## Sections 12.1 - 12.2 Overview

- Three-Dimensional Coordinates
  - Distance between points in 3D space
    $$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
  - Simple planes in 3D Space
    $$x = a,\ y = b,\ z = c$$
  - Spheres in 3D Space
    $$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$
- Vectors
  - Definition of a Vector
    * A vector $\mathbf{v} = \overrightarrow{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
    * Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
    * We often assume the initial point lays at the origin.
  - Component Form
    The vector with initial point at $(0, 0, 0)$ and terminal point at $(v_x, v_y, v_z)$ is represented by
    $$\langle v_x, v_y, v_z \rangle$$
  - 2D and 3D Vectors
    $$\langle a, b \rangle = \langle a, b, 0 \rangle$$
  - Position Vector
    If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c \rangle$ is its **position vector**.
    We assume $(a, b, c) = \langle a, b, c \rangle$.
  - Vector Between Points
    The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is
    $$\mathbf{P_1P_2} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \overrightarrow{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations
  * Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

  * Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties
  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  5. $0\mathbf{u} = \mathbf{0}$
  6. $1\mathbf{u} = \mathbf{u}$
  7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
  8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- Unit Vectors
  * A **unit vector** or **direction** is any vector whose length is 1.
  * Standard unit vectors
    · $\mathbf{i} = \langle 1, 0, 0 \rangle$
    · $\mathbf{j} = \langle 0, 1, 0 \rangle$
    · $\mathbf{k} = \langle 0, 0, 1 \rangle$
  * Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

  * Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

## 12.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- Angle between vectors

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

- Orthogonal Vectors
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if the angle between them is $\frac{\pi}{2} = 90°$
  - $\mathbf{0}$ is orthogonal to every vector
- Dot Product Properties
  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
  5. $\mathbf{0} \cdot \mathbf{u} = 0$
- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta$$

## 12.4 The Cross Product

- Determinants
  - 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - 3x3 Determinant

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3)
\end{aligned}$$

- Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \langle u_2v_3 - u_3v_2\, ,\, u_3v_1 - u_1v_3\, ,\, u_1v_2 - u_2v_1 \rangle$$

  Shortcut "long multiplication" method:

$$\begin{array}{rcccccl}
\langle & u_1 & , & u_2 & , & u_3 & \rangle \\
\times\langle & v_1 & , & v_2 & , & v_3 & \rangle \\
\hline
\langle & u_2v_3 - u_3v_2 & , & u_3v_1 - u_1v_3 & , & u_1v_2 - u_2v_1 & \rangle
\end{array}$$

- Right-Hand Rule
  - A method for determining a special orthogonal direction used throughout mathematics and physics in 3D space, with respect to an ordered pair of vectors $\mathbf{u}, \mathbf{v}$
  - $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$, $\mathbf{v}$ according to the Right-Hand Rule.

- Cross Product Magnitude

  $$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

  The area of the parallelogram determined by $\mathbf{u}, \mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$.

- Parallel Vectors
  - $\mathbf{u}, \mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are parallel if the angle between them is $0 = 0°$ or $\pi = 180°$
  - $\mathbf{0}$ is parallel to every vector

- Cross Product Properties
  1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
  4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
  5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
  6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products
  1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
  2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
  3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

  The standard unit vectors are known as a "right handed frame".

- Torque

  $$\overrightarrow{\tau} = \mathbf{r} \times \mathbf{F}$$
  $$|\overrightarrow{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

- Triple Scalar (or "Box") Product

  $$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

  Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelpiped determined by the three vectors.

## 12.5 Equations of Lines and Planes

- Vector Equation and Parametric Equations for a Line

$$\mathbf{r}(t) = \mathbf{P_0} + t\mathbf{v}$$

$$x = x_0 + At, y = y_0 + Bt, z = z_0 + Ct$$

for $-\infty < t < \infty$

- Symmetric Equations for a Line

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_1} - \mathbf{P_0}) = (1 - t)\mathbf{P_0} + t\mathbf{P_1}$$

for $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = D$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{n_1} \times \mathbf{n_2})$$

