AE353 Homework #3: Controllability and State Feedback

(due at the beginning of class on Friday, February 20)

1. The motion of a robot arm with one revolute joint can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -b/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements are the angle (x_1) and the angular velocity (x_2) of the joint, the input u is a torque applied to the arm at the joint, and

$$m = 5 \qquad\qquad b = 1$$

are parameters.

(a) Show that this system is controllable.

(b) Consider an input of the form

$$u = -Kx + k_{\text{reference}} r$$

where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

and r is a reference signal that you may assume is constant.

- *Feedback design.* Compute the gains k_1 and k_2 that would place the closed-loop eigenvalues at $-\sigma \pm j\omega$. You may assume that both $\sigma > 0$ and $\omega > 0$.
- *Feedforward design.* Compute the gain $k_{\text{reference}}$ so that $y = r$ in steady-state.

(c) **EXTRA CREDIT:**

- Show that the time to peak amplitude of the step response (i.e., the time that it takes for $y(t)$ to reach its *first* peak) is

$$t_p = \pi/\omega$$

- Show that the peak overshoot of the step response is

$$M_p = e^{-\pi\sigma/\omega}$$

(d) *Lines of constant time to peak.* Use part (c) to compute eigenvalue locations that would result in $t_p = 0.5$ and $M_p = 0.1$. Use part (b) to compute the corresponding gains. Use any method of simulation you want to compute the step response. Create a figure with two axes (e.g., `subplot(1,2,1)` and `subplot(1,2,2)` in MATLAB). On the first set of axes, plot the step response. On the second set of axes, plot the eigenvalue locations. Repeat for $M_p = 0.2$ and $M_p = 0.3$, keeping t_p constant and putting your results on the same two axes. What happens to the step response? What happens to the eigenvalue locations? (In what direction do they move with increasing M_p?)

(e) *Lines of constant peak overshoot.* Use part (c) to compute eigenvalue locations that would result in $t_p = 0.5$ and $M_p = 0.1$. Use part (b) to compute the corresponding gains. Use any method of simulation you want to compute the step response. Create a figure with two axes (e.g., `subplot(1,2,1)` and `subplot(1,2,2)` in MATLAB). On the first set of axes, plot the step response. On the second set of axes, plot the eigenvalue locations. Repeat for $t_p = 1.5$ and $t_p = 2.5$, keeping M_p constant and putting your results on the same two axes. What happens to the step response? What happens to the eigenvalue locations? (In what direction do they move with increasing t_p?)

(f) Suppose you are given a performance specification that requires $t_p < 1$ and $M_p < 0.15$.

- Sketch the region of the complex plane within which the eigenvalues must be located in order to meet this spec.
- Choose eigenvalue locations in this region (anywhere you want) and compute the corresponding gains.
- Compute *and visualize* the response of the closed-loop system to a reference signal

$$r(t) = \pi/2 \qquad\qquad \text{for all } t \geq 0$$

using the script `hw3prob01.m`. (This is exactly the same as the step response, but with a reference signal of magnitude $\pi/2$ instead of magnitude 1.) Submit only the lines of code you added to this script and a snapshot of the figure after the simulation has ended.

(You could, of course, use this same code to visualize the step responses you computed in the earlier parts of this problem, if you wanted.)

2. The rotational motion of an axisymmetric spacecraft about its yaw and roll axes can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

where the state elements x_1 and x_2 are the angular velocities about yaw and roll axes, the input u is an applied torque, and the parameter $\lambda = 9$ is the relative spin rate.

(a) Show that this system is controllable.

(b) Consider an input of the form

$$u = -Kx + k_{\text{reference}} r$$

where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

and r is a reference signal that you may assume is constant.

- *Feedback design.* Compute the gains k_1 and k_2 that would place the closed-loop eigenvalues at $-\sigma_1$ and $-\sigma_2$. You may assume that $\sigma_2 \geq \sigma_1 > 0$.
- *Feedforward design.* Compute the gain $k_{\text{reference}}$ so that $y = r$ in steady-state.

(c) *Dominant first-order response.* Use part (b) to compute the gains that place both closed-loop eigenvalues at -1. Use any method of simulation you want to compute and plot the step response. Repeat for eigenvalues at -1 and -5, then once more for eigenvalues at -1 and -10. Put all your results on the same axes. On these same axes, plot

$$y_{\text{firstorder}}(t) = 1 - e^{-t}$$

What happens to the step response as one of the eigenvalues moves farther out? Why would it make sense to say that, when $\sigma_1 = 1$ and $\sigma_2 = 10$, the system has a dominant first-order response with time constant $1/\sigma_1 = 1$?

