These notes were written to outline the major topics covered in Auburn University's Calculus III course based on Stewart's 7th Edition Calculus text. It progresses through most of the sections in Chapters 12 through 16, but Chapter 15 is reorganized slightly to introduce the Jacobian before introducing alternate coordinate systems. In addition, sections 15.5 and 15.6 are omitted entirely to match Auburn's course syllabus.

The purpose of these notes is not to replace any calculus or analysis textbook, but rather to be used as a guide/outline for students and instructors covering the topics in a Calculus III course.

As such, when deemed necessary, mathematical rigor is abandoned for the sake of simplicity or brevity. (Many theorems actually only apply to "nice" functions, usually requiring some level of continuity or differentiability.) Since for many applications of interest the relevant functions are "nice", students should be able to use these notes as a "good-enough" resource for working on computational problems, particularly the accompanying study problems.

12.1 Three-Dimensional Coordinate Systems

- Distance between points in 3D space

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Simple planes in 3D Space

$$x = a,\ y = b,\ z = c$$

- Spheres in 3D Space

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

12.2 Vectors

- Definition of a Vector
  - A vector $\mathbf{v} = \overrightarrow{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
  - Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
  - We often assume the initial point lays at the origin.
- Component Form

  The vector with initial point at $(0, 0, 0)$ and terminal point at (v_x, v_y, v_z) is represented by

$$\langle v_x, v_y, v_z \rangle$$

- 2D and 3D Vectors

$$\langle a, b \rangle = \langle a, b, 0 \rangle$$

- Position Vector

  If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c \rangle$ is its **position vector**.

  We assume $(a, b, c) = \langle a, b, c \rangle$.
- Vector Between Points

  The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is

$$\mathbf{P_1 P_2} = \overrightarrow{P_1 P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \overrightarrow{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations
  - Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

  - Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties
  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  5. $0\mathbf{u} = \mathbf{0}$
  6. $1\mathbf{u} = \mathbf{u}$
  7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
  8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- Unit Vectors
  - A **unit vector** or **direction** is any vector whose length is 1.
  - Standard unit vectors
    * $\mathbf{i} = \langle 1, 0, 0 \rangle$
    * $\mathbf{j} = \langle 0, 1, 0 \rangle$
    * $\mathbf{k} = \langle 0, 0, 1 \rangle$

- Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

- Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$$

12.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$$

- Angle between vectors

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

- Orthogonal Vectors
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if the angle between them is $\frac{\pi}{2} = 90°$
  - $\mathbf{0}$ is orthogonal to every vector

- Dot Product Properties
  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
  5. $\mathbf{0} \cdot \mathbf{u} = 0$

- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta$$

12.4 The Cross Product

- Determinants
  - 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - 3x3 Determinant

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3)
\end{aligned}$$

- Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \langle u_2v_3 - u_3v_2 \, , \, u_3v_1 - u_1v_3 \, , \, u_1v_2 - u_2v_1 \rangle$$

Shortcut "long multiplication" method:

$$\begin{array}{rcccccl}
\langle & u_1 & , & u_2 & , & u_3 & \rangle \\
\times\langle & v_1 & , & v_2 & , & v_3 & \rangle \\
\hline
\langle & u_2v_3 - u_3v_2 & , & u_3v_1 - u_1v_3 & , & u_1v_2 - u_2v_1 & \rangle
\end{array}$$

- Right-Hand Rule
  - A method for determining a special orthogonal direction used throughout mathematics and physics in 3D space, with respect to an ordered pair of vectors $\mathbf{u}, \mathbf{v}$
  - $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$, $\mathbf{v}$ according to the Right-Hand Rule.

- Cross Product Magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

  The area of the parallelogram determined by $\mathbf{u}, \mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$.

- Parallel Vectors
  - $\mathbf{u}, \mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are parallel if the angle between them is $0 = 0°$ or $\pi = 180°$
  - $\mathbf{0}$ is parallel to every vector

- Cross Product Properties
  1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
  4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
  5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
  6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products
  1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
  2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
  3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

  The standard unit vectors are known as a "right handed frame".

- Torque

$$\overrightarrow{\tau} = \mathbf{r} \times \mathbf{F}$$
$$|\overrightarrow{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

- Triple Scalar (or "Box") Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

  Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelpiped determined by the three vectors.