- Angle of Intersection of Two Planes

$$\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

## 12.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
  - To sketch a 3D surface, sketch planar cross-sections
    - $z = c$ is parallel to $xy$ plane
    - $y = b$ is parallel to $xz$ plane
    - $x = a$ is parallel to $yz$ plane
- Cylinders
  - A **cylinder** is any surface generated by considering parallel lines passing through a planar curve.
  - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
  - A **quadric surface** is any surface defined by a second degree equation of $x, y, z$.
  - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
  - Cross-sections in the coordinate planes include
    - Three ellipses
- Elliptical Cone
  - Cross-sections in the coordinate planes include
    - Two double-lines
    - One point (with parallel ellipses)
- Elliptical Paraboloid
  - Cross-sections in the coordinate planes include
    - Two parabolas
    - One point (with parallel ellipses)

- Hyperbolic Paraboloid
  - Cross-sections in the coordinate planes include
    - Two parabolas (with parallel parabolas)
    - One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
  - Cross-sections in the coordinate planes include
    - Two hyperbolas
    - One ellipsis (with parallel ellipses)
- Hyperboloid of Two Sheets
  - Cross-sections in the coordinate planes include
    - Two hyperbola
    - One empty cross-section (with parallel ellipses)

## 13.1 Vector Functions and Space Curves

- Curves, Paths, and Vector Functions
  - A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

    $$x = x(t), y = y(t), z = z(t)$$

    or with a **vector function**

    $$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

  - $x(t), y(t), z(t)$ are called **component functions**
- Vector Function Limits

  $$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

- Continuity of Vector Functions
  - The function $\mathbf{r}(t)$ is **continuous** if

    $$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

    for all $a$ in its domain.
  - $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

## 13.2 Derivatives and Integrals of Vector Functions

- Derivatives of Vector Functions

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \langle f'(t), g'(t), h'(t) \rangle$$

  - $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of $t$ is in its domain.
  - $\mathbf{r}'(a)$ is a **tangent vector** to the curve where $t = a$
  - The **tangent line** to a curve at $t = a$:

$$\mathbf{l}(t) = \mathbf{r}(a) + t\mathbf{r}'(a)$$

- Differentiation Rules for Vector Functions

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df}\frac{df}{dt}$$

- Derivative of a Constant Length Vector Function
  - If $|\mathbf{r}(t)| = c$ always, then
$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$
  - Thus the derivative of a constant length vector function is perpindicular to the original.

- Antiderivatives of Vector Functions
  - If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.
  - The **indefinite integral** $\displaystyle\int \mathbf{r}(t)\,dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

$$\int_a^b \mathbf{r}(t)dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Differential Vector Equations
  - If we know $\mathbf{r}'(t)$ and $\mathbf{r}(a)$ for some $t = a$, then

$$\mathbf{r}(t) = \int_a^t \mathbf{r}'(t)\,dt + \mathbf{r}(a)$$

## 13.3 Arc Length and Curvature

- Arc Length along a Space Curve

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{r}'(t)|\, dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{r}'(u)| du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

- Unit Tangent Vector

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$$

$$\mathbf{T}(t) = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

- Curvature

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{|d\mathbf{r}/dt|} = \frac{|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}|}{|\frac{d\mathbf{r}}{dt}|^3}$$

  For $y = f(x)$:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

- Principal Unit Normal Vector

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

  The triple $\mathbf{T}, \mathbf{N}, \mathbf{B}$ forms a right-handed frame.

## 13.4 Motion in Space: Velocity and Acceleration

- Position, Velocity, and Acceleration
  - Position: $\mathbf{r}(t)$
  - Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
  - Speed: $v(t) = |\mathbf{v}(t)| = \frac{ds}{dt}$
  - Direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
  - Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$
- Ideal Projectile Motion

$$\mathbf{a}(t) = \langle 0, -g \rangle$$

$$\mathbf{v}(t) = \langle v_0 \cos\alpha, -gt + v_0 \sin\alpha \rangle$$

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \right\rangle$$

- Tangental and Normal Components of Acceleration

$$\mathbf{a} = \left(\frac{d^2 s}{dt^2}\right)\mathbf{T} + \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{N} + 0\mathbf{B}$$

  - Tangental component

$$a_T = \frac{d^2 s}{dt^2} = v'$$

  - Normal component

$$a_N = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa v^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

