Online RL Algorithms that Pool Across Users

Tuesday Morning Session

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Digital Intervention Study Design Objectives

Within-Study Personalization

Maximize User Benefit

- Send messages at opportune moments

Use Online RL Algorithms

$$\mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

After-Study Analyses

Evaluate the Intervention

- Understand heterogeneity across user types and user states

Infer Treatment Effects

$$\mathbb{E}\left[Y_{t+1}(\bar{A}_{t-1},1) - Y_{t+1}(\bar{A}_{t-1},0) \mid X_t\right]$$

Digital Intervention Study Design Objectives

Within-Study Personalization

Maximize User Benefit

- Send messages at opportune moments

Use Online RL Algorithms

$$\mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

After-Study Analyses

Confidence Intervals Critical for

- Replicable science
- Publishing and sharing results

Infer Treatment Effects

$$\mathbb{E}\left[Y_{t+1}(\bar{A}_{t-1},1) - Y_{t+1}(\bar{A}_{t-1},0) \,|\, X_t\right]$$

To think about!

- How do we balance these objectives of within-study personalization and after-study analyses?
- How does using RL algorithms that learn across users make after study analyses more challenging?

Oralytics Study Overview

- **Total Decision Times:** 10 weeks with two decision times per day ($T = 140 = 10 \cdot 7 \cdot 2$)
- **Study Population:** $N \approx 70$ patients from dental clinics in Los Angeles
- **Data Collected After Study:** For each user $i \in [1 : N]$,

$$\underbrace{\left(O_{i,1}, A_{i,1}, Y_{i,2}\right)}_{D_{i,1}} \quad \underbrace{\left(O_{i,2}, A_{i,2}, Y_{i,3}\right)}_{D_{i,2}} \quad \ldots \quad \underbrace{\left(O_{i,T}, A_{i,T}, Y_{i,T+1}\right)}_{D_{i,T}}$$

$D_{i,t} \triangleq (O_{i,t}, A_{i,t}, Y_{i,t+1})$

Individual RL Algorithms

$\rightarrow$ Algorithm Update

$\rightarrow$ Data Collection

User 1 $\quad D_{1,1} \rightarrow \hat{\pi}_{1,2} \rightarrow D_{1,2} \rightarrow \hat{\pi}_{1,3} \rightarrow \dots \rightarrow \hat{\pi}_{1,T} \rightarrow D_{1,T}$

User 2 $\quad D_{2,1} \rightarrow \hat{\pi}_{2,2} \rightarrow D_{2,2} \rightarrow \hat{\pi}_{2,3} \rightarrow \dots \rightarrow \hat{\pi}_{2,T} \rightarrow D_{2,T}$

$\vdots$

User n $\quad D_{n,1} \rightarrow \hat{\pi}_{n,2} \rightarrow D_{n,2} \rightarrow \hat{\pi}_{n,3} \rightarrow \dots \rightarrow \hat{\pi}_{n,T} \rightarrow D_{n,T}$

Dependence Within a User

User states/rewards can be dependent over time

Limitations?

$D_{i,t} \triangleq (O_{i,t}, A_{i,t}, Y_{i,t+1})$

Individual RL Algorithms

$\rightarrow$ Algorithm Update

$\rightarrow$ Data Collection

User 1 $\quad D_{1,1} \rightarrow \hat{\pi}_{1,2} \rightarrow D_{1,2} \rightarrow \hat{\pi}_{1,3} \rightarrow \ldots \rightarrow \hat{\pi}_{1,T} \rightarrow D_{1,T}$

User 2 $\quad D_{2,1} \rightarrow \hat{\pi}_{2,2} \rightarrow D_{2,2} \rightarrow \hat{\pi}_{2,3} \rightarrow \ldots \rightarrow \hat{\pi}_{2,T} \rightarrow D_{2,T}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

User n $\quad D_{n,1} \rightarrow \hat{\pi}_{n,2} \rightarrow D_{n,2} \rightarrow \hat{\pi}_{n,3} \rightarrow \ldots \rightarrow \hat{\pi}_{n,T} \rightarrow D_{n,T}$

Dependence Within a User

User states/rewards can be dependent over time

Limitations

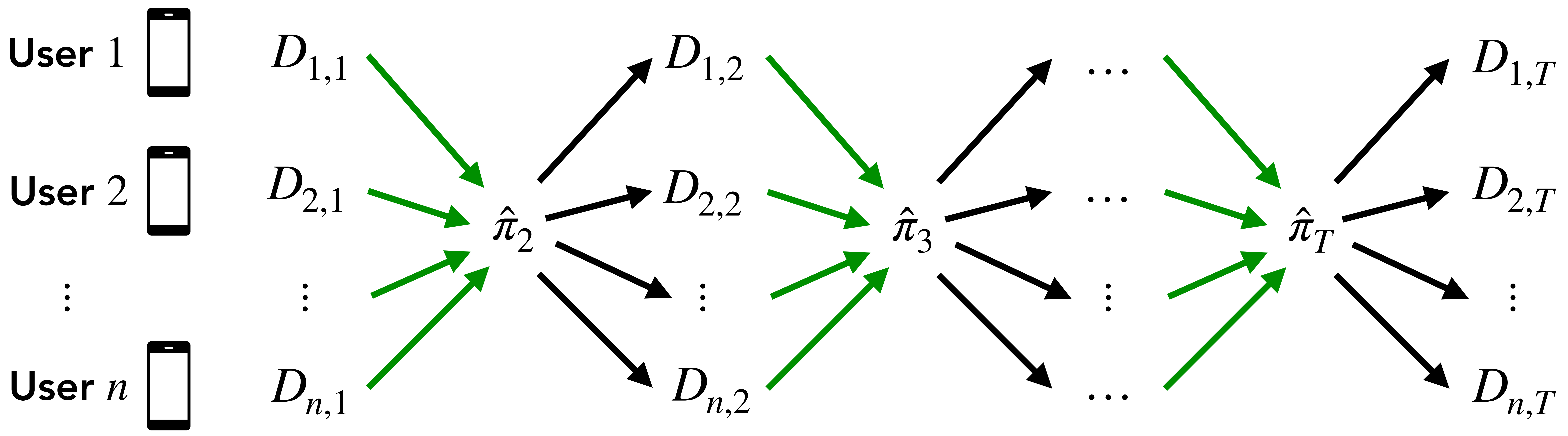
Rewards are noisy and few decision times per user → slow learning

$D_{i,t} \triangleq (O_{i,t}, A_{i,t}, Y_{i,t+1})$

Pooling RL Algorithm

Algorithm Update

Data Collection



Dependence Within a User

User states/rewards can be dependent over time

Dependence Between Users

Due to use of pooling algorithm

Related Work

Inference after Adaptive Sampling

- However, assume contextual bandit environment
- Hadad et al., 2021; Zhang et al. 2021; Bibaut et al. 2021

Inference for Longitudinal Data

- Does not allow for pooled RL algorithms (assumes i.i.d. user trajectories)
- Boruvka et al. 2016; Qian et al. 2020

This work

- Inference after using pooled RL algorithms for longitudinal data environments
- Make stronger assumptions on the pooled RL algorithm since we control it and have full knowledge of it

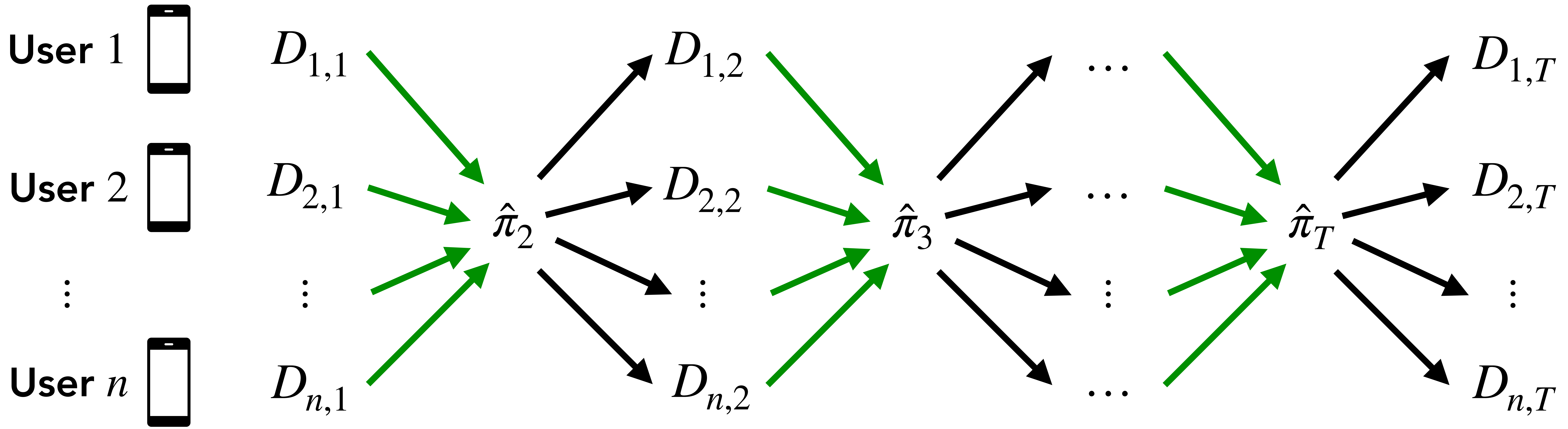
Overview

1. **Excursion effects after pooling**
2. Overview of Inferential Approach
3. Asymptotic Normality Proof Ideas

$D_{i,t} \triangleq (O_{i,t}, A_{i,t}, Y_{i,t+1})$

Pooling RL Algorithm

Algorithm Update
Data Collection



Dependence Within a User

User states/rewards can be dependent over time

Dependence Between Users

Due to use of pooling algorithm

Excursion Effects under Pooling

Recall the excursion effects we considered with no pooling:

$$\mathbb{E}\left[Y_{t+1}(\bar{A}_{t-1}, 1) - Y_{t+1}(\bar{A}_{t-1}, 0) \,|\, X_t = x\right]$$

- Randomness is over (i) potential outcomes, and (ii) $\bar{A}_{t-1}$, aka "behavior policy"

Why is the above problematic as n grows when there is pooling?

