AE 353 Homework 4 Solutions

1. State-space system:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 4 & -1 & 0.5 \\ 4 & -4 & 0.5 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u$$

a) Writing the cost function in terms of the standard LQR cost function:

$$\int_0^\infty (\|y\|^2 + \rho u^2)dt = \int_0^\infty (x^T C^T C x + u^T \rho u)dt$$

Therefore we have: $Q = C^T C$ and $R = \rho$.

b) Using the MATLAB function:

lqr(A,B,Q,R);

We can obtain the required gain matrix. Using $\rho = 10^{-6}$ and the following C matrices:

- C = [1 0 0 0]:

  K = [704.4826 143.7747 226.3255 31.0741]

- C = [0 1 0 0]:

  K = [8.0498 991.9831 1.0481 62.0161]

- C = [-1 1 0 0]:

  K = [-976.0928 984.0624 -59.7097 60.7886]

c) Using the formula for $k_{ref} = -(C(A - BK)^{-1}B)^{-1}$

- $k_{ref} = 1000$
- $k_{ref} = 1000$
- $k_{ref} = 1000$

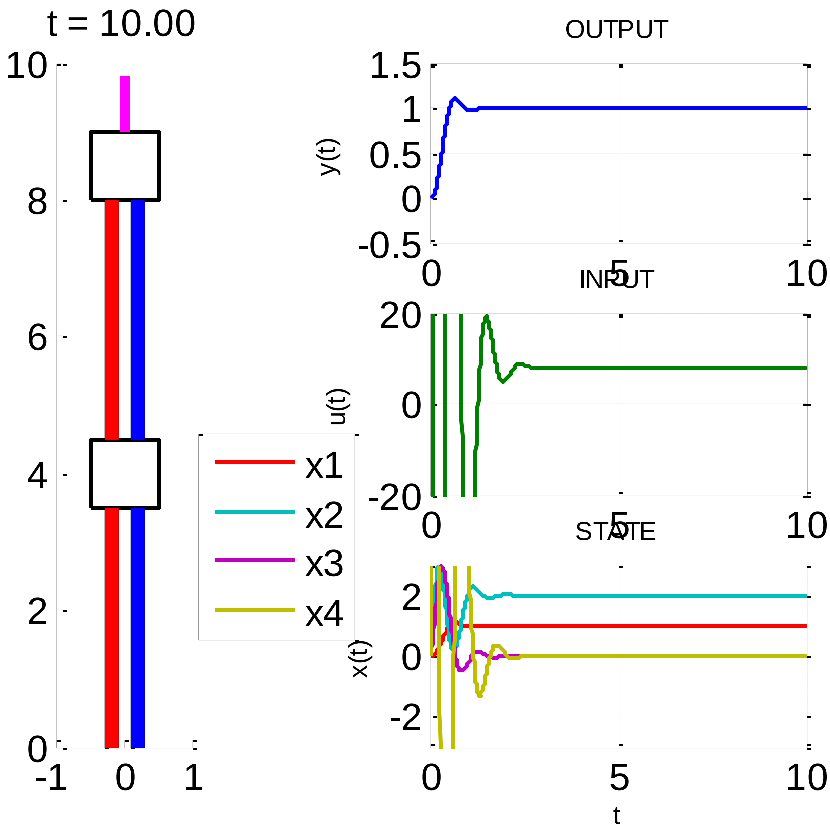
d) The following lines of code can be added:

```
x = zeros(4,size(t,2));
y = zeros(1,size(t,2));
u = zeros(1,size(t,2));

A = [0 0 1 0; 0 0 0 1; -8 4 -1 0.5; 4 -4 0.5 -0.5];
B = [0; 0; 0; 0.5];
C = [1 0 0 0];

Q = C'*C;   R = 10^-6;
```

