Sections 12.1 - 12.2 Overview

- Three-Dimensional Coordinates

  - Distance between points in 3D space

  $$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

  - Simple planes in 3D Space

  $$x = a, \; y = b, \; z = c$$

  - Spheres in 3D Space

  $$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

- Vectors

  - Definition of a Vector
    * A vector $\mathbf{v} = \vec{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
    * Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
    * We often assume the initial point lays at the origin.

  - Component Form
    The vector with initial point at $(0, 0, 0)$ and terminal point at (v_x, v_y, v_z) is represented by
    $$\langle v_x, v_y, v_z \rangle$$

  - 2D and 3D Vectors

    $$\langle a, b \rangle = \langle a, b, 0 \rangle$$

  - Position Vector
    If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c \rangle$ is its **position vector**.
    We assume $(a, b, c) = \langle a, b, c \rangle$.

  - Vector Between Points
    The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is

    $$\mathbf{P_1 P_2} = \overrightarrow{P_1 P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

– Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

– The Zero Vector

$$\mathbf{0} = \overrightarrow{0} = \langle 0, 0, 0 \rangle$$

– Vector Operations

 * Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

 * Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

– Vector Operation Properties

 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
 5. $0\mathbf{u} = \mathbf{0}$
 6. $1\mathbf{u} = \mathbf{u}$
 7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
 8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
 9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

– Unit Vectors

 * A **unit vector** or **direction** is any vector whose length is 1.
 * Standard unit vectors
   · $\mathbf{i} = \langle 1, 0, 0 \rangle$
   · $\mathbf{j} = \langle 0, 1, 0 \rangle$
   · $\mathbf{k} = \langle 0, 0, 1 \rangle$
 * Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

 * Length-Direction Form:
$$\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$$

12.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- Angle between vectors

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

- Orthogonal Vectors

  - $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if the angle between them is $\frac{\pi}{2} = 90°$
  - $\mathbf{0}$ is orthogonal to every vector

- Dot Product Properties

  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
  3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
  5. $\mathbf{0} \cdot \mathbf{u} = 0$

- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos \theta$$

12.4 The Cross Product

- Determinants

    – 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

    – 3x3 Determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_1 c_3)$$

- Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \langle u_2 v_3 - u_3 v_2 \,,\, u_3 v_1 - u_1 v_3 \,,\, u_1 v_2 - u_2 v_1 \rangle$$

Shortcut "long multiplication" method:

$$\begin{array}{r} \langle \quad u_1 \quad , \quad u_2 \quad , \quad u_3 \quad \rangle \\ \times \langle \quad v_1 \quad , \quad v_2 \quad , \quad v_3 \quad \rangle \\ \hline \langle \quad u_2 v_3 - u_3 v_2 \quad , \quad u_3 v_1 - u_1 v_3 \quad , \quad u_1 v_2 - u_2 v_1 \quad \rangle \end{array}$$

- Right-Hand Rule

    – A method for determining a special orthogonal direction used throughout mathematics and physics in 3D space, with respect to an ordered pair of vectors $\mathbf{u}, \mathbf{v}$

    – $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$, $\mathbf{v}$ according to the Right-Hand Rule.

- Cross Product Magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

  The area of the parallelogram determined by $\mathbf{u}, \mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$.

- Parallel Vectors

  - $\mathbf{u}, \mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are parallel if the angle between them is $0 = 0°$ or $\pi = 180°$
  - $\mathbf{0}$ is parallel to every vector

- Cross Product Properties

  1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
  4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
  5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
  6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products

  1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
  2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
  3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

  The standard unit vectors are known as a "right handed frame".

- Torque

$$\overrightarrow{\tau} = \mathbf{r} \times \mathbf{F}$$
$$|\overrightarrow{\tau}| = |\mathbf{r}||\mathbf{F}|\sin\theta$$

- Triple Scalar (or "Box") Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

  Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelpiped determined by the three vectors.

12.5 Equations of Lines and Planes

- Vector Equation and Parametric Equations for a Line

$$\mathbf{r}(t) = \mathbf{P_0} + t\mathbf{v}$$

$$x = x_0 + At, y = y_0 + Bt, z = z_0 + Ct$$

  for $-\infty < t < \infty$

- Symmetric Equations for a Line

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_0P_1}) = (1 - t)\mathbf{P_0} + t\mathbf{P_1}$$

  for $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = D$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{n_1} \times \mathbf{n_2})$$

- Angle of Intersection of Two Planes

$$\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

12.6 Cylinders and Quadratic Surfaces

- Sketching surfaces

    - To sketch a 3D surface, sketch planar cross-sections
        * $z = c$ is parallel to xy plane
        * $y = b$ is parallel to xz plane
        * $x = a$ is parallel to yz plane

- Cylinders

    - A **cylinder** is any surface generated by considering parallel lines passing through a planar curve.
    - A 3D surface defined by a function of only two variables results in a cylinder.

- Quadric Surfaces

    - A **quadric surface** is any surface defined by a second degree equation of x, y, z.
    - Most helpful to consider the cross-sections in each of the coordinate planes.

