1 Examples (2/4/15)

Consider the system

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + B_m u_m(t) \\ y_m(t) &= C_m x_m(t). \end{aligned} \tag{1}$$

As you know, the solution to this system is

$$y_m(t) = C_m e^{A_m t} x_m(0) + C_m \int_0^t e^{A_m(t-\tau)} B_m u_m(\tau) d\tau.$$

Note that, in this expression, $u_m(\tau)$ means "the input u_m at time τ" but $A_m(t-\tau)$ means "the matrix A_m multiplied by the quantity $t-\tau$." I acknowledge that this notation—although standard—is ambiguous. It requires you to understand which quantities are functions of time and which aren't—u_m is a function of time, while A_m is not. As you have seen, this solution simplifies to

$$y_m(t) = C_m e^{A_m t} x_m(0) + C_m A_m^{-1} \left(e^{A_m t} - I \right) B_m \bar{u}_m \tag{2}$$

in the particular case when A_m is invertible and $u_m(t) = \bar{u}_m$ is constant.

One thing to note, which we did not discuss in class, is that the formula (2) remains valid even when there are multiple inputs (i.e., $\bar{u}_m$ is a vector) and multiple outputs (i.e., $y_m(t)$ is a vector). In particular, consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with an input of the form

$$u = -Kx + k_{\text{reference}} r + d.$$

We can rewrite this input as

$$u = -Kx + \begin{bmatrix} k_{\text{reference}} & 1 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}.$$

Plugging in, the resulting system is

$$\begin{aligned} \dot{x} &= (A - BK)x + B \begin{bmatrix} k_{\text{reference}} & 1 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} \\ y &= Cx. \end{aligned} \tag{3}$$

Compare (1) with (3):

$$x_m = x \qquad u_m = \begin{bmatrix} r \\ d \end{bmatrix} \qquad y_m = y$$
$$A_m = A - BK \qquad B_m = B \begin{bmatrix} k_{\text{reference}} & 1 \end{bmatrix} \qquad C_m = C.$$

Plugging these things in, we can evaluate (2) to solve (3):

$$\begin{aligned} y(t) &= Ce^{(A-BK)t}x(0) + C\left(A - BK\right)^{-1}\left(e^{(A-BK)t} - I\right) B \begin{bmatrix} k_{\text{reference}} & 1 \end{bmatrix} \begin{bmatrix} \bar{r} \\ \bar{d} \end{bmatrix} \\ &= Ce^{(A-BK)t}x(0) + C\left(A - BK\right)^{-1}\left(e^{(A-BK)t} - I\right) B \left(k_{\text{reference}}\bar{r} + \bar{d}\right) \end{aligned}$$

Note that, at this point, it is very easy to solve (3) for the three cases that we have considered in class and on the homework—response to initial conditions ($r = 0$ and $d = 0$), response to a reference signal ($x(0) = 0$ and $d = 0$), and response to a disturbance load ($x(0) = 0$ and $r = 0$). Note also that, as was mentioned briefly in class, the "total" response to all three things taken at once is just the sum of the response to each thing taken separately.

As an example, suppose

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 7 & 10 \end{bmatrix}$$

and

$$\begin{aligned} k_{\text{reference}} &= -\left(C(A - BK)^{-1}B\right)^{-1} \\ &= -\left(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & -10 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{-1} \\ &= -\left(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1/10 & -7/10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{-1} \\ &= -\left(\begin{bmatrix} -1/10 & -7/10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{-1} \\ &= -\left(-1/10\right)^{-1} \\ &= 10 \end{aligned}$$

and

$$x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \bar{r} = 1 \qquad \bar{d} = 1.$$

We have seen previously (Section 2.1) that

$$(A - BK)V = V\Lambda$$

for

$$V = \begin{bmatrix} -5 & -2 \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}$$

and so

$$\begin{aligned} e^{(A-BK)t} &= V e^{\Lambda t} V^{-1} \\ &= \begin{bmatrix} -5 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1/3 & -2/3 \\ 1/3 & 5/3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 5e^{-5t} - 2e^{-2t} & 10e^{-5t} - 10e^{-2t} \\ -e^{-5t} + e^{-2t} & -2e^{-5t} + 5e^{-2t} \end{bmatrix}. \end{aligned}$$

