1 Examples (1/28/15)

Consider the system

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx.\end{aligned}$$

As you know, the solution to this system is

$$y(t) = Ce^{At}x(0),$$

where

$$e^{At}$$

is the matrix exponential function. In what follows, we will express this solution in terms of scalar exponential functions for three different values of A. We will also say whether the corresponding system is asymptotically stable. In all three cases, we will assume that

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and

$$x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

1.1 Stable system, real eigenvalues

Suppose

$$A = \begin{bmatrix} -7 & -10 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A

Eigenvalues of A are solutions to

$$0 = \det(\lambda I - A).$$

We have

$$\begin{aligned}0 &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda + 7 & 10 \\ -1 & \lambda \end{vmatrix} \\ &= \lambda^2 + 7\lambda + 10 \\ &= (\lambda + 5)(\lambda + 2).\end{aligned}$$

So, the eigenvalues of A are $\lambda_1 = -5$ and $\lambda_2 = -2$. Note that, at this point, we already know that the system is asymptotically stable—both eigenvalues have negative real part.

Find the eigenvectors of A

Eigenvectors of A are solutions v to

$$0 = (\lambda I - A)v.$$

First, we find the eigenvector

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_1 = -5$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_1 I - A)v_1 \\ &= \begin{bmatrix} 2 & 10 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \begin{bmatrix} 2\alpha + 10\beta \\ -\alpha - 5\beta \end{bmatrix}. \end{aligned}$$

As we have discussed, the "top" and "bottom" rows of this equation are redundant—they both require that

$$\alpha = -5\beta,$$

where we may choose β to be any non-zero real number. Suppose we choose $\beta = 1$. Then

$$v_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

Second, we find the eigenvector

$$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_2 = -2$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_2 I - A) v_2 \\ &= \begin{bmatrix} 5 & 10 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \begin{bmatrix} 5\alpha + 10\beta \\ -\alpha - 2\beta \end{bmatrix}, \end{aligned}$$

from which we conclude that

$$\alpha = -2\beta.$$

Choosing $\beta = 1$, we have

$$v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Find $y(t)$

Define

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}$$

and

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 1 & 1 \end{bmatrix}.$$

We compute $y(t)$ as follows:

$$\begin{aligned} y(t) &= C V e^{\Lambda t} V^{-1} x(0) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1/3 & -2/3 \\ 1/3 & 5/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{1}{3} \left(-4e^{-5t} + 7e^{-2t} \right). \end{aligned}$$

This result confirms our earlier statement that the system is asymptotically stable—as $t \to \infty$, both scalar exponential terms near zero, so $y(t) \to 0$.

1.2 Unstable system, real eigenvalues

Suppose

$$A = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A

We have

$$\begin{aligned} 0 &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda + 1 & -6 \\ -1 & \lambda \end{vmatrix} \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2). \end{aligned}$$

So, the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 2$. Note that, at this point, we already know that the system is *not* asymptotically stable—one eigenvalue does *not* have negative real part.

Find the eigenvectors of A

First, we find the eigenvector

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_1 = -3$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_1 I - A) v_1 \\ &= \begin{bmatrix} -2 & -6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \begin{bmatrix} -2\alpha - 6\beta \\ -\alpha - 3\beta \end{bmatrix}, \end{aligned}$$

from which we conclude

$$\alpha = -3\beta.$$

Choosing $\beta = 1$, we have

$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Second, we find the eigenvector

$$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_2 = 2$. We have

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_2 I - A) v_2 \\
&= \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= \begin{bmatrix} 3\alpha - 6\beta \\ -\alpha + 2\beta \end{bmatrix},
\end{aligned}$$

from which we conclude that

$$\alpha = 2\beta.$$

Choosing $\beta = 1$, we have

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find $y(t)$

Define

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}.$$

We compute $y(t)$ as follows:

$$\begin{aligned}
y(t) &= C V e^{\Lambda t} V^{-1} x(0) \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \frac{1}{5} \left(0 e^{-3t} + 5 e^{2t} \right) \\
&= e^{2t}.
\end{aligned}$$

