

Sections 12.1 - 12.2 Overview

- Three-Dimensional Coordinates

- Distance between points in 3D space

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Simple planes in 3D Space

$$x = a, y = b, z = c$$

- Spheres in 3D Space

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

- Vectors

- Definition of a Vector

- * A vector $\mathbf{v} = \overrightarrow{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
- * Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
- * We often assume the initial point lays at the origin.

- Component Form

The vector with initial point at $(0, 0, 0)$ and terminal point at (v_x, v_y, v_z) is represented by

$$\langle v_x, v_y, v_z \rangle$$

- 2D and 3D Vectors

$$\langle a, b \rangle = \langle a, b, 0 \rangle$$

- Position Vector

If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c \rangle$ is its **position vector**.

We assume $\langle a, b, c \rangle = \langle a, b, c \rangle$.

- Vector Between Points

The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is

$$\mathbf{P_1P_2} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \vec{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations

- * Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

- * Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

- Unit Vectors

- * A **unit vector** or **direction** is any vector whose length is 1.

- * Standard unit vectors

- $\mathbf{i} = \langle 1, 0, 0 \rangle$
- $\mathbf{j} = \langle 0, 1, 0 \rangle$
- $\mathbf{k} = \langle 0, 0, 1 \rangle$

- * Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

- * Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

10.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$$

- Angle between vectors

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

- Orthogonal Vectors

- \mathbf{u}, \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
- \mathbf{u}, \mathbf{v} are orthogonal if the angle between them is $\frac{\pi}{2} = 90^\circ$
- $\mathbf{0}$ is orthogonal to every vector

- Dot Product Properties

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$

- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos \theta$$

10.4 The Cross Product

- Right-hand rule

Any method for determining a special orthogonal direction used throughout mathematics and physics, with respect to an ordered pair of vectors \mathbf{u}, \mathbf{v}

- Unit Normal Vector

The vector \mathbf{n} orthogonal to an ordered pair of vectors \mathbf{u}, \mathbf{v} following the right-hand rule

- Cross Product

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta) \mathbf{n}$$

- Parallel Vectors

- \mathbf{u}, \mathbf{v} are parallel if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$
- \mathbf{u}, \mathbf{v} are parallel if the angle between them is $0 = 0^\circ$ or $\pi = 180^\circ$
- $\mathbf{0}$ is parallel to every vector

- Cross Product Properties

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products

1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

- Parallelogram Area The area of a parallelogram determined by \mathbf{u}, \mathbf{v} is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

- Determinants

- 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- 3x3 Determinant

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_1 c_3) \end{aligned}$$

- Computing Cross Products

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \end{aligned}$$

Shortcut “long multiplication” method:

$$\frac{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}{\begin{vmatrix} u_2 v_3 - u_3 v_2 & u_3 v_1 - u_1 v_3 & u_1 v_2 - u_2 v_1 \end{vmatrix}} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- Torque

$$\vec{\tau} = \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}| \sin \theta) \mathbf{n}$$

- Triple Scalar (or “Box”) Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelepiped determined by the three vectors.

- **Suggested Exercises for 10.4**

- Finding cross products: 1-14
- Finding areas and unit normal vectors using cross products: 15-18
- Finding volumes using cross products: 19-22
- Computing torque: 25-26

10.5 Lines and Planes in Space

- Vector Equation for a Line

$$\mathbf{r}(t) = \mathbf{P}_0 + t\mathbf{v}$$

for $-\infty < t < \infty$

- Parametric Equations for a Line

$$x = x_0 + tv_1, y = y_0 + tv_2, z = z_0 + tv_3$$

for $-\infty < t < \infty$

- Line Passing through a pair of points

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{P}_0\mathbf{P}_1) = (1 - t)\mathbf{P}_0 + t\mathbf{P}_1$$

for $-\infty < t < \infty$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{P}_0\mathbf{P}_1) = (1 - t)\mathbf{P}_0 + t\mathbf{P}_1$$

for $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$\mathbf{n} \cdot (\mathbf{P}_0\mathbf{P}) = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P} + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

- **Suggested Exercises for 10.5**

- Finding parametric equations for lines: 1-12
- Finding parametrizations for line segments: 13-20
- Finding equations for planes: 21-26
- Distance from a point to a line: 33-38
- Distance from a point to a plane: 39-44

10.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
 - To sketch a 3D surface, sketch planar cross-sections
 - * $z = c$ is parallel to xy plane
 - * $y = b$ is parallel to xz plane
 - * $x = a$ is parallel to yz plane
- Cylinders
 - A **cylinder** is any surface generated by moving a planar along a line normal to that plane.
 - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
 - A **quadric surface** is any surface defined by a second degree equation of x, y, z :
$$Ax^2 + Bx + Cy^2 + Dy + Ez^2 + Fz = G$$
 - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
 - Cross-sections in the coordinate planes include
 - * Three ellipses
- Elliptical Cone
 - Cross-sections in the coordinate planes include
 - * Two double-lines
 - * One point (with parallel ellipses)
 - Cross-sections parallel to the point cross-section are ellipses.

