# AE353 Homework #5: Observability and Output Feedback

(due at the beginning of class on Friday, March 13)

1. You have seen that the motion of a robot arm with one revolute joint can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1/5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements are the angle ($x_1$) and angular velocity ($x_2$) of the joint, the input $u$ is a torque applied to the arm at the joint, and the output $y$ is a measurement of the angle.

(a) Show that this system is observable. Please do so by hand.

(b) Design an observer

$$\dot{\widehat{x}} = A\widehat{x} + Bu - L(C\widehat{x} - y)$$

with both closed-loop eigenvalues at $-10$. Recall that by "closed-loop," we mean the system

$$\frac{d}{dt}(\widehat{x} - x) = (A - LC)(\widehat{x} - x)$$

where

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$$

is a gain matrix and where

$$\widehat{x} - x$$

is the error in our estimate of the state. Please design this observer "by hand" (i.e., compute the characteristic equation of the closed-loop system in terms of $\ell_1$ and $\ell_2$, then choose gains that make this characteristic equation look like what you want).

(c) Design a controller

$$u = -K\widehat{x} + k_{\text{reference}} r$$

with closed-loop eigenvalues at $-2 \pm j2$ and with zero steady-state error in reference tracking. Recall that by "closed-loop," we mean the system

$$\dot{x} = (A - BK)x + Bk_{\text{reference}} r,$$

which describes how the state would behave in closed-loop if our estimate of it were perfect, i.e., if $\widehat{x} = x$. Design this controller any way you like.

(d) Express the entire closed-loop system—with both observer and controller—in the form

$$\frac{d}{dt}\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} = A_{\text{cl}} \begin{bmatrix} x \\ \widehat{x} \end{bmatrix} + B_{\text{cl}} r$$

for some matrices $A_{\text{cl}}$ and $B_{\text{cl}}$. What are the eigenvalues of this system? Do you notice anything interesting?

(e) Compute the response of the entire closed-loop system to a reference signal

$$r(t) = \pi/2 \qquad\qquad \text{for all } t \geq 0$$

for the initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and for the initial state estimate

$$\widehat{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

over a suitable time horizon. Do so however you like. Plot the following:

- $x_1$ and $\widehat{x}_1$ as functions of time on the same figure;
- $x_2$ and $\widehat{x}_2$ as functions of time on the same figure;
- $u$ as a function of time.

(f) Implement your controller and observer by modifying the function

```
[u,userdata] = ControlLoop(y,r,userdata,params)
```

in the MATLAB script `hw5prob01.m`. As you know, this function should do two things:

- Choose the control input:

$$u(t) = -K\widehat{x}(t) + k_{\text{reference}} r(t)$$

- Update the state estimate:

$$\widehat{x}(t + \Delta t) = \widehat{x}(t) + \Delta t \left( A\widehat{x}(t) + Bu(t) - L(C\widehat{x}(t) - y(t)) \right)$$

You will recognize the update equation as a first-order approximation to

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) - L(C\widehat{x}(t) - y(t)).$$

Submit only your `ControlLoop` function (this should be the only part of the code that you change) and a snapshot of the figure after the simulation has ended.

2. You have seen that the rotational motion of an axisymmetric spacecraft about its yaw and roll axes can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

where the state elements $x_1$ and $x_2$ are the angular velocities about yaw and roll axes, the input $u$ is an applied torque, and the parameter $\lambda = 9$ is the relative spin rate.

(a) Show that this system is observable. Please do so by hand.

(b) Design an observer

$$\dot{\widehat{x}} = A\widehat{x} + Bu - L(C\widehat{x} - y)$$

with both closed-loop eigenvalues at $-15$. Please design this observer "by hand."

(c) Design a controller

$$u = -K\widehat{x}$$

with closed-loop eigenvalues at $-3 \pm j5$. Do so any way you like.

