AE403 Homework #4: Dynamics, Part 2

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(due at the beginning of class on Thursday, April 14)

1. (15 points) *Be able to derive criteria that guarantee passive spin stabilization under energy dissipation.* Consider a torque-free axisymmetric spacecraft. The rotational kinetic energy satisfies
$$T = \frac{1}{2}\left(J_t\omega_t^2 + J_a\omega_3^2\right), \tag{1}$$
the magnitude of the angular momentum satisfies
$$|h|^2 = J_t^2\omega_t^2 + J_a^2\omega_3^2, \tag{2}$$
and the nutation angle satisfies
$$\cos^2\theta = \left(\frac{J_a\omega_3}{|h|}\right)^2. \tag{3}$$
In class, we saw how to derive the following relationship, starting with (1)-(3):
$$\dot{\theta} = \left(\frac{J_tJ_a}{|h|^2\sin\theta\cos\theta}\right)\left(\frac{\dot{T}}{J_a - J_t}\right). \tag{4}$$
Assume that a particular spacecraft has principal moments of inertia $J_t = 10\ \mathrm{kg}\cdot\mathrm{m}^2$ and $J_a = 25\ \mathrm{kg}\cdot\mathrm{m}^2$ and is initially spinning about its own principal x axis (a transverse axis) with angular velocity $\omega_1(0) = 5$ rad/s. An onboard energy damper decreases the rotational kinetic energy. The energy dissipation rate is
$$\dot{T} = -10\sin\theta,$$
where θ is the nutation angle.

   (a) About which axis will the spacecraft rotate after energy dissipation?

   (b) What is the final nutation angle?

   (c) How long will it take the nutation angle to reach its final value?

   (d) What is the final angular velocity? (Does its magnitude go up or down?)

2. (20 points) *Be able to model and analyze the stability of a spinning spacecraft with a wheel nutation damper.* Consider a torque-free axisymmetric spacecraft with an axisymmetric damper wheel that spins about the body-fixed y axis, perpendicular to the axis of symmetry z. Let
$$\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \in \mathbb{R}^3$$

be the angular velocity of the spacecraft in body-fixed coordinates, and let $v \in \mathbb{R}$ be the spin rate of the damper wheel *relative* to the spacecraft. Let J_d be the moment of inertia of the damper wheel about its spin axis. Let $J_t = J_1 = J_2$ and $J_a = J_3$ be the transverse and axial moments of inertia of *both the spacecraft and the damper wheel together*—notice that these values do not change as the wheel rotates, and also notice that our definition implies $J_t > J_d$. Assume that the coefficient of friction between the spacecraft and the wheel is c.

(a) Write the angular momentum of the spacecraft/wheel system in body-fixed coordinates. Denote the resulting vector by h.

(b) Apply Euler's equation, stated as follows, to find three ordinary differential equations in four variables (ω and v) that govern the motion of the spacecraft/wheel system:

$$\tau = \dot{h} + \widehat{\omega} h$$

Everything in this equation is expressed in body-fixed coordinates. (Note that $\tau = 0$.)

(c) Apply Euler's equation again to find a fourth ordinary differential equation that governs the rotation of the wheel alone about its spin axis. (Note that $\tau \neq 0$ in this case, and remember that the wheel is symmetric about its spin axis.)

(d) Combine your answers from parts (b)-(c) and linearize these four ordinary differential equations about the nominal values $\omega_1 = \omega_2 = 0$, $\omega_3 = n$, and $v = 0$. You should find that ω_3 is now decoupled from the other state variables.

(e) Put your answer from part (d) in the form

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} \omega_1 \\ \omega_2 \\ v \end{bmatrix}$$

for some matrix $A \in \mathbb{R}^{3\times 3}$.

