AE 353 Homework #2 Solutions:

State Space and the Matrix Exponential (Part 2)

1. a) *Reference tracking.* $u = -3x + k_{reference}r$

At steady state, $\dot{x} = 0$. From the state space form, we get:

$$0 = \left[-\frac{b}{m}\right]x + \left[\frac{1}{m}\right]u$$

Substitute b = 1, m = 2 and $u = -3x + k_{reference}r$:

$$0 = \left[-\frac{1}{2}\right]x + [\frac{1}{2}](-3x + k_{reference}r)$$

From the state space form we know that: y = x. Further, we wish to have y = r. Thus substituting x = r in the above equation we get:

$$4r = k_{reference}r$$
$$k_{reference} = 4$$

We can also obtain $k_{reference}$ using the formula: $k_{reference} = -(C(A - BK)^{-1}B)^{-1}$

For our system we have: $A = -\frac{1}{2},\ B = \frac{1}{2},\ \ C = 1,\ \ K = 3$

Plugging the values in the above expression, we can verify that $k_{reference} = 4$

Using the above input, we have:

$$\dot{x} = \left[-\frac{b}{m}\right]x + \left[\frac{1}{m}\right](-3x + 4r)$$

$$\dot{x} = -2x + 2r$$

$$y = x$$

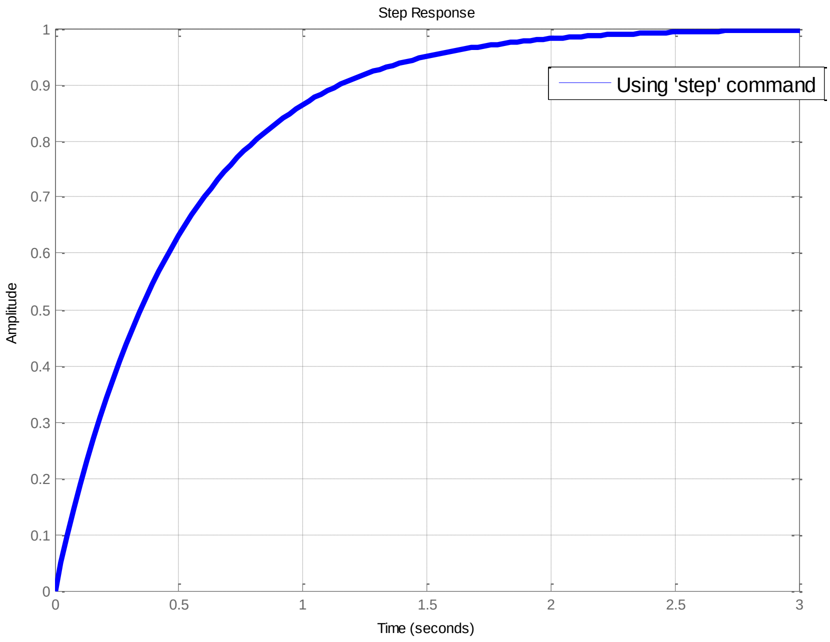
Therefore, $A_{cl} = -2,\ B_{cl} = 2\ and\ C_{cl} = 1.$

In MATLAB entering the following commands will help you display a step response for our system:

```
%Define our matrices:
A_cl = -2;
B_cl = 2;
C_cl = 1;
D_cl = 0;

%Create our state space system:
system = ss(A_cl, B_cl, C_cl, D_cl);
```

%Obtain step response:
step(system);
grid on;



Let us solve it by hand. The complete solution looks like:

$$y(t) = Ce^{(A-BK)t}x(0) + C(A-BK)^{-1}\left(e^{(A-BK)t} - I\right)B(k_{reference}\bar{r} + d)$$

For our system we have: $x(0) = 0,\ \ r = 1\ and\ d = 0$. Substituting these values and matrices A, B, C and K from above, we get:

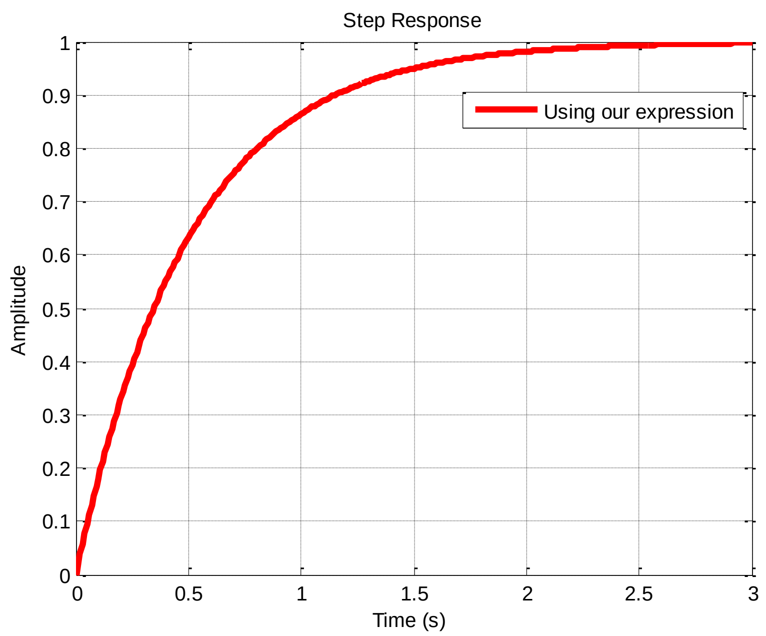
$$y(t) = -e^{-2t} + 1$$

To plot the expression in MATLAB we can use the following commands:

```
figure()
%Create the time vector:
t = 0:0.01:3;

%Evaluate the expression using the time vector:
y = -exp(-2.*t) + 1;  %We use the .* operator since t is a vector

%Create the plot with a red line of thickness 3:
plot(t, y, 'r ', 'LineWidth', 3);
title('Step Response');
xlabel('Time (s) ');   ylabel('Amplitude');
grid on;
```

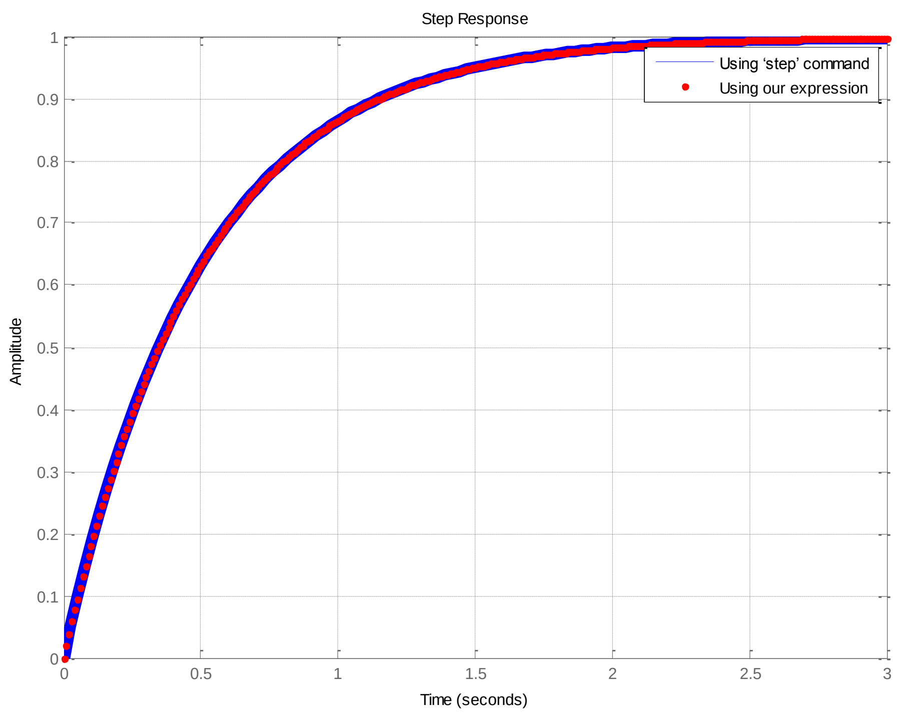


To plot them on the same figure:

```
%Create a new figure
figure()


step(system);
hold on;

plot(t, y, 'r.', 'MarkerSize', 10);
grid on;
legend('Using ‘step’ command', 'Using our expression');
```



Rise time, settling time, overshoot and steady state values can be found by observing the above plots.

From the plot we can see that the output tends to 1 as time increases. That is, $\lim_{t\to\infty} y(t) = 1$. Therefore, steady state value of step response = 1.

Rise time: The amount of time the output takes to go from 10% to 90% of the steady state value.

To find out the rise time you need to zoom in and note down the time when the output crosses 0.1 (10% of steady state value) and 0.9 (90% of steady state value).

Therefore for our case, Rise time = 1.1513 – 0.0527 = 1.0986 seconds

Settling time: Time from "zero" $t0$ until a time ts, where ts is defined as the epoch after which the system does not oscillate outside of the ± 2% window.

You can zoom in and observe the time after which the output always stays between 0.98 (2% below steady state) and 1.02 (2% above steady state).

Therefore for our case, Settling time = 1.956 seconds

Overshoot refers to how much the output exceeds the final steady state value.

For our plot you can see that the output goes from 0 to 1, without going above 1. Therefore there is no overshoot.

b) If r = 0, $u = -3x + d$

Substitute this to obtain the closed loop state space form:

$$\dot{x} = \left[-\frac{1}{2}\right]x + \left[\frac{1}{2}\right](-3x + d)$$

$$\dot{x} = -2x + \frac{d}{2}$$

And again: $y = x$

Therefore we have: $A_{cl} = -2,\ B_{cl} = \frac{1}{2}\ and\ C_{cl} = 1.$

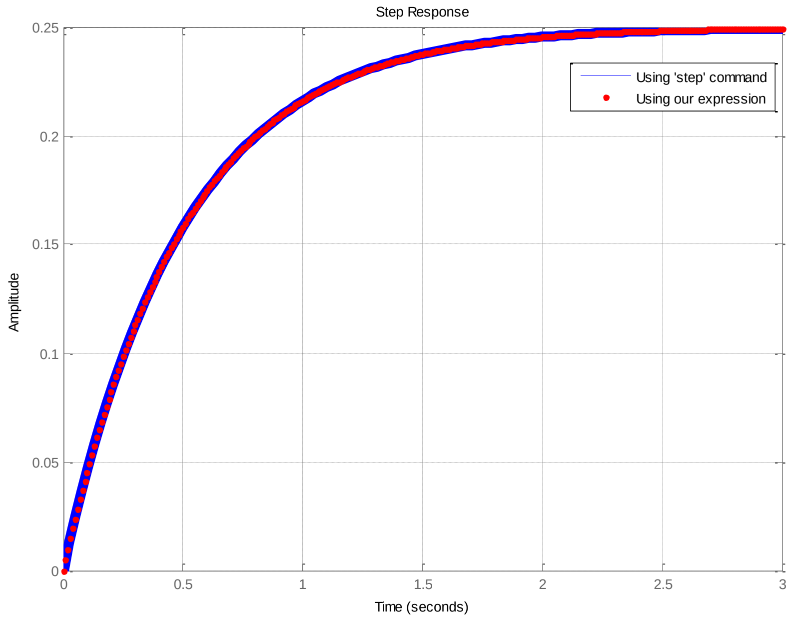
This is how the complete equation looks:

$$y(t) = Ce^{(A-BK)t}x(0) + C(A - BK)^{-1}\left(e^{(A-BK)t} - I\right)B(k_{reference}\bar{r} + d)$$

We have: $x(0) = 0,\ \ r = 0\ and\ d = 1$. Substituting the matrices from part(a) we get:

$$y(t) = \frac{1 - e^{-2t}}{4}$$

Using the same commands as in part (a), we can obtain the following combined plot:



Steady state error (note that r = 0):

$$\lim_{t\to\infty}(y(t) - r) = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{1 - e^{-2t}}{4} = \frac{1}{4}$$

We can see the same steady state value on the plot.

c) New input:

$$u = -3x + k_{reference} r + d - k_{integral} v$$

$$u = -3x + d - 34v$$

The state we want to observe is x. Therefore y = x. Writing down the differential equations:

$$\dot{x} = \left[-\frac{1}{2}\right] x + \left[\frac{1}{2}\right](-3x + d - 34v)$$

$$\dot{x} = -2x - 17v + \frac{d}{2}$$

$$\dot{v} = y - r = x$$

Writing it in the matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -2 & -17 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} d$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

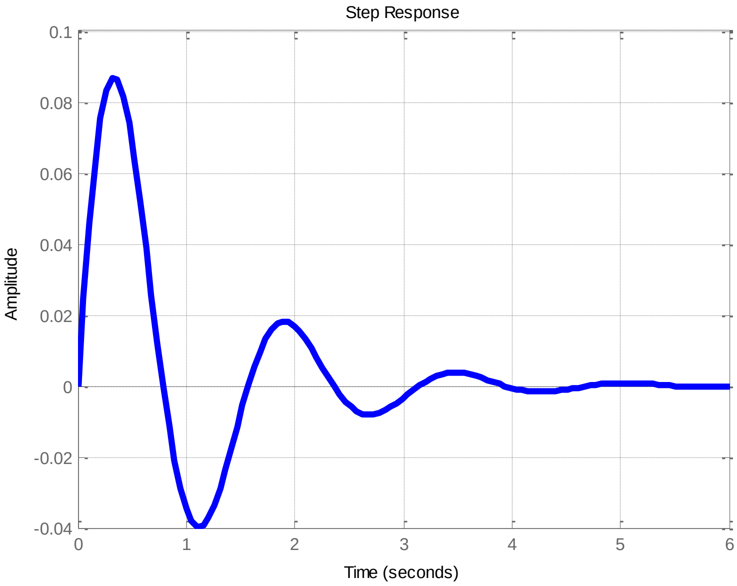
Therefore: $A_{cl} = \begin{bmatrix} -2 & -17 \\ 1 & 0 \end{bmatrix}$; $\quad B_{cl} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$; $\quad C_{cl} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

For asymptotic stability, let us check the eigenvalues of the system:

Characteristic polynomial: $\begin{vmatrix} \lambda+2 & 17 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 17$

On solving for eigenvalues, we get: $\lambda = -1 \pm j4$. The real part of the eigenvalues are negative, which suggests the system is asymptotically stable.

