

Sections 12.1 - 12.2 Overview

- Three-Dimensional Coordinates

- Distance between points in 3D space

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Simple planes in 3D Space

$$x = a, y = b, z = c$$

- Spheres in 3D Space

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

- Vectors

- Definition of a Vector

- * A vector $\mathbf{v} = \overrightarrow{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
- * Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
- * We often assume the initial point lays at the origin.

- Component Form

The vector with initial point at $(0, 0, 0)$ and terminal point at (v_x, v_y, v_z) is represented by

$$\langle v_x, v_y, v_z \rangle$$

- 2D and 3D Vectors

$$\langle a, b \rangle = \langle a, b, 0 \rangle$$

- Position Vector

If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c \rangle$ is its **position vector**.

We assume $\langle a, b, c \rangle = \langle a, b, c \rangle$.

- Vector Between Points

The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is

$$\mathbf{P_1P_2} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \vec{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations

- * Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

- * Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

- Unit Vectors

- * A **unit vector** or **direction** is any vector whose length is 1.

- * Standard unit vectors

- $\mathbf{i} = \langle 1, 0, 0 \rangle$
- $\mathbf{j} = \langle 0, 1, 0 \rangle$
- $\mathbf{k} = \langle 0, 0, 1 \rangle$

- * Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

- * Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

12.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$$

- Angle between vectors

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

- Orthogonal Vectors

- \mathbf{u}, \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
- \mathbf{u}, \mathbf{v} are orthogonal if the angle between them is $\frac{\pi}{2} = 90^\circ$
- $\mathbf{0}$ is orthogonal to every vector

- Dot Product Properties

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$

- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos \theta$$

12.4 The Cross Product

- Determinants

- 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- 3x3 Determinant

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_1 c_3) \end{aligned}$$

- Cross Product

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \end{aligned}$$

Shortcut “long multiplication” method:

$$\frac{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}{\begin{vmatrix} u_2 v_3 - u_3 v_2 & u_3 v_1 - u_1 v_3 & u_1 v_2 - u_2 v_1 \end{vmatrix}} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- Right-Hand Rule

- A method for determining a special orthogonal direction used throughout mathematics and physics in 3D space, with respect to an ordered pair of vectors \mathbf{u}, \mathbf{v}
- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u}, \mathbf{v} according to the Right-Hand Rule.

- Cross Product Magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

The area of the parallelogram determined by \mathbf{u}, \mathbf{v} is $|\mathbf{u} \times \mathbf{v}|$.

- Parallel Vectors

- \mathbf{u}, \mathbf{v} are parallel if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$
- \mathbf{u}, \mathbf{v} are parallel if the angle between them is $0 = 0^\circ$ or $\pi = 180^\circ$
- $\mathbf{0}$ is parallel to every vector

- Cross Product Properties

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products

1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

The standard unit vectors are known as a “right handed frame”.

- Torque

$$\begin{aligned}\vec{\tau} &= \mathbf{r} \times \mathbf{F} \\ |\vec{\tau}| &= |\mathbf{r}||\mathbf{F}| \sin \theta\end{aligned}$$

- Triple Scalar (or “Box”) Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelepiped determined by the three vectors.

12.5 Equations of Lines and Planes

- Vector Equation and Parametric Equations for a Line

$$\mathbf{r}(t) = \mathbf{P}_0 + t\mathbf{v}$$

$$x = x_0 + At, y = y_0 + Bt, z = z_0 + Ct$$

for $-\infty < t < \infty$

- Symmetric Equations for a Line

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{P}_0\mathbf{P}_1) = (1 - t)\mathbf{P}_0 + t\mathbf{P}_1$$

for $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = D$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

- Angle of Intersection of Two Planes

$$\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

12.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
 - To sketch a 3D surface, sketch planar cross-sections
 - * $z = c$ is parallel to xy plane
 - * $y = b$ is parallel to xz plane
 - * $x = a$ is parallel to yz plane
- Cylinders
 - A **cylinder** is any surface generated by considering parallel lines passing through a planar curve.
 - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
 - A **quadric surface** is any surface defined by a second degree equation of x, y, z .
 - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
 - Cross-sections in the coordinate planes include
 - * Three ellipses
- Elliptical Cone
 - Cross-sections in the coordinate planes include
 - * Two double-lines
 - * One point (with parallel ellipses)
- Elliptical Paraboloid
 - Cross-sections in the coordinate planes include
 - * Two parabolas
 - * One point (with parallel ellipses)

- Hyperbolic Paraboloid
 - Cross-sections in the coordinate planes include
 - * Two parabolas (with parallel parabolas)
 - * One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
 - Cross-sections in the coordinate planes include
 - * Two hyperbolas
 - * One ellipsis (with parallel ellipses)
- Hyperboloid of Two Sheets
 - Cross-sections in the coordinate planes include
 - * Two hyperbola
 - * One empty cross-section (with parallel ellipses)

13.1 Vector Functions and Space Curves

- Curves, Paths, and Vector Functions

- A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

$$x = x(t), y = y(t), z = z(t)$$

or with a **vector function**

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

- $x(t), y(t), z(t)$ are called **component functions**

- Vector Function Limits

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

- Continuity of Vector Functions

- The function $\mathbf{r}(t)$ is **continuous** if

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$$

for all a in its domain.