(f) We know that the system

$$\dot{x} = Ax$$

is stable if and only if all eigenvalues of $A \in \mathbb{R}^{m\times m}$ have strictly negative real part, i.e., if and only if A is *Hurwitz*. Routh's method is one way to check this condition. We begin by writing the characteristic polynomial in the form

$$\det(sI - A) = a_0 + sa_1 + \cdots + s^{m-1}a_{m-1} + s^m.$$

Then, depending on the degree m, Routh gives the following criteria for stability:

- $m = 1$: A is Hurwitz if and only if $a_0 > 0$.
- $m = 2$: A is Hurwitz if and only if $a_0, a_1 > 0$.
- $m = 3$: A is Hurwitz if and only if $a_0, a_1, a_2 > 0$ and $a_1 a_2 > a_0$.
- $m = 4$: A is Hurwitz if and only if $a_0, a_1, a_2, a_3 > 0$, $a_3 a_2 > a_1$, and

$$a_3 a_2 a_1 - a_3^2 a_0 > a_1^2.$$

(When $m \geq 5$, things get more complicated—and we don't usually need to solve these by hand, so don't worry about it.) Apply Routh's method to derive conditions on J_t, J_a, and J_d that guarantee the spacecraft/wheel system is stable. Verify that these conditions match the ones given by (4). In particular, explain any discrepancies.

3. (20 points) *Be able to model and analyze the stability of a dual-spin spacecraft with a wheel nutation damper on the de-spun platform.* Consider a torque-free axisymmetric dual-spin spacecraft with an axisymmetric damper wheel that is attached to the platform and that spins about the body-fixed y axis, perpendicular to the axis of symmetry z. Let $\omega \in \mathbb{R}^3$ be the angular velocity of the platform in body-fixed coordinates, let $v_r \in \mathbb{R}$ be the spin rate of the rotor *relative* to the platform, and let v_d be the spin rate of the damper wheel *relative* to the platform. Let J_r be the moment of inertia of the rotor about its spin axis (z), and let J_d be the moment of inertia of the damper wheel about its spin axis (y). Let $J_t = J_1 = J_2$ and $J_a = J_3$ be the transverse and axial moments of inertia of *everything together*—notice that our definition implies $J_t > J_d$ and $J_a > J_r$. Assume that the coefficient of friction between the spacecraft and the wheel is c. Finally, assume that—through active control—we maintain $\omega_3 = 0$, so also $\dot{\omega}_3 = 0$.

   (a) Write the angular momentum of the entire system in body-fixed coordinates. Denote the resulting vector by h.

   (b) Apply Euler's equation, stated as follows, to find three ordinary differential equations in four variables (ω_1, ω_2, v_r, and v_d) that govern the motion of the system:

$$\tau = \dot{h} + \widehat{\omega} h$$

   Everything in this equation is expressed in body-fixed coordinates. (Note that $\tau = 0$.)

   (c) Apply Euler's equation again to find a fourth ordinary differential equation that governs the rotation of the wheel alone about its spin axis. (Note that $\tau \neq 0$ in this case, and remember that the wheel is symmetric about its spin axis.)

   (d) Combine your answers from parts (b)-(c) and linearize these four ordinary differential equations about the nominal values $\omega_1 = \omega_2 = 0$, $v_r = n$, and $v_d = 0$. You should find that v_r is now decoupled from the other state variables.

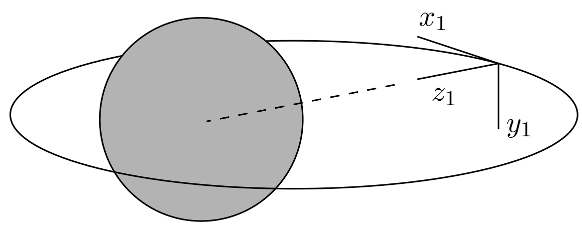
   (e) Put your answer from part (d) in the form

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{v}_d \end{bmatrix} = A \begin{bmatrix} \omega_1 \\ \omega_2 \\ v_d \end{bmatrix}$$

   for some matrix $A \in \mathbb{R}^{3\times 3}$.

   (f) Apply Routh's method (just as in Problem 2) to derive conditions on J_t, J_a, J_r, and J_d that guarantee the system is stable.

4. (15 points) *Be able to derive the kinematics of earth rotation.* Things are a bit different if we consider a spacecraft in orbit. Define an *orbit frame*, which we will denote 1, as follows:



Notice that we have defined this frame so that x_1 is always tangent to the orbit (i.e., to the velocity vector) and z_1 always points toward the center of the earth. Denote the inertial frame by 0 and the usual body-fixed frame by 2. It is common to use an XYZ body-axis Euler angle sequence to represent the orientation of the spacecraft with respect to the orbit frame—in other words, to represent R_2^1. We can express the angular velocity of the spacecraft as the sum

$$\omega_{0,2} = \omega_{0,1} + \omega_{1,2}.$$

If we assume a fixed orbital angular velocity n, then

$$\omega_{0,1} = -ny_1,$$

or in other words

$$\omega_{0,1}^1 = \begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix}.$$

(a) Compute $\omega_{0,1}^2$, in other words, to express the angular velocity of the orbit frame relative to the inertial frame in body-fixed coordinates.