(d) Suppose you are given a performance specification that requires a dominant first-order response with time constant $1/2$.

- Choose eigenvalues that meet this spec and compute the corresponding gains.
- Compute and visualize the response of the closed-loop system to an initial condition

$$x(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

using the script `hw3prob02.m`. Note that this script does everything for you—all you need to do is put `hw3prob02.m` in your working directory and call `hw3prob02(K,x0)` with an appropriate choice of gain matrix `K` and initial condition `x0`. Submit only a snapshot of the figure after the simulation has ended.

Although not required, you might consider the relationship between the step response (computed by you) and the response to initial conditions (computed by `hw3prob02.m`). How are these two things related?

3. Previously, we have studied the following model of the relationship between the applied torque τ and the pitch angle θ of a spacecraft:

$$\ddot{\theta} = \tau.$$

This week, you will begin by looking at the alternative model

$$J_{\text{pitch}}\ddot{\theta} = \tau, \tag{1}$$

which is exactly the same but which includes a parameter describing the moment of inertia about the pitch axis. Suppose $J_{\text{pitch}} = 15 \text{ kg} \cdot \text{m}^2$. Then, just like before, the system can be expressed in state space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/15 \end{bmatrix} u,$$

where

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \text{and} \qquad u = \begin{bmatrix} \tau \end{bmatrix}.$$

(a) Show that this system is controllable.

(b) Consider the application of state feedback

$$u = -Kx$$

where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}.$$

Compute the gains that would place the closed-loop eigenvalues at -10^{-1} and -10^{-2}.

(c) Suppose the torque is generated using a reaction wheel. Any torque applied to the spacecraft is also applied, in the opposite direction, to the wheel. The relationship between the torque and the angular velocity ν of the wheel can be approximated by

$$J_{\text{wheel}}\dot{\nu} = -\tau,$$

where τ is the *same torque* as in our model of the spacecraft and where $J_{\text{wheel}} = 1\ \text{kg}\cdot\text{m}^2$ is the moment of inertia of the wheel about its axis of rotation. Redefine the state as

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \nu \end{bmatrix}$$

and rewrite the system in state space form.

(d) Show that the system in part (c) is not controllable.

(e) Suppose the state feedback that you designed in part (b) were applied, unchanged, to the system that you derived in part (c):

$$K = \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix}$$

Use whatever method you like to simulate and plot $x(t)$ given an initial condition of $\theta(0) = 0$, $\dot{\theta}(0) = 0.01$, and $\nu(0) = 0$. Be sure to use a long enough time horizon. What happens to the pitch angle? What happens to the angular velocity of the wheel?

(f) If the spacecraft is in "low Earth orbit" (LEO), its motion about the pitch axis will be subject to a gravity gradient torque. Instead of (1), the dynamics become

$$J_{\text{pitch}}\ddot{\theta} = \tau + 3n^2\left(J_{\text{yaw}} - J_{\text{roll}}\right)\theta,$$

where $n = 0.0011$ rad/sec is the orbital angular velocity and where $J_{\text{roll}} = 15\ \text{kg}\cdot\text{m}^2$ and $J_{\text{yaw}} = 5\ \text{kg}\cdot\text{m}^2$ are moments of inertia about the roll and yaw axes, respectively. Modify your answer to part (c) and rewrite this system—including both the spacecraft and the reaction wheel—in state space form.

(g) Show that the system in part (f) is controllable.

(h) Consider the application of state feedback

$$u = -Kx$$

to the system in part (f), where

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}.$$

Compute the gains that would place the closed-loop eigenvalues at -10^{-1}, -10^{-2}, and -10^{-4}.

(i) Use whatever method you like to simulate and plot $x(t)$ given an initial condition of $\theta(0) = 0$, $\dot{\theta}(0) = 0.01$, and $\nu(0) = 0$. Be sure to use a long enough time horizon. (To get a sense of what this time horizon should be, ask yourself—how many seconds does it take a spacecraft in LEO to orbit the earth once?) What happens to the pitch angle? What happens to the angular velocity of the wheel?