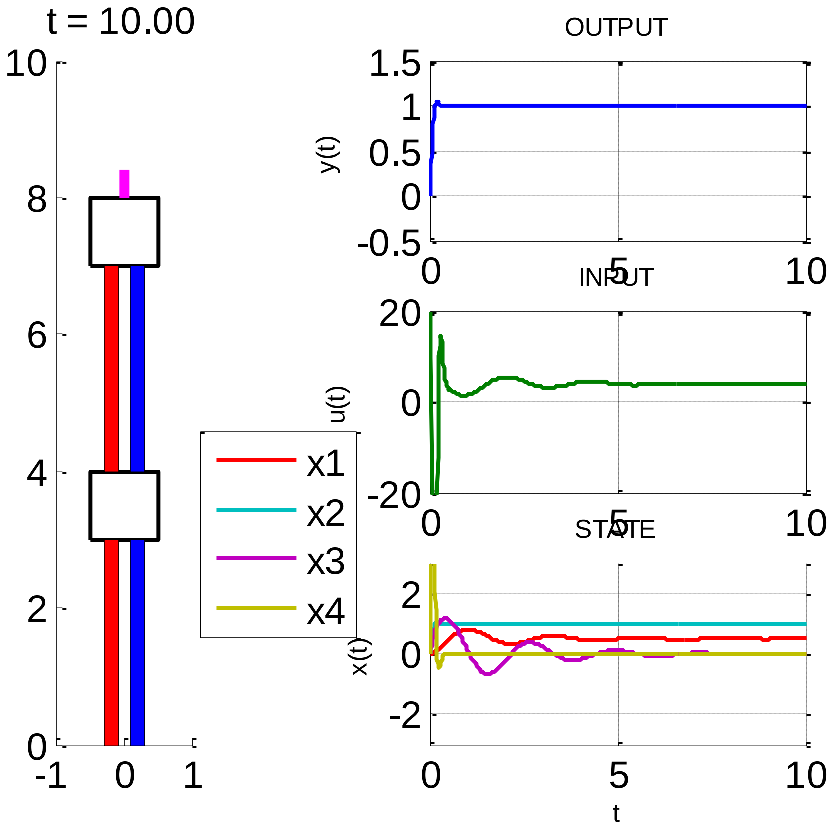
```
K = lqr(A,B,Q,R);

kref = - inv(C/(A-B*K)*B);

Acl = A - B*K;  Bcl = B*kref;   Ccl = C;

[y,~,x] = step(ss(Acl,Bcl,Ccl,0),t);

y = y'; x = x'; u = -K*x + kref*1;
```

The following plots are obtained for the three cases:

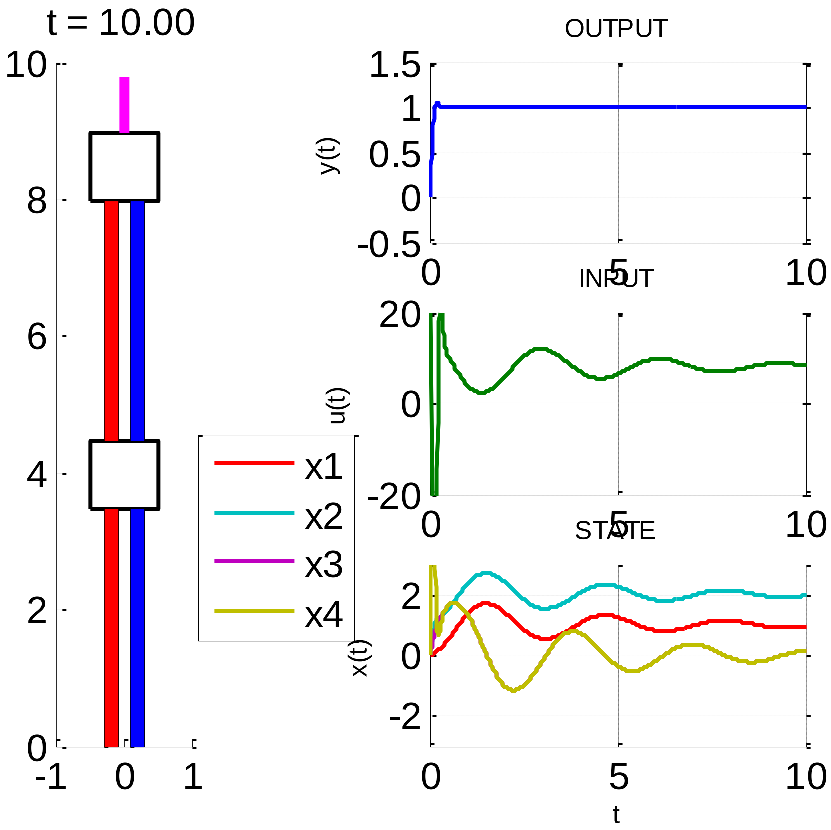
- C = [1 0 0 0]:



