AE403 Homework #2: Alternative Representations of Rotation

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(due at the beginning of class on Thursday, February 25)

1. (12 points) *Be able to construct a rotation matrix from Euler Angles.* Consider the XYZ body-axis Euler Angle sequence: frame 1 is generated by a rotation θ_1 about x_0, frame 2 is generated by a rotation θ_2 about y_1, and the body-fixed frame 3 is generated by a rotation θ_3 about z_2.

   (a) Write R_3^0 in terms of R_{x,θ_1}, R_{y,θ_2}, and R_{z,θ_3}.

   (b) Do the matrix multiplication and write R_3^0 in terms of θ_1, θ_2, and θ_3.

2. (6 points) *Be able to recover Euler Angles from a rotation matrix.* Consider again the XYZ body-axis Euler Angle sequence. Denote the resulting rotation matrix by

$$R_3^0 = A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

   (a) Find expressions for θ_1, θ_2, and θ_3 in terms of the elements of A, just like we did in class for the ZYZ Euler Angle sequence.

   (b) Find the XYZ Euler Angles for the following rotation matrix.

$$A = \begin{bmatrix} 0.3536 & -0.6124 & 0.7071 \\ 0.5732 & 0.7392 & 0.3536 \\ -0.7392 & 0.2803 & 0.6124 \end{bmatrix}$$

   Please express your answer in degrees.

3. (18 points) *Be able to construct a rotation matrix from a quaternion.* Any rotation matrix $R \in SO(3)$ can be generated by a *single* rotation of some angle μ about some fixed axis $a \in \mathbb{R}^3$, where $\|a\| = 1$. Exponential coordinates are one way to do this. First, recall that the scalar differential equation

$$\dot{x} = bx$$

   has the solution

$$x(t) = e^{bt}x(0).$$

   As you will learn in a course on linear systems, the analogous matrix differential equation

$$\dot{R} = \widehat{a}R$$

has the solution

$$R(t) = e^{\widehat{a}t} R(0),$$

where

$$e^{\widehat{a}t} = I + \widehat{a}t + \frac{(\widehat{a}t)^2}{2!} + \frac{(\widehat{a}t)^3}{3!} + \dots$$

is called the *matrix exponential*. We may interpret rotation by μ about a as the result after a body that is initially aligned with the base frame undergoes constant angular velocity a (of unit magnitude) for time $t = \mu$, or in other words simply as

$$R(\mu) = e^{\widehat{a}\mu} I = e^{\widehat{a}\mu}.$$

It is a remarkable fact that, in this case, the matrix exponential converges to

$$e^{\widehat{a}\mu} = I + \widehat{a} \sin \mu + \widehat{a}^2 (1 - \cos \mu) . \tag{1}$$

In particular, you can think of the matrix exponential as a map from skew-symmetric matrices to rotation matrices:

$$e^{\widehat{w}} \colon so(3) \to SO(3).$$

A quaternion is a number

$$Q = q_0 + iq_1 + jq_2 + kq_3,$$

where we refer to $q_0 \in \mathbb{R}$ as the "real part" and to

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3$$

as the "hyper-imaginary part." We will often use the compact notation

$$Q = (q_0, q) .$$

The terms q_0 and q here are also often referred to as *Euler parameters*. We use quaternions to describe orientation by associating

$$Q = (\cos (\mu/2) , a \sin (\mu/2)) \tag{2}$$

with a rotation by μ about the axis a. Notice that any quaternion that represents a rotation will satisfy

$$q_0^2 + q^T q = 1,$$

that is, it will be of unit length. Another way to say this is that every *unit quaternion* represents some rotation—conversely, it turns out that every rotation can be represented by exactly two unit quaternions. (Caveat: the zero rotation is represented by exactly one.)

(a) Using Eq. (2), prove that

$$\widehat{a} \sin \mu = 2q_0 \widehat{q}$$
$$\widehat{a}^2 (1 - \cos \mu) = 2 \left(qq^T - q^T qI \right) .$$

(b) Plugging these results into Eq. (1), prove that a rotation matrix $R \in SO(3)$ may be written as

$$R = \left(q_0^2 - q^T q\right) I + 2qq^T + 2q_0\widehat{q} \tag{3}$$

for quaternion $Q = (q_0, q)$.

(c) Write the quaternion that is generated by rotating an angle $\mu = 120°$ about an axis

$$a = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(d) Write the rotation matrix for the rotation in part (c).

(e) An *eigenvector* $v \in \mathbb{R}^3$ of a square matrix $A \in \mathbb{R}^{3\times3}$ is a column vector that satisfies $Av = \lambda v$ for some real number $\lambda \in \mathbb{R}$, which we call an *eigenvalue*. It is a fact that every rotation matrix has some eigenvector with eigenvalue $\lambda = 1$. Find this eigenvector for the rotation matrix you constructed in part (d). (What do you conclude about this eigenvector in general?)

