Problem 1

a) $$e^{At} = \mathbb{I} + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$$

$$Ae^{At} = A + A^2 t + \frac{1}{2!} A^3 t^2 + \ldots$$

$$e^{At}A = A + AtA + \frac{1}{2!} A^2 t A + \ldots$$

Since t is a scalar and A commutes with itself,

i.e. $$(At)A = tAA = tA^2 \quad \forall\, t \in \mathbb{R} \text{ and } A \in \mathbb{R}^{n \times m}$$

$$\Longrightarrow \quad Ae^{At} = e^{At}A \quad \forall\, A \in \mathbb{R}^{n \times m}$$

(See part b) for second part of proof)

b) $$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(\tau - t_0)} B\, u(\tau)\, d\tau + Du(t)$$

Recall that, for LTI systems, we can arbitrarily shift the limits on the convolution integral as (A, B, C, D) is not a time-dependent realisation.

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau + D\, u(t)$$

$\longrightarrow u(t) = e^{st}$

$$\therefore \; y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau + De^{st}$$

$\longrightarrow e^{st}$ is a scalar quantity so multiplication order is irrelevant

$$\therefore \; y(t) = Ce^{At}x_0 + C\int_0^t e^{s\tau} e^{A(t-\tau)} B\, d\tau + De^{st}$$

Problem 1 Continued

$$e^{s\tau} e^{A(t-\tau)} = e^{s\tau} e^{-st} e^{st} e^{-A(\tau-t)}$$

$$= e^{s(\tau-t)} e^{-A(\tau-t)} e^{st}$$

Since e^{st}, $e^{s\tau}$, e^{At} and $e^{A\tau}$ all commute.

$$e^{s(\tau-t)} e^{-A(\tau-t)} = e^{s(\tau-t)} \mathbb{I} e^{-A(\tau-t)}$$

$$= \left(\mathbb{I} + s(\tau-t) + \frac{1}{2!} s(\tau-t) + \ldots\right) \mathbb{I} e^{-A(\tau-t)} \quad (*)$$

$$= e^{s\mathbb{I}(\tau-t)} e^{-A(\tau-t)}$$

$$\therefore y(t) = Ce^{At} x_0 + C \int_0^t e^{(s\mathbb{I}-A)(\tau-t)} B e^{st} \, d\tau + De^{st}$$

$$= Ce^{At} x_0 + C \int_0^t e^{(s\mathbb{I}-A)(\tau-t)} d\tau \; Be^{st} + De^{st}$$

$$= Ce^{At} x_0 + C(s\mathbb{I}-A)^{-1} e^{(s\mathbb{I}-A)(\tau-t)} \Big|_0^t Be^{st} + De^{st}$$

$$= Ce^{At} x_0 + C(s\mathbb{I}-A)^{-1} Be^{st} - C(s\mathbb{I}-A)^{-1} e^{-(s\mathbb{I}-A)t} Be^{st} + De^{st}$$

$$e^{s\mathbb{I}t} = e^{st} \quad \text{as shown in } (*)$$

$$= Ce^{At} x_0 + C(s\mathbb{I}-A)^{-1} Be^{st} - C(s\mathbb{I}-A)^{-1} e^{At} Be^{st} + De^{st}$$

Now, let's show that $(sI-A)^{-1}e^{At} = e^{At}(sI-A)^{-1}$

$\forall$ invertible matrices M and N, $MN^{-1} = NM^{-1}$ holds.

Assume $(sI-A)^{-1}e^{At} = e^{At}(sI-A)^{-1}$ holds

$\therefore \quad e^{-At}(sI-A) = (sI-A)e^{-At}$

$\therefore \quad se^{-At} - e^{-At}A = se^{-At} - Ae^{-At}$

$\therefore \quad se^{-At} + e^{-At}(-A) = se^{-At} + (-A)e^{At}$

in a) we showed that $e^{At}A = Ae^{At}$

$\Longrightarrow \quad se^{-At} + (-A)e^{-At} = se^{-At} + (-A)e^{-At}$

$\therefore$ L.H.S. = R.H.S. ✓

$\therefore \quad y(t) = Ce^{At}\left[x_0 - (sI-A)^{-1}B\right] + \left[C(sI-A)^{-1}B + D\right]e^{st}$

Note that $s \neq \text{eig}(A)$ otherwise $y(t) \to \infty$

c) If the system is asymptotically stable $\Longrightarrow e^{At} \xrightarrow{t\to\infty} 0$

$\therefore \quad y(t) = \left[C(sI-A)^{-1}B + D\right]u(t)$

$\therefore \quad \dfrac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D = H(s)$

d) System is stable $\Longrightarrow$ $y_{ss}(t) = H(s)\, u(t)$

$u(t) = \sin \omega t$

$$= \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \Rightarrow y_{ss}(t) =$$

$$\therefore\ y_{ss}(t) = \left[C\,(sI - A)^{-1} B + D\right]\left(\frac{1}{2j} e^{j\omega t} - \frac{1}{2j} e^{-j\omega t}\right)$$

$$= \frac{1}{2j}\left(H(j\omega)\, e^{j\omega t} - H(-j\omega)\, e^{-j\omega t}\right)$$

$$H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)} \qquad H(-j\omega) = \overline{H(j\omega)}$$

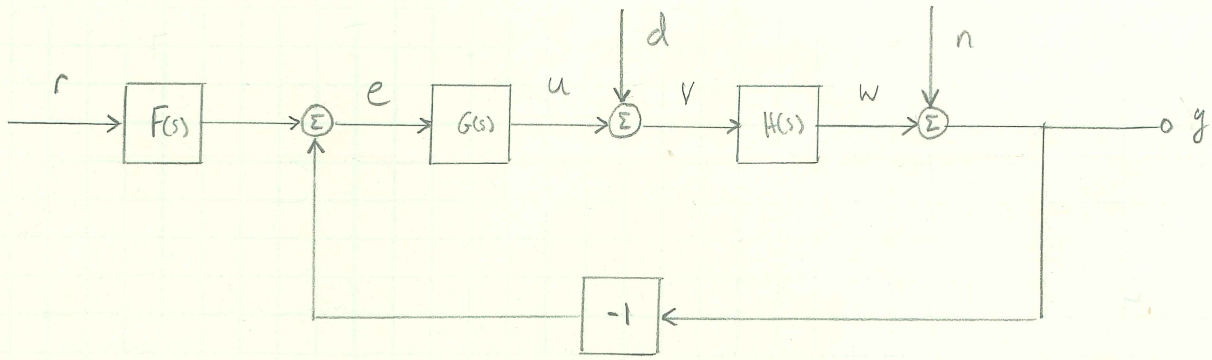
$$\overline{H}(j\omega) = |H(j\omega)|\, e^{-j\angle H(j\omega)}$$

$$\therefore\ y_{ss}(t) = \frac{1}{2j}\left(|H(j\omega)|\, e^{j\angle H(j\omega)}\, e^{j\omega t} - |H(j\omega)|\, e^{-j\angle H(j\omega)}\, e^{-j\omega t}\right)$$

$$= \frac{1}{2j}\left(|H(j\omega)|\, e^{j(\omega t + \angle H(j\omega))} - |H(j\omega)|\, e^{-j(\omega t + \angle H(j\omega))}\right)$$

$\therefore$ since $\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$ we have the following result

$$\Longrightarrow y_{ss}(t) = |H(j\omega)| \sin\left(\omega t + \angle H(j\omega)\right)$$



$$y = w + n = vH + n \implies y = (d+u)H + n$$

$$u = eG \qquad\qquad = Hd + GHe + n$$

$$e = rF - y$$

$$= rF - Hd - GHe - n$$

$$\therefore\ (1 + GH)e = rF - Hd - n$$

$$\therefore\ e = \frac{F}{1+GH} r - \frac{H}{1+GH} d - \frac{1}{1+GH} n$$

$$u = \frac{FG}{1+GH} r - \frac{GH}{1+GH} d - \frac{G}{1+GH} n$$

$$y = rF - e$$

$$= rF - \frac{F}{1+GH} r + \frac{H}{1+GH} d + \frac{1}{1+GH} n$$

$$y = \frac{FGH}{1+GH} r + \frac{H}{1+GH} d + \frac{1}{1+GH} n$$

$$\therefore \quad \frac{Y(s)}{R(s)} = T_{ry}(s) = \frac{FGH}{1+GH}$$

$$\frac{Y(s)}{D(s)} = T_{dy}(s) = \frac{H}{1+GH}$$

$$\frac{Y(s)}{N(s)} = T_{ny}(s) = \frac{1}{1+GH}$$

$$\frac{U(s)}{D(s)} = T_{du}(s) = \frac{-GH}{1+GH}$$

$$\frac{U(s)}{N(s)} = T_{nu}(s) = \frac{-G}{1+GH}$$

Problem 2

(a)

i) $\frac{Y(s)}{U(s)} = H(s) = C(s\mathbb{I} - A)^{-1}B + D$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s + 1/5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + 1/5\, s} \begin{bmatrix} s + 1/5 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$= \frac{1}{5s^2 + s}$$

$$H(s) = \frac{0.2}{s(s + 0.2)}$$

ii) See plot

iii) $|H(j\omega)| = \left| \frac{0.2}{j\omega(j\omega + 0.2)} \right| = 1 \Rightarrow 1 = \left| \frac{0.2}{j\omega(j\omega + 0.2)} \right| = \left| \frac{0.2}{-\omega^2 + 0.2 j\omega} \right|$

$\therefore \quad 1 = \frac{0.2}{[\omega^4 + 0.04\omega^2]^{1/2}} \longrightarrow \omega^4 + 0.04\omega^2 - 0.04 = 0$

$\omega = \pm 0.425438$ and $\pm j\, 0.470104$

Select the only real positive root

$\omega_c = 0.425438$ rad/s

v) $y(t) = Ce^{At}\left[x_0 - (s\mathbb{I} - A)^{-1} B\right] + \left[C(s\mathbb{I}-A)^{-1}B + D\right] e^{st}$

In an asymptotically stable system the Ce^{At} term will decay to zero as $t \to \infty$. This system, however, is not asymptotically stable as one of the eigenvalues of A is zero (the other is $-1/5$).

The dynamics associated with $\lambda = 0$ will result in a steady-state error. $x_0 = 0$, so we don't need to worry about the initial conditions.

Let's examine $\lim_{t\to\infty} \underbrace{-Ce^{At}(s\mathbb{I}-A)^{-1}B}_{\alpha}$

$$\alpha = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5-5e^{-t/5} \\ 0 & e^{-t/5} \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{5}{5s^2+5} \\ 0 & \frac{5}{5s+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$= \frac{5}{5s+1} e^{-t/5} - \frac{5}{5s+1} - \frac{1}{5s^2+s}$$

$$\lim_{t\to\infty} \alpha = -\frac{1}{s}$$

So the transfer function for the steady state error of the system is $-\frac{1}{s}$

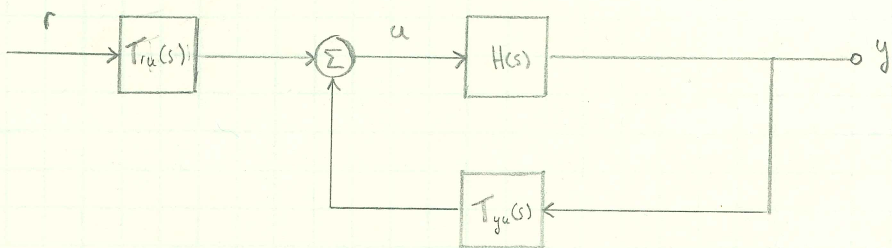
$e = r - y$

$y_{ss} = r - e \qquad y_{ss}(t) = r(t) + \frac{1}{\omega_c}$

(b)

i) $H(s)$ remains as it did in the open-loop case (we are not changing the physical device that we wish to control).

$F(s)$ and $G(s)$ are computed as follows:



Our controller/observer architecture is set-up according to the above block diagram. Once T_{ru} and T_{yu} have been computed, F and G may be determined using the following relationships:

$$F(s) = -T_{yu}^{-1} T_{ru} \qquad G(s) = -T_{yu}$$

T_{yu} and T_{ru} are computed by comparing our observer/controller system to the canonical linear state-space system and then applying the plant transfer function equation that was derived from the canonical system:

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \Longrightarrow H(s) = C(s\mathbb{I} - A)^{-1}B + D$$

$H(s)$ relates inputs (u) to outputs (y)

→

$$\dot{\hat{x}}(t) = \underbrace{(A - BK - LC)}_{\alpha}\,\hat{x}(t) + \underbrace{L}_{\gamma_1}\,y(t) + \underbrace{B\,k_{ref}}_{\gamma_2}\,r(t)$$

$$u(t) = \underbrace{-K}_{\delta}\,\hat{x}(t) + \underbrace{k_{ref}}_{\varepsilon}\,r(t)$$

$$T_{yu}(s) = -\delta\,(s\mathbb{I} - \alpha)^{-1}\,\gamma_1$$

$$T_{ru}(s) = -\delta\,(s\mathbb{I} - \alpha)^{-1}\,\gamma_2 + \varepsilon$$

$$T_{yu}(s) = \frac{-(l_1 k_1 + l_2 k_2)s - l_2 k_1 - 0.2\,l_1 k_1}{s^2 + (l_1 + 0.2k_2 + 0.2)s + l_2 + 0.2(l_1 + k_1 + l_1 k_2)}$$

$$T_{ru}(s) = \frac{k_1 s^2 + (l_1 k_1 + 0.2k_1)s + k_2 l_1 + 0.2\,l_1 k_1}{s^2 + (l_1 + 0.2k_2 + 0.2)s + l_2 + 0.2(l_1 + k_1 + l_1 k_2)}$$

$$F(s) = \frac{5k_1 s^2 + (k_1 + 5l_1 k_1)s + l_1 k_1 + 5 l_2 k_1}{5(l_1 k_1 + l_2 k_2)s + l_1 k_1 + 5\,l_2 k_1}$$

$$G(s) = \frac{(l_1 k_1 + l_2 k_2)s + l_2 k_1 + 0.2\,l_1 k_1}{s^2 + (l_1 + 0.2k_2 + 0.2)s + l_2 + 0.2(l_1 + k_1 + l_1 k_2)}$$

Problem 2 Continued

ii)

$$T_{ry}(s) = \frac{FGH}{1+GH} = \frac{0.2\,k_1}{s^2 + 0.2(k_2+1)s + 0.2k_1}$$

The denominator of the transfer function $T_{ry}(s)$ has the same eigenvalues as the controller.

The denominators of all of the other transfer functions have eigenvalues equal to the union of the controller and observer eigenvalues.

$$(1+GH) = 25s^4 + (25l_1 + 5k_2 + 10)s^3 + (10l_1 + 25l_2 + 5k_1 + k_2 + 5l_1k_2 + 1)s^2 + (l_1 + 5l_2 + k_1 + 5l_1k_1 + l_1k_2 + 5l_2k_2)s + l_1k_1 + 5l_2k_1$$

$$(1+GH)\,T_{dy}(s) = 5s^2 + (5l_1 + k_2 + 1)s + l_1 + 5l_2 + k_1 + l_1k_2$$

$$(1+GH)\,T_{ny}(s) = 25s^4 + (25l_1 + 5k_2 + 10)s^3 + (10l_1 + 25l_2 + 5k_1 + k_2 + 5l_1k_2 + 1)s^2 + (l_1 + 5l_2 + k_1 + l_1k_2)s$$

$$(1+GH)\,T_{du}(s) = (-5l_1k_1 - 5l_2k_2)s - l_1k_1 - 5l_2k_1$$

$$(1+GH)\,T_{nu}(s) = (-25l_1k_1 - 25l_2k_2)s^3 - (10l_1k_1 + 25l_2k_1 + 5l_2k_2)s^2 - (l_1k_1 + 5l_2k_1)s$$

iii) The overall closed-loop transfer function of the system $T_{ry}(s)$ is second order. We have analytical expressions to help us with performance specifications.

$$s^2 + 0.2(k_2+1)s + 0.2k_1$$

$$s^2 + 2\sigma s + (\sigma^2 + \omega^2) \qquad \omega = \frac{\pi}{t_p} \qquad \sigma = -\frac{\omega}{\pi}\ln(M_p)$$

$M_p = 0.1$

$t_p = 1.0$

$$\Longrightarrow \quad \omega = \pi \qquad \sigma = 23.025851$$

$$k_1 = 5(\sigma^2 + \omega^2) \qquad k_2 = 10\sigma - 1$$

$$K = [\,75.857512 \quad 22.025851\,]$$

The observer gains will affect the response but specific initial conditions will dictate the influence.

