AE353 Homework #3: Controllability and State Feedback

SOLUTIONS

(due at the beginning of class on Friday, February 20)

1. The motion of a robot arm with one revolute joint can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -b/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements are the angle (x_1) and the angular velocity (x_2) of the joint, the input u is a torque applied to the arm at the joint, and

$$m = 5 \qquad\qquad b = 1$$

are parameters.

(a) Show that this system is controllable.
The system is controllable iff its associated controllability matrix has full ROW rank.

$$\begin{aligned} \mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{b}{m^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{25} \end{bmatrix} \end{aligned}$$

To determine if $\mathcal{W}$ has full row rank, we apply row-column operations on its transpose:

$$RREF(\mathcal{W}^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad \mathcal{W} \quad$ is full row rank $\implies$ the system is controllable.

(b) Consider an input of the form

$$u = -Kx + k_{\text{reference}} r$$

where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

and r is a reference signal that you may assume is constant.

- *Feedback design.* Compute the gains k_1 and k_2 that would place the closed-loop eigenvalues at $-\sigma \pm j\omega$. You may assume that both $\sigma > 0$ and $\omega > 0$.

  For a second order system, in order to place eigenvalues at $-\sigma \pm j\omega$, the characteristic equation must be:

$$(s-(-\sigma-j\omega))\,(s-(-\sigma+j\omega))$$

If we expand these factors out we get:

$$s^2+2\sigma s+\left(\sigma^2+\omega^2\right)$$

The characteristic equation of $\mathbf{A}-\mathbf{BK}$ is:

$$s^2+\frac{b+k_2}{m}s+\frac{k_1}{m}=s^2+\frac{1+k_2}{5}s+\frac{k_1}{5}$$

If we equate coefficients of s in both characteristic equations, it follows that:

$$\begin{aligned}
k_1 &= m(\sigma^2+\omega^2)=5(\sigma^2+\omega^2)\\
k_2 &= 2m\sigma-b=10\sigma-b
\end{aligned}$$

We can also arrive at this answer by finding the eigenvalues of the closed-loop system. First, let's look at the state space system:

$$\begin{aligned}
\begin{bmatrix}\dot{x_1}\\ \ddot{x_2}\end{bmatrix} &= \begin{bmatrix}0 & 1\\ 0 & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}k_{reference}r\\
&= \begin{bmatrix}0 & 1\\ -\frac{k_1}{m} & -\frac{(b+k_2)}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}k_{reference}r
\end{aligned}$$

$$\begin{aligned}
&\begin{vmatrix}-s & 1\\ -\frac{k_1}{m} & -\frac{b+k_2}{m}-s\end{vmatrix}=0\\
&\therefore s^2+\frac{b+k_2}{m}s+\frac{k_1}{m}=0\\
&\therefore s=-\frac{b+k_2}{2m}\pm\frac{1}{2}\sqrt{\left(\frac{b+k_2}{m}\right)^2-\frac{4k_1}{m}}
\end{aligned}$$

Our goal is to place the closed-loop system eigenvalues at $s=\sigma\pm j\omega$. To do this, we must put the expression for the eigenvalues above in that form:

$$s=-\frac{b+k_2}{2m}\pm j\frac{1}{2m}\sqrt{4k_1m-b^2-2bk_2-{k_2}^2} \tag{1}$$

We can now immediately see that:

$$k_2=2m\sigma-b$$

Substituting this equation for k_2 into the imaginary part of Eq. (1) and setting it equal to ω yields:

$$\begin{aligned}
\omega &= \frac{1}{m}\sqrt{k_1m-m^2\sigma^2}\\
\therefore k_1 &= m(\omega^2+\sigma^2)
\end{aligned}$$

So, we have:

$$k_1 = 5(\sigma^2 + \omega^2) \quad k_2 = 10\sigma - 1$$

- *Feedforward design.* Compute the gain $k_{\text{reference}}$ so that $y = r$ in steady-state.

We start with the familiar equation: $k_{reference} = -\left[\mathbf{C}\left(\mathbf{A} - \mathbf{BK}\right)^{-1}\mathbf{B}\right]^{-1}$.
We can now substitute the state space matrices, as well as the results that we just obtained for k_1 and k_2:

$$\begin{aligned}
k_{reference} &= -\left[\begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 0 & -\frac{b}{m}\end{bmatrix} - \begin{bmatrix}0 & 0\\ \omega^2+\sigma^2 & -\frac{b-2m\sigma}{m}\end{bmatrix}\right)^{-1}\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\right]^{-1}\\
&= -\left[\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -\omega^2-\sigma^2 & -2\sigma\end{bmatrix}^{-1}\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\right]^{-1}\\
&= -\left[\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}-\frac{2\sigma}{\omega^2+\sigma^2} & -\frac{1}{\omega^2+\sigma^2}\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\right]^{-1}\\
&= -\left(-\frac{1}{m(\omega^2+\sigma^2)}\right)^{-1}
\end{aligned}$$

$$\therefore k_{reference} = k_1 = m(\omega^2 + \sigma^2) = 5(\omega^2 + \sigma^2)$$

(c) **EXTRA CREDIT:**

- Show that the time to peak amplitude of the step response (i.e., the time that it takes for $y(t)$ to reach its *first* peak) is

$$t_p = \pi/\omega$$

The output equation for the step response is:

$$y(t) = 1 - e^{\sigma t}\left[cos(\omega t) + \frac{\sigma}{\omega}sin(\omega t)\right]$$

This may be written as:

$$y(t) = 1 - e^{\sigma t}\left[\frac{\sigma^2+\omega^2}{\omega^2}sin(\omega t + \phi)\right] \quad \phi = tan^{-1}\left(\frac{\omega}{\sigma}\right)$$

by recognizing that:

$$\begin{aligned}
Asin\theta + Bcos\theta &= \sqrt{A^2+B^2}sin(\theta + \alpha) \quad \alpha = tan^{-1}\left(\frac{B}{A}\right)\\
Asin\theta - Bcos\theta &= \sqrt{A^2+B^2}sin(\theta - \alpha) \quad \alpha = tan^{-1}\left(\frac{B}{A}\right)
\end{aligned}$$

Taking the time derivative of the output equation, and setting it equal to zero, yields:

$$\dot{y}(t) = \sigma e^{-\sigma t}\left[\frac{\sigma^2+\omega^2}{\omega^2} sin(\omega t+\phi)\right] - e^{-\sigma t}\left[\frac{\sigma^2+\omega^2}{\omega} cos(\omega t+\phi)\right] = 0$$

$$\therefore \sigma\frac{\sigma^2+\omega^2}{\omega^2} sin(\omega t+\phi) - \frac{\sigma^2+\omega^2}{\omega} cos(\omega t+\phi) = 0$$

$$\therefore \sigma\frac{\sigma^2+\omega^2}{\omega^2}\left[sin(\omega t)cos(\phi)+cos(\omega t)sin(\phi)\right] - \frac{\sigma^2+\omega^2}{\omega}\left[cos(\omega t)sin(\phi)-sin(\omega t)cos(\phi)\right] = 0$$

Here, from the phase quantity, we note that $sin\phi = \omega$ and $cos\phi = \sigma$.

$$(\sigma^2+\omega^2)^2 sin(\omega t) = 0$$

This equation is only true when $\omega t = 0, \pi, 2\pi, ...$
We only want to consider the first zero of this function after $t = 0$, so we take $\omega t = \pi$

$$\therefore t_{peak} = \frac{\pi}{\omega}$$

- Show that the peak overshoot of the step response is

$$M_p = e^{-\pi\sigma/\omega}$$

This result follows from the one above. We can see that at $t = \frac{\pi}{\omega}$,

$$y(t) = 1 + e^{-\frac{\pi\sigma}{\omega}}$$

which is the percentage that the response rises above the steady-state value of 1 (the definition of overshoot).