12.5 Equations of Lines and Planes

- Vector Equation and Parametric Equations for a Line

$$\mathbf{r}(t) = \mathbf{P_0} + t\mathbf{v}$$

$$x = x_0 + At, y = y_0 + Bt, z = z_0 + Ct$$

for $-\infty < t < \infty$

- Symmetric Equations for a Line

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_1} - \mathbf{P_0}) = (1 - t)\mathbf{P_0} + t\mathbf{P_1}$$

for $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = D$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{n_1} \times \mathbf{n_2})$$

- Angle of Intersection of Two Planes

$$\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

12.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
  - To sketch a 3D surface, sketch planar cross-sections
    * $z = c$ is parallel to xy plane
    * $y = b$ is parallel to xz plane
    * $x = a$ is parallel to yz plane
- Cylinders
  - A **cylinder** is any surface generated by considering parallel lines passing through a planar curve.
  - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
  - A **quadric surface** is any surface defined by a second degree equation of x, y, z.
  - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
  - Cross-sections in the coordinate planes include
    * Three ellipses
- Elliptical Cone
  - Cross-sections in the coordinate planes include
    * Two double-lines
    * One point (with parallel ellipses)
- Elliptical Paraboloid
  - Cross-sections in the coordinate planes include
    * Two parabolas
    * One point (with parallel ellipses)

- Hyperbolic Paraboloid
  - Cross-sections in the coordinate planes include
    - Two parabolas (with parallel parabolas)
    - One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
  - Cross-sections in the coordinate planes include
    - Two hyperbolas
    - One ellipsis (with parallel ellipses)
- Hyperboloid of Two Sheets
  - Cross-sections in the coordinate planes include
    - Two hyperbola
    - One empty cross-section (with parallel ellipses)

13.1 Vector Functions and Space Curves

- Curves, Paths, and Vector Functions
  - A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

    $$x = x(t), y = y(t), z = z(t)$$

    or with a **vector function**

    $$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

  - $x(t), y(t), z(t)$ are called **component functions**
- Vector Function Limits

  $$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

- Continuity of Vector Functions
  - The function $\mathbf{r}(t)$ is **continuous** if

    $$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

    for all a in its domain.
  - $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

13.2 Derivatives and Integrals of Vector Functions

- Derivatives of Vector Functions

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \langle f'(t), g'(t), h'(t) \rangle$$

  - $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of t is in its domain.
  - $\mathbf{r}'(a)$ is a **tangent vector** to the curve where $t = a$
  - The **tangent line** to a curve at $t = a$:

$$\mathbf{l}(t) = \mathbf{r}(a) + t\mathbf{r}'(a)$$

- Differentiation Rules for Vector Functions

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df}\frac{df}{dt}$$

- Derivative of a Constant Length Vector Function
  - If $|\mathbf{r}(t)| = c$ always, then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

  - Thus the derivative of a constant length vector function is perpindicular to the original.

- Antiderivatives of Vector Functions
  - If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.
  - The **indefinite integral** $\displaystyle\int \mathbf{r}(t)\,dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

$$\int_a^b \mathbf{r}(t)dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Differential Vector Equations
  - If we know $\mathbf{r}'(t)$ and $\mathbf{r}(a)$ for some $t = a$, then

$$\mathbf{r}(t) = \int_a^t \mathbf{r}'(t)\,dt + \mathbf{r}(a)$$

13.3 Arc Length and Curvature

- Arc Length along a Space Curve

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{r}'(t)|\, dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{r}'(u)| du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

- Unit Tangent Vector

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$$

$$\mathbf{T}(t) = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

- Curvature

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{|d\mathbf{r}/dt|} = \frac{|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}|}{|\frac{d\mathbf{r}}{dt}|^3}$$

For $y = f(x)$:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

- Principal Unit Normal Vector

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

The triple $\mathbf{T}, \mathbf{N}, \mathbf{B}$ forms a right-handed frame.

