# AE403 Homework #3: Dynamics, Part 1

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(due at the beginning of class on Thursday, March 10)

1. (20 points) *Be able to derive Euler's equation.* This proof is the most difficult one that I will ask you to attempt all semester. It is a beautiful way of bringing together almost everything that you have learned about matrix notation as a way to represent both rotations and vector operations. Please note that we have already done most of this proof in class. Here, you will fill in the details.

   The following formulas (from class and HW1) will be helpful in solving this problem:

$$I = {R_1^0}^T R_1^0 = R_1^0 {R_1^0}^T \tag{1}$$

$$p^0 = R_1^0 p^1 \qquad \text{(and also } p^1 = {R_1^0}^T p^0\text{)} \tag{2}$$

$$\frac{d}{dt}(AB) = \dot{A}B + A\dot{B} \tag{3}$$

$$\widehat{\omega_{0,1}^0} = R_1^0 \widehat{\omega_{0,1}^1} {R_1^0}^T \tag{4}$$

$$\widehat{a}b = -\widehat{b}a \tag{5}$$

$$\widehat{a}^T = -\widehat{a} \tag{6}$$

$$\dot{R}_1^0 = \widehat{\omega_{0,1}^0} R_1^0 = R_1^0 \widehat{\omega_{0,1}^1} \tag{7}$$

$$(AB)^T = B^T A^T \tag{8}$$

$$\widehat{a}a = 0 \qquad \text{(i.e., the cross product of a vector with itself is zero)} \tag{9}$$

   OK, here we go:

   (a) The angular momentum of a homogenous rigid body (i.e., same density everywhere) about its center of mass is the vector

$$h = \int r \times v dm,$$

   where $r$ is the position and $v$ is the velocity of each piece of the body relative to its center of mass. We can express this equation in the coordinates of an inertial frame 0 as

$$h^0 = \int \widehat{r^0} \dot{r}^0 dm. \tag{10}$$

   Use (2), (3), and (7) to show that (10) can be rewritten as

$$h^0 = \int \widehat{r^0} \widehat{\omega_{0,1}^0} r^0 dm. \tag{11}$$

(b) Starting from (11), use (5), (6), (4), and (1)—in that order—to show that

$$h^0 = R_1^0 J^1 R_1^{0\,T} \omega_{0,1}^0, \tag{12}$$

where

$$J^1 = \int \widehat{r^1}^T \widehat{r^1} dm \tag{13}$$

is the *moment of inertia matrix* in body-fixed coordinates. In doing so, you will also have to use the fact that

$$\int A\,(\text{whatever})\,dx = A\left(\int (\text{whatever})\,dx\right)$$

and that

$$\int (\text{whatever})\,A dx = \left(\int (\text{whatever})\,dx\right) A$$

whenever $A$ does not depend on $x$. Note that this result can also be expressed as

$$h^1 = J^1 \omega_{0,1}^1.$$

(c) Starting from (12), use (3), (7), (2), (4), (8), (1), and (9) to show that

$$\dot{h}^0 = R_1^0 \left(\widehat{\omega_{0,1}^1} J^1 \omega_{0,1}^1 + J^1 \dot{\omega}_{0,1}^1\right). \tag{14}$$

In doing so, you will also have to use the fact that the mass moment of inertia matrix, when expressed as (13) in body-fixed coordinates, is constant in time.

(d) Euler tells us that the rate of change (as viewed from an inertial frame) of angular momentum about the center of mass is equal to the applied torque about the center of mass, or in other words that

$$\dot{h}^0 = \tau^0.$$

Combining this equation and (14), use (2) and (1) to show that

$$\tau^1 = J^1 \dot{\omega}_{0,1}^1 + \widehat{\omega_{0,1}^1} J^1 \omega_{0,1}^1. \tag{15}$$

You have finally arrived at Euler's equation! Everything in this equation is written in body-fixed coordinates. For convenience, we will often drop the notation and write Euler's equation in the simple form

$$\tau = J\dot{\omega} + \widehat{\omega} J \omega. \tag{16}$$

Now, kick back and relax for a little while. Bask in your own glory. You earned it!