(b) Compute $\omega_{1,2}^2$ in terms of $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$, in other words, to express the angular velocity of the body-fixed frame relative to the orbit frame in body-fixed coordinates.

(c) Combine parts (a)-(b) to find $\omega_{0,2}^2$. Express your answer in the form

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = A \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} - nb$$

for some matrix $A \in \mathbb{R}^{3\times 3}$ and vector $b \in \mathbb{R}^3$. (Notice that it would be easy to put this result in the form $\dot{\theta} = A^{-1}(\omega + nb)$, i.e., to find the angular rates in terms of the angular velocity.)

(d) Show that if you linearize, assuming small angles $\theta_1, \theta_2, \theta_3 \approx 0$ and angular rates $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3 \approx 0$, then your answer from part (c) can be approximated by

$$\begin{aligned} \omega_1 &= \dot{\theta}_1 - n\theta_3 \\ \omega_2 &= \dot{\theta}_2 - n \\ \omega_3 &= \dot{\theta}_3 + n\theta_1. \end{aligned}$$

5. (15 points) *Be able to express the gravity gradient torque in body-fixed coordinates for a spacecraft in low-earth orbit.* In class, we have seen that the gravity-gradient torque is given by

$$\tau_{\mathrm{gg}}^2 = \frac{3\mu}{|r^2|^5}\widehat{r^2} J^2 r^2,$$

where r is a vector from the Earth center to the spacecraft and μ is a gravitational constant. The superscripts in this expression can get a little confusing—the "2" refers to the body-fixed frame, while the "5" is a power. Notice that it is easy to express r in orbit frame coordinates:

$$r^1 = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix},$$

where d is the orbit radius.

(a) Assuming that

$$R_2^1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and that

$$J^2 = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix},$$

and noting that the orbit angular velocity n satisfies

$$n^2 = \frac{\mu}{d^3},$$

explicitly compute the vector τ_{gg}^2 in terms of n, the principal moments of inertia J_i, and the elements A_{ij} of R_2^1.

(b) Find the elements A_{ij} of R_2^1, assuming small angles $\theta_1, \theta_2, \theta_3 \approx 0$. Based on your answer, show that τ_{gg}^2 can be approximated by

$$-3n^2 \begin{bmatrix} (J_2 - J_3)\,\theta_1 \\ -(J_3 - J_1)\,\theta_2 \\ 0 \end{bmatrix}.$$

(c) Combining the results from Problem 4, show that the linearized equations of motion for the spacecraft can be written in the following way:

$$\begin{aligned} -3n^2 (J_2 - J_3)\,\theta_1 &= J_1\dot{\omega}_1 - (J_2 - J_3)(-n\omega_3) \\ 3n^2 (J_3 - J_1)\,\theta_2 &= J_2\dot{\omega}_2 \\ 0 &= J_3\dot{\omega}_3 - (J_1 - J_2)(-n\omega_1). \end{aligned}$$

Note that, with respect to the considered linearization in Problem 4 and Problem 5,

$$\begin{aligned} w_2 w_3 &\approx -n\dot{\theta}_3 = -n w_3 \\ w_3 w_1 &\approx 0 \\ w_1 w_2 &\approx -n\dot{\theta}_1 = -n w_1. \end{aligned}$$

(d) Notice that pitch motion (θ_2, w_2) is decoupled from roll/yaw motion. Combining the results from Problem 4 and 5, write the equations for pitch motion in the form

$$\begin{bmatrix} \dot{\theta}_2 \\ \dot{w}_2 \end{bmatrix} = A \begin{bmatrix} \theta_2 \\ w_2 \end{bmatrix} + b$$

for some $A \in \mathbb{R}^{2\times2}$ and $b \in \mathbb{R}^2$.

(e) In exactly the same way, write the equations for roll/yaw motion in the form

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_3 \\ \dot{w}_1 \\ \dot{w}_3 \end{bmatrix} = A \begin{bmatrix} \theta_1 \\ \theta_3 \\ w_1 \\ w_3 \end{bmatrix}$$

for some matrix $A \in \mathbb{R}^{4\times4}$.

