AE353 Homework #1: State Space and the Matrix Exponential (Part 1)

SOLUTIONS

(due at the beginning of class on Friday, January 30)

1. The relationship between the applied torque τ (in units of $\mathrm{N}\cdot\mathrm{m}$) and the angular velocity ω (in units of rad/s) of a control moment gyro on a spacecraft is

$$m\dot{\omega} + b\omega = \tau,$$

where

$m = 2 \text{ kg} \cdot \text{m}^2$	is the moment of inertia, and
$b = 1 \text{ kg} \cdot \text{m}^2/\text{s}$	is the coefficient of viscous friction.

Our goal in this problem is to spin down the gyro—i.e., for $\omega(t) \to 0$ as $t \to \infty$.

(a) Put this system in state-space form. Define states, inputs, and outputs as follows:

$$x = \begin{bmatrix}\omega\end{bmatrix}, \qquad u = \begin{bmatrix}\tau\end{bmatrix}, \qquad y = \begin{bmatrix}\omega\end{bmatrix}.$$

$$\dot{x} = \dot{\omega} = \frac{-b}{m}\omega + \frac{1}{m}\tau$$

$$\mathbf{A} = \frac{-b}{m} \qquad \mathbf{B} = \frac{1}{m} \qquad \mathbf{C} = 1$$

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}x(t)\end{aligned}$$

(b) Suppose we choose the input $u(t) = 0$ for all $t \geq 0$. Is the system asymptotically stable?

The degenerative matrix $\mathbf{A}$ has one eigenvalue $s = \frac{-b}{m} = -0.5$ that has a strictly negative real part. Therefore, the system is asymptotically stable.

(c) Suppose we choose the input $u(t) = -3x(t)$ for all $t \geq 0$. Is the system asymptotically stable?

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t)\\ \dot{x}(t) &= \frac{-b}{m}x(t) + \frac{1}{m}\left[-3x(t)\right]\\ \dot{x}(t) &= \left(\frac{-b}{m} - \frac{3}{m}\right)x(t)\\ \dot{x}(t) &= \left[\frac{-(b+3)}{m}\right]x(t)\end{aligned}$$

The degenerative matrix $\mathbf{A}$ has one eigenvalue $s = -2$ that has a strictly negative real part. Therefore, the system is asymptotically stable.

(d) Suppose $x(0) = 1$. For each of the inputs considered in (b) and (c), write an expression for $y(t)$ in terms of the scalar exponential function, and use MATLAB to evaluate this expression for $t \in [0, 10]$. Plot both results on the same figure. What is the difference?

For $u(t) = 0$

$$y(t) = x(t) = e^{\frac{-1}{2}t}x(0)$$
$$y(t) = x(t) = e^{\frac{-1}{2}t}$$

For $u(t) = -3x(t)$

$$y(t) = x(t) = e^{-2t}x(0)$$
$$y(t) = x(t) = e^{-2t}$$

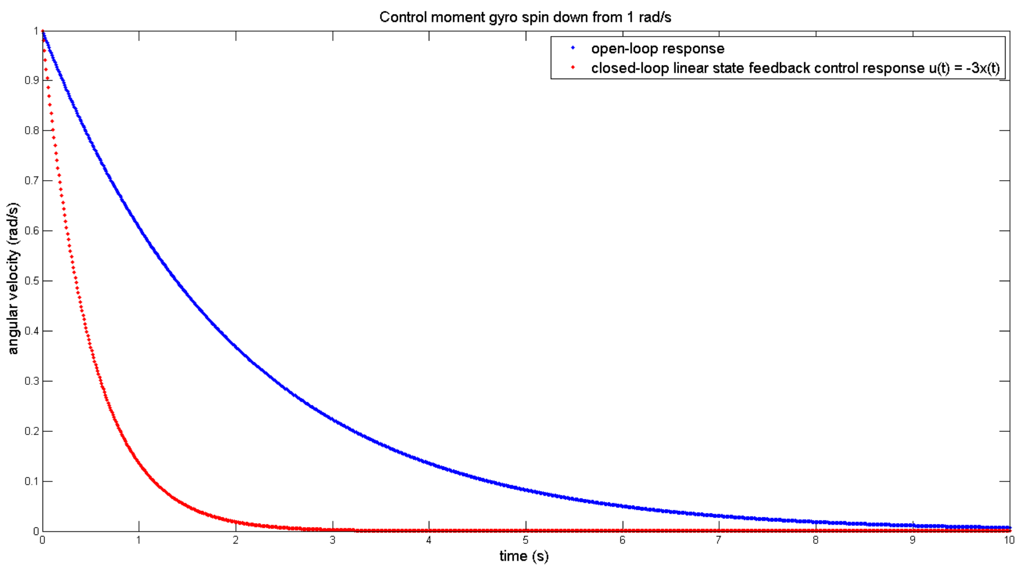

Figure 1: Comparison of a control moment gyro spin-down maneuver from 1 rad/s to 0 rad/s due to viscous friction alone and using angular velocity state feedback control.