2. (20 points) *Be able to compute principal axes.* Consider the moment of inertia matrix written in body-fixed coordinates $x_1, y_1, z_1$:

$$J^1 = \int \left(\widehat{r^1}^T \widehat{r^1}\right) dm.$$

As we have seen in class, $J^1$ is symmetric and positive definite, so the three eigenvalues of $J^1$ are all positive real numbers and the three eigenvectors can be chosen so that they form an orthogonal coordinate frame. Denote the eigenvalues by $\lambda_x, \lambda_y, \lambda_z$ and the corresponding eigenvectors by $x_2^1, y_2^1, z_2^1$. Notice that

$$J^1 \begin{bmatrix} x_2^1 & y_2^1 & z_2^1 \end{bmatrix} = \begin{bmatrix} x_2^1 & y_2^1 & z_2^1 \end{bmatrix} \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{bmatrix}.$$

We will define a "new" moment of inertia matrix

$$J^2 = \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{bmatrix}$$

and a "new" body-fixed frame

$$R_2^1 = \begin{bmatrix} x_2^1 & y_2^1 & z_2^1 \end{bmatrix}.$$

It is clear that $J^1$ and $J^2$ are related by a similarity transform:

$$J^2 = R_2^{1T} J^1 R_2^1.$$

We say that $x_2, y_2, z_2$ are the *principal axes of inertia*, that $\lambda_x, \lambda_y, \lambda_z$ are the *principal moments of inertia*, and that $J^2$ is the *principal moment of inertia matrix*. In order of smallest to largest, we call the principal moments of inertia the *minor*, *intermediate*, and *major* moments of inertia, respectively. It is almost always easier to work with a body-fixed frame that is aligned with the principal axes of inertia, so our first order of business will often be to find these principal axes.

In particular, consider a body with inertia matrix

$$J^1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and angular velocity

$$\omega_{0,1}^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find the angular momentum $h^1$.

(b) Find the principal moment of inertia. You are welcome to use Mathematica or MATLAB to check your work, but I recommend doing this by hand first, recalling that each eigenvalue $\lambda$ of a square matrix $A$ satisfies

$$\det(\lambda I - A) = 0$$

and that each corresponding eigenvector $v$ satisfies

$$(\lambda I - A)\, v = 0.$$

(c) Express both the angular velocity and the angular momentum in a coordinate frame aligned with the principal axes.

3. (20 points) *Be able to describe the characteristics of torque-free axisymmetric motion.* When written with respect to principle axes (where $J_1, J_2, J_3$ are the principal moments of inertia), Euler's equations (16) become

$$\begin{aligned}\tau_1 &= J_1\dot{\omega}_1 - (J_2 - J_3)\,\omega_2\omega_3 \\ \tau_2 &= J_2\dot{\omega}_2 - (J_3 - J_1)\,\omega_3\omega_1 \\ \tau_3 &= J_3\dot{\omega}_3 - (J_1 - J_2)\,\omega_1\omega_2.\end{aligned}$$

Consider an axisymmetric rigid body undergoing torque-free motion, for which

$$\tau_1 = \tau_2 = \tau_3 = 0$$

and

$$J_1 = J_2 = J_t$$

and

$$J_3 = J_a.$$

The subscripts $t$ and $a$ indicate the *transverse* and *axial* moments of inertia, respectively. Then, we have

$$\begin{aligned}0 &= J_t\dot{\omega}_1 - (J_t - J_a)\,\omega_2\omega_3 \\ 0 &= J_t\dot{\omega}_2 + (J_t - J_a)\,\omega_1\omega_3 \\ 0 &= J_a\dot{\omega}_3.\end{aligned}$$

