mathbb{P}\left[\left|\frac{\sum_{i=1}^{T} X_i}{T} - \mu\right| \leq \sqrt{\frac{\mathrm{Var}(X)}{\delta T}}\right] \geq 1 - \delta$$

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- **Chernoff-Hoeffding bound:** If X_i have mean μ and are $\gamma-$sub-Gaussian, i.e., $\mathbb{E}[e^{t(X-\mu)}] \leq e^{t^2\gamma^2/2}$ then

$$\mathbb{P}\left[\left|\frac{\sum_{i=1}^{T} X_i}{T} - \mu\right| \leq \gamma\sqrt{2\log(1/\delta)}\right] \geq 1 - \delta$$

  - Useful fact if $|X_i| \leq c$, then we can use $\gamma = c$

What do these three terms concentrate around?

Their means!

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \boxed{\frac{\sum_{t'=1}^{T} v_{t'}^2}{T}} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$

- $\left| \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} - \boxed{\mathbb{E}[v_{t'}^2]} \right| \lesssim \frac{c\sqrt{\mathrm{Var}(v_{t'}^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(v_{t'}^2) \cdot \log(1/\delta)}}{\sqrt{T}}$

What do these three terms concentrate around?

Their means!

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2}{T} + \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} + 2(u_i - u_j) \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}$
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- $\left| \frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T} - 2\sigma^2 \right| \lesssim \frac{c\sqrt{2\mathrm{Var}(\varepsilon^2)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(\varepsilon^2) \cdot \log(1/\delta)}}{\sqrt{T}}$

What do these three terms concentrate around?

Their means!

- $\rho_{i,j}^{(a)} = (u_i - u_j)^2 \boxed{\frac{\sum_{t'=1}^{T} v_{t'}^2}{T}} + \boxed{\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2}{T}} + 2(u_i - u_j) \boxed{\frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T}}$
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- $\left| \frac{\sum_{t'=1}^{T} v_t'(\varepsilon_{i,t'} - \varepsilon_{j,t'})}{T} - \boxed{0} \right| \lesssim \frac{c\sqrt{2\mathrm{Var}(v_t\varepsilon)}}{\sqrt{\delta T}}$ or $\frac{c\sqrt{\text{Sub-Gauss}(v_t\varepsilon) \cdot \log(1/\delta)}}{\sqrt{T}}$

Inverting the distance to get a control on $|u_i - u_j|$

- Assume v_t, ε are bounded and $\mathbb{E}[v_t^2] = v_\star^2$ then
  - $|\rho_{i,j}^{(a)} - (u_i - u_j)^2 v_\star^2 - 2\sigma^2| \leq \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$ with probability $1 - \delta$
  - Treat δ as a constant
  - Rearranging terms $|u_i - u_j|^2 \leq \dfrac{1}{v_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \dfrac{C}{\sqrt{T}}\right)$

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    - Treat δ as a constant
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- So if $\rho_{i,j}^{(a)} \leq \eta \Longrightarrow |u_i - u_j| \lesssim \sqrt{\eta - 2\sigma^2} + \dfrac{C}{T^{1/4}}$ if $v_\star^2 > 0$.

But how many users would satisfy $\rho_{i,j}^{(a)} \leq \eta$?

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq r\}|$

- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$

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- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \dfrac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}|}{|user\ nbrs|} \lesssim \dfrac{\sigma}{\sqrt{N_{i,\gamma}}}$

Univariate factors:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{T}}\right) + \frac{\sigma^2}{N_{i,\eta' - e_T}}$$

η'

e_T

NN bias
due to threshold

Error in
NN distance

NN noise
variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

Univariate factors + constant policy $\mathbb{P}(A_{i,t} = a) =$:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 \quad + \quad \frac{C}{\sqrt{p^2 T}}\right) \quad + \quad \frac{\sigma^2}{pN_{i,\eta'-e_T}}$$

η'

e_T

NN bias
due to threshold

Error in
NN distance

NN noise
variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

Multivariate factors + learning policy with $\mathbb{P}(A_{i,t} = a \mid History_{t-1}) \geq p$: A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{v_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η'

NN bias
due to threshold

e_T

Error in
NN distance

NN noise
variance

NN bias
inflation due to
learning policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^2 v_\star^2 \leq \gamma\}|$$

Further details

Scalings of $N_{i,\gamma}$

- When factors are sampled independently and uniformly from a discrete set $\{\Delta, \ldots, (M-1)\Delta\}$
  - $N_{i,r} \geq cN/M$ for any $r \geq 0$ if $v_\star > 0$.
- When factors are sampled independently and uniformly from a continuous set $[0,1]$
  - $N_{i,r} \geq c\sqrt{r/v_\star}$ for any $r \geq 0$.
- **HW:** You can now tune η to get refined error bounds.