- Under pooling, the distribution of $\bar{A}_{t-1}$ depends on how many other users are in the study!!

What can we do?

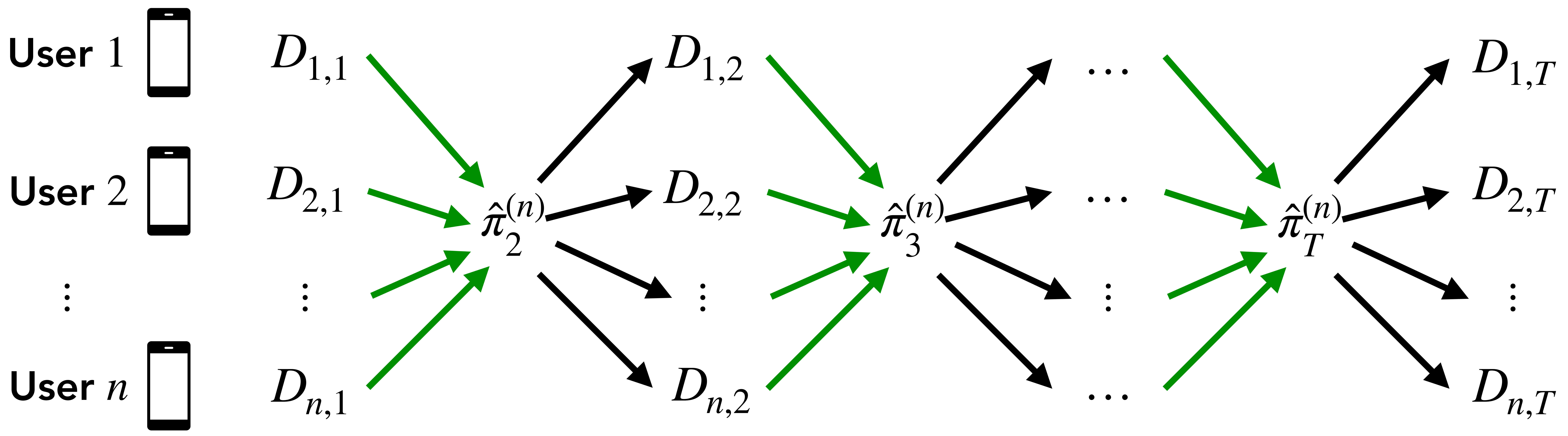
Idea: If the pooling policy converges to a limiting policy as $n \to \infty$, then we can consider excursions from that limiting policy

Excursion Effect from Limiting Policy:

$$\mathbb{E}_{\pi^\star}\left[Y_{t+1}(\bar{A}_{t-1}, 1) - Y_{t+1}(\bar{A}_{t-1}, 0) \,|\, X_t = x\right]$$

- Randomness is over (i) potential outcomes, and (ii) $\bar{A}_{t-1}$ chosen according to the limiting policy $\pi^\star$

Key Assumption: Convergence to Limiting Policies



$D_{i,t} \triangleq \left(O_{i,t}, A_{i,t}, Y_{i,t+1}\right)$

For each $\hat{\pi}_t^{(n)}$ as $n \to \infty$, $\hat{\pi}_t^{(n)} \to \pi_t^\star$ (limiting policy)

Limiting policy: the policy that would be learned if deployed on the whole population

Assumption: Parametric Policy Classes

Policy Class: $\left\{\pi(\,\cdot\,;\beta)\right\}_{\beta\in\mathbb{R}^d}$

- Estimated policy: $\hat{\pi}_t^{(n)}(s) \triangleq \pi\left(s;\hat{\beta}_{t-1}^{(n)}\right)$
- Limiting policy: $\pi_t^{\star}(s) \triangleq \pi\left(s;\beta_{t-1}^{\star}\right)$

Form $\hat{\beta}_{t-1}^{(n)}$ with $\left\{H_{i,t-1}\right\}_{i=1}^{n}$

(e.g. estimate of reward model parameters)

Assumption: Parametric Policy Classes

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Form $\hat{\beta}_{t-1}^{(n)}$ with $\{H_{i,t-1}\}_{i=1}^{n}$

(e.g. estimate of reward model parameters)

Key Assumptions

1. Convergence of $\hat{\beta}_t^{(n)} \xrightarrow{P} \beta_t^{\star}$ (for each t)
2. Policy class $\{\pi(\,\cdot\,;\beta)\}_{\beta\in\mathbb{R}^d}$ is smooth in β (Lipschitz)

Assumption: Parametric Policy Classes

Po

$\}_{i=1}^{n}$

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Example RL Algorithm: Boltzmann Sampling

$$\mathbb{P}\left(A_{i,t+1} = 1 \,|\, H_{1:n,t}, S_{i,t+1}\right) = \text{sigmoid}\left(\phi(S_{i,t+1})^\top \hat{\beta}_t\right)$$
$$= \frac{1}{1 + \exp\left(-\phi(S_{i,t+1})^\top \hat{\beta}_t\right)}$$

Key Assumptions

1. Convergence of $\hat{\beta}_t^{(n)} \xrightarrow{P} \beta_t^\star$ (for each t)
2. Policy class $\left\{\pi(\,\cdot\,;\beta)\right\}_{\beta \in \mathbb{R}^d}$ is smooth in β (Lipschitz)

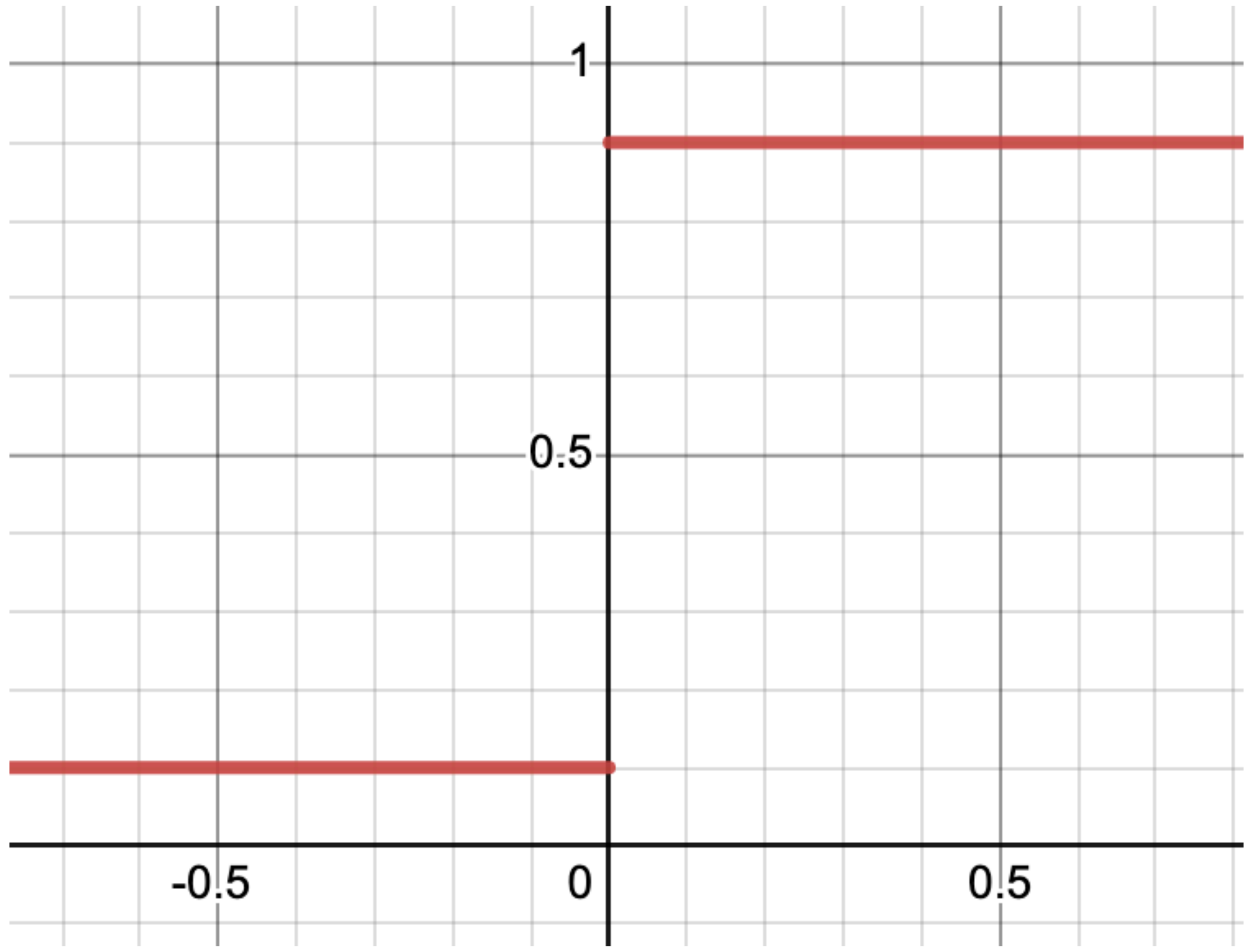
No assumption that RL algorithm's model is correct

Allocation Function: What probability should the limiting policy send a message?