- $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

13.2 Derivatives and Integrals of Vector Functions

- Derivatives of Vector Functions

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \langle f'(t), g'(t), h'(t) \rangle$$

- $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of t is in its domain.
- $\mathbf{r}'(a)$ is a **tangent vector** to the curve where $t = a$
- The **tangent line** to a curve at $t = a$:

$$\mathbf{l}(t) = \mathbf{r}(a) + t\mathbf{r}'(a)$$

- Differentiation Rules for Vector Functions

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df} \frac{df}{dt}$$

- Derivative of a Constant Length Vector Function

- If $|\mathbf{r}(t)| = c$ always, then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

- Thus the derivative of a constant length vector function is perpendicular to the original.

- Antiderivatives of Vector Functions

- If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.
- The **indefinite integral** $\int \mathbf{r}(t) dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}$$
$$\int \mathbf{r}(t) dt = \left\langle \int x(t) dt, \int y(t) dt, \int z(t) dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle$$
$$\int_a^b \mathbf{r}(t) dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Differential Vector Equations

- If we know $\mathbf{r}'(t)$ and $\mathbf{r}(a)$ for some $t = a$, then

$$\mathbf{r}(t) = \int_a^t \mathbf{r}'(t) dt + \mathbf{r}(a)$$

13.3 Arc Length and Curvature

- Arc Length along a Space Curve

$$L = \int_a^b \left| \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{r}'(t)| dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{r}'(u)| du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

- Unit Tangent Vector

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$$

$$\mathbf{T}(t) = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

- Curvature

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{|d\mathbf{r}/dt|} = \frac{|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}|}{|\frac{d\mathbf{r}}{dt}|^3}$$

For $y = f(x)$:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

- Principal Unit Normal Vector

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

The triple $\mathbf{T}, \mathbf{N}, \mathbf{B}$ forms a right-handed frame.

13.4 Motion in Space: Velocity and Acceleration

- Position, Velocity, and Acceleration

- Position: $\mathbf{r}(t)$
- Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
- Speed: $v(t) = |\mathbf{v}(t)| = \frac{ds}{dt}$
- Direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
- Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

- Ideal Projectile Motion

$$\begin{aligned}\mathbf{a}(t) &= \langle 0, -g \rangle \\ \mathbf{v}(t) &= \langle v_0 \cos \alpha, -gt + v_0 \sin \alpha \rangle \\ \mathbf{r}(t) &= \left\langle (v_0 \cos \alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t \right\rangle\end{aligned}$$

- Tangential and Normal Components of Acceleration

$$\mathbf{a} = \left(\frac{d^2 s}{dt^2} \right) \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N} + 0\mathbf{B}$$

- Tangential component

$$a_T = \frac{d^2 s}{dt^2} = v'$$

- Normal component

$$a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa v^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

14.1 Functions of Several Variables

- Functions of Two Variables

- A **function f of two variables** is a rule which assigns a real number $f(x, y)$ to each pair of real numbers (x, y) in its **domain**

$$\text{dom}(f) \subseteq \mathbb{R}^2$$

The set of values f takes on is its **range**

$$\text{ran}(f) = \{f(x, y) : (x, y) \in \text{dom}(f)\}$$

- The **level curve** for each $k \in \text{ran}(f)$ is given by the equation

$$f(x, y) = k$$

- The **graph** of f is a surface in 3D space which visualizes the function, given by the equation $z = f(x, y)$.

- Functions of Three Variables

- A **function f of three variables** is a rule which assigns a real number $f(x, y, z)$ to each pair of real numbers (x, y, z) in its **domain**

$$\text{dom}(f) \subseteq \mathbb{R}^3$$

The set of values f takes on is its **range**

$$\text{ran}(f) = \{f(x, y, z) : (x, y, z) \in \text{dom}(f)\}$$

- The **level surface** for each $k \in \text{ran}(f)$ is given by the equation

$$f(x, y, z) = k$$

- Alternate Forms

- We may also consider functions of the form $f(x_1, x_2, \dots) = f(P) = f(\mathbf{r})$.
- If $P = (x, y)$ and $\mathbf{r} = \langle x, y \rangle$, then $f(x, y) = f(P) = f(\mathbf{r})$.
- If $P = (x, y, z)$ and $\mathbf{r} = \langle x, y, z \rangle$, then $f(x, y, z) = f(P) = f(\mathbf{r})$.