## 14.1 Functions of Several Variables

- Functions of Two Variables
  - A **function** $f$ **of two variables** is a rule which assigns a real number $f(x, y)$ to each pair of real numbers $(x, y)$ in its **domain**

    $$\mathrm{dom}(f) \subseteq \mathbb{R}^2$$

    The set of values $f$ takes on is its **range**

    $$\mathrm{ran}(f) = \{f(x, y) : (x, y) \in \mathrm{dom}(f)\}$$

  - The **level curve** for each $k \in \mathrm{ran}(f)$ is given by the equation

    $$f(x, y) = k$$

  - The **graph** of $f$ is a surface in 3D space which visualizes the function, given by the equation $z = f(x, y)$.
- Functions of Three Variables
  - A **function** $f$ **of three variables** is a rule which assigns a real number $f(x, y, z)$ to each pair of real numbers $(x, y, z)$ in its **domain**

    $$\mathrm{dom}(f) \subseteq \mathbb{R}^3$$

    The set of values $f$ takes on is its **range**

    $$\mathrm{ran}(f) = \{f(x, y, z) : (x, y, z) \in \mathrm{dom}(f)\}$$

  - The **level surface** for each $k \in \mathrm{ran}(f)$ is given by the equation

    $$f(x, y, z) = k$$

- Alternate Forms
  - We may also consider functions of the form $f(x_1, x_2, \dots) = f(P) = f(\mathbf{r})$.
  - If $P = (x, y)$ and $\mathbf{r} = \langle x, y\rangle$, then $f(x, y) = f(P) = f(\mathbf{r})$.
  - If $P = (x, y, z)$ and $\mathbf{r} = \langle x, y, z\rangle$, then $f(x, y, z) = f(P) = f(\mathbf{r})$.

## 14.2 Limits and Continuity

- Limits
  - If the value of the function $f(P)$ becomes arbitrarily close to the number $L$ as vectors $P$ close to $P_0$ are plugged into the function, then the **limit of $f(P)$ as $P$ approaches $P_0$** is $L$:

    $$\lim_{P \to P_0} f(P) = L$$

  - For functions of two or three variables:

    $$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

    $$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = L$$

- Showing a Limit DNE
  - In order for a limit $\lim_{P \to P_0} f(x,y)$ to exist, the values of $f$ must approach $L$ no matter which direction we approach $P_0$.
  - Choose $y = g(x)$ and $y = h(x)$ where $P_0$ lays on both graphs. If

    $$\lim_{x \to x_0} f(x, g(x)) \neq \lim_{x \to x_0} f(x, h(x))$$

    then $\displaystyle\lim_{P \to P_0} f(x,y)$ DNE.
  - Or choose $x = g(y)$ and $x = h(y)$ where $P_0$ lays on both graphs. If

    $$\lim_{y \to y_0} f(g(y), y) \neq \lim_{y \to y_0} f(h(y), y)$$

    then $\displaystyle\lim_{P \to P_0} f(x,y)$ DNE.

- Limit Laws

$$\lim_{P \to P_0} (f(P) \pm g(P)) = \lim_{P \to P_0} f(P) \pm \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (f(P) \cdot g(P)) = \lim_{P \to P_0} f(P) \cdot \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (kf(P)) = k \lim_{P \to P_0} f(P)$$

$$\lim_{P\to P_0} \frac{f(P)}{g(P)} = \frac{\lim\limits_{P\to P_0} f(P)}{\lim\limits_{P\to P_0} g(P)}$$

$$\lim_{P\to P_0} (f(P))^{r/s} = \left(\lim_{P\to P_0} f(P)\right)^{r/s}$$

- Computing Limits
  - Variables not involved in a limit may be eliminated:

    $$\lim_{P\to P_0} f(x) = \lim_{x\to x_0} f(x)$$

  - Due to the Limit Laws, many limits follow the "just plug it in" rule.
  - If plugging in results in a zero in a denominator, use factoring, perhaps with conjugates.
  - L'Hopital's Rule does not apply for multiple variable limits.
- Continuity
  - A function $f(P)$ is **continuous** if $\lim\limits_{P\to P_0} f(P) = f(P_0)$ for all points $P_0$ in its domain.
  - If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

## 14.3 Partial Derivatives

- Partial Derivatives
  - For a function $f$ of two variables $(x, y)$:

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

  - To compute partial derivatives with respect to a variable, treat all other variables as constants and differentiate as normal.
  - Functions of more than two variables behave similarly. For $T(x, y, z)$:

$$\frac{\partial T}{\partial z} = T_z(x, y, z) = \lim_{h \to 0} \frac{T(x, y, z+h) - T(x, y, z)}{h}$$

- Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 g}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial g}{\partial z}\right] = (g_z)_z = g_{zz}$$

- Mixed Derivative Theorem
  - For many naturally occuring functions:

$$f_{xy} = f_{yx}$$

## 14.4 Tangent Planes and Linear Approximations

- Tangent Plane to $z = f(x, y)$ at $(a, b, f(a, b))$

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Linearization of $f(x, y)$ at $(a, b)$

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Differentiability and a Sufficient Condition
  - A multi-variable function $f$ is **differentiable** at a point if its linearizaration approximates the value of the function near that point.
  - If $f_x$, $f_y$ exist near $(a, b)$ and are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$.