If we place the observer poles at $s = -40$ and $s = -40$ we get

$$L = \begin{bmatrix} 79.8 \\ 1584.04 \end{bmatrix}$$

and satisfy the specs. in the context of V) as well

iv) $$|T_{ry}(s)| = \left| \frac{0.2k_1}{s^2 + 0.2(k_2+1)s + 0.2k_1} \right| = \frac{0.2k_1}{(j\omega_{bw})^2 + 0.2(k_2+1)(j\omega_{bw}) + 0.2k_1}$$

$$= \left| \frac{0.2k_1}{(-\omega^2 + 0.2k_1) + (0.2(k_2+1)\omega)j} \right|$$

$$= \frac{0.2k_1}{\left[\omega_{bw}^4 + \left(\frac{k_2^2}{25} + \frac{2k_2}{25} + \frac{2k_1}{5} + \frac{1}{25}\right)\omega_{bw}^2 + \frac{k_1^2}{25}\right]^{1/2}} = \frac{1}{\sqrt{2}}$$

Select the real, positive root

$$\omega_{bw} = 4.517953 \text{ rad/s}$$

vi) $T_{dy}(s)$ - low pass

$T_{ny}(s)$ - high pass

$T_{du}(s)$ - low pass

$T_{nu}(s)$ - band pass

vii) Using the second order expressions we arrive at

$$K = [\ 7585.751256 \quad 229.258509\ ]$$

We can keep the observer poles where they were for the first design.

We note that the Bode plot for T_{ry} is shifted to the right (higher frequencies) by one decade. So now, for example, the bandwidth frequency is now $\omega_{bw} = 45.18$ rad/s.

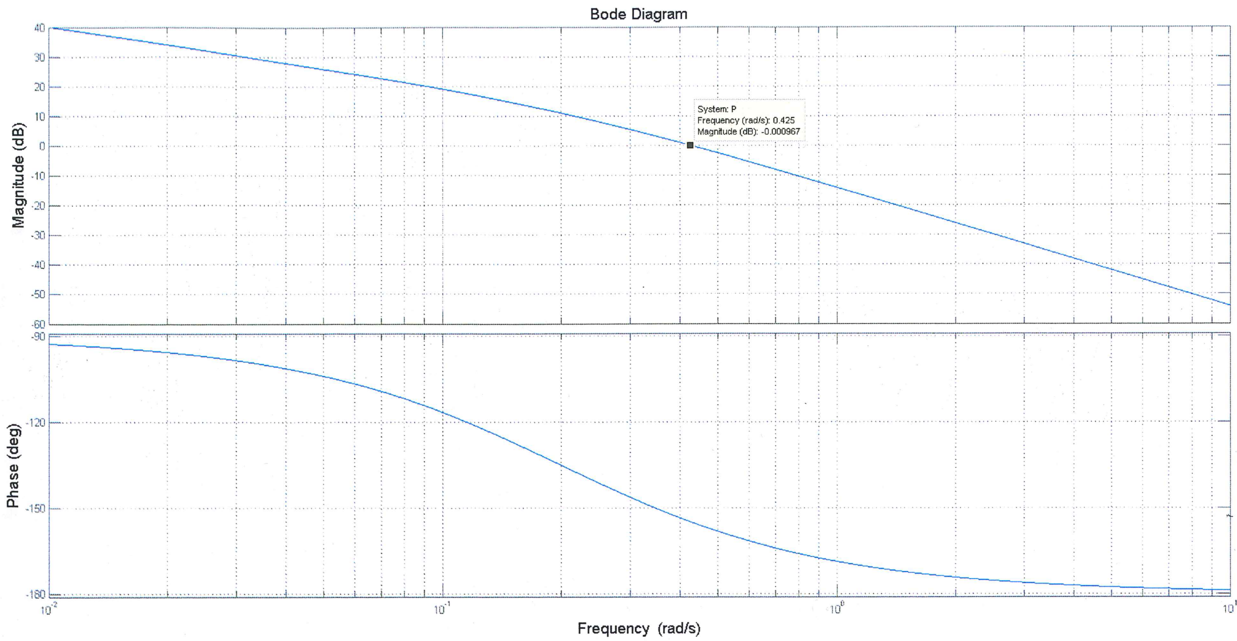
$T_{dy}(s)$ - shifted down and to the right

$T_{ny}(s)$ - shifted to the right

$T_{du}(s)$ - shifted to the right

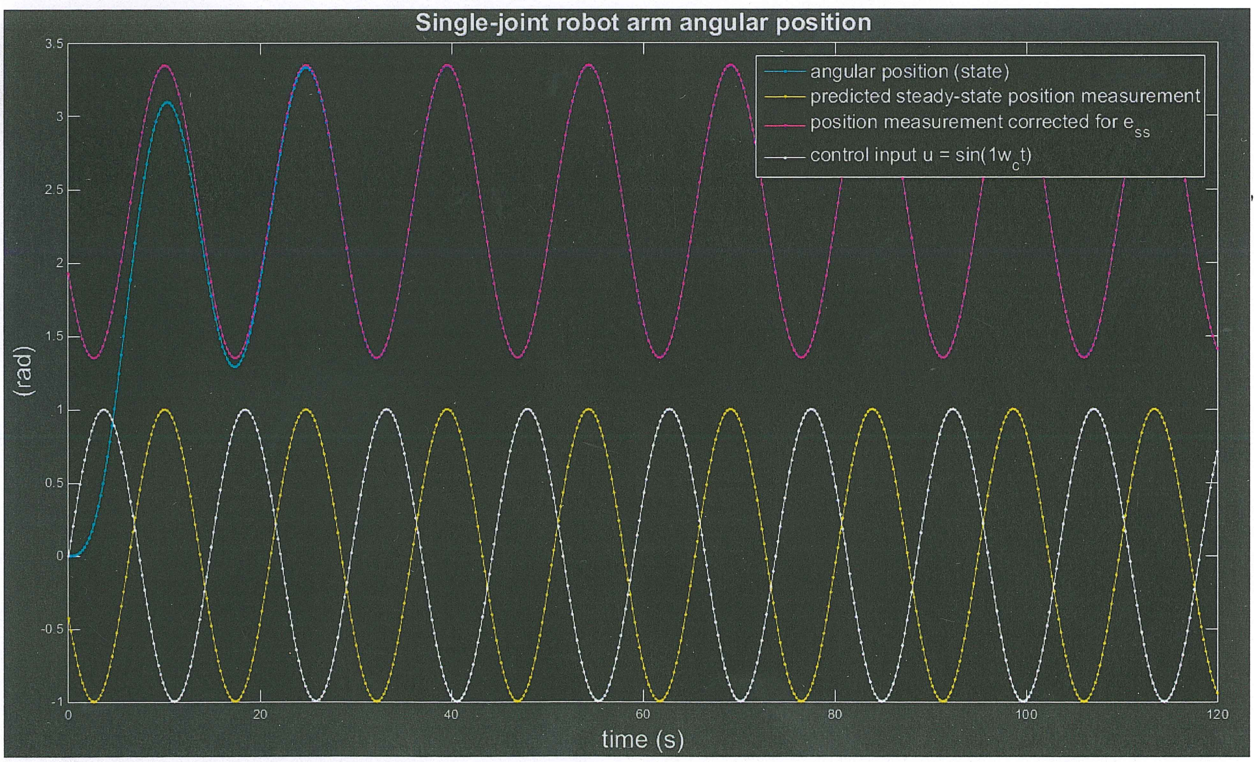
$T_{nu}(s)$ - shifted up and to the right

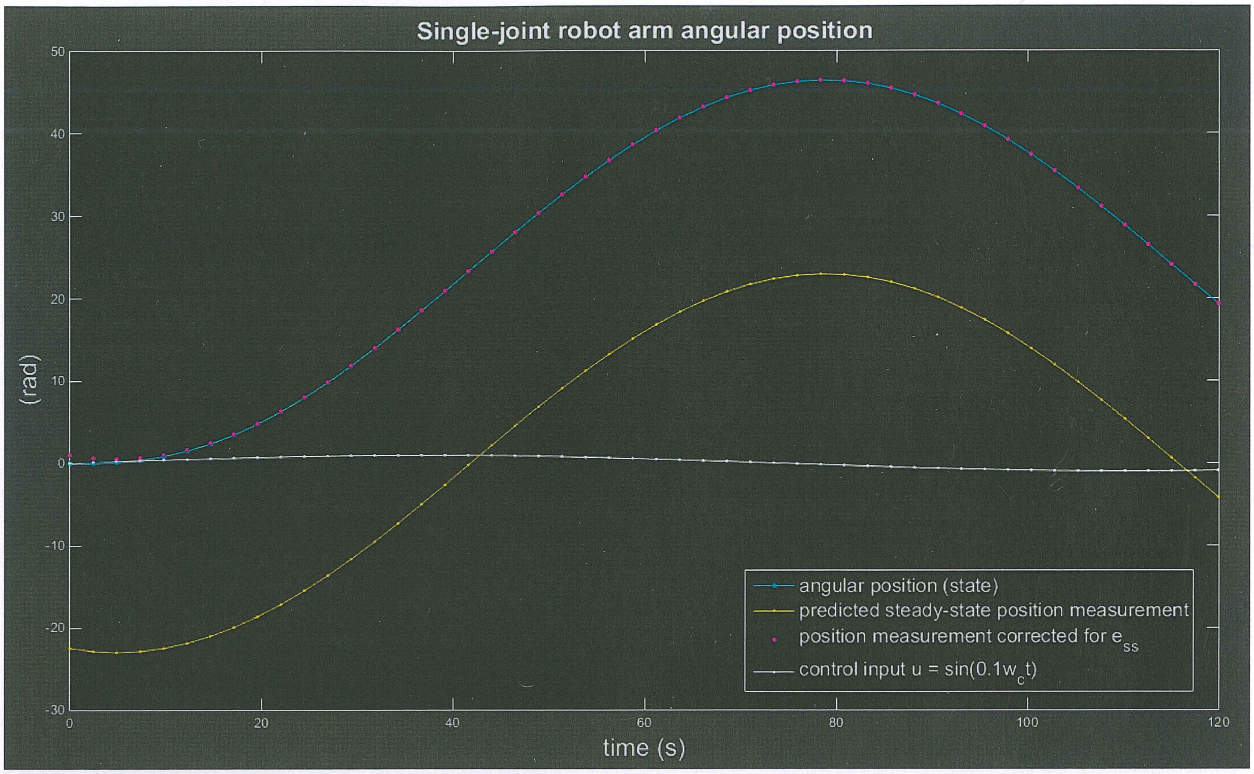
Problem 2 a) ii)

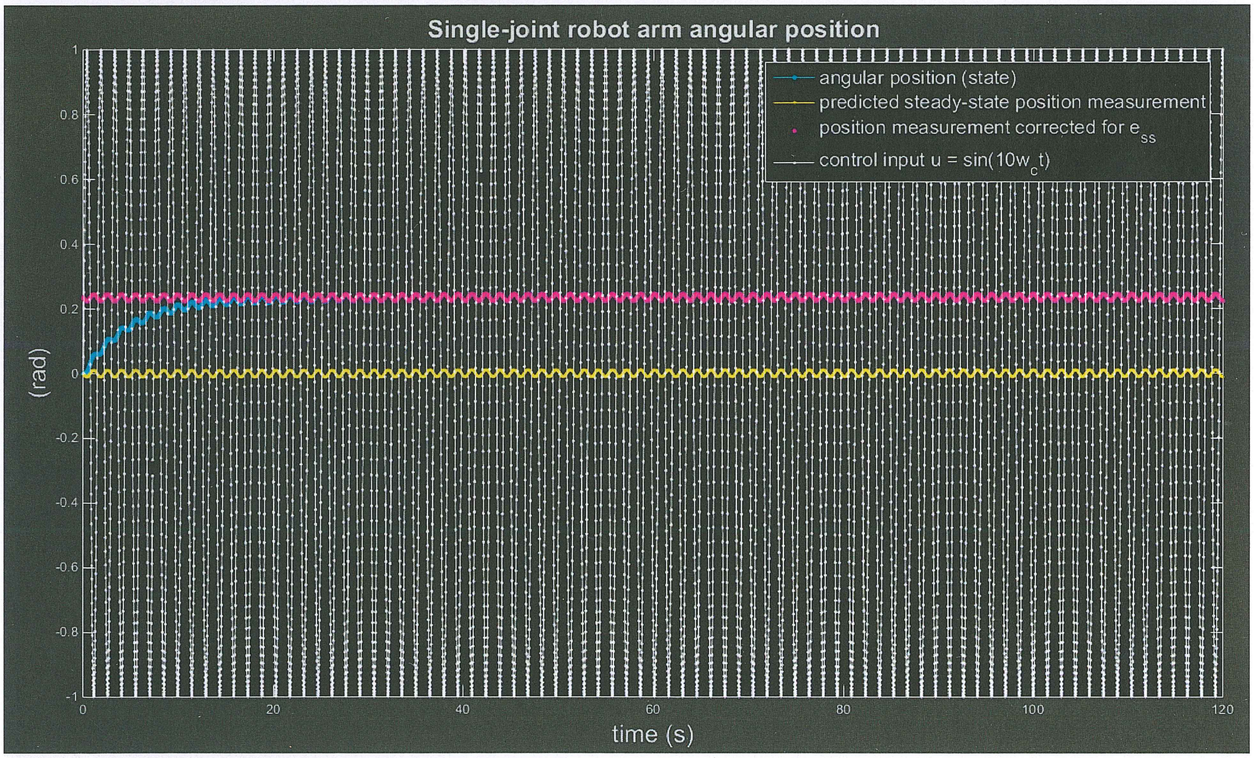


We can see that the open-loop plant transfer function acts as a low-pass filter, attenuating high frequencies. The cut-off frequency is $\omega_c = 0.425\ rad/s$.