2. The relationship between the applied torque τ (in units of $\mathrm{N} \cdot \mathrm{m}$) and the angle θ (in units of rad) of an antenna on a spacecraft is

$$m\ddot{\theta} + b\dot{\theta} = \tau,$$

where

$m = 0.1 \text{ kg} \cdot \text{m}^2$	is the moment of inertia, and
$b = 0.5 \text{ kg} \cdot \text{m}^2/\text{s}$	is the coefficient of viscous friction.

Our goal here is to keep the antenna angle zero—i.e., for $\theta(t) \to 0$ as $t \to \infty$.

(a) Put this system in state-space form. Define states, inputs, and outputs as follows:

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \qquad u = \begin{bmatrix} \tau \end{bmatrix}, \qquad y = \begin{bmatrix} \theta \end{bmatrix}.$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \tau$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) \end{aligned}$$

(b) Suppose we choose the input $u(t) = 0$ for all $t \geq 0$. Is the system asymptotically stable?

$$det\left(\mathbf{A} - s\mathbb{I}\right) = 0$$

$$\begin{aligned} &\begin{vmatrix} -s & 1 \\ 0 & -\frac{b}{m} - s \end{vmatrix} = 0 \\ &\therefore s^2 + \frac{b}{m}s = 0 \\ &\therefore s = 0 \quad \text{and} \quad s = -\frac{b}{m} = -5 \end{aligned}$$

Since one of the eigenvalues has a not strictly negative real part (i.e. zero), the system is not asymptotically stable. Technically, this system is marginally stable since none of the eigenvalues are strictly in the right-half plane, but any physically realizable system would be considered unstable.

(c) Suppose we choose the input

$$u(t) = -\begin{bmatrix} 5 & 1 \end{bmatrix} x(t)$$

for all $t \geq 0$. Is the system asymptotically stable?

$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} 5 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{5}{m} & -\frac{(b+1)}{m} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&\begin{vmatrix} -s & 1 \\ -15 & -50-s \end{vmatrix} = 0 \\
&\therefore s^2 + 15s + 50 = 0 \\
&\therefore s = -10 \quad \text{and} \quad s = -5
\end{aligned}
$$

Since both eigenvalues have strictly negative real parts, the system is asymptotically stable.

(d) Suppose

$$x(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

For each of the inputs considered in (b) and (c), please do the following:

- Write an expression for $y(t)$ in terms of the matrix exponential function, and use MATLAB to evaluate this expression for $t \in [0, 2]$.
- Write an expression for $y(t)$ in terms of *scalar* exponential functions—i.e., diagonalize to simplify the matrix exponential—and use MATLAB to evaluate this expression for $t \in [0, 2]$. If you arrive at an expression with complex exponentials, please rewrite them in terms of sines and cosines.

Plot all four results—two for the input of (b), two for the input of (c)—on the same figure. What differences do you see?

i) <u>For $u(t) = 0$</u>

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x(t)$$

For $s = 0$

$$
\begin{aligned}
\begin{bmatrix} -s & 1 \\ 0 & -5-s \end{bmatrix} \mathbf{v}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\therefore \mathbf{v}_1 &= \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \alpha \in \mathbb{R}
\end{aligned}
$$

For $s = -5$

$$\mathbf{v}_2 = \beta \begin{bmatrix} -\frac{m}{b} \\ 1 \end{bmatrix} = \beta \begin{bmatrix} -0.2 \\ 1 \end{bmatrix} \qquad \beta \in \mathbb{R}$$

From the eigenvector information we can form the eigenvector matrix, and then diagonalize $\mathbf{A}$

$$\mathbf{V} = \begin{bmatrix} 1 & -0.2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & -0.2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}$$

Finally, the output $y(t)$ is computed as follows:

$$\begin{aligned} y(t) &= \mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}x(0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= \frac{4}{5}\left(1 - e^{-5t}\right) \end{aligned}$$

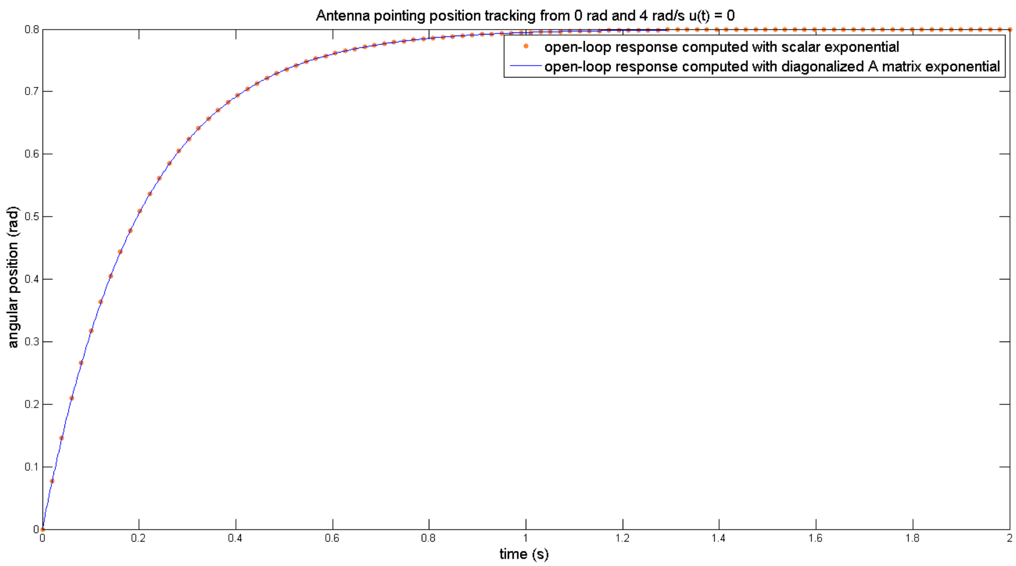

Figure 2: Open-loop spacecraft antenna pointing angle as a function of time with viscous friction effects.