Multivariate factors: Bias analysis

- Assume v_t, ε are bounded and $\mathbb{E}[v_t v_t^\top] = \Sigma_v$ then
- $|\rho_{i,j}^{(a)} - (u_i - u_j)^\top \Sigma_v (u_i - u_j) - 2\sigma^2| \leq \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$ with probability $1 - \delta$
- Treat δ as a constant
- Rearranging terms $\|u_i - u_j\|_2^2 \leq \dfrac{1}{\lambda_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \dfrac{C}{\sqrt{T}}\right)$ where $\lambda_\star = \lambda_{\min}(\Sigma_v)$
- So if $\rho_{i,j}^{(a)} \leq \eta \implies \|u_i - u_j\|_2 \lesssim \sqrt{\eta - 2\sigma^2} + \dfrac{C}{T^{1/4}}$ if $\lambda_\star^2 > 0$.

Multivariate factors: Variance analysis

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^\top \Sigma_v (u_i - u_j) \leq r\}|$
- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \dfrac{C\sqrt{\log(c/\delta)}}{\sqrt{T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \dfrac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)}|}{|user\ nbrs|} \lesssim \dfrac{\sigma}{\sqrt{N_{i,\gamma}}}$

Multivariate factors:
A general error bound for user NN when $A_{i,t}$ is always a

$$\left(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t}\right)^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{T}}\right) + \frac{\sigma^2}{N_{i,\eta' - e_T}}$$

η'

e_T

NN bias
due to threshold

Error in
NN distance

NN noise
variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Multivariate factors + learning policy with exploration p:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η' — NN bias due to threshold

e_T — Error in NN distance

NN noise variance

NN bias inflation due to **learning** policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \mathbf{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Constant policy: Bias analysis

- Assume $A_{j,t}$ are iid Bernoulli random variables p – constant MRT – Like in HeartSteps V1
    - Let $a = 1$, then what is the distribution of $B_{i,j,t'} \triangleq \mathbf{1}(A_{i,t'}{=}A_{j,t'}{=}a)$?
    - $\rho_{i,j}^{(a)} = \dfrac{\sum_{t'=1}^{T} (R_{i,t'} - R_{j,t'})^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}}$

$$= \frac{\sum_{t'=1}^{T} \left[(u_i v_{t'} - u_j v_{t'})^2 + (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 + 2(u_i v_{t'} - u_j v_{t'})(\varepsilon_{i,t'} - \varepsilon_{j,t'}) \right] \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}}$$

Bias analysis: The denominator changes

- $\left|(u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - (u_i - u_j)^2\, \mathbb{E}[v_{t'}^2]\right| \lesssim \frac{c_v^2 \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$

- $\left|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - 2\sigma^2\right| \lesssim \frac{c_\varepsilon \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$

- $|\frac{T_{i,j}}{T} - p^2| \lesssim \frac{c\sqrt{\log(1/\delta)}}{\sqrt{T}}$

    - A better bound available: $T_{i,j} \geq cp^2T$ with probability $\geq 1 - e^{-cp^2T}$.

Bias analysis: The denominator changes

- Assume v_t, ε are bounded and $\mathbb{E}[v_t v_t^\top] = \Sigma_v$ then

- Hence $|\rho_{i,j}^{(a)} - (u_i - u_j)^\top \Sigma_v (u_i - u_j) - 2\sigma^2| \leq \frac{C\sqrt{\log(c/\delta)}}{\sqrt{p^2 T}}$ with probability $1 - \delta$

- Treat δ as a constant

- Rearranging terms $\|u_i - u_j\|_2^2 \leq \frac{1}{\lambda_\star^2}\left(\rho_{i,j}^{(a)} - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right)$ where $\lambda_\star = \lambda_{\min}(\Sigma_v)$

- So if $\rho_{i,j}^{(a)} \leq \eta \implies \|u_i - u_j\|_2 \lesssim \sqrt{\eta - 2\sigma^2} + \frac{C}{p^{1/2} T^{1/4}}$ if $\lambda_\star^2 > 0$.