Problem 2 a) iv)

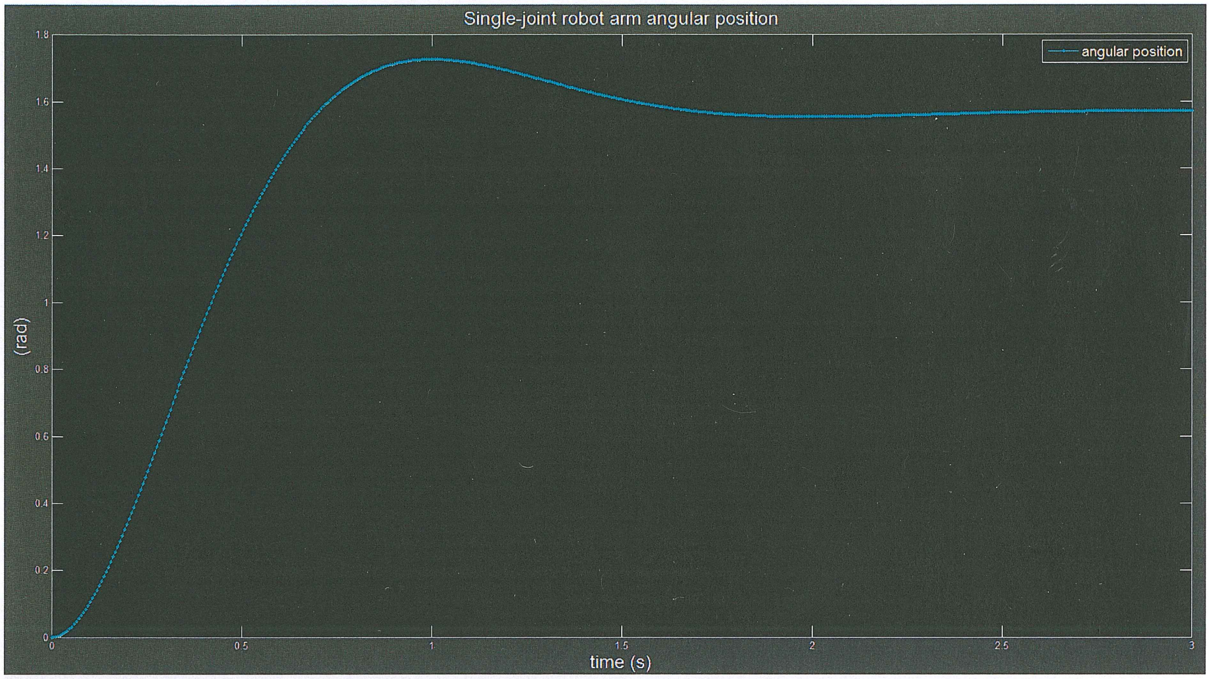






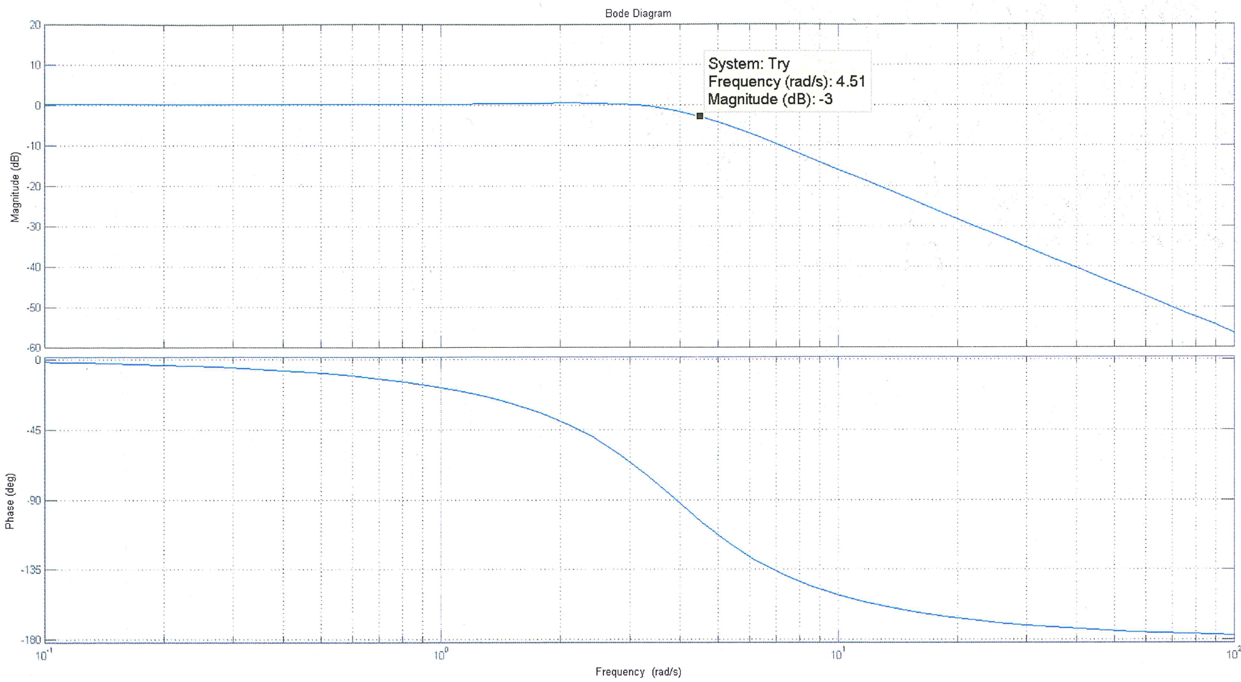
We can see that, in open-loop, the plant amplifies low frequencies and attenuates high ones. There is a steady-state error due to the fact that the state matrix (A) has zero as one of its eigenvalues. This means that the system is not asymptotically stable, rather it is Lyapunov stable and the transient dynamics associated with the zero eigenvalue do not decay as $t \to \infty$.

Problem 2 b) iii)



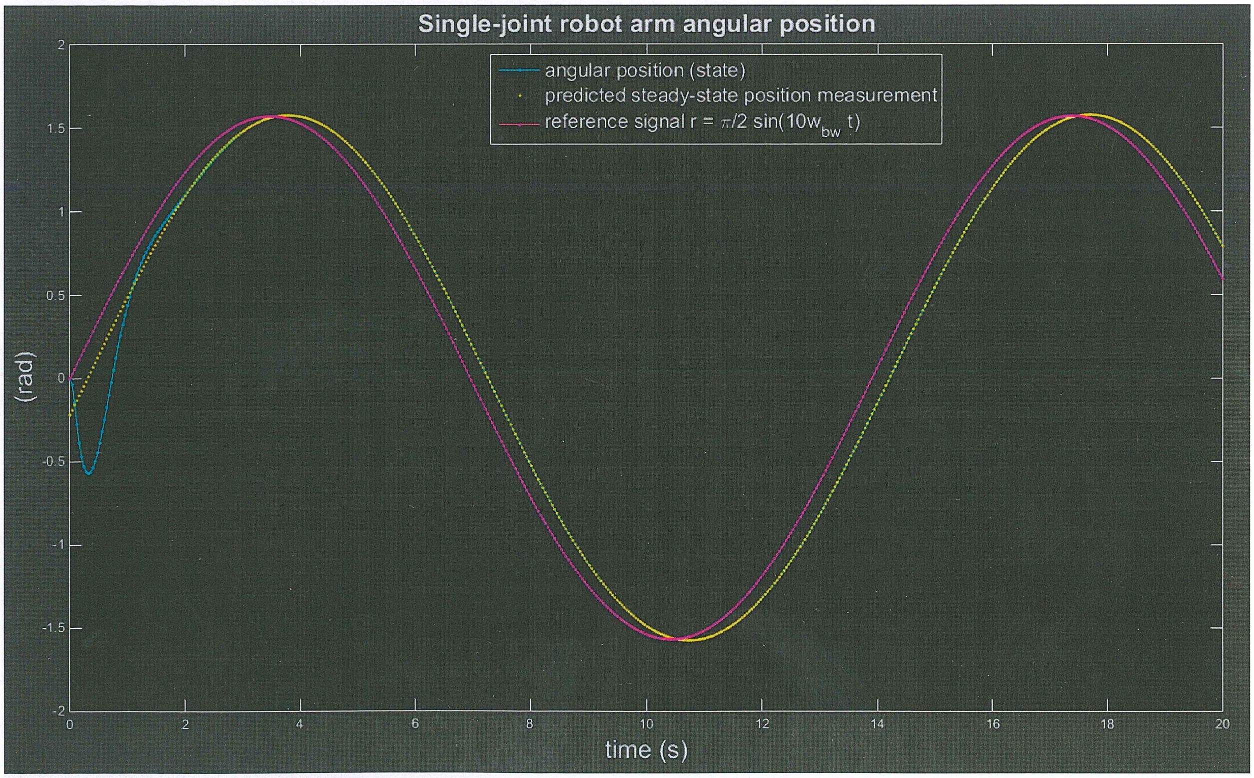
Here we see the closed-loop system responding to a constant reference signal of $r(t) = \frac{\pi}{2}$. The peak time is approximately 1 second and the overshoot is less than 10%.

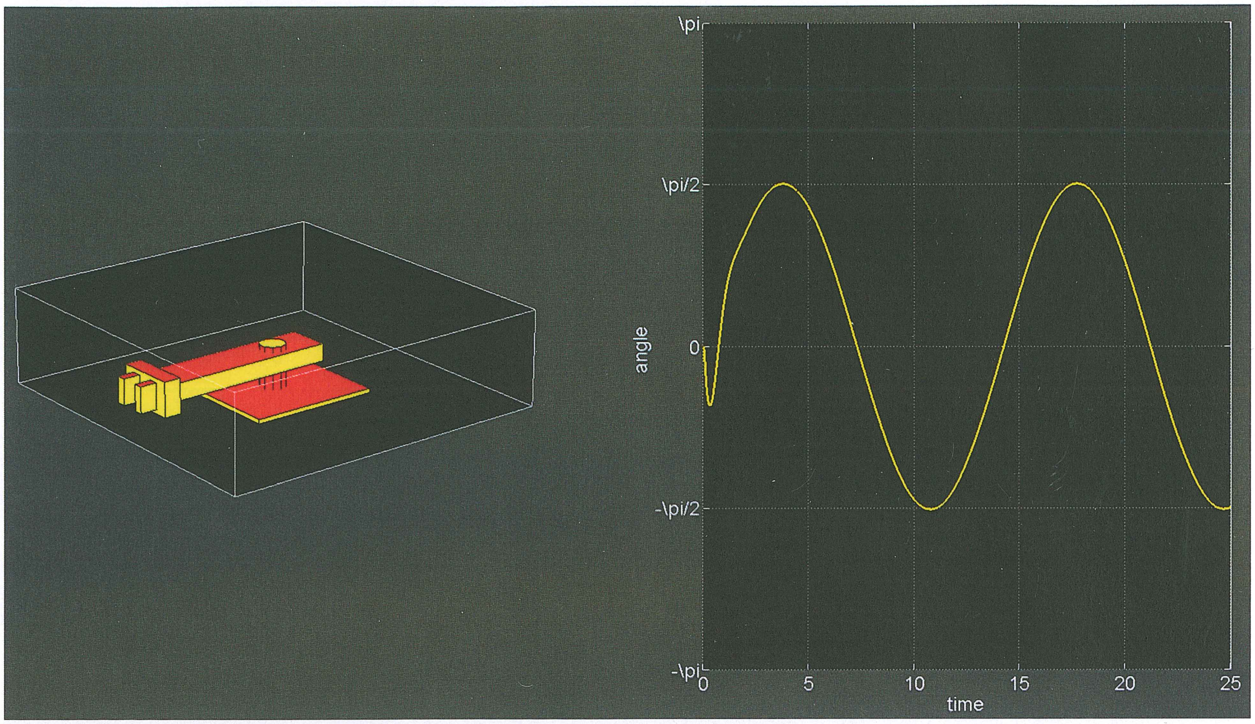
Problem 2 b) iv)



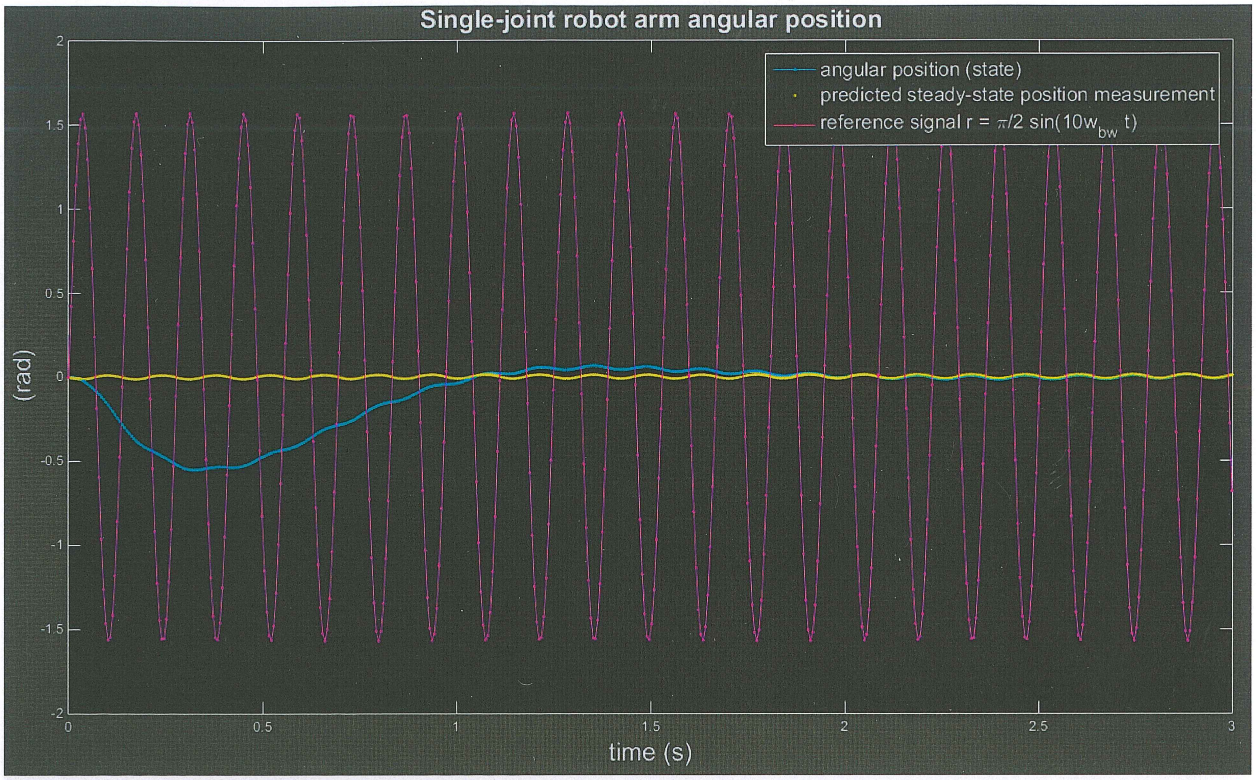
This Bode plot shows the bandwidth frequency for the closed-loop transfer function.