(d) Express the entire closed-loop system—with both observer and controller—in the form

$$\frac{d}{dt}\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} = A_{\text{cl}} \begin{bmatrix} x \\ \widehat{x} \end{bmatrix}$$

for some matrix $A_{\text{cl}}$. What are the eigenvalues of this system? Do you notice anything interesting?

(e) Compute the response of the entire closed-loop system for the initial condition

$$x(0) = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

and for the initial state estimate

$$\widehat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

over a suitable time horizon. Doing so is easy using the matrix exponential, but you may use any other approach you like. Plot the following:

- $x_1$ and $\widehat{x}_1$ as functions of time on the same figure;
- $x_2$ and $\widehat{x}_2$ as functions of time on the same figure;
- $u$ as a function of time.

(f) Implement your controller and observer by modifying the function

```
[u,userdata] = ControlLoop(y,userdata,params)
```

in the MATLAB script `hw5prob02.m`. Submit only your `ControlLoop` function (this should be the only part of the code that you change) and a snapshot of the figure after the simulation has ended.

3. You have seen that the pitch motion of a spacecraft in LEO with a reaction wheel can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/J_{\text{pitch}} \\ -1/J_{\text{wheel}} \end{bmatrix} u,$$

where:

- $x_1$ is the pitch angle;
- $x_2$ is the pitch angular velocity;
- $x_3$ is the spin rate of the reaction wheel relative to the spacecraft;
- $u$ is the applied torque (between the wheel and the spacecraft);
- $\lambda = 3n^2(J_{\text{yaw}} - J_{\text{roll}})/J_{\text{pitch}}$ is a constant;
- $n = 0.0011$ rad/sec is the orbital angular velocity;
- $J_{\text{pitch}} = 25 \text{ kg} \cdot \text{m}^2$, $J_{\text{roll}} = 15 \text{ kg} \cdot \text{m}^2$, $J_{\text{yaw}} = 5 \text{ kg} \cdot \text{m}^2$, and $J_{\text{wheel}} = 1 \text{ kg} \cdot \text{m}^2$ are moments of inertia.

In this problem, you will see how useful it is to be able to test that a system is observable before you actually try to implement an observer.

(a) Suppose you can measure the pitch angle:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

Show, any way you like, that the resulting system is not observable.

(b) Suppose you can measure the pitch angular velocity:

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x.$$

Show, any way you like, that the resulting system is not observable.

(c) Suppose you can measure the spin rate of the reaction wheel relative to the spacecraft:

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x.$$

Show, any way you like, that the resulting system is not observable.

(d) Suppose you can measure the *absolute* spin rate of the reaction wheel (i.e., the angular velocity of the wheel with respect to an inertial reference frame):

$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x.$$

Show, any way you like, that the resulting system *is* observable. (Can you explain this?)

(e) For the output in part (d), design an observer

$$\dot{\widehat{x}} = A\widehat{x} + Bu - L(C\widehat{x} - y)$$

with closed-loop eigenvalues at $-5$, $-0.05$, and $-0.0005$. Do so any way you like.

(f) Design a controller

$$u = -K\widehat{x}$$

with closed-loop eigenvalues at $-1$, $-0.01$, and $-0.0001$. Do so any way you like.

(g) Express the entire closed-loop system—with both observer and controller—in the form

$$\frac{d}{dt}\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} = A_{\text{cl}} \begin{bmatrix} x \\ \widehat{x} \end{bmatrix}$$

for some matrix $A_{\text{cl}}$. What are the eigenvalues of this system? (You should use MATLAB to verify your answer, but by this time I think you see the pattern.)

