Sections 12.1 - 12.2 Overview

- Three-Dimensional Coordinates
  - Distance between points in 3D space
    $$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
  - Simple planes in 3D Space
    $$x = a,\ y = b,\ z = c$$
  - Spheres in 3D Space
    $$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$
- Vectors
  - Definition of a Vector
    * A vector $\mathbf{v} = \overrightarrow{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
    * Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
    * We often assume the initial point lays at the origin.
  - Component Form
    The vector with initial point at $(0, 0, 0)$ and terminal point at (v_x, v_y, v_z) is represented by
    $$\langle v_x, v_y, v_z \rangle$$
  - 2D and 3D Vectors
    $$\langle a, b \rangle = \langle a, b, 0 \rangle$$
  - Position Vector
    If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c \rangle$ is its **position vector**.
    We assume $(a, b, c) = \langle a, b, c \rangle$.
  - Vector Between Points
    The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is
    $$\mathbf{P_1P_2} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \overrightarrow{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations
  * Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

  * Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties
  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  5. $0\mathbf{u} = \mathbf{0}$
  6. $1\mathbf{u} = \mathbf{u}$
  7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
  8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- Unit Vectors
  * A **unit vector** or **direction** is any vector whose length is 1.
  * Standard unit vectors
    · $\mathbf{i} = \langle 1, 0, 0 \rangle$
    · $\mathbf{j} = \langle 0, 1, 0 \rangle$
    · $\mathbf{k} = \langle 0, 0, 1 \rangle$
  * Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

  * Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

12.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- Angle between vectors

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

- Orthogonal Vectors
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if the angle between them is $\frac{\pi}{2} = 90°$
  - $\mathbf{0}$ is orthogonal to every vector

- Dot Product Properties
  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
  5. $\mathbf{0} \cdot \mathbf{u} = 0$

- Projection Vector

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos\theta$$

12.4 The Cross Product

- Determinants
  - 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - 3x3 Determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3)$$

- Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \langle u_2v_3 - u_3v_2 \, , \, u_3v_1 - u_1v_3 \, , \, u_1v_2 - u_2v_1 \rangle$$

  Shortcut "long multiplication" method:

$$\begin{array}{rcccccl} \langle & u_1 & , & u_2 & , & u_3 & \rangle \\ \times\langle & v_1 & , & v_2 & , & v_3 & \rangle \\ \hline \langle & u_2v_3 - u_3v_2 & , & u_3v_1 - u_1v_3 & , & u_1v_2 - u_2v_1 & \rangle \end{array}$$

- Right-Hand Rule
  - A method for determining a special orthogonal direction used throughout mathematics and physics in 3D space, with respect to an ordered pair of vectors $\mathbf{u}, \mathbf{v}$
  - $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$, $\mathbf{v}$ according to the Right-Hand Rule.

- Cross Product Magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

The area of the parallelogram determined by $\mathbf{u}, \mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$.

- Parallel Vectors
  - $\mathbf{u}, \mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are parallel if the angle between them is $0 = 0°$ or $\pi = 180°$
  - $\mathbf{0}$ is parallel to every vector
- Cross Product Properties
  1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
  4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
  5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
  6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
- Standard Unit Vector Cross Products
  1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
  2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
  3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

  The standard unit vectors are known as a "right handed frame".
- Torque

$$\overrightarrow{\tau} = \mathbf{r} \times \mathbf{F}$$
$$|\overrightarrow{\tau}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

- Triple Scalar (or "Box") Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelpiped determined by the three vectors.

12.5 Equations of Lines and Planes

- Vector Equation and Parametric Equations for a Line

$$\mathbf{r}(t) = \mathbf{P_0} + t\mathbf{v}$$

$$x = x_0 + At, y = y_0 + Bt, z = z_0 + Ct$$

for $-\infty < t < \infty$

- Symmetric Equations for a Line

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_0 P_1}) = (1 - t)\mathbf{P_0} + t\mathbf{P_1}$$

for $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = D$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{n_1} \times \mathbf{n_2})$$

- Angle of Intersection of Two Planes

$$\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

12.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
  - To sketch a 3D surface, sketch planar cross-sections
    - $z = c$ is parallel to xy plane
    - $y = b$ is parallel to xz plane
    - $x = a$ is parallel to yz plane
- Cylinders
  - A **cylinder** is any surface generated by considering parallel lines passing through a planar curve.
  - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
  - A **quadric surface** is any surface defined by a second degree equation of x, y, z.
  - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
  - Cross-sections in the coordinate planes include
    - Three ellipses
- Elliptical Cone
  - Cross-sections in the coordinate planes include
    - Two double-lines
    - One point (with parallel ellipses)
- Elliptical Paraboloid
  - Cross-sections in the coordinate planes include
    - Two parabolas
    - One point (with parallel ellipses)

