AE353 Homework #2:
State Space and the Matrix Exponential (Part 2)

(due at the beginning of class on Friday, February 6)

1. Last week, you found that the rotational motion of a control moment gyro on a spacecraft could be described in state-space form as

$$\dot{x} = \left[-b/m\right] x + \left[1/m\right] u$$
$$y = \left[1\right] x$$

where the state x is angular velocity, the input u is applied torque, and

$$m = 2 \qquad\qquad b = 1$$

are parameters. In what follows, please assume the input has the form

$$u = -3x + k_{\text{reference}} r + d$$

where r is a known reference signal and d is an unknown disturbance load. You may assume that both r and d are constant.

(a) *Reference tracking.* Suppose $d = 0$. Please do the following:

- Find $k_{\text{reference}}$ so that $y = r$ in steady-state.
- Rewrite the system as

$$\dot{x} = A_{cl} x + B_{cl} r$$
$$y = C_{cl} x$$

for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.

- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if $x(0) = 0$ and $r = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.
- Find the rise time, settling time, overshoot, and steady-state value of the step response from your MATLAB plot. Your answers can be approximate.

(b) *Disturbance rejection.* Suppose $r = 0$. Please do the following:

- Rewrite the system as

$$\dot{x} = A_{cl} x + B_{cl} d$$
$$y = C_{cl} x$$

for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.

- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if $x(0) = 0$ and $d = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.

- Find the steady-state error in response to a unit disturbance load. In other words, find the steady-state value of $y(t) - r$ if $x(0) = 0$ and $d = 1$. Do so by hand, verifying your result with the plot.

(c) *Disturbance rejection with integral action.* Again, suppose $r = 0$. But this time, consider the alternative input

$$u = -3x + k_{\text{reference}} r + d - k_{\text{integral}} v$$

where $k_{\text{integral}} = 34$ and where we define

$$\dot{v} = y - r.$$

Please do the following:

- Define

$$z = \begin{bmatrix} x \\ v \end{bmatrix}.$$

  Rewrite the system as

$$\begin{aligned} \dot{z} &= A_{cl} z + B_{cl} d \\ y &= C_{cl} z \end{aligned}$$

  for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.
- Is this system asymptotically stable? (Please do all computation by hand.)
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if

$$z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $d = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.
- Find the steady-state error in response to a unit disturbance load. In other words, find the steady-state value of $y(t) - r$ if

$$z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $d = 1$. Do so by hand, verifying your result with the plot.
- What differences are there between the results here and the results in (b)? Why?

2. Last week, you found that the rotational motion of an antenna on a spacecraft could be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -b/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements are angle (x_1) and angular velocity (x_2), the input u is an applied torque, and

$$m = 0.1 \qquad\qquad b = 0.5$$

are parameters. In what follows, please assume the input has the form

$$u = -\begin{bmatrix} 5 & 1 \end{bmatrix} x + k_{\text{reference}} r + d$$

where r is a known reference signal and d is an unknown disturbance load. You may assume that both r and d are constant.

(a) *Reference tracking.* Suppose $d = 0$. Please do the following:

- Find $k_{\text{reference}}$ so that $y = r$ in steady-state.
- Rewrite the system as

$$\dot{x} = A_{cl} x + B_{cl} r$$
$$y = C_{cl} x$$

  for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $r = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.
- Find the rise time, settling time, overshoot, and steady-state value of the step response from your MATLAB plot. Your answers can be approximate.

(b) *Disturbance rejection.* Suppose $r = 0$. Please do the following:

- Rewrite the system as

$$\dot{x} = A_{cl} x + B_{cl} d$$
$$y = C_{cl} x$$

  for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $d = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.

- Find the steady-state error in response to a unit disturbance load. In other words, find the steady-state value of $y(t) - r$ if

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $d = 1$. Do so by hand, verifying your result with the plot.