Maximize Rewards

$$\pi^\star(s) = \mathbf{1}\{\text{Treatment Effect}(s) > 0\}$$

Probability of Sending a Message

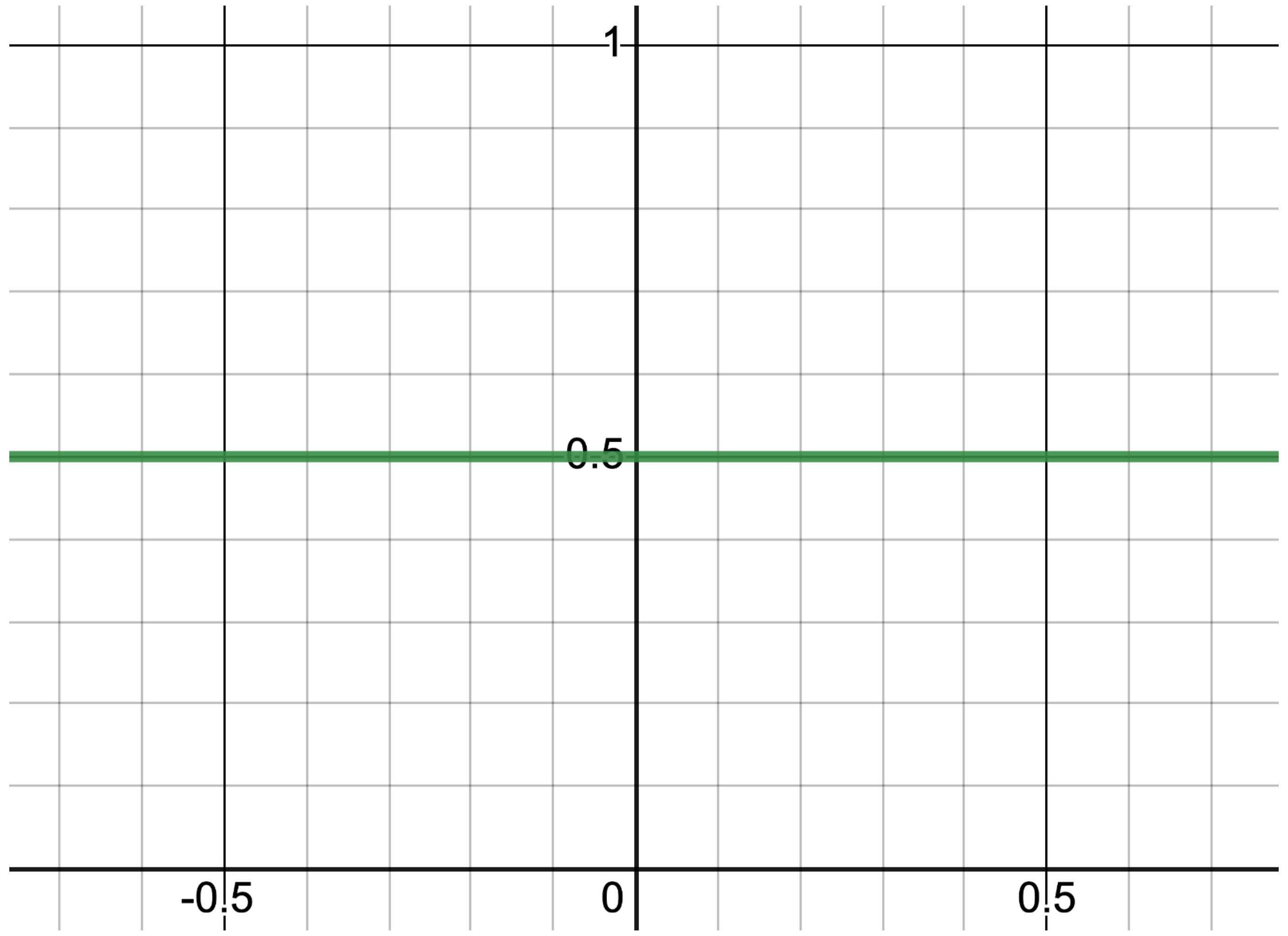


Treatment Effect in State s

Accurately Infer Treatment Effects

$$\pi^\star(s) = 0.5$$

Probability of Sending a Message



Treatment Effect in State s

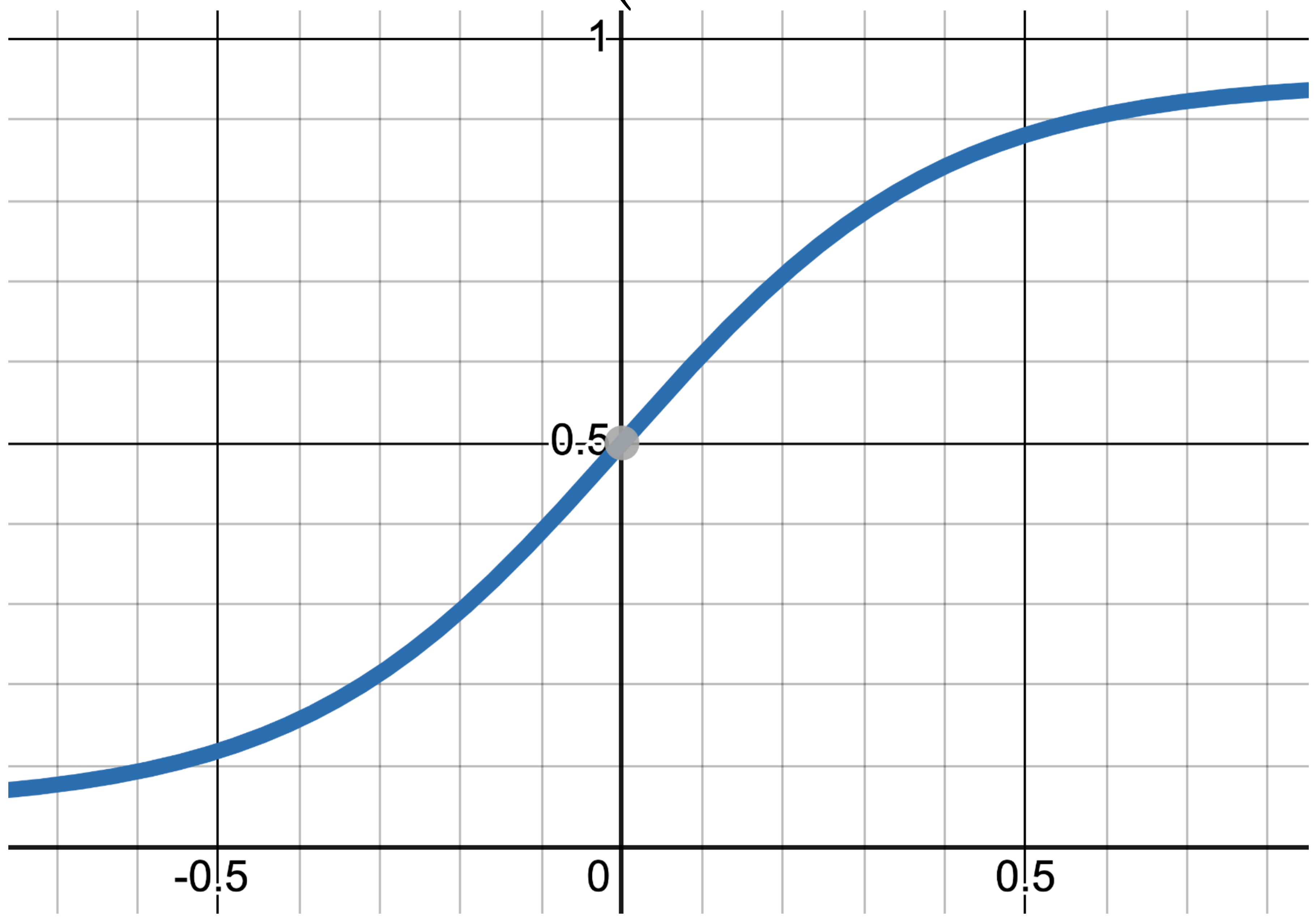
Allocation Function: What probability should the limiting policy send a message?