i) <u>For $u(t) = -\begin{bmatrix} 5 & 1 \end{bmatrix} x(t)$</u>

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{5}{m} \end{bmatrix} x(t) = \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} x(t)$$

For $s = -10$

$$\begin{bmatrix} -s & 1 \\ -50 & -15-s \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \mathbf{v}_1 = \alpha \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

For $s = -5$

$$\mathbf{v}_2 = \beta \begin{bmatrix} -0.2 \\ 1 \end{bmatrix} \qquad \beta \in \mathbb{R}$$

From the eigenvector information we can form the eigenvector matrix, and then diagonalize $\mathbf{A}$

$$\mathbf{V} = \begin{bmatrix} -0.1 & -0.2 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 10 & 2 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} \begin{bmatrix} -0.1 & -0.2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 0 & -5 \end{bmatrix}$$

Finally, the output $y(t)$ is computed as follows:

$$\begin{aligned} y(t) &= \mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}x(0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -0.1 & -0.2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-10t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} 10 & 2 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ &= \frac{4}{5}\left(e^{-5t} - e^{-10t}\right) \end{aligned}$$

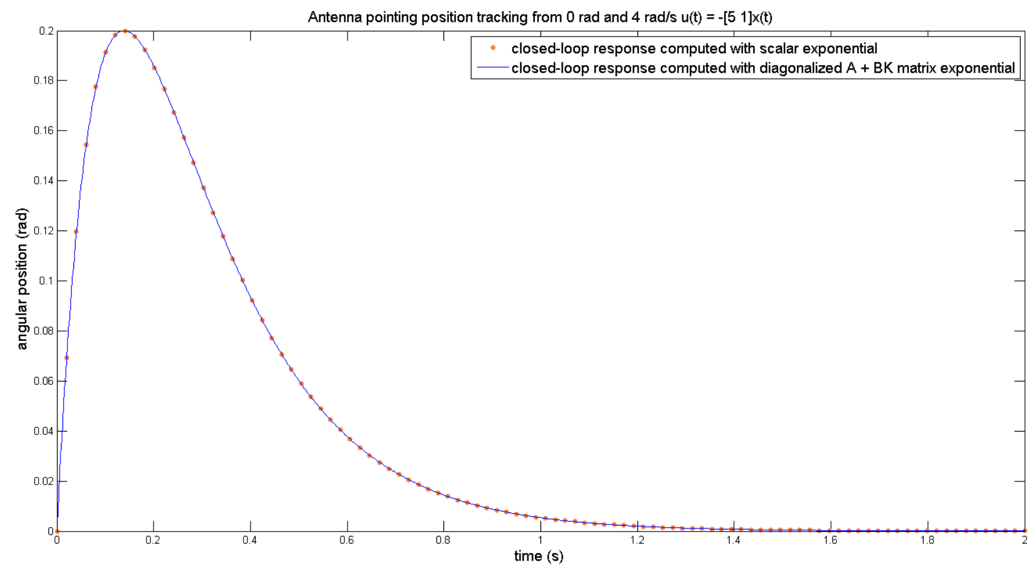

Figure 3: Spacecraft antenna pointing angle as a function of time using position and angular velocity state feedback control.

3. Euler's equations describe the rotational motion of a rigid body subject to applied torques. When written with respect to principle axes (where J_1, J_2, J_3 are the principal moments of inertia), these equations can be written

$$\begin{aligned} \tau_1 &= J_1\dot{\omega}_1 - (J_2 - J_3)\,\omega_2\omega_3 \\ \tau_2 &= J_2\dot{\omega}_2 - (J_3 - J_1)\,\omega_3\omega_1 \\ \tau_3 &= J_3\dot{\omega}_3 - (J_1 - J_2)\,\omega_1\omega_2. \end{aligned}$$

Consider an axisymmetric rigid body, for which

$$J_1 = J_2 = J_t$$

and

$$J_3 = J_a.$$

The subscripts t and a indicate the *transverse* and *axial* moments of inertia, respectively. Suppose $\tau_2 = \tau_3 = 0$, so we restrict the applied torque to be about only one of the transverse axes. Then, we have

$$\begin{aligned}
\tau_1 &= J_t\dot{\omega}_1 - (J_t - J_a)\,\omega_2\omega_3 \\
0 &= J_t\dot{\omega}_2 + (J_t - J_a)\,\omega_1\omega_3 \\
0 &= J_a\dot{\omega}_3.
\end{aligned}$$