- Elliptical Paraboloid
 - Cross-sections in the coordinate planes include
 - * Two parabolas
 - * One point (with parallel ellipses)
 - Cross-sections parallel to the point cross-section are ellipses.
- Hyperbolic Paraboloid
 - Cross-sections in the coordinate planes include
 - * Two parabolas (with parallel parabolas)
 - * One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
 - Cross-sections in the coordinate planes include
 - * Two hyperbolas
 - * One ellipsis (with parallel hyperbolas)
- Hyperboloid of Two Sheets
 - Cross-sections in the coordinate planes include
 - * Two hyperbola
 - * One empty cross-section (with parallel hyperbolas)
 - Cross-sections parallel to the empty cross-section are ellipses.
- **Suggested Exercises for 10.6**
 - Identify surfaces from equations: 1-12
 - Sketching surfaces: 13-44

11.1 Vector Functions and their Derivatives

- Curves, Paths, and Vector Functions

- A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

or with a **vector function**

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

- $x(t), y(t), z(t)$ are called **component functions**

- Vector Function Limits

- If the value of the vector function $\mathbf{r}(t)$ becomes arbitrarily close to the vector \mathbf{L} as values of t close to t_0 are plugged into the function, then the **limit of $\mathbf{r}(t)$ as t approaches t_0** is \mathbf{L} , or

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$$

* (Precise definition:) Suppose there exists a function $\delta(\epsilon)$ so that for all positive numbers $\epsilon > 0$ and all numbers t where $|\mathbf{r}(t) - \mathbf{L}| < \epsilon$, it follows that $|t - t_0| < \delta(\epsilon)$. Then we say that $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$.

- Note that

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow t_0} f(t), \lim_{t \rightarrow t_0} g(t), \lim_{t \rightarrow t_0} h(t) \right\rangle$$

- Continuity of Vector Functions

- The function $\mathbf{r}(t)$ is **continuous at a point t_0** if

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

- The function $\mathbf{r}(t)$ is **continuous** if

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

for all t_0 in its domain.

– $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

- Derivatives of Vector Functions

- $\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$
- $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
- $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of t is in its domain.
- $\mathbf{r}(t)$ is **smooth** if $\mathbf{r}(t)$ is differentiable, $\mathbf{r}'(t)$ is continuous, and $\mathbf{r}'(t) \neq 0$
- $\mathbf{r}'(t_0)$ is a **tangent vector** to the curve where $t = t_0$
- The **tangent line** to a curve given by the vector function:

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

- Vectors and Physics

- Position: $\mathbf{r}(t)$
- Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
- Speed: $|\mathbf{v}(t)|$
- Direction: $\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
 - * (Remember that $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$)
- Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

- Differentiation Rules for Vector Functions

1. Constant Function Rule

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

2. Constant Multiple Rules

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

3. Sum and Difference Rules

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

4. Scalar Product Rule

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

5. Dot Product Rule

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

6. Cross Product Rule

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

7. Chain Rule

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df} \frac{df}{dt}$$

• Derivative of a Constant Length Vector Function

- If $|\mathbf{r}(t)| = c$ always, then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

- Thus the derivative of a constant length vector function is perpendicular to the original.

• Suggested Exercises for 11.1

- Position/Velocity/Acceleration Vectors: 1-14

11.2 Integrals of Vector Functions

- Antiderivatives of Vector Functions

- If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.
- The **indefinite integral** $\int \mathbf{r}(t) dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t) dt = \left\langle \int x(t) dt, \int y(t) dt, \int z(t) dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle$$

$$\int_a^b \mathbf{r}(t) dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Initial Value Problems

- If we know $\mathbf{r}'(t)$ and $\mathbf{r}(t_0)$, then

$$\mathbf{r}(t) = \mathbf{R}'(t) + \mathbf{r}(t_0) - \mathbf{R}'(t_0)$$

- Ideal Projectile Motion

- Assume the following:
 - * The acceleration acting on a projectile is $\langle 0, -g \rangle$
 - * The launch position is the origin $\langle 0, 0 \rangle = \mathbf{0}$
 - * The launch angle is α
 - * The initial velocity is \mathbf{v}_0 , and initial speed is $v_0 = |\mathbf{v}_0|$
- This results in the initial value problem:

$$\mathbf{a}(t) = \langle 0, -g \rangle$$

$$\mathbf{v}(0) = \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle$$

$$\mathbf{r}(0) = \langle 0, 0 \rangle$$

- The velocity function solves to

$$\mathbf{v}(t) = \langle v_0 \cos \alpha, -gt + v_0 \sin \alpha \rangle$$

- The position function solves to

$$\mathbf{r}(t) = \left\langle (v_0 \cos \alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t \right\rangle$$

with parametric equations

$$x = (v_0 \cos \alpha)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t$$

- The parabolic position curve can be expressed as

$$y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x$$

- Properties of ideal projectile motion beginning at origin:

$$y_{max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$t_{tot} = \frac{2v_0 \sin \alpha}{g}$$

$$R = \frac{v_0^2}{g} \sin 2\alpha$$

- If we assume the initial position is instead $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, then the position function changes to

$$\mathbf{r}(t) = \left\langle (v_0 \cos \alpha)t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + y_0 \right\rangle$$

• Suggested Exercises for 11.2

- Vector function integrals: 1-6
- Vector function initial value problems: 7-12
- Ideal projectile motion: 15-21

11.3 Arc Length in Space

- Arc Length along a Space Curve

- Approximation

$$L \approx \sum_{i=0}^n |\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)| = \sum_{i=1}^n \left| \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \right| \Delta t$$

- Definition

$$L = \int_a^b \left| \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{v}(t)| dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau$$

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \text{speed}$$

- Unit Tangent Vector

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Suggested Exercises for 11.3

- Unit tangent vectors and arc length: 1-8
 - Arc length parameter: 11-14

11.4 Curvature of a Curve

- Curvature

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

- Curvature of a Circle

- The curvature of a circle with radius a is constantly

$$\kappa = \frac{1}{a}$$

- Principal Unit Normal Vector

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}$$

- Circles of Curvature

- The circle which:

1. is tangent to a curve at a point
2. has the same curvature as the curve at that point
3. lies on the concave side of the curve, in the direction of \mathbf{N}

- Radius: $a = \frac{1}{\kappa}$.