- Linear Approximation

  If $f$ is differentiable at $(a, b)$, then

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

## 14.5 The Chain Rule

- Gradient Vector Function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Nested Functions
  - If $f$ is a function of $\mathbf{r} = \langle x,y,z\rangle$, and $x,y,z$ are functions of $t$, then we say $x,y,z$ are itermediate variables and may consider the following composed function of $t$:

$$f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

  - If $f$ is a function of $\mathbf{r} = \langle x,y,z\rangle$, and $x,y,z$ are functions of $\mathbf{s} = \langle t,u,v\rangle$, then we say $x,y,z$ are itermediate variables and may consider the following composed function of $t,u,v$:

$$f(\mathbf{r}(\mathbf{s})) = f(x(t,u,v), y(t,u,v), z(t,u,v))$$

- Chain Rule
  - For functions of the form $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

  - For functions of the form $f(\mathbf{r}(\mathbf{s})) = f(x(t,u,v), y(t,u,v), z(t,u,v))$:

$$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

- Differentiation by Substitution
  - The multi-variable Chain Rule can be avoided by "plugging in" functions and using single-variable calculus.

- Total Derivative
  - If $f$ is a function of $x, y, z$, and $y, z$ are also functions of $x$, then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

- Implicit Differentiation
  - If $f(x, y) = c$ defines $y$ as a function of $x$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}$$

- Tree Diagram for the Chain Rule
  - The tree diagram for the chain rule can be used to generate the chain rule.
  - It also holds for multiple levels of intermediate variables.

## 14.6 Directional Derivatives and the Gradient Vector

- Directional Derivative
  - The **directional derivative** of $f$ for the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

  - The maximum value of $D_{\mathbf{u}}f$ at a fixed point $P_0$ is $|\nabla f(P_0)|$, which occurs when $\mathbf{u} = \frac{\nabla f(P_0)}{|\nabla f(P_0)|}$.
- Normal Vector to Level Curves and Surfaces
  - The gradient vectors $\nabla f$ are normal vectors to the level curves $f(x, y) = k$ for every $(x, y)$ in the domain of $f$.
  - The gradient vectors $\nabla f$ are normal vectors to the level surfaces $f(x, y, z) = k$ for every $(x, y, z)$ in the domain of $f$.

## 14.7 Maximum and Minimum Values

- Local Maximum and Minimum Values
  - Let $f$ be a function of many variables defined near the point $P_0$.
    * $f$ has a **local maximum** $f(P_0)$ at $P_0$ if $f(P_0)$ is the largest value of $f$ near $P_0$
    * $f$ has a **local minimum** $f(P_0)$ at $P_0$ if $f(P_0)$ is the smallest value of $f$ near $P_0$
- Critical Points
  - If $P_0$ is a point in the domain of $f$ and

    $$\nabla f(P_0) = 0 \text{ or } \nabla f(P_0) \text{ DNE}$$

    then $P_0$ is called a **critical point**.
  - Critical points occur when the tangent plane is horizontal or DNE.
  - The local maximum and minimum values of a function always occur at critical points.
- Saddle Points
  - Not every critical point gives a local extreme value.
  - The **saddle points** of $f$ are the critical points which don't yield local extreme values.
- Discriminant Function
  - The **discriminant** of $f$ with variables $x, y$ is the function

    $$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$
- Second Derivative Test for Local Extreme Values of $f(x, y)$

  Let $(a, b)$ be a critical point of of $f$ where $\nabla f$ is defined.
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  - If $f_D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.
  - If $f_D(a, b) = 0$, then the test is inconclusive.

- Absolute Maximum and Minimum Values
  - Let $f$ be a function of many variables.
    - $f$ has an **absolute maximum** $f(P_0)$ at $P_0$ if $f(P_0)$ is the largest value in the range of $f$
    - $f$ has an **absolute minimum** $f(P_0)$ at $P_0$ if $f(P_0)$ is the smallest value in the range of $f$
  - Every continuous function of many variables with a closed and bounded domain has an absolute maximum and minimum value.
- Finding Absolute Max/Min of $f(x, y)$ on a Closed and Bounded Region $D$
  - The following points are candidates for giving the absolute extrema:
    - Critical points of $f$ within $D$.
    - Critical points for a function which gives part of the boundary of $D$.
    - Corners of $D$.
  - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.