- C = [0 1 0 0]:


t = 10.00
OUTPUT
y(t)
INPUT
u(t)
STATE
x(t)
t
x1
x2
x3
x4

- C = [-1 1 0 0]:


t = 10.00
OUTPUT
y(t)
INPUT
u(t)
STATE
x(t)
t
x1
x2
x3
x4

2. State-space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{5} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} u$$

$$y = [1\ 0]x$$

a) We have $Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = [0.1]$

Using the lqr command in MATLAB, we obtain the following gain matrix:

K = [31.6228   17.0894]

b) Using the formula: $k_{ref} = -(C(A - BK)^{-1}B)^{-1}$, we get $k_{ref} = 31.6228$

c) Closed loop matrix:

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ -6.3246 & -3.6179 \end{bmatrix}$$

Eigenvalues for this matrix are: $\lambda = -1.8089 \pm 1.7471j = -\sigma \pm \omega j$

From the previous homework, the time to peak can be approximately taken as:

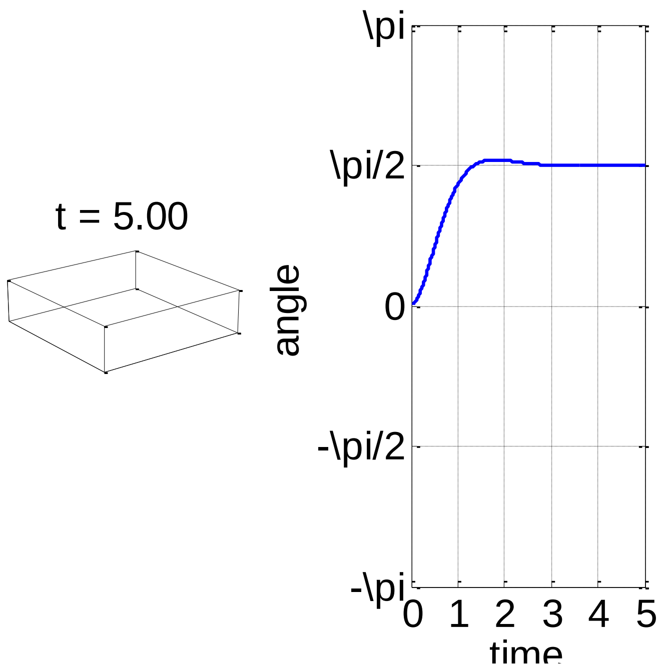
$$t_p = \frac{\pi}{\omega} = \frac{\pi}{1.7471} = 1.7982\ s$$

The peak overshoot can be taken as:

$$M_p = e^{-\frac{\pi\sigma}{\omega}} = 0.0387$$

d) Lines of code added to the script:

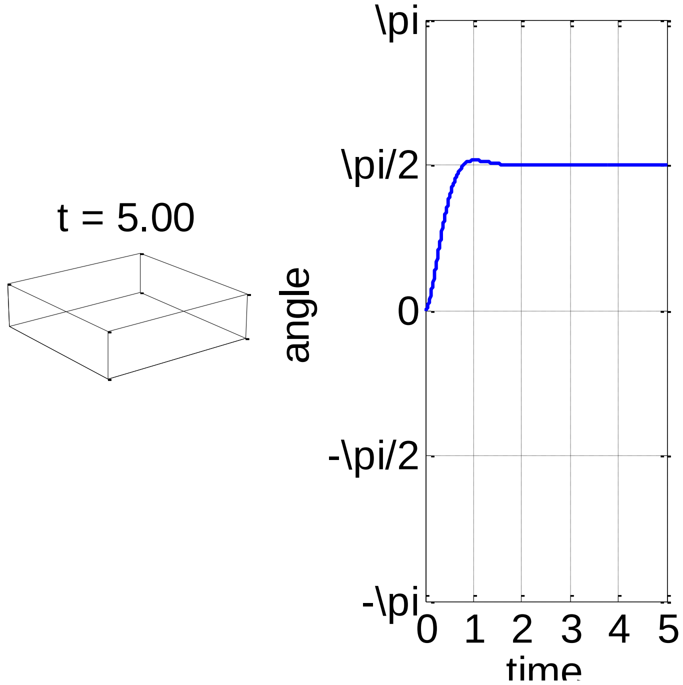
```
t = 0:1e-2:5;

A = [0 1; 0 -0.2];   B = [0; 0.2];
C = [1 0];

Q = [100 0; 0 1];    R = 0.1;

K = lqr(A,B,Q,R);

kref = - inv(C/(A-B*K)*B);

Acl = A-B*K;     Bcl = B*kref*pi/2;    Ccl = C;

[y,~,~] = step(ss(Acl,Bcl,Ccl,0),t);
```



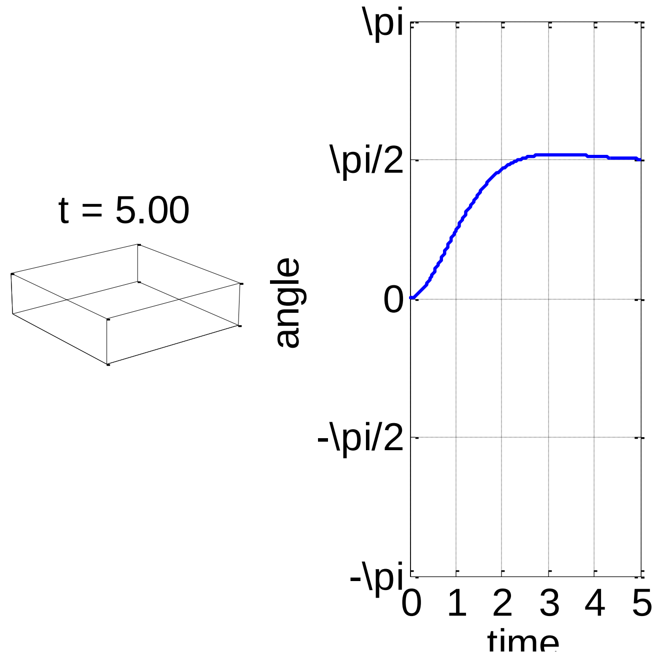
Yes, the plot seems to match the time to peak and peak magnitude that we obtained earlier.

e) A decrease in R should reduce the time to peak. While solving the LQR, we find a u which minimises the integral.

If we wish to get the same integral value, while decreasing R, u should have a larger value. This implies the system is given a larger input magnitude. As a result the system should also rise quicker. Below is the plot for R = 0.01:



The plot with R = 1:



You can see from the plots that we guessed correctly.

3. State-space form:

$$\dot{x} = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

a) The first term in the LQR cost function can be written as:

$$\|x\|^2 = (x_1^2 + x_2^2) = [x_1 \quad x_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore we have:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \rho$$

b) We have the A, B and Q matrices. We can try different values of ρ and use the lqr command in MATLAB to find the corresponding K matrix. Then we have to check for the closed loop eigenvalues, which means: eig(A-B*K).