Constant policy: Variance analysis: denominator changes

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^\top \Sigma_v (u_i - u_j) \leq r\}|$
- $|$User-nbrs$| = |\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \frac{C\sqrt{\log(c/\delta)}}{\sqrt{p^2 T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)|}{|user\ nbrs\ with\ A_{j,t} = a|} \lesssim \frac{\sigma}{\sqrt{pN_{i,\gamma}}}$

Multivariate factors + constant policy: A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}}$$

η' e_T

NN bias due to threshold

Error in NN distance

NN noise variance

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Multivariate factors + learning policy with exploration p:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\eta - 2\sigma^2 + \frac{C}{\sqrt{p^2 T}}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

η' — NN bias due to threshold

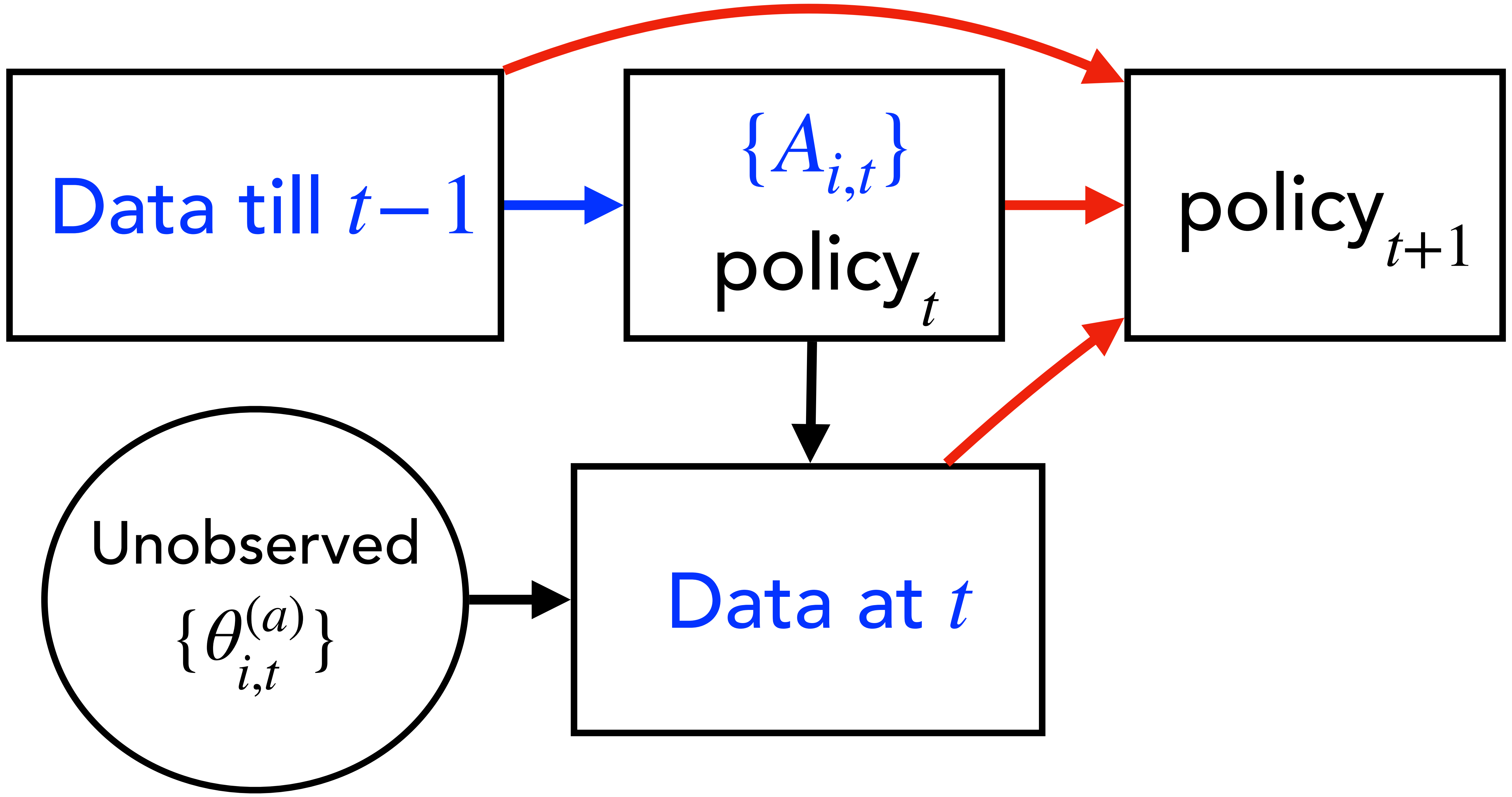
e_T — Error in NN distance

NN noise variance

NN bias inflation due to **learning** policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \boldsymbol{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$