## 14.8 Lagrange Multipliers

- The Method of Lagrange Multipliers
  - The **Method of Lagrange Multipliers** says that if $f$ is a function of many variables which has an absolute max/min value on the restriction $g(P) = k$ where $\nabla g \neq 0$, then the absolute max/min occurs at a point $P$ where
  $$\nabla f(P) = \lambda \nabla g(P) \text{ and } g(P) = k$$
  for some real number $\lambda$.
  - If two constraints $g(P) = k$ and $h(P) = l$ are given, then the absolute max/min occurs where
  $$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P) \text{ and } g(P) = k \text{ and } h(P) = l$$
  for some real numbers $\lambda, \mu$.

## 15.1 Double Integrals over Rectangles

- Double Integral
  - We define the **double integral** of a function $f(x, y)$ over a region $R$ to be

    $$\iint\limits_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

    where for each positive integer $n$ we've defined a way to partition $R$ into $n$ pieces

    $$\Delta R_{n,1}, \Delta R_{n,2}, \dots, \Delta R_{n,n}$$

    where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

    $$\lim_{n\to\infty} \max(\Delta A_{n,i}) = 0$$

  - Since for $f(x, y) \geq 0$,

    $$\sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

    is an approximation of the volume under $z = f(x, y)$ and over $R$, the double integral is used to define the precise volume.
  - If $f$ is not always positive, then the double integral represents **net volume**: volume above the $xy$-plane minus volume below the $xy$-plane.
- Midpoint Rule for Approximating Rectangular Double Integrals
  - For the rectangle

    $$R : a \leq x \leq b, c \leq y \leq d$$

    we may approximate the double integral by partitioning the rectangle into a grid of $m \times n$ rectangular pieces all with area $\Delta A$ and evaluating:

    $$\iint\limits_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

    where $(\overline{x_i}, \overline{y_j})$ is the midpoint of the $i \times j$ rectangle.

## 15.2 Iterated Integrals

- Volume as Integral of Area
  - If $A(x)$ is the area of a solid's cross-section, then the solid's volume is

$$V = \int_a^b A(x)\,dx$$

- Iterated Integrals over Rectangles
  - A double integral over a rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

    can be expressed as the **iterated integrals**:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

## 15.3 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions
  - For **Type I** regions which may be expressed as

    $$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

    a double integral over $R$ may be expressed as the iterated integral:

    $$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

  - For **Type II** regions which may be expressed as

    $$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

    a double integral over $R$ may be expressed as the iterated integral:

    $$\iint_R f(x, y)\, dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

- Finding Limits of Integration
  1. Sketch the region and label bounding curves
  2. Determine if the region is Type I or Type II by identifying (I) bottom/top curves $y = g_1(x), y = g_2(x)$ or (II) left/right curves $x = h_1(y), x = h_2(y)$.

  For Type I:

  3. Use the leftmost and rightmost $x$-values in the region $a, b$ to complete the iterated integral:

  $$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

  For Type II:

  3. Use the bottommost and topmost $y$-values in the region $c, d$ to complete the iterated integral:

  $$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

- Swapping Variables of Integration
  - You can only swap the order of integration of an iterated integral by drawing the region and reinterpreting it as a region of the opposite Type.
- Additivity

  If $R$ can be split into two regions $R_1, R_2$, then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

- Average Value of Two-Variable Functions
  - The average value of a two-variable function $f$ over a region $R$ is defined to be

$$\frac{1}{\text{Area of } R}\iint\limits_R f(x,y)\,dA$$

- Area as a Double Integral
  - The area of a region $R$ in the plane is

$$A = \iint\limits_R dA = \iint\limits_R 1\,dA$$

## 15.7 Triple Integrals

- Triple Integral
  - We define the **triple integral** of a function $f(x, y, z)$ over a solid $D$ to be

$$\iiint_D f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i}, z_{n,i})\Delta V_{n,i}$$

    where for each positive integer $n$ we've defined a way to partition $D$ into $n$ pieces

$$\Delta D_{n,1}, \Delta D_{n,2}, \ldots, \Delta D_{n,n}$$

    where $\Delta D_{n,i}$ has volume $\Delta V_{n,i}$, contains the point $(x_{n,i}, y_{n,i}, z_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta V_{n,i}) = 0$$

