Updating a Movements Account

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1 Definitions

Assume that account contains age, and that eage age group or period has length 1.

- Let $\boldsymbol{N}$ be population.
- Let $\boldsymbol{C}_1, \dots, \boldsymbol{C}_K$ be components of the demographic accounts such as migration, births, or deaths. All $\boldsymbol{C}_k$ are compatible with $\boldsymbol{N}$.
- Let $\boldsymbol{Q} = \{\boldsymbol{N}, \boldsymbol{C}_1, \dots, \boldsymbol{C}_K\}$ denote the demographic account.
- Let $\boldsymbol{A}$ be accession. $\boldsymbol{A}$ does not include age 0 (i.e. births).
- Let $\boldsymbol{E}$ be exposure, derived deterministically from $\boldsymbol{N}$.
- Let $\boldsymbol{\gamma}^N$ be expected counts from the system model for $\boldsymbol{N}$.
- Let $\boldsymbol{\gamma}_k^C$ be rates or expected counts from the system model for $\boldsymbol{C}_k$.
- Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_M$ be the datasets used to estimate the demographic account. If $\boldsymbol{X}_m$ is used to estimate $\boldsymbol{N}$, then $\boldsymbol{N}$ can be collapsed to be exactly compatible with $\boldsymbol{X}_m$; let $\boldsymbol{N}_m = \boldsymbol{N}^{[X_m]}$ denote the collapsed version of $\boldsymbol{N}$. If $\boldsymbol{X}_m$ is used to estimate $\boldsymbol{C}_k$, then $\boldsymbol{C}_k$ can be collapsed to be exactly compatible with $\boldsymbol{X}_m$; let $\boldsymbol{C}_{km} = \boldsymbol{C}_k^{[X_m]}$ denote the collapsed version of $\boldsymbol{C}_k$.
- Age group a is the age interval between exact ages a and $a + 1$.
- Period t is the interval between exact times $t - 1$ and t.
- Let $\hat{\boldsymbol{N}} = \boldsymbol{\gamma}^N$ be expected population.
- Let $\hat{\boldsymbol{E}}$ be expected exposure. $\hat{\boldsymbol{E}}$ is calculated from $\hat{\boldsymbol{N}}$ using the formulas for calculating $\boldsymbol{E}$ from $\boldsymbol{N}$.

2 System model

$$p(\boldsymbol{Q}) = p(\boldsymbol{N}) \prod_{k=1}^{K} p(\boldsymbol{C}_k | \boldsymbol{N}) \tag{1}$$

Population: Poisson, no exposure

Net migration, internal or external: Normal integer-only

Everything else: Poisson, typically with exposure, but can be without exposure

3 Quantities Required

3.1 Updating starting values for $\boldsymbol{N}$

Let $\boldsymbol{N} = (\boldsymbol{N}_0, \boldsymbol{N}_{1+})$, where $\boldsymbol{N}_0$ is $\boldsymbol{N}$ at time 0. Proposals for $\boldsymbol{N}_0$ do not depend on the current value of $\boldsymbol{N}_0$, so

$$\frac{J(\boldsymbol{Q}^{(h)} | \boldsymbol{Q}^*)}{J(\boldsymbol{Q}^* | \boldsymbol{Q}^{(h)})} = \frac{p(n^{(h)})}{p(n^*)}. \tag{2}$$

We then have

$$\frac{p(\boldsymbol{Q}^*)}{p(\boldsymbol{Q}^{(h)})} \frac{J(\boldsymbol{Q}^{(h)} | \boldsymbol{Q}^*)}{J(\boldsymbol{Q}^* | \boldsymbol{Q}^{(h)})} = \frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} \left(\prod_{k=1}^{K} \frac{p(\boldsymbol{C}_k^{(h)} | \boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)} | \boldsymbol{N}^{(h)})} \right) \frac{p(n^{(h)})}{p(n^*)} \tag{3}$$

$$= \frac{p(\boldsymbol{N}_{1+}^*)}{p(\boldsymbol{N}_{1+}^{(h)})} \prod_{k=1}^{K} \frac{p(\boldsymbol{C}_k^{(h)} | \boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)} | \boldsymbol{N}^{(h)})}. \tag{4}$$

3.2 Updating component $\boldsymbol{C}_u$

Suppress dependence on $\boldsymbol{\gamma}$.

Use expected population $\hat{\boldsymbol{\gamma}}^N$, rather than actual population $\boldsymbol{N}$, to generate proposals:

$$\frac{J(\boldsymbol{Q}^{(h)} | \boldsymbol{Q}^*)}{J(\boldsymbol{Q}^* | \boldsymbol{Q}^{(h)})} = \frac{p(c^{(h)} | \hat{\boldsymbol{N}})}{p(c^* | \hat{\boldsymbol{N}})}. \tag{5}$$