- Center: $\langle x_0, y_0 \rangle = \mathbf{r}(t_0) + a\mathbf{N}$.

- Equations:

$$(x - x_0)^2 + (y - y_0)^2 = a^2$$

$$\mathbf{c}(t) = \langle a \sin t + x_0, a \cos t + y_0 \rangle, 0 \leq t \leq 2\pi$$

- Suggested Exercises for 11.4:

- Find $\mathbf{T}, \mathbf{N}, \kappa$: 1-4, 9-16
- Circles of Curvature: 21-22

11.5 Tangential and Normal Components of Acceleration

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

- Right-handed vector frames

- $\mathbf{i}, \mathbf{j}, \mathbf{k}$
- $\mathbf{T}, \mathbf{N}, \mathbf{B}$

- Tangential and Normal Components of Acceleration

$$\mathbf{a} = \left(\frac{d^2 s}{dt^2} \right) \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N} + 0\mathbf{B}$$

- Tangential component

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} |\mathbf{v}|$$

- Normal component

$$a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

- Torsion

- Magnitude of torsion

$$|\tau| = \left| \frac{d\mathbf{B}}{ds} \right|$$

- Signed torsion

$$\frac{d\mathbf{B}}{ds} = (-\tau)\mathbf{N}$$

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{1}{|\mathbf{v}|} \left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N} \right)$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

- Suggested Exercises for 11.5:

- Finding tangential and normal components of acceleration: 1-6
- Finding \mathbf{B} and τ : 9-16

11.6 Velocity and Acceleration in Polar Coordinates

- Polar Coordinates (r, θ)

- Cartesian to Polar

$$r = \sqrt{x^2 + y^2}, \theta = \text{Arctan}\left(\frac{y}{x}\right)$$

- Polar to Cartesian

$$x = r \cos \theta, y = r \sin \theta$$

- Cylindrical Coordinates (r, θ, z)

- Cartesian to Cylindrical

$$r = \sqrt{x^2 + y^2}, \theta = \text{Arctan}\left(\frac{y}{x}\right), z = z$$

- Cylindrical to Cartesian

$$x = r \cos \theta, y = r \sin \theta, z = z$$

- Polar/Cylindrical Unit Vectors

$$\mathbf{u}_r = \langle \cos \theta, \sin \theta \rangle, \mathbf{u}_\theta = \langle -\sin \theta, \cos \theta \rangle$$

- Cylindrical Right-handed frame

$$\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{k}$$

- Derivatives

$$\frac{d}{dt} [\mathbf{u}_r] = \dot{\mathbf{u}}_r = \dot{\theta} \mathbf{u}_\theta$$

$$\frac{d}{dt} [\mathbf{u}_\theta] = \dot{\mathbf{u}}_\theta = -\dot{\theta} \mathbf{u}_r$$

- Polar Position/Velocity/Acceleration

$$\mathbf{r} = r \mathbf{u}_r$$

$$\mathbf{v} = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{u}_\theta$$

- Cylindrical Position/Velocity/Acceleration

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$$

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

- **Suggested Exercises for 11.6:**

- Expressing \mathbf{v} and \mathbf{a} in terms of \mathbf{u}_r and \mathbf{u}_θ : 1-5

12.1 Functions of Several Variables

- Real-Valued Functions

- A **real-valued function** f on with **domain** $D \subset \mathbb{R}^n$ is a rule that assigns a real number

$$f(x_1, x_2, \dots, x_n) \in \mathbb{R}$$

to each $(x_1, x_2, \dots, x_n) \in D$.

- The domain of a function is assumed to be all of \mathbb{R}^n except where the function is not well-defined.
- The **range** of the function is

$$R = \{f(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in D\}$$

- Regions

- A subset of the xy -plane (\mathbb{R}^2) or xyz -space (\mathbb{R}^3) is known as a **region**.
- The **ball** $B(p, \epsilon)$ is the set of points

$$B(p, \epsilon) = \{q \in \mathbb{R}^2 : \text{the distance between } p \text{ and } q \text{ is less than } \epsilon\}$$

Its **center** is the point p and its **radius** is ϵ .

- A point $p \in \mathbb{R}^2$ is known as an **interior point** of a region R if *there exists some ball* containing p that lies inside R .
- A point $p \in \mathbb{R}^2$ is known as a **boundary point** of a region R if *every ball* containing p contains some points in R and some points not in R .
- A point $p \in \mathbb{R}^2$ is known as an **exterior point** of a region R if *there exists some ball* containing p that lies outside R .
- The **interior** of R is the set

$$\text{int}(R) = \{p : p \text{ is an interior point of } R\}$$

- The **boundary** of R is the set

$$\text{bd}(R) = \{p : p \text{ is a boundary point of } R\}$$

- The **exterior** of R is the set

$$\text{ext}(R) = \{p : p \text{ is an exterior point of } R\}$$

- A region R is **open** if it doesn't contain any of its boundary.
- A region R is **closed** if it contains all of its boundary.
- A region R is **bounded** if it can be contained within a ball.
- A region R is **unbounded** if it cannot be contained within a ball.