- Hyperbolic Paraboloid
  - Cross-sections in the coordinate planes include
    - Two parabolas (with parallel parabolas)
    - One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
  - Cross-sections in the coordinate planes include
    - Two hyperbolas
    - One ellipsis (with parallel ellipses)
- Hyperboloid of Two Sheets
  - Cross-sections in the coordinate planes include
    - Two hyperbola
    - One empty cross-section (with parallel ellipses)

13.1 Vector Functions and Space Curves

- Curves, Paths, and Vector Functions
  - A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

    $$x = x(t), y = y(t), z = z(t)$$

    or with a **vector function**

    $$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

  - $x(t), y(t), z(t)$ are called **component functions**
- Vector Function Limits

  $$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

- Continuity of Vector Functions
  - The function $\mathbf{r}(t)$ is **continuous** if

    $$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

    for all a in its domain.
  - $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

13.2 Derivatives and Integrals of Vector Functions

- Derivatives of Vector Functions

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \langle f'(t), g'(t), h'(t) \rangle$$

  - $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of t is in its domain.
  - $\mathbf{r}'(a)$ is a **tangent vector** to the curve where $t = a$
  - The **tangent line** to a curve at $t = a$:

$$\mathbf{l}(t) = \mathbf{r}(a) + t\mathbf{r}'(a)$$

- Differentiation Rules for Vector Functions

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df}\frac{df}{dt}$$

- Derivative of a Constant Length Vector Function
  - If $|\mathbf{r}(t)| = c$ always, then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

  - Thus the derivative of a constant length vector function is perpindicular to the original.

- Antiderivatives of Vector Functions
  - If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.
  - The **indefinite integral** $\displaystyle\int \mathbf{r}(t)\,dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

$$\int_a^b \mathbf{r}(t)dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Differential Vector Equations
  - If we know $\mathbf{r}'(t)$ and $\mathbf{r}(a)$ for some $t = a$, then

$$\mathbf{r}(t) = \int_a^t \mathbf{r}'(t)\,dt + \mathbf{r}(a)$$

13.3 Arc Length and Curvature

- Arc Length along a Space Curve

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{r}'(t)|\, dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{r}'(u)| du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

- Unit Tangent Vector

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$$

$$\mathbf{T}(t) = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

- Curvature

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{|d\mathbf{r}/dt|} = \frac{|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}|}{|\frac{d\mathbf{r}}{dt}|^3}$$

For $y = f(x)$:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

- Principal Unit Normal Vector

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

The triple $\mathbf{T}, \mathbf{N}, \mathbf{B}$ forms a right-handed frame.

13.4 Motion in Space: Velocity and Acceleration

- Position, Velocity, and Acceleration
  - Position: $\mathbf{r}(t)$
  - Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
  - Speed: $v(t) = |\mathbf{v}(t)| = \frac{ds}{dt}$
  - Direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
  - Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$
- Ideal Projectile Motion

$$\mathbf{a}(t) = \langle 0, -g \rangle$$

$$\mathbf{v}(t) = \langle v_0 \cos\alpha, -gt + v_0 \sin\alpha \rangle$$

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \right\rangle$$

- Tangental and Normal Components of Acceleration

$$\mathbf{a} = \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} + 0\mathbf{B}$$

  - Tangental component

$$a_T = \frac{d^2s}{dt^2} = v'$$

  - Normal component

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa v^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

14.1 Functions of Several Variables

- Functions of Two Variables
  - A **function** f **of two variables** is a rule which assigns a real number $f(x, y)$ to each pair of real numbers (x, y) in its **domain**
  $$\mathrm{dom}(f) \subseteq \mathbb{R}^2$$
  The set of values f takes on is its **range**
  $$\mathrm{ran}(f) = \{f(x, y) : (x, y) \in \mathrm{dom}(f)\}$$
  - The **level curve** for each $k \in \mathrm{ran}(f)$ is given by the equation
  $$f(x, y) = k$$
  - The **graph** of f is a surface in 3D space which visualizes the function, given by the equation $z = f(x, y)$.
- Functions of Three Variables
  - A **function** f **of three variables** is a rule which assigns a real number $f(x, y, z)$ to each pair of real numbers (x, y, z) in its **domain**
  $$\mathrm{dom}(f) \subseteq \mathbb{R}^3$$
  The set of values f takes on is its **range**
  $$\mathrm{ran}(f) = \{f(x, y, z) : (x, y, z) \in \mathrm{dom}(f)\}$$
  - The **level surface** for each $k \in \mathrm{ran}(f)$ is given by the equation
  $$f(x, y, z) = k$$
- Alternate Forms
  - We may also consider functions of the form $f(x_1, x_2, \dots) = f(P) = f(\mathbf{r})$.
  - If $P = (x, y)$ and $\mathbf{r} = \langle x, y\rangle$, then $f(x, y) = f(P) = f(\mathbf{r})$.
  - If $P = (x, y, z)$ and $\mathbf{r} = \langle x, y, z\rangle$, then $f(x, y, z) = f(P) = f(\mathbf{r})$.