Then

$$\frac{p(\boldsymbol{Q}^*)}{p(\boldsymbol{Q}^{(h)})}\frac{J(\boldsymbol{Q}^{(h)}|\boldsymbol{Q}^*)}{J(\boldsymbol{Q}^*|\boldsymbol{Q}^{(h)})} = \frac{p(\boldsymbol{N}^*)\prod_{k=1}^{K} p(\boldsymbol{C}_k^*|\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})\prod_{k=1}^{K} p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^{(h)})} \tag{6}$$

$$= \frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})}\frac{p(\boldsymbol{C}_u^*|\boldsymbol{N}^*)}{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^{(h)})}\left(\prod_{k\neq u}\frac{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^{(h)})}\right)\frac{p(c^{(h)}|\hat{\boldsymbol{N}})}{p(c^*|\hat{\boldsymbol{N}})} \tag{7}$$

$$= \frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})}\frac{p(\boldsymbol{C}_u^*|\boldsymbol{N}^*)}{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^{(h)})}\frac{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^{(h)})}{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^*)}\frac{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^*)}{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^{(h)})} \times \left(\prod_{k\neq u}\frac{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^{(h)})}\right)\frac{p(c^{(h)}|\hat{\boldsymbol{N}})}{p(c^*|\hat{\boldsymbol{N}})} \tag{8}$$

$$= \frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})}\frac{p(\boldsymbol{C}_u^*|\boldsymbol{N}^*)p(c^{(h)}|\hat{\boldsymbol{N}})}{p(\boldsymbol{C}_u^{(h)}|\boldsymbol{N}^*)p(c^*|\hat{\boldsymbol{N}})}\prod_{k=1}^{K}\frac{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^{(h)})} \tag{9}$$

4 Cells updated directly in proposals

4.1 Starting population

Cell updated is $\boldsymbol{N}[i,a,0]$. Let $n^{(h)}$ denote current value and n^* the proposed value, and define $\Delta = n^* - n^{(h)}$.

4.2 Generic $\boldsymbol{C}_k$

Cell updated is $\boldsymbol{C}_k[i,a,l,t]$. Let $c^{(h)}$ denote the current value and c^* the proposed value. Let $\Delta = c^* - c^{(h)}$.

4.3 Births

Cell updated is $\boldsymbol{B}[i,a,l,t]$. Let $c^{(h)}$ denote the current value and c^* the proposed value. Let $\Delta = c^* - c^{(h)}$.

4.4 Internal migration with origin-destination format

Cell updated is $\boldsymbol{M}[i,r_{\text{orig}},r_{\text{dest}},a,l,t]$. Let $c^{(h)}$ denote the current value and c^* the proposed value, and let $\Delta = c^* - c^{(h)}$.

4.5 Internal migration with pool format

Cells updated are $\boldsymbol{M}[i,r,\text{Out},a,l,t]$ and $\boldsymbol{M}[i,r',\text{In},a,l,t]$, $r' \neq r$. Let $c_{\text{out}}^{(h)}$ and $c_{\text{in}}^{(h)}$ denote the current values and c_{out}^* and c_{in}^* the proposed values. Let $\Delta = c_{\text{out}}^* - c_{\text{out}}^{(h)} = c_{\text{in}}^* - c_{\text{in}}^{(h)}$.

4.6 Internal migration with net format

Cells updated are $\boldsymbol{M}[i, r, a, l, t]$ and $\boldsymbol{M}[i, r', a, l, t]$, $r' \neq r$. Let $c_1^{(h)}$ and $c_2^{(h)}$ denote the current values and c_1^* and c_2^* the proposed values. Let $\Delta = c_1^* - c_1^{(h)} = -(c_2^* - c_2^{(h)})$.

5 Calculating $\frac{p(\boldsymbol{N}_{1+}^*)}{p(\boldsymbol{N}_{1+}^{(h)})}$

Define cohort quantities

$$n^{(h)}(s) = \boldsymbol{N}^{(h)}[i, \min(a+s, A), s] \tag{10}$$

$$n^*(s) = \boldsymbol{N}^*[i, \min(a+s, A), s] = n^{(h)}(s) + \Delta \tag{11}$$

$$\lambda(s) = \boldsymbol{\gamma}_N[i, \min(a+s, A), s] \tag{12}$$

Then

$$\frac{p(\boldsymbol{N}_{1+}^*)}{p(\boldsymbol{N}_{1+}^{(h)})} = \prod_{s=1}^{T} \frac{\text{Poisson}\left(n^*(s) | \lambda(s)\right)}{\text{Poisson}\left(n^{(h)}(s) | \lambda(s)\right)} \tag{13}$$

6 Calculating $\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})}$

6.1 Generic component $\boldsymbol{C}_k$

Define cohort quantities

$$n^{(h)}(s) = \boldsymbol{N}^{(h)}[i, \min(a + 1(l = \text{U}) + s, A), t+s] \tag{14}$$

$$n^*(s) = n^{(h)}(s) + S_k \Delta \tag{15}$$

$$\lambda(s) = \boldsymbol{\gamma}_N[i, \min(a + 1(l = \text{U}) + s, A), t+s] \tag{16}$$

Then

$$\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} = \prod_{s=0}^{T-t} \frac{\text{Poisson}\left(n^*(s) | \lambda(s)\right)}{\text{Poisson}\left(n^{(h)}(s) | \lambda(s)\right)} \tag{17}$$

6.2 Births

Define cohort quantities starting at age 0

$$n^{(h)}(s) = \boldsymbol{N}^{(h)}[i, \min(s, A), t+s] \tag{18}$$

$$n^*(s) = n^{(h)}(s) + \Delta \tag{19}$$

$$\lambda(s) = \boldsymbol{\gamma}_N[i, \min(s, A), t+s] \tag{20}$$

Then

$$\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} = \prod_{s=0}^{T-t} \frac{\text{Poisson}(n^*(s) | \lambda(s))}{\text{Poisson}(n^{(h)}(s) | \lambda(s))} \tag{21}$$