Learning policy:
Sequential dependence between observations



Learning policy: Bias analysis
Similar except now with Martingales

- Still goes through using "Azuma-Hoeffing bounds" and careful Martingale construction
- $\left|(u_i - u_j)^2 \frac{\sum_{t'=1}^{T} v_{t'}^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - (u_i - u_j)^2 \, \mathbb{E}[v_{t'}^2]\right| \lesssim \frac{c_v^2 \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$
- $\left|\frac{\sum_{t'=1}^{T} (\varepsilon_{i,t'} - \varepsilon_{j,t'})^2 \cdot B_{i,j,t'}}{\sum_{t'=1}^{T} B_{i,j,t'}} - 2\sigma^2\right| \lesssim \frac{c_\varepsilon \sqrt{\log(1/\delta)}}{\sqrt{T_{i,j}}}$
- $|\frac{T_{i,j}}{T} - p^2| \lesssim \frac{c\sqrt{\log(1/\delta)}}{\sqrt{T}}$
    - A better bound available: $T_{i,j} \geq cp^2T$ with probability $\geq 1 - e^{-cp^2T}$.

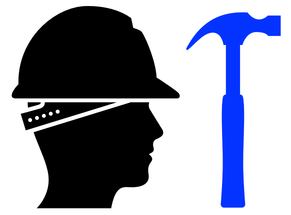
Learning policy: Bias bounds
Essentially same as the MRT bound

- If $\rho_{i,j}^{(a)} \leq \eta \quad \Longrightarrow \quad \|u_i - u_j\|_2 \lesssim \frac{1}{\lambda_\star}\left(\sqrt{\eta - 2\sigma^2} + \frac{C}{p^{1/2}T^{1/4}}\right)$ if $\lambda_\star^2 > 0.$

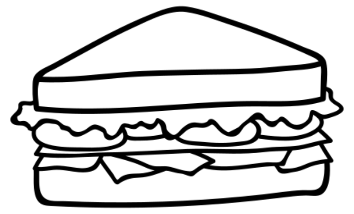
Learning policy: Variance analysis — Non-trivial changes

- $N_{i,r} \triangleq |\{j \neq i : (u_i - u_j)^\top \Sigma_v (u_i - u_j) \leq r\}|$
- |User-nbrs| = $|\rho_{i,j}^{(a)} \leq \eta| \geq N_{i,\gamma}$ for $\gamma = \eta - 2\sigma^2 - \frac{C\sqrt{\log(c/\delta)}}{\sqrt{p^2 T}}$
- Why do we care? Variance
  - $|\bar{\varepsilon}_t| = \frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)|}{|user\ nbrs\ with\ A_{j,t} = a|}$

? noise at t **correlated** with user neighbors (learning policy)



Martingale concentration, new sandwich argument for NN



Learning policy: Variance bounds — Has a "bias" like term

- $$\bar{\varepsilon}_t^2 = \left(\frac{|\sum_{j \in user\ nbrs} \varepsilon_{j,t}^{(a)} \mathbf{1}(A_{j,t} = a)|}{|user\ nbrs\ with\ A_{j,t} = a|}\right)^2$$

$$\lesssim \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

Martingale concentration, **new sandwich argument** for NN

Multivariate factors + learning policy with exploration p:
A general error bound for user NN

$$(\widehat{\theta}^{(a)}_{i,t,\eta} - \theta^{(a)}_{i,t})^2 \lesssim \frac{1}{\lambda_\star^2}\left(\underbrace{\eta - 2\sigma^2}_{\eta'} + \underbrace{\frac{C}{\sqrt{p^2 T}}}_{e_T}\right) + \frac{\sigma^2}{pN_{i,\eta'-e_T}} + c_{noise}\left[\frac{N_{i,\eta'+e_T} - N_{i,\eta'-e_T}}{pN_{i,\eta'-e_T}}\right]^2$$

NN bias due to threshold

Error in NN distance

NN noise variance

NN bias inflation due to **learning** policy

$$N_{i,\gamma} \triangleq |\{j \neq i : (u_i - u_j)^\top \mathbf{\Sigma}_v (u_i - u_j) \leq \gamma\}|$$