13.4 Motion in Space: Velocity and Acceleration

- Position, Velocity, and Acceleration
  - Position: $\mathbf{r}(t)$
  - Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
  - Speed: $v(t) = |\mathbf{v}(t)| = \frac{ds}{dt}$
  - Direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
  - Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$
- Ideal Projectile Motion

$$\mathbf{a}(t) = \langle 0, -g \rangle$$

$$\mathbf{v}(t) = \langle v_0 \cos\alpha, -gt + v_0 \sin\alpha \rangle$$

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \right\rangle$$

- Tangental and Normal Components of Acceleration

$$\mathbf{a} = \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} + 0\mathbf{B}$$

  - Tangental component

$$a_T = \frac{d^2s}{dt^2} = v'$$

  - Normal component

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa v^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

14.1 Functions of Several Variables

- Functions of Two Variables
  - A **function** f **of two variables** is a rule which assigns a real number $f(x,y)$ to each pair of real numbers (x,y) in its **domain**
    $$\mathrm{dom}(f) \subseteq \mathbb{R}^2$$
    The set of values f takes on is its **range**
    $$\mathrm{ran}(f) = \{f(x,y) : (x,y) \in \mathrm{dom}(f)\}$$
  - The **level curve** for each $k \in \mathrm{ran}(f)$ is given by the equation
    $$f(x,y) = k$$
  - The **graph** of f is a surface in 3D space which visualizes the function, given by the equation $z = f(x,y)$.
- Functions of Three Variables
  - A **function** f **of three variables** is a rule which assigns a real number $f(x,y,z)$ to each pair of real numbers (x,y,z) in its **domain**
    $$\mathrm{dom}(f) \subseteq \mathbb{R}^3$$
    The set of values f takes on is its **range**
    $$\mathrm{ran}(f) = \{f(x,y,z) : (x,y,z) \in \mathrm{dom}(f)\}$$
  - The **level surface** for each $k \in \mathrm{ran}(f)$ is given by the equation
    $$f(x,y,z) = k$$
- Alternate Forms
  - We may also consider functions of the form $f(x_1, x_2, \dots) = f(P) = f(\mathbf{r})$.
  - If $P = (x,y)$ and $\mathbf{r} = \langle x, y\rangle$, then $f(x,y) = f(P) = f(\mathbf{r})$.
  - If $P = (x,y,z)$ and $\mathbf{r} = \langle x, y, z\rangle$, then $f(x,y,z) = f(P) = f(\mathbf{r})$.

14.2 Limits and Continuity

- Limits
  - If the value of the function $f(P)$ becomes arbitrarily close to the number L as vectors P close to P_0 are plugged into the function, then the **limit of $f(P)$ as P approaches P_0** is L:

$$\lim_{P \to P_0} f(P) = L$$

  - For functions of two or three variables:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

- Showing a Limit DNE
  - In order for a limit $\lim_{P\to P_0} f(x,y)$ to exist, the values of f must approach L no matter which direction we approach P_0.
  - Choose $y = g(x)$ and $y = h(x)$ where P_0 lays on both graphs. If

$$\lim_{x\to x_0} f(x, g(x)) \neq \lim_{x\to x_0} f(x, h(x))$$

    then $\lim\limits_{P\to P_0} f(x,y)$ DNE.
  - Or choose $x = g(y)$ and $x = h(y)$ where P_0 lays on both graphs. If

$$\lim_{y\to y_0} f(g(y), y) \neq \lim_{y\to y_0} f(h(y), y)$$

    then $\lim\limits_{P\to P_0} f(x,y)$ DNE.

- Limit Laws

$$\lim_{P\to P_0}(f(P) \pm g(P)) = \lim_{P\to P_0} f(P) \pm \lim_{P\to P_0} g(P)$$

$$\lim_{P\to P_0}(f(P) \cdot g(P)) = \lim_{P\to P_0} f(P) \cdot \lim_{P\to P_0} g(P)$$

$$\lim_{P\to P_0}(kf(P)) = k \lim_{P\to P_0} f(P)$$

$$\lim_{P \to P_0} \frac{f(P)}{g(P)} = \frac{\lim_{P \to P_0} f(P)}{\lim_{P \to P_0} g(P)}$$

$$\lim_{P \to P_0} (f(P))^{r/s} = \left(\lim_{P \to P_0} f(P) \right)^{r/s}$$

- Computing Limits
  - Variables not involved in a limit may be eliminated:

    $$\lim_{P \to P_0} f(x) = \lim_{x \to x_0} f(x)$$

  - Due to the Limit Laws, many limits follow the "just plug it in" rule.
  - If plugging in results in a zero in a denominator, use factoring, perhaps with conjugates.
  - L'Hopital's Rule does not apply for multiple variable limits.
- Continuity
  - A function $f(P)$ is **continuous** if $\lim_{P \to P_0} f(P) = f(P_0)$ for all points P_0 in its domain.
  - If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

14.3 Partial Derivatives

- Partial Derivatives
  - For a function f of two variables (x, y):