Balance Maximizing Rewards and Inferring Treatment Effects

- Between trial learning / Continual learning

$$\pi^\star(s) = \text{Softmax}\Big(\text{Treatment Effect}(s)\Big)$$

Probability of Sending a Message



Treatment Effect in State s

No longer have issue of unstable learned policies from taking a "hardmax"

Discussion Question

Explain the following:

- **(Our setting)** When a pooling RL algorithm that forms policies $\{\hat{\pi}_t\}_{t=2}^{T}$ is used the resulting data trajectories $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ **are not independent** across people.
- **("Oracle" Setting)** When the target policies $\{\pi_t^{\star}\}_{t=2}^{T}$ are used the resulting data trajectories $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ **are independent** across people.

Overview

1. Excursion effects after pooling
2. **Overview of Inferential Approach**
3. Asymptotic Normality Proof Ideas

Estimating Excursion Effects under Pooling

Excursion Effect from Limiting Policy:

$$\mathbb{E}_{\pi^\star}\left[Y_{t+1}(\bar{A}_{t-1},1) - Y_{t+1}(\bar{A}_{t-1},0) \,|\, X_t = x\right]$$

Significant Challenges

- The data was collected under estimated policies $\{\hat{\pi}_t\}_{t=1}^T$, but we are interested in excursions from $\{\pi_t^\star\}_{t=1}^T$
  - We do not know the limiting policy $\{\pi_t^\star\}_{t=1}^T$!!
- Our data trajectories are not independent across patients

Estimating Excursion Effects under Pooling

Inferential target $\theta^\star$ solves:

$$0 = \mathbb{E}_{\pi^\star}\left[\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\theta^\star\right)\right]$$

Set derivative of loss equal to zero to solve for minimizer

Estimator $\hat{\theta}$ solves:

$$0 = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\hat{\theta}\right)$$

Example Least Squares Loss: $\ell\left(H_{i,T};\theta\right) = \sum_{t=1}^{T}\left(Y_{i,t+1} - X_{i,t}^\top\theta_0 - A_{i,t}\theta_1\right)^2$

Estimating Excursion Effects under Pooling

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Estimator $\hat{\theta}$ solves:

$$0 = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\hat{\theta}\right)$$

Intuitively, why is forming estimators like this reasonable?

- As $n \to \infty$, the policies $\{\hat{\pi}_t\}_{t=1}^{T}$ will converge to the limiting policies $\{\pi_t^\star\}_{t=1}^{T}$
- Can include action centering, but omit for simplicity

What if you use standard inference approach on pooling online RL data?

Can get confidence intervals that extremely overconfident!

$$\ell\left(H_{i,T};\theta\right) = \sum_{t=1}^{T}\left(Y_{i,t+1} - X_{i,t}^{\top}\theta_0 - A_{i,t}\theta_1\right)^2$$

Coverage of 95% Confidence Intervals for Treatment Effect $\theta_1^\star$

$\hat{\theta}_1$ Variance Estimators	$n = 50$	$n = 100$
Standard Sandwich	75.8%	77.6%

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Coverage of 95% Confidence Intervals for Treatment Effect $\theta_1^\star$

$\hat{\theta}_1$ Variance Estimators	$n = 50$	$n = 100$
Standard Sandwich	75.8%	77.6%
"Adaptive" Sandwich	95.4%	96.5%

Our "Adaptive" Sandwich Variance Estimator

- Data from pooling online RL algorithms → valid confidence intervals
- Applicable to inference for minimizers of general loss functions

Coverage of 95% Confidence Intervals for Treatment Effect $\theta_1^\star$

$\hat{\theta}_1$ Variance Estimators	$n = 50$	$n = 100$
Standard Sandwich	75.8%	77.6%
"Adaptive" Sandwich	95.4%	96.5%