6.3 Internal migration with origin-destination format

$$n_{\mathrm{orig}}^{(h)}(s) = \boldsymbol{N}^{(h)}[i, r_{\mathrm{orig}}, \min(a + 1(l = \mathrm{U}) + s, A), t + s] \tag{22}$$
$$n_{\mathrm{dest}}^{(h)}(s) = \boldsymbol{N}^{(h)}[i, r_{\mathrm{dest}}, \min(a + 1(l = \mathrm{U}) + s, A), t + s] \tag{23}$$
$$n_{\mathrm{orig}}^{*}(s) = n^{(h)}(s) - \Delta \tag{24}$$
$$n_{\mathrm{dest}}^{*}(s) = n^{(h)}(s) + \Delta \tag{25}$$

$$\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} = \prod_{s=0}^{T-t} \frac{\mathrm{Poisson}\left(n_{\mathrm{orig}}^{*}(s)|\lambda(s)\right)}{\mathrm{Poisson}\left(n_{\mathrm{orig}}^{(h)}(s)|\lambda(s)\right)} \frac{\mathrm{Poisson}\left(n_{\mathrm{dest}}^{*}(s)|\lambda(s)\right)}{\mathrm{Poisson}\left(n_{\mathrm{dest}}^{(h)}(s)|\lambda(s)\right)} \tag{26}$$

6.4 Internal migration with pool format

$$n_{\mathrm{out}}^{(h)}(s) = \boldsymbol{N}^{(h)}[i, r, \min(a + 1(l = \mathrm{U}) + s, A), t + s] \tag{27}$$
$$n_{\mathrm{in}}^{(h)}(s) = \boldsymbol{N}^{(h)}[i, r', \min(a + 1(l = \mathrm{U}) + s, A), t + s] \tag{28}$$
$$n_{\mathrm{out}}^{*}(s) = n^{(h)}(s) - \Delta \tag{29}$$
$$n_{\mathrm{in}}^{*}(s) = n^{(h)}(s) + \Delta \tag{30}$$

$$\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} = \prod_{s=0}^{T-t} \frac{\mathrm{Poisson}\left(n_{\mathrm{out}}^{*}(s)|\lambda(s)\right)}{\mathrm{Poisson}\left(n_{\mathrm{out}}^{(h)}(s)|\lambda(s)\right)} \frac{\mathrm{Poisson}\left(n_{\mathrm{in}}^{*}(s)|\lambda(s)\right)}{\mathrm{Poisson}\left(n_{\mathrm{in}}^{(h)}(s)|\lambda(s)\right)} \tag{31}$$

6.5 Internal migration with net format

$$n_{1}^{(h)}(s) = \boldsymbol{N}^{(h)}[i, r, \min(a + 1(l = \mathrm{U}) + s, A), t + s] \tag{32}$$
$$n_{2}^{(h)}(s) = \boldsymbol{N}^{(h)}[i, r', \min(a + 1(l = \mathrm{U}) + s, A), t + s] \tag{33}$$
$$n_{1}^{*}(s) = n^{(h)}(s) + \Delta \tag{34}$$
$$n_{2}^{*}(s) = n^{(h)}(s) - \Delta \tag{35}$$

$$\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} = \prod_{s=0}^{T-t} \frac{\mathrm{Poisson}\left(n_{1}^{*}(s)|\lambda(s)\right)}{\mathrm{Poisson}\left(n_{1}^{(h)}(s)|\lambda(s)\right)} \frac{\mathrm{Poisson}\left(n_{2}^{*}(s)|\lambda(s)\right)}{\mathrm{Poisson}\left(n_{2}^{(h)}(s)|\lambda(s)\right)} \tag{36}$$

7 Calculating $\frac{p(\boldsymbol{C}_k^*|\boldsymbol{N}^*)p(c^{(h)}|\hat{\boldsymbol{N}})}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)p(c^*|\hat{\boldsymbol{N}})}$

7.1 Generic $\boldsymbol{C}_k$

No exposure Cancels

With exposure Let

$$e^{(h)} = \boldsymbol{E}^{(h)}[i, a, l, t] \tag{37}$$
$$e^* = e^{(h)} + 1(l = \mathrm{L}) S_k \tfrac{1}{2} \Delta \tag{38}$$
$$\gamma = \boldsymbol{\gamma}_k[i, a, l, t] \tag{39}$$

Then

$$\frac{p(\boldsymbol{C}_k^*|\boldsymbol{N}^*)p(c^*|\hat{\boldsymbol{N}})}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)p(c^{(h)}|\hat{\boldsymbol{N}})} = \frac{\text{Poisson}\,(c^*|\gamma e^*)\,\text{Poisson}\left(c^{(h)}|\gamma \hat{e}\right)}{\text{Poisson}\left(c^{(h)}|\gamma e^*\right)\text{Poisson}\,(c^*|\gamma \hat{e})} \tag{40}$$
$$= (e^*/\hat{e})^{(c^* - c^{(h)})} \tag{41}$$

7.2 Internal migration with origin-destination format

No exposure Cancels

With exposure

$$e^{(h)} = \boldsymbol{E}^{(h)}[i, r_{\text{orig}}, a, l, t] \tag{42}$$
$$e^* = e^{(h)} - 1(l = \mathrm{L}) \tfrac{1}{2} \Delta \tag{43}$$
$$\gamma = \boldsymbol{\gamma}_k[i, r_{\text{orig}}, a, l, t] \tag{44}$$

Remainder as for generic $\boldsymbol{C}_k$.