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

  - To compute partial derivatives with respect to a variable, treat all other variables as constants and differentiate as normal.
  - Functions of more than two variables behave similarly. For $T(x, y, z)$:

$$\frac{\partial T}{\partial z} = T_z(x,y,z) = \lim_{h\to 0} \frac{T(x,y,z+h)-T(x,y,z)}{h}$$

- Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 g}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial g}{\partial z}\right] = (g_z)_z = g_{zz}$$

- Mixed Derivative Theorem
  - For many naturally occuring functions:

$$f_{xy} = f_{yx}$$

14.4 Tangent Planes and Linear Approximations

- Tangent Plane to $z = f(x, y)$ at $(a, b, f(a, b))$

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Linearization of $f(x, y)$ at (a, b)

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Differentiability and a Sufficient Condition
  - A multi-variable function f is **differentiable** at a point if its linearizaration approximates the value of the function near that point.
  - If f_x, f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

- Linear Approximation

  If f is differentiable at (a, b), then

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

14.5 The Chain Rule

- Gradient Vector Function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Nested Functions
  - If f is a function of $\mathbf{r} = \langle x,y,z\rangle$, and x,y,z are functions of t, then we say x,y,z are itermediate variables and may consider the following composed function of t:

$$f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

  - If f is a function of $\mathbf{r} = \langle x,y,z\rangle$, and x,y,z are functions of $\mathbf{s} = \langle t,u,v\rangle$, then we say x,y,z are itermediate variables and may consider the following composed function of t,u,v:

$$f(\mathbf{r}(\mathbf{s})) = f(x(t,u,v), y(t,u,v), z(t,u,v))$$

- Chain Rule
  - For functions of the form $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

  - For functions of the form $f(\mathbf{r}(\mathbf{s})) = f(x(t,u,v), y(t,u,v), z(t,u,v))$:

$$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

- Differentiation by Substitution
  - The multi-variable Chain Rule can be avoided by "plugging in" functions and using single-variable calculus.

- Total Derivative
  - If f is a function of x, y, z, and y, z are also functions of x, then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

- Implicit Differentiation
  - If $f(x, y) = c$ defines y as a function of x, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}$$

- Tree Diagram for the Chain Rule
  - The tree diagram for the chain rule can be used to generate the chain rule.
  - It also holds for multiple levels of intermediate variables.

14.6 Directional Derivatives and the Gradient Vector

- Directional Derivative
  - The **directional derivative** of f for the unit vector $\mathbf{u}$ is

    $$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

  - The maximum value of $D_{\mathbf{u}}f$ at a fixed point P_0 is $|\nabla f(P_0)|$, which occurs when $\mathbf{u} = \frac{\nabla f(P_0)}{|\nabla f(P_0)|}$.
- Normal Vector to Level Curves and Surfaces
  - The gradient vectors ∇f are normal vectors to the level curves $f(x, y) = k$ for every (x, y) in the domain of f.
  - The gradient vectors ∇f are normal vectors to the level surfaces $f(x, y, z) = k$ for every (x, y, z) in the domain of f.

14.7 Maximum and Minimum Values

- Local Maximum and Minimum Values
  - Let f be a function of many variables defined near the point P_0.
    - f has a **local maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value of f near P_0
    - f has a **local minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value of f near P_0
- Critical Points
  - If P_0 is a point in the domain of f and
    $$\nabla f(P_0) = 0 \text{ or } \nabla f(P_0) \text{ DNE}$$
    then P_0 is called a **critical point**.
  - Critical points occur when the tangent plane is horizontal or DNE.
  - The local maximum and minimum values of a function always occur at critical points.
- Saddle Points
  - Not every critical point gives a local extreme value.
  - The **saddle points** of f are the critical points which don't yield local extreme values.
- Discriminant Function
  - The **discriminant** of f with variables x, y is the function
    $$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$
- Second Derivative Test for Local Extreme Values of $f(x, y)$

  Let (a, b) be a critical point of of f where ∇f is defined.
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  - If $f_D(a, b) < 0$, then f has a saddle point at (a, b).
  - If $f_D(a, b) = 0$, then the test is inconclusive.

- Absolute Maximum and Minimum Values
  - Let f be a function of many variables.
    - f has an **absolute maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value in the range of f
    - f has an **absolute minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value in the range of f
  - Every continuous function of many variables with a closed and bounded domain has an absolute maximum and minimum value.
- Finding Absolute Max/Min of $f(x, y)$ on a Closed and Bounded Region D
  - The following points are candidates for giving the absolute extrema:
    - Critical points of f within D.
    - Critical points for a function which gives part of the boundary of D.
    - Corners of D.
  - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.