The third equation implies that $\omega_3$ is constant. We call it the *spin rate* and denote it by

$$n = \omega_3.$$

What remains is

$$\begin{aligned}0 &= J_t\dot{\omega}_1 - (J_t - J_a)\,n\omega_2 \\ 0 &= J_t\dot{\omega}_2 + (J_t - J_a)\,n\omega_1.\end{aligned}$$

If we define the *relative spin rate* as

$$\lambda = \left(\frac{J_t - J_a}{J_t}\right) n,$$

then we can simplify our equations of motion further to

$$\begin{aligned}0 &= \dot{\omega}_1 - \lambda\omega_2 \\ 0 &= \dot{\omega}_2 + \lambda\omega_1.\end{aligned} \tag{17}$$

These differential equations have the general solution

$$\begin{aligned}\omega_1(t) &= \omega_1(0)\cos\lambda t + \omega_2(0)\sin\lambda t \\ \omega_2(t) &= \omega_2(0)\cos\lambda t - \omega_1(0)\sin\lambda t.\end{aligned} \tag{18}$$

Three constants of motion are of particular interest to us:

- *Transverse angular velocity.* Note that

$$\omega_t = \sqrt{\omega_1(t)^2 + \omega_2(t)^2}$$

is constant. We call this the transverse angular velocity.

- *Nutation angle.* The angle $\theta$ between the angular momentum $h$ and the axis of symmetry $z$ is constant. Denote the elements of $h$ by $(h_1, h_2, h_3)$. Let

$$h_t = \sqrt{h_1(t)^2 + h_2(t)^2} = J_t \omega_t$$

and

$$h_a = h_3 = J_a n.$$

Both of these things are constant. At any instant in time,

$$\begin{aligned} \sin\theta &= \frac{h_t}{\|h\|} \\ \cos\theta &= \frac{h_a}{\|h\|}. \end{aligned}$$

We call $\theta$ the nutation angle.

- *Wobble angle.* The angle $\gamma$ between the angular velocity $\omega$ and the axis of symmetry $z$ is also constant. Just as for the nutation angle, we have

$$\begin{aligned} \sin\gamma &= \frac{\omega_t}{\|\omega\|} \\ \cos\gamma &= \frac{n}{\|\omega\|}. \end{aligned}$$

We call $\gamma$ the wobble angle.

Finally, notice the following relationship between the nutation angle and the wobble angle:

$$\tan\theta = \left(\frac{J_t}{J_a}\right)\tan\gamma.$$

So, in fact, something fundamentally different happens depending on the relative magnitude of $J_t$ and $J_a$:

- If $J_t > J_a$, then $\tan\theta > \tan\gamma$, and so the nutation angle is bigger than the wobble angle. In the body frame, the angular momentum vector sweeps out a bigger cone than the angular velocity vector about $z$. In this case, the spacecraft must be long and skinny, a "soup can," a *prolate* body of revolution.
- If $J_t < J_a$, then the wobble angle is bigger than the nutation angle. That is, the angular velocity vector sweeps out the bigger cone. In this case, the spacecraft must be short and fat, a "tuna can," an *oblate* body of revolution.

Consider an axisymmetric satellite that is deployed from the Space Shuttle. Designed to rotate about the $z$ body axis, the actual angular velocity is $\omega = \begin{bmatrix} 0.05 \\ 0 \\ 1 \end{bmatrix}$. The principal moments of inertia of the satellite are $J_t = 50000\ \mathrm{kg}\cdot\mathrm{m}^2$ and $J_a = 4000\ \mathrm{kg}\cdot\mathrm{m}^2$.

(a) Is the satellite prolate or oblate?

(b) Find the transverse angular velocity.

(c) Find the spin rate and the relative spin rate.

(d) Find the nutation angle and the wobble angle.

(e) We have already seen that (18) is one way to write the solution to (17). It is also possible to write this solution as

$$\begin{aligned}\omega_1(t) &= a\sin(bt+c)\\ \omega_2(t) &= a\cos(bt+c),\end{aligned}$$

for appropriately chosen constants $a, b, c$. Find the values of these constants using any method that you wish (e.g., plug into (17) and express them in terms of initial conditions), and say how they are related to the constants of motion defined above.

