AE353 Homework #7:
Frequency Response

(due at the beginning of class on Friday, April 10)

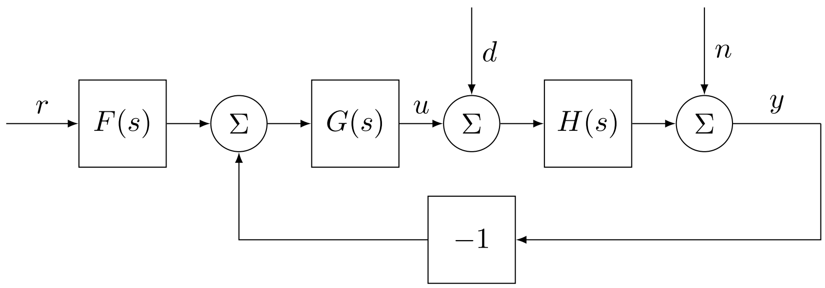


Figure 1: A standard control system.

1. Consider the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

which has solution

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Assume that there is only one input and one output.

(a) Prove that

$$e^{At}A = Ae^{At}$$

and, as a consequence, that

$$e^{At}(sI - A)^{-1} = (sI - A)^{-1}e^{At}.$$

Hint: for the first part of this problem, use the definition of the matrix exponential; for the second part of this problem, use the fact that

$$MN^{-1} = (NM^{-1})^{-1}$$

for any two invertible matrices M and N.

(b) Suppose that $u(t) = e^{st}$, where s is not an eigenvalue of A. Prove that

$$y(t) = Ce^{At}\left(x(0) - (sI - A)^{-1}B\right) + \left(C(sI - A)^{-1}B + D\right)e^{st}.$$

(c) Suppose that the system is stable. Prove that the transfer function from u to y is

$$H(s) = C(sI - A)^{-1}B + D.$$

(d) Suppose that the system is stable and that $u(t) = \sin(\omega t)$. Prove that the steady-state output is

$$y_{ss}(t) = |H(j\omega)|\sin(\omega t + \angle H(j\omega)).$$

(e) For the standard control system in Figure 1, compute the closed-loop transfer functions $T_{ry}(s)$, $T_{dy}(s)$, $T_{ny}(s)$, $T_{du}(s)$, and $T_{nu}(s)$ in terms of $F(s)$, $G(s)$, and $H(s)$.

2. You have seen that the motion of a robot arm with one revolute joint can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1/5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where the state elements are the angle (x_1) and angular velocity (x_2) of the joint, the input u is a torque applied to the arm at the joint, and the output y is a measurement of the angle.

(a) *Open Loop*

i. Compute (by hand) the transfer function $H(s)$ from u to y.

ii. Compute the Bode plot of $H(s)$, both gain and phase. You should discover that the open-loop system acts as a low-pass filter—high-frequency inputs are attenuated.

iii. Compute the crossover frequency, i.e., the frequency ω_c at which $|H(j\omega_c)| = 1$. Please do so by hand, but check your answer with the Bode plot.

iv. Compute the response $y(t)$ to the input $u(t) = \sin(\omega_c t)$ for the initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Plot $u(t)$, $y(t)$, and $y_{ss}(t)$—computed as in Problem 1(d)—on the same figure. Verify your result by implementing $u(t)$ in `ControlLoop` in `hw7prob02.m`. Submit your `ControlLoop` function and a snapshot of the figure after the simulation has ended. Repeat for $u(t) = \sin(0.1\omega_c t)$ and for $u(t) = \sin(10\omega_c t)$. What do you notice?

v. **(EXTRA CREDIT)** You should have found that $y(t)$ and $y_{ss}(t)$ don't quite match—$y(t)$ converges to something of the form $y_{ss}(t) + m$. In other words, the formula from Problem 1d doesn't quite work in this case. Explain why. Derive an expression for $y_{ss}(t)$ that does work, and verify your answer by plotting $u(t)$, $y(t)$, and $y_{ss}(t)$ on the same figure.

(b) *Closed Loop*

i. Suppose you apply a controller of the form

$$u = -K\hat{x} + k_{\text{reference}} r$$

and an observer of the form

$$\dot{\widehat{x}} = A\widehat{x} + Bu - L(C\widehat{x} - y)$$

for some choice of

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

and

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix},$$

where $k_{\text{reference}}$ is chosen so that $y = r$ in steady-state. Compute the transfer functions $F(s)$ and $G(s)$ that, along with $H(s)$ from part a.i, would express the closed-loop system in the standard form of Figure 1. Leave each transfer function in terms of k_1, k_2, ℓ_1, and ℓ_2, but simplify as much as possible (i.e., eliminate common factors from each numerator and denominator).