14.2 Limits and Continuity

- Limits
  - If the value of the function $f(P)$ becomes arbitrarily close to the number L as vectors P close to P_0 are plugged into the function, then the **limit of $f(P)$ as P approaches P_0** is L:

$$\lim_{P \to P_0} f(P) = L$$

  - For functions of two or three variables:

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

$$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = L$$

- Showing a Limit DNE
  - In order for a limit $\lim_{P \to P_0} f(x,y)$ to exist, the values of f must approach L no matter which direction we approach P_0.
  - Choose $y = g(x)$ and $y = h(x)$ where P_0 lays on both graphs. If

$$\lim_{x \to x_0} f(x, g(x)) \neq \lim_{x \to x_0} f(x, h(x))$$

    then $\lim\limits_{P \to P_0} f(x,y)$ DNE.
  - Or choose $x = g(y)$ and $x = h(y)$ where P_0 lays on both graphs. If

$$\lim_{y \to y_0} f(g(y), y) \neq \lim_{y \to y_0} f(h(y), y)$$

    then $\lim\limits_{P \to P_0} f(x,y)$ DNE.

- Limit Laws

$$\lim_{P \to P_0} (f(P) \pm g(P)) = \lim_{P \to P_0} f(P) \pm \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (f(P) \cdot g(P)) = \lim_{P \to P_0} f(P) \cdot \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (kf(P)) = k \lim_{P \to P_0} f(P)$$

$$\lim_{P\to P_0}\frac{f(P)}{g(P)}=\frac{\lim_{P\to P_0}f(P)}{\lim_{P\to P_0}g(P)}$$

$$\lim_{P\to P_0}(f(P))^{r/s}=\left(\lim_{P\to P_0}f(P)\right)^{r/s}$$

- Computing Limits
  - Variables not involved in a limit may be eliminated:

    $$\lim_{P\to P_0}f(x)=\lim_{x\to x_0}f(x)$$

  - Due to the Limit Laws, many limits follow the "just plug it in" rule.
  - If plugging in results in a zero in a denominator, use factoring, perhaps with conjugates.
  - L'Hopital's Rule does not apply for multiple variable limits.
- Continuity
  - A function $f(P)$ is **continuous** if $\lim_{P\to P_0}f(P)=f(P_0)$ for all points P_0 in its domain.
  - If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

14.3 Partial Derivatives

- Partial Derivatives
  - For a function f of two variables (x, y):

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

  - To compute partial derivatives with respect to a variable, treat all other variables as constants and differentiate as normal.
  - Functions of more than two variables behave similarly. For $T(x, y, z)$:

$$\frac{\partial T}{\partial z} = T_z(x, y, z) = \lim_{h \to 0} \frac{T(x, y, z+h) - T(x, y, z)}{h}$$

- Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 g}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial g}{\partial z}\right] = (g_z)_z = g_{zz}$$

- Mixed Derivative Theorem
  - For many naturally occuring functions:

$$f_{xy} = f_{yx}$$

14.4 Tangent Planes and Linear Approximations

- Tangent Plane to $z = f(x, y)$ at $(a, b, f(a, b))$

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Linearization of $f(x, y)$ at (a, b)

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Differentiability and a Sufficient Condition
  - A multi-variable function f is **differentiable** at a point if its linearizaration approximates the value of the function near that point.
  - If f_x, f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

- Linear Approximation

  If f is differentiable at (a, b), then

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

14.5 The Chain Rule

- Gradient Vector Function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Nested Functions
  - If f is a function of $\mathbf{r} = \langle x, y, z\rangle$, and x, y, z are functions of t, then we say x, y, z are itermediate variables and may consider the following composed function of t:
  $$f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$
  - If f is a function of $\mathbf{r} = \langle x, y, z\rangle$, and x, y, z are functions of $\mathbf{s} = \langle t, u, v\rangle$, then we say x, y, z are itermediate variables and may consider the following composed function of t, u, v:
  $$f(\mathbf{r}(\mathbf{s})) = f(x(t,u,v), y(t,u,v), z(t,u,v))$$
- Chain Rule
  - For functions of the form $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$:
  $$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$
  - For functions of the form $f(\mathbf{r}(\mathbf{s})) = f(x(t,u,v), y(t,u,v), z(t,u,v))$:
  $$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$
- Differentiation by Substitution
  - The multi-variable Chain Rule can be avoided by "plugging in" functions and using single-variable calculus.