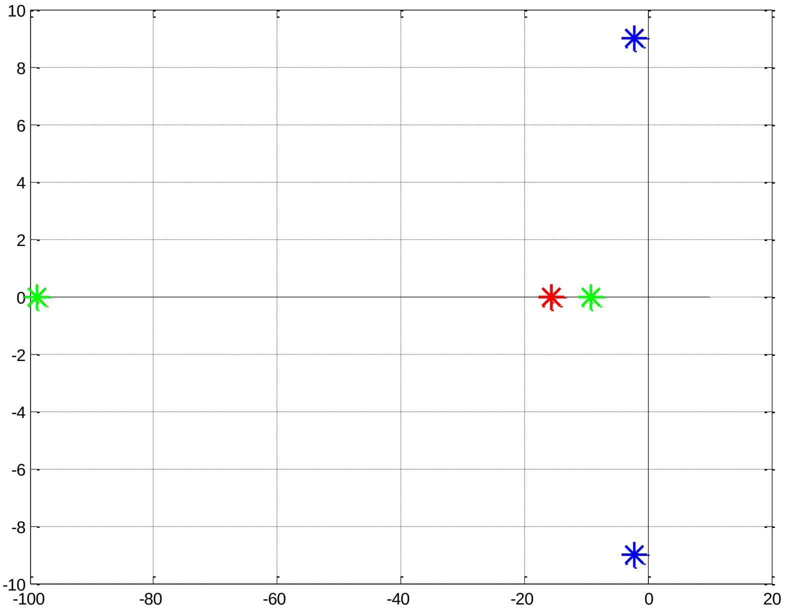
The eigenvalues can be checked by using the following command in MATLAB, where you keep varying R:

eig(A-B*lqr(A,B,Q,R))

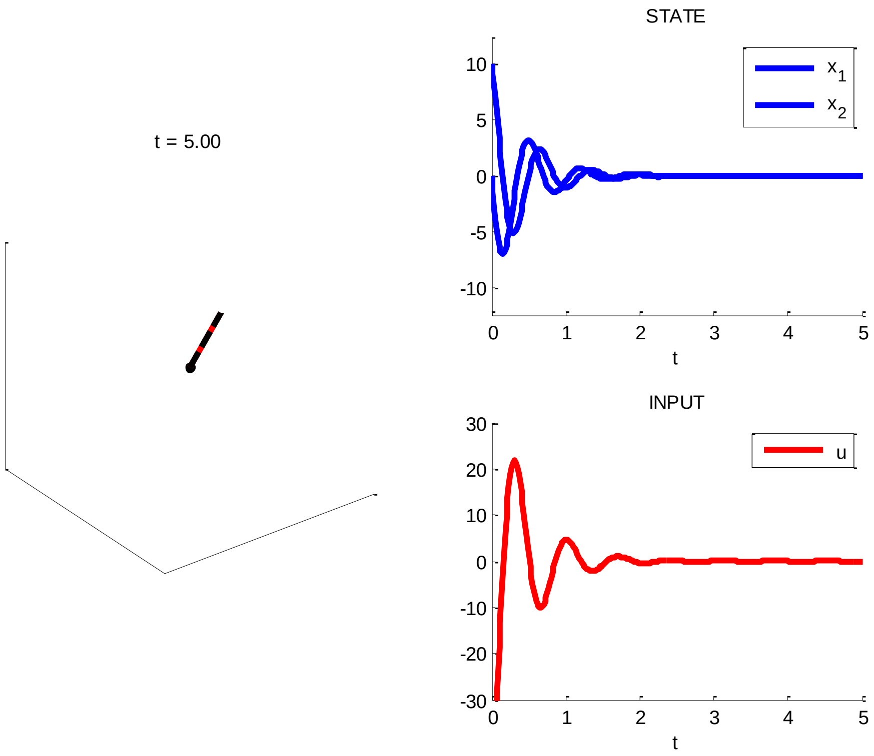
- For R = 0.1, we get eigenvalues: $\lambda = -2.2197 \pm 8.9960j$
- For R = 1.5432e-3, we get eigenvalues: $\lambda = -15.6148, -15.5622$
- For R = 10^{-4}, we get eigenvalues: $\lambda = -9.1495, -98.7638$