(h) Compute the response of the entire closed-loop system for the initial condition

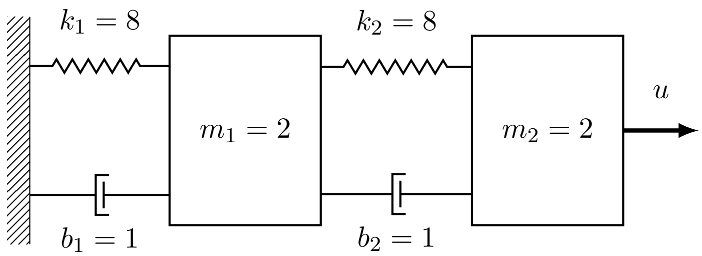
$$x(0) = \begin{bmatrix} 0.00 \\ 0.01 \\ 0.00 \end{bmatrix}$$

and for the initial state estimate

$$\widehat{x}(0) = \begin{bmatrix} 0.500 \\ 0.009 \\ 0.001 \end{bmatrix}$$

over a suitable time horizon. Doing so is easy using the matrix exponential, but you may use any other approach you like. Plot $x$ and $\widehat{x}$ as functions of time on the same figure. It may be easier for you to make sense of these results if you plot them over two different time horizons, one short (e.g., 1 hour) and one long (e.g., 1 day).

Although certainly not required, you might be interested by what happens if the initial condition and initial state estimate are changed, even by a little tiny bit. Try it!




4. You have seen that the spring-mass-damper system shown above can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 4 & -1 & 0.5 \\ 4 & -4 & 0.5 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u$$

where $x_1$ and $x_2$ are the absolute displacements of each mass from their equilibrium positions, $x_3$ and $x_4$ are the corresponding velocities of each mass, and $u$ is the applied force.

(a) Suppose you can measure the difference $x_2 - x_1$, i.e., the amount of stretch in the second spring. Express this measurement as

$$y = Cx$$

for an appropriate choice of $C$. Show that the resulting system is observable.

(b) Design an observer

$$\dot{\widehat{x}} = A\widehat{x} + Bu - L(C\widehat{x} - y)$$

with all closed-loop eigenvalues at $-25$. Do so any way you like.

(c) Design a controller

$$u = -K\widehat{x}$$

with closed-loop eigenvalues at $-5$. Do so any way you like.

(d) Express the entire closed-loop system—with both observer and controller—in the form

$$\frac{d}{dt}\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} = A_{\text{cl}} \begin{bmatrix} x \\ \widehat{x} \end{bmatrix}$$

for some matrix $A_{\text{cl}}$. What are the eigenvalues of this system? (You should use MATLAB to verify your answer, but by this time I think you see the pattern.)

(e) Compute the response of the entire closed-loop system for the initial condition and initial state estimate

$$x(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \qquad \widehat{x}(0) = \begin{bmatrix} 1.1 \\ 2.9 \\ -0.1 \\ 0.1 \end{bmatrix}$$

over a suitable time horizon. Doing so is easy using the matrix exponential, but you may use any other approach you like. Plot the following:

- $y$ as a function of time;
- $u$ as a function of time.
- $x_1$, $\widehat{x}_1$, $x_2$, and $\widehat{x}_2$ as functions of time on the same figure.

(f) Implement your controller and observer by modifying the function

```
[u,userdata] = ControlLoop(y,userdata,params)
```

in the MATLAB script `hw5prob04.m`. Submit only your `ControlLoop` function (this should be the only part of the code that you change) and a snapshot of the figure after the simulation has ended. Why are your results so terrible? Why are they so much different from what you obtained in part (e)? Briefly explain.

**HINT:** run `hw5prob04.m` again with zero input (i.e., with `u=0` in `ControlLoop`). What do you notice about the output response?

(g) Redesign your observer to get better performance. In particular, repeat the following steps until you are satisfied:

- choose different locations for the observer eigenvalues;
- compute the gain matrix $L$ that results in these eigenvalues;
- implement your observer and test using `hw5prob04.m`.

Briefly explain the design process that you used (e.g., how did you define "better" performance? how did you choose the eigenvalue locations?). You need only submit the `ControlLoop` function and the figure for your final design.