Using the step command in MATLAB, we get the following plot:



For the general state space form:

$$\dot{x}_m(t) = A_m x_m(t) + B_m u_m(t)$$
$$y_m(t) = C_m x_m(t).$$

The solution looks like:

$$y_m(t) = C_m e^{A_m t} x_m(0) + C_m A_m^{-1} \left(e^{A_m t} - I\right) B_m \bar{u}_m$$

Relating this to our state space form, we have: $x_m = z,\ A_m = A_{cl},\ B_m = B_{cl},\ u_m = d,$ $and\ C_m = C_{cl}$

$$\dot{z} = A_{cl} z + B_{cl} d$$
$$y = C_{cl} x$$

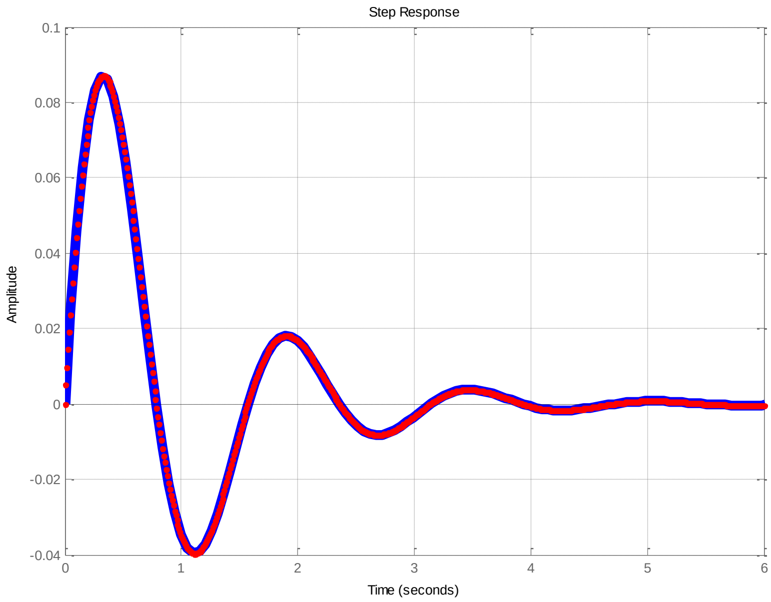
The solution to the above system looks like:

$$y(t) = C_{cl} e^{A_{cl} t} z(0) + C_{cl} A_{cl}^{-1} (e^{A_{cl} t} - I) B_{cl} \bar{d}$$

Writing $e^{A_{cl} t} = V e^{\Lambda t} V^{-1}$ and substituting the matrices, $z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\bar{d} = 1$:

$$y(t) = \frac{e^{-t}}{8} \sin(4t)$$

The combined plot:



For steady state error (note r = 0):

$$\lim_{t\to\infty}(y(t) - r) = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{e^{-t}}{8}\sin(4t) = 0$$

The same steady state value can be observed in the plot.

Here the system is able to tolerate disturbances and return to the reference point. This is because of the additional integral term in the control input.

2. We have the following state space model:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -b/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$$y = [1\ 0]x$$

$$u = -[5\ 1]x + k_{reference} r + d$$

Substituting b = 0.5 and m = 0.1:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, C = [1\ 0], K = [5\ 1]$$

a) We have the following expression for $k_{reference}$:

$$k_{reference} = -(C(A - BK)^{-1}B)^{-1} = 5$$

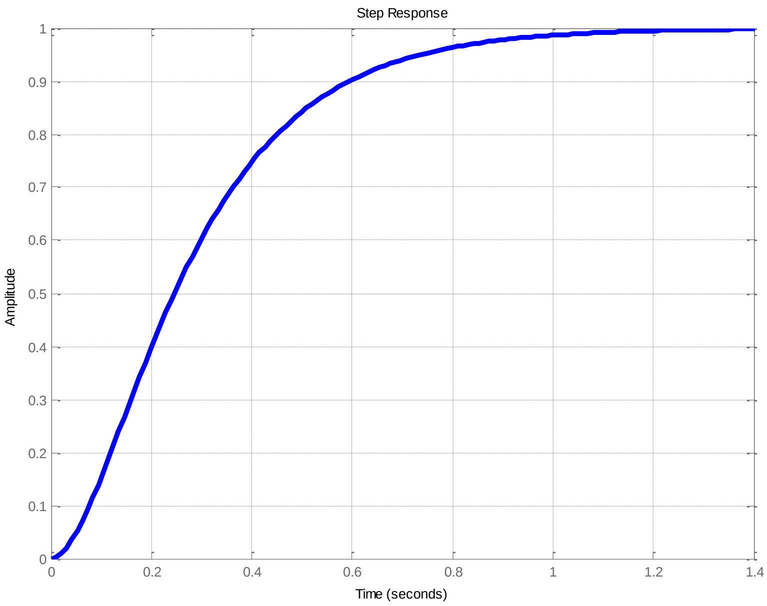
Substituting $k_{reference}$ and u we get the following closed loop state space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} x + \begin{bmatrix} 0 \\ 50 \end{bmatrix} r$$

$$y = [1\ 0]x$$

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix},\ \ B_{cl} = \begin{bmatrix} 0 \\ 50 \end{bmatrix}\ \ and\ C_{cl} = [1\ 0]$$

Using the ‘step’ command in MATLAB, we get the following plot:



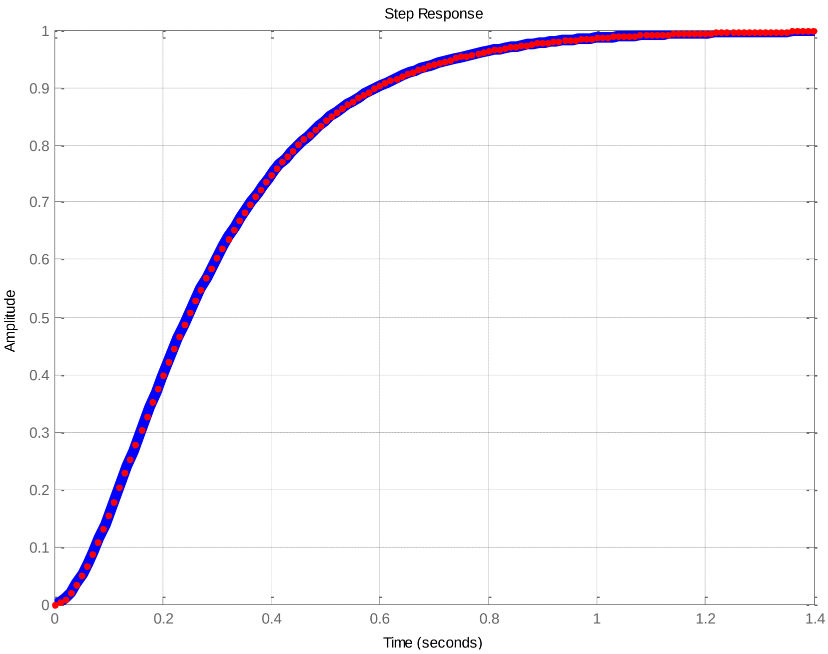
Let us solve it by hand. The complete solution looks like:

$$y(t) = Ce^{(A-BK)t}x(0) + C(A-BK)^{-1}\left(e^{(A-BK)t} - I\right)B(k_{reference}\bar{r} + d)$$

For our system we have: $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \ r = 1\ and\ d = 0$. Substituting these values and matrices A, B, C and K from above, we get:

$$y(t) = e^{-10t} - 2e^{-5t} + 1$$

Plotting it on the same figure, we get:



From the graph, we can observe the following quantities:

Rise time = 0.594 – 0.0759 = 0.5181 seconds
Settling time = 0.9198 seconds
Overshoot = 0
Steady state value = 1.

b) With r = 0, the input becomes:

$$u = \ -[5\ 1]x + d$$

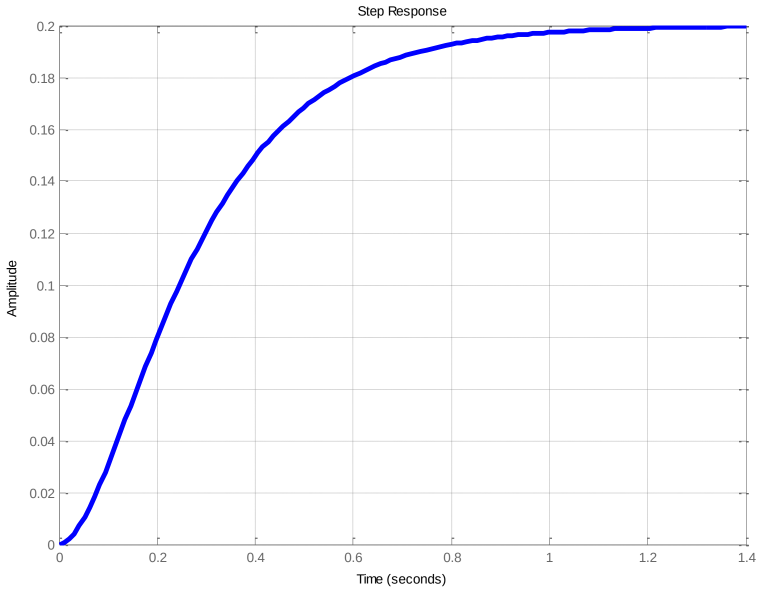
Plugging this in the state space form, we can obtain the closed loop state space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} d$$

$$y = [1\ 0]x$$

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix},\ B_{cl} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}\ and\ C_{cl} = [1\ 0]$$

Using ‘step’ command in MATLAB, we get:



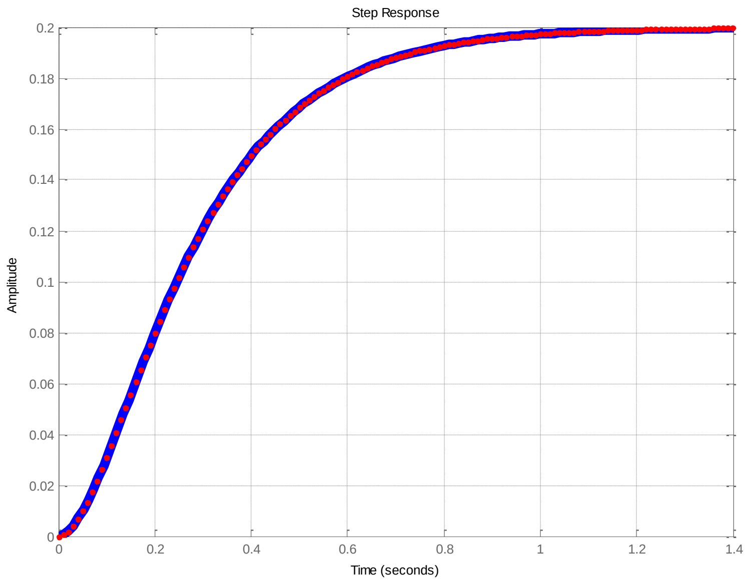
The total response is:

$$y(t) = Ce^{(A-BK)t}x(0) + C(A-BK)^{-1}\left(e^{(A-BK)t} - I\right)B(k_{reference}\bar{r} + d)$$