(f) Apply the results of Problem 2 to find a set of XYZ Euler Angles (body-fixed) that generate the rotation matrix you constructed in part (d). How many other sets of XYZ Euler Angles would give the same answer?

4. (10 points)*Be able to recover a quaternion from a rotation matrix.* Consider an arbitrary rotation matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \tag{4}$$

(a) By equating Eq. (4) with Eq. (1), prove that

$$\mu = \pm \cos^{-1}\left(\frac{A_{11} + A_{22} + A_{33} - 1}{2}\right) \tag{5}$$

$$a = \frac{1}{2\sin\mu} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}. \tag{6}$$

Note the singularity at $\mu = 0$, for which a can be chosen arbitrarily. Note also that (a, μ) and $(-a, -\mu)$ produce exactly the same rotation matrix.

(b) Using the identities

$$\sin\mu = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$$
$$\cos\mu = 2\cos^2\frac{\mu}{2} - 1$$

derive formulas for q_0 and q that do not involve trigonometric functions. Your formula for q_0 should depend only on the elements of A. Your formula for q should depend both on the elements of A and on q_0. (The formulas you find here will only be valid for $q_0 \neq 0$. From Eq. (3), it is possible to derive three other analogous formulas that are valid for $q_1 \neq 0$, $q_2 \neq 0$, and $q_3 \neq 0$, respectively. You are not required to derive these formulas here, but it is important to keep them in mind.)

(c) A spacecraft rotates by $90°$ about its own x-axis, $120°$ about its own y axis, then by $270°$ about its own z-axis. Find the rotation matrix expressing its orientation with respect to the base frame.

(d) Find the equivalent angle and axis for the rotation matrix in part (c).

(e) Find the equivalent quaternion for the rotation matrix in part (c).

5. (6 points) *Be able to relate the time derivative of Euler Angles with angular velocity.* Consider, again, the XYZ body-axis Euler Angle sequence.

   (a) Angular velocities add, so that we may write the angular velocity of the body in body-fixed coordinates as
   $$\omega^3_{0,3} = \omega^3_{0,1} + \omega^3_{1,2} + \omega^3_{2,3}.$$
   Express $\omega^3_{0,3}$ in terms of $\dot\theta_1$, $\dot\theta_2$, and $\dot\theta_3$. Write your answer as
   $$\omega^3_{0,3} = A \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix}$$
   for some matrix $A \in \mathbb{R}^{3\times 3}$.

   (b) Starting with the results of part (a), express $\dot\theta_1$, $\dot\theta_2$, and $\dot\theta_3$ in terms of $\omega^3_{0,3}$. Write your answer as
   $$\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix} = B\omega^3_{0,3}$$
   for some matrix $B \in \mathbb{R}^{3\times 3}$.

   In this problem, the matrices A and B should depend only on the Euler Angles, not on their angular rates or on components of the angular velocity.

6. (16 points) *Be able to relate the time derivative of a quaternion with angular velocity.* Denote the body-fixed angular velocity by
   $$\omega^1_{0,1} = \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$
   Find the relationship between $\dot q_0$, $\dot q$, and ω in the following steps:

   (a) In Problem 8 of HW1, you showed that
   $$\dot R = R\widehat{\omega},$$
   or equivalently that
   $$\widehat{\omega} = R^T \dot R.$$
   By computing R and $\dot R$ from Eq. (3) and plugging them into the above expression, prove that
   $$\begin{aligned} \omega_1 &= 2\left(-q_1\dot q_0 + q_0\dot q_1 + q_3\dot q_2 - q_2\dot q_3\right) \\ \omega_2 &= 2\left(-q_2\dot q_0 - q_3\dot q_1 + q_0\dot q_2 + q_1\dot q_3\right) \\ \omega_3 &= 2\left(-q_3\dot q_0 + q_2\dot q_1 - q_1\dot q_2 + q_0\dot q_3\right). \end{aligned} \tag{7}$$

(b) By differentiating the constraint

$$q_0^2 + q^T q = 1$$

derive one more equation

$$0 = 2\left(q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3\right). \tag{8}$$

(c) From the above, we can write Eqs. (7)-(8) in matrix notation as

$$2\begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \\ q_0 & q_1 & q_2 & q_3 \end{bmatrix}\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix}.$$

Prove that the matrix in the above expression is orthonormal (just like a rotation matrix).