Plugging in, we have

$$\begin{aligned} y(t) = &\frac{1}{3} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 5e^{-5t} - 2e^{-2t} & 10e^{-5t} - 10e^{-2t} \\ -e^{-5t} + e^{-2t} & -2e^{-5t} + 5e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1/10 & -7/10 \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 5e^{-5t} - 2e^{-2t} & 10e^{-5t} - 10e^{-2t} \\ -e^{-5t} + e^{-2t} & -2e^{-5t} + 5e^{-2t} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} (10 + 1) \\ = &\frac{1}{3} \left(e^{-5t} - e^{-2t} \right) + \frac{11}{30} \left(3 + 2e^{-5t} - 5e^{-2t} \right) \\ = &\frac{11}{10} + \frac{1}{30} \left(32e^{-5t} - 65e^{-2t} \right). \end{aligned}$$

In the script `ex02042015.m`, available on the course website, this solution is compared to what would have been obtained (1) by evaluating the matrix exponential function in MATLAB and (2) by using the functions “initial” and “step” in MATLAB. Results are shown in Figure 1.

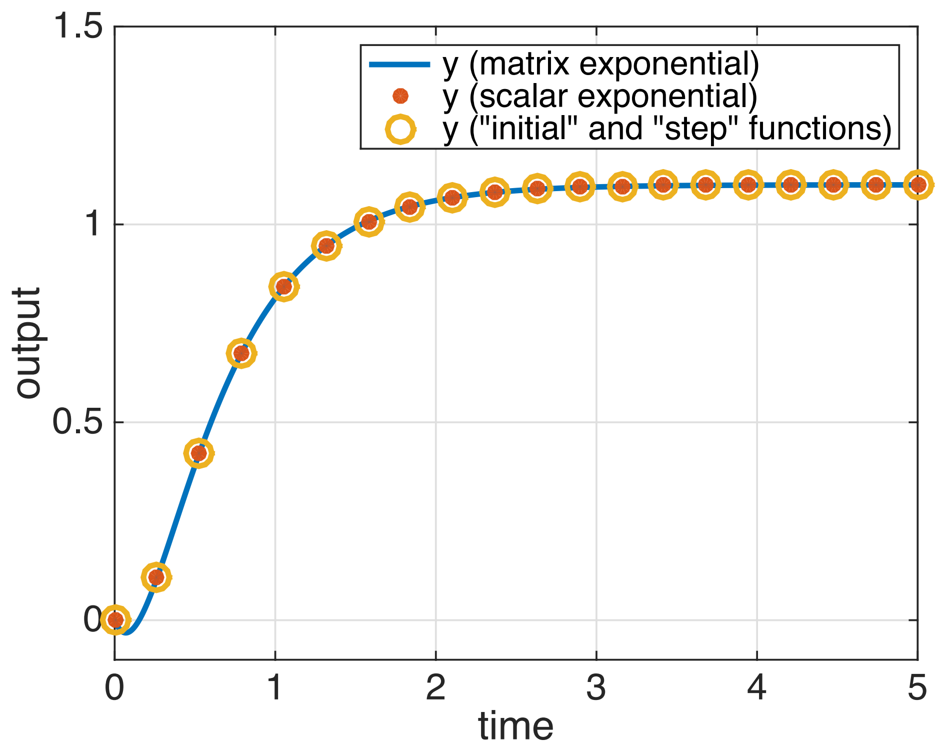

Figure 1: Agreement between three different ways of computing the output in response to a non-zero initial condition, a non-zero reference signal, and a non-zero disturbance load.