14.8 Lagrange Multipliers

- The Method of Lagrange Multipliers
  - The **Method of Lagrange Multipliers** says that if f is a function of many variables which has an absolute max/min value on the restriction $g(P) = k$ where $\nabla g \neq 0$, then the absolute max/min occurs at a point P where
    $$\nabla f(P) = \lambda \nabla g(P) \text{ and } g(P) = k$$
    for some real number λ.
  - If two constraints $g(P) = k$ and $h(P) = l$ are given, then the absolute max/min occurs where
    $$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P) \text{ and } g(P) = k \text{ and } h(P) = l$$
    for some real numbers λ, μ.

15.1 Double Integrals over Rectangles

- Double Integral
  - We define the **double integral** of a function $f(x, y)$ over a region R to be

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

    where for each positive integer n we've defined a way to partition R into n pieces

$$\Delta R_{n,1}, \Delta R_{n,2}, \ldots, \Delta R_{n,n}$$

    where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta A_{n,i}) = 0$$

  - Since for $f(x, y) \geq 0$,

$$\sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

    is an approximation of the volume under $z = f(x, y)$ and over R, the double integral is used to define the precise volume.
  - If f is not always positive, then the double integral represents **net volume**: volume above the xy-plane minus volume below the xy-plane.

- Midpoint Rule for Approximating Rectangular Double Integrals
  - For the rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

    we may approximate the double integral by partitioning the rectangle into a grid of $m \times n$ rectangular pieces all with area ΔA and evaluating:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

    where $(\overline{x_i}, \overline{y_j})$ is the midpoint of the $i \times j$ rectangle.

15.2 Iterated Integrals

- Volume as Integral of Area
  - If $A(x)$ is the area of a solid's cross-section, then the solid's volume is

$$V = \int_a^b A(x)\,dx$$

- Iterated Integrals over Rectangles
  - A double integral over a rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

    can be expressed as the **iterated integrals**:

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

15.3 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions
  - For **Type I** regions which may be expressed as

    $$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

    a double integral over R may be expressed as the iterated integral:

    $$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  - For **Type II** regions which may be expressed as

    $$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

    a double integral over R may be expressed as the iterated integral:

    $$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Finding Limits of Integration

  1. Sketch the region and label bounding curves
  2. Determine if the region is Type I or Type II by identifying (I) bottom/top curves $y = g_1(x), y = g_2(x)$ or (II) left/right curves $x = h_1(y), x = h_2(y)$.

  For Type I:

  3. Use the leftmost and rightmost x-values in the region a, b to complete the iterated integral:

     $$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  For Type II:

  3. Use the bottommost and topmost y-values in the region c, d to complete the iterated integral:

     $$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Swapping Variables of Integration
  - You can only swap the order of integration of an iterated integral by drawing the region and reinterpreting it as a region of the opposite Type.
- Additivity

  If R can be split into two regions R_1, R_2, then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

- Average Value of Two-Variable Functions
  - The average value of a two-variable function f over a region R is defined to be

$$\frac{1}{\text{Area of } R}\iint\limits_R f(x,y)\,dA$$

- Area as a Double Integral
  - The area of a region R in the plane is

$$A = \iint\limits_R dA = \iint\limits_R 1\,dA$$

15.7 Triple Integrals

- Triple Integral
  - We define the **triple integral** of a function $f(x, y, z)$ over a solid D to be

$$\iiint_D f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i}, z_{n,i})\Delta V_{n,i}$$

where for each positive integer n we've defined a way to partition D into n pieces

$$\Delta D_{n,1}, \Delta D_{n,2}, \dots, \Delta D_{n,n}$$

where $\Delta D_{n,i}$ has volume $\Delta V_{n,i}$, contains the point $(x_{n,i}, y_{n,i}, z_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta V_{n,i}) = 0$$

- Iterated Integral for Rectangular Boxes
  - The triple integral over the rectangular box

$$D : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2, c_1 \leq z \leq c_2$$

can be expressed as the **iterated integrals**:

$$\iiint_D f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dy\, dx$$

$$= \int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dz\, dy = \int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dz\, dx = \dots$$

- Iterated Integral for Generated Solids
  - If the solid D is determined by the bottom/top surfaces

$$h_1(x, y) \leq z \leq h_2(x, y)$$

and has shadow R in the xy-plane, then a triple integral over D can be expressed as:

$$\iiint_D f(x, y, z)\, dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz \right] dA$$

- In general:

$$\iiint\limits_D f(x,y,z)\,dV = \iint\limits_R \left[\int_{\text{bottom surface}}^{\text{top surface}} f(x,y,z)\,d\square\right] dA$$

  where $\square$ is chosen from x, y, z to be the "up" orientation.