1.3 Stable system, complex eigenvalues

Suppose

$$A = \begin{bmatrix} -10 & -169 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A

We have

$$\begin{aligned} 0 &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda + 10 & 169 \\ -1 & \lambda \end{vmatrix} \\ &= \lambda^2 + 10\lambda + 169. \end{aligned}$$

This expression is not easily factored, so we solve it using the quadratic formula:

$$\begin{aligned} \lambda &= \frac{-10 \pm \sqrt{10^2 - 4(169)}}{2} \\ &= -5 \pm \sqrt{25 - 169} \\ &= -5 \pm \sqrt{-144} \\ &= -5 \pm j12. \end{aligned}$$

So, the eigenvalues of A are $\lambda_1 = -5 - j12$ and $\lambda_2 = -5 + j12$. Note that, at this point, we already know that the system is asymptotically stable—both eigenvalues have negative real part.

Find the eigenvectors of A

First, we find the eigenvector

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_1 = -5 - j12$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_1 I - A)v_1 \\ &= \begin{bmatrix} 5 - j12 & -169 \\ -1 & -5 - j12 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \end{aligned}$$

from which we conclude

$$\alpha = (-5 - j12)\beta.$$

Choosing $\beta = 1$, we have

$$v_1 = \begin{bmatrix} -5 - j12 \\ 1 \end{bmatrix}.$$

Second, we find the eigenvector

$$v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

that is associated with $\lambda_2 = -5 + j12$. We have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= (\lambda_2 I - A)v_2 \\ &= \begin{bmatrix} 5 + j12 & -169 \\ -1 & -5 + j12 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \end{aligned}$$

from which we conclude that

$$\alpha = (-5 + j12)\beta.$$

Choosing $\beta = 1$, we have

$$v_2 = \begin{bmatrix} -5 + j12 \\ 1 \end{bmatrix}.$$

Find $y(t)$

Define

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 - j12 & 0 \\ 0 & -5 + j12 \end{bmatrix}$$

and

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -5 - j12 & -5 + j12 \\ 1 & 1 \end{bmatrix}.$$

We compute $y(t)$ as follows:

$$\begin{aligned} y(t) &= CVe^{\Lambda t}V^{-1}x(0) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -5 - j12 & -5 + j12 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(-5-j12)t} & 0 \\ 0 & e^{(-5+j12)t} \end{bmatrix} \begin{bmatrix} j/24 & (12 + j5)/24 \\ -j/24 & (12 - j5)/24) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{1}{24}\left((12 + j7)e^{(-5-j12)t} + (12 - j7)e^{(-5+j12)t}\right) \\ &= \frac{1}{24}e^{-5t}\left((12 + j7)e^{-j12t} + (12 - j7)e^{j12t}\right). \end{aligned}$$

We can simplify this expression further by using the identity

$$e^{j\omega} = \cos\omega + j\sin\omega.$$

In particular, we have

$$e^{-j12t} = \cos(12t) - j\sin(12t)$$

and

$$e^{j12t} = \cos(12t) + j\sin(12t).$$

Consequently,

$$\begin{aligned}
&(12 + j7)e^{-j12t} + (12 - j7)e^{j12t} \\
&\qquad = (12 + j7)(\cos(12t) - j\sin(12t)) + (12 - j7)(\cos(12t) + j\sin(12t)) \\
&\qquad = (12\cos(12t) + 7\sin(12t)) + j(7\cos(12t) - 12\sin(12t)) \\
&\qquad\qquad + (12\cos(12t) + 7\sin(12t)) + j(-7\cos(12t) + 12\sin(12t)) \\
&\qquad = (24\cos(12t) + 14\sin(12t)),
\end{aligned}$$

and so

$$y(t) = \frac{1}{24}e^{-5t}\left(24\cos(12t) + 14\sin(12t)\right).$$