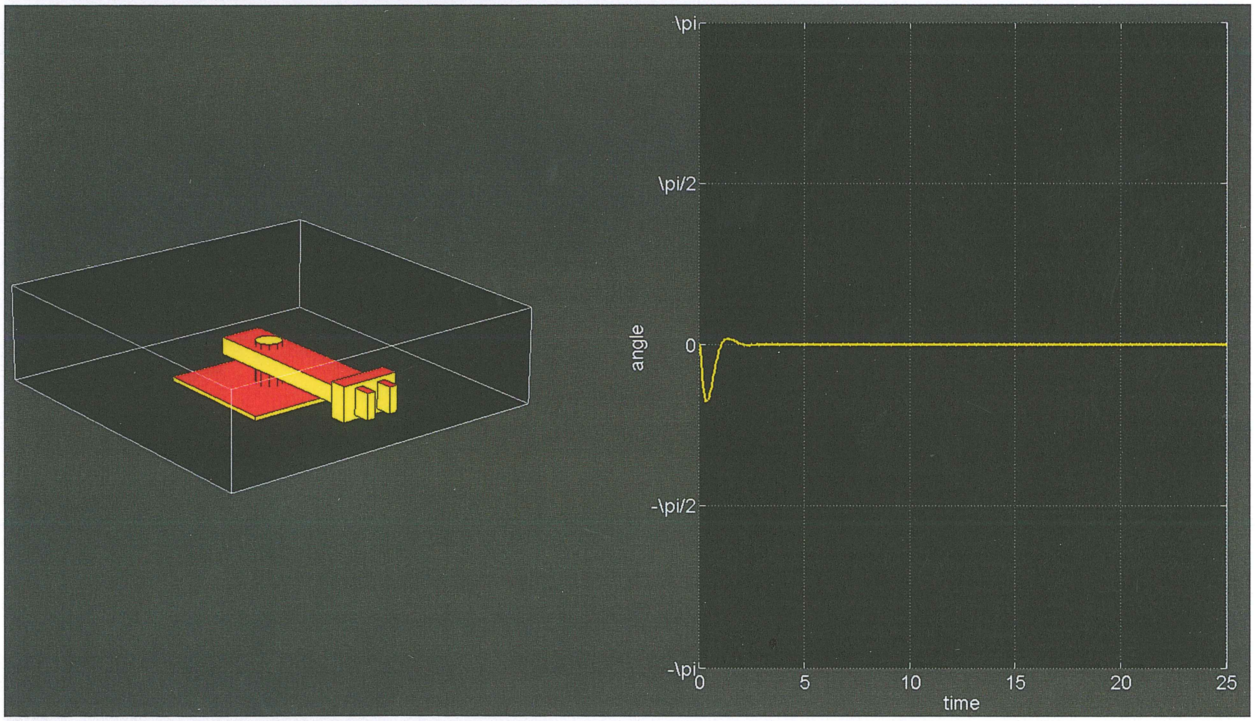
Problem 2 b) v)





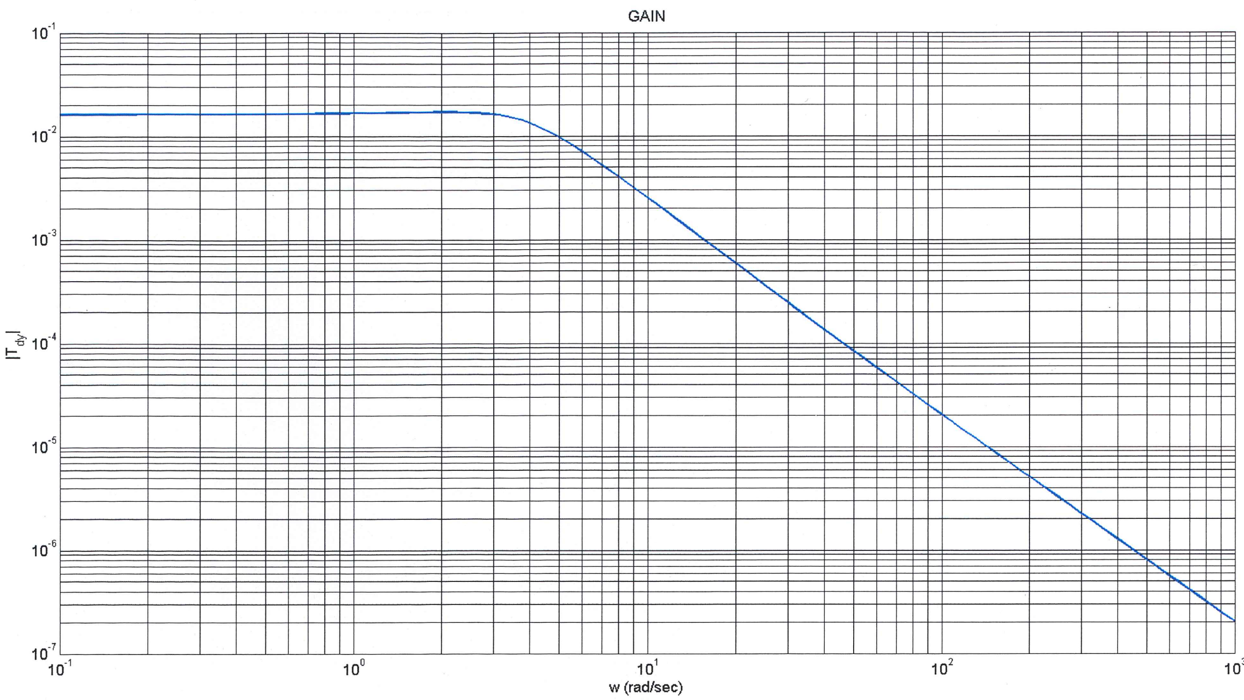
We see here that the steady-state output has a slight phase offset compared to the reference signal. This is expected after referring to the Bode plot where we see that at a frequency of $\omega = 0.1\omega_{bw}$ we expect to see a phase difference of about eight degrees.



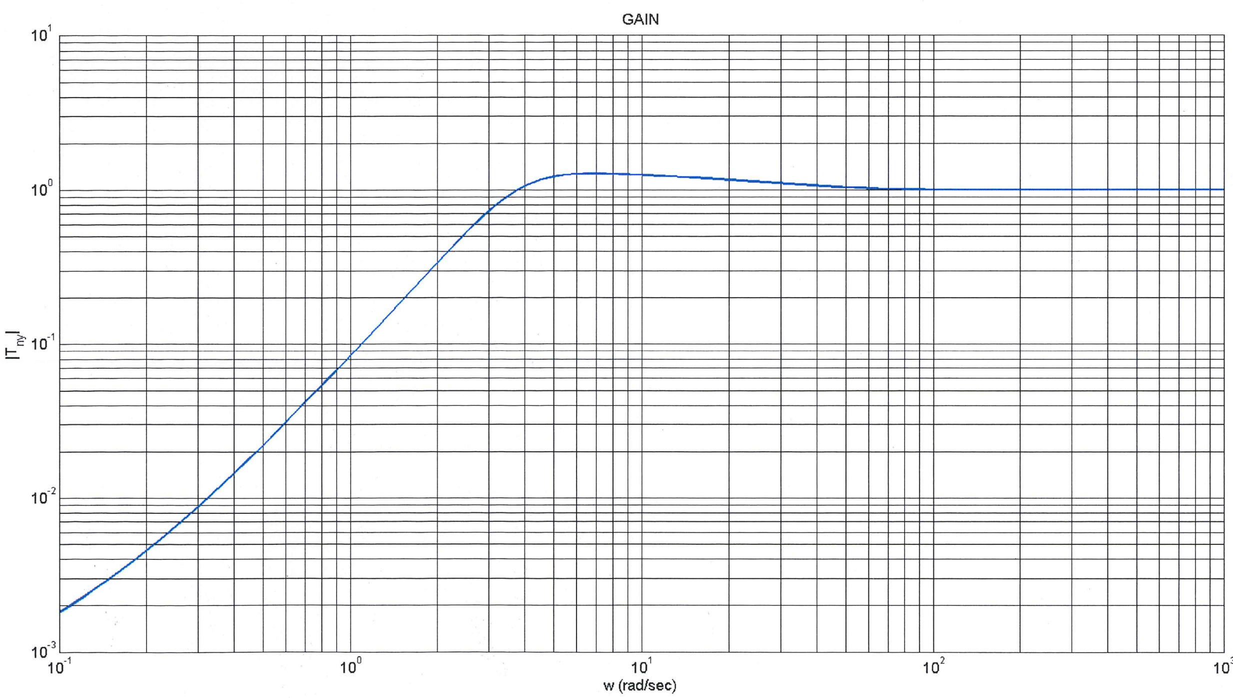


At the higher reference signal frequency of $\omega = 10\omega_{bw}$ we see that the amplitude of the output has been attenuated a great deal and has a phase offset of almost 180 degrees (also expected when one refers to the Bode plot).

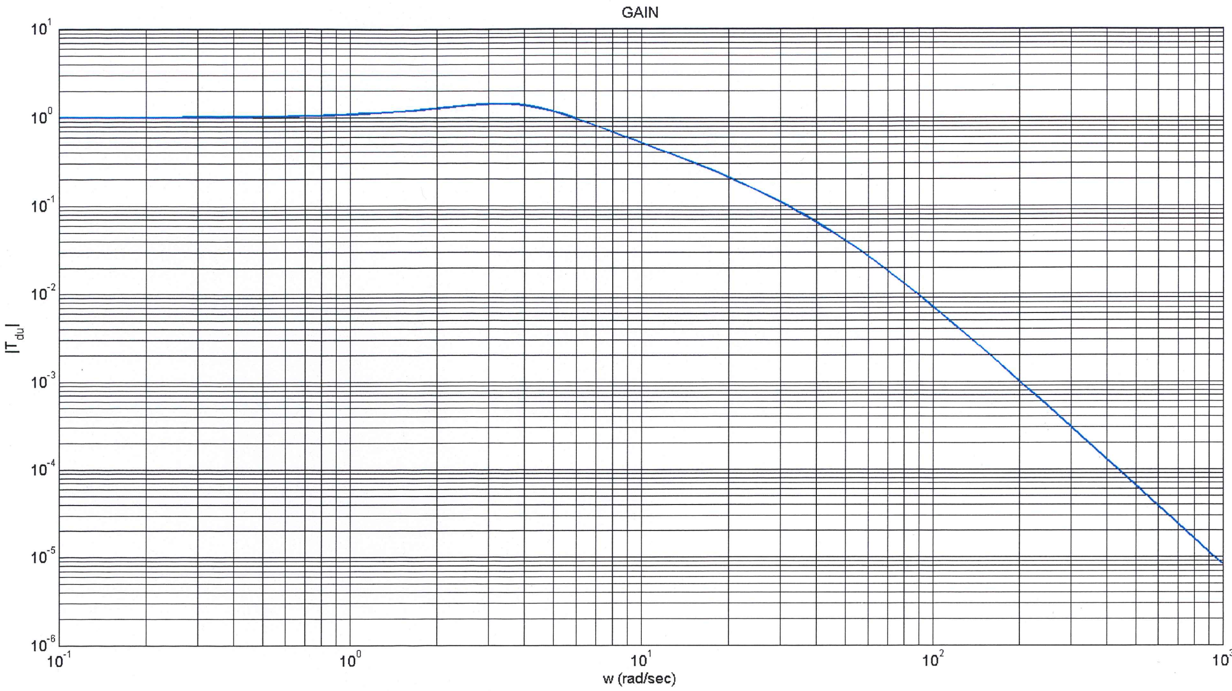
Problem 2 b) vi)



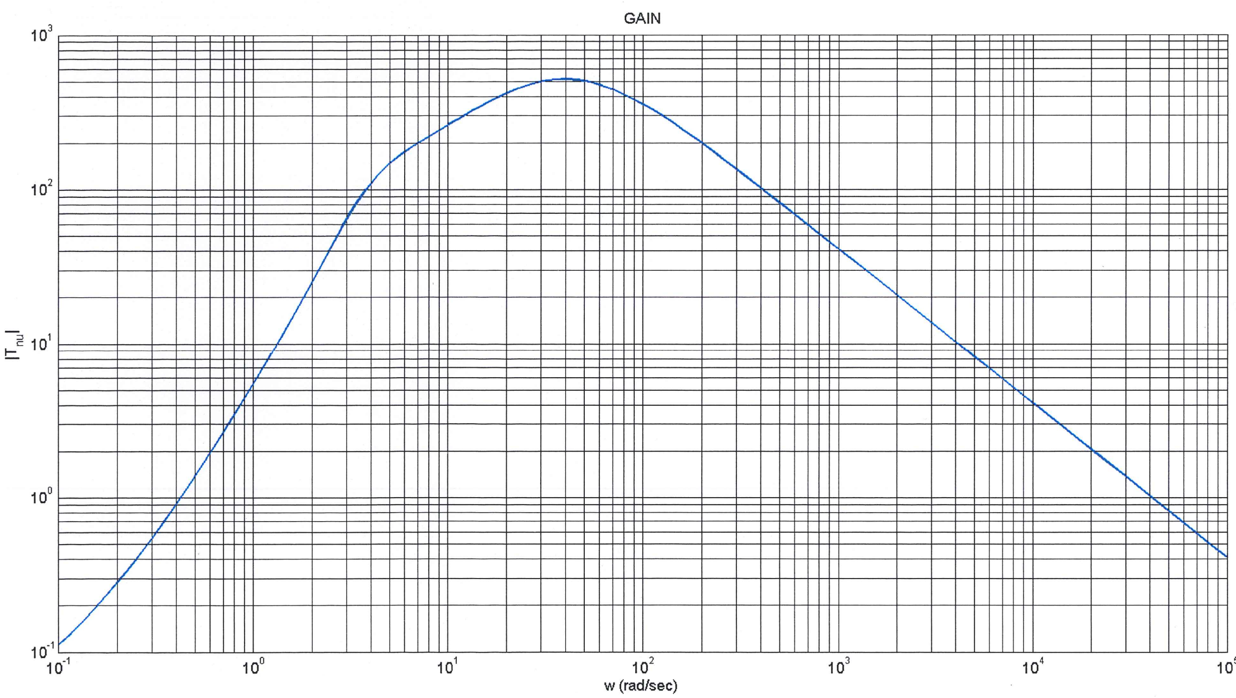
Bode plot for $T_{dy}(s)$ – low pass



Bode plot for $T_{ny}(s)$ – high pass

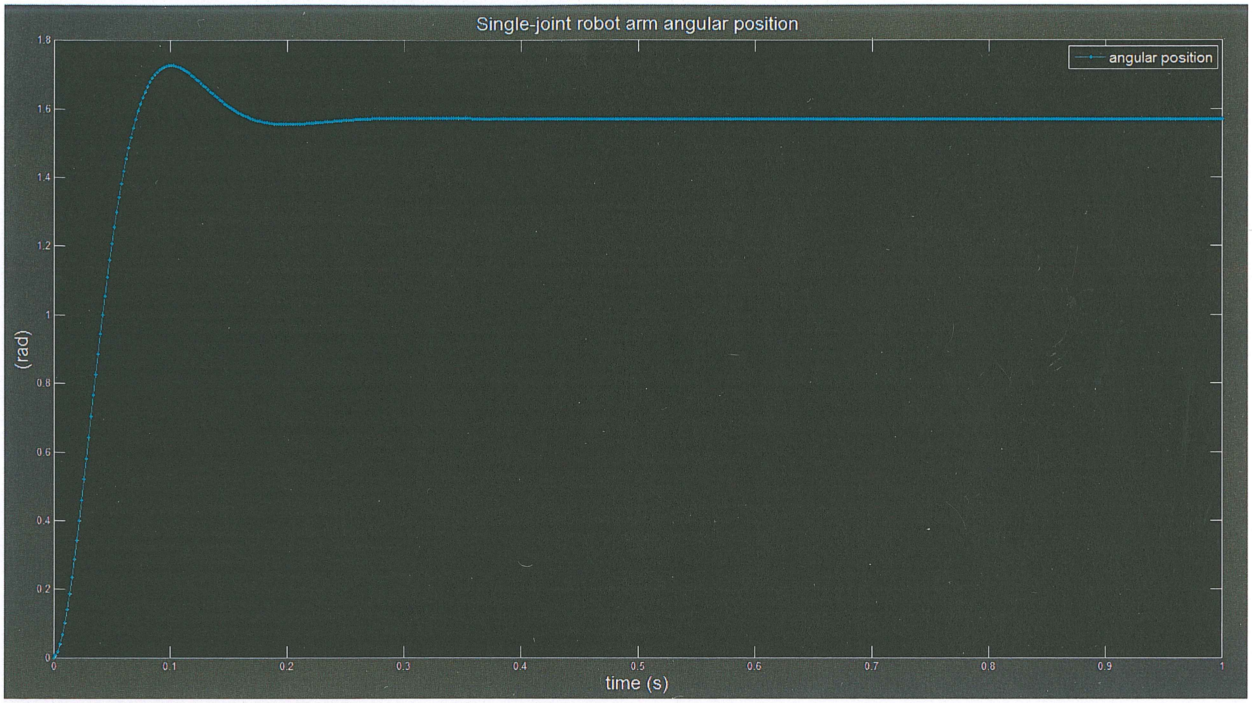


Bode plot for $T_{du}(s)$ – high pass



Bode plot for $T_{nu}(s)$ – band pass

Problem 2 b) vii)



When we redesign to achieve a time-to-peak a factor of ten smaller, the Bode plots shift to the right by one decade.