Impact of Adaptive Sandwich Variance Approach

Enables the use of pooling RL algorithms in digital intervention studies

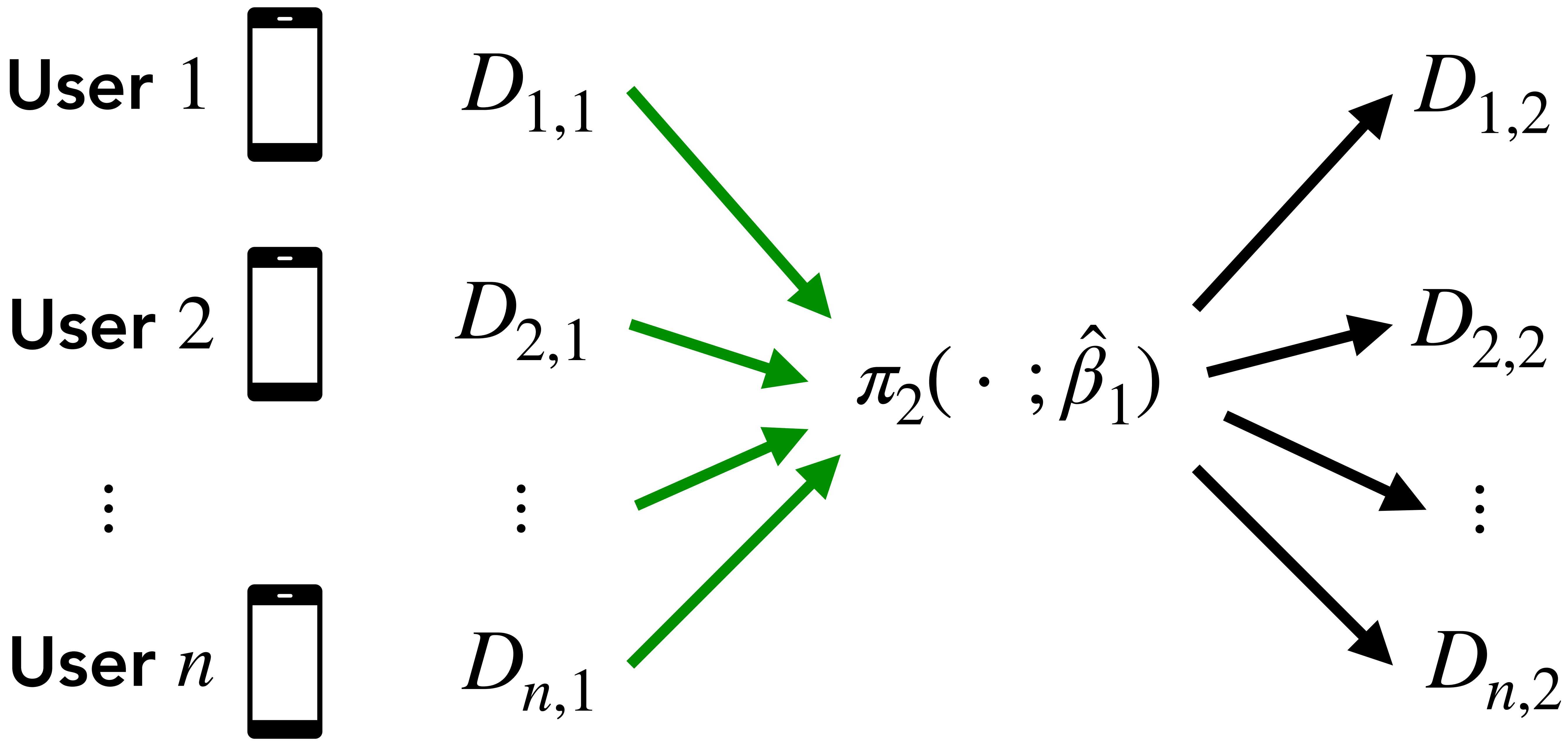
Oralytics:
Oral Health Coaching

MiWaves:
Curbing Adolescent Marijuana Use

Inference Challenges

(1) Dependencies both **within** and **between** users

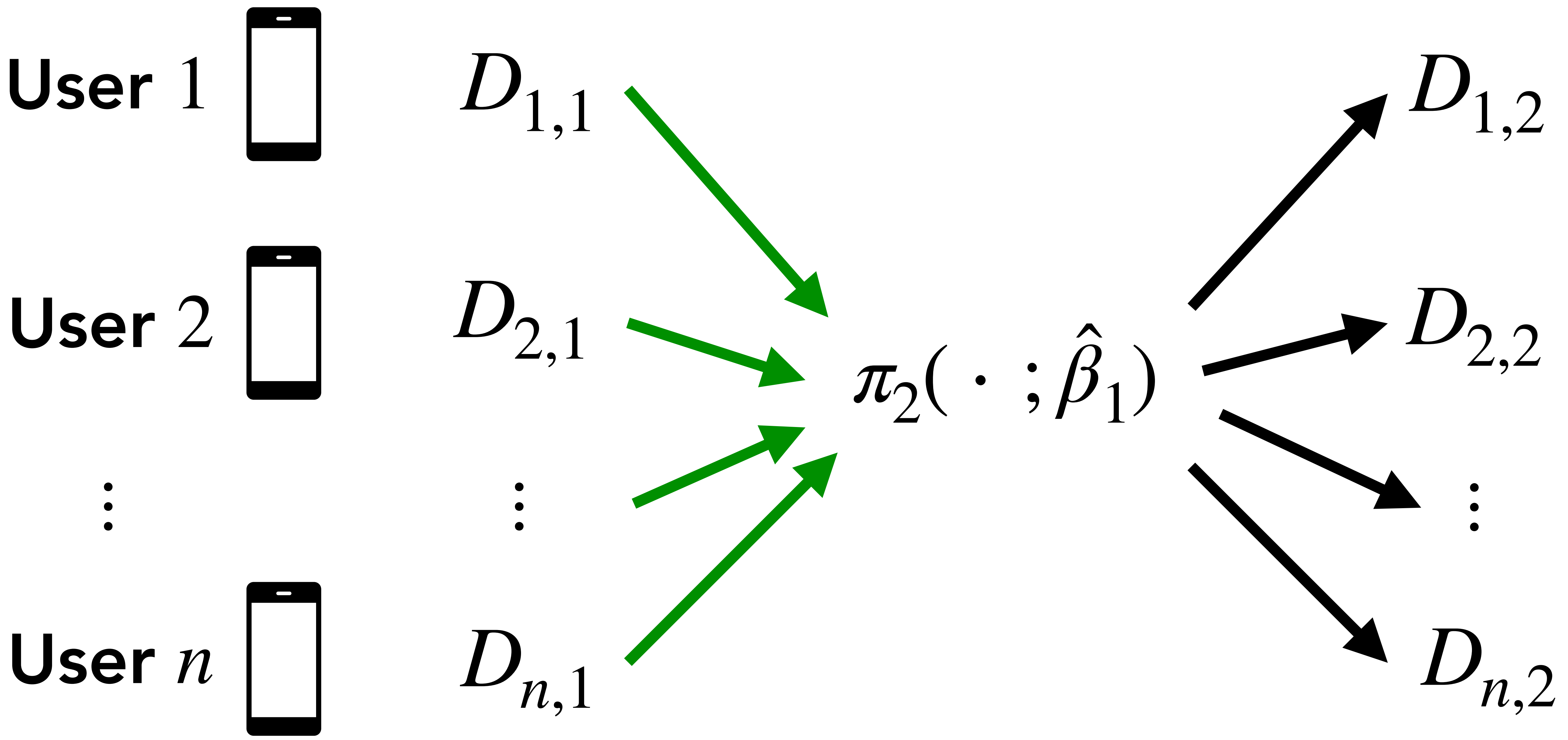
(2) Error of $\hat{\theta}$ implicitly depends on how the algorithm forms and updates policies $\hat{\pi}_t = \pi_t(\cdot\, ; \hat{\beta}_{t-1})$



Variability in $\hat{\beta}_1$ learned
→ Variability in data collected
→ Variability in $\hat{\theta}$

Key Insight

Even though $\{\hat{\beta}_t\}_{t=1}^{T-1}$ affect *data collection* if framed properly they can be mathematically treated like plug-in estimates of nuisance parameters that are used for *data analysis*.



Variability in $\hat{\beta}_1$ learned
$\rightarrow$ Variability in data collected
$\rightarrow$ Variability in $\hat{\theta}$

Standard Inference with "Plug-in" Nuisance Parameters

Given a dataset $\{H_{i,T}\}_{i=1}^{n}$ where $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are i.i.d.

(1) **Form a nuisance estimator $\hat{\beta}$ that solves:** $0 = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \beta} g\left(H_{i,T}; \hat{\beta}\right)$

(2) **"Plug-in" $\beta = \hat{\beta}$ to solve for $\hat{\theta}$ (data reuse):** $0 = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \theta} \ell\left(H_{i,T}; \hat{\theta}, \beta\right)$

Knowledge of how $\hat{\theta}$ changes with different values of β allows us to derive

joint asymptotic distribution: $\sqrt{n} \begin{pmatrix} \hat{\beta} - \beta^{\star} \\ \hat{\theta} - \theta^{\star} \end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0, \Sigma_{\theta,\beta}\right)$

Example: Observational Data Setting

Given a dataset $\{H_{i,T}\}_{i=1}^{n}$ where $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are i.i.d. collected by some unknown fixed policy

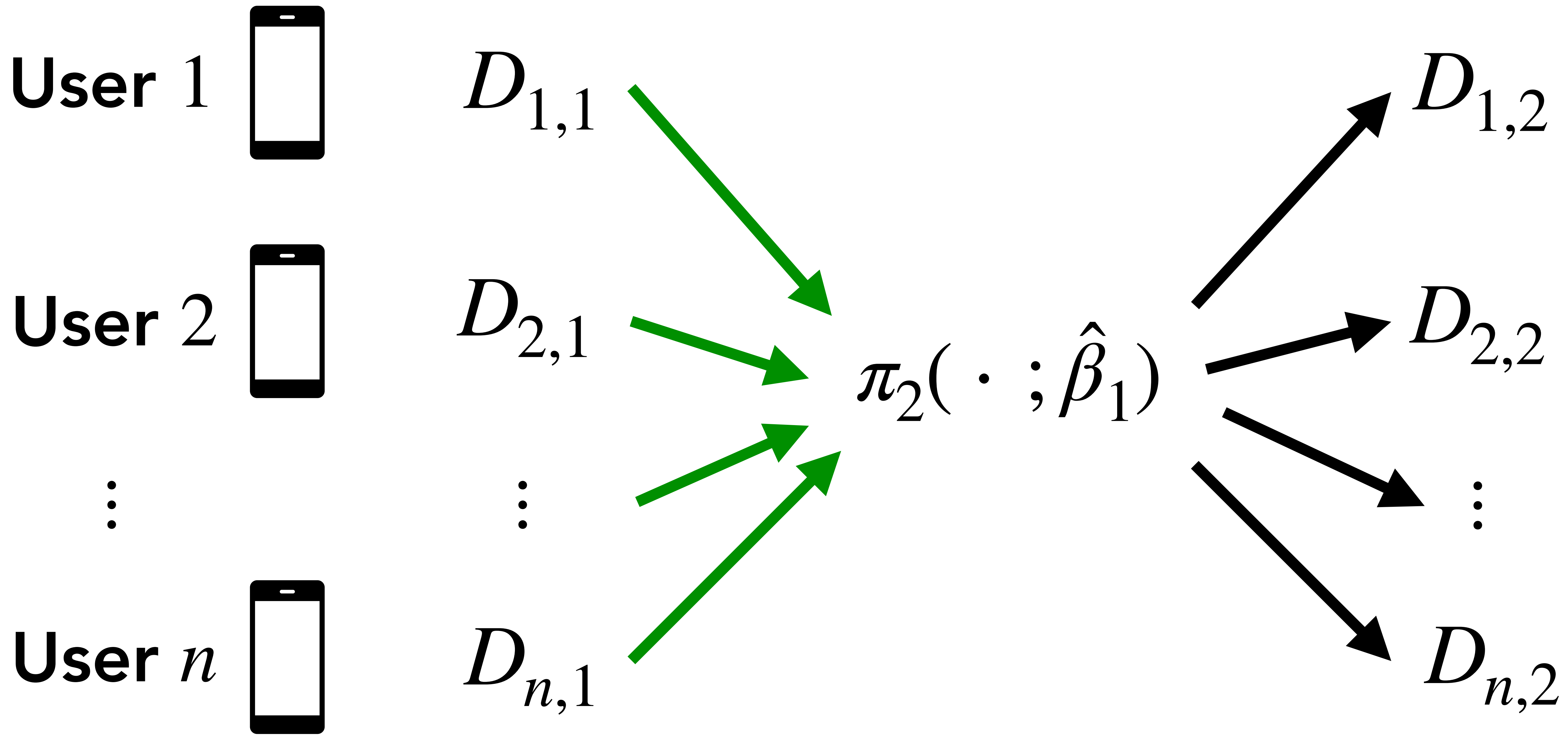
(1) **Form a nuisance estimator $\hat{\beta}$ that solves:** $0 = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\beta}g\big(H_{i,T};\hat{\beta}\big)$

$\hat{\beta}$ from fitted logistic regression model for $\mathbb{P}(A_{i,t} = 1 \mid S_{i,t}) \approx \mathrm{sigmoid}(S_{i,t}^{\top}\hat{\beta})$

(2) **"Plug-in" $\beta = \hat{\beta}$ to solve for $\hat{\theta}$ (data reuse):** $0 = \cancel{\frac{1}{N}\sum_{i=1}^{N}\frac{\partial}{\partial\theta}\ell\big(H_{i,T};\hat{\theta},\beta\big)}$

Forming estimator $\hat{\theta}$ involves using the estimated action selection probabilities $\mathrm{sigmoid}(S_{i,t}^{\top}\hat{\beta})$

In online RL setting…



$\hat{\beta}_1$ is not a plug-in estimator used to form $\hat{\theta}$.