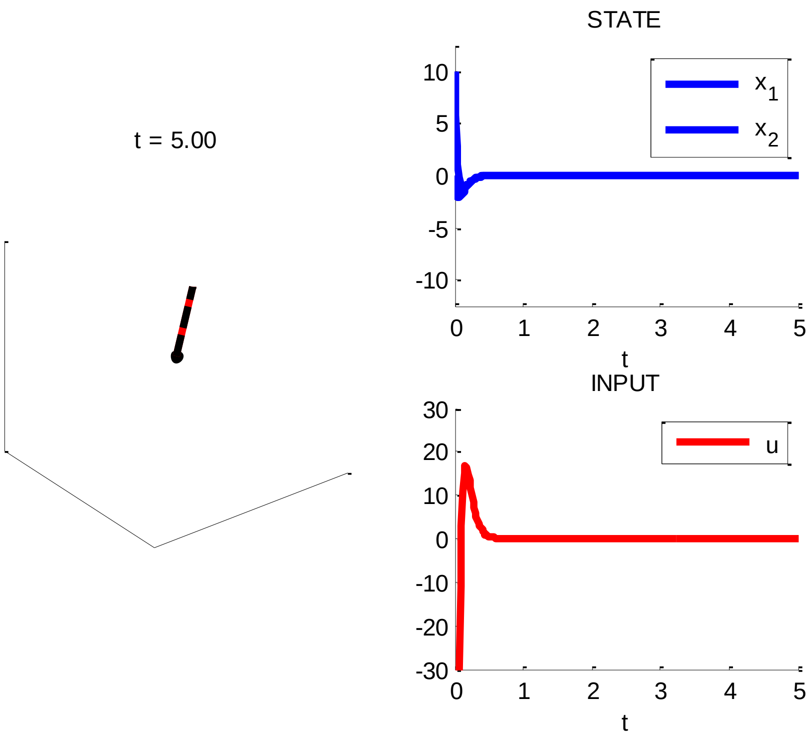
Figure showing all the eigenvalues:

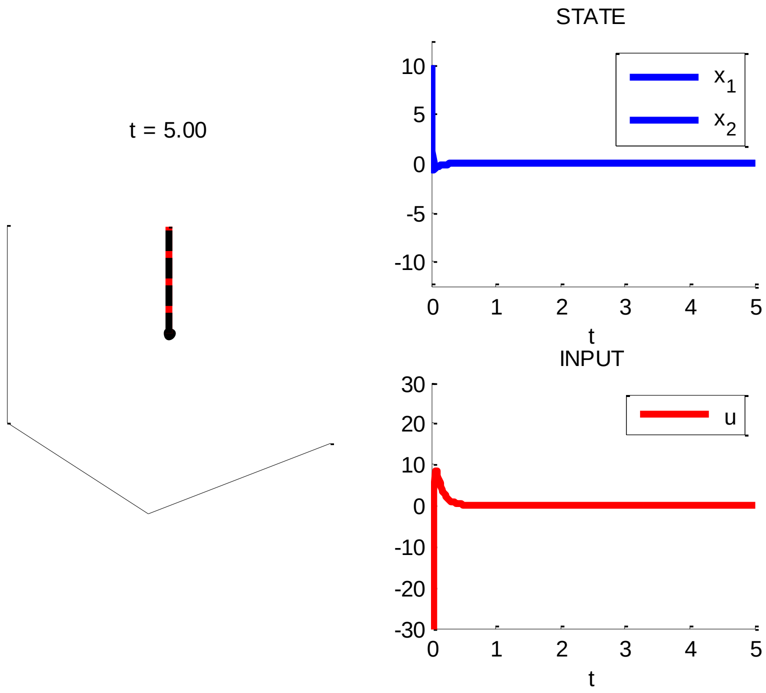
Blue: Under-damped system
Red: Critically damped system
Green: Over-damped system



c) The closed loop response for each of the cases above looks like:







From the plots we can see that as R decreases the response of the system becomes faster.