2 Examples (1/28/15)

Consider the system

$$\begin{aligned}\dot{x} &= Ax\\ y &= Cx.\end{aligned}$$

As you know, the solution to this system is

$$y(t) = Ce^{At}x(0),$$

where

$$e^{At}$$

is the matrix exponential function. In what follows, we will express this solution in terms of scalar exponential functions for three different values of A. We will also say whether the corresponding system is asymptotically stable. In all three cases, we will assume that

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and

$$x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

2.1 Stable system, real eigenvalues

Suppose

$$A = \begin{bmatrix} -7 & -10 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A

Eigenvalues of A are solutions to

$$0 = \det\left(\lambda I - A\right).$$

We have

$$\begin{aligned}0 &= \det\left(\lambda I - A\right)\\ &= \begin{vmatrix} \lambda + 7 & 10 \\ -1 & \lambda \end{vmatrix}\\ &= \lambda^2 + 7\lambda + 10\\ &= (\lambda + 5)(\lambda + 2).\end{aligned}$$

So, the eigenvalues of A are $\lambda_1 = -5$ and $\lambda_2 = -2$. Note that, at this point, we already know that the system is asymptotically stable—both eigenvalues have negative real part.

Find the eigenvectors of A

Eigenvectors of A are solutions v to

$$0 = (\lambda I - A)v.$$

First, we find the eigenvector

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_1 = -5$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_1 I - A)v_1 \\ &= \begin{bmatrix} 2 & 10 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \begin{bmatrix} 2\alpha + 10\beta \\ -\alpha - 5\beta \end{bmatrix}. \end{aligned}$$

As we have discussed, the "top" and "bottom" rows of this equation are redundant—they both require that

$$\alpha = -5\beta,$$

where we may choose β to be any non-zero real number. Suppose we choose $\beta = 1$. Then

$$v_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

Second, we find the eigenvector

$$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_2 = -2$. We have

$$\begin{aligned}\begin{bmatrix}0\\0\end{bmatrix} &= (\lambda_2 I - A)v_2\\ &= \begin{bmatrix}5 & 10\\ -1 & -2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\\ &= \begin{bmatrix}5\alpha + 10\beta\\ -\alpha - 2\beta\end{bmatrix},\end{aligned}$$

from which we conclude that

$$\alpha = -2\beta.$$

Choosing $\beta = 1$, we have

$$v_2 = \begin{bmatrix}-2\\1\end{bmatrix}.$$

Find $y(t)$

Define

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}-5 & 0\\ 0 & -2\end{bmatrix}$$

and

$$V = \begin{bmatrix}v_1 & v_2\end{bmatrix} = \begin{bmatrix}-5 & -2\\ 1 & 1\end{bmatrix}.$$

We compute $y(t)$ as follows:

$$\begin{aligned}y(t) &= CVe^{\Lambda t}V^{-1}x(0)\\ &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-5 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{-5t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}-1/3 & -2/3\\ 1/3 & 5/3\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\\ &= \frac{1}{3}\left(-4e^{-5t} + 7e^{-2t}\right).\end{aligned}$$

This result confirms our earlier statement that the system is asymptotically stable—as $t \to \infty$, both scalar exponential terms near zero, so $y(t) \to 0$.

2.2 Unstable system, real eigenvalues

Suppose

$$A = \begin{bmatrix}-1 & 6\\ 1 & 0\end{bmatrix}.$$

Find the eigenvalues of A

We have

$$\begin{aligned} 0 &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda + 1 & -6 \\ -1 & \lambda \end{vmatrix} \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2). \end{aligned}$$

So, the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 2$. Note that, at this point, we already know that the system is *not* asymptotically stable—one eigenvalue does *not* have negative real part.

Find the eigenvectors of A

First, we find the eigenvector

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_1 = -3$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_1 I - A) v_1 \\ &= \begin{bmatrix} -2 & -6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \begin{bmatrix} -2\alpha - 6\beta \\ -\alpha - 3\beta \end{bmatrix}, \end{aligned}$$

from which we conclude

$$\alpha = -3\beta.$$

Choosing $\beta = 1$, we have

$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Second, we find the eigenvector

$$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_2 = 2$. We have

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_2 I - A) v_2 \\
&= \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= \begin{bmatrix} 3\alpha - 6\beta \\ -\alpha + 2\beta \end{bmatrix},
\end{aligned}$$

from which we conclude that

$$\alpha = 2\beta.$$

Choosing $\beta = 1$, we have

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find $y(t)$

Define

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}.$$

We compute $y(t)$ as follows:

$$\begin{aligned}
y(t) &= CVe^{\Lambda t}V^{-1}x(0) \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \frac{1}{5}\left(0e^{-3t} + 5e^{2t}\right) \\
&= e^{2t}.
\end{aligned}$$