(d) *Lines of constant time to peak.* Use part (c) to compute eigenvalue locations that would result in $t_p = 0.5$ and $M_p = 0.1$. Use part (b) to compute the corresponding gains. Use any method of simulation you want to compute the step response. Create a figure with two axes (e.g., `subplot(1,2,1)` and `subplot(1,2,2)` in MATLAB). On the first set of axes, plot the step response. On the second set of axes, plot the eigenvalue locations. Repeat for $M_p = 0.2$ and $M_p = 0.3$, keeping t_p constant and putting your results on the same two axes. What happens to the step response? What happens to the eigenvalue locations? (In what direction do they move with increasing M_p?)

$$\omega = \frac{\pi}{t_p}$$

$$\sigma = -\left(\frac{\omega}{\pi}\right) ln(M_p)$$

Using the above relations, as well as the equations for the corresponding feedback and feedforward gains that were derived in part b), yields the following three cases:

$t_p = 0.5$	$M_p = 0.1$	$s = -4.605170 \pm j6.283185$	$k_1 = 303.430050$	$k2 = 45.051702$
$t_p = 0.5$	$M_p = 0.2$	$s = -3.218876 \pm j6.283185$	$k_1 = 249.197896$	$k2 = 31.188758$
$t_p = 0.5$	$M_p = 0.3$	$s = -2.407946 \pm j6.283185$	$k_1 = 226.383098$	$k2 = 23.079456$

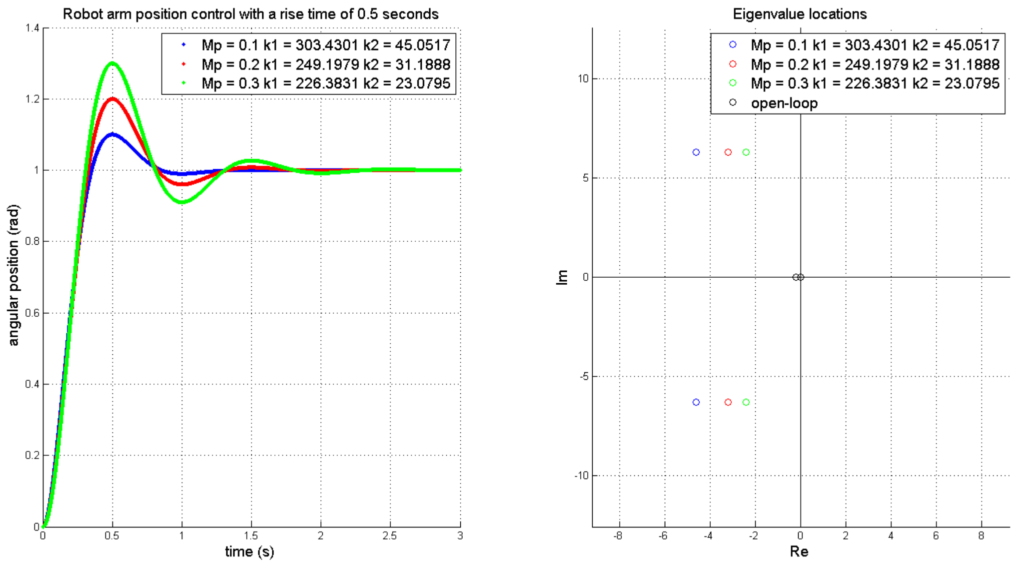


Figure 1: Robotic arm position as a function of time for a peak time of $t_p = 0.5s$ and overshoot values of $M_p = 10\%, 20\%$ and 30%

Note that as we move the eigenvalues closer to the imaginary axis ($j\omega$ axis), we see the overshoot increase. Also note that, in general, decreasing the value of the gains results in the eigenvalues moving closer to the imaginary axis as well as causing the amount of overshoot to increase.

(e) *Lines of constant peak overshoot.* Use part (c) to compute eigenvalue locations that would result in $t_p = 0.5$ and $M_p = 0.1$. Use part (b) to compute the corresponding gains. Use any method of simulation you want to compute the step response. Create a figure with two axes (e.g., `subplot(1,2,1)` and `subplot(1,2,2)` in MATLAB). On the first set of axes, plot the step response. On the second set of axes, plot the eigenvalue locations. Repeat for $t_p = 1.5$ and $M_p = 2.5$, keeping M_p constant and putting your results on the same two axes. What happens to the step response? What happens to the eigenvalue locations? (In what direction do they move with increasing t_p?)

$$
\begin{array}{lllll}
t_p = 0.5 & M_p = 0.1 & s = -4.605170 \pm j6.283185 & k_1 = 303.430050 & k2 = 45.051702 \\
t_p = 1.5 & M_p = 0.1 & s = -1.535057 \pm j2.094395 & k_1 = 33.714450 & k2 = 14.350567 \\
t_p = 2.5 & M_p = 0.1 & s = -0.921034 \pm j1.256637 & k_1 = 12.137202 & k2 = 8.210340
\end{array}
$$

Here we see that increasing peak time, in general, brings the eigenvalues closer to the origin. Lines of constant peak overshoot pass through the origin.

(f) Suppose you are given a performance specification that requires $t_p < 1$ and $M_p < 0.15$.

- Sketch the region of the complex plane within which the eigenvalues must be located in order to meet this spec.

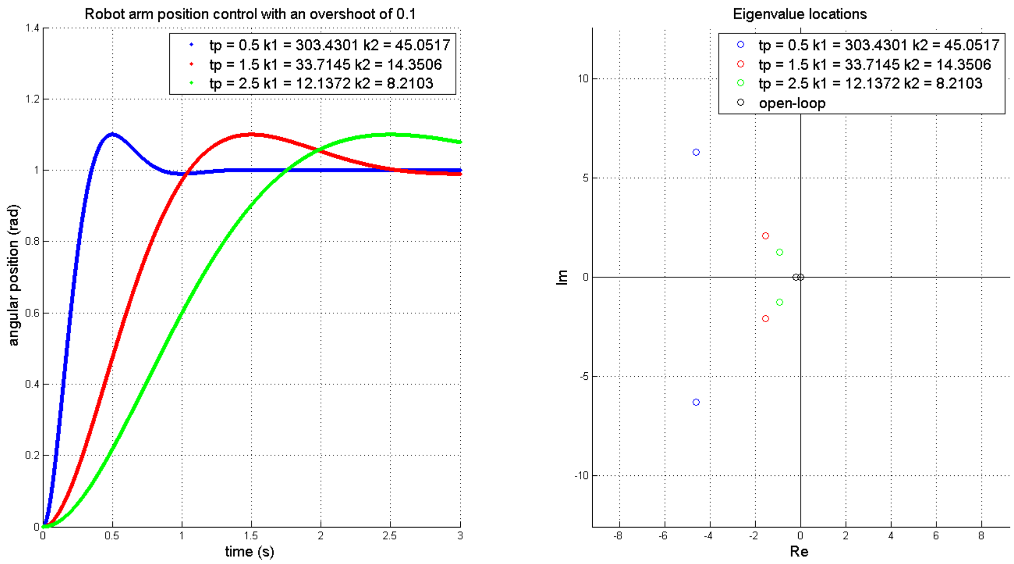


Figure 2: Robotic arm position as a function of time for an overshoot of $M_p = 0.1s$ and peak times of $t_p = 0.5, 1.5$ and $2.5s$

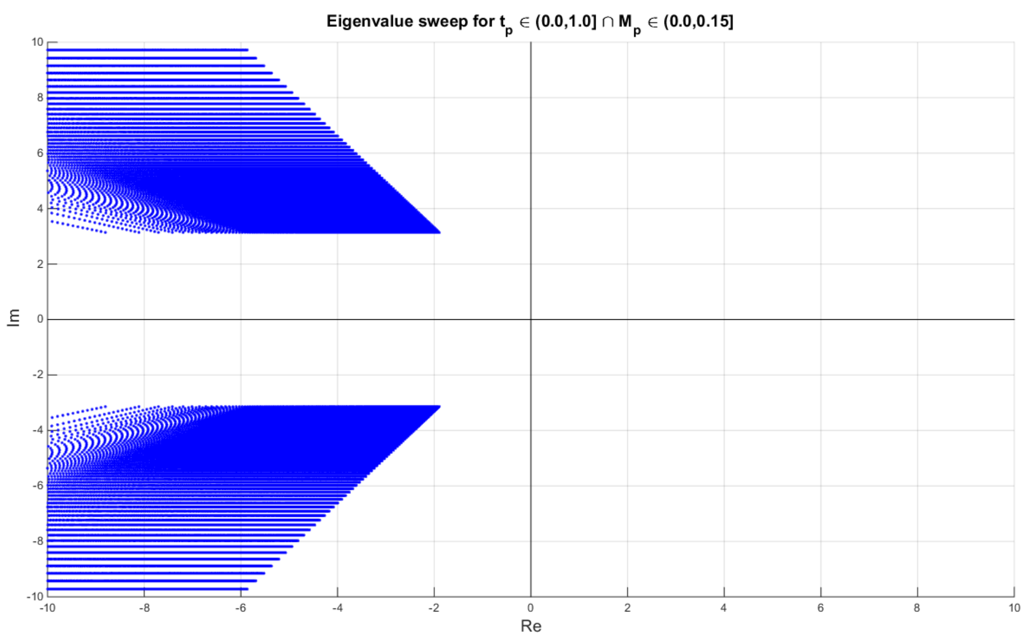


Figure 3: Eigenvalue sweep of for $t_p \in (0.0, 1.0]$ and $M_p \in (0.0, 0.15]$

Figure 3 is an eigenvalue sweep of the half-plane intersections of the peak time and overshoot "allowable" regions on either side of the real axis for this system. All of the blue eigenvalues will result in a response on the intervals $t_p \in (0.0, 1.0]$ and $M_p \in (0.0, 0.15]$.