(d) Prove that

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2}\omega^T q \\ \dot{q} &= \frac{1}{2}\left(\omega q_0 - \widehat{\omega} q\right). \end{aligned} \tag{9}$$

(e) Prove that

$$\omega^T\omega = 4\left(\dot{q}_0^2 + \dot{q}^T\dot{q}\right).$$

(f) As an exercise, consider a spacecraft that is initially oriented so that its body frame is rotated by 45° about the z_0-axis. This spacecraft begins rotating at a constant rate of 1 radian per second about its own x-axis: Write the quaternion $Q(0) = (q_0(0), q(0))$ that represents the orientation of the spacecraft at time $t = 0$.

(g) Write the angular velocity of the above spacecraft in body-fixed coordinates.

(h) Integrate Eq. (9) to find $q_0(t)$ and $q(t)$. Verify that $q_0(2\pi) = -q_0(0)$ and that $q(2\pi) = -q(0)$. What has the spacecraft done?

7. (12 points) *Be able to perform coordinate transformation and sequential rotation with quaternions.* If Q is the quaternion representation of R_1^0 and P is the quaternion representation of R_2^1, then the quaternion representation of R_2^0 is given by the multiplication QP.

Multiplying two quaternions is like multiplying two complex numbers, but in this case order matters (i.e., multiplication is not commutative):

$$\begin{aligned} i^2 &= -1 & j^2 &= -1 & k^2 &= -1 \\ ij &= k & jk &= i & ki &= j \\ ji &= -k & kj &= -i & ik &= -j. \end{aligned}$$

It is also easy to verify that

$$QP = \left(q_0p_0 - q^T p, q_0p + p_0q + \widehat{q}p\right), \tag{10}$$

so quaternion multiplication can be done without keeping track of i, j, k. We can represent the coordinates of a point (or a vector) as a quaternion with zero real part. In particular, we can represent the coordinates

$$p^1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

as the quaternion

$$P^1 = (0, p^1)$$

It is also possible to perform coordinate transformation with quaternions. Let $Q = (q_0, q)$ be the quaternion representing R_1^0, the orientation of frame 1 with respect to frame 0. Define the *conjugate* of Q as

$$Q^* = (q_0, -q)\,.$$

Then, the quaternion P^0 representing the coordinates p^0 are given by the *quaternion rotation operator* as

$$P^0 = QP^1Q^*.$$

There is a beautiful symmetry to these formulas, in particular if you compare them to the ones for rotation matrices.

Now, consider again the spacecraft in Problem 4c:

(a) Find the quaternions Q_x, Q_y and Q_z corresponding to the three sequential rotations.

(b) Find the quaternion $Q_xQ_yQ_z$ both by direct multiplication and by using the formula (10), and verify that your two answers are the same.

(c) Find the rotation matrix corresponding to $Q_xQ_yQ_z$ and check that it is the same one you found in Problem 4.

(d) A point p on the spacecraft has coordinates

$$p^1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Use the quaternion rotation operator to find the coordinates p^0 describing how p looks from the base frame. Do the multiplication directly or use Eq. (10), as you like.

8. (20 points)*Be able to plot orientation as a function of time using rotation matrices, Euler Angles, and quaternions—and be able to explain the pros and cons of each representation.* By integrating the angular velocity as a function of time, we can find the orientation as a function of time, e.g., to find $R_1^0(t)$. As a capstone to this homework assignment, you will integrate the output of a "strapped-down rate gyro" to find and plot the orientation of a rigid body. The rate gyro is fixed to the body and records the angular velocity

$$w_{0,1}^1 = 10e^{-t} \begin{bmatrix} \sin t \\ \sin 2t \\ \sin 3t \end{bmatrix}.$$

As usual, the base frame is frame 0 and the body-fixed frame is frame 1. Assume that these frames are initially aligned. Integrate from $t = 0$ to $t = 6$ seconds. To complete this problem using rotation matrix, Euler angles, and quaternions, you will be using the MATLAB script `hw2_prob8.m`, available on the course website.

(a) Integrate the equations for $\dot{R}$, for $\dot{\theta}$, and for $(\dot{q}_0, \dot{q})$ from $t = 0$ to $t = 6$ seconds. Plot R, $(\theta_1, \theta_2, \theta_3)$, and (q_0, q) as functions of time.

(b) Plot the rates of change $\dot{R}$, $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$, and $(\dot{q}_0, \dot{q})$ as functions of time.

(c) Compute the rotation matrix corresponding to $(\theta_1, \theta_2, \theta_3)$ and to (q_0, q) at the final time $t = 6$. Compare your results to the value of R at time $t = 6$, and comment on any discrepancies.

Please be careful to properly define your initial conditions! Note that, although you are not required to submit an animation, it may be extremely useful to visualize your results in order to debug them. It would be relatively easy to modify the code in order to do this.