- Iterated Integral for Rectangular Boxes
  - The triple integral over the rectangular box

$$D : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$$

    can be expressed as the **iterated integrals**:

$$\iiint_D f(x,y,z)\,dV = \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{b_1}^{b_2}\int_{c_1}^{c_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dz\,dy = \int_{a_1}^{a_2}\int_{c_1}^{c_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dz\,dx = \ldots$$

- Iterated Integral for Generated Solids
  - If the solid $D$ is determined by the bottom/top surfaces

$$h_1(x,y) \le z \le h_2(x,y)$$

    and has shadow $R$ in the $xy$-plane, then a triple integral over $D$ can be expressed as:

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right] dA$$

- In general:

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{\text{bottom surface}}^{\text{top surface}} f(x,y,z)\,d\square\right] dA$$

where $\square$ is chosen from $x, y, z$ to be the "up" orientation.

- Additivity

If $D$ can be split into two regions $D_1, D_2$, then

$$\iint_D f(x,y,z)\,dV = \iint_{D_1} f(x,y,z)\,dV + \iint_{D_2} f(x,y,z)\,dV$$

- Average Value of Three-Variable Functions

  - The average value of a three-variable function $f$ over a solid $D$ is defined to be

$$\frac{1}{\text{Volume of } D}\iiint_R f(x,y,z)\,dV$$

- Volume as a Triple Integral

  - The volume of a solid $D$ in space is

$$V = \iiint_D dV = \iiint_D 1\,dV$$

## 15.10 Change of Variables in Multiple Integrals

- Transformations
  - Two similar regions in 2D space can be transformed by a "nice" pair of functions

$$\mathbf{r}(u,v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s})\rangle = \langle x(u,v), y(u,v)\rangle$$

    that map points in a $uv$ plane to the $xy$ plane.
  - Two similar solids in 3D space can be transformed by a "nice" triple of functions

$$\mathbf{r}(u,v,w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s})\rangle = \langle x(u,v,w), y(u,v,w), z(u,v,w)\rangle$$

    that map points in a $uvw$ space to the $xyz$ space.

- The Jacobian
  - The Jacobian of a 2D transformation given by $\mathbf{r}(u,v)$ is the determinant

$$\mathbf{r}_J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

  - The Jacobian of a 3D transformation given by $\mathbf{r}(u,v,w)$ is the determinant

$$\mathbf{r}_J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- 2D Substitution
  - Suppose that the region $R$ in the $xy$-plane is the result of applying the transformation $\mathbf{r}(u,v)$ to the region $G$ in the $uv$-plane.
  - Then it follows that

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_G f(x(u,v), y(u,v))|\mathbf{r}_J(u,v)|\,du\,dv$$

- Unit Square and Triangle
  - The unit square in the $uv$ plane with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ is useful for substitution problems involving parallelograms.
  - The unit triangle in the $uv$ plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$ is useful for substitution problems involving triangles.
- 3D Substitution
  - Suppose that the solid $D$ in $xyz$ space is the result of applying the transformation $\mathbf{r}(u,v,w)$ to the region $H$ in $uvw$ space.
  - Then it follows that

$$\iiint_D f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_H f(x(u,v,w), y(u,v,w), z(u,v,w))|\mathbf{r}_J(u,v,w)|\,du\,dv\,dw$$

### 15.4 Double Integrals in Polar Coordinates

- Integrating over Regions expressed using Polar Coordinates
  - The polar coordinate transformation

    $$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$

    from polar $G$ into Cartesian $R$ yields

    $$\iint\limits_R f(x,y)\,dA = \iint\limits_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

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### 15.8 Triple Integrals in Cylindrical Coordinates

- Cylindrical Coordinates
  - The cylindrical coordinate transformation

    $$\mathbf{r}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

    from cylindrical $H$ into Cartesian $D$ yields

    $$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

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### 15.9 Triple Integrals in Spherical Coordinates

- Spherical Coordinates
  - The spherical coordinate transformation

    $$\mathbf{r}(\rho,\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

    from spherical $H$ into Cartesian $D$ yields

    $$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

## 16.1 Vector Fields

- Vector Fields
  - A **vector field** assigns a vector to each point in 2D or 3D space.