- Choose eigenvalue locations in this region (anywhere you want) and compute the corresponding gains.

  If we select a peak time of $t_p = 0.5s$ and an overshoot of $M_p = 0.075$, the corresponding gains will be:

$$k_1 = 331.581768$$
$$k_2 = 50.805343$$

- Compute *and visualize* the response of the closed-loop system to a reference signal

$$r(t) = \pi/2 \qquad \text{for all } t \geq 0$$

  using the script `hw3prob01.m`. (This is exactly the same as the step response, but with a reference signal of magnitude $\pi/2$ instead of magnitude 1.) Submit only the lines of code you added to this script and a snapshot of the figure after the simulation has ended.

(You could, of course, use this same code to visualize the step responses you computed in the earlier parts of this problem, if you wanted.)

```
m = 5;
b = 1;

r = pi/2;

A = [0 1;0 -b/m];
B = [0;1/m];
C = [1 0];
D = 0;

tp = 0.5;
Mp = 0.075;

omega = pi/tp;
sig = -omega/pi*log(Mp);

X0 = [0;0];

k1 = m*(omega^2 + sig^2)
k2 = 2*m*sig - b

K = [k1 k2];

kreference = -inv(C*(inv(A-B*K))*B);

% Compute step response using ode45
```

```
tspan = linspace(0,10,1000);
[tMATLAB,yMATLAB] = ode45(@(t,X)prob1EOM(t,X,K,kreference,r,A,B,C,D),tspan,X0);


function Xdot = prob1EOM(t,X,K,kreference,r,A,B,C,D)

    u = -K*X + kreference*r;
    Xdot = A*X + B*u;
```

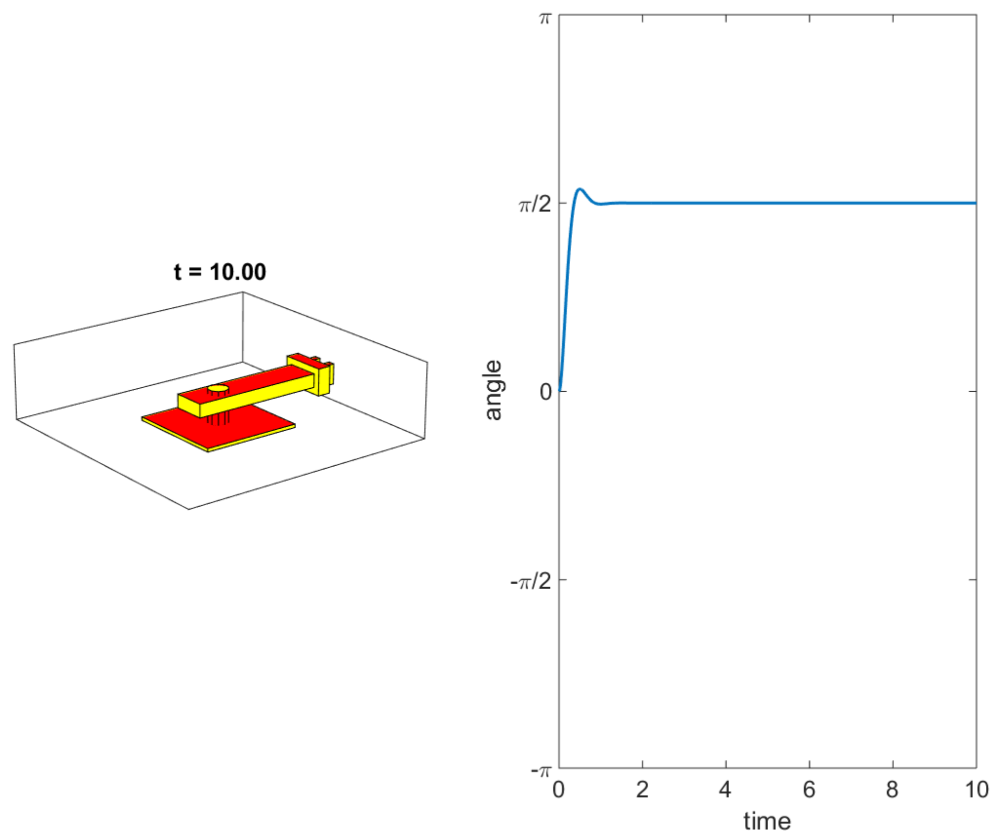


Figure 4: Robot arm position tracking simulation with $t_p = 0.5$ and $M_p = 0.075$

2. The rotational motion of an axisymmetric spacecraft about its yaw and roll axes can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

where the state elements x_1 and x_2 are the angular velocities about yaw and roll axes, the input u is an applied torque, and the parameter $\lambda = 9$ is the relative spin rate.

(a) Show that this system is controllable.

The system is controllable iff its associated controllability matrix has full ROW rank.

$$\begin{aligned}\mathcal{W} &= \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 \\ 0 & -\lambda\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 \\ 0 & -9\end{bmatrix}\end{aligned}$$

To determine if $\mathcal{W}$ has full row rank, we apply row-column operations on its transpose:

$$RREF(\mathcal{W}^T) = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

$\therefore \quad \mathcal{W} \quad$ is full row rank $\implies$ the system is controllable.

(b) Consider an input of the form

$$u = -Kx + k_{\text{reference}} r$$

where

$$K = \begin{bmatrix}k_1 & k_2\end{bmatrix}$$

and r is a reference signal that you may assume is constant.

- *Feedback design.* Compute the gains k_1 and k_2 that would place the closed-loop eigenvalues at $-\sigma_1$ and $-\sigma_2$. You may assume that $\sigma_2 \geq \sigma_1 > 0$.

$$\begin{aligned}\dot{x} &= \begin{bmatrix}0 & \lambda \\ -\lambda & 0\end{bmatrix} x - \begin{bmatrix}1 \\ 0\end{bmatrix} \begin{bmatrix}k_1 & k_2\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 0\end{bmatrix} k_{reference} r \\ &= \begin{bmatrix}-k_1 & \lambda - k_2 \\ -\lambda & 0\end{bmatrix} x + \begin{bmatrix}1 \\ 0\end{bmatrix} k_{reference} r\end{aligned}$$

$$\therefore \begin{vmatrix}-k_1 - s & \lambda - k_2 \\ -\lambda & -s\end{vmatrix} = s^2 + k_1 s + (\lambda^2 - \lambda k_2) = s^2 + k_1 s + (81 - 9k_2)$$

We want to place eigenvalues at $-\sigma_1$ and $-\sigma_2$, which results in a characteristic equation of:

$$(s + \sigma_1)(s + \sigma_2) = s^2 + (\sigma_1 + \sigma_2)s + \sigma_1 \sigma_2$$

Matching coefficients of s, we arrive at the following results for the gains:

$$k_1 = \sigma_1 + \sigma_2 \quad k_2 = \frac{\lambda^2 - \sigma_1 \sigma_2}{\lambda} = \frac{81 - \sigma_1 \sigma_2}{9}$$

- *Feedforward design.* Compute the gain $k_{\text{reference}}$ so that $y = r$ in steady-state.