7.2.1 Internal migration with pool format

Two cells change, so don't get complete cancellations.

Without exposure

$$\gamma_{\text{out}} = \boldsymbol{\gamma}_k[i, r, \text{Out}, a, l, t] \tag{45}$$
$$\gamma_{\text{in}} = \boldsymbol{\gamma}_k[i, r', \text{In}, a, l, t] \tag{46}$$

$$\frac{p(\boldsymbol{C}_k^*|\boldsymbol{N}^*)p(c^{(h)}|\hat{\boldsymbol{N}})}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)p(c^*|\hat{\boldsymbol{N}})} = \frac{\text{Poisson}\,(c^*_{\text{out}}|\gamma_{\text{out}})\,\text{Poisson}\,(c^*_{\text{in}}|\gamma_{\text{in}})\,\text{Poisson}\left(c^{(h)}_{\text{out}}|\gamma_{\text{out}}\right)}{\text{Poisson}\left(c^{(h)}_{\text{out}}|\gamma_{\text{out}}\right)\text{Poisson}\left(c^{(h)}_{\text{in}}|\gamma_{\text{in}}\right)\text{Poisson}\,(c^*_{\text{out}}|\gamma_{\text{out}})} \tag{47}$$
$$= \frac{\text{Poisson}\,(c^*_{\text{in}}|\gamma_{\text{in}})}{\text{Poisson}\left(c^{(h)}_{\text{in}}|\gamma_{\text{in}}\right)} \tag{48}$$

With exposure

$$e^{(h)}_{\text{out}} = \boldsymbol{E}^{(h)}[i, r, a, l, t] \tag{49}$$
$$e^{(h)}_{\text{in}} = \boldsymbol{E}^{(h)}[i, r', a, l, t] \tag{50}$$
$$e^{*}_{\text{out}} = e^{(h)}_{\text{out}} - 1(l = \text{L})\tfrac{1}{2}\Delta \tag{51}$$
$$e^{*}_{\text{in}} = e^{(h)}_{\text{in}} + 1(l = \text{L})\tfrac{1}{2}\Delta \tag{52}$$
$$\hat{e}_{\text{out}} = \hat{\boldsymbol{E}}[i, r, a, l, t] \tag{53}$$
$$\hat{e}_{\text{in}} = \hat{\boldsymbol{E}}[i, r', a, l, t] \tag{54}$$
$$\gamma_{\text{out}} = \boldsymbol{\gamma}_k[i, r, \text{Out}, a, l, t] \tag{55}$$
$$\gamma_{\text{in}} = \boldsymbol{\gamma}_k[i, r', \text{In}, a, l, t] \tag{56}$$

$$\frac{p(\boldsymbol{C}^{*}_k|\boldsymbol{N}^{*})p(c^{(h)}|\hat{\boldsymbol{N}})}{p(\boldsymbol{C}^{(h)}_k|\boldsymbol{N}^{*})p(c^{*}|\hat{\boldsymbol{N}})} = \frac{\text{Poisson}\left(c^{*}_{\text{out}}|\gamma_{\text{out}}e^{*}_{\text{out}}\right)\text{Poisson}\left(c^{*}_{\text{in}}|\gamma_{\text{in}}e^{*}_{\text{in}}\right)\text{Poisson}\left(c^{(h)}_{\text{out}}|\gamma_{\text{out}}\hat{e}_{\text{out}}\right)}{\text{Poisson}\left(c^{(h)}_{\text{out}}|\gamma_{\text{out}}e^{*}_{\text{out}}\right)\text{Poisson}\left(c^{(h)}_{\text{in}}|\gamma_{\text{in}}e^{*}_{\text{in}}\right)\text{Poisson}\left(c^{*}_{\text{out}}|\gamma_{\text{out}}\hat{e}_{\text{out}}\right)} \tag{57}$$

$$= \left(e^{*}_{\text{out}}/\hat{e}_{\text{out}}\right)^{\left(c^{*}_{\text{out}} - c^{(h)}_{\text{out}}\right)} \frac{\text{Poisson}\left(c^{*}_{\text{in}}|\gamma_{\text{in}}e^{*}_{\text{in}}\right)}{\text{Poisson}\left(c^{(h)}_{\text{in}}|\gamma_{\text{in}}e^{*}_{\text{in}}\right)} \tag{58}$$

7.2.2 Internal migration with net format

Never uses exposure, but does sometimes use weights.

$$w_{\text{out}} = \boldsymbol{W}[i, r, a, l, t] \tag{59}$$
$$w_{\text{in}} = \boldsymbol{W}[i, r', a, l, t] \tag{60}$$
$$\gamma_{\text{out}} = \boldsymbol{\gamma}_k[i, r, a, l, t] \tag{61}$$
$$\gamma_{\text{in}} = \boldsymbol{\gamma}_k[i, r', a, l, t] \tag{62}$$