14.2 Limits and Continuity

- Limits

- If the value of the function $f(P)$ becomes arbitrarily close to the number L as vectors P close to P_0 are plugged into the function, then the **limit of $f(P)$ as P approaches P_0** is L :

$$\lim_{P \rightarrow P_0} f(P) = L$$

- For functions of two or three variables:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

$$\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x,y,z) = L$$

- Showing a Limit DNE

- In order for a limit $\lim_{P \rightarrow P_0} f(x,y)$ to exist, the values of f must approach L no matter which direction we approach P_0 .
- Choose $y = g(x)$ and $y = h(x)$ where P_0 lays on both graphs. If

$$\lim_{x \rightarrow x_0} f(x, g(x)) \neq \lim_{x \rightarrow x_0} f(x, h(x))$$

then $\lim_{P \rightarrow P_0} f(x,y)$ DNE.

- Or choose $x = g(y)$ and $x = h(y)$ where P_0 lays on both graphs. If

$$\lim_{y \rightarrow y_0} f(g(y), y) \neq \lim_{y \rightarrow y_0} f(h(y), y)$$

then $\lim_{P \rightarrow P_0} f(x,y)$ DNE.

- Limit Laws

$$\lim_{P \rightarrow P_0} (f(P) \pm g(P)) = \lim_{P \rightarrow P_0} f(P) \pm \lim_{P \rightarrow P_0} g(P)$$

$$\lim_{P \rightarrow P_0} (f(P) \cdot g(P)) = \lim_{P \rightarrow P_0} f(P) \cdot \lim_{P \rightarrow P_0} g(P)$$

$$\lim_{P \rightarrow P_0} (kf(P)) = k \lim_{P \rightarrow P_0} f(P)$$

$$\lim_{P \rightarrow P_0} \frac{f(P)}{g(P)} = \frac{\lim_{P \rightarrow P_0} f(P)}{\lim_{P \rightarrow P_0} g(P)}$$

$$\lim_{P \rightarrow P_0} (f(P))^{r/s} = \left(\lim_{P \rightarrow P_0} f(P) \right)^{r/s}$$

- Computing Limits

- Variables not involved in a limit may be eliminated:

$$\lim_{P \rightarrow P_0} f(x) = \lim_{x \rightarrow x_0} f(x)$$

- Due to the Limit Laws, many limits follow the “just plug it in” rule.
- If plugging in results in a zero in a denominator, use factoring, perhaps with conjugates.
- L'Hopital's Rule does not apply for multiple variable limits.

- Continuity

- A function $f(P)$ is **continuous** if $\lim_{P \rightarrow P_0} f(P) = f(P_0)$ for all points P_0 in its domain.
- If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

14.3 Partial Derivatives

- Partial Derivatives

- For a function f of two variables (x, y) :

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

- To compute partial derivatives with respect to a variable, treat all other variables as constants and differentiate as normal.
- Functions of more than two variables behave similarly. For $T(x, y, z)$:

$$\frac{\partial T}{\partial z} = T_z(x, y, z) = \lim_{h \rightarrow 0} \frac{T(x, y, z + h) - T(x, y, z)}{h}$$

- Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 g}{\partial z^2} = \frac{\partial}{\partial z} \left[\frac{\partial g}{\partial z} \right] = (g_z)_z = g_{zz}$$

- Mixed Derivative Theorem

- For many naturally occurring functions:

$$f_{xy} = f_{yx}$$

14.4 Tangent Planes and Linear Approximations

- Tangent Plane to $z = f(x, y)$ at $(a, b, f(a, b))$

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Linearization of $f(x, y)$ at (a, b)

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Differentiability and a Sufficient Condition

- A multi-variable function f is **differentiable** at a point if its linearization approximates the value of the function near that point.
- If f_x, f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

- Linear Approximation

If f is differentiable at (a, b) , then

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

14.5 The Chain Rule

- Gradient Vector Function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Nested Functions

- If f is a function of $\mathbf{r} = \langle x, y, z \rangle$, and x, y, z are functions of t , then we say x, y, z are intermediate variables and may consider the following composed function of t :

$$f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

- If f is a function of $\mathbf{r} = \langle x, y, z \rangle$, and x, y, z are functions of $\mathbf{s} = \langle t, u, v \rangle$, then we say x, y, z are intermediate variables and may consider the following composed function of t, u, v :

$$f(\mathbf{r}(\mathbf{s})) = f(x(t, u, v), y(t, u, v), z(t, u, v))$$

- Chain Rule

- For functions of the form $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

- For functions of the form $f(\mathbf{r}(\mathbf{s})) = f(x(t, u, v), y(t, u, v), z(t, u, v))$:

$$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$$

- Differentiation by Substitution

- The multi-variable Chain Rule can be avoided by “plugging in” functions and using single-variable calculus.

