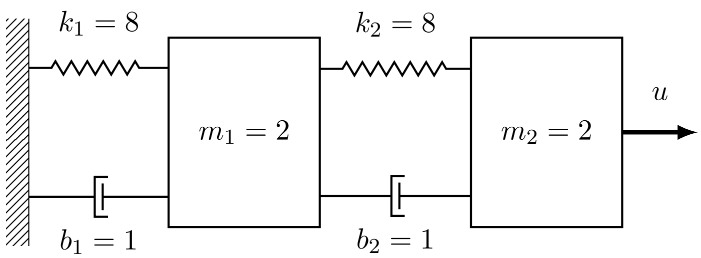
AE353 Homework #4: Optimal Control Design

(due at the beginning of class on Friday, February 27)



1. You have seen that the spring-mass-damper system shown above can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 4 & -1 & 0.5 \\ 4 & -4 & 0.5 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u$$

where x_1 and x_2 are the absolute displacements of each mass from their equilibrium positions, x_3 and x_4 are the corresponding velocities of each mass, and u is the applied force. In what follows, consider an input of the form

$$u = -Kx + k_{\text{reference}} r$$

where K and $k_{\text{reference}}$ are gains and r is a reference signal.

(a) Suppose

$$y = Cx$$

and we want to design optimal state feedback that minimizes the cost

$$\int_0^\infty \left(\|y\|^2 + \rho u^2 \right) dt. \tag{1}$$

Find the matrices Q and R for which (1) is equivalent to the standard LQR cost

$$\int_0^\infty \left(x^T Q x + u^T R u \right) dt.$$

(Your answer will be in terms of C and ρ.)

(b) Find the optimal choice of K for Q and R as defined in part (a), given $\rho = 10^{-6}$ and ...

- the output is the displacement x_1 of the first mass;
- the output is the displacement x_2 of the second mass;
- the output is the difference $x_2 - x_1$, i.e., the amount of stretch in the second spring.

(c) Compute the gain $k_{\text{reference}}$ so that $y = r$ in steady-state for each of the three choices of K that you found in part (b).

(d) Compute and visualize the step response of the closed-loop system for each of the three choices of K and $k_{\text{reference}}$ that you found in parts (b)-(c) using the script `hw4prob01.m`. Submit only the lines of code you added to this script and a snapshot of the figure after each simulation has ended. Briefly explain what you observed.

2. You have seen that the motion of a robot arm with one revolute joint can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1/5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements are the angle (x_1) and the angular velocity (x_2) of the joint, and the input u is a torque applied to the arm at the joint. In what follows, consider an input of the form

$$u = -Kx + k_{\text{reference}} r$$

where K and $k_{\text{reference}}$ are gains and r is a reference signal.

(a) *Feedback design.* Choose K to minimize the standard LQR cost

$$\int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

where

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 0.1 \end{bmatrix}.$$

(b) *Feedforward design.* Compute the gain $k_{\text{reference}}$ so that $y = r$ in steady-state.

(c) *Analysis.* Compute the closed-loop eigenvalues. Using this result, predict the time to peak and the peak overshoot of the unit step response.

(d) *Simulation.* Compute the response of the closed-loop system to a reference signal

$$r(t) = \pi/2 \qquad \text{for all } t \geq 0$$

using the script `hw4prob02.m`. Submit only the lines of code you added to this script and a snapshot of the figure after the simulation has ended. Are your results consistent with the prediction you made in part (c)?

(e) *Design Iteration.* Will an increase or a decrease in R reduce the time to peak? Check your guess with analysis and simulation.

3. You have seen that the rotational motion of an axisymmetric spacecraft about its yaw and roll axes can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

where the state elements x_1 and x_2 are the angular velocities about yaw and roll axes, the input u is an applied torque, and the parameter $\lambda = 9$ is the relative spin rate. Last week, you showed that this system was controllable, and applied state feedback

$$u = -Kx$$

to place the closed-loop eigenvalues in various locations. This week, you will apply *optimal* state feedback, where K is chosen to minimize the cost

$$\int_0^\infty \left(\|x\|^2 + \rho u^2 \right) dt \tag{2}$$

for some $\rho > 0$, where $\|x\|$ is the length of x (i.e., the standard Euclidean "2-norm").