The third equation implies that ω_3 is constant. We call it the *spin rate* and denote it by

$$n = \omega_3.$$

What remains is

$$\begin{aligned}
\tau_1 &= J_t\dot{\omega}_1 - (J_t - J_a)\,n\omega_2 \\
0 &= J_t\dot{\omega}_2 + (J_t - J_a)\,n\omega_1.
\end{aligned}$$

If we define the *relative spin rate* as

$$\lambda = \left(\frac{J_t - J_a}{J_t}\right) n,$$

then we can simplify our equations of motion further to

$$\begin{aligned}
(\tau_1/J_t) &= \dot{\omega}_1 - \lambda\omega_2 \\
0 &= \dot{\omega}_2 + \lambda\omega_1.
\end{aligned} \tag{1}$$

In what follows, we will assume that $J_t = 5000\ \text{kg}\cdot\text{m}^2$, $J_a = 2000\ \text{kg}\cdot\text{m}^2$, and $n = 15$ rad/sec. Our goal is to eliminate roll and pitch motion—i.e., for $\omega_1(t) \to 0$ and $\omega_2(t) \to 0$ as $t \to \infty$.

(a) Put this system in state-space form. Define states, inputs, and outputs as follows:

$$x = \begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}, \qquad u = \begin{bmatrix}\tau_1/J_t\end{bmatrix}, \qquad y = \begin{bmatrix}\omega_1\end{bmatrix}.$$

$$\begin{bmatrix}\dot{\omega}_1 \\ \dot{\omega}_2\end{bmatrix} = \begin{bmatrix}0 & \lambda \\ -\lambda & 0\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix} + \begin{bmatrix}1 \\ 0\end{bmatrix}\frac{\tau_1}{J_t}$$

$$\mathbf{A} = \begin{bmatrix}0 & \lambda \\ -\lambda & 0\end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix}1 \\ 0\end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\begin{aligned}
\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\
y(t) &= \mathbf{C}x(t)
\end{aligned}$$

(b) Suppose we choose the input $u(t) = 0$ for all $t \geq 0$. Is the system asymptotically stable?

$$det\left(\mathbf{A} - s\mathbb{I}\right) = 0$$

$$\begin{vmatrix} -s & \lambda \\ -\lambda & -s \end{vmatrix} = 0$$
$$\therefore s^2 + \lambda^2 = 0$$
$$\therefore s = \pm j\lambda = \pm j9$$

Since the eigenvalues do not have strictly negative real parts, the system is not asymptotically stable. Since both eigenvalues are exactly on the imaginary axis the system is marginally stable, but any physically realizable system would be considered unstable.

(c) Suppose we choose the input

$$u(t) = -\begin{bmatrix} 6 & -1 \end{bmatrix} x(t)$$

for all $t \geq 0$. Is the system asymptotically stable?

$$\begin{aligned}
\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \dot{\omega}_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 6 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \dot{\omega}_2 \end{bmatrix} \\
&= \begin{bmatrix} -6 & \lambda + 1 \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \dot{\omega}_2 \end{bmatrix} \\
&= \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \dot{\omega}_2 \end{bmatrix}
\end{aligned}$$

$$\begin{vmatrix} -6 - s & 10 \\ -9 & -s \end{vmatrix} = 0$$
$$\therefore s^2 + 6s + 90 = 0$$
$$\therefore s = -3 \pm j9$$

Since both eigenvalues have strictly negative real parts, the system is asymptotically stable.

(d) Suppose

$$x(0) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

For each of the inputs considered in (b) and (c), please do the following:

- Write an expression for $y(t)$ in terms of the matrix exponential function, and use MATLAB to evaluate this expression for $t \in [0, 2]$.
- Write an expression for $y(t)$ in terms of *scalar* exponential functions—i.e., diagonalize to simplify the matrix exponential—and use MATLAB to evaluate this expression for $t \in [0, 2]$. If you arrive at an expression with complex exponentials, please rewrite them in terms of sines and cosines.

Plot all four results—two for the input of (b), two for the input of (c)—on the same figure. What differences do you see?

i) <u>For $u(t) = 0$</u>

$$\dot{x}(t) = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} x(t) = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} x(t)$$

For $s = j9$

$$\begin{bmatrix} -s & 9 \\ -9 & -s \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\therefore \mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ j \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

For $s = -j9$

$$\mathbf{v}_2 = \beta \begin{bmatrix} 1 \\ -j \end{bmatrix} \qquad \beta \in \mathbb{R}$$

From the eigenvector information we can form the eigenvector matrix, and then diagonalize $\mathbf{A}$

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 0.5 & -j0.5 \\ 0.5 & j0.5 \end{bmatrix} \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} = \begin{bmatrix} j9 & 0 \\ 0 & -j9 \end{bmatrix}$$