$k_{reference} = -\left[\mathbf{C}\left(\mathbf{A} - \mathbf{BK}\right)^{-1}\mathbf{B}\right]^{-1}.$

We can now substitute the state space matrices into this equation, as well as the results that we just obtained for k_1 and k_2:

$$\begin{aligned}
k_{reference} &= -\left[\begin{bmatrix}0 & 1\end{bmatrix}\left(\begin{bmatrix}-k_1 & \lambda - k_2\\ -\lambda & 0\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\0\end{bmatrix}\right]^{-1}\\
&= -\left[\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-(\sigma_1+\sigma_2) & \lambda - \frac{\lambda^2-\sigma_1\sigma_2}{\lambda}\\ -\lambda & 0\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}\right]^{-1}\\
&= -\left[\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0 & -\frac{1}{\lambda}\\ \frac{\lambda}{\sigma_1+\sigma_2} & -\frac{\sigma_1+\sigma_2}{\sigma_1\sigma_2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right]^{-1}\\
&= -\left(\frac{\lambda}{\sigma_1\sigma_2}\right)^{-1}
\end{aligned}$$

$$\therefore k_{reference} = -\frac{\sigma_1\sigma_2}{\lambda} = -\frac{\sigma_1\sigma_2}{9}$$

(c) *Dominant first-order response.* Use part (b) to compute the gains that place both closed-loop eigenvalues at -1. Use any method of simulation you want to compute and plot the step response. Repeat for eigenvalues at -1 and -5, then once more for eigenvalues at -1 and -10. Put all your results on the same axes. On these same axes, plot

$$y_{\text{firstorder}}(t) = 1 - e^{-t}$$

What happens to the step response as one of the eigenvalues moves farther out? Why would it make sense to say that, when $\sigma_1 = 1$ and $\sigma_2 = 10$, the system has a dominant first-order response with time constant $1/\sigma_1 = 1$?

$$s_1 = -1 \quad s_2 = -1 \quad k_1 = 2 \quad k_2 = \frac{80}{9} \quad k_{reference} = -\frac{1}{9}$$
$$s_1 = -1 \quad s_2 = -5 \quad k_1 = 6 \quad k_2 = \frac{76}{9} \quad k_{reference} = -\frac{5}{9}$$
$$s_1 = -1 \quad s_2 = -10 \quad k_1 = 11 \quad k_2 = \frac{71}{9} \quad k_{reference} = -\frac{10}{9}$$

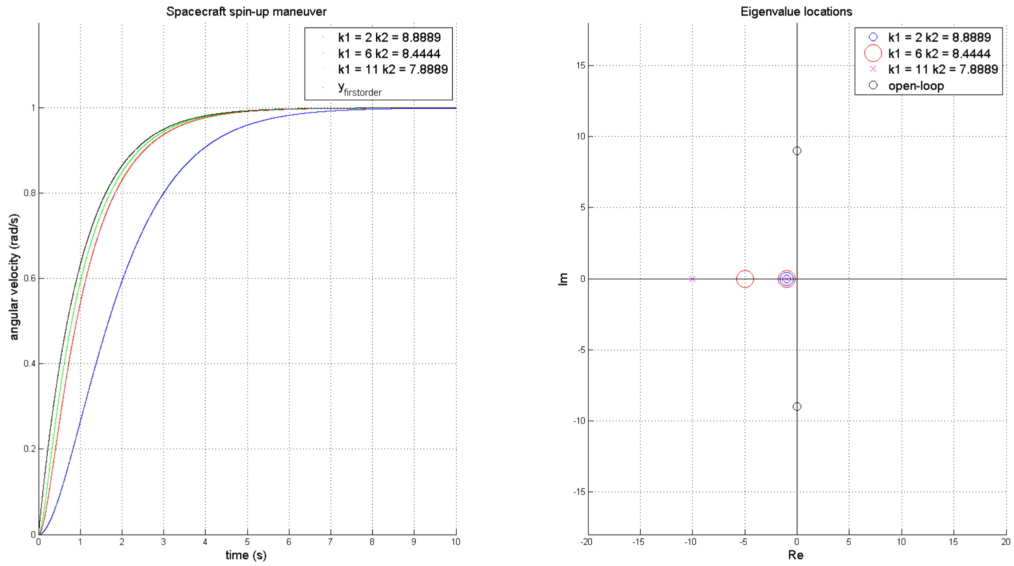


Figure 5: Spacecraft spin-up maneuver.

As the second eigenvalue moves farther out, the system responds much more quickly (the rise time decreases). When the eigenvalues are at -1 and -10, the system's response is dominated by the eigenvalue at $-\sigma_1 = -1$. The eigenvalue located at -10 does not influence the system's response as much as when it is located closer to the origin.

(d) Suppose you are given a performance specification that requires a dominant first-order response with time constant $1/2$.

- Choose eigenvalues that meet this spec and compute the corresponding gains.

  In order to meet this specification, we can place one eigenvalue at -2 and the other can remain at -10, so as not to significantly impact the response (but also not so far into the negative half-plane that the system experiences a violent response).
  This eigenvalue placement is achieved using the following gain settings:

$$k_1 = 12 \quad k_2 = \frac{61}{9}$$

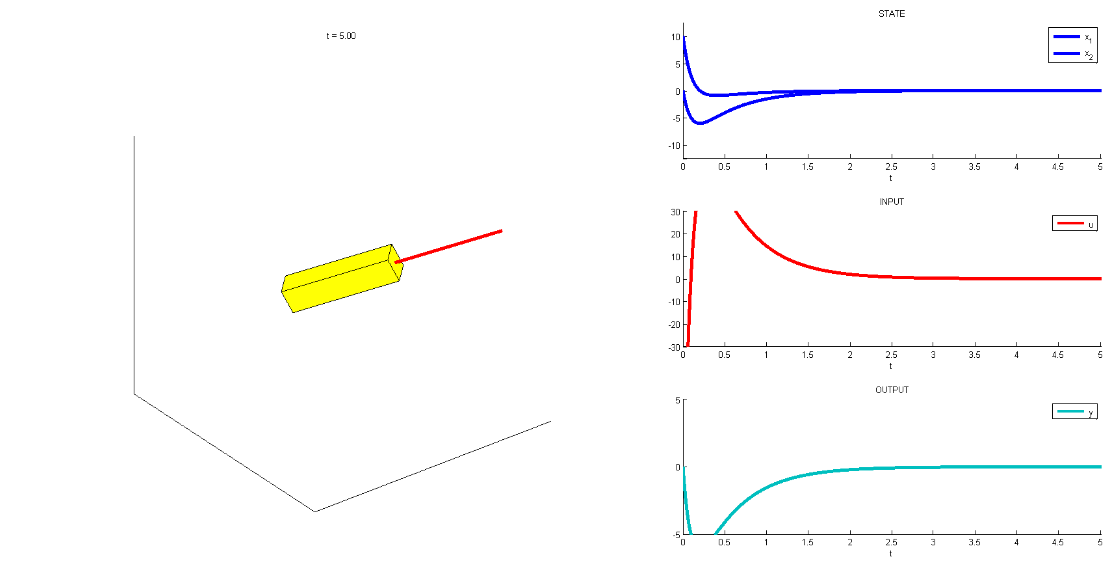


Figure 6: Spacecraft yaw-to-roll spin-axis realignment from an initial yaw rate of 10 rad/s.

- Compute and visualize the response of the closed-loop system to an initial condition

  $$x(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

  using the script `hw3prob02.m`. Note that this script does everything for you—all you need to do is put `hw3prob02.m` in your working directory and call `hw3prob02(K,x0)` with an appropriate choice of gain matrix `K` and initial condition `x0`. Submit only a snapshot of the figure after the simulation has ended.

Although not required, you might consider the relationship between the step response (computed by you) and the response to initial conditions (computed by `hw3prob02.m`). How are these two things related?

3. Previously, we have studied the following model of the relationship between the applied torque τ and the pitch angle θ of a spacecraft:

   $$\ddot{\theta} = \tau.$$

   This week, you will begin by looking at the alternative model

   $$J_{\text{pitch}}\ddot{\theta} = \tau, \tag{2}$$

   which is exactly the same but which includes a parameter describing the moment of inertia about the pitch axis. Suppose $J_{\text{pitch}} = 15\ \text{kg} \cdot \text{m}^2$. Then, just like before, the system can be expressed in state space form as

   $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/15 \end{bmatrix} u,$$

where

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \text{and} \qquad u = \begin{bmatrix} \tau \end{bmatrix}.$$

(a) Show that this system is controllable.