- Total Derivative
  - If f is a function of x, y, z, and y, z are also functions of x, then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

- Implicit Differentiation
  - If $f(x, y) = c$ defines y as a function of x, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}$$

- Tree Diagram for the Chain Rule
  - The tree diagram for the chain rule can be used to generate the chain rule.
  - It also holds for multiple levels of intermediate variables.

14.6 Directional Derivatives and the Gradient Vector

- Directional Derivative
  - The **directional derivative** of f for the unit vector $\mathbf{u}$ is
$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$
  - The maximum value of $D_{\mathbf{u}}f$ at a fixed point P_0 is $|\nabla f(P_0)|$, which occurs when $\mathbf{u} = \frac{\nabla f(P_0)}{|\nabla f(P_0)|}$.
- Normal Vector to Level Curves and Surfaces
  - The gradient vectors ∇f are normal vectors to the level curves $f(x, y) = k$ for every (x, y) in the domain of f.
  - The gradient vectors ∇f are normal vectors to the level surfaces $f(x, y, z) = k$ for every (x, y, z) in the domain of f.

12.6 Tangent Planes and Differentials

- Normal Vector to a Level Surface
  - ∇f is normal to the level surface $f(x, y, z) = c$ for every point (x, y, z) in the domain of f.
- Normal Vector to the Surface $z = f(x, y)$
  - If $g(x, y, z) = f(x, y) - z$, then

    $$\nabla g = \langle f_x, f_y, -1\rangle$$

    is normal to the surface $z = f(x, y)$ for every point (x, y) in the domain of f.
- Tangent Line to Curve of Intersection of Two Surfaces
  - If P_0 is a point on two surfaces with normal vectors $\mathbf{n_1}, \mathbf{n_2}$, then the tangent line to the curve of intersection is given by

    $$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{n_1} \times \mathbf{n_2})$$

- **Suggested Exercises for 12.6:**
  - Finding tangent planes & normal lines to surfaces of the form $f(x, y, z) = c$: 1-8
  - Finding tangent planes & normal lines to surfaces of the form $z = f(x, y)$: 9-12
  - Finding tangent lines to curves of intersection: 13-18

12.7 Extreme Values and Saddle Points

- Local Extreme Values
  - Let f be a function of many variables defined on a region containing the point P_0.
    * $f(P_0)$ is a **local maximum** if it is the largest nearby value (there exists an open region around P_0 over which no greater value of f exists)
    * $f(P_0)$ is a **local minimum** if it is the smallest nearby value (there exists an open region around P_0 over which no lesser value of f exists)
  - Local max/mins are also known as **local extrema**.
- Critical Points
  - The **critical points** for a function f of many variables are the points in the domain where
    $$\nabla f = 0 \text{ or } \nabla f \text{ DNE}$$
  - Critical points occur when there is a horizontal tangent plane or no tangent plane.
- First Derivative Test for Local Extreme Values
  - The local extreme values of a function always occur at critical points.
- Saddle Points
  - Not every critical point gives a local extreme value.
  - The **saddle points** of f are the critical points which don't yield local extreme values.
- Discriminant Function
  - The **discriminant** (sometimes called "Hessian") of $f(x, y)$ is the function
    $$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second Derivative Test for Local Extreme Values of $f(x, y)$
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  - If $f_D(a, b) < 0$, then f has a saddle point at (a, b).
  - If $f_D(a, b) = 0$, then the test is inconclusive.
- Absolute Extrema on Closed and Bounded Regions
  - Let f be a function of many variables defined on a region containing the point P_0.
    * $f(P_0)$ is the **absolute maximum** of f if it is the largest value in the range of f
    * $f(P_0)$ is the **absolute minimum** of f if it is the smallest value in the range of f
  - Absolute max/mins are also known as **absolute extrema**.
  - Every continuous function of many variables with a closed and bounded domain has absolute extrema.
- Finding Absolute Extrema of $f(x, y)$ on a Closed and Bounded Region D
  - The following points are candidates for giving the absolute extrema:
    * Critical points within D.
    * Critical points on any of D's boundary curves. (Find a relation of x and y and use that to make f a function of a single variable.)
    * Corners of D.
  - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.
- **Suggested Exercises for 12.7:**
  - Finding local max/min and saddle points: 1-30
  - Finding absolute max/min: 31-36

12.8 Lagrange Multipliers

- The Method of Lagrange Multipliers
  - The **Method of Lagrange Multipliers** says that if $f(P)$ is a function of many variables which has an absolute extreme value on the restricted domain $\{P : g(P) = c\}$, and f, g are differentiable functions such that $\nabla g \neq \mathbf{0}$, then the absolute extreme value occurs satisfies

    $$\nabla f = \lambda \nabla g \text{ and } g = c$$

    for some real number λ.