Finally, the output $y(t)$ is computed as follows:

$$\begin{aligned} y(t) &= \mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}x(0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{j9t} & 0 \\ 0 & e^{-j9t} \end{bmatrix} \begin{bmatrix} 0.5 & -j0.5 \\ 0.5 & j0.5 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} \\ &= cos(9t) - 4sin(9t) \end{aligned}$$

i) <u>For $u(t) = -\begin{bmatrix} 6 & -1 \end{bmatrix} x(t)$</u>

$$\dot{x}(t) = \begin{bmatrix} -6 & \lambda + 1 \\ -\lambda & 0 \end{bmatrix} x(t) = \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} x(t)$$

For $s = -3 + j9$

$$\begin{bmatrix} -6 - s & 10 \\ -9 & -s \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\therefore \mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ \frac{3}{10} + j\frac{9}{10} \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

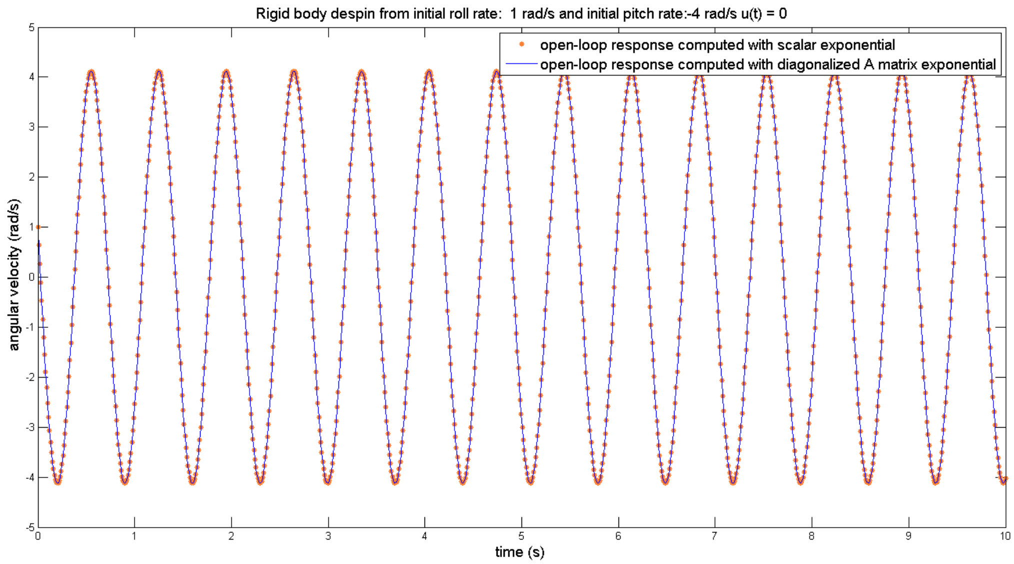

Figure 4: Rigid body angular velocity about its roll axis as a function of time.

For $s = -3 - j9$

$$\mathbf{v}_2 = \beta \begin{bmatrix} 1 \\ \frac{3}{10} - j\frac{9}{10} \end{bmatrix} \qquad \beta \in \mathbb{R}$$

From the eigenvector information we can form the eigenvector matrix, and then diagonalize $\mathbf{A}$

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ \frac{3}{10} + j\frac{9}{10} & \frac{3}{10} - j\frac{9}{10} \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \frac{1}{2} + \frac{j}{6} & -j\frac{5}{9} \\ \frac{1}{2} - \frac{j}{6} & j\frac{5}{9} \end{bmatrix} \begin{bmatrix} -6 & 10 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{3}{10} + j\frac{9}{10} & \frac{3}{10} - j\frac{9}{10} \end{bmatrix} = \begin{bmatrix} -3 + j9 & 0 \\ 0 & -3 - j9 \end{bmatrix}$$

Finally, the output $y(t)$ is computed as follows:

$$\begin{aligned} y(t) &= \mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}x(0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{3}{10} + j\frac{9}{10} & \frac{3}{10} - j\frac{9}{10} \end{bmatrix} \begin{bmatrix} e^{(-3+j9)t} & 0 \\ 0 & e^{(-3-j9)t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{j}{6} & -j\frac{5}{9} \\ \frac{1}{2} - \frac{j}{6} & j\frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} \\ &= e^{-3t}\left[cos(9t) - \frac{43}{9}sin(9t)\right] \end{aligned}$$