It is a property of the data collection procedure!!

$$\hat{\theta} \text{ solves} \quad 0 = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\hat{\theta}\right)$$

Importance Weights as a Theoretical Tool

"Simple" Solution: $\hat{\theta}$ solves for $\{\beta_t\}_{t=1}^{T-1} = \{\hat{\beta}_t\}_{t=1}^{T-1}$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\left\{\prod_{t=2}^{T}\left(\frac{\pi_t(S_{i,t};\beta_{t-1})}{\pi_t(S_{i,t};\hat{\beta}_{t-1})}\right)^{A_{i,t}}\left(\frac{1-\pi_t(S_{i,t};\beta_{t-1})}{1-\pi_t(S_{i,t};\hat{\beta}_{t-1})}\right)^{1-A_{i,t}}\right\}\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\hat{\theta}\right)$$

- Weights are not used to form $\hat{\theta}$!!
- Allows us to capture how changes in $\{\beta_t\}_{t=2}^{T}$ affect errors in $\hat{\theta}$

→ analyze similarly to plug-in estimators of nuisance parameters

Adaptive Sandwich Variance

$$\sqrt{n}\left(\hat{\theta}^{(n)} - \theta^\star\right) \xrightarrow{D} \mathcal{N}\left(0,\, \ddot{L}_\theta^{-1} \Sigma^{\text{adapt}} \ddot{L}_\theta^{-1}\right)$$

$$\Sigma^{\text{adapt}} = \mathbb{E}_{\pi^\star}\left[\left\{ \frac{\partial}{\partial \theta} \ell\left(H_{i,T}; \theta^\star\right) + \underbrace{\sum_{t=1}^{T-1} M_t \dot{g}_t\left(H_{i,t}; \beta_t^\star\right)}_{\text{Correction in Variance Due to Pooled RL Algorithm}} \right\}^{\otimes 2}\right]$$

$$\ddot{L}_\theta = \mathbb{E}_{\pi^\star}\left[\frac{\partial^2}{\partial\theta\partial\theta} \ell\left(H_{i,T}; \theta\right)\right]$$

Correction in Variance Due to Pooled RL Algorithm

M_t **given in paper:** Statistical Inference After Adaptive Sampling for Longitudinal Data (https://arxiv.org/abs/2202.07098)

Discussion Question

Why are the number of nuisance parameters increasing with the number of update times? What are potential concerns and how could we manage this?

Asymptotic Normality Proof Ideas

Overview

1. Proving normality in "oracle" setting ($H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are i.i.d.)
2. Proving normality when $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are collected with a pooling RL algorithm ($T = 2$ case)

"Oracle" Setting with i.i.d. Data Trajectories

- Given a dataset $\left\{H_{i,T}\right\}_{i=1}^{n}$ where $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are i.i.d. collected by known target policies $\{\pi_t^\star\}_{t=1}^{T}$
- Estimand $\theta^\star$ where $\theta = \theta^\star$ solves

$$0 = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right] \triangleq \mathbb{E}_{\pi^\star}\left[\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\theta\right)\right]$$

- Estimator $\hat{\theta}$ where $\theta = \hat{\theta}$ solves

$$0 = \mathbb{P}_n\left[\dot{\ell}_i(\theta)\right] \triangleq \mathbb{E}_{\pi^\star}\left[\frac{\partial}{\partial\theta}\ell\left(H_{i,T};\theta\right)\right]$$

Example Least Squares Loss: $\ell\left(H_{i,T};\theta\right) = \sum_{t=1}^{T}\left(Y_{i,t+1} - X_{i,t}^{\top}\theta_0 - A_{i,t}\theta_1\right)^2$

Normality Result (Standard Sandwich Variance)

$$\sqrt{n}\left(\hat{\theta} - \theta^\star\right) \xrightarrow{D} \mathcal{N}\left(0, \ddot{L}^{-1}\Sigma(\ddot{L}^{-1})^\top\right)$$

where

$$\Sigma = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\dot{\ell}_i(\theta^\star)^\top\right]$$

Following Theorem 5.21 of Van Der Vaart, Asymptotic Statistics

and

$$\ddot{L} = \frac{\partial}{\partial\theta}\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right]\Big|_{\theta=\theta^\star} = \mathbb{E}_{\pi^\star}\left[\tfrac{\partial^2}{\partial\theta\partial\theta}\ell_i(\theta)\right]\Big|_{\theta=\theta^\star}$$

Proof Outline

$$\Sigma = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\dot{\ell}_i(\theta^\star)^\top\right]$$

$$\sqrt{n}\left(\hat{\theta} - \theta^\star\right) \xrightarrow{D} \mathcal{N}\left(0, \ddot{L}^{-1}\Sigma(\ddot{L}^{-1})^\top\right)$$

(1) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right) \xrightarrow{D} \mathcal{N}(0,\Sigma)$

(2) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right)$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

(3) $\sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) = o_P(1)$

Proof Outline

$$\Sigma = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\dot{\ell}_i(\theta^\star)^\top\right]$$

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(1) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right) \xrightarrow{D} \mathcal{N}(0,\Sigma)$

Central Limit Theorem

(2) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right)$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

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(1) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right) \xrightarrow{D} \mathcal{N}(0,\Sigma)$

(2) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right)$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

(3) $\sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) = o_P(1)$

Notation Slide

- A sequence of random variables $Z_n = o_P(1)$ if for any $\epsilon > 0$, as $n \to \infty$, $P(\|Z_n\|_2 > \epsilon) \to 0$
- More generally, $Z_n = o_P(B_n)$ for some random sequence B_n, if for any $\epsilon > 0$, as $n \to \infty$, $P\left(\frac{\|Z_n\|_2}{\|B_n\|_2} > \epsilon\right) \to 0$
- See Van der Vaart, Asymptotic Statistics, Chapter 2.2

Step (2) Outline

$$\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\right]\right)$$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

Step (2) Outline

Asymptotic Equicontinuity

$$\sqrt{n}\left(\mathbb{P}_n \dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\right]\right)$$

$$= \sqrt{n}\left(\mathbb{P}_n \dot{\ell}_i(\hat{\theta}) - \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\hat{\theta})\right]\right) + o_P(1)$$

$$= \sqrt{n}\left(\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\right] - \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\hat{\theta})\right]\right) + o_P(1)$$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

- Use $\hat{\theta} \xrightarrow{P} \theta^\star$
- Show that random mapping is continuous in θ
- Apply continuous mapping theorem

Why does the following equality hold?

$$\sqrt{n}\Big(\mathbb{P}_n \dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}\big[\dot{\ell}_i(\theta^\star)\big]\Big)$$

$$= \sqrt{n}\Big(\mathbb{P}_n \dot{\ell}_i(\hat{\theta}) - \mathbb{E}_{\pi^\star}\big[\dot{\ell}_i(\hat{\theta})\big]\Big) + o_P(1)$$

$$= \sqrt{n}\Big(\mathbb{E}_{\pi^\star}\big[\dot{\ell}_i(\theta^\star)\big] - \mathbb{E}_{\pi^\star}\big[\dot{\ell}_i(\hat{\theta})\big]\Big) + o_P(1)$$

$$= -\ddot{L}\sqrt{n}\big(\hat{\theta} - \theta^\star\big) + \sqrt{n}\, o_P\big(\|\hat{\theta} - \theta^\star\|_2\big) + o_P(1)$$

Using Definitions of $\hat{\theta}$, $\theta^\star$

$\hat{\theta}$ solves $0 = \mathbb{P}_n \dot{\ell}_i(\hat{\theta})$

$\theta^\star$ solves

$0 = \mathbb{E}_{\pi^\star}\big[\dot{\ell}_i(\theta^\star)\big]$

Why does the following equality hold?

$$\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right)$$

$$= \sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\hat{\theta}) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\hat{\theta})]\right) + o_P(1)$$

Differentiability

$$\ddot{L} = \frac{\partial}{\partial\theta}\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right]\Big|_{\theta=\theta^\star}$$

$$= \sqrt{n}\left(\mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)] - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\hat{\theta})]\right) + o_P(1)$$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

Differentiability

$$\ddot{L} = \frac{\partial}{\partial\theta}\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right]\Big|_{\theta=\theta^\star}$$

$$\lim_{\theta\to\theta^\star} \frac{\left\|\mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta)] - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)] - \ddot{L}(\theta-\theta^\star)\right\|_2}{\|\theta-\theta^\star\|_2} = 0$$

$$\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\hat{\theta})\right] - \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\right] - \ddot{L}\left(\hat{\theta} - \theta^\star\right) = o_P\left(\|\hat{\theta} - \theta^\star\|_2\right)$$

$$\begin{aligned}\sqrt{n}\left(\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\right] - \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\hat{\theta})\right]\right) + o_P(1) \\ = -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)\end{aligned}$$

Proof Outline

$$\Sigma = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\dot{\ell}_i(\theta^\star)^\top\right]$$

$$\sqrt{n}\left(\hat{\theta} - \theta^\star\right) \xrightarrow{D} \mathcal{N}\left(0, \ddot{L}^{-1}\Sigma(\ddot{L}^{-1})^\top\right)$$

(1) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right) \xrightarrow{D} \mathcal{N}(0,\Sigma)$