ii. Compute the closed-loop transfer functions $T_{ry}(s)$, $T_{dy}(s)$, $T_{ny}(s)$, $T_{du}(s)$, and $T_{nu}(s)$. Leave each transfer function in terms of k_1, k_2, ℓ_1, and ℓ_2, but simplify as much as possible (i.e., eliminate common factors from each numerator and denominator). Compare the denominator of each transfer function to the characteristic equations associated with $A - BK$ and $A - LC$. How are the roots of each denominator (i.e., the "poles") related to the eigenvalues of the closed-loop system?

iii. Compute K and L so that the closed-loop system exhibits a time-to-peak of approximately 1.0 seconds and a peak overshoot of less than 10%.

iv. Compute $T_{ry}(s)$ for the K and L you chose. Compute the Bode plot (gain and phase). Find the bandwidth, i.e., the frequency ω_{bw} at which

$$|T_{ry}(j\omega_{bw})| = 1/\sqrt{2}.$$

Please do so by hand, but check your answer with the Bode plot.

v. Simulate the closed-loop system in response to the reference signal

$$r(t) = (\pi/2)\sin(0.1\omega_{bw}t)$$

for the initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \widehat{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

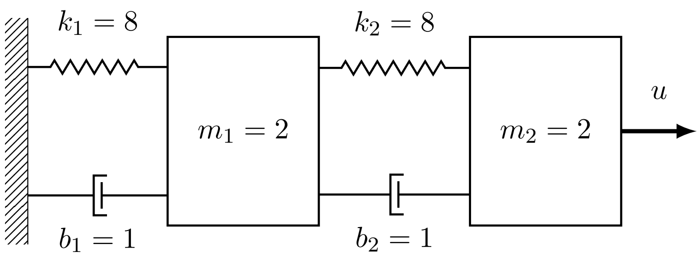
Plot $r(t)$, $y(t)$, and $y_{ss}(t)$ on the same figure. Verify your result by implementing your controller and observer in `ControlLoop` in `hw7prob02.m`. You should call

```
hw7prob02(rmag,rfreq)
```

from the MATLAB prompt to set the magnitude and frequency of $r(t)$. Submit your `ControlLoop` function and a snapshot of the figure after the simulation has ended. Repeat for $u(t) = \sin(10\omega_{bw}t)$. What do you notice?

vi. Compute $T_{dy}(s)$, $T_{ny}(s)$, $T_{du}(s)$, and $T_{nu}(s)$ for the K and L you chose. Draw a Bode plot for each transfer function (gain only). In each case, say whether the transfer function is low-pass, band-pass, or high-pass.

vii. Redesign K and L so that the closed-loop system exhibits a time-to-peak of approximately 0.1 seconds (i.e., ten times smaller) and a peak overshoot of less than 10%. How do the Bode plots in parts iv and vi change?



3. You have seen that the spring-mass-damper system shown above can be described in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 4 & -1 & 0.5 \\ 4 & -4 & 0.5 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x$$

where x_1 and x_2 are the absolute displacements of each mass from their equilibrium positions, x_3 and x_4 are the corresponding velocities of each mass, u is the applied force, and y is a measurement of x_2.

(a) *Open Loop*

i. Compute the transfer function $H(s)$ from u to y.

ii. Compute the Bode plot of $H(s)$, both gain and phase. You should discover that the open-loop system acts as a band-pass filter—low-frequency and high-frequency inputs are attenuated.

iii. Compute the resonant peak, i.e., the frequency ω_{max} at which $|H(j\omega_{\text{max}})|$ is biggest.

iv. Compute the response $y(t)$ to the input $u(t) = \sin(\omega_{\text{max}} t)$ for the initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Plot $u(t)$, $y(t)$, and $y_{ss}(t)$—computed as in Problem 1(d)—on the same figure. Verify your result by implementing $u(t)$ in `ControlLoop` in `hw7prob03.m`. Submit your `ControlLoop` function and a snapshot of the figure after the simulation has ended.

(b) *Closed Loop*

i. Design a controller (with integral action) of the form

$$u = -K\hat{x} - k_{\text{integral}} v$$

where

$$\dot{v} = y - r$$

and an observer of the form

$$\dot{\widehat{x}} = A\widehat{x} + Bu - L(C\widehat{x} - y)$$

so the closed-loop system exhibits a 5%-settling time of approximately 2 seconds.

ii. Compute the transfer functions $F(s)$ and $G(s)$ that, along with $H(s)$ from part a.i, would express the closed-loop system in the standard form of Figure 1.

iii. Compute $T_{ry}(s)$ and draw the Bode plot (gain and phase). Find the bandwidth and the zero-frequency gain. What kind of filter does the closed-loop system act like?

iv. Compute $T_{dy}(s)$, $T_{ny}(s)$, $T_{du}(s)$, and $T_{nu}(s)$. Draw the Bode plot for each transfer function (gain only). In each case, say whether the transfer function is low-pass, band-pass, or high-pass.