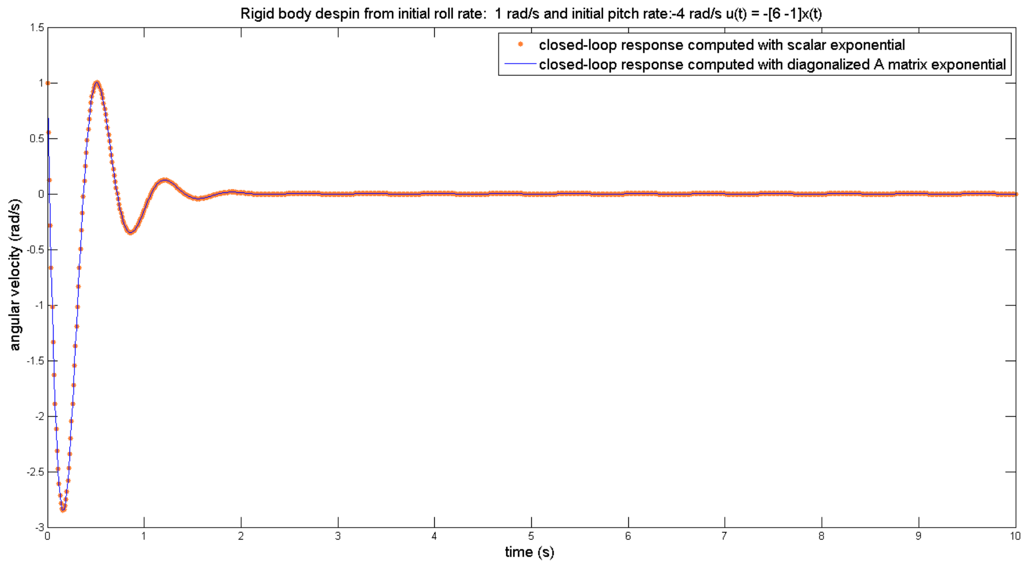

Figure 5: Rigid body angular velocity about its roll axis as a function of time during a despin maneuver accomplished using angular velocity state feedback control.

4. The approximate relationship between the force f applied laterally to the base of an upright rocket (e.g., with a gimbaled thruster) and the pitch angle θ of this rocket is

$$\ddot{\theta} = \left(\frac{mg\ell}{J_t}\right)\theta - \left(\frac{\gamma}{J_t}\right)\dot{\theta} + \left(\frac{\ell}{J_t}\right)f,$$

where we will assume:

- $m = 1$ is the mass of the rocket,
- $g = 10$ is the acceleration of gravity,
- $\ell = 5/2$ is the length of the rocket,
- $J_t = 5$ is the transverse moment of inertia of the rocket,
- $\gamma = 20$ is a coefficient of viscous friction.

The goal is to keep the rocket upright—i.e., for $\theta(t) \to 0$ as $t \to \infty$.

(a) Put this system in state-space form. Define states, inputs, and outputs as follows:

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \qquad u = \begin{bmatrix} f \end{bmatrix}, \qquad y = \begin{bmatrix} \theta \end{bmatrix}.$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{l}{J_t} \end{bmatrix} f$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{l}{J_t} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t)$$

(b) Suppose we choose the input $u(t) = 0$ for all $t \geq 0$. Is the system asymptotically stable?

$$det\left(\mathbf{A} - s\mathbb{I}\right) = 0$$

$$\begin{vmatrix} -s & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} - s \end{vmatrix} = \begin{vmatrix} -s & 1 \\ 5 & -4 - s \end{vmatrix} = 0$$
$$\therefore s^2 + 4s - 5 = 0$$
$$\therefore s = -5 \quad \text{or} \quad s = 1$$

Since one of the eigenvalues does not have a strictly negative real part, the system is asymptotically unstable.

(c) Suppose we choose the input

$$u(t) = -\begin{bmatrix} 28 & 4 \end{bmatrix} x(t)$$

for all $t \geq 0$. Is the system asymptotically stable?

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 28 & 4 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \dot{\omega_2} \end{bmatrix}$$

$$\begin{vmatrix} -s & 1 \\ -9 & -6 - s \end{vmatrix} = 0$$
$$\therefore s^2 + 6s + 9 = 0$$
$$\therefore s = -3 \quad \text{or} \quad s = -3$$

Since both eigenvalues have strictly negative real parts, the system is asymptotically stable.

(d) Suppose

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For each of the inputs considered in (b) and (c), please do the following:

- Write an expression for $y(t)$ in terms of the matrix exponential function, and use MATLAB to evaluate this expression for $t \in [0, 3]$.
- Write an expression for $y(t)$ in terms of *scalar* exponential functions—i.e., diagonalize to simplify the matrix exponential—and use MATLAB to evaluate this expression for $t \in [0, 3]$. If you arrive at an expression with complex exponentials, please rewrite them in terms of sines and cosines.

Plot all four results—two for the input of (b), two for the input of (c)—on the same figure. What differences do you see?

HINT: You may have trouble solving this problem at first, since you will likely find that the closed-loop "A" matrix A_{cl} is not diagonalizable for the input proposed in part (c). Instead, try the coordinate transformation $x = Vz$, where V is the matrix produced in MATLAB as follows:

```
[V, J] = jordan(Acl)
```

Then, use the fact that

$$e^{Jt} = \begin{bmatrix} e^{ht} & te^{ht} \\ 0 & e^{ht} \end{bmatrix}$$

whenever

$$J = \begin{bmatrix} h & 1 \\ 0 & h \end{bmatrix}.$$

Note that J is an example of a matrix in so-called "Jordan form."

i) For $u(t) = 0$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} x(t)$$

For $s = -5$

$$\begin{bmatrix} -s & 1 \\ 5 & -s-4 \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ -5 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

For $s = 1$

$$\mathbf{v}_2 = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \beta \in \mathbb{R}$$

From the eigenvector information we can form the eigenvector matrix, and then diagonalize $\mathbf{A}$

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

Finally, the output $y(t)$ is computed as follows:

$$\begin{aligned} y(t) &= \mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}x(0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{6}e^{-5t} + \frac{5}{6}e^{t} \end{aligned}$$