- Sketching Functions

- Level curve

$$\{(x, y) : f(x, y) = c\}$$

- Surface $z = f(x, y)$

$$\{(x, y, f(x, y)) : (x, y) \in \text{Dom}(f)\}$$

- Contour curve

$$\{(x, y, c) : f(x, y) = c\}$$

- Level surface

$$\{(x, y, z) : f(x, y, z) = c\}$$

- Suggested Exercises for 12.1:

- Identifying and describing domains, ranges, level curves, boundaries: 1-12
- Relating level curves to graphs: 13-18
- Sketching surfaces and level curves: 19-28
- Finding level curves through a point: 29-32
- Sketching level surfaces: 33-40
- Finding level surfaces through a point: 41-44

12.2 Limits and Continuity in Higher Dimensions

- Limits

- If the value of the vector function $f(P)$ becomes arbitrarily close to the number L as points P close to P_0 are plugged into the function, then the **limit of $f(P)$ as P approaches P_0 is L** :

$$\lim_{P \rightarrow P_0} f(P) = L$$

- * Precise definition:

Suppose there exists a function $\delta(\epsilon)$ so that for all positive numbers $\epsilon > 0$ and all numbers P where $|f(P) - L| < \epsilon$, it follows that $|\mathbf{P} - \mathbf{P}_0| < \delta(\epsilon)$. Then we say that $\lim_{P \rightarrow P_0} f(P) = L$.

- Note that values of f must approach L no matter which direction we approach p_0 .

- Limit Laws

1. Sum/Difference Law

$$\lim_{p \rightarrow p_0} (f(p) \pm g(p)) = \lim_{p \rightarrow p_0} f(p) \pm \lim_{p \rightarrow p_0} g(p)$$

2. Product Law

$$\lim_{p \rightarrow p_0} (f(p) \cdot g(p)) = \lim_{p \rightarrow p_0} f(p) \cdot \lim_{p \rightarrow p_0} g(p)$$

3. Constant Multiple Law

$$\lim_{p \rightarrow p_0} (kf(p)) = k \lim_{p \rightarrow p_0} f(p)$$

4. Quotient Law

$$\lim_{p \rightarrow p_0} \frac{f(p)}{g(p)} = \frac{\lim_{p \rightarrow p_0} f(p)}{\lim_{p \rightarrow p_0} g(p)}$$

5. Power Law (for $r, s \in \mathbb{Z}$)

$$\lim_{p \rightarrow p_0} (f(p))^{r/s} = \left(\lim_{p \rightarrow p_0} f(p) \right)^{r/s}$$

- Computing Limits

- $\lim_{P \rightarrow P_0} f(x) = \lim_{x \rightarrow x_0} f(x)$
- Factoring and cancelling is a useful strategy.
- L'Hopital's Rule does not apply for multiple variable limits.

- Showing a Limit DNE

- If

$$\lim_{x \rightarrow x_0} h(x, f(x)) \neq \lim_{x \rightarrow x_0} h(x, g(x))$$

then $\lim_{P \rightarrow P_0} h(x, y)$ DNE.

- Continuity

- A function $f(p)$ is **continuous** if $\lim_{p \rightarrow p_0} f(p) = f(p_0)$ for all points p_0 in its domain.
- If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

- **Suggested Exercises for 12.2:**

- Computing limits: 1-26
- Showing limits don't exist: 35-42

12.3 Partial Derivatives

- Partial Derivatives

- The **partial derivative of f with respect to x_i** is the limit

$$\frac{\partial f}{\partial x_i} = f_{x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

- By the definition, we can see that to compute partial derivatives with respect to a variable, we can treat all other variables as constants and differentiate as normal.

- Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 g}{\partial z \partial z} = \frac{\partial^2 g}{\partial z^2} = g_{zz}$$

- Mixed Derivative Theorem:

$$f_{xy} = f_{yx}$$

- Suggested Exercises for 12.3:

- Finding first-order partial derivatives: 1-38
- Finding second-order partial derivatives: 41-50
- Finding partial derivatives from the limit definition: 53-56

12.4 The Chain Rule

- Gradient Vector Function

$$\nabla f = \langle f_{x_1}, \dots, f_{x_n} \rangle = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

- Chain Rule

- For single variable functions:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \dots$$

- For multi-variable functions:

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots$$

- Differentiation by Substitution

- The Chain Rule can be avoided by “plugging in” functions and using single-variable calculus.

- Total Derivative

- If the variables x, y, z of a function f are dependent on each other, then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial z} \frac{dz}{dx}$$

- Implicit Differentiation

- If $f(x, y) = c$ defines y as a function of x , then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

- Suggested Exercises for 12.4:

- Finding $\frac{dw}{dt}$ for $w = f(x(t), y(t), z(t))$: 1-6
- Finding partial derivatives for compositions of multi-variable functions: 7-12, 33-38
- Using partial derivatives for implicit differentiation: 25-28

12.5 Directional Derivatives and Gradient Vectors

- Directional Derivative

- The **directional derivative of f in the direction of \mathbf{u}** is

$$\frac{df}{ds_{\mathbf{u}}} = \left(\frac{df}{ds} \right)_{\mathbf{u}} = D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

- If θ is the angle between ∇f and \mathbf{u} , then

$$\frac{df}{ds_{\mathbf{u}}} = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

- The angle θ between ∇f and \mathbf{u} determines the value of the directional derivative at a fixed point p_0 :

- * Max: $|\nabla f_{p_0}|$ at $\theta = 0$
- * Zero: 0 at $\theta = \frac{\pi}{2}$
- * Min: $-|\nabla f_{p_0}|$ at $\theta = \pi$

- Normal Vector to a Level Curve

- ∇f is normal to the level curve $f(x, y) = c$ for every point (x, y) in the domain of f .