v. Simulate the closed-loop system in response to:

$$\begin{aligned} r(t) &= 3 \\ d(t) &= \sin(t/50) \\ n(t) &= (1/5)\sin(50t) \end{aligned}$$

for the initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \widehat{x}(0) = \begin{bmatrix} -0.4 \\ 0.3 \\ 0.5 \\ -0.2 \end{bmatrix}.$$

Plot $r(t)$, $y(t)$, and $y_{ss}(t)$ on the same figure. Verify your result by implementing your controller and observer in `ControlLoop` in `hw7prob03.m`. You should call

```
hw7prob03(rmag,rfreq,dmag,dfreq,nmag,nfreq)
```

from the MATLAB prompt to set the magnitude and frequency of $r(t)$, $d(t)$, and $n(t)$. Submit your `ControlLoop` function and a snapshot of the figure after the simulation has ended.

vi. **(EXTRA CREDIT)** Adjust your controller and observer design to reduce the sensitivity of the closed-loop system to low-frequency disturbance loads and high-frequency measurement noise. How much can you do this without sacrificing performance? (I.e., without increasing the 5%-settling time?) Repeat parts iii-v with your new design and comment on the difference.

4. **(EXTRA CREDIT)**

(a) Draw the Bode plot—both gain and phase—for the following transfer function:

$$H(s) = \frac{2000(s+2)}{(s+0.01)(s^2+40s+40000)}$$

Do this first by hand, then confirm your result with MATLAB. Turn in both your sketch and the MATLAB plot.

(b) What transfer function would have produced the Bode plot shown in Figure 2? Use the smallest number of poles and zeros that give a reasonable fit.

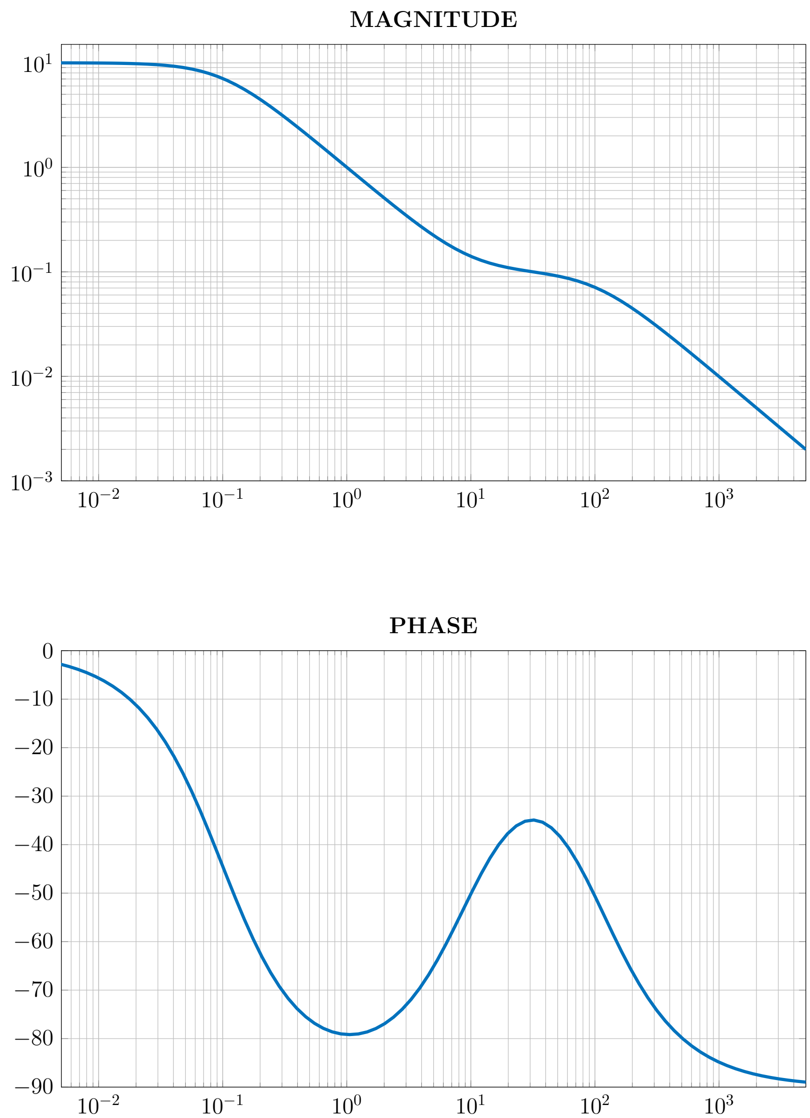


Figure 2: Bode plot (frequency in radians, magnitude in absolute, phase in degrees).