Problem 3

ⓐ

i)

$$H(s) = \frac{0.5s^2 + 0.5s + 4}{s^4 + 1.5s^3 + 12.25s^2 + 4s + 16}$$

iii)

$$|H(j\omega)| = \left| \frac{0.5(j\omega)^2 + 0.5(j\omega) + 4}{(j\omega)^4 + 1.5(j\omega)^3 + 12.25(j\omega)^2 + 4(j\omega) + 16} \right|$$

$$-1.5\omega^3 j \quad - 12.25\omega^2 \quad + 4\omega j$$

$$= \left| \frac{(-0.5\omega^2 + 4) + (0.5\omega)j}{(\omega^4 - 12.25\omega^2 + 16) + (4\omega - 1.5\omega^3)j} \right|$$

To find ω_{max}, differentiate $|H(\omega)|$ w.r.t. ω, set equal to zero and solve. This is most easily achieved in MATLAB

$$\frac{d}{d\omega}\left\{|H(\omega)|\right\} = 0 \Rightarrow \left.\begin{aligned} \omega_{max} &= 0.0 \\ \omega_{max} &= 1.227778 \\ \omega_{max} &= 2.845822 \\ \omega_{max} &= 3.306984 \end{aligned}\right\}$$

The resulting function has 11 roots, 4 of which are real and positive

These 4 solutions are all stationary points on the Bode plot, but evaluating $|H(\omega)|$ at each reveals that $\omega_{max} = 1.22778$ is the actual maximum corresponding to the resonant peak.

(b) i) $K = [6.166554 \quad 37.872256 \quad 2.412718 \quad 12.315948]$ $\quad k_{int} = 100$

$L = [2842.5 \quad 78.5 \quad 3546.75 \quad 2270]^T$

iii) $\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly - Bk_{int} v$

$u = -K\hat{x} - k_{int} v$

$\dot{v} = y - r$

$$\dot{z} = \begin{bmatrix} \dot{\hat{x}} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK-LC & -Bk_{int} \\ 0 & 0 \end{bmatrix}}_{A_m} \begin{bmatrix} \hat{x} \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} L \\ \vdots \\ 1 \end{bmatrix}}_{\gamma} y + \underbrace{\begin{bmatrix} 0 \\ --- \\ -1 \end{bmatrix}}_{\delta} r$$

$$u = \overbrace{\begin{bmatrix} -K & -k_{int} \end{bmatrix}}^{C_m} z + 0$$

So $D_m = 0$ for both cases

$T_{ru} = C_m(sI - A_m)^{-1}\delta$

$T_{yu} = C_m(sI - A_m)^{-1}\gamma$

v) The steady state response will be determined by superposition of the reference, disturbance and noise signals scaled by their respective transfer functions i.e.

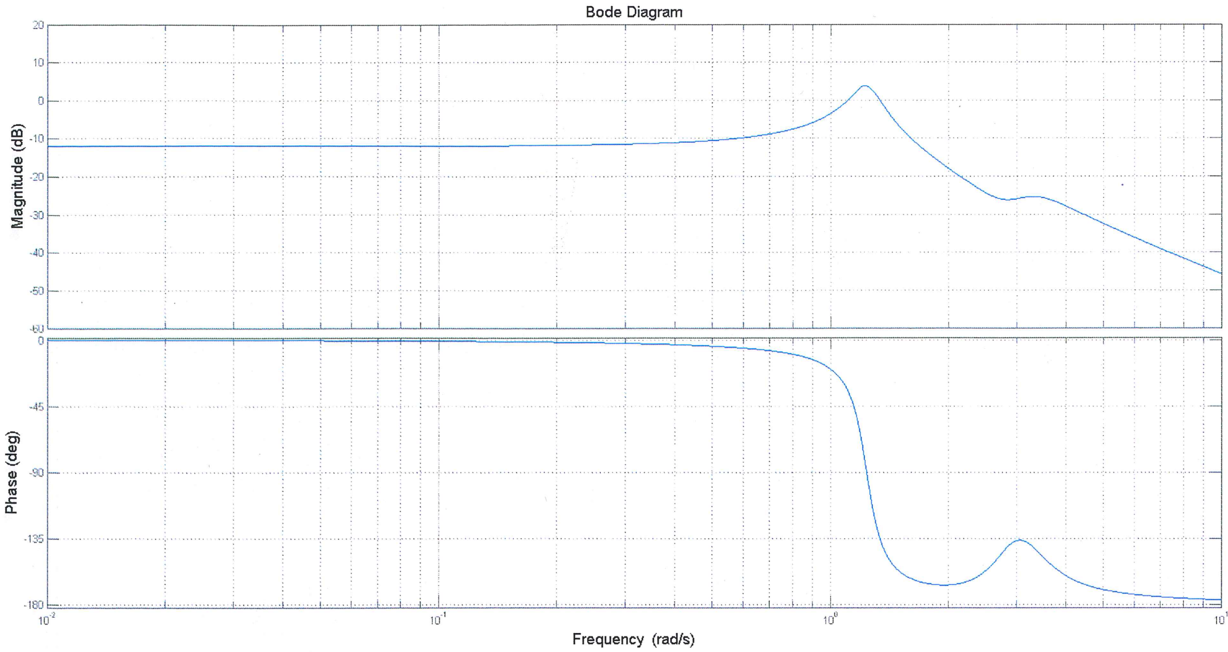
$$y_{ss}(t) = T_{ry}(s)\, r(t) + T_{ny}(s)\, n(t) + T_{dy}(s)\, d(t)$$

Specifically, T_{ny} should be evaluated at $\omega = 50$ rad/s, T_{dy} at $\frac{1}{50}$ rad/s and T_{ry} at 0 rad/s.

$$y_{ss}(t) = |T_{ry}(0)| \sin(0t + \angle T_{ry}(0)) + |T_{ny}(50)| \sin(50t + \angle T_{ny}(50))$$

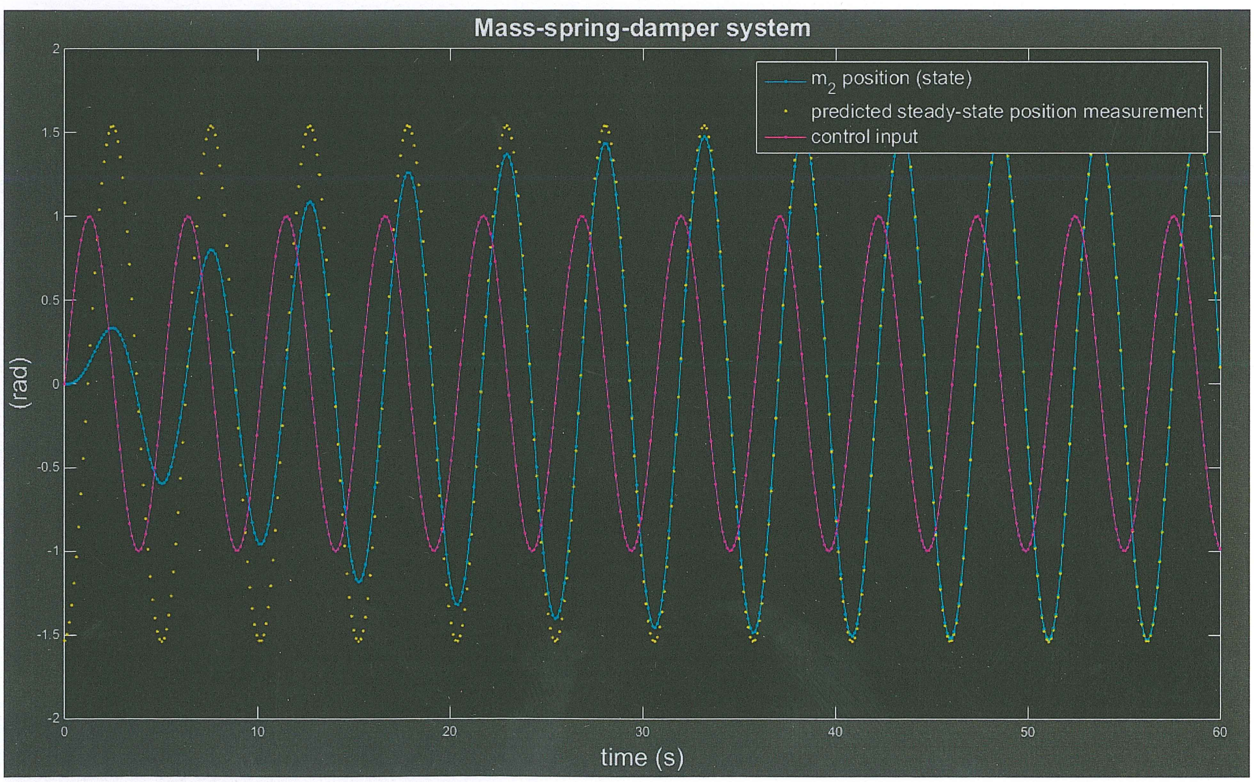
$$+ |T_{dy}(1/50)| \sin(1/50\, t + \angle T_{dy}(1/50))$$

Problem 3 a) ii)

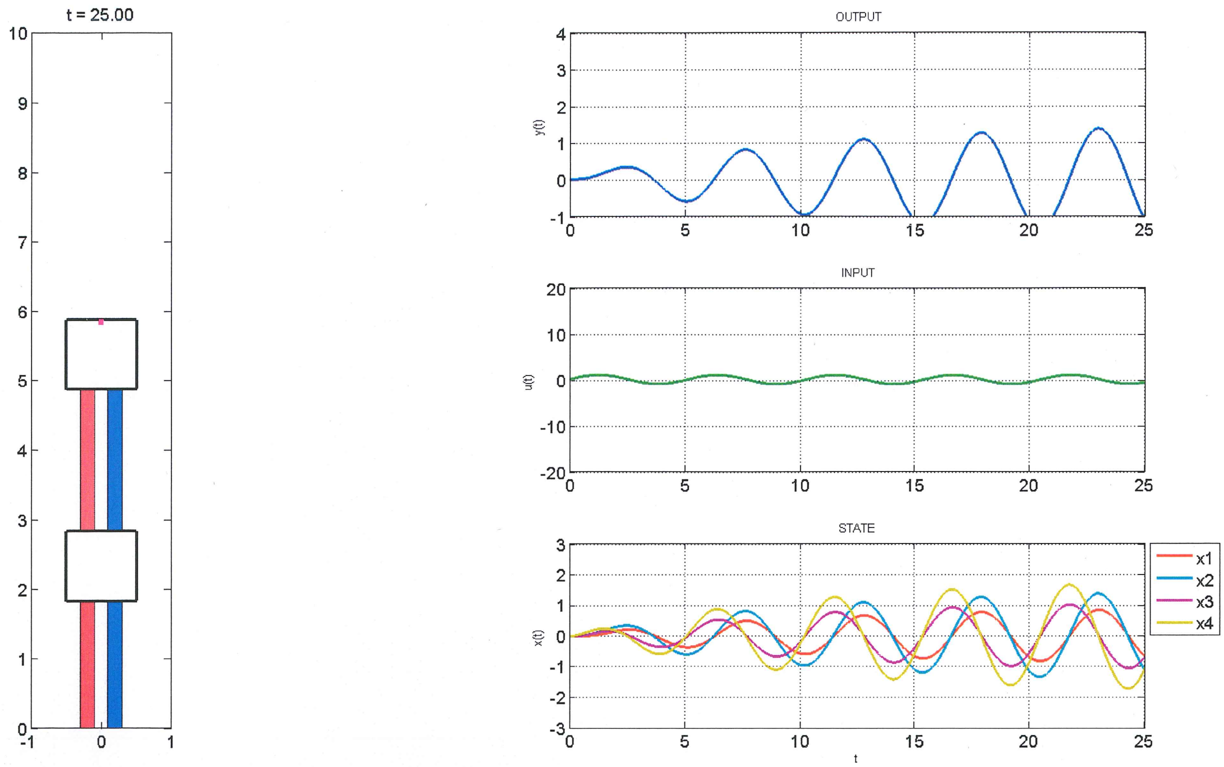


The open-loop plant transfer function acts like a band-pass filter. There is a prominent resonant peak, with frequencies attenuated above and below it.

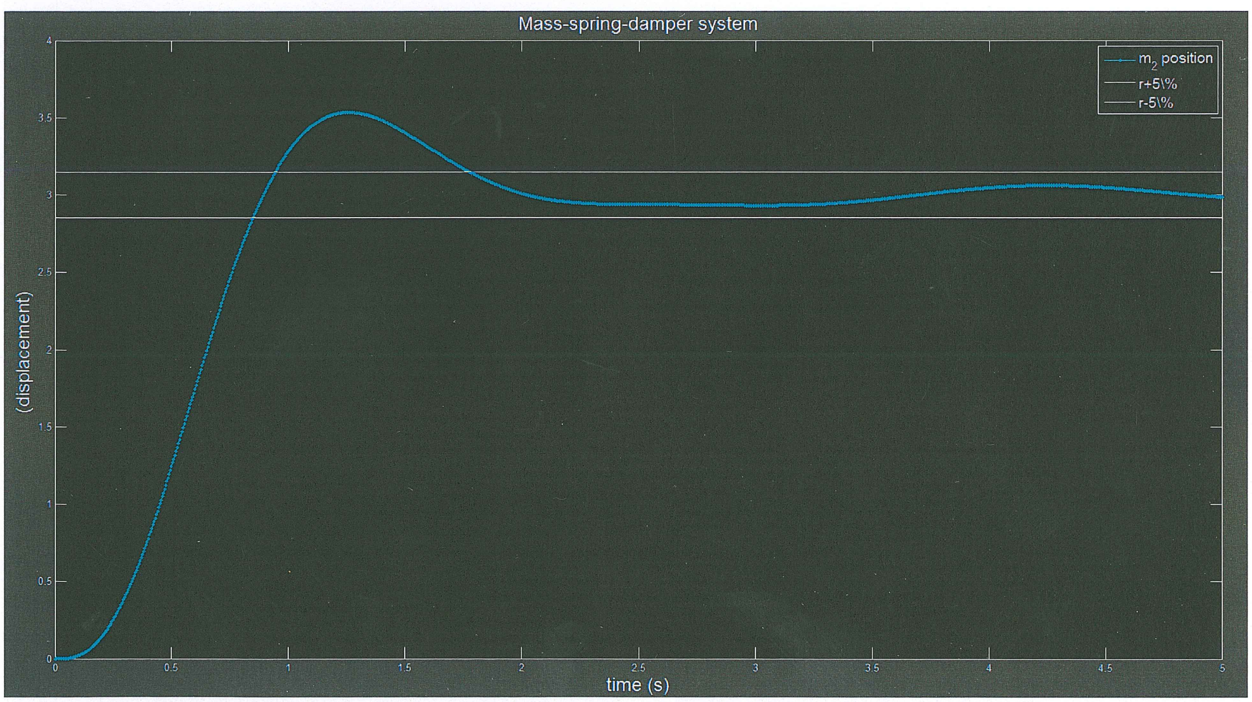
Problem 3 a) iv)



We can see that when the system is driven at its resonant frequency, the output is about 80 degrees out of phase with the input, as expected from examining the Bode plot above.