- Gradient Rules

1. Constant Multiple Rule

$$\nabla(kf) = k\nabla f$$

2. Sum Rule

$$\nabla(f + g) = \nabla f + \nabla g$$

3. Difference Rule

$$\nabla(f - g) = \nabla f - \nabla g$$

4. Product Rule

$$\nabla(fg) = g(\nabla f) + f(\nabla g)$$

5. Quotient Rule

$$\nabla \left(\frac{f}{g} \right) = \frac{g(\nabla f) - f(\nabla g)}{g^2}$$

- **Suggested Exercises for 12.5:**

- Finding ∇f at a point: 1-8
- Finding directional derivatives: 9-16
- Finding the direction of maximal/minimal rate of change: 17-22
- Finding the direction of no instantaneous change: 27-28

12.6 Tangent Planes and Differentials

- Normal Vector to a Level Surface
 - ∇f is normal to the level surface $f(x, y, z) = c$ for every point (x, y, z) in the domain of f .
- Normal Vector to the Surface $z = f(x, y)$
 - If $g(x, y, z) = f(x, y) - z$, then

$$\nabla g = \langle f_x, f_y, -1 \rangle$$

is normal to the surface $z = f(x, y)$ for every point (x, y) in the domain of f .

- Tangent Line to Curve of Intersection of Two Surfaces
 - If P_0 is a point on two surfaces with normal vectors $\mathbf{n}_1, \mathbf{n}_2$, then the tangent line to the curve of intersection is given by

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

- **Suggested Exercises for 12.6:**

- Finding tangent planes & normal lines to surfaces of the form $f(x, y, z) = c$: 1-8
- Finding tangent planes & normal lines to surfaces of the form $z = f(x, y)$: 9-12
- Finding tangent lines to curves of intersection: 13-18

12.7 Extreme Values and Saddle Points

- Local Extreme Values

- Let f be a function of many variables defined on a region containing the point P_0 .
 - * $f(P_0)$ is a **local maximum** if it is the largest nearby value (there exists an open region around P_0 over which no greater value of f exists)
 - * $f(P_0)$ is a **local minimum** if it is the smallest nearby value (there exists an open region around P_0 over which no lesser value of f exists)
- Local max/mins are also known as **local extrema**.

- Critical Points

- The **critical points** for a function f of many variables are the points in the domain where

$$\nabla f = 0 \text{ or } \nabla f \text{ DNE}$$

- Critical points occur when there is a horizontal tangent plane or no tangent plane.

- First Derivative Test for Local Extreme Values

- The local extreme values of a function always occur at critical points.

- Saddle Points

- Not every critical point gives a local extreme value.
- The **saddle points** of f are the critical points which don't yield local extreme values.

- Discriminant Function

- The **discriminant** (sometimes called “Hessian”) of $f(x, y)$ is the function

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second Derivative Test for Local Extreme Values of $f(x, y)$
 - If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
 - If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
 - If $f_D(a, b) < 0$, then f has a saddle point at (a, b) .
 - If $f_D(a, b) = 0$, then the test is inconclusive.
- Absolute Extrema on Closed and Bounded Regions
 - Let f be a function of many variables defined on a region containing the point P_0 .
 - * $f(P_0)$ is the **absolute maximum** of f if it is the largest value in the range of f
 - * $f(P_0)$ is the **absolute minimum** of f if it is the smallest value in the range of f
 - Absolute max/mins are also known as **absolute extrema**.
 - Every continuous function of many variables with a closed and bounded domain has absolute extrema.
- Finding Absolute Extrema of $f(x, y)$ on a Closed and Bounded Region D
 - The following points are candidates for giving the absolute extrema:
 - * Critical points within D .
 - * Critical points on any of D 's boundary curves. (Find a relation of x and y and use that to make f a function of a single variable.)
 - * Corners of D .
 - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.
- Suggested Exercises for 12.7:
 - Finding local max/min and saddle points: 1-30
 - Finding absolute max/min: 31-36

12.8 Lagrange Multipliers

- The Method of Lagrange Multipliers
 - The **Method of Lagrange Multipliers** says that if $f(P)$ is a function of many variables which has an absolute extreme value on the restricted domain $\{P : g(P) = c\}$, and f, g are differentiable functions such that $\nabla g \neq \mathbf{0}$, then the absolute extreme value occurs satisfies

$$\nabla f = \lambda \nabla g \text{ and } g = c$$

for some real number λ .

- **Suggested Exercises for 12.8:**
 - Finding absolute extrema using the Method of Lagrange Multipliers: 1-30

13.1 Double and Iterated Integrals over Rectangles

- Volume as Integral of Area

- If $A(x)$ is the area of a solid's cross-section, then its volume is

$$V = \int_a^b A(x) dx$$

- Double Integrals over Rectangles

- For a solid bounded above by $z = f(x, y) \geq 0$ over the rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

its cross-sectional area at x is given by:

$$A(x) = \int_c^d f(x, y) dy$$

- Thus its volume is the **iterated integral**:

$$V = \int_a^b A(x) dx = \int_a^b \int_c^d f(x, y) dy dx$$

- Similarly, its cross-sectional area at y and volume may be given by:

$$A(y) = \int_a^b f(x, y) dx$$

$$V = \int_c^d A(y) dy = \int_c^d \int_a^b f(x, y) dx dy$$

- We also represent its volume as a **double integral**:

$$V = \iint_R f(x, y) dA$$

- If $f(x, y) \not\geq 0$, then the double integral represents **net volume**: volume above the xy -plane minus volume below the xy -plane.