$$\frac{p(\boldsymbol{C}^{*}_k|\boldsymbol{N}^{*})p(c^{(h)}|\hat{\boldsymbol{N}})}{p(\boldsymbol{C}^{(h)}_k|\boldsymbol{N}^{*})p(c^{*}|\hat{\boldsymbol{N}})} = \frac{\text{N}\left(c^{*}_{\text{out}}|\gamma_{\text{out}}, \frac{\phi^2}{w_{\text{out}}}\right)\text{N}\left(c^{*}_{\text{in}}|\gamma_{\text{in}}, \frac{\phi^2}{w_{\text{in}}}\right)\text{N}\left(c^{(h)}_{\text{out}}|\gamma_{\text{out}}, \frac{\phi^2}{w_{\text{out}}}\right)}{\text{N}\left(c^{(h)}_{\text{out}}|\gamma_{\text{out}}, \frac{\phi^2}{w_{\text{out}}}\right)\text{N}\left(c^{(h)}_{\text{in}}|\gamma_{\text{in}}, \frac{\phi^2}{w_{\text{in}}}\right)\text{N}\left(c^{*}_{\text{out}}|\gamma_{\text{out}}, \frac{\phi^2}{w_{\text{out}}}\right)} \tag{63}$$

$$= \frac{\text{N}\left(c^{*}_{\text{in}}|\gamma_{\text{in}}, \frac{\phi^2}{w_{\text{in}}}\right)}{\text{N}\left(c^{(h)}_{\text{in}}|\gamma_{\text{in}}, \frac{\phi^2}{w_{\text{in}}}\right)} \tag{64}$$

8 Calculating $\prod_{k=1}^{K} \frac{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^{(h)})}$

Does not depend on which component is being updated. Instead look at form of individual $\frac{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^*)}{p(\boldsymbol{C}_k^{(h)}|\boldsymbol{N}^{(h)})}$

8.1 Generic $\boldsymbol{C}_k$

No exposure Drops out

With exposure Define cohort quantities

$$c^{(h)}(s) = \boldsymbol{C}_k^{(h)}[i, \min(a+s, A), t+s] \tag{65}$$

$$e^*(0) = e^{(h)}(s) + 1(l = \mathrm{L}) S_k \tfrac{1}{2}\Delta \tag{66}$$

$$e^*(s) = e^{(h)}(s) + S_k \tfrac{1}{2}\Delta, \quad s > 0 \tag{67}$$

$$\lambda(s) = \boldsymbol{\gamma}_N[i, \min(a+s, A), t+s] \tag{68}$$

Then

$$\frac{p(\boldsymbol{N}^*)}{p(\boldsymbol{N}^{(h)})} = \prod_{s=0}^{T-t} \frac{\text{Poisson}\,(n^*(s)|\lambda(s))}{\text{Poisson}\,\left(n^{(h)}(s)|\lambda(s)\right)} \tag{69}$$

8.2 Algorithm for updating starting population

8.2.1 Choose cell to update

Choose a cell u in $\boldsymbol{N}$ from the first time point. All valid cells have equal selection probabilities.

8.2.2 Identify subsequent population cells changed by update

Let $\mathcal{N}$ be a set of indices for $\boldsymbol{N}$ consisting of u plus indices for all cells for later time points whose values would need to change to satisfy the accounting identities if the value of cell u changed. For all $i \in \mathcal{N}$, let $n_i^{(h)} = \boldsymbol{N}[i]$ and $\lambda_i = \boldsymbol{\gamma}^N[i]$. Also let $\lambda = \lambda_u$.

8.2.3 Identify accession cells changed by update

If the account uses age, then let $\mathcal{A}$ be the indices for cells in $\boldsymbol{A}$ whose values would need to change to satisfy the accounting identities if the value of cell u in $\boldsymbol{N}$ changed. For all $i \in \mathcal{A}$, let $a_i^{(h)} = \boldsymbol{A}[i]$.

8.2.4 Identify exposure cells changed by update

Let $\mathcal{E}$ be the indices for cells in $\boldsymbol{E}$ whose values would need to change to satisfy the accounting identities if the values of cell u in $\boldsymbol{N}$ changed. For all $i \in \mathcal{E}$, let $e_i^{(h)} = \boldsymbol{E}[i]$.

8.2.5 Identify cells of components where expected values changed by update

If the system model for $\boldsymbol{C}_k$ uses exposure, and if at least one cell in $\mathcal{E}$ maps on to $\boldsymbol{C}_k$, then let $\mathcal{C}_k$ be the indices for cells in $\boldsymbol{C}$ whose expected values were to change if the value of cell u in $\boldsymbol{N}$ changed; otherwise let $\mathcal{C}_k = \{\}$. For all $i \in \mathcal{C}_k$, let $c_{ik} = \boldsymbol{C}_k[i]$ and $\lambda_{ik} = \boldsymbol{\gamma}_k^C[i]$. Let $e_{ik}^{(h)}$ denote the element of $\boldsymbol{E}$ associated with cell i of $\boldsymbol{C}_k$. One element of $\boldsymbol{E}$ may be associated with more than one cell of $\boldsymbol{C}_k$, such as when $\boldsymbol{C}_k$ measures migrations using an origin-destination format.

8.2.6 Identify corresponding cells in datasets measuring population

If $\boldsymbol{X}_m$ is used to estimate $\boldsymbol{N}$, and at least one cell in $\mathcal{N}$ maps on to a cell in $\boldsymbol{N}_m$, then let $\mathcal{X}_m$ be the cells that $\mathcal{N}$ maps on to; otherwise let $\mathcal{X}_m = \{\}$. For all $i \in \mathcal{X}_m$, let $x_{im}^{(h)} = \boldsymbol{X}_m[i]$.