- Total Derivative

- If f is a function of x, y, z , and y, z are also functions of x , then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial z} \frac{dz}{dx}$$

- Implicit Differentiation

- If $f(x, y) = c$ defines y as a function of x , then

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{f_x}{f_y}$$

- Tree Diagram for the Chain Rule

- The tree diagram for the chain rule can be used to generate the chain rule.
- It also holds for multiple levels of intermediate variables.

14.6 Directional Derivatives and the Gradient Vector

- Directional Derivative

- The **directional derivative** of f for the unit vector \mathbf{u} is

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

- The maximum value of $D_{\mathbf{u}}f$ at a fixed point P_0 is $|\nabla f(P_0)|$, which occurs when $\mathbf{u} = \frac{\nabla f(P_0)}{|\nabla f(P_0)|}$.

- Normal Vector to Level Curves and Surfaces

- The gradient vectors ∇f are normal vectors to the level curves $f(x, y) = k$ for every (x, y) in the domain of f .
- The gradient vectors ∇f are normal vectors to the level surfaces $f(x, y, z) = k$ for every (x, y, z) in the domain of f .

14.7 Maximum and Minimum Values

- Local Maximum and Minimum Values

- Let f be a function of many variables defined near the point P_0 .
 - f has a **local maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value of f near P_0
 - f has a **local minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value of f near P_0

- Critical Points

- If P_0 is a point in the domain of f and

$$\nabla f(P_0) = 0 \text{ or } \nabla f(P_0) \text{ DNE}$$

then P_0 is called a **critical point**.

- Critical points occur when the tangent plane is horizontal or DNE.
- The local maximum and minimum values of a function always occur at critical points.

- Saddle Points

- Not every critical point gives a local extreme value.
- The **saddle points** of f are the critical points which don't yield local extreme values.

- Discriminant Function

- The **discriminant** of f with variables x, y is the function

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second Derivative Test for Local Extreme Values of $f(x, y)$

Let (a, b) be a critical point of f where ∇f is defined.

- If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $f_D(a, b) < 0$, then f has a saddle point at (a, b) .
- If $f_D(a, b) = 0$, then the test is inconclusive.

- Absolute Maximum and Minimum Values
 - Let f be a function of many variables.
 - * f has an **absolute maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value in the range of f
 - * f has an **absolute minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value in the range of f
 - Every continuous function of many variables with a closed and bounded domain has an absolute maximum and minimum value.
- Finding Absolute Max/Min of $f(x, y)$ on a Closed and Bounded Region D
 - The following points are candidates for giving the absolute extrema:
 - * Critical points of f within D .
 - * Critical points for a function which gives part of the boundary of D .
 - * Corners of D .
 - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.

14.8 Lagrange Multipliers

- The Method of Lagrange Multipliers

- The **Method of Lagrange Multipliers** says that if f is a function of many variables which has an absolute max/min value on the restriction $g(P) = k$ where $\nabla g \neq 0$, then the absolute max/min occurs at a point P where

$$\nabla f(P) = \lambda \nabla g(P) \text{ and } g(P) = k$$

for some real number λ .

- If two constraints $g(P) = k$ and $h(P) = l$ are given, then we find where

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P) \text{ and } g(P) = k \text{ and } h(P) = l$$

for some real numbers λ, μ .

15.1 Double Integrals over Rectangles

- Double Integral

- We define the **double integral** of a function $f(x, y)$ over a region R to be

$$\iint_R f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{n,i}, y_{n,i}) \Delta A_{n,i}$$

where for each positive integer n we've defined a way to partition R into n pieces

$$\Delta R_{n,1}, \Delta R_{n,2}, \dots, \Delta R_{n,n}$$

where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

$$\lim_{n \rightarrow \infty} \max(\Delta A_{n,i}) = 0$$

- Since for $f(x, y) \geq 0$,

$$\sum_{i=1}^n f(x_{n,i}, y_{n,i}) \Delta A_{n,i}$$

is an approximation of the volume under $z = f(x, y)$ and over R , the double integral is used to define the precise volume.

- If f is not always positive, then the double integral represents **net volume**: volume above the xy -plane minus volume below the xy -plane.

- Midpoint Rule for Approximating Rectangular Double Integrals

- For the rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

we may approximate the double integral by partitioning the rectangle into a grid of $m \times n$ rectangular pieces all with area ΔA and evaluating:

$$\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A$$

where (\bar{x}_i, \bar{y}_j) is the midpoint of the $i \times j$ rectangle.

15.2 Iterated Integrals

- Volume as Integral of Area

- If $A(x)$ is the area of a solid's cross-section, then the solid's volume is

$$V = \int_a^b A(x) dx$$

- Iterated Integrals over Rectangles

- A double integral over a rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

can be expressed as the **iterated integrals**:

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

15.3 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions

- For **Type I** regions which may be expressed as

$$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

a double integral over R may be expressed as the iterated integral:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

- For **Type II** regions which may be expressed as

$$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

a double integral over R may be expressed as the iterated integral:

$$\iint_R f(x, y) dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- Finding Limits of Integration

1. Sketch the region and label bounding curves
2. Determine if the region is Type I or Type II by identifying (I) bottom/top curves $y = g_1(x), y = g_2(x)$ or (II) left/right curves $x = h_1(y), x = h_2(y)$.