- Suggested Exercises for 13.1:

- Evaluating iterated integrals with constant bounds: 1-12
- Evaluating double integrals over rectangles: 13-28

13.2 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions

- For a solid bounded above by $z = f(x, y) \geq 0$ over the region

$$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

its cross-sectional area at x is given by:

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy$$

- Thus its volume is the **iterated integral**:

$$V = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

- Similarly, for a solid bounded above by $z = f(x, y) \geq 0$ over the region

$$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

its cross-sectional area at x is given by:

$$A(y) = \int_{h_1(y)}^{h_2(y)} f(x, y) dx$$

$$V = \int_a^b A(y) dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- We also represent its volume as a **double integral**:

$$V = \iint_R f(x, y) dA$$

- If $f(x, y) \not\geq 0$, then the double integral represents **net volume**: volume above the xy -plane minus volume below the xy -plane.

- Finding Limits of Integration

1. Sketch the region and label bounding curves
2. Determine if it is easier to describe bottom/top bounds

$$g_1(x) \leq y \leq g_2(x)$$

or left/right bounds

$$h_1(y) \leq x \leq h_2(y)$$

For $g_1(x) \leq y \leq g_2(x)$:

3. Find the x -limits of integration a, b by finding the leftmost, rightmost x -values in the region:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

For $h_1(y) \leq x \leq h_2(y)$:

3. Find the y -limits of integration c, d by finding the bottommost, topmost y -values in the region:

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- Swapping Variables of Integration

- You can only swap the order of integration of an iterated integral by first converting to a double-integral, and using the above steps.

- Properties of Double Integrals

1. Zero Integral

$$\iint_R 0 dA = 0$$

2. Constant Multiple

$$\iint_R c f(x, y) dA = c \iint_R f(x, y) dA$$

3. Sum/Difference

$$\iint_R f(x, y) \pm g(x, y) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$$

4. Domination

If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then

$$\iint_R f(x, y) dA \leq \iint_R g(x, y) dA$$

5. Additivity

If R can be split into two regions R_1, R_2 , then

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

- **Suggested Exercises for 13.2:**

- Evaluating nonrectangular double integrals: 1-6, 11-14
- Finding limits of integration: 7-10, 33-44
- Swapping order of integration: 25-32

13.3 Area by Double Integration

- Areas of Regions in the Plane

- The area of a region R in the plane is

$$A = \iint_R dA = \iint_R 1 \, dA$$

- Average Value of a Function of Two Variables

- The average value of $f(x, y)$ over the region R is defined to be

$$\text{Avg Val} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA$$

- **Suggested Exercises for 13.3:**

- Finding areas of regions: 1-8
- Finding average values of functions: 15-18

13.5 Triple Integrals in Rectangular Coordinates

- Hypervolume as Integral of Volume

- A hypersolid is a region of \mathbb{R}^4 , that is, a set of ordered 4-tuples (x, y, z, w) .
- If $V(x)$ is the volume of a four-dimensional hypersolid's cross-section, then its hypervolume is

$$HV = \int_a^b V(x) dx$$

- Applications include modeling density within 3D space: (x, y, z, δ) .
- Hypervolume in $xyz\delta$ -space represents mass.

- Triple Integrals over Rectangular Boxes

- For a hypersolid bounded above by $w = f(x, y, z) \geq 0$ over the rectangular box

$$D : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3$$

its cross-sectional volume at x is given by:

$$V(x) = \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z) dz dy$$

- Thus its hypervolume is the iterated integral:

$$HV = \int_{a_1}^{b_1} V(x) dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z) dz dy dx$$

- The *constant bounds* of this iterated integral and differentials may be swapped around.
- We also represent its hypervolume as the **triple integral**

$$HV = \iiint_D f(x, y, z) dV$$

- If $w = f(x, y, z) \not\geq 0$, then the triple integral represents net hypervolume.

- Triple Integrals over Other Solids

- For a general solid with bottom/top surface

$$h_1(x, y) \leq z \leq h_2(x, y)$$

and shadow in the xy plane bounded by

$$a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

the triple integral over the solid may be expressed by the iterated integral:

$$\iiint_D f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$$

- Other orders of integration can be attained by using shadows in other coordinate planes and/or swapping order of integration for the shadow.

- Volumes of Regions in Space

- The volume of a solid D in space is

$$V = \iiint_D dV = \iiint_D 1 dV$$

- Average Value of a Function of Three Variables

- The average value of $f(x, y, z)$ over the solid D is defined to be

$$\text{Avg Val} = \frac{1}{\text{volume of } D} \iiint_D f(x, y, z) dV$$

- Triple Integral Properties

- The properties for double integrals in Section 13.2 similarly hold for triple integrals.

- **Suggested Exercises for 13.5:**

- Evaluating triple integrals: 7-20
- Finding volumes of solids: 23-36
- Finding the average value of functions: 37-40

13.8 Substitution in Multiple Integrals

- Transformations

- Two similar regions in 2D space can be transformed by a “nice” pair of functions

$$\mathbf{r}(u, v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}) \rangle = \langle x(u, v), y(u, v) \rangle$$

that map points in a uv plane to the xy plane.

- Two similar solids in 3D space can be transformed by a “nice” triple of functions

$$\mathbf{r}(u, v, w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}) \rangle = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$$

that map points in a uvw space to the xyz space.