8.2.7 Find minimum value for population and accession

Let

$$v = \begin{cases} \min\left(\min_i n_i^{(h)}, \min_i a_i^{(h)}\right) & \text{if } \boldsymbol{N} \text{ has age} \\ \min_i n_i^{(h)} & \text{otherwise.} \end{cases} \tag{70}$$

8.2.8 Generate proposed values for cell

Draw proposed value n^* from the left-truncated Poisson distribution

$$p(y) \propto \begin{cases} e^{-\lambda}\lambda^y & \text{if } y \geq n^{(h)} - v \\ 0 & \text{otherwise.} \end{cases} \tag{71}$$

Let

$$\Delta = n^* - n^{(h)}. \tag{72}$$

8.2.9 Calculate ratio of likelihoods

For all $i \in \mathcal{N}_m$, let $n_{km}^* = n_{km}^{(h)} + \Delta$ and let $x_{km} = X_m[i]$. Then

$$r_{X|N} = \prod_{m \in \mathcal{M}_X} \prod_{i \in \mathcal{N}_m} \frac{p(x_{km}|c_{km}^*)}{p(x_{km}|c_{km}^{(h)})}. \tag{73}$$

8.2.10 Calculate ratio of prior probabilities

For all $i \in \mathcal{N}$, let $n_i^* = n_i^{(h)} + \Delta$. Then

$$r_{N|\lambda} = \prod_{i \in \mathcal{N}} \frac{p(n_i^*|\lambda_i)}{p(n_i^{(h)}|\lambda_i)}. \tag{74}$$

Let e^*_{ik} denote an updated version of $e^{(h)}_{ik}$. For all age group other than the final age group, $e^*_{ik} = e^{(h)}_{ik} + \Delta/2$ (REFERENCE TO DATA STRUCTURES CHAPTER). Then

$$r_{C_k|\lambda,E} = \prod_{i \in \mathcal{C}_k} \frac{p(c_{ik}|\lambda_{ik} e^*_{ik})}{p(c_{ik}|\lambda_i e^{(h)}_{ik})}. \tag{75}$$

The term inside the multiplication simplifies, on a log scale, to

$$-\lambda_{ik}(e^*_{ik} - e^{(h)}_{ik}) + c_{ik}(\log e^*_{ik} - \log e^{(h)}_{ik}). \tag{76}$$

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8.2.11 Calculate ratio of proposal probabilities

Let

$$r_J = \frac{p(n^{(h)}|\lambda)}{p(n^*|\lambda)}. \tag{77}$$

8.2.12 Calculate combined Metropolis-Hastings ratio, determine acceptance, and update

Let

$$r = r_{X|N} \times \times r_{N|\lambda} \times r_J. \tag{78}$$

If $r > 1$ or if $U \sim \text{Unif}(0,1) < r$, then accept the proposal, and update $\boldsymbol{N}$, $\boldsymbol{E}$, and, if $\boldsymbol{N}$ has age, $\boldsymbol{A}$. Otherwise leave all arrays at their current values.

The calculation of the Metropolis-Hasting ratio can be simplified slightly by taking advantage of the fact that r_J and the first element of $r_{N|\lambda}$ cancel.

8.3 Algorithm for updating component $\boldsymbol{C}$

8.3.1 Choose cells in component to update

Choose a cell u in $\boldsymbol{C}$ to update. If $\boldsymbol{C}$ has a pool format, then u must come from the 'ins' part of the array, with every 'ins' cell having an equal selection probability; otherwise it can come from any cell in $\boldsymbol{C}$, with every cell having an equal selection probability. If $\boldsymbol{C}$ is internal migration has a pool format, then select a second cell u_{oth} that is identical to u except that it comes from the 'outs' part of $\boldsymbol{C}$, and belongs to different categories for the 'between' dimensions. If $\boldsymbol{C}$ is internal migration and has a net format, then choose a second cell u_{oth} that is identical to u except that belongs to different categories for the 'between' dimensions. In either case, all valid cells have equal selection probabilities. Let $\mathcal{J}$ denote the set of cells in $\boldsymbol{C}$ that are being updated. Let $c^{(h)} = \boldsymbol{C}[u]$ and $c^{(h)}_{\text{oth}} = \boldsymbol{C}[u_{\text{oth}}]$. It is also convenient, for all $j \in \mathcal{J}$, to define $c^{(h)}_j = \boldsymbol{C}[j]$.

Let $\mathcal{Q}$ denote the set of m such that (i) $\boldsymbol{X}_m$ is used to estimate $\boldsymbol{C}$, and (ii) at least one cell in $\mathcal{J}$ maps on to a cell in $\boldsymbol{C}_m$. If $m \in \mathcal{Q}$, then let $\mathcal{J}_m$ be the cells that $\mathcal{J}$ maps on to. If $\boldsymbol{C}$ is internal migration, or if $\boldsymbol{C}$ has parent-child dimensions (implying that it is births), then $\mathcal{J}_m$ contains one or two cells; otherwise it contains one cell. For all $j \in \mathcal{J}_m$, let $c^{(h)}_{jm} = \boldsymbol{C}_m[j]$.