The system is controllable iff its associated controllability matrix has full ROW rank.

$$\begin{aligned} \mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{15} \\ \frac{1}{15} & 0 \end{bmatrix} \end{aligned}$$

To determine if $\mathcal{W}$ has full row rank, we apply row-column operations on its transpose:

$$RREF(\mathcal{W}^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad \mathcal{W} \quad$ is full row rank $\implies$ the system is controllable.

(b) Consider the application of state feedback

$$u = -Kx$$

where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}.$$

Compute the gains that would place the closed-loop eigenvalues at -10^{-1} and -10^{-2}.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ \frac{1}{J_{pitch}} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -\frac{k_1}{J_{pitch}} & -\frac{k_2}{J_{pitch}} \end{bmatrix} x \end{aligned}$$

$$\therefore \begin{vmatrix} s & 1 \\ -\frac{k_1}{J_{pitch}} & -\frac{k_2}{J_{pitch}} - s \end{vmatrix} = s^2 + \frac{k_2}{J_{pitch}} s + \frac{k_1}{J_{pitch}} = s^2 + \frac{k_2}{15} s + \frac{k_1}{15} = 0$$

We want to place eigenvalues at $-\sigma_1 = -0.1$ and $-\sigma_2 = -0.01$, which results in a characteristic equation of:

$$(s + \sigma_1)(s + \sigma_2) = s^2 + (\sigma_1 + \sigma_2)s + \sigma_1 \sigma_2$$

Matching coefficients of s, we arrive at the following results for the gains:

$$k_1 = J\sigma_1\sigma_2 = 0.015 \quad k_2 = J(\sigma_1 + \sigma_2) = 1.65$$

(c) Suppose the torque is generated using a reaction wheel. Any torque applied to the spacecraft is also applied, in the opposite direction, to the wheel. The relationship between the torque and the angular velocity ν of the wheel can be approximated by

$$J_{\text{wheel}}\dot{\nu} = -\tau,$$

where τ is the *same torque* as in our model of the spacecraft and where $J_{\text{wheel}} = 1\ \text{kg}\cdot\text{m}^2$ is the moment of inertia of the wheel about its axis of rotation. Redefine the state as

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \nu \end{bmatrix}$$

and rewrite the system in state space form.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{J_{pitch}} \\ -\frac{1}{J_{wheel}} \end{bmatrix} u$$

(d) Show that the system in part (c) is not controllable.

$$\begin{aligned} \mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{15} & 0 \\ \frac{1}{15} & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\ RREF(\mathcal{W}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -15 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$\therefore\ \ rank(\mathcal{W}) = 2 < 3 \implies$ the system is not controllable.

(e) Suppose the state feedback that you designed in part (b) were applied, unchanged, to the system that you derived in part (c):

$$K = \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix}$$

Use whatever method you like to simulate and plot $x(t)$ given an initial condition of $\theta(0) = 0$, $\dot{\theta}(0) = 0.01$, and $\nu(0) = 0$. Be sure to use a long enough time horizon. What happens to the pitch angle? What happens to the angular velocity of the wheel?

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ \frac{1}{J_{pitch}} \\ -\frac{1}{J_{wheel}} \end{bmatrix} \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\nu} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1}{J_{pitch}} & -\frac{k_2}{J_{pitch}} & 0 \\ \frac{k_1}{J_{wheel}} & \frac{k_2}{J_{wheel}} & 0 \end{bmatrix} x \end{aligned}$$

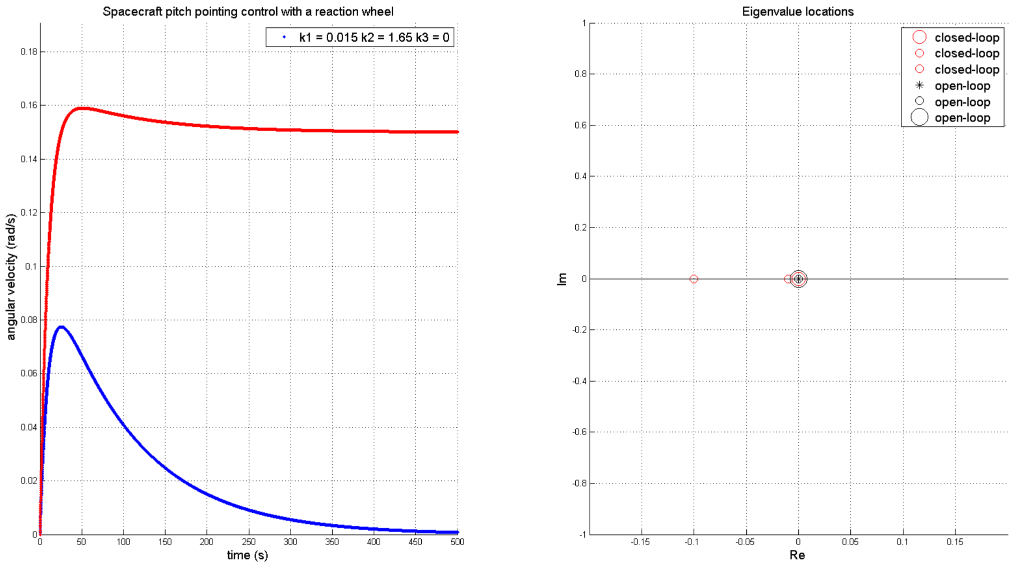


Figure 7: Spacecraft pitch control with a reaction wheel using partial state feedback (spacecraft angular position and angular velocity).

(f) If the spacecraft is in "low Earth orbit" (LEO), its motion about the pitch axis will be subject to a gravity gradient torque. Instead of (2), the dynamics become

$$J_{\text{pitch}}\ddot{\theta} = \tau + 3n^2\left(J_{\text{yaw}} - J_{\text{roll}}\right)\theta,$$

where $n = 0.0011$ rad/sec is the orbital angular velocity and where $J_{\text{roll}} = 15\ \text{kg}\cdot\text{m}^2$ and $J_{\text{yaw}} = 5\ \text{kg}\cdot\text{m}^2$ are moments of inertia about the roll and yaw axes, respectively. Modify your answer to part (c) and rewrite this system—including both the spacecraft and the reaction wheel—in state space form.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3n^2(J_{yaw}-J_{roll})}{J_{pitch}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ \frac{1}{J_{pitch}} \\ -\frac{1}{J_{wheel}} \end{bmatrix} u$$

(g) Show that the system in part (f) is controllable.

$$
\begin{aligned}
\mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{J_{pitch}} & 0 \\ \frac{1}{J_{pitch}} & 0 & \frac{3n^2(J_{yaw}-J_{roll})}{J_{pitch}^2} \\ -\frac{1}{J_{wheel}} & 0 & 0 \end{bmatrix} \\
RREF(\mathcal{W}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$\therefore \quad rank(\mathcal{W}) = 3 \quad \Longrightarrow$ the system is controllable.

(h) Consider the application of state feedback

$$u = -Kx$$

to the system in part (f), where

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}.$$

Compute the gains that would place the closed-loop eigenvalues at -10^{-1}, -10^{-2}, and -10^{-4}.