4. (20 points) *Be able to integrate Euler's equations, written with respect to principle axes, for an axisymmetric spacecraft during a spin-up maneuver.* Consider an axisymmetric spacecraft with moments of inertia $J_t$ and $J_a$ as usual. A constant torque $\tau_3$ is applied about the $z$ axis. You may assume that $\omega_3(0) = 0$, but both $\omega_1(0)$ and $\omega_2(0)$ may be non-zero.

(a) Write the equations of motion.

(b) Integrate the third equation to find $\omega_3(t)$.

(c) Using your answer for $\omega_3(t)$, the parameter

$$\sigma = \frac{J_t - J_a}{J_t}\frac{\tau_3}{J_a},$$

and the scaled time variable

$$s = \frac{\sigma t^2}{2},$$

rewrite the remaining two equations of motion in terms of $d\omega_1/ds$, $d\omega_2/ds$, $\omega_1$, and $\omega_2$ only. You may find it useful to apply the chain rule:

$$\dot{f} = \frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt}.$$

(d) Solve these two equations to find $\omega_1(s)$ and $\omega_2(s)$.

(e) Express the result in terms of $\sigma$ and $t$ to find $\omega_1(t)$ and $\omega_2(t)$.

(f) Answer the following questions: (i) Does the wobble angle change during the spin-up maneuver? If so, does it increase or decrease? (ii) Does the nutation angle change during the spin-up maneuver? If so, does it increase or decrease?

(g) How long must the torque be applied in order to reach a spin rate of $n$?

5. (20 points) *Be able to design and to simulate a spin-up maneuver.* As a capstone to this homework assignment, you will design and simulate a spin-up maneuver for a spacecraft that is not *quite* axisymmetric. In particular, consider a satellite with principal moments of inertia

$$J_1 = 4600 \text{ kg} \cdot \text{m}^2, J_2 = 4400 \text{ kg} \cdot \text{m}^2, \text{ and } J_3 = 750 \text{ kg} \cdot \text{m}^2.$$

The goal is to achieve a spin rate of 60 rpm about the minor axis $z$. Spin thrusters are available that exert a constant torque of 100 N $\cdot$ m about $z$. The satellite begins with an angular velocity of $\omega = 0.0001x$.

(a) Assume the satellite is axisymmetric, so $J_1 = J_2 = 4500 \text{ kg} \cdot \text{m}^2$. Apply your result from Prob. 4 to find the length of time for which the spin thrusters should be fired.

(b) Recall that

$$\dot{R} = R\widehat{\omega},$$

where $R$ is the orientation of the body with respect to the base frame and $\omega$ is written in body-fixed coordinates. Assume that the body is initially aligned with the base frame. Simulate the spin-up maneuver, by simultaneously integrating both Euler's equations (to find the angular velocity) and the above equations (to find the orientation). Animate your results.

(c) Repeat part (b) but for larger values of $\omega_1(0)$, e.g., 1e-3, 1e-2, 1e-1, and 1e-0. How do your results change? Explain. In particular, interpret your results in terms of the relative motion of the body-fixed $z$ axis, the angular velocity vector, and the angular momentum vector.

To complete this problem, please start with the MATLAB script `hw3_prob5.m`, available on the course website. All you have to do is (1) define the initial conditions, the moment of inertia matrix, and the time required for spin-up; and (2) compute $\dot{R}$ and $\dot{\omega}$. In the display, the green side of the box is facing the positive $x$ axis, the blue side is facing the positive $y$ axis, and the red side is facing the positive $z$ axis. Also shown is the body-fixed $z$ axis (red), a unit vector in the direction of $\omega$ (green), and a unit vector in the direction of $h$ (blue). Note that the box will size itself according to the inertia matrix that you specify—what *should* it look like for the principal moments of inertia given above? (Like a soup can or a tuna can?)