(c) *Disturbance rejection with integral action.* Again, suppose $r = 0$. But this time, consider the alternative input

$$u = -\begin{bmatrix} 5 & 1 \end{bmatrix} x + k_{\text{reference}} r + d - k_{\text{integral}} v$$

where $k_{\text{integral}} = 10$ and where we define

$$\dot{v} = y - r.$$

Please do the following:

- Define

$$z = \begin{bmatrix} x \\ v \end{bmatrix}.$$

Rewrite the system as

$$\begin{aligned} \dot{z} &= A_{cl} z + B_{cl} d \\ y &= C_{cl} z \end{aligned}$$

for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.

- Is this system asymptotically stable? (You may use MATLAB for the computation.)
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the steady-state error in response to a unit disturbance load. In other words, find the steady-state value of $y(t) - r$ if

$$z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $d = 1$. Do so by hand, verifying your result with the plot.

- What differences are there between the results here and the results in (b)? Why?

3. Last week, you found that the rotational motion of an axisymmetric spacecraft about its yaw and roll axes could be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements x_1 and x_2 are the angular velocities about yaw and roll axes, the input u is an applied torque, and the parameter $\lambda = 9$ is the relative spin rate. In what follows, please assume the input has the form

$$u = -\begin{bmatrix} 6 & -1 \end{bmatrix} x + k_{\text{reference}} r + d$$

where r is a known reference signal and d is an unknown disturbance load. You may assume that both r and d are constant.

(a) *Reference tracking.* Suppose $d = 0$. Please do the following:

- Prove that there exists no choice of $k_{\text{reference}}$ for which $y = r$ in steady-state.

  HINT: try to find $k_{\text{reference}}$ in the normal way and see what happens.

- Consider the alternative output

  $$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

  Find $k_{\text{reference}}$ so that $y = r$ in steady-state.

  Please continue to use this new output for the rest of the problem, including parts (a), (b), and (c).

- Rewrite the system as

  $$\dot{x} = A_{cl} x + B_{cl} r$$
  $$y = C_{cl} x$$

  for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if

  $$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $r = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.
- Find the rise time, settling time, overshoot, and steady-state value of the step response from your MATLAB plot. Your answers can be approximate.

(b) *Disturbance rejection.* Suppose $r = 0$. Please do the following:

- Rewrite the system as

$$\begin{aligned}\dot{x} &= A_{cl}x + B_{cl}d \\ y &= C_{cl}x\end{aligned}$$

  for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the step response of this system by hand. In other words, express $y(t)$ in terms of scalar exponential functions if

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $d = 1$. Use MATLAB to evaluate this expression, plotting it on the same figure as before.
- Find the steady-state error in response to a unit disturbance load. In other words, find the steady-state value of $y(t) - r$ if

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $d = 1$. Do so by hand, verifying your result with the plot.

(c) *Disturbance rejection with integral action.* Again, suppose $r = 0$. But this time, consider the alternative input

$$u = -\begin{bmatrix} 6 & -1 \end{bmatrix} x + k_{\text{reference}} r + d - k_{\text{integral}} v$$

where $k_{\text{integral}} = -5$ and where we define

$$\dot{v} = y - r.$$

Please do the following:

- Define

$$z = \begin{bmatrix} x \\ v \end{bmatrix}.$$

  Rewrite the system as

$$\begin{aligned}\dot{z} &= A_{cl}z + B_{cl}d \\ y &= C_{cl}z\end{aligned}$$

  for an appropriate choice of A_{cl}, B_{cl}, and C_{cl}.
- Is this system asymptotically stable? (You may use MATLAB for the computation.)
- Find and plot the step response of this system in MATLAB using "`step`."
- Find the steady-state error in response to a unit disturbance load. In other words, find the steady-state value of $y(t) - r$ if

$$z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and $d = 1$. Do so by hand, verifying your result with the plot.
- What differences are there between the results here and the results in (b)? Why?