- **Suggested Exercises for 12.8:**
  - Finding absolute extrema using the Method of Lagrange Multipliers: 1-30

13.1 Double and Iterated Integrals over Rectangles

- Volume as Integral of Area
  - If $A(x)$ is the area of a solid's cross-section, then its volume is

$$V = \int_a^b A(x)\,dx$$

- Double Integrals over Rectangles
  - For a solid bounded above by $z = f(x, y) \geq 0$ over the rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

    its cross-sectional area at x is given by:

$$A(x) = \int_c^d f(x, y)\,dy$$

  - Thus its volume is the **iterated integral**:

$$V = \int_a^b A(x)\,dx = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

  - Similarly, its cross-sectional area at y and volume may be given by:

$$A(y) = \int_a^b f(x, y)\,dx$$

$$V = \int_c^d A(y)\,dy = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

  - We also represent its volume as a **double integral**:

$$V = \iint_R f(x, y)\,dA$$

  - If $f(x, y) \ngeq 0$, then the double integral represents **net volume**: volume above the xy-plane minus volume below the xy-plane.

- **Suggested Exercises for 13.1:**
  - Evaluating iterated integrals with constant bounds: 1-12
  - Evaluating double integrals over rectangles: 13-28

13.2 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions
  - For a solid bounded above by $z = f(x, y) \geq 0$ over the region

    $$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

    its cross-sectional area at x is given by:

    $$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

  - Thus its volume is the **iterated integral**:

    $$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

  - Similarly, for a solid bounded above by $z = f(x, y) \geq 0$ over the region

    $$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

    its cross-sectional area at x is given by:

    $$A(y) = \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx$$

    $$V = \int_a^b A(y)\, dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

  - We also represent its volume as a **double integral**:

    $$V = \iint_R f(x, y)\, dA$$

  - If $f(x, y) \ngeq 0$, then the double integral represents **net volume**: volume above the xy-plane minus volume below the xy-plane.

- Finding Limits of Integration

  1. Sketch the region and label bounding curves
  2. Determine if it is easier to describe bottom/top bounds

  $$g_1(x) \leq y \leq g_2(x)$$

  or left/right bounds

  $$h_1(y) \leq x \leq h_2(y)$$

  For $g_1(x) \leq y \leq g_2(x)$:

  3. Find the x-limits of integration a, b by finding the leftmost, rightmost x-values in the region:

  $$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  For $h_1(y) \leq x \leq h_2(y)$:

  3. Find the y-limits of integration c, d by finding the bottommost, topmost y-values in the region:

  $$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Swapping Variables of Integration

  - You can only swap the order of integration of an iterated integral by first converting to a double-integral, and using the above steps.

- Properties of Double Integrals

  1. Zero Integral

  $$\iint\limits_R 0\,dA = 0$$

  2. Constant Multiple

  $$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$$

3. Sum/Difference

$$\iint_R f(x,y) \pm g(x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

4. Domination
   If $f(x,y) \leq g(x,y)$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\,dA \leq \iint_R g(x,y)\,dA$$

5. Additivity
   If R can be split into two regions R_1, R_2, then

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

- **Suggested Exercises for 13.2:**
  - Evaluating nonrectangular double integrals: 1-6, 11-14
  - Finding limits of integration: 7-10, 33-44
  - Swapping order of integration: 25-32

13.3 Area by Double Integration

- Areas of Regions in the Plane
  - The area of a region R in the plane is

$$A = \iint_R dA = \iint_R 1\, dA$$

- Average Value of a Function of Two Variables
  - The average value of $f(x, y)$ over the region R is defined to be

$$\text{Avg Val} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA$$

- **Suggested Exercises for 13.3:**
  - Finding areas of regions: 1-8
  - Finding average values of functions: 15-18

13.5 Triple Integrals in Rectangular Coordinates

- Hypervolume as Integral of Volume
  - A hypersolid is a region of $\mathbb{R}^4$, that is, a set of ordered 4-tuples (x, y, z, w).
  - If $V(x)$ is the volume of a four-dimensional hypersolid's cross-section, then its hypervolume is
$$HV = \int_a^b V(x)\,dx$$
  - Applications include modeling density within 3D space: (x, y, z, δ).
  - Hypervolume in $xyz\delta$-space represents mass.
- Triple Integrals over Rectangular Boxes
  - For a hypersolid bounded above by $w = f(x, y, z) \geq 0$ over the rectangular box
$$D : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3$$
its cross-sectional volume at x is given by:
$$V(x) = \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z)\,dz\,dy$$
  - Thus its hypervolume is the iterated integral:
$$HV = \int_{a_1}^{b_1} V(x)\,dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z)\,dz\,dy\,dx$$
  - The *constant bounds* of this iterated integral and differentials may be swapped around.
  - We also represent its hypervolume as the **triple integral**
$$HV = \iiint\limits_D f(x, y, z)\,dV$$
  - If $w = f(x, y, z) \ngeq 0$, then the triple integral represents net hypervolume.