(a) Find the matrices Q and R for which (2) is equivalent to the standard LQR cost

$$\int_0^\infty \left(x^T Q x + u^T R u\right) dt.$$

(b) Find a value of ρ for which the optimal K results in a closed-loop system that is:

- under-damped (i.e., has eigenvalues that are complex conjugates);
- critically-damped (i.e., has eigenvalues that are approximately equal);
- over-damped (i.e., has eigenvalues that are real and distinct).

On a single figure, plot the three sets of closed-loop eigenvalues.

(c) For each of the cases in part (b), plot the closed-loop state response $x(t)$ and input response $u(t)$ for the initial condition

$$x_0 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

using the script `hw4prob03.m`. This script does everything for you—all you need to do is put it in your working directory and call `hw4prob03(K,x0)` with an appropriate choice of gain matrix `K` and initial condition `x0`. Submit only a snapshot of the figure after each simulation has ended. Briefly explain how the response changes with ρ.

4. In this problem, you will apply the Hamilton-Jacobi-Bellman equation to derive an optimal controller for a scalar state-space system. As you have seen in class, this controller can be found by solving the following optimal control problem:

$$\begin{aligned} \underset{u_{[t_0,t_1]}}{\text{minimize}} \quad & m x(t_1)^2 + \int_{t_0}^{t_1} \left(q x(t)^2 + r u(t)^2\right) dt \\ \text{subject to} \quad & \frac{dx(t)}{dt} = a x(t) + b u(t) \\ & x(t_0) = x_0 \end{aligned} \tag{3}$$

where $q \geq 0$, $r > 0$, and $m \geq 0$. This optimal control problem has the general form

$$\begin{aligned} \underset{u_{[t_0,t_1]}}{\text{minimize}} \quad & h(x(t_1)) + \int_{t_0}^{t_1} g(x(t), u(t)) dt \\ \text{subject to} \quad & \frac{dx(t)}{dt} = f(x(t), u(t)) \\ & x(t_0) = x_0. \end{aligned} \tag{4}$$

The Hamilton-Jacobi-Bellman equation for a problem of this form is

$$-\frac{\partial v(t,x)}{\partial t} = \min_u \left\{ \frac{\partial v(t,x)}{\partial x} f(x,u) + g(x,u) \right\}, \tag{5}$$

where $v(t,x)$ is the value function. Since at time t_1 we clearly have

$$v(t_1, x) = h(x) = m x^2, \tag{6}$$

it makes sense to guess a value function of the form

$$v(t,x) = p(t) x^2, \tag{7}$$

where p is some function of time that remains to be derived. Proceed as follows:

(a) Find f, g, and h by comparing (3) with (4).

(b) Find $\partial v/\partial t$ and $\partial v/\partial x$ by taking partial derivatives of (7).

(c) Plug f, g, h, $\partial v/\partial t$, and $\partial v/\partial x$ into (5).

(d) Minimize the right-hand side of (5) with respect to u, either by "completing the square" or by applying the "first and second derivative test."

(e) Use your result from part (d) to find the gain k for which $u = -kx$ is the optimal input. Your answer should depend on p.

(f) Use your result from part (d) to write an ordinary differential equation that must be satisfied by p. (Hint: equate coefficients.) Write the boundary condition for this ODE by equating coefficients of (6) with $h(t_1)$.

(g) If t_1 is finite, then the gain k is time-varying. Suppose $t_1 \to \infty$. It is a remarkable fact that p—hence, k—tends to a steady-state value. Write an equation that characterizes this steady-state value. It should be quadratic in p, and should look familiar to you. You have now found the "steady-state" or "infinite-horizon" optimal controller—it is characterized by the quadratic equation that you derived just now for p and by the expression for k in terms of p that you derived in part (e).

CONGRATULATIONS! THIS IS A HUGE RESULT.

(h) What happens to k as $(q/r) \to 0$ and how will the controller behave?

(i) What happens to k as $(q/r) \to \infty$ and how will the controller behave?