AE403 Homework #1: Rotation Matrices

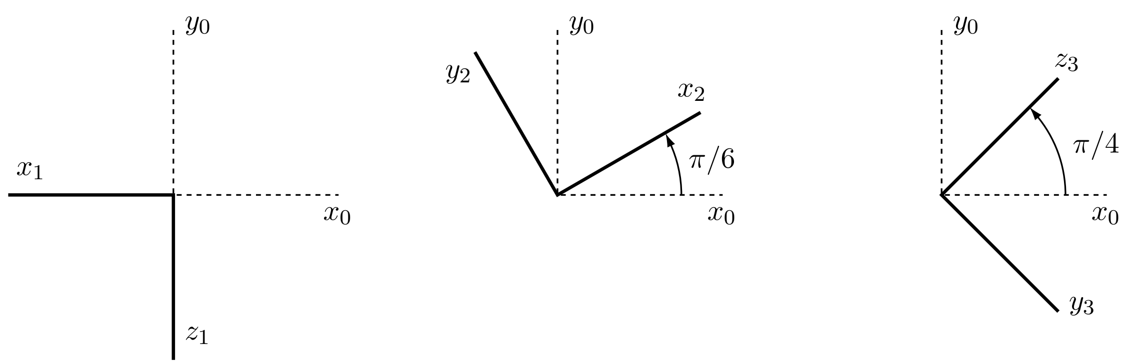
Koki Ho

(due at the beginning of class on Tuesday, Feb 9)

1. (15 points) *Be able to write a rotation matrix by inspection.* The orientation of frame k written in the coordinates of frame j is the rotation matrix

$$R_k^j = \begin{bmatrix} x_k^j & y_k^j & z_k^j \end{bmatrix}.$$

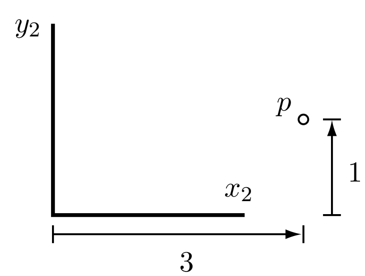
Express the orientation of each frame shown in the coordinates of the base frame 0. As always, assume that frames satisfy the right-hand rule.



2. (10 points) *Be able to perform coordinate transformations.* If a point p has coordinates p^k in frame k, then it also has coordinates

$$p^j = R_k^j p^k$$

in frame j. Consider the point p shown. Write the coordinates of p in frame 2, then use the rotation matrix you found in Problem 1 to write the coordinates of p in frame 0.



3. (10 points) *Be able to perform sequential rotations.* For any collection of frames i, j, and k, we have

$$R^i_k = R^i_j R^j_k.$$

Consider the same frames as in Problem 1 and the same point p as in Problem 2. Find the coordinates of p in frame 3 in the following two ways:

   (a) using the coordinates of p in frame 0 (Problem 2) and the rotation matrix relating frame 0 with frame 3 (Problem 1);

   (b) using the coordinates of p in frame 2 and the rotation matrix relating frame 2 with frame 3, computed using the above formula.

   Check that your answers are the same.

4. (15 points) *Be able to take dot products and cross products using matrix multiplication.* The dot product $v \cdot w$ between two vectors v and w can be found by writing the coordinates of both v and w with respect to the same coordinate frame, for example as v^0 and w^0, and then taking the matrix multiplication

$$v \cdot w = (v^0)^T w^0.$$

Similarly, the cross product $v \times w$ can be found by taking the matrix multiplication

$$v \times w = \widehat{(v^0)} w^0,$$

where $\widehat{a}$ (or $[a\times]$) is the "wedge" operation on $a \in \mathbb{R}^3$ defined by

$$\widehat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

In this expression, a_1, a_2, a_3 are the elements of a. It is easy to verify that these formulas give the same result as however you normally compute dot products and cross products. For the frames defined in Problem 1, compute the following:

   (a) the dot product $z_1 \cdot x_2$

   (b) the cross product $z_1 \times x_2$

   (c) the value of $z_1 \cdot (z_1 \times x_2)$ and of $x_2 \cdot (z_1 \times x_2)$.

   In each case, also give a geometric interpretation of the result, including a picture.

5. (10 points) *Be able to check if a matrix is a rotation matrix.* The space of all possible rotation matrices is the subset of $\mathbb{R}^{3\times 3}$ that is given by

$$SO(3) = \{\, R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I \text{ and } \det R = +1 \,\}.$$

So, the inverse of a rotation matrix is given by its transpose:

$$R^{-1} = R^T$$

for any $R \in SO(3)$. Notice in particular that

$$R^1_0 = \left(R^0_1\right)^T.$$

Another direct consequence is that $SO(3)$ is *closed* under matrix multiplication, so if $R_1 \in SO(3)$ and $R_2 \in SO(3)$, then

$$R_1 R_2 \in SO(3) \qquad \text{and} \qquad R_2 R_1 \in SO(3).$$

(Notice, however, that in general $R_1 R_2 \neq R_2 R_1$, i.e., rotations do not commute.) Verify by checking the constraints on $SO(3)$ that each of the matrices you found in Problem 1 is, in fact, a rotation matrix. Also give one example of a 3×3 matrix, any one you want, that is *not* a rotation matrix, and prove that it is not.