- Triple Integrals over Other Solids
  - For a general solid with bottom/top surface
    $$h_1(x,y) \le z \le h_2(x,y)$$
    and shadow in the xy plane bounded by
    $$a \le x \le b, g_1(x) \le y \le g_2(x)$$
    the triple integral over the solid may be expressed by the iterated integral:
    $$\iiint\limits_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$
  - Other orders of integration can be attained by using shadows in other coordinate planes and/or swapping order of integration for the shadow.
- Volumes of Regions in Space
  - The volume of a solid D in space is
    $$V = \iiint\limits_D dV = \iiint\limits_D 1\,dV$$
- Average Value of a Function of Three Variables
  - The average value of $f(x,y,z)$ over the solid D is defined to be
    $$\text{Avg Val} = \frac{1}{\text{volume of } D} \iiint\limits_D f(x,y,z)\,dV$$
- Triple Integral Properties
  - The properties for double integrals in Section 13.2 similarly hold for triple integrals.
- **Suggested Exercises for 13.5:**
  - Evaluating triple integrals: 7-20
  - Finding volumes of solids: 23-36
  - Finding the average value of functions: 37-40

13.8 Substitution in Multiple Integrals

- Transformations
  - Two similar regions in 2D space can be transformed by a "nice" pair of functions
    $$\mathbf{r}(u,v)=\mathbf{r}(\mathbf{s})=\langle x(\mathbf{s}),y(\mathbf{s})\rangle=\langle x(u,v),y(u,v)\rangle$$
    that map points in a uv plane to the xy plane.
  - Two similar solids in 3D space can be transformed by a "nice" triple of functions
    $$\mathbf{r}(u,v,w)=\mathbf{r}(\mathbf{s})=\langle x(\mathbf{s}),y(\mathbf{s}),z(\mathbf{s})\rangle=\langle x(u,v,w),y(u,v,w),z(u,v,w)\rangle$$
    that map points in a uvw space to the xyz space.
- The Jacobian
  - The Jacobian of a 2D transformation given by $\mathbf{r}(u,v)$ is the determinant
    $$\mathbf{r}_J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\frac{\partial\mathbf{r}}{\partial\mathbf{s}}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$
  - The Jacobian of a 3D transformation given by $\mathbf{r}(u,v,w)$ is the determinant
    $$\mathbf{r}_J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\frac{\partial\mathbf{r}}{\partial\mathbf{s}}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$
- 2D Substitution
  - Suppose that the region R in the xy-plane is the result of applying the transformation $\mathbf{r}(u,v)$ to the region G in the uv-plane.
  - Then it follows that
    $$\iint_R f(x,y)\,dx\,dy=\iint_G f(x(u,v),y(u,v))|\mathbf{r}_J(u,v)|\,du\,dv$$

- 3D Substitution
  - Suppose that the solid D in xyz space is the result of applying the transformation $\mathbf{r}(u, v, w)$ to the region H in uvw space.
  - Then it follows that
$$\iiint_D f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_H f(x(u, v, w), y(u, v, w), z(u, v, w))|\mathbf{r}_J(u, v, w)|\, du\, dv\, dw$$

- **Suggested Exercises for 13.8:**
  - 2D Jacobians, Transformations, and substitutions: 1-10

13.4 Double Integrals in Polar Form

- Integrating over Regions expressed using Polar Coordinates
  - The polar coordinate transformation

    $$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$

    from polar G into Cartesian R yields

    $$\iint_R f(x,y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- **Suggested Exercises for 13.4:**
  - Changing Cartesian integrals to polar integrals: 1-16
  - Finding integrals over polar regions: 17-22

13.7 Triple Integrals in Cylindrical and Spherical Coordinates

- Cylindrical Coordinates
  - The cylindrical coordinate transformation

    $$\mathbf{r}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

    from cylindrical H into Cartesian D yields

    $$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

- Spherical Coordinates
  - The spherical coordinate transformation

    $$\mathbf{r}(\rho,\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

    from spherical H into Cartesian D yields

    $$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

- **Suggested Exercises for 13.7:**
  - Cylindrical coordinate integrals: 1-20
  - Finding integrals over polar regions: 21-38

14.1 Line Integrals

- Line Integrals with Respect to Arclength
  - The area of the ribbon with base along the curve C in xyz space and height given by $f(x, y, z)$ is given by the **line integral of** $f(x, y, z)$ **over** C **with respect to arclength** s:

$$\int\limits_C f(x, y, z)\,ds$$

  - Arclength line integrals can be evaulated by finding a smooth parametrization $\mathbf{r}(s)$ of the curve C with respect to arclength s for $a \leq s \leq b$:

$$\int\limits_C f(x, y, z)\,ds = \int_{s=a}^{s=b} f(x(s), y(s), z(s))\,ds$$

  - If $\mathbf{r}(t)$ is an arbitrary parametrization of C for $a \leq t \leq b$, then

$$\int\limits_C f(x, y, z)\,ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t))|\mathbf{v}(t)|\,dt$$