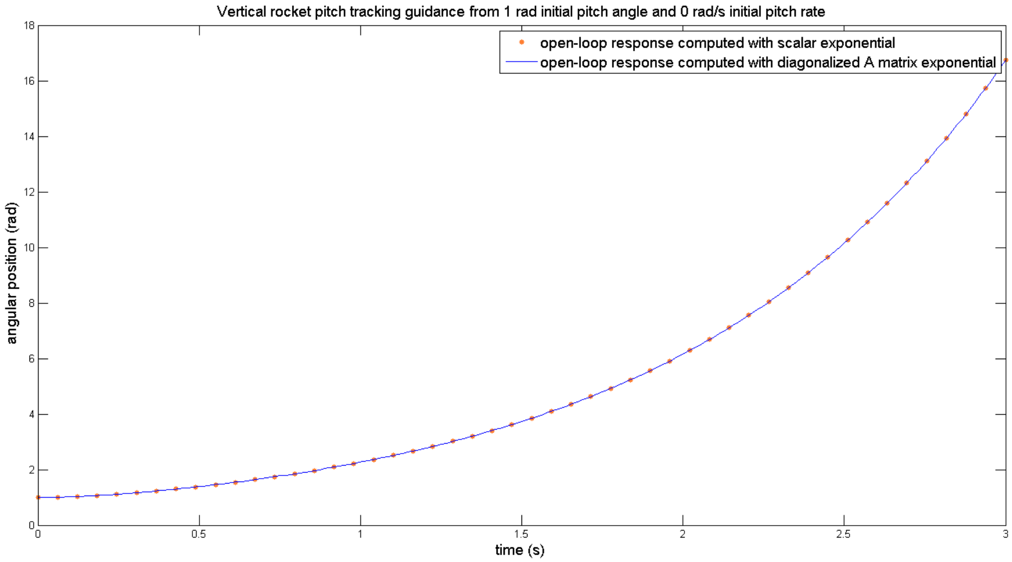

Figure 6: Launch vehicle pitch angle as a function of time.

i) For $u(t) = -\begin{bmatrix} 28 & 4 \end{bmatrix} x(t)$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} x(t)$$

For $s = -3$

$$\begin{bmatrix} -s & 1 \\ -9 & -6-s \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

From the eigenvector information we can form the eigenvector matrix, and then diagonalize $\mathbf{A}$

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}$$

Unfortunately, $\mathbf{V}$ is not diagonalizable as its determinant is zero (also its eigenvectors are not all linearly independent). To remedy this, we revert to the Jordan form of $\mathbf{V}$, which is:

$$\mathbf{V}_{jordan} = \begin{bmatrix} 3 & 1 \\ -9 & 0 \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 0 & -\frac{1}{9} \\ 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -9 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$$

Finally, the output $y(t)$ is computed as follows:

$$
\begin{aligned}
y(t) &= \mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}x(0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{9} \\ 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= e^{-3t} + 3te^{-3t}
\end{aligned}
$$

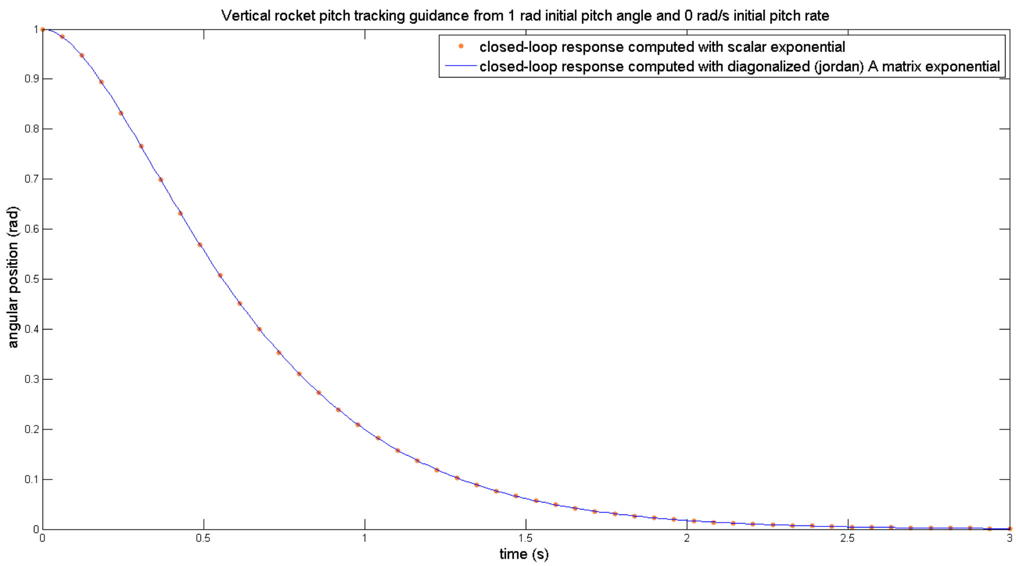

Figure 7: Launch vehicle pitch angle as a function of time using angular position and velocity state feedback control.