(2) $\sqrt{n}\left(\mathbb{P}_n\dot{\ell}_i(\theta^\star) - \mathbb{E}_{\pi^\star}[\dot{\ell}_i(\theta^\star)]\right)$

$$= -\ddot{L}\sqrt{n}\left(\hat{\theta} - \theta^\star\right) + \sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) + o_P(1)$$

(3) $\sqrt{n}\,o_P\left(\|\hat{\theta} - \theta^\star\|_2\right) = o_P(1)$

HW Exercise (uses above two steps)

Overview

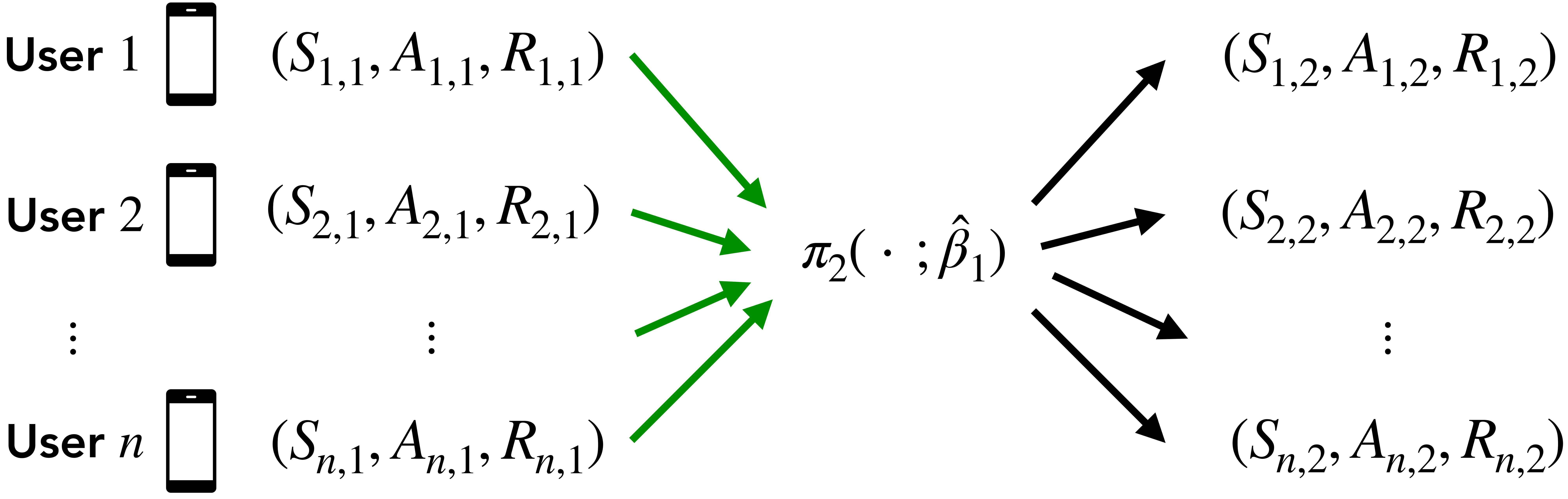
1. Proving normality in "oracle" setting

   ($H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are i.i.d.)

2. Proving normality when $H_{1,T}, H_{2,T}, \ldots, H_{n,T}$ are

   collected with a pooling RL algorithm ($T = 2$ case)

Recap of Problem Setting $T = 2$



- Use π_1 to collect $\left\{(S_{i,1}, A_{i,1}, R_{i,1})\right\}_{i=1}^{n}$
- Use $\left\{(S_{i,1}, A_{i,1}, R_{i,1})\right\}_{i=1}^{n}$ to form $\hat{\beta}_1^{(n)}$
- Use $\hat{\pi}_2(\cdot) = \pi\left(\cdot \, ; \hat{\beta}_1^{(n)}\right)$ to collect $\left\{(S_{i,2}, A_{i,2}, R_{i,2})\right\}_{i=1}^{n}$

Pooling Setting with non-i.i.d. Data Trajectories

- Given a dataset $\{H_{i,2}\}_{i=1}^{n}$ collected by estimated policy $\hat{\pi}_2$
- Estimand $\theta^\star$ where $\theta = \theta^\star$ solves

$$0 = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right] \triangleq \mathbb{E}_{\pi^\star}\left[\frac{\partial}{\partial\theta}\ell\left(H_{i,2};\theta\right)\right]$$

- Estimator $\hat{\theta}$ where $\theta = \hat{\theta}$ solves

$$0 = \mathbb{P}_n\left[\dot{\ell}_i(\theta)\right] \triangleq \mathbb{E}_{\pi^\star}\left[\frac{\partial}{\partial\theta}\ell\left(H_{i,2};\theta\right)\right]$$

Example Least Squares Loss: $\ell\left(H_{i,T};\theta\right) = \sum_{t=1}^{T}\left(Y_{i,t+1} - X_{i,t}^\top\theta_0 - A_{i,t}\theta_1\right)^2$

$\hat{\beta}_1$ as Estimator of Nuisance

Inferential target $\theta^\star$ solves:

$$0 = \mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta^\star)\right]$$

- $\pi_2^\star = \pi_2(\,\cdot\,; \beta_1^\star)$ is a nuisance function estimated with $\hat{\pi}_2^{(n)} = \pi_2(\,\cdot\,; \hat{\beta}_1^{(n)})$
- **However, nuisance function is estimated by the RL algorithm rather than the data analyst**

$T = 2$ Case: Loss Function for β_1

- Formed by the RL algorithm
- Limiting $\beta_1^\star$

$$0 = \mathbb{E}_{\pi^\star}\left[\dot{g}_{i,1}\left(\beta_1^\star\right)\right] = \mathbb{E}_{\pi^\star}\left[\frac{\partial}{\partial\beta} g_1\left(H_{i,1};\beta\right)\Big|_{\beta=\beta_1^\star}\right]$$

- Estimator $\hat{\beta}_1$

$$0 = \mathbb{P}_n \, \dot{g}_{i,1}\left(\hat{\beta}_1\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial}{\partial\beta} g_1\left(H_{i,1};\beta\right)\Big|_{\beta=\beta_1^\star}$$

Normality Result (Adaptive Sandwich Variance)

$$\sqrt{n}\left(\hat{\theta}-\theta^\star\right) \xrightarrow{D} \mathcal{N}\left(0, \ddot{L}^{-1}\Sigma^{\text{adapt}}(\ddot{L}^{-1})^\top\right)$$

where

$$\Sigma^{\text{adapt}} = \mathbb{E}_{\pi^\star}\left[\left\{\dot{\ell}_i(\theta^\star) - V_1\ddot{G}_1^{-1}\dot{g}_{i,1}(\beta_1^\star)\right\}\left\{\dot{\ell}_i(\theta^\star) - V_1\ddot{G}_1^{-1}\dot{g}_{i,1}(\beta_1^\star))\right\}^\top\right]$$

$$\ddot{G}_1 = \frac{\partial}{\partial\beta_1}\mathbb{E}\left[\dot{g}_{1,i}(\beta_1)\right]\Big|_{\beta_1=\beta_1^\star} \quad \text{and} \quad \ddot{L} = \frac{\partial}{\partial\theta}\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right]\Big|_{\theta=\theta^\star}$$

Joint Asymptotic Normality Result

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0,\ \begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1}\Sigma_{1:2}\begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

$$\Sigma_{1:2} \triangleq \mathbb{E}_{\pi^\star}\left[\begin{pmatrix}\dot{g}_i(\beta_1^\star)\\ \dot{\ell}_i(\theta^\star)\end{pmatrix}\begin{pmatrix}\dot{g}_i(\beta_1^\star)\\ \dot{\ell}_i(\theta^\star)\end{pmatrix}^\top\right] \text{ and } V_1 = \frac{\partial}{\partial\beta_1}\mathbb{E}_{\pi_2(\beta_1)}\left[\dot{\ell}_i(\theta^\star)\right]\Big|_{\beta_1=\beta_1^\star}$$

Joint Asymptotic Normality Result

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0, \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:2} \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

$$\Sigma_{1:2} \triangleq \mathbb{E}_{\pi^\star}\left[\begin{pmatrix}\dot{g}_i(\beta_1^\star) \\ \dot{\ell}_i(\theta^\star)\end{pmatrix}\begin{pmatrix}\dot{g}_i(\beta_1^\star) \\ \dot{\ell}_i(\theta^\star)\end{pmatrix}^\top\right] \text{ and } V_1 = \frac{\partial}{\partial \beta_1}\mathbb{E}_{\pi_2(\beta_1)}\left[\dot{\ell}_i(\theta^\star)\right]\Big|_{\beta_1=\beta_1^\star}$$

Interpretation of V_1: Change in criterion for $\theta^\star$ with little changes in policy used to collect data (i.e. β_1)

$T = 2$ Setting: Naive Approach

Joint Criteria for $\hat{\beta}_1, \hat{\theta}$:

$$0 = \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}\dot{g}_{i,1}(\hat{\beta}_1)\\ \dot{\ell}_i(\hat{\theta})\end{pmatrix}$$

- **Issue:** Above the relationship between $\hat{\theta}$ and $\hat{\beta}_1$ is not explicit
  - We use weighting to represent how $\hat{\theta}$ is affected by estimation of the nuisance $\pi_2^\star = \pi_2(\,\cdot\,; \beta_1^\star)$

$T = 2$ Setting: Radon-Nikodym Weights

Joint Criteria for $\hat{\beta}_1, \hat{\theta}$:

$$0 = \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} \dot{g}_{i,1}(\hat{\beta}_1) \\ W_{i,2}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta}) \end{pmatrix}$$

$\hat{\beta}_1$ is also a plug-in into the Radon-Nikodym weight!

where

$$W_{i,2}(\beta_1) \triangleq \left(\frac{\pi_{i,2}(\beta_1)}{\pi_{i,2}(\hat{\beta}_1)}\right)^{A_{i,2}}\left(\frac{1-\pi_{i,2}(\beta_1)}{1-\pi_{i,2}(\hat{\beta}_1)}\right)^{1-A_{i,2}}$$

$$\pi_{i,2}(\beta_1) \triangleq \pi_2(S_{i,2};\beta_1)$$

Proof Key Steps

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0,\ \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:2} \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

Proof Key Steps

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0,\ \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:2} \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

(1) $\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\beta_1^\star) \\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star) \\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}\right) \xrightarrow{D} \mathcal{N}\left(0,\Sigma_{1:2}\right)$

Our data is no longer independent across users!