2.3 Stable system, complex eigenvalues

Suppose

$$A = \begin{bmatrix} -10 & -169 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A

We have

$$\begin{aligned} 0 &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda + 10 & 169 \\ -1 & \lambda \end{vmatrix} \\ &= \lambda^2 + 10\lambda + 169. \end{aligned}$$

This expression is not easily factored, so we solve it using the quadratic formula:

$$\begin{aligned} \lambda &= \frac{-10 \pm \sqrt{10^2 - 4(169)}}{2} \\ &= -5 \pm \sqrt{25 - 169} \\ &= -5 \pm \sqrt{-144} \\ &= -5 \pm j12. \end{aligned}$$

So, the eigenvalues of A are $\lambda_1 = -5 - j12$ and $\lambda_2 = -5 + j12$. Note that, at this point, we already know that the system is asymptotically stable—both eigenvalues have negative real part.

Find the eigenvectors of A

First, we find the eigenvector

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_1 = -5 - j12$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_1 I - A) v_1 \\ &= \begin{bmatrix} 5 - j12 & -169 \\ -1 & -5 - j12 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \end{aligned}$$

from which we conclude

$$\alpha = (-5 - j12)\beta.$$

Choosing $\beta = 1$, we have

$$v_1 = \begin{bmatrix} -5 - j12 \\ 1 \end{bmatrix}.$$

Second, we find the eigenvector

$$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_2 = -5 + j12$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_2 I - A) v_2 \\ &= \begin{bmatrix} 5 + j12 & -169 \\ -1 & -5 + j12 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \end{aligned}$$

from which we conclude that

$$\alpha = (-5 + j12)\beta.$$

Choosing $\beta = 1$, we have

$$v_2 = \begin{bmatrix} -5 + j12 \\ 1 \end{bmatrix}.$$

Find $y(t)$

Define

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 - j12 & 0 \\ 0 & -5 + j12 \end{bmatrix}$$

and

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -5 - j12 & -5 + j12 \\ 1 & 1 \end{bmatrix}.$$

We compute $y(t)$ as follows:

$$\begin{aligned} y(t) &= C V e^{\Lambda t} V^{-1} x(0) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -5 - j12 & -5 + j12 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(-5-j12)t} & 0 \\ 0 & e^{(-5+j12)t} \end{bmatrix} \begin{bmatrix} j/24 & (12 + j5)/24 \\ -j/24 & (12 - j5)/24) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{1}{24} \left((12 + j7) e^{(-5-j12)t} + (12 - j7) e^{(-5+j12)t} \right) \\ &= \frac{1}{24} e^{-5t} \left((12 + j7) e^{-j12t} + (12 - j7) e^{j12t} \right). \end{aligned}$$

We can simplify this expression further by using the identity

$$e^{j\omega} = \cos\omega + j\sin\omega.$$

In particular, we have

$$e^{-j12t} = \cos(12t) - j\sin(12t)$$

and

$$e^{j12t} = \cos(12t) + j\sin(12t).$$

Consequently,

$$\begin{aligned}
&(12 + j7)e^{-j12t} + (12 - j7)e^{j12t} \\
&\qquad = (12 + j7)(\cos(12t) - j\sin(12t)) + (12 - j7)(\cos(12t) + j\sin(12t)) \\
&\qquad = (12\cos(12t) + 7\sin(12t)) + j(7\cos(12t) - 12\sin(12t)) \\
&\qquad\qquad + (12\cos(12t) + 7\sin(12t)) + j(-7\cos(12t) + 12\sin(12t)) \\
&\qquad = (24\cos(12t) + 14\sin(12t)),
\end{aligned}$$

and so

$$y(t) = \frac{1}{24}e^{-5t}\left(24\cos(12t) + 14\sin(12t)\right).$$