We have: $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ r = 0\ and\ d = 1$. Substituting the matrices from part(a) we get:

$$y(t) = \frac{1 + e^{-10t} - 2e^{-5t}}{5}$$

Plotting it on the same figure, we get:



For steady state error (note r = 0):

$$\lim_{t\to\infty}(y(t)-r) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\frac{1+e^{-10t}-2e^{-5t}}{5} = \frac{1}{5}$$

The same steady state value can be observed in the plot.

c) New input:

$$u = -[5\ 1]x + k_{reference}r + d - k_{integral}v$$

$$u = -[5\ 1]x + d - 10v$$

We know: y = Cx. Writing down the equations:

$$\dot{v} = y - r = [1\ 0]x$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix}x + \begin{bmatrix} 0 \\ 10 \end{bmatrix}d - \begin{bmatrix} 0 \\ 100 \end{bmatrix}v$$

$$y = [1\ 0]x$$

Incorporating v in the state vector, we get the following state space form:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -50 & -15 & -100 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}d$$

$$y = [1\ 0\ 0]x$$

Therefore:

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 \\ -50 & -15 & -100 \\ 1 & 0 & 0 \end{bmatrix}, B_{cl} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, C_{cl} = [1\ 0\ 0]$$

Using MATLAB to compute the eigenvalues of A_{cl}:

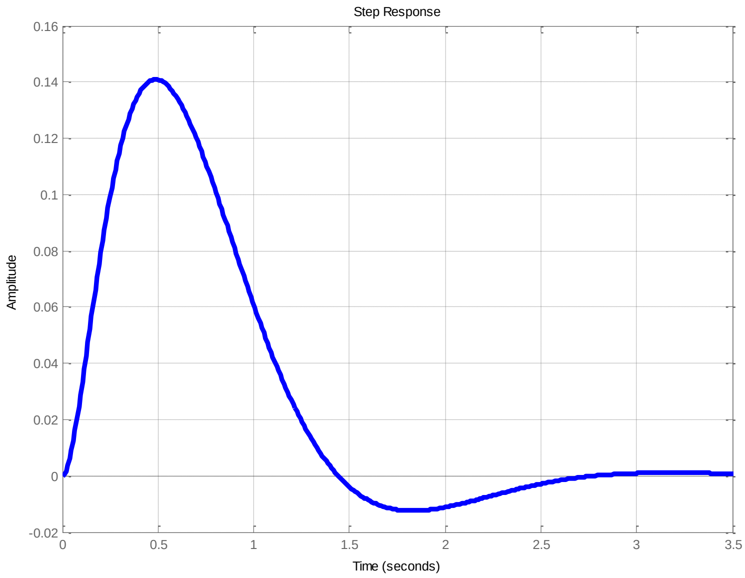
eig(A_cl)

We get:

$$\lambda = -11.378, -1.811 \pm j2.3472$$

All 3 eigenvalues have a negative real part, and hence the system is stable.

Using the ‘step’ command in MATLAB we get:



At steady state, the rate of change of the state vector should be 0 (since our system is asymptotically stable).

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = 0$$

This implies that $\dot{v} = 0$. We know that $\dot{v} = y - r = y$

Therefore at steady state, $y = 0$.

The same steady state value can be observed in the plot.

Here the system is able to tolerate disturbances and return to the reference point. This is because of the additional integral term in the control input.

3. State space form:

$$\dot{x} = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [1\ 0]x$$

Input:

$$u = -[6 - 1]x + k_{reference} r + d$$

a) Since d = 0, the input becomes:

$$u = -[6 - 1]x + k_{reference} r$$

Therefore:

$$\dot{x} = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} x + \begin{bmatrix} k_{reference} \\ 0 \end{bmatrix} r$$

At steady state, the state vector must be constant. Looking at the second equation in the state-space form above:

$$\dot{x_2} = -9x_1 = 0 \qquad \text{(at steady state)}$$

Therefore, irrespective of what $k_{reference}$ we choose, at steady state x_1 will always go to 0 for the above system. From $y = [1\ 0]x$, we know that $y = x_1$. Thus there exists no choice of $k_{reference}$ for which y will always go to r at steady state.

Also, in the formula:

$$k_{reference} = -\frac{1}{(C(A - BK)^{-1}B)}$$

we cannot compute $k_{reference}$, since the denominator goes to 0.