- Additivity

  If D can be split into two regions D_1, D_2, then

$$\iint\limits_D f(x,y,z)\,dV = \iint\limits_{D_1} f(x,y,z)\,dV + \iint\limits_{D_2} f(x,y,z)\,dV$$

- Average Value of Three-Variable Functions

  - The average value of a three-variable function f over a solid D is defined to be

$$\frac{1}{\text{Volume of } D}\iiint\limits_R f(x,y,z)\,dV$$

- Volume as a Triple Integral

  - The volume of a solid D in space is

$$V = \iiint\limits_D dV = \iiint\limits_D 1\,dV$$

15.10 Change of Variables in Multiple Integrals

- Transformations
  - Two similar regions in 2D space can be transformed by a "nice" pair of functions

    $$\mathbf{r}(u, v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s})\rangle = \langle x(u, v), y(u, v)\rangle$$

    that map points in a uv plane to the xy plane.
  - Two similar solids in 3D space can be transformed by a "nice" triple of functions

    $$\mathbf{r}(u, v, w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s})\rangle = \langle x(u, v, w), y(u, v, w), z(u, v, w)\rangle$$

    that map points in a uvw space to the xyz space.
- The Jacobian
  - The Jacobian of a 2D transformation given by $\mathbf{r}(u, v)$ is the determinant

    $$\mathbf{r}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

  - The Jacobian of a 3D transformation given by $\mathbf{r}(u, v, w)$ is the determinant

    $$\mathbf{r}_J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- 2D Substitution
  - Suppose that the region R in the xy-plane is the result of applying the transformation $\mathbf{r}(u, v)$ to the region G in the uv-plane.
  - Then it follows that

    $$\iint_R f(x, y)\, dx\, dy = \iint_G f(x(u, v), y(u, v))|\mathbf{r}_J(u, v)|\, du\, dv$$

- Unit Square and Triangle
  - The unit square in the uv plane with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ is useful for substitution problems involving parallelograms.
  - The unit triangle in the uv plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$ is useful for substitution problems involving triangles.
- 3D Substitution
  - Suppose that the solid D in xyz space is the result of applying the transformation $\mathbf{r}(u,v,w)$ to the region H in uvw space.
  - Then it follows that

$$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint\limits_H f(x(u,v,w),y(u,v,w),z(u,v,w))|\mathbf{r}_J(u,v,w)|\,du\,dv\,dw$$

15.4 Double Integrals in Polar Coordinates

- Integrating over Regions expressed using Polar Coordinates
  - The polar coordinate transformation

    $$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$

    from polar G into Cartesian R yields

    $$\iint_R f(x,y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

15.8 Triple Integrals in Cylindrical Coordinates

- Cylindrical Coordinates
  - The cylindrical coordinate transformation

    $$\mathbf{r}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

    from cylindrical H into Cartesian D yields

    $$\iiint_D f(x,y,z)\,dV = \iiint_H f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

15.9 Triple Integrals in Spherical Coordinates

- Spherical Coordinates
  - The spherical coordinate transformation

    $$\mathbf{r}(\rho,\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

    from spherical H into Cartesian D yields

    $$\iiint_D f(x,y,z)\,dV = \iiint_H f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

16.1 Vector Fields

- Vector Fields
  - A **vector field** assigns a vector to each point in 2D or 3D space.

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle P(\mathbf{r}), Q(\mathbf{r}) \rangle = \langle P, Q \rangle$$

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle = \langle P(\mathbf{r}), Q(\mathbf{r}), R(\mathbf{r}) \rangle = \langle P, Q, R \rangle$$

- Gradient Vector Field
  - The gradient vector field $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ assigns vectors whose directions are normal to level curves and whose magnitudes are equal to the maximal directional derivative at the point.
  - The gradient vector field $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ assigns vectors whose directions are normal to level surfaces and whose magnitudes are equal to the maximal directional derivative at the point.