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{3n^2(J_{yaw}-J_{roll})}{J_{pitch}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ \frac{1}{J_{pitch}} \\ -\frac{1}{J_{wheel}} \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\nu} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ \frac{3n^2(J_{yaw}-J_{roll})-k_1}{J_{pitch}} & -\frac{k_2}{J_{pitch}} & -\frac{k_3}{J_{pitch}} \\ \frac{k_1}{J_{wheel}} & \frac{k_2}{J_{wheel}} & \frac{k_3}{J_{wheel}} \end{bmatrix} x
\end{aligned}
$$

$$
\therefore \begin{vmatrix} -s & 1 & 0 \\ \frac{3n^2(J_{yaw}-J_{roll})-k_1}{J_{pitch}} & -\frac{k_2}{J_{pitch}} - s & -\frac{k_3}{J_{pitch}} \\ \frac{k_1}{J_{wheel}} & \frac{k_2}{J_{wheel}} & \frac{k_3}{J_{wheel}} - s \end{vmatrix} = 0
$$

$$
\therefore s^3 + \frac{J_{wheel}k_3 - J_{pitch}k_2}{J_{pitch}J_{wheel}} s^2 + \frac{3n^2\left(J_{roll} - J_{yaw}\right) + k_1}{J_{pitch}} s + \frac{3n^2 k_3 \left(J_{yaw} - J_{roll}\right)}{J_{pitch}J_{wheel}} = 0
$$

The characteristic equation for placing three eigenvalues on the real axis of the left half-plane is:

$$s^3 + (\sigma_1 + \sigma_2 + \sigma_3)s^2 + (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)s + \sigma_1\sigma_2\sigma_3 = 0$$

By equating coefficients and solving we arrive at the following equations for the gain values:

$$k_1 = 3n^2\left(J_{yaw} - J_{roll}\right) + J_{pitch}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)$$
$$k_2 = J_{pitch}(\sigma_1 + \sigma_2 + \sigma_3) - \frac{J_{pitch}^2(\sigma_1\sigma_2\sigma_3)}{3n^2\left(J_{roll} - J_{yaw}\right)}$$
$$k_3 = -\frac{J_{pitch}J_{wheel}\sigma_1\sigma_2\sigma_3}{3n^2\left(J_{roll} - J_{yaw}\right)}$$

These equations yield gain values of:

$$k_1 = 0.0151287 \quad k_2 = 1.031665 \quad k_3 = -0.0413223$$

(i) Use whatever method you like to simulate and plot $x(t)$ given an initial condition of $\theta(0) = 0$, $\dot{\theta}(0) = 0.01$, and $\nu(0) = 0$. Be sure to use a long enough time horizon. (To get a sense of what this time horizon should be, ask yourself—how many seconds does it take a spacecraft in LEO to orbit the earth once?) What happens to the pitch angle? What happens to the angular velocity of the wheel?

```
% Part i)
Jp = 15;
Jr = 15;
Jy = 5;
Jw = 1;

n = 0.0011;

X0 = [0;0.01;0];

kreference = 0;
r = 0;

A = [0 1 0;3*n^2*(Jy - Jr)/Jp 0 0;0 0 0];
B = [0;1/Jp;-1/Jw];
C = [1 0 0];
D = 0;

sig1 = 0.1;
sig2 = 0.01;
sig3 = 0.0001;
k1 = 3*Jy*n^2 - 3*Jr*n^2 + Jp*sig1*sig2 + Jp*sig1*sig3 + Jp*sig2*sig3;
k2 = Jp*(sig1 + sig2 + sig3 - (Jp*sig1*sig2*sig3)/(3*n^2*(Jr - Jy)));
k3 = -(Jp*Jw*sig1*sig2*sig3)/(3*n^2*(Jr - Jy));
K = [k1 k2 k3];

% Compute step response using ode45 function
tspan = linspace(0,70000,10000);
[T,ytemp] = ode45(@(t,X)stateSpaceEOM(t,X,K,kreference,r,A,B,C,D),tspan,X0);
```

```
% store the angular velocity data
ySpacecraft = ytemp(:,1);
yWheel = ytemp(:,3);

function Xdot = stateSpaceEOM(t,X,K,kreference,r,A,B,C,D)

    u = -K*X + kreference*r;
    Xdot = A*X + B*u;
```

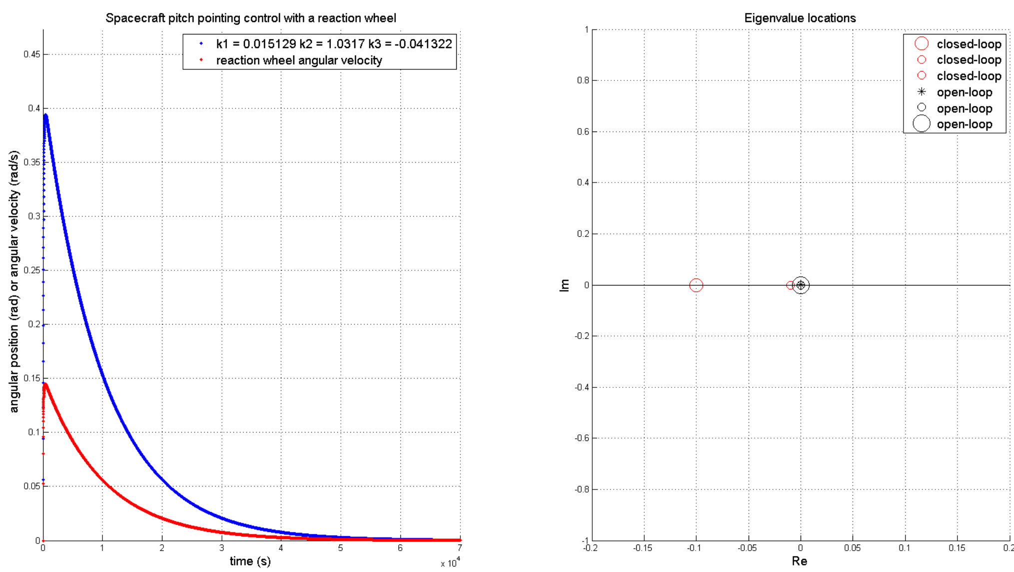


Figure 8: Spacecraft pitch control with a reaction wheel using full state feedback.

Now that we are using FULL state feedback, we see that the angular velocity of the spacecraft AND the reaction wheel are controllable and can be brought to zero over time.

4. In this problem, you will see how to discretize a state space system and how to use this discretization to do trajectory generation.

   (a) **EXTRA CREDIT:** Consider a general state space system

$$\dot{x} = Ax + Bu. \tag{3}$$

   Assume that the matrix A is invertible. Suppose the input is constant on time intervals of length $h > 0$. In other words, for each non-negative integer $k \in \{0, 1, 2, \dots\}$, we have $u(t) = u(kh)$ for all $t \in [kh, (k+1)h)$. Prove that

$$x((k+1)h) = A_d x(kh) + B_d u(kh).$$

   for

$$A_d = e^{Ah} \qquad B_d = A^{-1}\left(e^{Ah} - I\right)B.$$

Note that A_d and B_d depend on h, A, and B, but not on k. Finding A_d and B_d is called "discretizing" the system (3). We often write the resulting "discrete-time" system as

$$x_{k+1} = A_d x_k + B_d u_k.$$

We start with the general linear time-invariant (LTI) system:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

The continuous-time solution to the above equation is:

$$x(t) = e^{\mathbf{A}t}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

Next, define the value of the state vector at the end of the k^{th} discretization step to be:

$$x[k] = x(kh)$$

where h is the sampling time, or time between the measurements of the state vector taken by some sensor.

$$x[k] = e^{\mathbf{A}kh}x(0) + \int_0^{kh} e^{\mathbf{A}(kh-\tau)}\mathbf{B}u(\tau)d\tau$$
$$x[k+1] = e^{\mathbf{A}(k+1)h}x(0) + \int_0^{(k+1)h} e^{\mathbf{A}((k+1)h-\tau)}\mathbf{B}u(\tau)d\tau$$
$$x[k+1] = e^{\mathbf{A}kh+\mathbf{A}h}x(0) + \int_0^{kh} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau)d\tau + \int_{kh}^{(k+1)h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau)d\tau$$

k and h are just constants, so we can break up the exponential in front of the initial condition and factor $e^{\mathbf{A}h}$ out of the first integrand. Note, that this would NOT be possible for a linear time-varying system (LTV) where $\mathbf{A} = \mathbf{A}(t)$, but we are dealing with an LTI system where $e^{\mathbf{A}h}$ is a constant, so this is OK.

$$x[k+1] = e^{\mathbf{A}h}\left[e^{\mathbf{A}kh}x(0) + \int_0^{kh} e^{\mathbf{A}(kh-\tau)}\mathbf{B}u(\tau)d\tau\right] + \int_{kh}^{(k+1)h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau)d\tau$$

Now, we observe that the bracketed portion of the above equation is equivalent to $x[k]$.

$$x[k+1] = e^{\mathbf{A}h}x[k] + \int_{kh}^{(k+1)h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau)d\tau$$

For our discrete model of the continuous system, we are assuming that the control is held constant over one discretization sample period h (also known as zero-order hold). That is to say, $u[k] = u(kh)$ is constant. Also, because our system is LTI, $\mathbf{B}$ is a constant matrix.

$$\therefore x[k+1] = e^{\mathbf{A}h}x[k] + \left(\int_{kh}^{(k+1)h} e^{\mathbf{A}(kh+h-\tau)}d\tau\right)\mathbf{B}u[k]$$

$$x[k+1] = e^{\mathbf{A}h}x[k] + \mathbf{A}^{-1}\left[e^{\mathbf{A}(kh+h-kh-h)} - e^{\mathbf{A}(kh+h-kh)}\right]Bu[k]$$
$$x[k+1] = e^{\mathbf{A}h}x[k] - \mathbf{A}^{-1}\left[\mathbb{I} - e^{\mathbf{A}h}\right]Bu[k]$$

$$\therefore x[k+1] = e^{\mathbf{A}h}x[k] + \mathbf{A}^{-1}\left(e^{\mathbf{A}h} - \mathbb{I}\right)Bu[k]$$

Note, that we used the fact that $\mathbf{A}$ is invertible when we evaluated the integral

(b) Suppose that $x \in \mathbb{R}^n$ (i.e., that the state has n elements). Assuming $x_0 = 0$, find the matrix W_{discrete} for which

$$x_n = W_{\text{discrete}} u_{\text{discrete}}, \tag{4}$$

where

$$u_{\text{discrete}} = \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}.$$

Note that (4) computes the state of the system (3) after n time intervals of length h, assuming the constant input u_k is applied during each interval $k+1$.