6. (15 points) *Be able to prove the number of the degrees of freedom of a rotation.* Using the property of a rotation matrix, find how many components of a rotation matrix are independent.

   (a) How many independent equations can we derive from the property $R^T R = I$?

   (b) How many independent equations can we derive from the property $RR^T = I$? Are they independent from the equations you found in part (a)?

   (c) Prove that only three of the nine components of a rotation matrix are independent. Note that these three components do not uniquely define the orientation of two reference frames, which will be covered later in the semester.

7. (10 points) *Be able to prove that rotations are rigid-body transformations.* A map $g \colon \mathbb{R}^3 \to \mathbb{R}^3$ is called a rigid-body transformation if it satisfies the following two properties:

   - length is preserved, so that $\|g(v)\| = \|v\|$ for all vectors $v \in \mathbb{R}^3$
   - orientation is preserved, so that $g(v \times w) = g(v) \times g(w)$ for all vectors $v, w \in \mathbb{R}^3$.

   Prove that matrix multiplication by any rotation matrix $R \in SO(3)$ is a rigid-body transformation, or in other words that the map $g(v) = Rv$ for $v \in \mathbb{R}^3$ is a rigid-body transformation. To show that length is preserved, use the fact that

$$\|v\|^2 = v^T v$$

   for any $v \in \mathbb{R}^3$. To show that orientation is preserved, use the fact that

$$\widehat{Rv} = R\hat{v}R^T \tag{1}$$

   for any $v \in \mathbb{R}^3$ and $R \in SO(3)$. This last formula is an example of a *similarity transformation*, which arises when we ask the following question: if two points p and q are related by a linear transformation, and if we can express this transformation in the coordinates of frame 1 as

$$p^1 = Aq^1$$

   for some $A \in \mathbb{R}^{3\times3}$, then for what matrix $B \in \mathbb{R}^{3\times3}$ do we also have

$$p^0 = Bq^0$$

   in the coordinates of frame 0? It is easy to answer this question. You know that

$$\begin{aligned} p^0 &= R_1^0 p^1 \\ q^0 &= R_1^0 q^1 \end{aligned}$$

as in Problem 2, so that

$$\left(R_1^0 p^1\right) = B\left(R_1^0 q^1\right).$$

Since $\left(R_1^0\right)^{-1} = \left(R_1^0\right)^T$ as in Problem 5, we have

$$p^1 = \left(R_1^0\right)^T B\left(R_1^0\right) q^1.$$

As an immediate consequence,

$$A = \left(R_1^0\right)^T B\left(R_1^0\right),$$

so the answer to our question is

$$B = \left(R_1^0\right) A \left(R_1^0\right)^T. \tag{2}$$

To prove Eq. (1), we consider the cross product

$$u = v \times w,$$

which (as we know from Problem 4) is a linear transformation taking w to u. In frame 1, this transformation is expressed as

$$u^1 = \widehat{(v^1)} w^1.$$

In frame 0, this same transformation is expressed as

$$u^0 = \widehat{(v^0)} w^0.$$

From Eq. (2), we know the relationship between $\widehat{(v^0)}$ and $\widehat{(v^1)}$ is

$$\widehat{(v^0)} = \left(R_1^0\right)\widehat{(v^1)}\left(R_1^0\right)^T.$$

Noting that

$$v^0 = R_1^0 v^1,$$

we may equivalently write

$$\widehat{\left(R_1^0 v^1\right)} = \left(R_1^0\right)\widehat{(v^1)}\left(R_1^0\right)^T.$$

This is exactly the formula given in Eq. (1), but here we have made the frames explicit. You should take the time to understand this derivation.

8. (15 points) *Be able to compute the time derivative of a rotation matrix.* For any $R \in SO(3)$, we have

$$\dot{R} = \widehat{w} R$$

for some $w \in \mathbb{R}^3$. To be more precise, we say

$$\dot{R}_1^0 = \widehat{\left(w_{0,1}^0\right)} R_1^0, \tag{3}$$

where R_1^0 is the rotation matrix expressing the orientation of the body-fixed frame 1 in the coordinates of the base frame 0, $\dot{R}_1^0$ is the time derivative of this rotation matrix, and $w_{0,1}^0$ is the angular velocity of frame 1 with respect to frame 0, written in the coordinates of frame 0. We often call $w_{0,1}^0$ the *spatial angular velocity* to indicate that it is written in the coordinates of the base frame. A similar formula may be expressed in terms of the *body angular velocity*, which we denote by $w_{0,1}^1$, written in the coordinates of frame 1. We emphasize that the spatial angular velocity and the body angular velocity are the *same thing*, they are both $w_{0,1}$, just written in different coordinates.

Starting from Eq. (3) and using some of your results from Problem 7, prove that

$$\dot{R}_1^0 = R_1^0 \widehat{\left(w_{0,1}^1\right)}. \tag{4}$$