16.2 Line Integrals

- Common Curve Parametrizations
  - A line segment beginning at P_0 and ending at P_1

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_1} - \mathbf{P_0}), 0 \leq t \leq 1$$

  - A circle centered at the origin with radius a

$$\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle, 0 \leq t \leq 2\pi \text{ (counter-clockwise)}$$

$$\mathbf{r}(t) = \langle a\sin t, a\cos t\rangle, 0 \leq t \leq 2\pi \text{ (clockwise)}$$

  - A planar curve given by $y = f(x)$ from (x_0, y_0) to (x_1, y_1)

$$\mathbf{r}(t) = \langle t, f(t)\rangle, x_0 \leq t \leq x_1 \text{ (for } x_0 \leq x_1)$$

$$\mathbf{r}(t) = \langle -t, f(-t)\rangle, -x_0 \leq t \leq -x_1 \text{ (for } x_0 \leq x_1)$$

- Line Integrals with Respect to Arclength
  - We define the **line integral with respect to arclength** of a function of many variables $f(\mathbf{r})$ along a curve C to be

$$\int_C f(\mathbf{r})\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\mathbf{r}_{n,i})\Delta s_{n,i}$$

    where for each positive integer n we've defined a way to partition C into n pieces

$$\Delta C_{n,1}, \Delta C_{n,2}, \dots, \Delta C_{n,n}$$

    where $\Delta C_{n,i}$ has length $\Delta s_{n,i}$, contains the position vector $\mathbf{r}_{n,i}$, and

$$\lim_{n\to\infty} \max(\Delta s_{n,i}) = 0$$

  - If $\mathbf{r}(t)$ is a parametrization of C for $a \leq t \leq b$, then

$$\int_C f(\mathbf{r})\,ds = \int_{t=a}^{t=b} f(\mathbf{r}(t))\frac{ds}{dt}\,dt$$

- Line Integrals with Respect to Variables
  - Similarly, we can find the **line integral with respect to a variable** for a function of many variables $f(\mathbf{r})$ along a curve C:

$$\int\limits_C f(\mathbf{r})\,dx = \int_{t=a}^{t=b} f(\mathbf{r}(t))\frac{dx}{dt}\,dt$$

  - Similar defintions hold for y, z.

- Line Integrals of Vector Fields
  - The **line integral of a vector field** is defined to be the line integral with respect to arclength of the dot product of the vector field $\mathbf{F}(\mathbf{r}) = \langle P(\mathbf{r}), Q(\mathbf{r}), R(\mathbf{r})\rangle$ with the unit tangent vector $\mathbf{T}(\mathbf{r})$ to the curve.

$$\int_C \mathbf{F}(x,y,z)\cdot\mathbf{T}(x,y,z)\,ds$$

  - There are several ways to write and evaluate line integrals of vector fields:

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_C \langle P,Q,R\rangle\cdot\langle\,dx,\,dy,\,dz\rangle$$

$$= \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} + R(\mathbf{r}(t))\frac{dz}{dt}\right)\,dt$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$$

- Additivity

  Let $C_1 + C_2$ represent the curve taken by moving along C_1 followed by moving along C_2.

$$\int\limits_{C_1+C_2} f(\mathbf{r})\,ds = \int\limits_{C_1} f(\mathbf{r})\,ds + \int\limits_{C_2} f(\mathbf{r})\,ds$$

$$\int\limits_{C_1+C_2} f(\mathbf{r})\,dx = \int\limits_{C_1} f(\mathbf{r})\,dx + \int\limits_{C_2} f(\mathbf{r})\,dx$$

$$\int\limits_{C_1+C_2} \mathbf{F}\cdot\,d\mathbf{r} = \int\limits_{C_1} \mathbf{F}\cdot\,d\mathbf{r} + \int\limits_{C_2} \mathbf{F}\cdot\,d\mathbf{r}$$

- Effects of Curve Orientation

  Let $-C$ represent the curve taken by moving along C in the opposite direction.

$$\int\limits_C f(\mathbf{r})\,ds = +\int\limits_{-C} f(\mathbf{r})\,ds$$

$$\int\limits_C f(\mathbf{r})\,dx = -\int\limits_{-C} f(\mathbf{r})\,dx$$

$$\int\limits_C \mathbf{F}\cdot\,d\mathbf{r} = -\int\limits_{-C} \mathbf{F}\cdot\,d\mathbf{r}$$