For Type I:

3. Use the leftmost and rightmost x -values in the region a, b to complete the iterated integral:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

For Type II:

3. Use the bottommost and topmost y -values in the region c, d to complete the iterated integral:

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- Swapping Variables of Integration

- You can only swap the order of integration of an iterated integral by drawing the region and reinterpreting it as a region of the opposite Type.

- Additivity

If R can be split into two regions R_1, R_2 , then

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

- Average Value of Two-Variable Functions

- The average value of a two-variable function f over a region R is defined to be

$$\frac{1}{\text{Area of } R} \iint_R f(x, y) dA$$

- Area as a Double Integral

- The area of a region R in the plane is

$$A = \iint_R dA = \iint_R 1 dA$$

15.7 Triple Integrals

- Triple Integral

- We define the **triple integral** of a function $f(x, y, z)$ over a solid D to be

$$\iiint_D f(x, y, z) dV = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{n,i}, y_{n,i}, z_{n,i}) \Delta V_{n,i}$$

where for each positive integer n we've defined a way to partition D into n pieces

$$\Delta D_{n,1}, \Delta D_{n,2}, \dots, \Delta D_{n,n}$$

where $\Delta D_{n,i}$ has volume $\Delta V_{n,i}$, contains the point $(x_{n,i}, y_{n,i}, z_{n,i})$, and

$$\lim_{n \rightarrow \infty} \max(\Delta V_{n,i}) = 0$$

- Iterated Integral for Rectangular Boxes

- The triple integral over the rectangular box

$$D : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2, c_1 \leq z \leq c_2$$

can be expressed as the **iterated integrals**:

$$\begin{aligned} \iiint_D f(x, y, z) dV &= \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z) dz dy dx \\ &= \int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z) dx dz dy = \int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z) dy dz dx = \dots \end{aligned}$$

- Iterated Integral for Generated Solids

- If the solid D is determined by the bottom/top surfaces

$$h_1(x, y) \leq z \leq h_2(x, y)$$

and has shadow R in the xy -plane, then a triple integral over D can be expressed as:

$$\iiint_D f(x, y, z) dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z) dz \right] dA$$

- In general:

$$\iiint_D f(x, y, z) dV = \iint_R \left[\int_{\text{bottom surface}}^{\text{top surface}} f(x, y, z) d\Box \right] dA$$

where \Box is chosen from x, y, z to be the “up” orientation.

- Additivity

If D can be split into two regions D_1, D_2 , then

$$\iint_D f(x, y, z) dV = \iint_{D_1} f(x, y, z) dV + \iint_{D_2} f(x, y, z) dV$$

- Average Value of Three-Variable Functions

- The average value of a three-variable function f over a solid D is defined to be

$$\frac{1}{\text{Volume of } D} \iiint_R f(x, y, z) dV$$

- Volume as a Triple Integral

- The volume of a solid D in space is

$$V = \iiint_D dV = \iiint_D 1 dV$$

15.10 Change of Variables in Multiple Integrals

- Transformations

- Two similar regions in 2D space can be transformed by a “nice” pair of functions

$$\mathbf{r}(u, v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}) \rangle = \langle x(u, v), y(u, v) \rangle$$

that map points in a uv plane to the xy plane.

- Two similar solids in 3D space can be transformed by a “nice” triple of functions

$$\mathbf{r}(u, v, w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}) \rangle = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$$

that map points in a uvw space to the xyz space.

- The Jacobian

- The Jacobian of a 2D transformation given by $\mathbf{r}(u, v)$ is the determinant

$$\mathbf{r}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- The Jacobian of a 3D transformation given by $\mathbf{r}(u, v, w)$ is the determinant

$$\mathbf{r}_J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- 2D Substitution

- Suppose that the region R in the xy -plane is the result of applying the transformation $\mathbf{r}(u, v)$ to the region G in the uv -plane.
- Then it follows that

$$\iint_R f(x, y) dx dy = \iint_G f(x(u, v), y(u, v)) |\mathbf{r}_J(u, v)| du dv$$

- 3D Substitution

- Suppose that the solid D in xyz space is the result of applying the transformation $\mathbf{r}(u, v, w)$ to the region H in uvw space.