4. Optimal control problem:

$$\begin{aligned} \underset{u_{[t_0,t_1]}}{\text{minimize}} \quad & m x(t_1)^2 + \int_{t_0}^{t_1} \left(q x(t)^2 + r u(t)^2 \right) dt \\ \text{subject to} \quad & \frac{dx(t)}{dt} = a x(t) + b u(t) \\ & x(t_0) = x_0 \end{aligned}$$

a) On comparing the above form to:

$$\begin{aligned} \underset{u_{[t_0,t_1]}}{\text{minimize}} \quad & h(x(t_1)) + \int_{t_0}^{t_1} g(x(t), u(t)) dt \\ \text{subject to} \quad & \frac{dx(t)}{dt} = f(x(t), u(t)) \\ & x(t_0) = x_0. \end{aligned}$$

We get:

$$f\big(x(t), u(t)\big) = a x(t) + b u(t)$$

$$g\big(x(t), u(t)\big) = q x(t)^2 + r u(t)^2$$

$$h\big(x(t_1)\big) = m x(t_1)^2$$

b) Partial derivatives of the expression $v(t, x) = p(t) x^2$ gives:

$$\frac{\partial v}{\partial t} = \frac{\partial p(t)}{\partial t} x^2 = \dot{p} x^2$$

$$\frac{\partial v}{\partial x} = 2 p(t) x$$

c) Plugging all the above into:

$$-\frac{\partial v(t,x)}{\partial t} = \min_u \left\{ \frac{\partial v(t,x)}{\partial x} f(x,u) + g(x,u) \right\}$$

We get:

$$-\dot{p} x^2 = \min_u \{ 2px(ax + bu) + qx^2 + ru^2 \}$$

d) The right-hand side of the above equation is a quadratic function in u. Let's differentiate w.r.t. u to get the minimum:

$$\frac{d\,\{ 2px(ax + bu) + qx^2 + ru^2 \}}{du} = 2pxb + 2ru$$

Equating this to 0, we get: $u = -\frac{pxb}{r}$. At this u, the above function has a maxima/minima.

We must check the 2nd derivative to determine whether this is the minima or maxima:

$$\frac{d\,\{ 2pxb + 2ru \}}{du} = 2r > 0$$

Since r > 0, we can say that the function is minimum at that point.

e) From the optimal u we obtained above, we can say $k = \frac{pb}{r}$

f) We have the expression for u which minimises the right-hand side of (5).

$$u = -\frac{pb}{r}x$$

Substituting this in equation (5), we get:

$$-\dot{p}x^2 = 2pax^2 + 2pbx\left(-\frac{pb}{r}x\right) + qx^2 + r\left(-\frac{pb}{r}x\right)^2$$

Cancelling the common x^2:

$$-\dot{p} = 2ap - \frac{2b^2}{r}p^2 + q + \frac{b^2}{r}p^2$$

$$\dot{p} = \frac{b^2}{r}p^2 - 2ap - q$$

For the boundary condition, we know that $v(t,x) = p(t)x^2$ and $v(t_1,x) = mx^2$.

$$v(t_1,x) = p(t_1)x^2 = mx^2$$

Therefore, $p(t_1) = m$.

g) At steady-state, we know that $\dot{p} = 0$. Thus we have the following quadratic equation for p:

$$\frac{b^2}{r}p^2 - 2ap - q = 0$$

$$p^2 - \frac{2ar}{b^2}p - \frac{qr}{b^2} = 0$$

Which gives:

$$p = \frac{a \pm \sqrt{a^2 + \frac{q}{r}b^2}}{\frac{b^2}{r}}$$

Thus:

$$k = \frac{b}{r}p = \frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 + \frac{q}{r}}$$

h) As $(q/r) \to 0$, we get:

$$k = \frac{2a}{b}, 0$$

For k = 0, the controller will not provide any input at all. For $k = \frac{2a}{b}$, the controller should provide control of the form: $u = -\frac{2a}{b}x$ which should stabilise the system.

i) As $(q/r) \rightarrow \infty$, we get:

$$k = \pm\infty$$

An infinite gain is not realizable in practical cases. However, theoretically the controller will provide an infinite correction and bring the system to steady state instantly. We can observe this in plots in Q3. As we keep reducing R, the system takes a smaller amount of time to reach steady state.