- Work
  - If $\mathbf{F}$ is a vector field representing the force applied to an object as it is moved over a smooth curve C, then the **work** done by the force over that curve is given by

$$\int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$$

16.3 The Fundamental Theorem for Line Integrals

- The Fundamental Theorem
  - If C is any smooth curve beginning at the point A and ending at the point B, then
$$\int_C \nabla f \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$$
  - If C is any smooth curve which is **closed** (begins and ends at the same point), then
$$\int_C \nabla f \cdot d\mathbf{r} = 0$$
- Conservative Fields
  - We say $\mathbf{F} = \langle M, N, P \rangle$ is a conservative field if there is a **potential function** f such that $\nabla f = \mathbf{F}$.
  - Line integrals of conservative fields are said to be path independent since for any curve C beginning at A and ending at B:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$$
  - We can prove a field is conservative by finding its potential function or showing it satisfies the **Component Test**:
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

16.4 Green's Theorem

- Green's Theorem
  - Let C be the boundary of the region R oriented counter-clockwise, and $\mathbf{F}(x, y)$ be a two-dimensional vector field.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  - Due to Green's Theorem, we can find the area of R using a line integral:

$$A = \iint\limits_R 1 - 0 \, dA = \int_C x \, dy$$

$$A = \iint\limits_R 0 - (-1) \, dA = \int_C -y \, dx$$

$$A = \iint\limits_R \frac{1}{2} - \left(-\frac{1}{2} \right) dA = \int_C \frac{1}{2} x \, dy - \frac{1}{2} y \, dx$$

16.5 Curl and Divergence

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Curl
  - The **curl** of a vector field is another vector field:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

  - By the Component Test, if $\mathbf{F}$ is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.
- Divergence
  - The **divergence** of a vector field is the scalar function:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

  - For any vector field, the divergence of curl is always zero.

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

- Green's Theorem Alternate Forms
  - If $\mathbf{F}$ is a two-dimensional vector field, and $\mathbf{n}$ is the outward unit normal vector field for a counter-clockwise closed curve C:

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F} \, dA$$

16.6 Parametric Surfaces and Their Areas

- Parametric Surface Equations

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

- Common Parametric Surfaces
  - The plane determined by the point P_0 and vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ can be parametrized by
  $$\mathbf{r} = \mathbf{P_0} + u\mathbf{v_1} + v\mathbf{v_2}$$
  - The surface $z = f(x,y)$ can be parametrized by
  $$\mathbf{r} = \langle x, y, f(x,y)\rangle$$
  - A surface determined by a cylindrical coordinate equation can be parametrized by substituting into
  $$\mathbf{r} = \langle r\cos\theta, r\sin\theta, z\rangle$$
  - A surface determined by a spherical coordinate equation can be parametrized by substituting into
  $$\mathbf{r} = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

- Surface Area
  - If G is the region in the uv plane which maps onto the surface S by the parametric equations $\mathbf{r}(u,v)$, then the surface area of S is:
  $$\iint_G |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$
  where $\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$.

16.7 Surface Integrals

- Surface Integral
  - If G is the region in the uv plane which maps onto the surface S by the parametric equations $\mathbf{r}(u,v)$, then the surface integral of $f(\mathbf{r})$ along S is:

$$\iint_S f(\mathbf{r})\,d\sigma = \iint_G f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

- Surface Orientation
  - The orientation of a surface is determined by a continuous unit normal vector field $\mathbf{n}$ on the surface.
  - The Möbius strip is an example of a non-orientable surface.
- Surface Integral of Vector Field
  - If G is the region in the uv plane which maps onto the surface S by the parametric equations $\mathbf{r}(u,v)$, and $\mathbf{n}$ is the unit normal vector field giving the orientation of S, then the surface integral of the vector field $\mathbf{F}$ along S is:

$$\iint_S \mathbf{F}\cdot d\vec{\sigma} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_G \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

16.8 Stokes' Theorem

- Stokes' Theorem
  - Let C give the counter-clockwise oriented boundary of a surface S.

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\, d\vec{\sigma} = \int_C \mathbf{F}\cdot\, d\mathbf{r}$$

16.9 Divergence Theorem

- Divergence Theorem
  - Let S give the outward-oriented boundary surface of the solid D.

$$\iint_S \mathbf{F}\cdot\, d\vec{\sigma} = \iiint_D \operatorname{div}\mathbf{F}\, dV$$