- Then it follows that

$$\begin{aligned} & \iiint_D f(x, y, z) \, dx \, dy \, dz \\ &= \iiint_H f(x(u, v, w), y(u, v, w), z(u, v, w)) |\mathbf{r}_J(u, v, w)| \, du \, dv \, dw \end{aligned}$$

15.4 Double Integrals in Polar Coordinates

- Integrating over Regions expressed using Polar Coordinates
 - The polar coordinate transformation

$$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta \rangle$$

from polar G into Cartesian R yields

$$\iint_R f(x, y) dA = \iint_G f(r \cos \theta, r \sin \theta) r dr d\theta$$

15.8 Triple Integrals in Cylindrical Coordinates

- Cylindrical Coordinates
 - The cylindrical coordinate transformation

$$\mathbf{r}(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle$$

from cylindrical H into Cartesian D yields

$$\iiint_D f(x, y, z) dV = \iiint_H f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

15.9 Triple Integrals in Spherical Coordinates

- Spherical Coordinates
 - The spherical coordinate transformation

$$\mathbf{r}(\rho, \phi, \theta) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$$

from spherical H into Cartesian D yields

$$\iiint_D f(x, y, z) dV = \iiint_H f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

16.1 Vector Fields

- Vector Fields

- A **vector field** assigns a vector to each point in 2D or 3D space.

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle = \langle M(\mathbf{r}), N(\mathbf{r}) \rangle$$

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle = \langle M(\mathbf{r}), N(\mathbf{r}), P(\mathbf{r}) \rangle$$

- Gradient Vector Field

- The gradient vector field $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ assigns vectors whose directions are normal to level curves and whose magnitudes are equal to the maximal directional derivative at the point.
- The gradient vector field $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ assigns vectors whose directions are normal to level surfaces and whose magnitudes are equal to the maximal directional derivative at the point.

16.2 Line Integrals

- Common Curve Parametrizations

- A line segment beginning at P_0 and ending at P_1

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{P}_1 - \mathbf{P}_0), 0 \leq t \leq 1$$

- A circle centered at the origin with radius a

$$\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle, 0 \leq t \leq 2\pi \text{ (counter-clockwise)}$$

$$\mathbf{r}(t) = \langle a \sin t, a \cos t \rangle, 0 \leq t \leq 2\pi \text{ (clockwise)}$$

- A planar curve given by $y = f(x)$ from (x_0, y_0) to (x_1, y_1)

$$\mathbf{r}(t) = \langle t, f(t) \rangle, x_0 \leq t \leq x_1 \text{ (for } x_0 \leq x_1 \text{)}$$

$$\mathbf{r}(t) = \langle -t, f(-t) \rangle, x_1 \leq t \leq x_0 \text{ (for } x_1 \leq x_0 \text{)}$$

- Line Integrals with Respect to Arclength

- We define the **line integral with respect to arclength** of a function of many variables $f(\mathbf{r})$ along a curve C to be

$$\int_C f(\mathbf{r}) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\mathbf{r}_{n,i}) \Delta s_{n,i}$$

where for each positive integer n we've defined a way to partition C into n pieces

$$\Delta C_{n,1}, \Delta C_{n,2}, \dots, \Delta C_{n,n}$$

where $\Delta C_{n,i}$ has length $\Delta s_{n,i}$, contains the position vector $\mathbf{r}_{n,i}$, and

$$\lim_{n \rightarrow \infty} \max(\Delta s_{n,i}) = 0$$

- If $\mathbf{r}(t)$ is a parametrization of C for $a \leq t \leq b$, then

$$\int_C f(\mathbf{r}) ds = \int_{t=a}^{t=b} f(\mathbf{r}(t)) \frac{ds}{dt} dt$$

- Line Integrals with Respect to Variables

- Similarly, we can find the **line integral with respect to a variable** for a function of many variables $f(\mathbf{r})$ along a curve C :

$$\int_C f(\mathbf{r}) dx = \int_{t=a}^{t=b} f(\mathbf{r}(t)) \frac{dx}{dt} dt$$

- Similar definitions hold for y, z .

- Line Integrals of Vector Fields

- The **line integral of a vector field** is defined to be the line integral with respect to arclength of the dot product of the vector field $\mathbf{F}(\mathbf{r}) = \langle M(\mathbf{r}), N(\mathbf{r}), P(\mathbf{r}) \rangle$ with the unit tangent vector $\mathbf{T}(\mathbf{r})$ to the curve.

$$\int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) ds$$

- There are several ways to write and evaluate line integrals of vector fields:

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle M, N, P \rangle \cdot \langle dx, dy, dz \rangle \\ &= \int_C M dx + N dy + P dz = \int_a^b \left(M(\mathbf{r}(t)) \frac{dx}{dt} + N(\mathbf{r}(t)) \frac{dy}{dt} + P(\mathbf{r}(t)) \frac{dz}{dt} \right) dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt \end{aligned}$$

- Additivity

Let $C_1 + C_2$ represent the curve taken by moving along C_1 followed by moving along C_2 .