- Additivity

$$\int\limits_{C_1+C_2} f\,ds = \int\limits_{C_1} f\,ds + \int\limits_{C_2} f\,ds$$

- Reversing Arclength Line Integrals

$$\int\limits_C f\,ds = \int\limits_{-C} f\,ds$$

- **Suggested Exercises for 14.1:**
  - Identifying vector equations for graphs: 1-8
  - Evaluating line integrals: 9-22

14.2 Vector Fields, Work, Circulation, and Flux

- Line Integrals with Respect to Variables
  - The net projected area of the ribbon with base curve C and height $f(x, y, z)$ with respect to the x-axis is given by the **line integral of** $f(x, y, z)$ **over** C **with respect to** x:
    $$\int_C f(x, y, z)\, dx$$
    (similar for y, z)
  - Line integrals with respect to variables can be evaluated by finding a parametrization $\mathbf{r}(t)$ for the curve C:
    $$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t)) \frac{dx}{dt}\, dt$$
  - Such integrals have the property
    $$\int_{-C} f\, dx = -\int_C f\, dx$$
- Vector Fields
  - A **vector field** is a function
    $$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$$
    ($\mathbf{F} = \langle M, N, P\rangle$ for short) which assigns a vector to each point in its domain.
  - Gradient functions $\nabla f = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$ and transformations $\langle x(u, v, w), y(u, v, w), z(u, v, w)\rangle$ are examples of vector fields.
- Line Integrals of Vector Fields
  - The **line integral of** $\mathbf{F} = \langle M, N, P\rangle$ **over** C is given by
    $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dx + P\, dz$$
    gives the sum of the line integrals of each component of $\mathbf{F}$ with respect to each variable x, y, z.

- These line integrals can be calculated by using parametrizations of C:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dx + P\,dz = \int_a^b \left(M\frac{dx}{dt} + N\frac{dx}{dt} + P\frac{dz}{dt}\right)\,dt$$

$$= \int_a^b \mathbf{F}\cdot\mathbf{v}\,dt = \int_a^b \mathbf{F}\cdot\mathbf{T}\,ds$$

- It follows that

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$$

- Work over a Smooth Curve
  - Work is given by the product of force and displacement:

$$W = \mathbf{F}\cdot\mathbf{D}$$

  - So work over a smooth curve can be approximated by the Riemann sum:

$$W \approx \sum_{i=1}^{n} \mathbf{F}(x_i, y_i, z_i)\cdot\Delta\mathbf{r_i}$$

  - We limit this sum to infinity to define work over a smooth curve:

$$W = \int_C \mathbf{F}\cdot d\mathbf{r}$$

- Flow
  - The **flow** of a fluid along a curve C is defined to be the line integral

$$\text{Flow} = \int_C \mathbf{F}\cdot d\mathbf{r}$$

  - If C is closed (its starting point and ending point are the same), then the flow is also known as the **circulation**.

- Flux (2D)
  - The two-dimenstional **flux** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds$$

    where $\mathbf{n}$ is the outward unit normal vector to C.
  - If C is oriented counter-clockwise, then

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx = \int_a^b \left(M \frac{dy}{dt} - N \frac{dx}{dt} \right) dt$$

- **Suggested Exercises for 14.2:**
  - Work over a curve: 7-22
  - Circulation, flow, and flux: 23-28, 37-40

14.3 Path Independence, Potential Functions, and Conservative Fields

- Several Equivalencies for Conservative Fields

  The following are all equivalent for piecewise smooth curves and vector fields with continuous first derivatives:

  - $\mathbf{F} = \langle M, N, P \rangle$ is a **conservative field**.
  - $\mathbf{F} \cdot d\mathbf{r} = M\,dx + N\,dy + P\,dz$ is **exact**.
  - $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent**: the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the endpoints of the curve C.
  - There exists a **potential function** f such that $\nabla f = \mathbf{F}$.
  - (Closed Loop Property of Conservative Fields)
    $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed loop C in D.
  - (Fundamental Theorem of Line Integrals)
    $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$ for every path C connecting A to B.
  - (Component Test for Conservative Fields)
    $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$, $\dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$, and $\dfrac{\partial N}{\partial x} = \dfrac{\partial N}{\partial y}$.

- **Suggested Exercises for 14.3:**

  - Determining if a field is conservative: 1-6
  - Finding potential functions: 7-12
  - Evaluating integrals of differential forms: 13-22

14.4 Green's Theorem in the Plane

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Divergence
  - The **divergence** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by

  $$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \mathbf{F}$$

    In physics, divergence is often called the **flux density**.