- The Jacobian

- The Jacobian of a 2D transformation given by $\mathbf{r}(u, v)$ is the determinant

$$\mathbf{r}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- The Jacobian of a 3D transformation given by $\mathbf{r}(u, v, w)$ is the determinant

$$\mathbf{r}_J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- 2D Substitution

- Suppose that the region R in the xy -plane is the result of applying the transformation $\mathbf{r}(u, v)$ to the region G in the uv -plane.
- Then it follows that

$$\iint_R f(x, y) dx dy = \iint_G f(x(u, v), y(u, v)) |\mathbf{r}_J(u, v)| du dv$$

- 3D Substitution

- Suppose that the solid D in xyz space is the result of applying the transformation $\mathbf{r}(u, v, w)$ to the region H in uvw space.
- Then it follows that

$$\begin{aligned} & \iiint_D f(x, y, z) \, dx \, dy \, dz \\ &= \iiint_H f(x(u, v, w), y(u, v, w), z(u, v, w)) |\mathbf{r}_J(u, v, w)| \, du \, dv \, dw \end{aligned}$$

- Suggested Exercises for 13.8:

- 2D Jacobians, Transformations, and substitutions: 1-10

13.4 Double Integrals in Polar Form

- Integrating over Regions expressed using Polar Coordinates

- The polar coordinate transformation

$$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta \rangle$$

from polar G into Cartesian R yields

$$\iint_R f(x, y) dA = \iint_G f(r \cos \theta, r \sin \theta) r dr d\theta$$

- **Suggested Exercises for 13.4:**

- Changing Cartesian integrals to polar integrals: 1-16
- Finding integrals over polar regions: 17-22

13.7 Triple Integrals in Cylindrical and Spherical Coordinates

- Cylindrical Coordinates

- The cylindrical coordinate transformation

$$\mathbf{r}(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle$$

from cylindrical H into Cartesian D yields

$$\iiint_D f(x, y, z) dV = \iiint_H f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

- Spherical Coordinates

- The spherical coordinate transformation

$$\mathbf{r}(\rho, \phi, \theta) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$$

from spherical H into Cartesian D yields

$$\iiint_D f(x, y, z) dV = \iiint_H f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

- **Suggested Exercises for 13.7:**

- Cylindrical coordinate integrals: 1-20
- Finding integrals over polar regions: 21-38

14.1 Line Integrals

- Line Integrals with Respect to Arclength

- The area of the ribbon with base along the curve C in xyz space and height given by $f(x, y, z)$ is given by the **line integral of $f(x, y, z)$ over C with respect to arclength s** :

$$\int_C f(x, y, z) ds$$

- Arclength line integrals can be evaluated by finding a smooth parametrization $\mathbf{r}(s)$ of the curve C with respect to arclength s for $a \leq s \leq b$:

$$\int_C f(x, y, z) ds = \int_{s=a}^{s=b} f(x(s), y(s), z(s)) ds$$

- If $\mathbf{r}(t)$ is an arbitrary parametrization of C for $a \leq t \leq b$, then

$$\int_C f(x, y, z) ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t)) |\mathbf{v}(t)| dt$$

- Additivity

$$\int_{C_1+C_2} f ds = \int_{C_1} f ds + \int_{C_2} f ds$$

- Reversing Arclength Line Integrals

$$\int_C f ds = \int_{-C} f ds$$

- Suggested Exercises for 14.1:

- Identifying vector equations for graphs: 1-8
- Evaluating line integrals: 9-22

14.2 Vector Fields, Work, Circulation, and Flux

- Line Integrals with Respect to Variables

- The net projected area of the ribbon with base curve C and height $f(x, y, z)$ with respect to the x -axis is given by the **line integral of $f(x, y, z)$ over C with respect to x** :

$$\int_C f(x, y, z) dx$$

(similar for y, z)

- Line integrals with respect to variables can be evaluated by finding a parametrization $\mathbf{r}(t)$ for the curve C :

$$\int_C f(x, y, z) dx = \int_a^b f(x(t), y(t), z(t)) \frac{dx}{dt} dt$$

- Such integrals have the property

$$\int_{-C} f dx = - \int_C f dx$$

- Vector Fields

- A **vector field** is a function

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$$

($\mathbf{F} = \langle M, N, P \rangle$ for short) which assigns a vector to each point in its domain.

- Gradient functions $\nabla f = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ and transformations $\langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$ are examples of vector fields.

- Line Integrals of Vector Fields

- The **line integral of $\mathbf{F} = \langle M, N, P \rangle$ over C** is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz$$

gives the sum of the line integrals of each component of \mathbf{F} with respect to each variable x, y, z .

- These line integrals can be calculated by using parametrizations of C :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz = \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_a^b \mathbf{F} \cdot \mathbf{v} dt = \int_a^b \mathbf{F} \cdot \mathbf{T} ds$$

- It follows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_{-C} \mathbf{F} \cdot d\mathbf{r}$$

- Work over a Smooth Curve

- Work is given by the product of force and displacement:

$$W = \mathbf{F} \cdot \mathbf{D}$$

- So work over a smooth curve can be approximated by the Riemann sum:

$$W \approx \sum_{i=1}^n \mathbf{F}(x_i, y_i, z_i) \cdot \Delta \mathbf{r}_i$$

- We limit this sum to infinity to define work over a smooth curve:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Flow

- The **flow** of a fluid along a curve C is defined to be the line integral

$$\text{Flow} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- If C is closed (its starting point and ending point are the same), then the flow is also known as the **circulation**.

- Flux (2D)

- The two-dimensional **flux** of \mathbf{F} across C is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds$$

where \mathbf{n} is the outward unit normal vector to C .