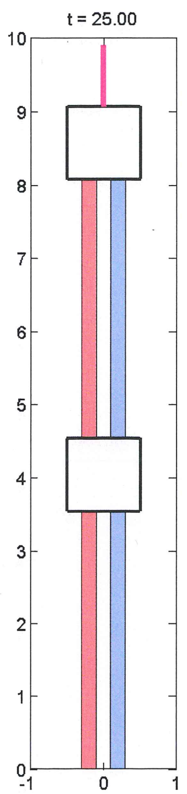
Problem 3 b) i)

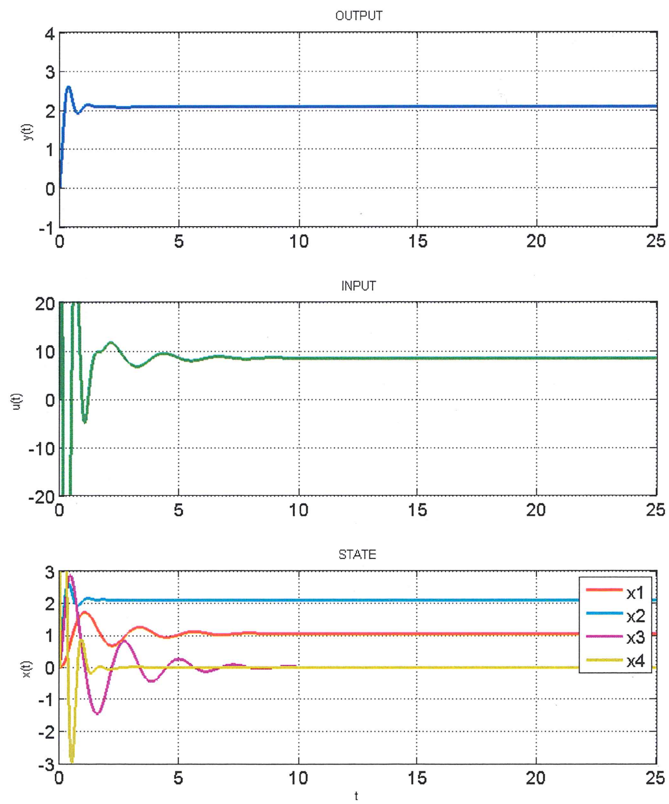


This was achieved using LQR control using the following weights/gains:

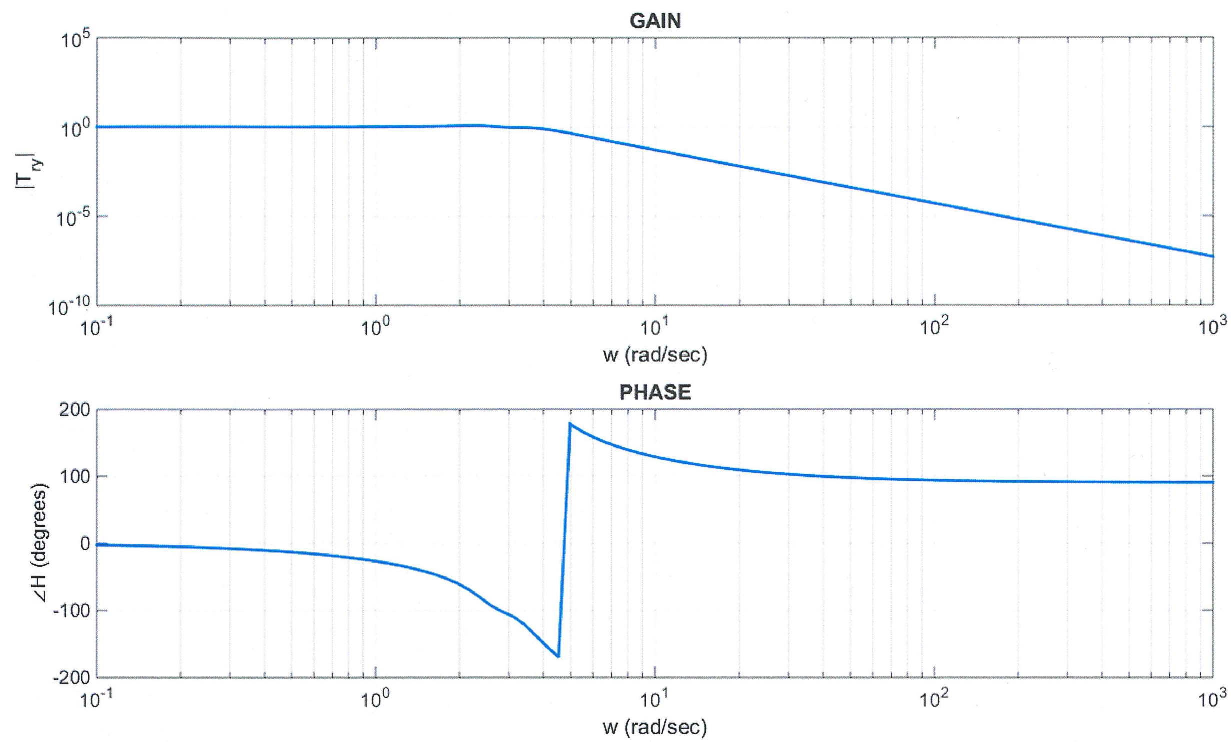
```
Q = [1 0 0 0;0 100 0 0;0 0 1 0;0 0 0 1];
K = lqr(A,B,Q,0.05);
L = acker(A',C',[-20 -20 -20 -20])';
kint = 100;
```

Note that this was simulated from $x_0 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$ so the weights of the observer are irrelevant (perfect state knowledge was assumed for the estimator).



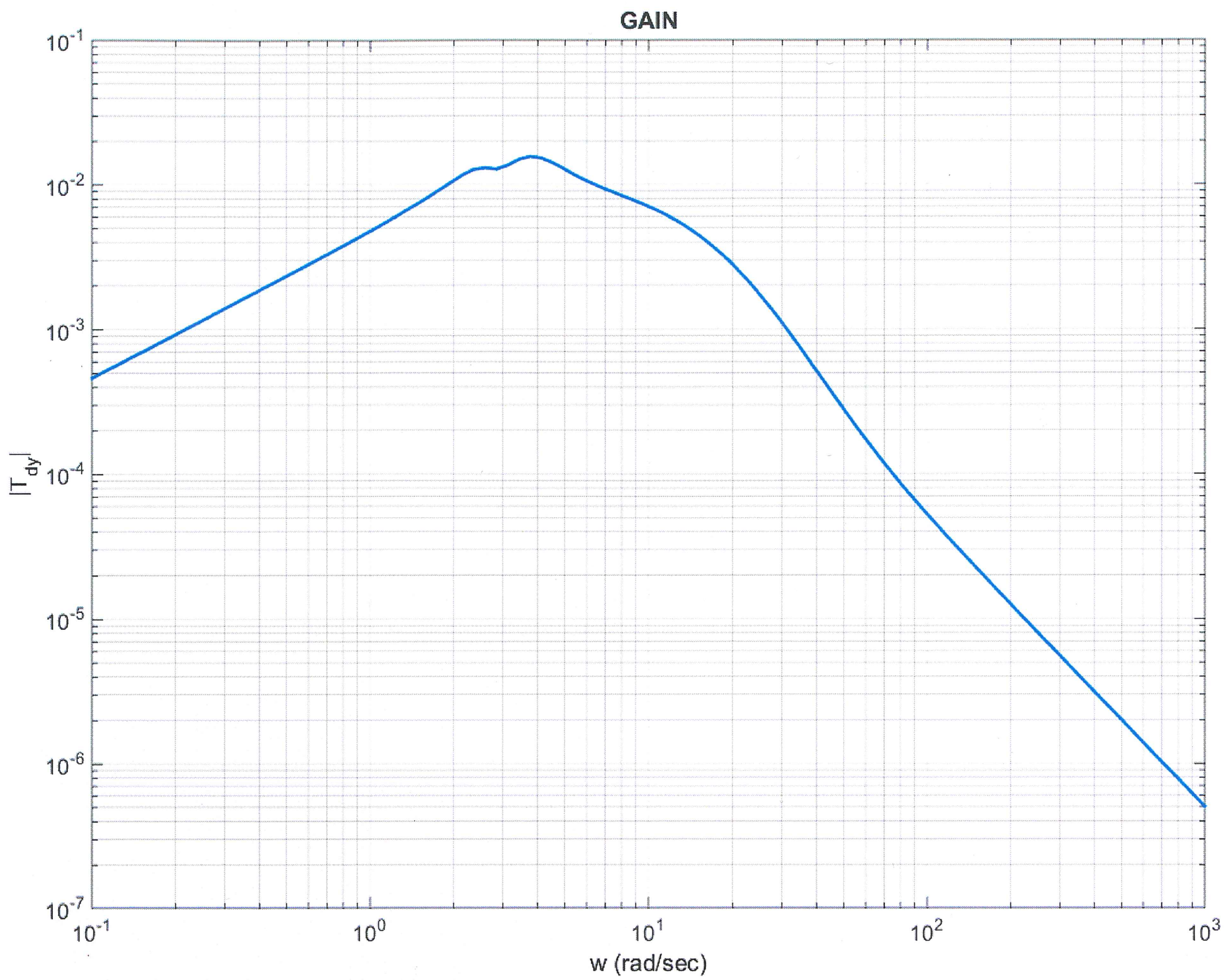


Problem 3 b) iii)

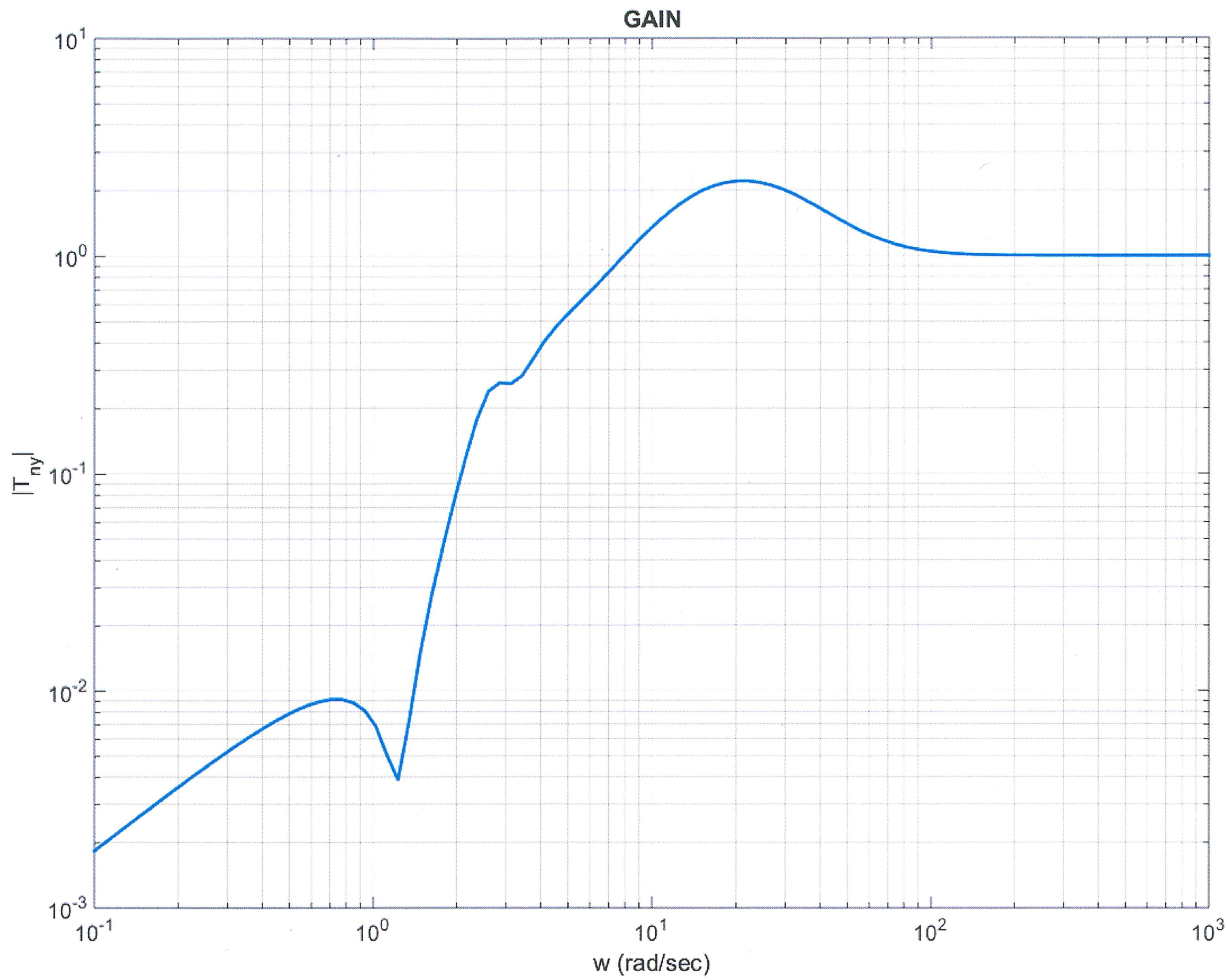


The zero-frequency gain is 1 (0 dB) as it should be if we expect to track a constant reference signal. The bandwidth is $\omega_{bw} = 4.132\ rad/s$. The closed-loop system acts like low-pass filter, high frequency reference signals are attenuated.

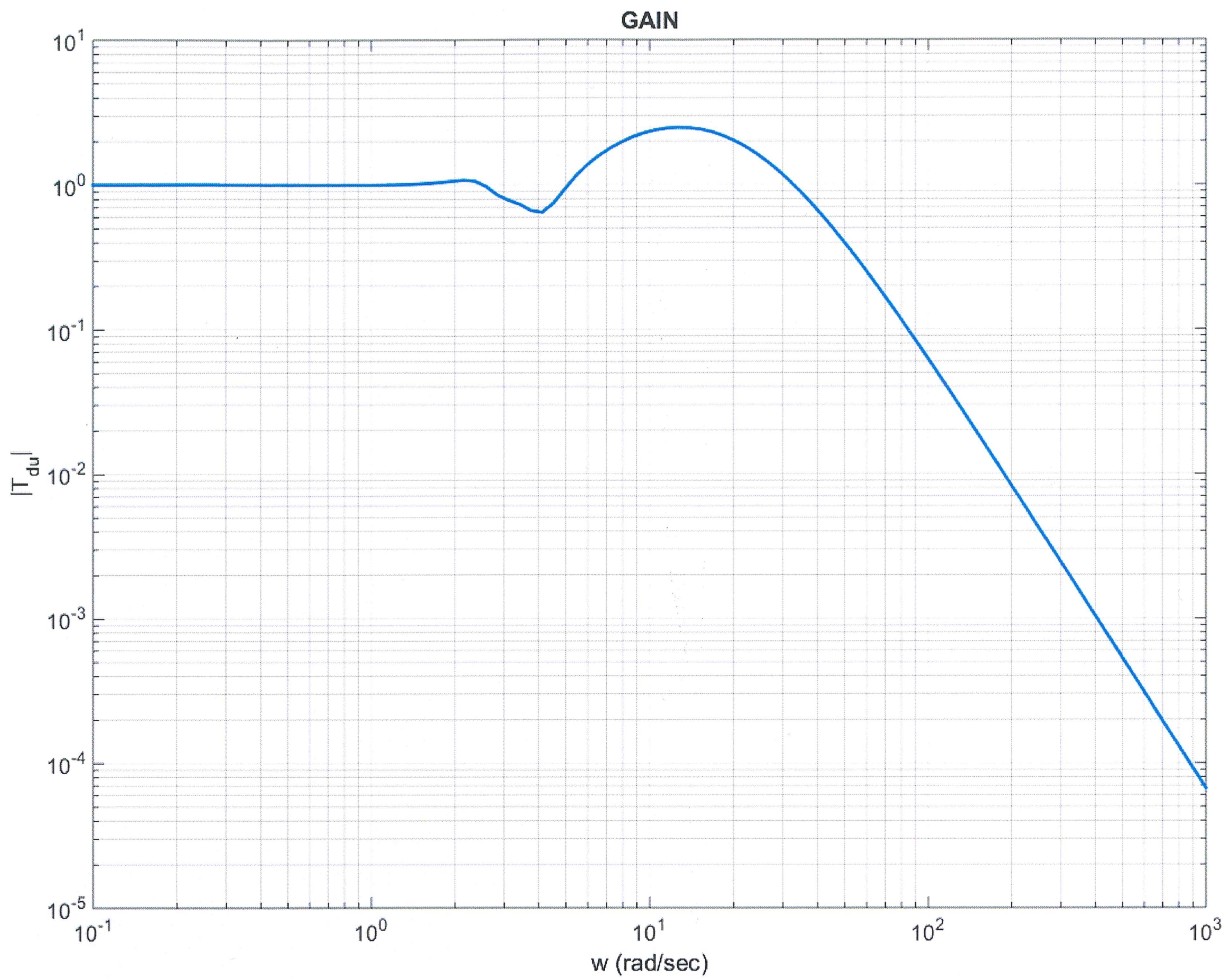
Problem 3 b) iv)