5. *Extra Credit (worth +25% — no partial credit, but you may iterate with Tim, Donald, or Akshay until your answer is acceptable).* In this problem, you will explore some properties of the matrix exponential and its application to analysis of volume-preserving flows.

   (a) Suppose that $A \in \mathbb{R}^{n\times n}$ is diagonalizable and that it has eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that the eigenvalues of e^A are $e^{\lambda_1}, \dots, e^{\lambda^n}$.

   Hints:

   - the eigenvalues of $M \in \mathbb{R}^{n\times n}$ satisfy $\det(\lambda I - M) = 0$
   - if $M \in \mathbb{R}^{n\times n}$ is invertible, then $\det(M^{-1}) = (\det M)^{-1}$

   If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is diagonalizable, then $\exists$ an invertible matrix $\mathbf{V} \in \mathbb{R}^{n\times n}$, and a diagonal matrix $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ such that:

$$\Lambda = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

   Where $\lambda_1, ..., \lambda_n$ are the eigenvalues of the matrix $\mathbf{A}$.

$$\begin{aligned} \therefore \mathbf{A} &= \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \\ \therefore \mathbf{A}^2 &= \mathbf{A}\mathbf{A} = \left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\right)\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\right) = \mathbf{V}\boldsymbol{\Lambda}^2\mathbf{V}^{-1} \end{aligned}$$

   Now assume $\mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}^n\mathbf{V}^{-1}$

$$\therefore \mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A} = \left(\mathbf{V}\boldsymbol{\Lambda}^n\mathbf{V}^{-1}\right)\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\right) = \left(\mathbf{V}\boldsymbol{\Lambda}^{n+1}\mathbf{V}^{-1}\right)$$

   By induction, we have $\mathbf{A}^n = \mathbf{V}\boldsymbol{\Lambda}^n\mathbf{V}^{-1}$.
   Substitute this result into the definition of the matrix exponential:

$$\begin{aligned} e^{\mathbf{A}} &= \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{V}\boldsymbol{\Lambda}^n\mathbf{V}^{-1} \\ &= \mathbf{V}\left(\sum_{n=0}^{\infty}\frac{\boldsymbol{\Lambda}^n}{n!}\right)\mathbf{V}^{-1} \\ \therefore e^{\mathbf{A}} &= \mathbf{V}\mathbf{e}^{\Lambda}\mathbf{V}^{-1} \end{aligned}$$

   Therefore $e^{\mathbf{A}}$ is also a diagonalizable matrix whose eigenvalues are $e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_n}$

(b) Consider a set $Z_0 \subseteq \mathbb{R}^n$ and a state-space system

$$\dot{x} = Ax,$$

where $A \in \mathbb{R}^{n\times n}$ is diagonalizable. We can propagate the set Z_0 along the flow induced by the system as

$$Z(t) = e^{At} Z_0 = \left\{e^{At} z : z \in Z_0\right\}.$$

Denote the volume of the set Z_0 by $\operatorname{vol}(Z_0)$. We say that the system preserves volume if $\operatorname{vol}(Z(t)) = \operatorname{vol}(Z_0)$ for all t. Is it possible for a stable system to preserve volume? Prove your answer.

Hints:

- if $M \in \mathbb{R}^{n\times n}$ and $Z \subseteq \mathbb{R}^n$, then

$$\operatorname{vol}(MZ) = |\det M| \operatorname{vol}(Z),$$

where

$$MZ = \{Mz : z \in Z\}$$

- if $M \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\det M = \lambda_1 \cdots \lambda_n$

$$z(t) = e^{At} z_0$$

$$\begin{aligned} vol(z(t)) &= vol\left(e^{At} z_0\right) \\ &= \|det\left(e^{At}\right)\| vol(z_0) \end{aligned}$$

$$det\left(e^{At}\right) = e^{\lambda_1 t} \cdot e^{\lambda_2 t} \cdots e^{\lambda_n t}$$

If $Re(\lambda_i) = 0$, then $det\left(e^{At}\right) = 1$ even if $Im(\lambda_i) \neq 0$ because, provided that A has real-valued entries, if any of its eigenvalues are complex, they will always occur in complex conjugate pairs.

To see this we note that if $\lambda_i \in \mathbb{C}$ is a complex eigenvalue of A, with a non-zero eigenvector $v \in \mathbb{C}$, by definition this means:

$$Av = \lambda_i v, \quad v \neq 0.$$

If we take complex conjugates of this equation, we obtain:

$$A\bar{v} = \bar{A}\bar{v} = \bar{\lambda}_i \bar{v}$$

The first equality follows from the fact that A has real entries (so A is equal to its own complex conjugate, i.e. $\bar{A} = A$). It follows that $\bar{\lambda}_i$ is an eigenvalue of A, with $\bar{v}$ as an eigenvector.

$$e^{(a+jb)t} \cdot e^{(a-jb)t} = e^{2at}$$
$$\therefore a = 0 \implies e^{2at} = 0$$

$\therefore$ If $Re(\lambda_i) = 0, \quad i = 1, \ldots, n$, then

$$vol(z(t)) = 1 \cdot vol(z_0)$$

$\therefore$ A marginally stable system can preserve volume.