$$\begin{aligned} \int_{C_1+C_2} f(\mathbf{r}) ds &= \int_{C_1} f(\mathbf{r}) ds + \int_{C_2} f(\mathbf{r}) ds \\ \int_{C_1+C_2} f(\mathbf{r}) dx &= \int_{C_1} f(\mathbf{r}) dx + \int_{C_2} f(\mathbf{r}) dx \\ \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \end{aligned}$$

- Effects of Curve Orientation

Let $-C$ represent the curve taken by moving along C in the opposite direction.

$$\int_C f(\mathbf{r}) \, ds = + \int_{-C} f(\mathbf{r}) \, ds$$

$$\int_C f(\mathbf{r}) \, dx = - \int_{-C} f(\mathbf{r}) \, dx$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_{-C} \mathbf{F} \cdot d\mathbf{r}$$

- Work

- If \mathbf{F} is a vector field representing the force applied to an object as it is moved over a smooth curve C , then the **work** done by the force over that curve is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

16.3 The Fundamental Theorem for Line Integrals

- The Fundamental Theorem

- If C is any smooth curve beginning at the point A and ending at the point B , then

$$\int_C \nabla f \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$$

- If C is any smooth curve which is **closed** (begins and ends at the same point), then

$$\int_C \nabla f \cdot d\mathbf{r} = 0$$

- Conservative Fields

- We say $\mathbf{F} = \langle M, N, P \rangle$ is a conservative field if there is a **potential function** f such that $\nabla f = \mathbf{F}$.
- Line integrals of conservative fields are said to be path independent since for any curve C beginning at A and ending at B :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$$

- We can prove a field is conservative by finding its potential function or showing it satisfies the **Component Test**:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$$

14.2 Vector Fields, Work, Circulation, and Flux

- Line Integrals with Respect to Variables

- The net projected area of the ribbon with base curve C and height $f(x, y, z)$ with respect to the x -axis is given by the **line integral of $f(x, y, z)$ over C with respect to x** :

$$\int_C f(x, y, z) dx$$

(similar for y, z)

- Line integrals with respect to variables can be evaluated by finding a parametrization $\mathbf{r}(t)$ for the curve C :

$$\int_C f(x, y, z) dx = \int_a^b f(x(t), y(t), z(t)) \frac{dx}{dt} dt$$

- Such integrals have the property

$$\int_{-C} f dx = - \int_C f dx$$

- Vector Fields

- A **vector field** is a function

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$$

($\mathbf{F} = \langle M, N, P \rangle$ for short) which assigns a vector to each point in its domain.

- Gradient functions $\nabla f = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ and transformations $\langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$ are examples of vector fields.

- Line Integrals of Vector Fields

- The **line integral of $\mathbf{F} = \langle M, N, P \rangle$ over C** is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz$$

gives the sum of the line integrals of each component of \mathbf{F} with respect to each variable x, y, z .

- These line integrals can be calculated by using parametrizations of C :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz = \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_a^b \mathbf{F} \cdot \mathbf{v} dt = \int_a^b \mathbf{F} \cdot \mathbf{T} ds$$

- It follows that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Work over a Smooth Curve

- Work is given by the product of force and displacement:

$$W = \mathbf{F} \cdot \mathbf{D}$$

- So work over a smooth curve can be approximated by the Riemann sum:

$$W \approx \sum_{i=1}^n \mathbf{F}(x_i, y_i, z_i) \cdot \Delta \mathbf{r}_i$$

- We limit this sum to infinity to define work over a smooth curve:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Flow

- The **flow** of a fluid along a curve C is defined to be the line integral

$$\text{Flow} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- If C is closed (its starting point and ending point are the same), then the flow is also known as the **circulation**.

- Flux (2D)

- The two-dimensional **flux** of \mathbf{F} across C is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds$$

where \mathbf{n} is the outward unit normal vector to C .

- If C is oriented counter-clockwise, then

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx = \int_a^b \left(M \frac{dy}{dt} - N \frac{dx}{dt} \right) dt$$

- **Suggested Exercises for 14.2:**

- Work over a curve: 7-22
- Circulation, flow, and flux: 23-28, 37-40

14.3 Path Independence, Potential Functions, and Conservative Fields

- Several Equivalencies for Conservative Fields

The following are all equivalent for piecewise smooth curves and vector fields with continuous first derivatives:

- $\mathbf{F} = \langle M, N, P \rangle$ is a **conservative field**.
- $\mathbf{F} \cdot d\mathbf{r} = M dx + N dy + P dz$ is **exact**.
- $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent**: the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the endpoints of the curve C .
- There exists a **potential function** f such that $\nabla f = \mathbf{F}$.
- (Closed Loop Property of Conservative Fields)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$
for every closed loop C in D .
- (Fundamental Theorem of Line Integrals)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$
for every path C connecting A to B .
- (Component Test for Conservative Fields)

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \text{ and } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

- **Suggested Exercises for 14.3:**

- Determining if a field is conservative: 1-6
- Finding potential functions: 7-12
- Evaluating integrals of differential forms: 13-22

14.4 Green's Theorem in the Plane

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Divergence

- The **divergence** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \mathbf{F}$$

In physics, divergence is often called the **flux density**.