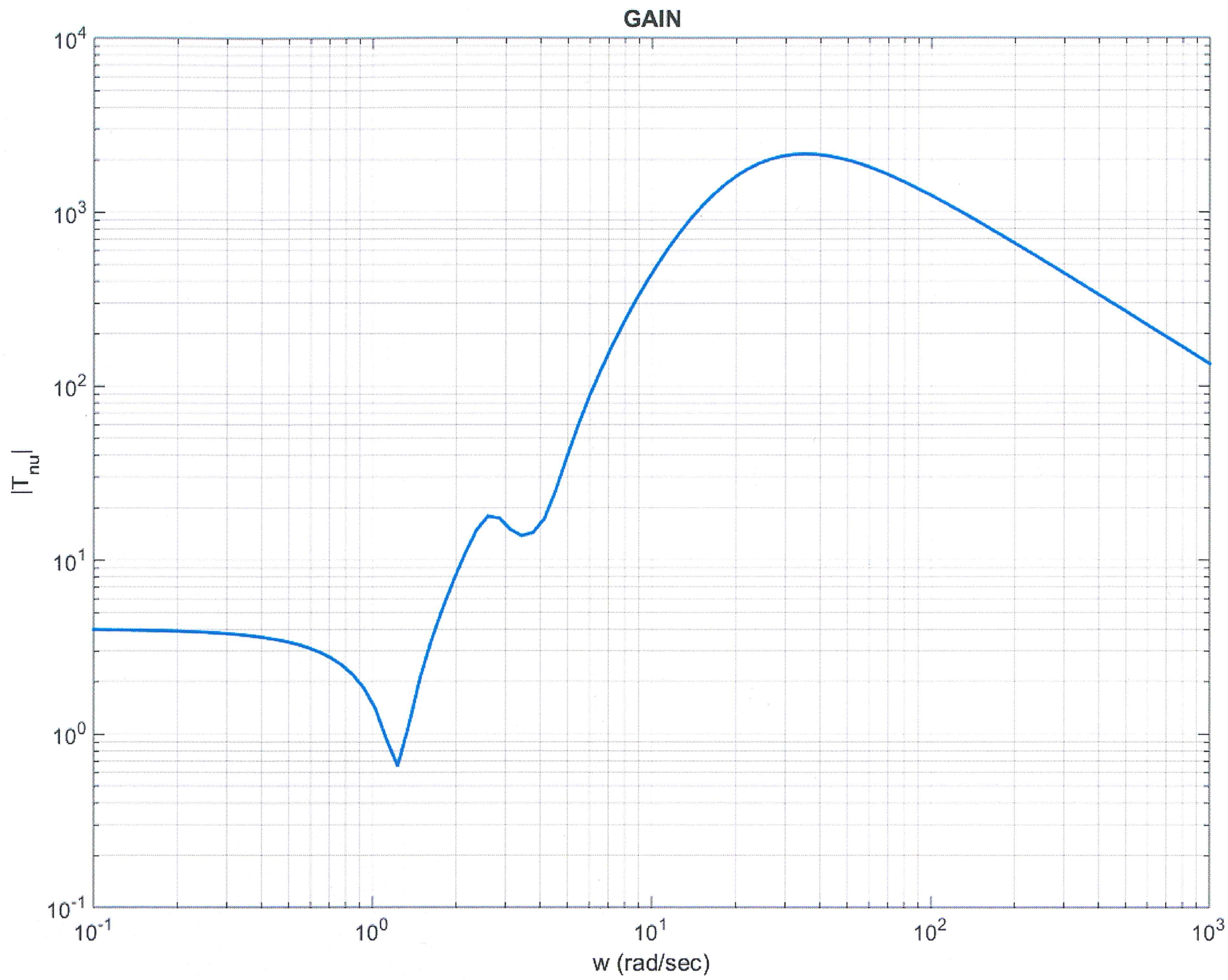
Bode plot for $T_{dy}(s)$ – band pass



Bode plot for $T_{ny}(s)$ – low-pass

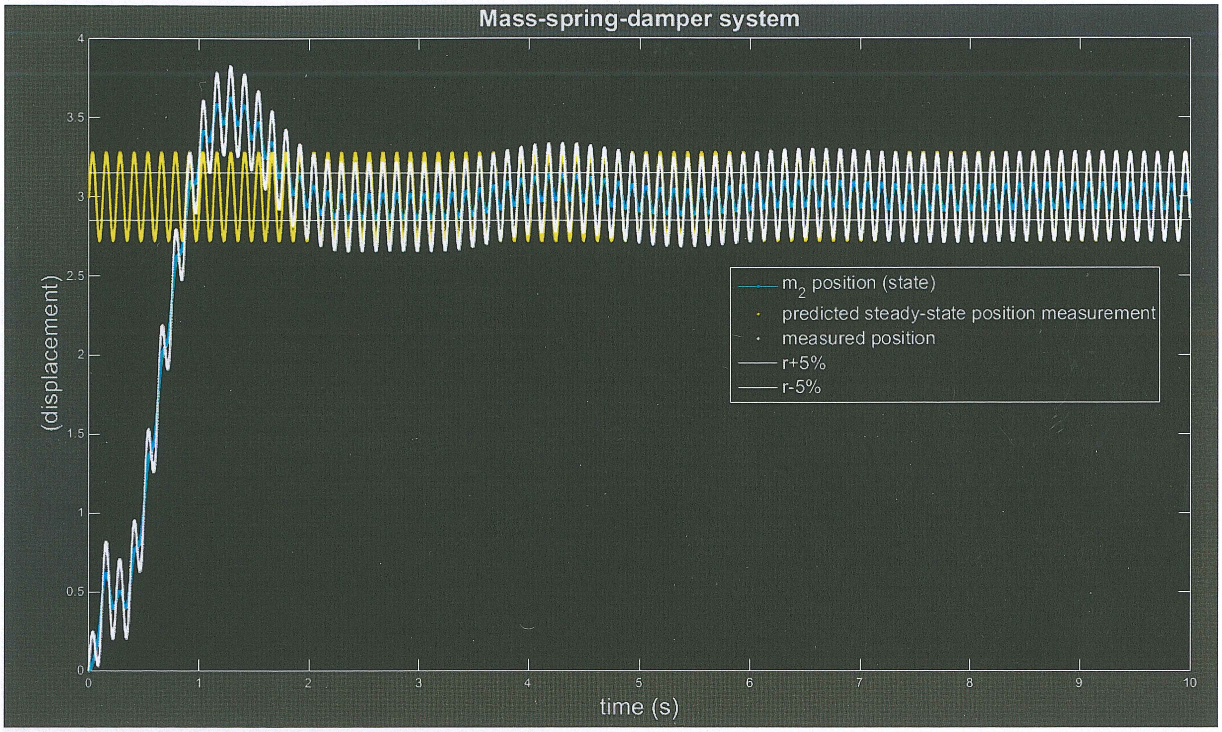


Bode plot for $T_{du}(s)$ – low pass

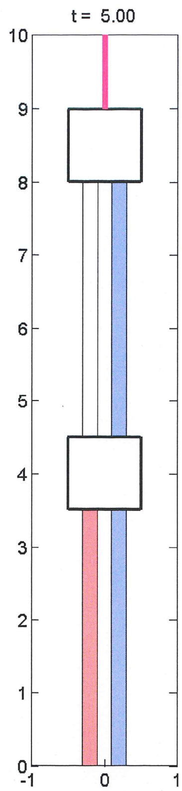


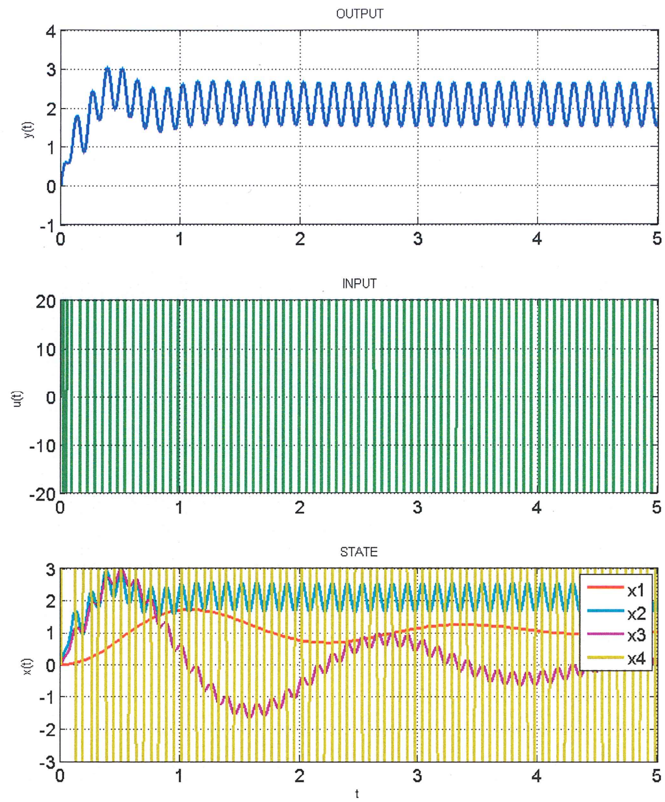
Bode plot for $T_{nu}(s)$ – high-pass/notch

Problem 3 b) v)



We can see that our design is fairly robust to the low-frequency disturbance load and this makes sense because our Bode plot suggests that all disturbances regardless of frequency are attenuated (because the whole curve is vertically translated well below 0 dB). Our design is, however, a bit sensitive to high frequency measurement noise. Examining the plot $T_{ny}(s)$ we see that noise is amplified by a factor of 1.396. So our noise amplitude of 0.2 is being amplified as can be seen in the output figure above. To remedy this we should find a way to reduce the gain of the $T_{ny}(s)$ at a frequency of 50 rad/s. We could also include more filters to help attenuate high-frequency noise, but that would involve reconstructing our control architecture.





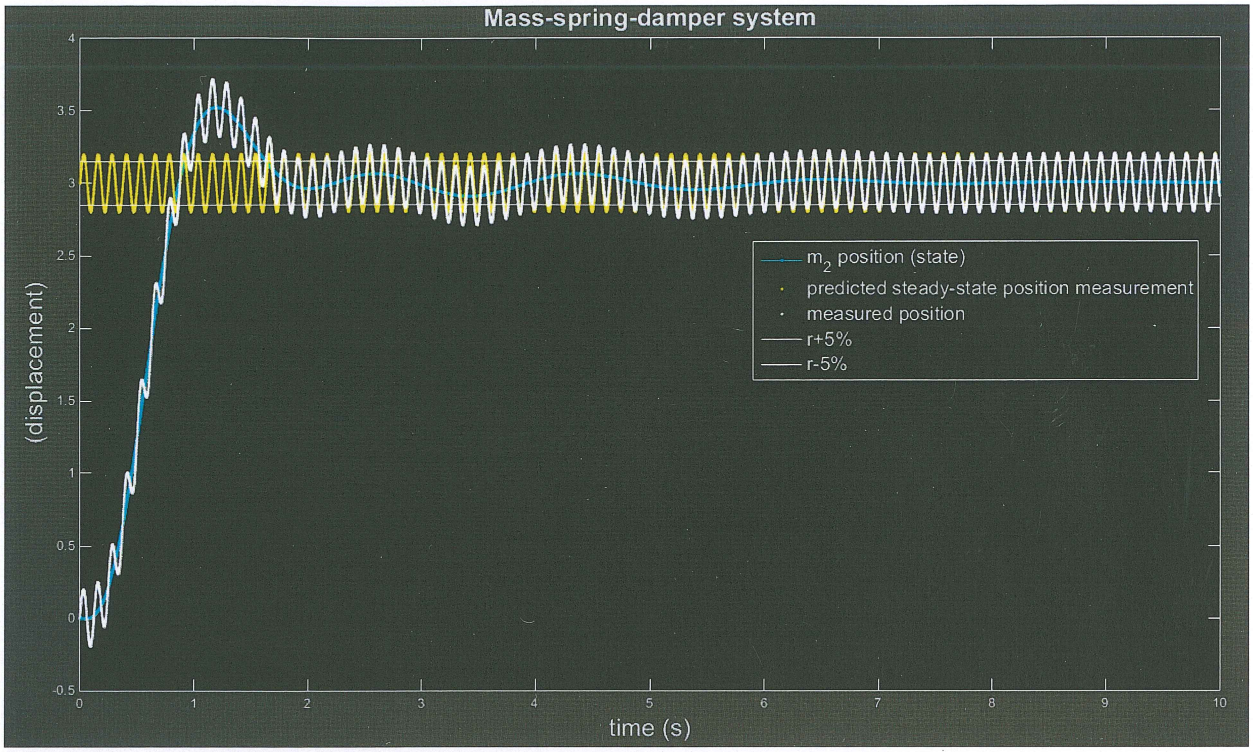

```
persistent isFirstTime
if isempty(isFirstTime)
    isFirstTime = false;
    fprintf(1,'initialize control loop\n');
    userdata.xhat = params.xhat0;
    userdata.errorInt = 0.0;
end
%u=sin(params.wmax*t);

error = y-r;
userdata.errorInt = userdata.errorInt + error*params.dt;

v = userdata.errorInt;

u = -params.K*userdata.xhat - params.kint*v;
userdata.xhat = userdata.xhat + params.dt*(params.A*userdata.xhat +
params.B*u - params.L*(params.C*userdata.xhat - y));
```

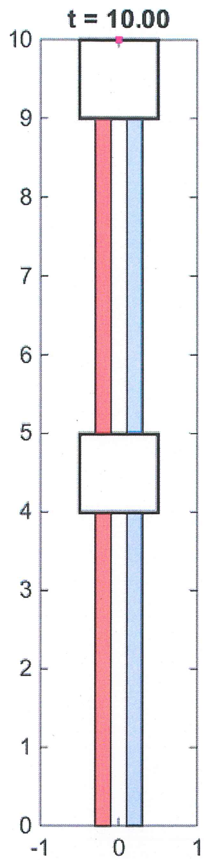
Problem 3 b) vi)