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle P(\mathbf{r}), Q(\mathbf{r})\rangle = \langle P, Q\rangle$$

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \langle P(\mathbf{r}), Q(\mathbf{r}), R(\mathbf{r})\rangle = \langle P, Q, R\rangle$$

- Gradient Vector Field
  - The gradient vector field $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ assigns vectors whose directions are normal to level curves and whose magnitudes are equal to the maximal directional derivative at the point.
  - The gradient vector field $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$ assigns vectors whose directions are normal to level surfaces and whose magnitudes are equal to the maximal directional derivative at the point.

## 16.2 Line Integrals

- Common Curve Parametrizations
  - A line segment beginning at $P_0$ and ending at $P_1$

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_1} - \mathbf{P_0}), 0 \leq t \leq 1$$

  - A circle centered at the origin with radius $a$

$$\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle, 0 \leq t \leq 2\pi \text{ (counter-clockwise)}$$

$$\mathbf{r}(t) = \langle a\sin t, a\cos t\rangle, 0 \leq t \leq 2\pi \text{ (clockwise)}$$

  - A planar curve given by $y = f(x)$ from $(x_0, y_0)$ to $(x_1, y_1)$

$$\mathbf{r}(t) = \langle t, f(t)\rangle, x_0 \leq t \leq x_1 \text{ (for } x_0 \leq x_1)$$

$$\mathbf{r}(t) = \langle -t, f(-t)\rangle, x_1 \leq t \leq x_0 \text{ (for } x_1 \leq x_0)$$

- Line Integrals with Respect to Arclength
  - We define the **line integral with respect to arclength** of a function of many variables $f(\mathbf{r})$ along a curve $C$ to be

$$\int_C f(\mathbf{r})\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\mathbf{r}_{n,i})\Delta s_{n,i}$$

    where for each positive integer $n$ we've defined a way to partition $C$ into $n$ pieces

$$\Delta C_{n,1}, \Delta C_{n,2}, \ldots, \Delta C_{n,n}$$

    where $\Delta C_{n,i}$ has length $\Delta s_{n,i}$, contains the position vector $\mathbf{r}_{n,i}$, and

$$\lim_{n\to\infty} \max(\Delta s_{n,i}) = 0$$

  - If $\mathbf{r}(t)$ is a parametrization of $C$ for $a \leq t \leq b$, then

$$\int_C f(\mathbf{r})\,ds = \int_{t=a}^{t=b} f(\mathbf{r}(t))\frac{ds}{dt}\,dt$$

- Line Integrals with Respect to Variables
  - Similarly, we can find the **line integral with respect to a variable** for a function of many variables $f(\mathbf{r})$ along a curve $C$:

$$\int_C f(\mathbf{r})\,dx = \int_{t=a}^{t=b} f(\mathbf{r}(t))\frac{dx}{dt}\,dt$$

  - Similar defintions hold for $y, z$.
- Line Integrals of Vector Fields
  - The **line integral of a vector field** is defined to be the line integral with respect to arclength of the dot product of the vector field $\mathbf{F}(\mathbf{r}) = \langle P(\mathbf{r}), Q(\mathbf{r}), R(\mathbf{r})\rangle$ with the unit tangent vector $\mathbf{T}(\mathbf{r})$ to the curve.

$$\int_C \mathbf{F}(x,y,z)\cdot\mathbf{T}(x,y,z)\,ds$$

  - There are several ways to write and evaluate line integrals of vector fields:

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_C \langle P,Q,R\rangle\cdot\langle\,dx,\,dy,\,dz\rangle$$

$$= \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} + R(\mathbf{r}(t))\frac{dz}{dt}\right)\,dt$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$$

- Additivity

  Let $C_1 + C_2$ represent the curve taken by moving along $C_1$ followed by moving along $C_2$.

$$\int_{C_1+C_2} f(\mathbf{r})\,ds = \int_{C_1} f(\mathbf{r})\,ds + \int_{C_2} f(\mathbf{r})\,ds$$

$$\int_{C_1+C_2} f(\mathbf{r})\,dx = \int_{C_1} f(\mathbf{r})\,dx + \int_{C_2} f(\mathbf{r})\,dx$$

$$\int_{C_1+C_2} \mathbf{F}\cdot\,d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\,d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot\,d\mathbf{r}$$

- Effects of Curve Orientation

  Let $-C$ represent the curve taken by moving along $C$ in the opposite direction.