6. (15 points) *Be able to simulate a spin satellite.* In class, we discussed a famous spin satellite Explorer-I, which was launched in 1958 and was the first satellite sent to space by the US. Explorer-I was designed to spin about its minor axis. Unfortunately, it resulted in a flat spin due to energy dissipation that was caused by its flexible antenna.

   Now, you are an engineer whose task is to design the attitude control system of a similar satellite as Explorer-I. This time, the science instrument requires the spacecraft to spin at a rate of approximately 100 rpm (slower than Explorer-I) about its minor axis (again!). However, now we do not have the flexible antenna. Instead, the scientists want to have an axisymmetric wheel that spins about the body-fixed y-axis. It turns out that there is friction between the spacecraft and the wheel, and its coefficient of friction is $c = 1.0\ \mathrm{N \cdot m/(rad/s)}$.

   Your job is to use your results from Problem 2 to check the stability of this spacecraft. The spacecraft is axisymmetric and its principal moments of inertia of both the spacecraft and the wheel together are $J_t = J_1 = J_2 = 4.83\ \mathrm{kg \cdot m^2}$, $J_a = J_3 = 0.044\ \mathrm{kg \cdot m^2}$ (i.e., same as the original Explorer-I). The moment of inertia of the wheel about its spin axis is $J_d = 1\ \mathrm{kg \cdot m^2}$. Assume that the spacecraft is initially spinning at 100 rpm about its z-axis, but is not spinning about its x- and y-axes. Also assume that the spin rate of the wheel relative to the spacecraft is 0.01 rpm initially.

   (a) Using MATLAB, integrate the linearized equations of motion you found in Problem 2 (d) and simulate the motion of the spacecraft over 500 seconds (with a time step of 1 second). Plot the angular rates.

   (b) Using MATLAB, integrate the original nonlinear equations of motion you found in Problem 2 (b)-(c) and simulate the motion of the spacecraft over 500 seconds. Plot the angular rates. Do the results match the results from part (a)? Is linearization a valid approach for stability analysis? Hint: The spin rate ω_3 is not constant any more!

   (c) Now consider an oblate spacecraft. $J_3 = 10\ \mathrm{kg \cdot m^2}$, with everything else remaining the same. Repeat parts (a) and (b) to simulate the motion of the spacecraft over 500 seconds. Plot the angular rates for both cases of the linearized and nonlinear equations of motion.

   (d) Compare results from both parts (a) and (c) against your results from Problem 2 (f) (based on Routh's criteria). Discuss the results and compile your response as an attitude control systems engineer to the scientists.

For Problem 6, you will need the results from Problem 2. Hence, I reveal the solution to Problem 2 (e) for your verification purposes.

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-n(J_a-J_t)}{J_t} & \frac{nJ_d}{J_t} \\ \frac{n(J_a-J_t)}{(J_t-J_d)} & 0 & \frac{c}{(J_t-J_d)} \\ \frac{-n(J_a-J_t)}{(J_t-J_d)} & 0 & \frac{-cJ_t}{J_d(J_t-J_d)} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ v \end{bmatrix}$$

7. (Extra Credit: 10 points) *Be able to derive criteria on J_1, J_2, J_3 that guarantee (marginal) passive gravity-gradient stabilization for a spacecraft in low-earth orbit.* Consider again the spacecraft in low-earth orbit with gravity gradient considered in Problem 4 and 5. Define the following three constants:

$$a_1 = \frac{J_2 - J_3}{J_1} \qquad a_2 = \frac{J_1 - J_3}{J_2} \qquad a_3 = \frac{J_2 - J_1}{J_3}.$$

Find six inequalities involving a_1 and a_3 that guarantee marginal passive gravity-gradient stabilization for a spacecraft in low-earth orbit. (No a_2 should appear in them.) One inequality comes from Problem 5 (d) (with some additional manipulations to eliminate a_2), three more inequalities come from Problem 5 (e), and two final inequalities ($|a_1| < 1$ and $|a_3| < 1$) can be derived easily from the definition of moments of inertia.

Using either MATLAB or Mathematica, plot all of these inequalities on the same figure. Indicate—by shading regions of your plot—which values of a_1 and a_3 lead to which types of instability (e.g., pitch or roll/yaw).

What are the implications for the design of a spacecraft?