HINT: the matrix W_{discrete} should look very familiar to you.

$$x_1 = \mathbf{A}_d \cancelto{0}{x_0} + \mathbf{B}_d u_0$$
$$x_2 = \mathbf{A}_d x_1 + \mathbf{B}_d u_1 = \mathbf{A}_d\left(\mathbf{A}_d \cancelto{0}{x_0} + \mathbf{B}_d u_0\right) + \mathbf{B}_d u_1 = \mathbf{A}_d^2 \cancelto{0}{x_0} + \mathbf{A}_d \mathbf{B}_d u_0 + \mathbf{B}_d u_1$$
$$x_3 = \mathbf{A}_d x_2 + \mathbf{B}_d u_2 = \mathbf{A}_d^3 \cancelto{0}{x_0} + \mathbf{A}_d^2 \mathbf{B}_d u_0 + \mathbf{A}_d \mathbf{B}_d u_1 + \mathbf{B}_d u_2$$
$$\vdots$$
$$x_n = \begin{bmatrix} \mathbf{B}_d & \mathbf{A}_d\mathbf{B}_d & \mathbf{A}_d^2\mathbf{B}_d & \cdots & \mathbf{A}_d^{n-1}\mathbf{B}_d \end{bmatrix} u$$

This is the controllability matrix for the discrete time system $\mathcal{W}_d$.

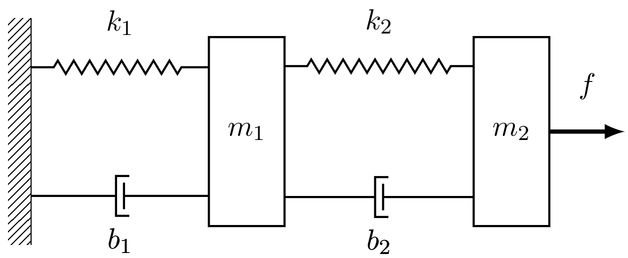
(c) Suppose you want to achieve a given state $x_n = x_{\text{goal}}$, starting from $x_0 = 0$. What must be true of W_{discrete} in order for this to be possible, and what sequence of inputs u_{discrete} should you choose?

Using the relationship between the discrete time controllability matrix $\mathcal{W}_d$ and the state vector at the boundary of the n^{th} time step, we can left-multiply x_{goal} by the inverse of

$\mathcal{W}_d$ to recover the discrete time control history that transfers the state from $x_0 = 0$ to the goal state, i.e.

$$u_{discrete} = \mathcal{W}_d^{-1} x_{goal}$$

Clearly, we require that $\mathcal{W}_d$ be invertible to do this (have full rank/det = 0).



(d) A model of the spring-mass-damper system shown above is

$$\begin{aligned} m_1 \ddot{p}_1 &= -k_1 p_1 - b_1 \dot{p}_1 + k_2(p_2 - p_1) + b_2(\dot{p}_2 - \dot{p}_1) \\ m_2 \ddot{p}_2 &= -k_2(p_2 - p_1) - b_2(\dot{p}_2 - \dot{p}_1) + f, \end{aligned}$$

where f is an applied force and where p_1 and p_2 are absolute displacements of each mass from their equilibrium positions. Put this system in state space form, assuming that

$$m_1 = m_2 = 2 \qquad k_1 = k_2 = 8 \qquad b_1 = b_2 = 1.$$

You may choose states however you like, but make sure the inputs and outputs are

$$u = \begin{bmatrix} f \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

$$\begin{bmatrix} \dot{p}_1 \\ \ddot{p}_1 \\ \dot{p}_2 \\ \ddot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix} \begin{bmatrix} p_1 \\ \dot{p}_1 \\ p_2 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

(e) Show that the system in part (d) is controllable.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -8 & -1 & 4 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 4 & \frac{1}{2} & -4 & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} \\ &= \begin{bmatrix} 0.000000 & 0.000000 & 0.250000 & 1.625000 \\ 0.000000 & 0.250000 & 1.625000 & -5.500000 \\ 0.000000 & 0.500000 & -0.250000 & -1.750000 \\ 0.500000 & -0.250000 & -1.750000 & 3.687500 \end{bmatrix} \end{aligned}$$

$$RREF(\mathcal{W}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, $det(\mathcal{W}) = 1 \implies$ the system is controllable.

(f) Suppose

$$p_1(0) = 0 \qquad p_2(0) = 0 \qquad \dot{p}_1(0) = 0 \qquad \dot{p}_2(0) = 0$$

and you want to achieve

$$p_1(4) = 1 \qquad p_2(4) = 4 \qquad \dot{p}_1(4) = 0 \qquad \dot{p}_2(4) = 0$$

with inputs that are constant over time intervals of length $h = 1$. Apply your results from parts (a)-(c) in order to find these inputs (i.e., in order to find u_{discrete}).

$$\mathbf{A}_d = \begin{bmatrix} -0.276289173264692 & 0.189014656460671 & 0.398314397165350 & 0.313153230557574 \\ -0.259504329455073 & -0.308727214446576 & -0.496554296387611 & 0.336245110116899 \\ 0.398314397165350 & 0.313153230557574 & 0.122025223900658 & 0.502167887018244 \\ -0.496554296387611 & 0.336245110116898 & -0.756058625842683 & 0.027517895670322 \end{bmatrix}$$

$$\mathbf{B}_d = \begin{bmatrix} 0.059957547366749 \\ 0.156576615278787 \\ 0.169704394379167 \\ 0.251083943509122 \end{bmatrix}$$

$$\begin{aligned} \mathcal{W}_d &= \begin{bmatrix} \mathbf{B}_d & \mathbf{A}_d\mathbf{B}_d & \mathbf{A}_d^2\mathbf{B}_d & \mathbf{A}_d^3\mathbf{B}_d \end{bmatrix} \\ &= \begin{bmatrix} 0.059957547366749 & 0.159253105548222 & 0.000613213434109 & -0.107198536480720 \\ 0.156576615278787 & -0.063740403318695 & -0.163873170462724 & -0.050461679262089 \\ 0.169704394379167 & 0.219708937336450 & 0.020808127925814 & -0.183785003909440 \\ 0.251083943509122 & -0.098521225951477 & -0.269334146770827 & -0.078549719569998 \end{bmatrix} \end{aligned}$$

$$u_d = \mathcal{W}_d^{-1} x_{goal}$$
$$= \begin{bmatrix} 39.861218398928386 \\ 15.054069103644826 \\ 21.313974484218342 \\ 35.452519095860822 \end{bmatrix}$$

(g) Simulate the response of the spring-mass-damper system to the inputs you chose in part (f) using the script `hw3prob04.m`. Submit only the lines of code you added to this script and a snapshot of the figure after the simulation has ended.

```
% - initial condition
x0 = [0;0;0;0];

% - goal
xgoal = [1;0;4;0];

% - length of time interval
h = 1;

% - continuous-time state space system
m1 = 2;
m2 = 2;
k1 = 8;
k2 = 8;
b1 = 1;
b2 = 1;
A = [0 1 0 0;...
    -(k1+k2)/m1 -(b1+b2)/m1 k2/m1 b2/m1;...
    0 0 0 1;...
    k2/m2 b2/m2 -k2/m2 -b2/m2];
B = [0;0;0;1/m2];
C = [1 0 0 0;0 0 1 0];

% - inputs (a column vector of length n=4)
Ad = expm(A*h);
Bd = inv(A)*(expm(A*h) - eye(4))*B;
Wd = [Bd Ad*Bd Ad^2*Bd Ad^3*Bd];

udiscrete = inv(Wd)*xgoal
```