$$\int\limits_C f(\mathbf{r})\,ds = +\int\limits_{-C} f(\mathbf{r})\,ds$$

$$\int\limits_C f(\mathbf{r})\,dx = -\int\limits_{-C} f(\mathbf{r})\,dx$$

$$\int\limits_C \mathbf{F}\cdot\,d\mathbf{r} = -\int\limits_{-C} \mathbf{F}\cdot\,d\mathbf{r}$$

- Work

  - If $\mathbf{F}$ is a vector field representing the force applied to an object as it is moved over a smooth curve $C$, then the **work** done by the force over that curve is given by

$$\int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$$

## 16.3 The Fundamental Theorem for Line Integrals

- The Fundamental Theorem
  - If $C$ is any smooth curve beginning at the point $A$ and ending at the point $B$, then
    $$\int_C \nabla f \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$$
  - If $C$ is any smooth curve which is **closed** (begins and ends at the same point), then
    $$\int_C \nabla f \cdot d\mathbf{r} = 0$$
- Conservative Fields
  - We say $\mathbf{F} = \langle M, N, P \rangle$ is a conservative field if there is a **potential function** $f$ such that $\nabla f = \mathbf{F}$.
  - Line integrals of conservative fields are said to be path independent since for any curve $C$ beginning at $A$ and ending at $B$:
    $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$$
  - We can prove a field is conservative by finding its potential function or showing it satisfies the **Component Test**:
    $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

## 16.4 Green's Theorem

- Green's Theorem
  - Let $C$ be the boundary of the region $R$ oriented counter-clockwise, and $\mathbf{F}(x, y)$ be a two-dimensional vector field.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  - Due to Green's Theorem, we can find the area of $R$ using a line integral:

$$A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx$$

## 16.5 Curl and Divergence

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Curl
  - The **curl** of a vector field is another vector field:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

  - By the Component Test, if $\mathbf{F}$ is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.
- Divergence
  - The **divergence** of a vector field is the scalar function:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

  - For any vector field, the divergence of curl is always zero.

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

- Green's Theorem Alternate Forms
  - If $\mathbf{F}$ is a two-dimensional vector field, and $\mathbf{n}$ is the outward unit normal vector field for a counter-clockwise closed curve $C$:

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F} \, dA$$

### 16.6 Parametric Surfaces and Their Areas

- Parametric Surface Equations

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

- Common Parametric Surfaces
  - The plane determined by the point $P_0$ and vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ can be parametrized by
  $$\mathbf{r} = \mathbf{P_0} + u\mathbf{v_1} + v\mathbf{v_2}$$
  - The surface $z = f(x,y)$ can be parametrized by
  $$\mathbf{r} = \langle x, y, f(x,y)\rangle$$
  - A surface determined by a cylindrical coordinate equation can be parametrized by substituting into
  $$\mathbf{r} = \langle r\cos\theta, r\sin\theta, z\rangle$$
  - A surface determined by a spherical coordinate equation can be parametrized by substituting into
  $$\mathbf{r} = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

- Surface Area
  - If $G$ is the region in the $uv$ plane which maps onto the surface $S$ by the parametric equations $\mathbf{r}(u,v)$, then the surface area of $S$ is:
  $$\iint_G |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$
  where $\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$.

## 16.7 Surface Integrals

- Surface Orientation
  - The orientation of a surface is determined by a continuous unit normal vector field on the surface.
- Surface Integral
  - If $G$ is the region in the $uv$ plane which maps onto the surface $S$ by the parametric equations $\mathbf{r}(u, v)$, then the surface integral of $f(\mathbf{r})$ along $S$ is:

$$\iint_S f(\mathbf{r})\, d\sigma = \iint_G f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

- Surface Integral of Vector Field
  - If $G$ is the region in the $uv$ plane which maps onto the surface $S$ by the parametric equations $\mathbf{r}(u, v)$, and $\mathbf{n}$ is the unit normal vector field giving the orientation of $S$, then the surface integral of the vector field $\mathbf{F}$ along $S$ is:

$$\iint_S \mathbf{F} \cdot d\vec{\sigma} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_G \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

## 16.8 Stokes' Theorem

- Stokes' Theorem
  - Let $C$ give the counter-clockwise oriented boundary of a surface $S$.

$$\iint_S \mathbf{F} \cdot d\vec{\sigma} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

## 16.9 Divergence Theorem

- Divergence Theorem
  - Let $S$ give the outward-oriented boundary surface of the solid $D$.

$$\iint_S \mathbf{F} \cdot d\vec{\sigma} = \iiint_D \operatorname{div} \mathbf{F} \, dV$$