Considering the new output:

$$y = [0\ 1]x$$

We have:

$$A = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [0\ 1], K = [6 - 1]$$

We get:

$$k_{reference} = -(C(A - BK)^{-1}B)^{-1} = -10$$

The closed loop system looks like:

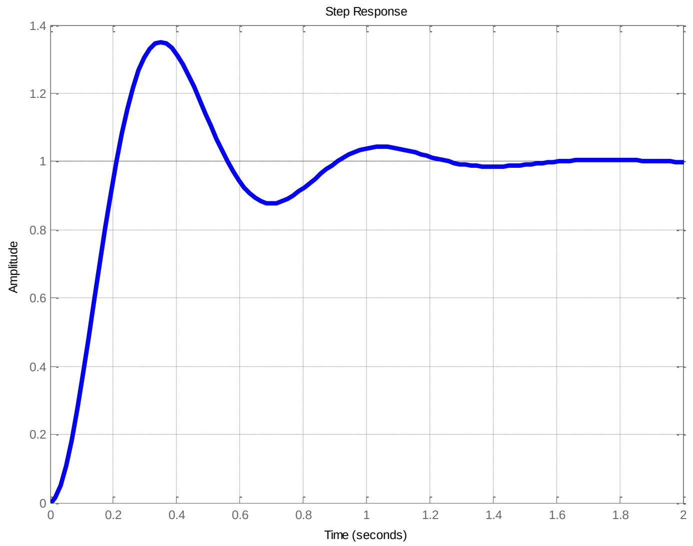
$$\dot{x} = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} x + \begin{bmatrix} -10 \\ 0 \end{bmatrix} r$$

$$y = [0\ 1]x$$

Where:

$$A_{cl} = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix}, B_{cl} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, C_{cl} = [0\ 1]$$

Using the 'step' command in MATLAB, we get the following plot:

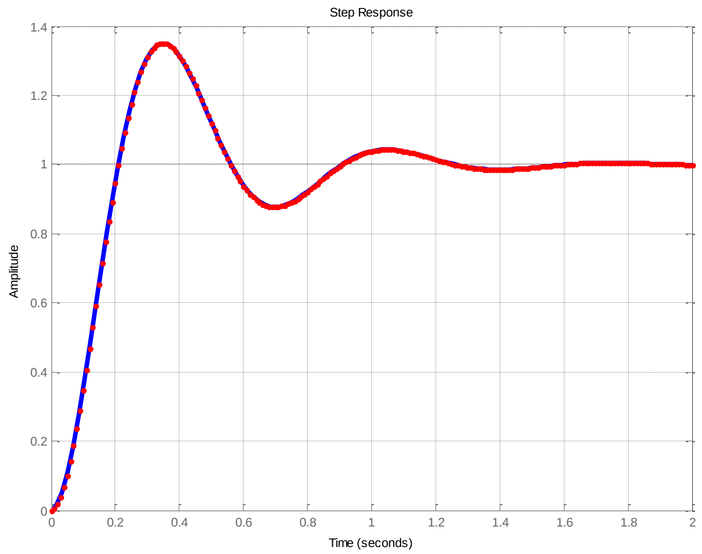


This is how the complete equation looks:

$$y(t) = Ce^{(A-BK)t}x(0) + C(A-BK)^{-1}\left(e^{(A-BK)t} - I\right)B(k_{reference}\bar{r} + d)$$

Substituting the matrices and $x(0) = \begin{bmatrix}0\\0\end{bmatrix}, d = 0 \; and \; \bar{r} = 1$:

$$y(t) = 1 - \frac{e^{-3t}}{3}\sin(9t) - e^{-3t}\cos(9t)$$



From the figure we can obtain the following quantities:

Rise time = 0.1419 seconds
Settling time = 1.1785 seconds
Overshoot = 35%
Steady state value = 1

b) For disturbance rejection:

$$u = -[6 - 1]x + d$$

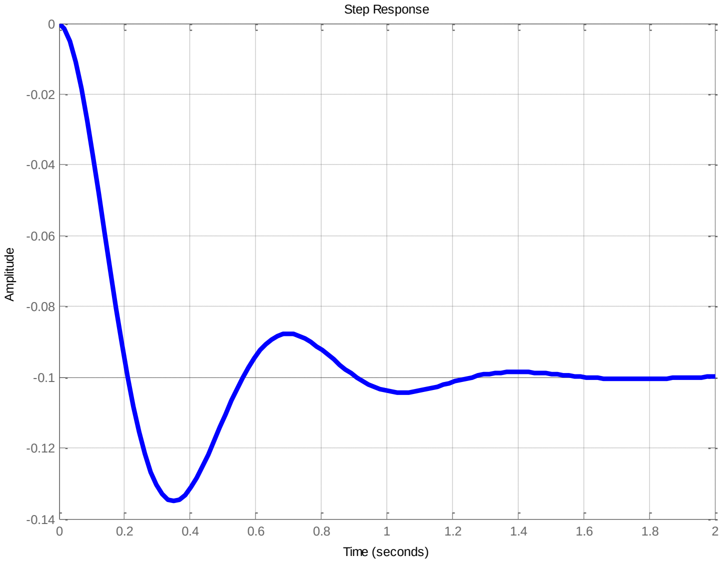
$$\dot{x} = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d$$

$$y = [0\ 1]x$$

Where:

$$A_{cl} = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix}, B_{cl} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_{cl} = [0\ 1]$$

Using the ‘step’ command in MATLAB we get the following plot:

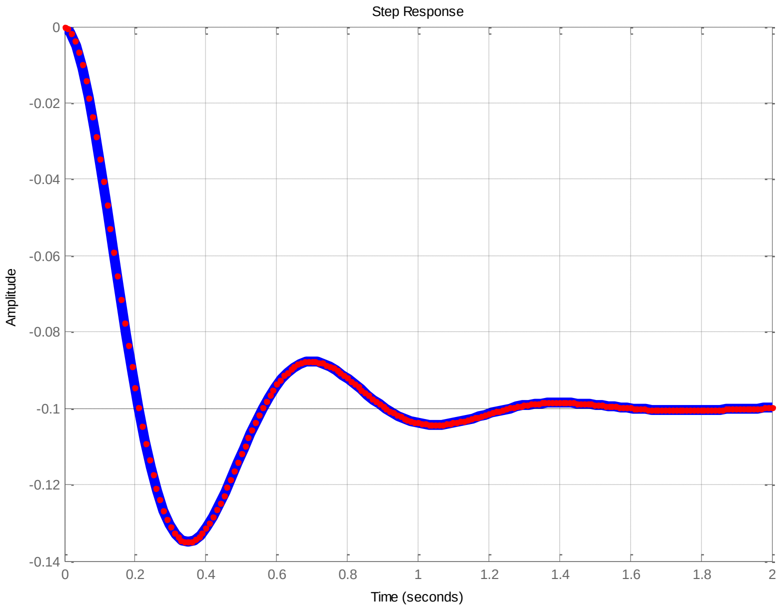


Total response:

$$y(t) = Ce^{(A-BK)t}x(0) + C(A-BK)^{-1}\left(e^{(A-BK)t} - I\right)B(k_{reference}\bar{r} + d)$$

Substituting the matrices and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d = 1\ and\ \bar{r} = 0$:

$$y(t) = \frac{e^{-3t}}{10}\cos(9t) + \frac{e^{-3t}}{30}\sin(9t) - \frac{1}{10}$$



For steady state error (note $r = 0$):

$$\lim_{t\to\infty}(y(t)-r) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{e^{-3t}}{10}\cos(9t) + \frac{e^{-3t}}{30}\sin(9t) - \frac{1}{10}\right) = -\frac{1}{10}$$

The same steady state value can be observed in the plot.

c) New input:

$$u = -[6\ -1]x + k_{reference}r + d - k_{integral}v$$

$$u = -[6\ -1]x + d + 5v$$

We know: y = Cx. Writing down the equations:

$$\dot{v} = y - r = [0\ 1]x$$

$$\dot{x} = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d + \begin{bmatrix} 5 \\ 0 \end{bmatrix} v$$

$$y = [0\ 1]x$$

Incorporating v in the state vector, we get the following state space form:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -6 & 10 & 5 \\ -9 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d \qquad \text{and} \qquad y = [0\ 1\ 0]x$$

Here:

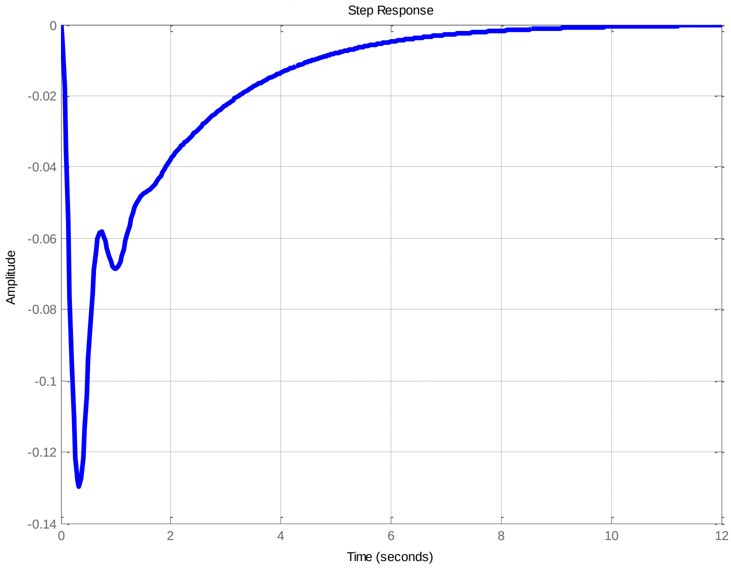
$$A_{cl} = \begin{bmatrix} -6 & 10 & 5 \\ -9 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_{cl} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_{cl} = [0\ 1\ 0]$$

Using MATLAB to find the eigenvalues of A_{cl}, we get:

$$\lambda = -0.5162, -2.7419 \pm j8.9247$$

All 3 eigenvalues have a negative real part, and hence the system is stable.

Using the ‘step’ command in MATLAB we get:



At steady state, the rate of change of the state vector should be 0 (since our system is asymptotically stable).

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = 0$$

This implies that $\dot{v} = 0$. We know that $\dot{v} = y - r = y$

Therefore at steady state, $y = 0$.

The same steady state value can be observed in the plot.

Here the system is able to tolerate disturbances and return to the reference point. This is because of the additional integral term in the control input.