Table 1: Cells involved in update

Component	Format	Update two cells in $\boldsymbol{C}$	Update two states in $\boldsymbol{N}$
Internal migration	Origin-destination	No	Yes
Internal migration	Pool	Yes	Yes
Internal migration	Net	Yes	Yes
All other	–	No	No

8.3.2 Identify population cells involved in update

Let $\mathcal{K}$ denote the set of cells in $\boldsymbol{N}$ whose values would need to change to satisfy the accounting identities if the values of the cells in $\mathcal{J}$ changed. If $\boldsymbol{C}$ is internal migration, then $\mathcal{K} = \{\mathcal{K}^{\text{orig}}, \mathcal{K}^{\text{dest}}\}$, where $\mathcal{K}^{\text{orig}}$ contains cells in the origin and $\mathcal{K}^{\text{dest}}$ contains cells in the destination. Similarly, if $\boldsymbol{C}$ has parent-child dimensions, then $\mathcal{K} = \{\mathcal{K}^{\text{parent}}, \mathcal{K}^{\text{child}}\}$. For all $i \in \mathcal{K}$, let $n_i^{(h)} = \boldsymbol{N}[i]$. If $\boldsymbol{C}$ is internal migration, then, for all $i \in \mathcal{K}^{\text{orig}}$, let $n_i^{\text{orig}(h)} = \boldsymbol{N}[i]$. Define $n_i^{\text{dest}(h)}$, $n_i^{\text{parent}(h)}$, and $n_i^{\text{child}(h)}$ similarly.

Let $\mathcal{M}_X$ denote the set of m such that (i) $\boldsymbol{X}_m$ is used to estimate $\boldsymbol{N}$, and (ii) at least one cell in $\mathcal{K}$ maps on to a cell in $\boldsymbol{N}_m$. If $m \in \mathcal{M}_X$, then let $\mathcal{K}_m$ be the cells that $\mathcal{K}$ maps on to. If $\boldsymbol{C}$ is internal migration, then $\mathcal{K}_m = \{\mathcal{K}_m^{\text{orig}}, \mathcal{K}_m^{\text{dest}}\}$. Similarly, if $\boldsymbol{C}$ has parent-child dimensions, then $\mathcal{K}_m = \{\mathcal{K}_m^{\text{parent}}, \mathcal{K}_m^{\text{child}}\}$. For all $i \in \mathcal{K}_m$, let $n_{km}^{(h)} = \boldsymbol{C}_m[i]$. If $\boldsymbol{C}$ is internal migration, then for all $i \in \mathcal{K}_m^{\text{orig}}$, let $n_{km}^{\text{orig}(h)} = \boldsymbol{N}[i]$. Define $n_{km}^{\text{dest}(h)}$, $n_{km}^{\text{parent}(h)}$, and $n_{km}^{\text{child}(h)}$ similarly.

The scope of the updates of $\boldsymbol{C}$ and $\boldsymbol{N}$ is summarised in Table 1.

8.3.3 Identify accession cells involved in update

If $\boldsymbol{C}$ has an age dimension, then let $\mathcal{K}_A$ be the set of cells in $\boldsymbol{A}$ whose values would need to change to satisfy the accounting identities if the values of the cells in $\mathcal{J}$ changed. Sets $\mathcal{K}_A^{\text{orig}}$, $\mathcal{K}_A^{\text{dest}}$, $\mathcal{K}_A^{\text{parent}}$, and $\mathcal{K}_A^{\text{child}}$ are defined analogously to their equivalents for population, as are values $a_i^{\text{orig}(h)}$, $a_i^{\text{dest}(h)}$, $a_i^{\text{parent}(h)}$, and $a_i^{\text{child}(h)}$.

8.3.4 Find minimum values for population and accession

If $\boldsymbol{C}$ is not internal migration and does not have parent-child dimensions, then let

$$v = \begin{cases} \min\left(\min_i n_i^{(h)}, \min_i a_i^{(h)}\right) & \text{if } \boldsymbol{C} \text{ has age} \\ \min_i n_i^{(h)} & \text{otherwise.} \end{cases} \tag{79}$$

If $\boldsymbol{C}$ is internal migration, then let

$$v^{\text{orig}} = \begin{cases} \min\left(\min_i n_i^{\text{orig}(h)}, \min_i a_i^{\text{orig}(h)}\right) & \text{if } \boldsymbol{C} \text{ has age} \\ \min_i n_i^{\text{orig}(h)} & \text{otherwise.} \end{cases} \tag{80}$$

$$v^{\text{dest}} = \begin{cases} \min\left(\min_i n_i^{\text{dest}(h)}, \min_i a_i^{\text{dest}(h)}\right) & \text{if } \boldsymbol{C} \text{ has age} \\ \min_i n_i^{\text{dest}(h)} & \text{otherwise.} \end{cases} \tag{81}$$

If $\boldsymbol{C}$ has parent-child dimensions, then define v^{parent} and v^{child} similarly.

8.3.5 Obtain current exposures

If the system model for $\boldsymbol{C}$ uses exposure, then, for $j \in \mathcal{J}$, let $e_j^{(h)} = \boldsymbol{E}[j]$.

8.3.6 Obtain current expected values for component

If the demographic model for $\boldsymbol{C}$ uses exposure, then, for $j \in \mathcal{J}$, let $\lambda_j^{C(h)} = \boldsymbol{\gamma}^C[j]e_j^{(h)}$. Otherwise, for $j \in \mathcal{J}$, let $\lambda_j^{C(h)} = \boldsymbol{\gamma}^C[j]$.

8.3.7 Obtain expected counts for population

For $i \in \mathcal{K}$, let $\lambda_i^N = \boldsymbol{\gamma}^N[i]$.