- Spin
  - The **spin** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by

  $$\operatorname{spin} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

    In physics, spin is often called the **circulation density**.
  - Spin is also the **k-component of curl**, defined in a later section.

- Simple Curves
  - A curve which does not cross itself is said to be **simple**.

- Green's Theorem in the Plane
  - Let C be a piecewise smooth, simple closed curve enclosing the region R and oriented counter-clockwise. Let $\mathbf{F} = \langle M, N \rangle$ be a vector field for which M, N have continuous first partial derivatives in an open region containing R. Then:

  $$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \operatorname{div} \mathbf{F} \, dA$$

  $$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R \operatorname{spin} \mathbf{F} \, dA$$

- **Suggested Exercises for 14.4:**
  - Using Green's Theorem to find circulation and flux: 5-14
  - Using Green's Theorem to evaluate line integrals: 17-20

14.5 Surfaces and Area

- Parametrization of Surfaces
  - Vector functions of two variables
    $$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$
    may be used to parametrize surfaces in xyz space.
- Smooth Vector Functions
  - A surface parametrized by $\mathbf{r}(u,v)$ is called **smooth** if
    $$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$
    are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ on the interior of the surface.
- Surface Area of a Parametrized Surface
  - The area of a smooth surface with parametrizing vector function $\mathbf{r}(u,v)$ for a region R in the uv plane is given by
    $$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$
- Implicit Surface
  - Level surfaces $F(x,y,z) = c$ are sometimes called **implicit surfaces**.
  - If $\mathbf{p}$ is a unit vector normal a coordinate plane, then the surface area defined by $F(x,y,z)$ bounded by the cylinder given by a region R in that coordinate plane is
    $$\iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA$$
- Surface Area Differential
  - The integral $\iint_S d\sigma$ is used to represent surface area, and $d\sigma$ is known as the surface area differential.
    $$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA$$

- **Suggested Exercises for 14.5:**
  - Finding parametrizations of surfaces: 1-16
  - Finding surface area: 17-26

14.6 Surface Integrals and Flux

- Surface Integrals
  - The **surface integral** of a function $G(x, y, z)$ over a surface S is given by

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x(u,v), y(u,v), z(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

$$= \iint_R G(x,y,z)\frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\,dA$$

- Orientable Surfaces
  - A surface is said to be **orientable** if it is "two-sided" - there exists a continuous normal unit vector field $\mathbf{n}$ to the surface.
- Flux in Three Dimensions
  - The flux of a three dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is given by the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$

- **Suggested Exercises for 14.6:**
  - Evaluating surface integrals: 1-14
  - Three-dimensional flux: 15-24

14.7 Stokes' Theorem

- Curl
  - The **curl** of a vector field $\mathbf{F}$ is defined as
    $$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \operatorname{curl}\mathbf{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$
  - The counterclockwise spin with respect to a unit vector $\mathbf{u}$ is given by
    $$\operatorname{spin}_{\mathbf{u}}\mathbf{F} = \operatorname{curl}\mathbf{F} \cdot \mathbf{u} = \nabla \times \mathbf{F} \cdot \mathbf{u}$$
  - In particular, in 2D:
    $$\operatorname{spin}\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \operatorname{spin}_{\mathbf{k}}\mathbf{F}$$
    and in 3D:
    $$\operatorname{curl}\mathbf{F} = \left\langle \operatorname{spin}_{\mathbf{i}}\mathbf{F}, \operatorname{spin}_{\mathbf{j}}\mathbf{F}, \operatorname{spin}_{\mathbf{k}}\mathbf{F} \right\rangle$$
- Stokes' Theorem
  - If a curve C in $\mathbb{R}^3$ is the boundary of a surface S, and we want to compute the counter-clockwise circulation with respect to unit normal vectors $\mathbf{n}$ on the surface, we may use
    $$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \operatorname{spin}_{\mathbf{n}}\mathbf{F}\, d\sigma = \iint_S (\operatorname{curl}\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$
- Identities and Properties
  - Due to the Mixed Derivative Theorem,
    $$\operatorname{curl}\nabla f = \nabla \times \nabla f = \mathbf{0}$$
  - If $\nabla \times \mathbf{F} = \mathbf{0}$ for every point in a region D, then
    $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$$
    for every curve C and surface S within D.
- **Suggested Exercises for 14.7:**
  - Using Stokes' Theorem: 1-10

14.8 Divergence Theorem and a Unified Theory

- Divergence Theorem
  - Divergence in $\mathbb{R}^3$ is defined as

$$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$

  - The Divergence Theorem lets us measure the flux on a closed surface S by integrating over the divergence within its bounded region D:

$$\text{Flux} = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint\limits_D \operatorname{div}\mathbf{F}\, dV = \iiint\limits_D \nabla \cdot \mathbf{F}\, dV$$

- **Suggested Exercises for 14.8:**
  - Using the Divergence Theorem: 5-16