4. In this problem, you will see how to discretize a state space system and how to use this discretization to do trajectory generation.

   (a) **EXTRA CREDIT:** Consider a general state space system

   $$\dot{x} = Ax + Bu. \tag{2}$$

   Assume that the matrix A is invertible. Suppose the input is constant on time intervals of length $h > 0$. In other words, for each non-negative integer $k \in \{0, 1, 2, \dots\}$, we have $u(t) = u(kh)$ for all $t \in [kh, (k+1)h)$. Prove that

   $$x((k+1)h) = A_d x(kh) + B_d u(kh).$$

   for

   $$A_d = e^{Ah} \qquad\qquad B_d = A^{-1}\left(e^{Ah} - I\right)B.$$

   Note that A_d and B_d depend on h, A, and B, but not on k. Finding A_d and B_d is called "discretizing" the system (2). We often write the resulting "discrete-time" system as

   $$x_{k+1} = A_d x_k + B_d u_k.$$

   (b) Suppose that $x \in \mathbb{R}^n$ (i.e., that the state has n elements). Assuming $x_0 = 0$, find the matrix W_{discrete} for which

   $$x_n = W_{\text{discrete}} u_{\text{discrete}}, \tag{3}$$

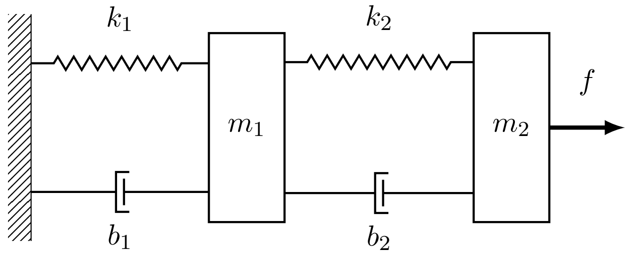
   where

   $$u_{\text{discrete}} = \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}.$$

   Note that (3) computes the state of the system (2) after n time intervals of length h, assuming the constant input u_k is applied during each interval $k+1$.

   HINT: the matrix W_{discrete} should look very familiar to you.

   (c) Suppose you want to achieve a given state $x_n = x_{\text{goal}}$, starting from $x_0 = 0$. What must be true of W_{discrete} in order for this to be possible, and what sequence of inputs u_{discrete} should you choose?



(d) A model of the spring-mass-damper system shown above is

$$\begin{aligned} m_1\ddot{p}_1 &= -k_1 p_1 - b_1\dot{p}_1 + k_2(p_2 - p_1) + b_2(\dot{p}_2 - \dot{p}_1) \\ m_2\ddot{p}_2 &= -k_2(p_2 - p_1) - b_2(\dot{p}_2 - \dot{p}_1) + f, \end{aligned}$$

where f is an applied force and where p_1 and p_2 are absolute displacements of each mass from their equilibrium positions. Put this system in state space form, assuming that

$$m_1 = m_2 = 2 \qquad k_1 = k_2 = 8 \qquad b_1 = b_2 = 1.$$

You may choose states however you like, but make sure the inputs and outputs are

$$u = \begin{bmatrix} f \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

(e) Show that the system in part (d) is controllable.

(f) Suppose

$$p_1(0) = 0 \qquad p_2(0) = 0 \qquad \dot{p}_1(0) = 0 \qquad \dot{p}_2(0) = 0$$

and you want to achieve

$$p_1(4) = 1 \qquad p_2(4) = 4 \qquad \dot{p}_1(4) = 0 \qquad \dot{p}_2(4) = 0$$

with inputs that are constant over time intervals of length $h = 1$. Apply your results from parts (a)-(c) in order to find these inputs (i.e., in order to find u_{discrete}).

(g) Simulate the response of the spring-mass-damper system to the inputs you chose in part (f) using the script `hw3prob04.m`. Submit only the lines of code you added to this script and a snapshot of the figure after the simulation has ended.

5. **EXTRA CREDIT:** In the first three problems, you were asked to compute gains that placed eigenvalues in specific locations "by hand" (i.e., by computing the characteristic equation and by choosing gains that make this equation look like what you want). This process can be made systematic. In particular, go back and repeat every such computation *by transformation to controllable canonical form* (see videos on piazza). Note that the MATLAB function `acker` uses exactly this method, and is a good way to check your work, whether or not you do this extra credit problem.