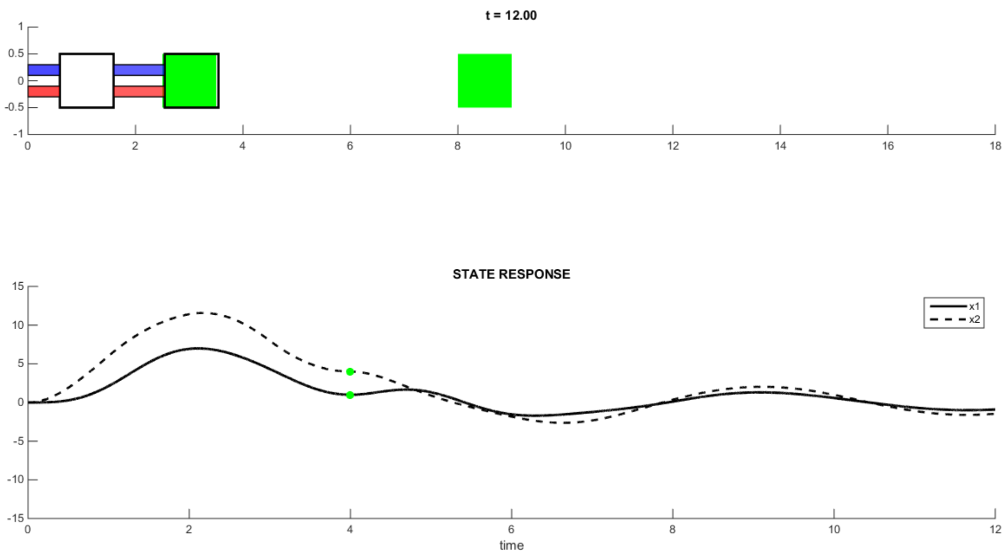


Figure 9: Spring-mass-damper system discrete time simulation.

5. **EXTRA CREDIT:** In the first three problems, you were asked to compute gains that placed eigenvalues in specific locations "by hand" (i.e., by computing the characteristic equation and by choosing gains that make this equation look like what you want). This process can be made systematic. In particular, go back and repeat every such computation *by transformation to controllable canonical form* (see videos on piazza). Note that the MATLAB function `acker` uses exactly this method, and is a good way to check your work, whether or not you do this extra credit problem.

   For problem 1, we have:

$$\begin{aligned}\mathcal{W} &= \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{b}{m^2}\end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{25}\end{bmatrix}\end{aligned}$$

   The closed-loop system characteristic equation for placing eigenvalues at $\sigma \pm j\omega$ is:

$$s^2 + 2\sigma s + \left(\sigma^2 + \omega^2\right) \implies r_1 = 2\sigma \quad r_2 = \sigma^2 + \omega^2$$

$$det(A - s\mathbb{I}) = s^2 + \frac{1}{5}s \implies a_1 = \frac{1}{5} \quad a_2 = 0$$

$$
\begin{aligned}
\mathbf{K}_{CCF} &= \begin{bmatrix} r_1 - a_1 & r_2 - a_2 \end{bmatrix} = \begin{bmatrix} 2\sigma - \frac{1}{5} & \sigma^2 + \omega^2 \end{bmatrix} \\
\mathbf{A}_{CCF} &= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & 0 \\ 1 & 0 \end{bmatrix} \\
\mathbf{B}_{CCF} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathcal{W}_{CCF} &= \begin{bmatrix} \mathbf{B}_{CCF} & \mathbf{A}_{CCF}\mathbf{B}_{CCF} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{bmatrix} \\
\mathbf{K} &= \mathbf{K}_{CCF}\mathcal{W}_{CCF}\mathcal{W}^{-1} = \begin{bmatrix} 10\sigma - 1 & 5(\sigma^2 + \omega^2) \end{bmatrix}
\end{aligned}
$$

For problem 2, we have:

$$
\begin{aligned}
\mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -9 \end{bmatrix}
\end{aligned}
$$

The closed-loop system characteristic equation for placing eigenvalues at $\sigma \pm j\omega$ is:

$$
(s + \sigma_1)(s + \sigma_2) = s^2 + (\sigma_1 + \sigma_2)s + \sigma_1\sigma_2 \implies r_1 = \sigma_1 + \sigma_2 \quad r_2 = \sigma_1\sigma_2
$$

$$
det(A - s\mathbb{I}) = s^2 + 81 \implies a_1 = 0 \quad a_2 = 81
$$

$$
\begin{aligned}
\mathbf{K}_{CCF} &= \begin{bmatrix} r_1 - a_1 & r_2 - a_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 + \sigma_2 & \sigma_1\sigma_2 - 81 \end{bmatrix} \\
\mathbf{A}_{CCF} &= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -81 \\ 1 & 0 \end{bmatrix} \\
\mathbf{B}_{CCF} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathcal{W}_{CCF} &= \begin{bmatrix} \mathbf{B}_{CCF} & \mathbf{A}_{CCF}\mathbf{B}_{CCF} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{K} &= \mathbf{K}_{CCF}\mathcal{W}_{CCF}\mathcal{W}^{-1} = \begin{bmatrix} \sigma_1 + \sigma_2 & \frac{81 - \sigma_1\sigma_2}{9} \end{bmatrix}
\end{aligned}
$$

For problem 3, we have:

$$
\begin{aligned}
\mathcal{W} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{J_{pitch}} & 0 \\ \frac{1}{J_{pitch}} & 0 & \frac{3n^2(J_{yaw}-J_{roll})}{J_{pitch}^2} \\ -\frac{1}{J_{wheel}} & 0 & 0 \end{bmatrix}
\end{aligned}
$$

The closed-loop system characteristic equation for placing eigenvalues at $\sigma \pm j\omega$ is:

$$
\begin{aligned}
&s^3 + (\sigma_1+\sigma_2+\sigma_3)s^2 + (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)s + \sigma_1\sigma_2\sigma_3 \implies \\
&r_1 = \sigma_1+\sigma_2+\sigma_3 \quad r_2 = \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3 \quad r_3 = \sigma_1\sigma_2\sigma_3
\end{aligned}
$$

$$
det(A - s\mathbb{I}) = s^3 + \frac{3n^2\left(J_{roll} - J_{yaw}\right)}{J_{pitch}}s \implies a_1 = 0 \quad a_2 = \frac{3n^2\left(J_{roll} - J_{yaw}\right)}{J_{pitch}} \quad a_3 = 0
$$

$$
\begin{aligned}
\mathbf{K}_{CCF} &= \begin{bmatrix} r_1 - a_1 & r_2 - a_2 & r_3 - a_3 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1+\sigma_2+\sigma_3 & \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3 - \frac{3n^2(J_{roll}-J_{yaw})}{J_{pitch}} & \sigma_1\sigma_2\sigma_3 \end{bmatrix} \\
\mathbf{A}_{CCF} &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3n^2(J_{roll}-J_{yaw})}{J_{pitch}} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
\mathbf{B}_{CCF} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{W}_{CCF} &= \begin{bmatrix} \mathbf{B}_{CCF} & \mathbf{A}_{CCF}\mathbf{B}_{CCF} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{3n^2(J_{roll}-J_{yaw})}{J_{pitch}} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{K} = \mathbf{K}_{CCF}\mathcal{W}_{CCF}\mathcal{W}^{-1} &= \begin{bmatrix} 3n^2\left(J_{yaw} - J_{roll}\right) + J_{pitch}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3) \\ J_{pitch}(\sigma_1+\sigma_2+\sigma_3) - \frac{J_{pitch}^2(\sigma_1\sigma_2\sigma_3)}{3n^2(J_{roll}-J_{yaw})} \\ -\frac{J_{pitch}J_{wheel}\sigma_1\sigma_2\sigma_3}{3n^2(J_{roll}-J_{yaw})} \end{bmatrix}^T
\end{aligned}
$$