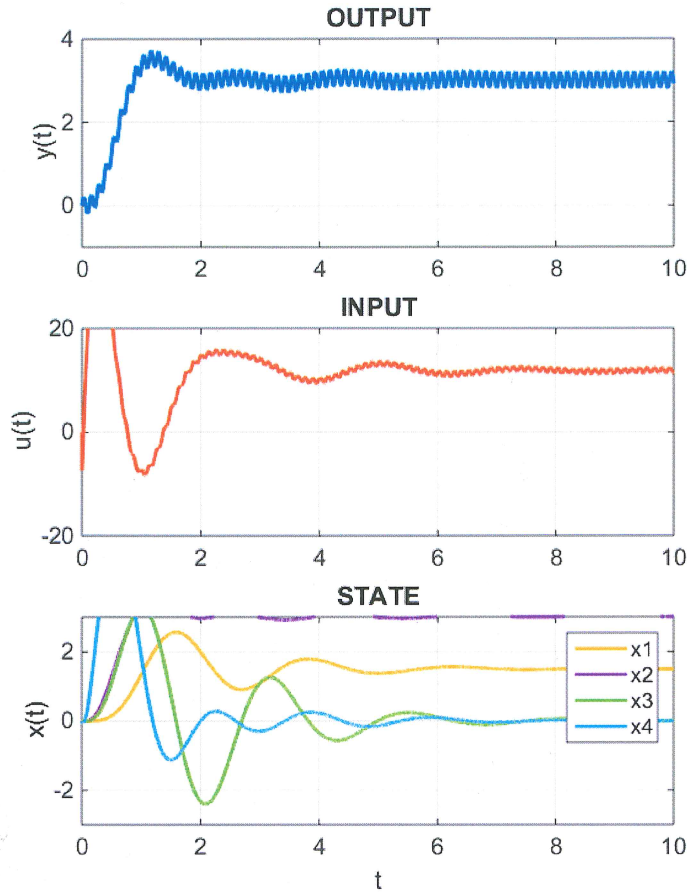


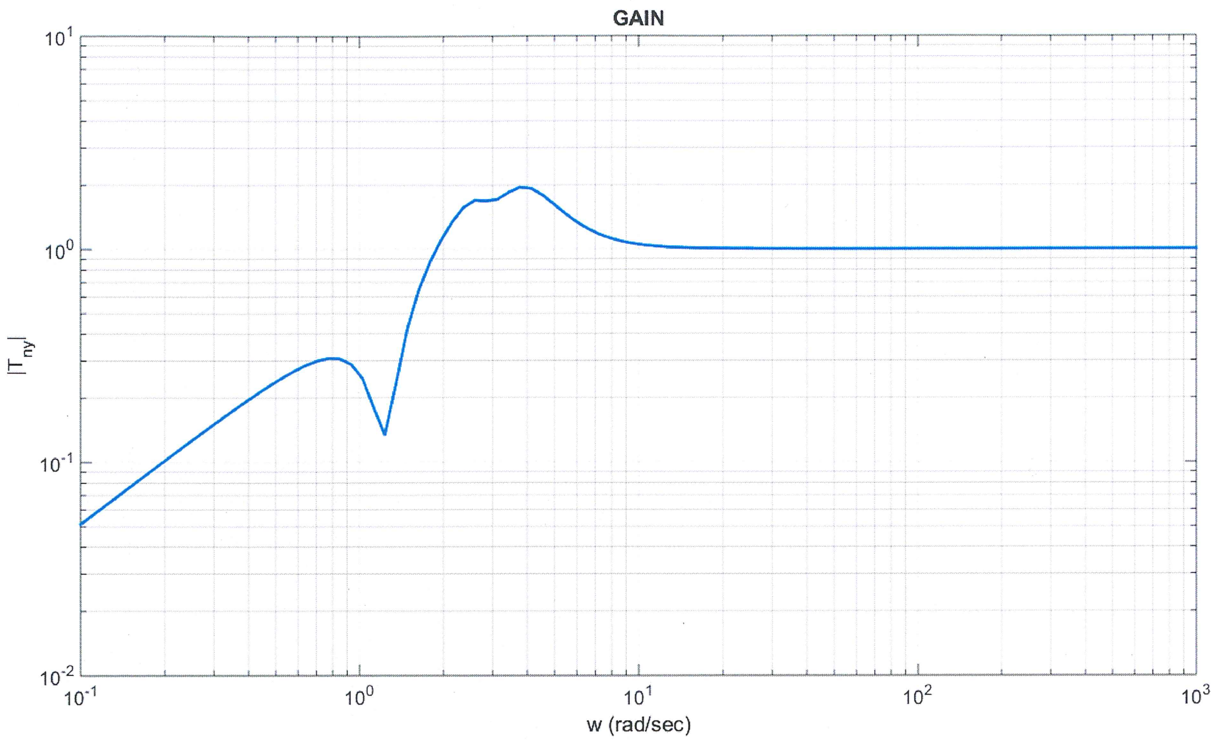
Here, we redesign our system using a Kalman filter:

```
Qo = 1.0;
Ro = eye(4);
L = lqr(A',C',inv(Ro),inv(Qo))';
kint = 100;
```

Not only have we reduced the measurement noise, but we have also caused the state to settle to the reference signal whereas in the previous design, the state retained some oscillatory behavior in steady-state.


t = 10.00


OUTPUT
y(t)
INPUT
u(t)
STATE
x(t)
x1
x2
x3
x4
t



The bode plot that we really care about in this redesign is $T_{ny}(s)$ which determines how much noise impacts our state measurements. We can see that with our new design we have shifted this Bode plot to the left and decreased the gain of the transfer function at the frequency at which the noise is occurring ($\omega = 50\, rad/s$).

Problem 4

$$H(s) = \frac{2000s + 4000}{s^3 + 40.01s^2 + 40000.4s + 400} = \frac{2000(s+2)}{(s+0.01)(s^2+40s+40000)}$$

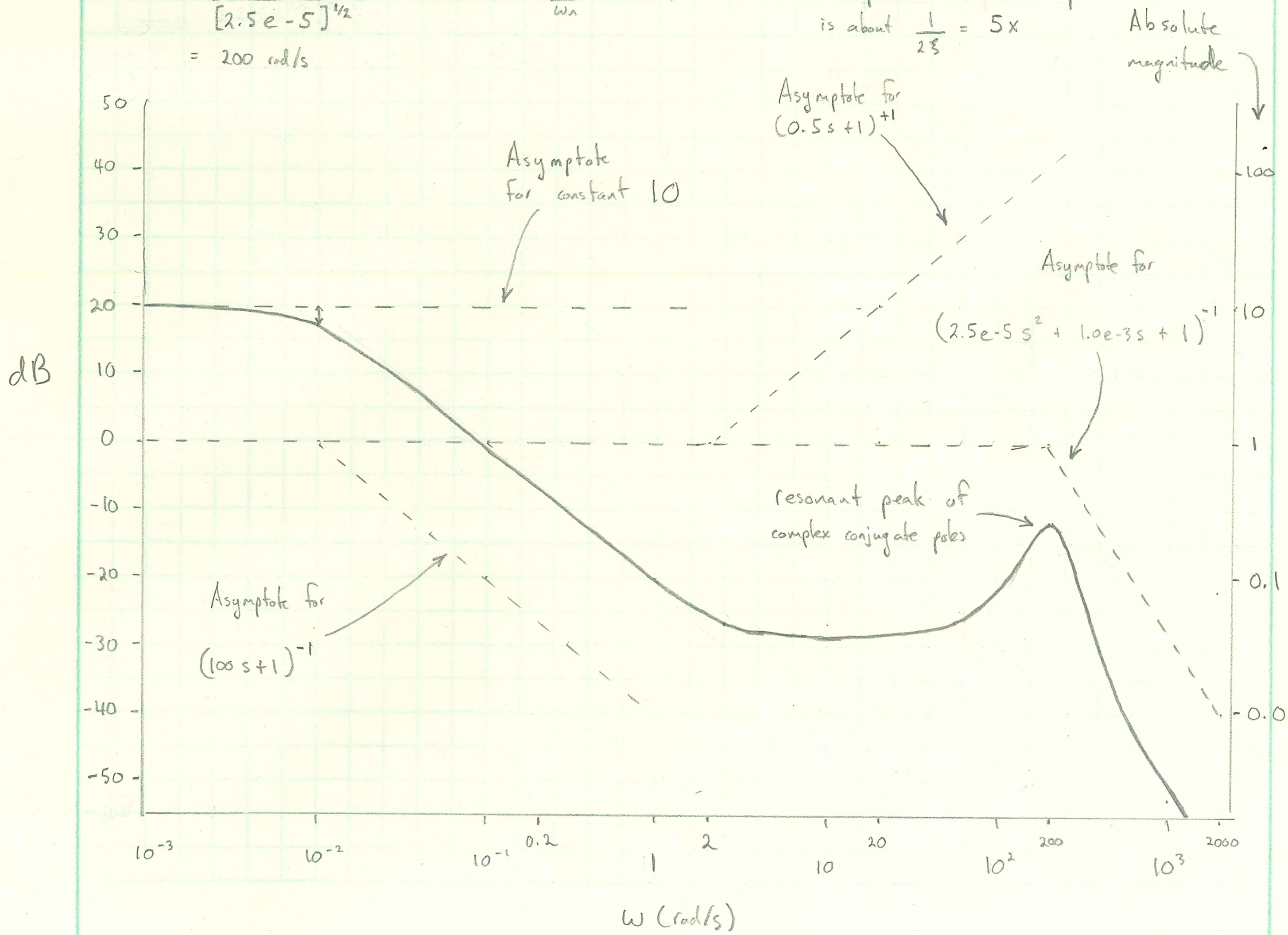
$$= \frac{2000(2)}{(0.01)(40000)} \quad \frac{(0.5s+1)}{(100s+1)(2.5e\text{-}5\, s^2 + 1.0e\text{-}3s + 1)}$$

Low frequency gain is 10 which is $20 \log_{10}(10) = 20\,dB$

Real pole/zero: $(j\omega\tau+1)^{\pm 1}$ enters at 1 (0 dB), breaks at $\omega = 1/\tau$ for $(j\omega\tau+1)$ and increases at a rate of 10/decade for a zero (decreases for a pole).

Second order term: $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\xi\left(\frac{j\omega}{\omega_n}\right) + 1\right]^{\pm 1}$: enters at 1 (0 dB), breaks at ω_n, then increases/decreases at 100/decade depending on whether the term is in the numerator or denominator.

$\omega_n = \frac{1}{[2.5e\text{-}5]^{1/2}} = 200$ rad/s $\qquad 1.0e\text{-}3 = \frac{2\xi}{\omega_n} \Rightarrow \xi = 0.1 \Rightarrow$ amplification at resonant peak is about $\frac{1}{2\xi} = 5\times$

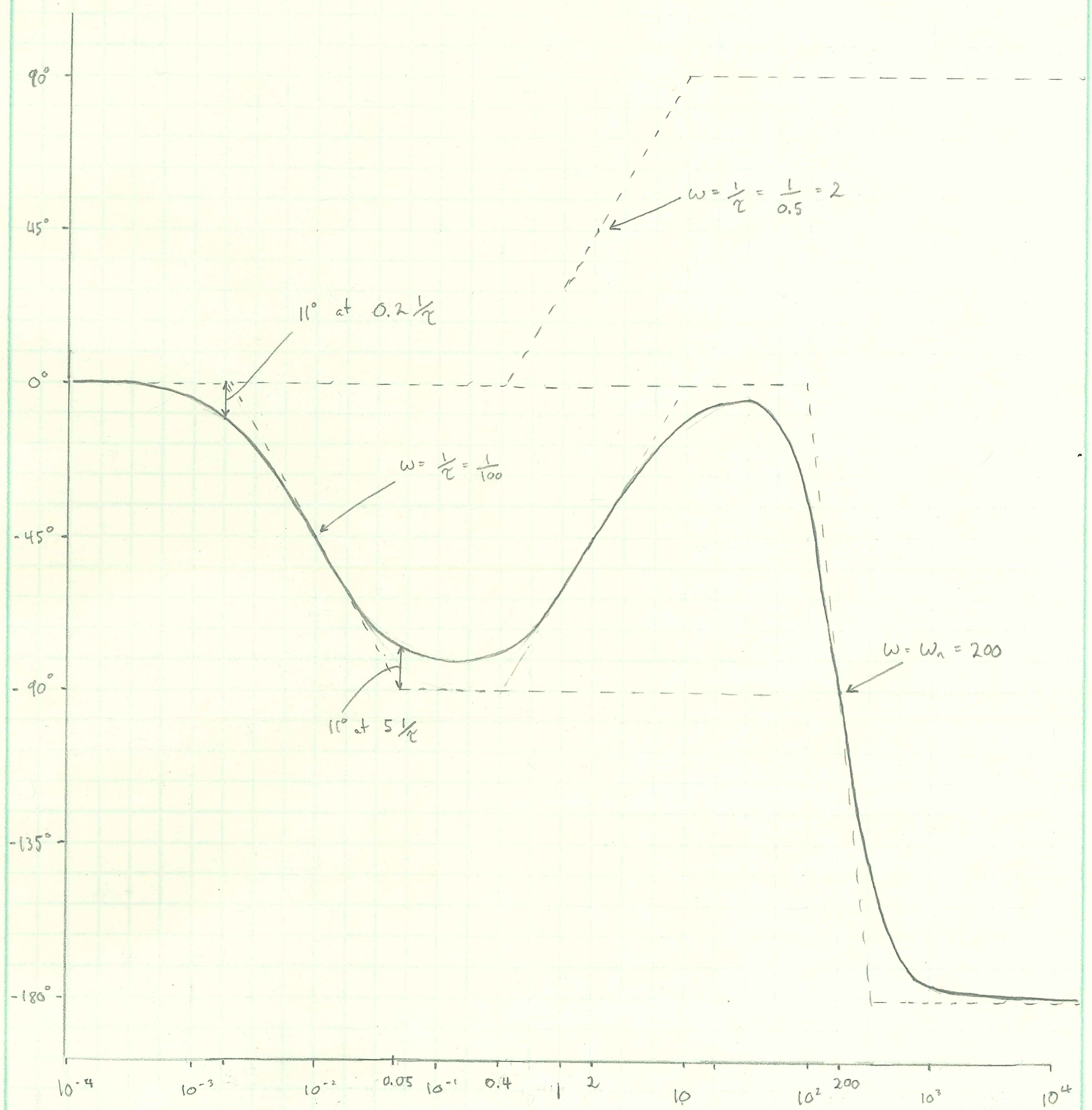


Phase

Constant: phase is 0°

Real pole/zero: phase is 0° at low frequencies, then 45° at break point, then +90° for zero, −90° for pole

Complex poles/zeros: phase is 0° at low frequencies, ±90° at ω_n, then ±180° at high frequencies.



b) • Examining the gain curve we can see that the DC gain is 10 (20 dB) so if our transfer function is written in proper form, the constant will be 10.

• Between $\omega = 10^{-1}$ and $\omega = 10^{1}$ we see that the slope of the gain curve is -10/decade (-20 dB/decade). We also note that the phase plot begins at 0° for low frequencies, transitions to -45° at $\omega = 10^{-1}$, then heads towards -90° at $\omega = 1$. Both of these behaviours can be explained by the presence of a real pole in the denominator of the transfer function. In addition, we know that the pole's break frequency must be at $\omega = 1/\tau = 10^{-1}$ since this is where the gain plot begins decreasing and where the phase reaches -45°.

• At $\omega = 10^{1}$ we see that the gain plot begins to level off, suggesting that there is a real zero in the numerator whose asymptote is counter acting the denominator pole's asymptote (i.e. the net asymptote's slope is zero in this region). We also note that the phase curve begins increasing in this regime as well, so we conclude that there is a real zero with a break frequency of $\omega = 1/\tau = 10^{1}$

• Finally, we see that around $\omega = 10^{2}$, the gain and phase begin decreasing in the same way they did after the low break frequency pole broke. So, at this point the effects of the first pole and the zero are negating one another and this high frequency behaviour can be accounted for by another real pole with break frequency $\omega = 1/\tau = 10^{2}$.

$$H(s) = \frac{10\,(0.1s+1)}{(10s+1)(0.01s+1)}$$