Weighted Martingale CLT

Proof Key Steps

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0, \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:2} \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

(2) $$\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\beta_1^\star) \\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star) \\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}\right)$$

$$= \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} + \sqrt{n}\, o_P\left(\left\| \begin{matrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{matrix} \right\|\right) + o_P(1)$$

Proof Outline of Step (2)

$$\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i\left(\beta_1^\star\right)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i\left(\theta^\star\right)\end{bmatrix}-\mathbb{E}\begin{bmatrix}\dot{g}_i\left(\beta_1^\star\right)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i\left(\theta^\star\right)\end{bmatrix}\right)$$

$$=\begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1}\sqrt{n}\begin{pmatrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{pmatrix}++\sqrt{n}o_P\left(\left\|\begin{matrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{matrix}\right\|\right)+o_P(1)$$

Proof Outline of Step (2)

$$\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}\right)$$

$$= \sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix}\right) + o_P(1)$$

$$= \begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1}\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star\\ \hat{\theta} - \theta^\star\end{pmatrix} + \left(\left\|\begin{matrix}\hat{\beta}_1 - \beta_1^\star\\ \hat{\theta} - \theta^\star\end{matrix}\right\|\right) + o_P(1)$$

Asymptotic Equicontinuity

- Use that $(\hat{\beta}_1, \hat{\theta}) \xrightarrow{P} (\beta_1^\star, \theta^\star)$
- Show that random mapping is continuous in (β_1, θ)
- Apply continuous mapping theorem

Why does this equality hold?

$$\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\beta^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}\right)$$

$$= \sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix}\right) + o_P(1)$$

$$= \sqrt{n}\left(\mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix}\right) + o_P(1)$$

$$= \begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1}\sqrt{n}\begin{pmatrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{pmatrix} + \left(\left\|\begin{matrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{matrix}\right\|\right) + o_P(1)$$

Using Definitions of $\hat{\beta}_1, \hat{\theta}$, $\beta_1^\star, \theta^\star$

$\hat{\beta}_1, \hat{\theta}$ solves

$$0 = \mathbb{P}_n\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix}$$

$\theta^\star$ solves

$$0 = \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}$$

Why does this equality hold?

$$\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\beta^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}\right)$$

$$= \sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix}\right) + o_P(1)$$

$$= \sqrt{n}\left(\mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\hat{\beta}_1)\\ W_{i,1}(\hat{\beta}_1)\dot{\ell}_i(\hat{\theta})\end{bmatrix}\right) + o_P(1)$$

$$= \begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1}\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star\\ \hat{\theta} - \theta^\star\end{pmatrix} + \left(\left\|\begin{matrix}\hat{\beta}_1 - \beta_1^\star\\ \hat{\theta} - \theta^\star\end{matrix}\right\|\right) + o_P(1)$$

Differentiability

$$\ddot{L} = \frac{\partial}{\partial\theta}\mathbb{E}_{\pi^\star}\left[\dot{\ell}_i(\theta)\right]\Big|_{\theta=\theta^\star}$$

$$\ddot{G}_1 = \frac{\partial}{\partial\beta}\mathbb{E}\left[\dot{g}_{1,i}(\beta)\right]\Big|_{\beta=\beta^\star}$$

$$V_1 = \frac{\partial}{\partial\beta_1}\mathbb{E}_{\pi_2(\beta_1)}\left[\dot{\ell}_i(\theta^\star)\right]\Big|_{\beta_1=\beta_1^\star}$$

Differentiability

$$\frac{\partial}{\partial(\beta_1, \theta)} \mathbb{E} \begin{bmatrix} \dot{g}_i(\beta_1) \\ W_{i,1}(\beta_1)\dot{\ell}_i(\theta) \end{bmatrix} \Bigg|_{(\beta_1,\theta)=(\beta_1^\star,\theta^\star)}$$

$$= \mathbb{E} \begin{bmatrix} \frac{\partial}{\partial \beta_1} \dot{g}_i(\beta_1) \Big|_{\beta_1=\beta_1^\star} & \frac{\partial}{\partial \theta} \dot{g}_i(\beta_1^\star) \Big|_{\theta=\theta^\star} \\ \frac{\partial}{\partial \beta_1} W_{i,1}(\beta_1) \Big|_{\beta_1=\beta_1^\star} \dot{\ell}_i(\theta^\star) & W_{i,1}(\beta_1^\star) \frac{\partial}{\partial \theta} \dot{\ell}_i(\theta) \Big|_{\theta=\theta^\star} \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{G}_1 & 0 \\ V_1 & \ddot{L} \end{bmatrix}$$

We've now outlined how to show step (2)

(2) $$\sqrt{n}\left(\mathbb{P}_n\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix} - \mathbb{E}\begin{bmatrix}\dot{g}_i(\beta_1^\star)\\ W_{i,1}(\beta_1^\star)\dot{\ell}_i(\theta^\star)\end{bmatrix}\right)$$

$$= \begin{bmatrix}\ddot{G}_1 & 0\\ V_1 & \ddot{L}\end{bmatrix}^{-1}\sqrt{n}\begin{pmatrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{pmatrix} + \left(\left\|\begin{matrix}\hat{\beta}_1-\beta_1^\star\\ \hat{\theta}-\theta^\star\end{matrix}\right\|\right) + o_P(1)$$

Proof Key Steps

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0, \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:2} \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

(3) $\sqrt{n}\, o_P\left(\left\| \begin{matrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\theta} - \theta^\star\end{matrix} \right\|\right) = o_P(1)$

HW Exercise (uses previous two steps)

Joint Asymptotic Normality Result $T = 2$

$$\sqrt{n}\begin{pmatrix}\hat{\beta} - \beta^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0,\ \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:2} \begin{bmatrix}\ddot{G}_1 & 0 \\ V_1 & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

Due to lower-triangular structure of "bread" matrices,

$$\sqrt{n}\left(\hat{\theta} - \theta^\star\right) \xrightarrow{D} \mathcal{N}\left(0, \ddot{L}^{-1}\Sigma^{\text{adapt}}(\ddot{L}^{-1})^\top\right)$$

$$\Sigma^{\text{adapt}} = \mathbb{E}_{\pi^\star}\left[\left\{\dot{\ell}_i(\theta^\star) - V_1\ddot{G}_1^{-1}\dot{g}_{i,1}(\beta_1^\star)\right\}\left\{\dot{\ell}_i(\theta^\star) - V_1\ddot{G}_1^{-1}\dot{g}_{i,1}(\beta_1^\star))\right\}^\top\right]$$

Slides for Reference

Joint Asymptotic Normality Result (General T)

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_{1:T-1} - \beta^{\star}_{1:T-1} \\ \hat{\theta} - \theta^{\star}\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0,\ \begin{bmatrix}\ddot{G}_{1:T-1} & 0 \\ V_{1:T-1} & \ddot{L}\end{bmatrix}^{-1} \Sigma_{1:T} \begin{bmatrix}\ddot{G}_{1:T-1} & 0 \\ V_{1:T-1} & \ddot{L}\end{bmatrix}^{-1,\top}\right)$$

$$\ddot{G}_{1:T-1} = \frac{\partial}{\partial \beta_{1:T-1}} \mathbb{E}_{\pi(\beta_{1:T-1})} \begin{bmatrix}\dot{g}_{1,i}(\beta_1) \\ \dot{g}_{2,i}(\beta_2) \\ \vdots \\ \dot{g}_{T-1,i}(\beta_{T-1})\end{bmatrix} \Bigg|_{\beta_{1:T-1} = \beta^{\star}_{1:T-1}}$$

$$V_{1:T-1} = \frac{\partial}{\partial \beta_{1:T-1}} \mathbb{E}_{\pi(\beta_{1:T-1})} \left[\dot{\ell}_i(\theta^{\star})\right] \Big|_{\beta_{1:T-1} = \beta^{\star}_{1:T-1}}$$