8.3.8 Generate proposed values for cells in component

Draw proposed value c^*. If $\boldsymbol{C}$ is a type of increment and does not have parent-child dimensions, then the value is drawn from the left-truncated Poisson distribution

$$p(y) \propto \begin{cases} e^{-\lambda}\lambda^{(h)y} & \text{if } y \geq c^{(h)} - v \\ 0 & \text{otherwise.} \end{cases} \tag{82}$$

If $\boldsymbol{C}$ a type of decrement, then c^* is drawn from the right-truncated Poisson distribution

$$p(y) \propto \begin{cases} e^{-\lambda}\lambda^{(h)y} & \text{if } y \leq v - c^{(h)} \\ 0 & \text{otherwise.} \end{cases} \tag{83}$$

If $\boldsymbol{C}$ is internal migration, then c^* is drawn from the truncated Poisson distribution

$$p(y) \propto \begin{cases} e^{-\lambda}\lambda^{(h)y} & \text{if } c^{(h)} - v^{\text{dest}} \leq y \leq v^{\text{orig}} - c^{(h)} \\ 0 & \text{otherwise.} \end{cases} \tag{84}$$

If $\boldsymbol{C}$ has parent-child dimensions, then c^* is drawn from the truncated Poisson distribution

$$p(y) \propto \begin{cases} e^{-\lambda}\lambda^{(h)y} & \text{if } c^{(h)} - v^{\text{parent}} \leq y \leq v^{\text{child}} - c^{(h)} \\ 0 & \text{otherwise.} \end{cases} \tag{85}$$

Let

$$\Delta_c = c^* - c^{(h)}. \tag{86}$$

If $\boldsymbol{C}$ is internal migration, or if $\boldsymbol{C}$ has parent-child dimensions, then let $c^*_{\text{oth}} = c^{(h)}_{\text{oth}} + \Delta_c$.

8.3.9 Calculate ratio of likelihoods for component $\boldsymbol{C}$

For all $j \in \mathcal{J}_m$, let $c^*_{jm} = c^{(h)}_{jm} + \Delta_c$, and let $x_{jm} = X_m[j]$. The ratio of the likelihoods for component $\boldsymbol{C}$ can then be calculated as

$$r_{X|C} = \prod_{m \in \mathcal{Q}} \prod_{j \in \mathcal{J}_m} \frac{p(x_{jm}|c^*_{jm})}{p(x_{jm}|c^{(h)}_{jm})}. \tag{87}$$

8.3.10 Calculate ratio of likelihoods for population $\boldsymbol{N}$

If $\boldsymbol{C}$ is an increment and does not have parent-child dimensions, then, for all $i \in \mathcal{K}_m$, let

$$n^*_{km} = n^{(h)}_{km} + \Delta_c. \tag{88}$$

If $\boldsymbol{C}$ is a decrement, then, for all $i \in \mathcal{K}_m$, let

$$n^*_{km} = n^{(h)}_{km} - \Delta_c. \tag{89}$$

If $\boldsymbol{C}$ is internal migration, then for all $i \in \mathcal{K}^{\text{orig}}_m$, let

$$n^{\text{orig}*}_{km} = n^{\text{orig}(h)}_{km} - \Delta_c, \tag{90}$$

and for all $i \in \mathcal{K}^{\text{dest}}_m$, let

$$n^{\text{dest}*}_{km} = n^{\text{dest}(h)}_{km} + \Delta_c. \tag{91}$$

If $\boldsymbol{C}$ has parent-child dimensions, then for all $i \in \mathcal{K}^{\text{parent}}_m$, let

$$n^{\text{parent}*}_{km} = n^{\text{parent}(h)}_{km} - \Delta_c, \tag{92}$$

and for all $i \in \mathcal{K}^{\text{child}}_m$, let

$$n^{\text{child}*}_{km} = n^{\text{child}(h)}_{km} + \Delta_c. \tag{93}$$

In addition, for all $i \in \mathcal{K}_m$ let $x_{km} = X_m[i]$. The ratio of likelihoods for population $\boldsymbol{N}$ can then be calculated as

$$r_{X|N} = \prod_{m \in \mathcal{M}_X} \prod_{i \in \mathcal{K}_m} \frac{p(x_{km}|c^*_{km})}{p(x_{km}|c^{(h)}_{km})}. \tag{94}$$

8.3.11 Calculate ratio of prior probabilities for component $\boldsymbol{C}$

Let

$$r_{C|\lambda} = \prod_{j \in \mathcal{J}} \frac{p(c_j^* | \lambda_j^{C*})}{p(c_j^{(h)} | \lambda_j^{C(h)})}. \tag{95}$$

8.3.12 Calculate ratio of prior probabilities for population $\boldsymbol{N}$

Let

$$r_{N|\lambda} = \prod_{i \in \mathcal{K}} \frac{p(n_i^* | \lambda_i^N)}{p(n_i^{(h)} | \lambda_i^N)}. \tag{96}$$

8.3.13 Calculate ratio of proposal probabilities

Let

$$r_J = \frac{p(c_u^{(h)} | \lambda_u^{C*})}{p(c_u^* | \lambda_u^{C(h)})}. \tag{97}$$

8.3.14 Calculate combined ratio, determine acceptance, and update

Let

$$r = r_{X|C} \times r_{X|N} \times r_{C|\lambda} \times r_{N|\lambda} \times r_J. \tag{98}$$

If $r > 1$ or if $U \sim \text{Unif}(0, 1) < r$, then accept the proposal, and update $\boldsymbol{C}$ and $\boldsymbol{N}$, and possibly $\boldsymbol{A}$ and $\boldsymbol{E}$. Otherwise leave all elements at their current values.