- Spin

- The **spin** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by

$$\operatorname{spin} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

In physics, spin is often called the **circulation density**.

- Spin is also the **k-component of curl**, defined in a later section.

- Simple Curves

- A curve which does not cross itself is said to be **simple**.

- Green's Theorem in the Plane

- Let C be a piecewise smooth, simple closed curve enclosing the region R and oriented counter-clockwise. Let $\mathbf{F} = \langle M, N \rangle$ be a vector field for which M, N have continuous first partial derivatives in an open region containing R . Then:

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n} \, ds &= \iint_R \operatorname{div} \mathbf{F} \, dA \\ \int_C \mathbf{F} \cdot \mathbf{T} \, ds &= \iint_R \operatorname{spin} \mathbf{F} \, dA \end{aligned}$$

- Suggested Exercises for 14.4:

- Using Green's Theorem to find circulation and flux: 5-14
- Using Green's Theorem to evaluate line integrals: 17-20

14.5 Surfaces and Area

- Parametrization of Surfaces

- Vector functions of two variables

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

may be used to parametrize surfaces in xyz space.

- Smooth Vector Functions

- A surface parametrized by $\mathbf{r}(u, v)$ is called **smooth** if

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ on the interior of the surface.

- Surface Area of a Parametrized Surface

- The area of a smooth surface with parametrizing vector function $\mathbf{r}(u, v)$ for a region R in the uv plane is given by

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| dA$$

- Implicit Surface

- Level surfaces $F(x, y, z) = c$ are sometimes called **implicit surfaces**.
- If \mathbf{p} is a unit vector normal a coordinate plane, then the surface area defined by $F(x, y, z)$ bounded by the cylinder given by a region R in that coordinate plane is

$$\iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA$$

- Surface Area Differential

- The integral $\iint_S d\sigma$ is used to represent surface area, and $d\sigma$ is known as the surface area differential.

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| dA = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA$$

- **Suggested Exercises for 14.5:**

- Finding parametrizations of surfaces: 1-16
- Finding surface area: 17-26

14.6 Surface Integrals and Flux

- Surface Integrals

- The **surface integral** of a function $G(x, y, z)$ over a surface S is given by

$$\begin{aligned}\iint_S G(x, y, z) d\sigma &= \iint_R G(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA \\ &= \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA\end{aligned}$$

- Orientable Surfaces

- A surface is said to be **orientable** if it is “two-sided” - there exists a continuous normal unit vector field \mathbf{n} to the surface.

- Flux in Three Dimensions

- The flux of a three dimensional vector field \mathbf{F} across an oriented surface S in the direction of \mathbf{n} is given by the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$$

- Suggested Exercises for 14.6:

- Evaluating surface integrals: 1-14
- Three-dimensional flux: 15-24

14.7 Stokes' Theorem

- Curl

- The **curl** of a vector field \mathbf{F} is defined as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \text{curl } \mathbf{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

- The counterclockwise spin with respect to a unit vector \mathbf{u} is given by

$$\text{spin}_{\mathbf{u}} \mathbf{F} = \text{curl } \mathbf{F} \cdot \mathbf{u} = \nabla \times \mathbf{F} \cdot \mathbf{u}$$

- In particular, in 2D:

$$\text{spin } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \text{spin}_{\mathbf{k}} \mathbf{F}$$

and in 3D:

$$\text{curl } \mathbf{F} = \langle \text{spin}_{\mathbf{i}} \mathbf{F}, \text{spin}_{\mathbf{j}} \mathbf{F}, \text{spin}_{\mathbf{k}} \mathbf{F} \rangle$$

- Stokes' Theorem

- If a curve C in \mathbb{R}^3 is the boundary of a surface S , and we want to compute the counter-clockwise circulation with respect to unit normal vectors \mathbf{n} on the surface, we may use

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S \text{spin}_{\mathbf{n}} \mathbf{F} \, d\sigma = \iint_S (\text{curl } \mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

- Identities and Properties

- Due to the Mixed Derivative Theorem,

$$\text{curl } \nabla f = \nabla \times \nabla f = \mathbf{0}$$

- If $\nabla \times \mathbf{F} = \mathbf{0}$ for every point in a region D , then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0$$

for every curve C and surface S within D .

- Suggested Exercises for 14.7:

- Using Stokes' Theorem: 1-10

14.8 Divergence Theorem and a Unified Theory

- Divergence Theorem

- Divergence in \mathbb{R}^3 is defined as

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$

- The Divergence Theorem lets us measure the flux on a closed surface S by integrating over the divergence within its bounded region D :

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \operatorname{div} \mathbf{F} \, dV = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

- Suggested Exercises for 14.8:

- Using the Divergence Theorem: 5-16