- If C is oriented counter-clockwise, then

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx = \int_a^b \left(M \frac{dy}{dt} - N \frac{dx}{dt} \right) dt$$

- **Suggested Exercises for 14.2:**

- Work over a curve: 7-22
- Circulation, flow, and flux: 23-28, 37-40

14.3 Path Independence, Potential Functions, and Conservative Fields

- Several Equivalencies for Conservative Fields

The following are all equivalent for piecewise smooth curves and vector fields with continuous first derivatives:

- $\mathbf{F} = \langle M, N, P \rangle$ is a **conservative field**.
- $\mathbf{F} \cdot d\mathbf{r} = M dx + N dy + P dz$ is **exact**.
- $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent**: the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the endpoints of the curve C .
- There exists a **potential function** f such that $\nabla f = \mathbf{F}$.
- (Closed Loop Property of Conservative Fields)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$
for every closed loop C in D .
- (Fundamental Theorem of Line Integrals)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$
for every path C connecting A to B .
- (Component Test for Conservative Fields)

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

- **Suggested Exercises for 14.3:**

- Determining if a field is conservative: 1-6
- Finding potential functions: 7-12
- Evaluating integrals of differential forms: 13-22

14.4 Green's Theorem in the Plane

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Divergence

- The **divergence** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \mathbf{F}$$

In physics, divergence is often called the **flux density**.

- Spin

- The **spin** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by

$$\operatorname{spin} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

In physics, spin is often called the **circulation density**.

- Spin is also the **k-component of curl**, defined in a later section.

- Simple Curves

- A curve which does not cross itself is said to be **simple**.

- Green's Theorem in the Plane

- Let C be a piecewise smooth, simple closed curve enclosing the region R and oriented counter-clockwise. Let $\mathbf{F} = \langle M, N \rangle$ be a vector field for which M, N have continuous first partial derivatives in an open region containing R . Then:

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n} \, ds &= \iint_R \operatorname{div} \mathbf{F} \, dA \\ \int_C \mathbf{F} \cdot \mathbf{T} \, ds &= \iint_R \operatorname{spin} \mathbf{F} \, dA \end{aligned}$$

- Suggested Exercises for 14.4:

- Using Green's Theorem to find circulation and flux: 5-14
- Using Green's Theorem to evaluate line integrals: 17-20

14.5 Surfaces and Area

- Parametrization of Surfaces

- Vector functions of two variables

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

may be used to parametrize surfaces in xyz space.

- Smooth Vector Functions

- A surface parametrized by $\mathbf{r}(u, v)$ is called **smooth** if

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ on the interior of the surface.

- Surface Area of a Parametrized Surface

- The area of a smooth surface with parametrizing vector function $\mathbf{r}(u, v)$ for a region R in the uv plane is given by

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| dA$$

- Implicit Surface

- Level surfaces $F(x, y, z) = c$ are sometimes called **implicit surfaces**.
- If \mathbf{p} is a unit vector normal a coordinate plane, then the surface area defined by $F(x, y, z)$ bounded by the cylinder given by a region R in that coordinate plane is

$$\iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA$$

- Surface Area Differential

- The integral $\iint_S d\sigma$ is used to represent surface area, and $d\sigma$ is known as the surface area differential.

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| dA = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA$$

- **Suggested Exercises for 14.5:**

- Finding parametrizations of surfaces: 1-16
- Finding surface area: 17-26

14.6 Surface Integrals and Flux

- Surface Integrals

- The **surface integral** of a function $G(x, y, z)$ over a surface S is given by

$$\begin{aligned}\iint_S G(x, y, z) d\sigma &= \iint_R G(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA \\ &= \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA\end{aligned}$$

- Orientable Surfaces

- A surface is said to be **orientable** if it is “two-sided” - there exists a continuous normal unit vector field \mathbf{n} to the surface.

- Flux in Three Dimensions

- The flux of a three dimensional vector field \mathbf{F} across an oriented surface S in the direction of \mathbf{n} is given by the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$$

- Suggested Exercises for 14.6:

- Evaluating surface integrals: 1-14
- Three-dimensional flux: 15-24

14.7 Stokes' Theorem

- Curl

- The **curl** of a vector field \mathbf{F} is defined as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \text{curl } \mathbf{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

- The counterclockwise spin with respect to a unit vector \mathbf{u} is given by

$$\text{spin}_{\mathbf{u}} \mathbf{F} = \text{curl } \mathbf{F} \cdot \mathbf{u} = \nabla \times \mathbf{F} \cdot \mathbf{u}$$

- In particular, in 2D:

$$\text{spin } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \text{spin}_{\mathbf{k}} \mathbf{F}$$

and in 3D:

$$\text{curl } \mathbf{F} = \langle \text{spin}_{\mathbf{i}} \mathbf{F}, \text{spin}_{\mathbf{j}} \mathbf{F}, \text{spin}_{\mathbf{k}} \mathbf{F} \rangle$$

- Stokes' Theorem

- If a curve C in \mathbb{R}^3 is the boundary of a surface S , and we want to compute the counter-clockwise circulation with respect to unit normal vectors \mathbf{n} on the surface, we may use

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S \text{spin}_{\mathbf{n}} \mathbf{F} \, d\sigma = \iint_S (\text{curl } \mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

- Identities and Properties

- Due to the Mixed Derivative Theorem,

$$\text{curl } \nabla f = \nabla \times \nabla f = \mathbf{0}$$

- If $\nabla \times \mathbf{F} = \mathbf{0}$ for every point in a region D , then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0$$

for every curve C and surface S within D .

- Suggested Exercises for 14.7:

- Using Stokes' Theorem: 1-10

14.8 Divergence Theorem and a Unified Theory

- Divergence Theorem

- Divergence in \mathbb{R}^3 is defined as

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$

- The Divergence Theorem lets us measure the flux on a closed surface S by integrating over the divergence within its bounded region D :

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \operatorname{div} \mathbf{F} \, dV = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

- Suggested Exercises for 14.8:

- Using the Divergence Theorem: 5-16