- Ellipsoids

    - Cross-sections in the coordinate planes include
        * Three ellipses

- Elliptical Cone

    - Cross-sections in the coordinate planes include
        * Two double-lines
        * One point (with parallel ellipses)

- Elliptical Paraboloid

    - Cross-sections in the coordinate planes include
        * Two parabolas
        * One point (with parallel ellipses)

- Hyperbolic Paraboloid

  - Cross-sections in the coordinate planes include
    * Two parabolas (with parallel parabolas)
    * One double line (with parallel hyperbolas)

- Hyperboloid of One Sheet

  - Cross-sections in the coordinate planes include
    * Two hyperbolas
    * One ellipsis (with parallel ellipses)

- Hyperboloid of Two Sheets

  - Cross-sections in the coordinate planes include
    * Two hyperbola
    * One empty cross-section (with parallel ellipses)

13.1 Vector Functions and Space Curves

- Curves, Paths, and Vector Functions

  - A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

  $$x = x(t), y = y(t), z = z(t)$$

  or with a **vector function**

  $$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

  - $x(t), y(t), z(t)$ are called **component functions**

- Vector Function Limits

  $$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

- Continuity of Vector Functions

  - The function $\mathbf{r}(t)$ is **continuous** if

  $$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

  for all a in its domain.
  - $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

13.2 Derivatives and Integrals of Vector Functions

- Derivatives of Vector Functions

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \langle f'(t), g'(t), h'(t) \rangle$$

  - $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of t is in its domain.
  - $\mathbf{r}'(a)$ is a **tangent vector** to the curve where $t = a$
  - The **tangent line** to a curve at $t = a$:

$$\mathbf{l}(t) = \mathbf{r}(a) + t\mathbf{r}'(a)$$

- Differentiation Rules for Vector Functions

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df}\frac{df}{dt}$$

- Derivative of a Constant Length Vector Function

  - If $|\mathbf{r}(t)| = c$ always, then
$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

  - Thus the derivative of a constant length vector function is perpindicular to the original.

- Antiderivatives of Vector Functions

    – If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.

    – The **indefinite integral** $\displaystyle\int \mathbf{r}(t)\, dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t)\, dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t)\, dt = \left\langle \int x(t)\, dt, \int y(t)\, dt, \int z(t)\, dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t)\, dt = \left\langle \int_a^b x(t)\, dt, \int_a^b y(t)\, dt, \int_a^b z(t)\, dt \right\rangle$$

$$\int_a^b \mathbf{r}(t)\, dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Differential Vector Equations

    – If we know $\mathbf{r}'(t)$ and $\mathbf{r}(a)$ for some $t = a$, then

$$\mathbf{r}(t) = \int_a^t \mathbf{r}'(t)\, dt + \mathbf{r}(a)$$

13.3 Arc Length and Curvature

- Arc Length along a Space Curve

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{r}'(t)| \, dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{r}'(u)| du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

- Unit Tangent Vector

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$$

$$\mathbf{T}(t) = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

- Curvature

$$\kappa(s) = \left| \frac{d\mathbf{T}}{ds} \right|$$

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{|d\mathbf{r}/dt|} = \frac{|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}|}{|\frac{d\mathbf{r}}{dt}|^3}$$

For $y = f(x)$:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

- Principal Unit Normal Vector

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$$

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

The triple $\mathbf{T}, \mathbf{N}, \mathbf{B}$ forms a right-handed frame.

13.4 Motion in Space: Velocity and Acceleration

- Position, Velocity, and Acceleration

    - Position: $\mathbf{r}(t)$
    - Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
    - Speed: $v(t) = |\mathbf{v}(t)| = \frac{ds}{dt}$
    - Direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
    - Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

- Ideal Projectile Motion

$$\mathbf{a}(t) = \langle 0, -g \rangle$$

$$\mathbf{v}(t) = \langle v_0 \cos\alpha, -gt + v_0 \sin\alpha \rangle$$

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \right\rangle$$

- Tangental and Normal Components of Acceleration

$$\mathbf{a} = \left(\frac{d^2s}{dt^2} \right) \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N} + 0\mathbf{B}$$

    - Tangental component

$$a_T = \frac{d^2s}{dt^2} = v'$$

    - Normal component

$$a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa v^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

14.1 Functions of Several Variables

- Functions of Two Variables

  - A **function f of two variables** is a rule which assigns a real number $f(x, y)$ to each pair of real numbers (x, y) in its **domain**

  $$\text{dom}(f) \subseteq \mathbb{R}^2$$

  The set of values f takes on is its **range**

  $$\text{ran}(f) = \{f(x, y) : (x, y) \in \text{dom}(f)\}$$

  - The **level curve** for each $k \in \text{ran}(f)$ is given by the equation

  $$f(x, y) = k$$

  - The **graph** of f is a surface in 3D space which visualizes the function, given by the equation $z = f(x, y)$.

- Functions of Three Variables

  - A **function f of three variables** is a rule which assigns a real number $f(x, y, z)$ to each pair of real numbers (x, y, z) in its **domain**

  $$\text{dom}(f) \subseteq \mathbb{R}^3$$

  The set of values f takes on is its **range**

  $$\text{ran}(f) = \{f(x, y, z) : (x, y, z) \in \text{dom}(f)\}$$

  - The **level surface** for each $k \in \text{ran}(f)$ is given by the equation

  $$f(x, y, z) = k$$

- Alternate Forms

  - We may also consider functions of the form $f(x_1, x_2, \dots) = f(P) = f(\mathbf{r})$.
  - If $P = (x, y)$ and $\mathbf{r} = \langle x, y \rangle$, then $f(x, y) = f(P) = f(\mathbf{r})$.
  - If $P = (x, y, z)$ and $\mathbf{r} = \langle x, y, z \rangle$, then $f(x, y, z) = f(P) = f(\mathbf{r})$.

14.2 Limits and Continuity

- Limits

  - If the value of the function $f(P)$ becomes arbitrarily close to the number L as vectors P close to P_0 are plugged into the function, then the **limit of $f(P)$ as P approaches P_0** is L:

  $$\lim_{P \to P_0} f(P) = L$$

  - For functions of two or three variables:

  $$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

  $$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = L$$

- Showing a Limit DNE

  - In order for a limit $\lim_{P \to P_0} f(x,y)$ to exist, the values of f must approach L no matter which direction we approach P_0.

  - Choose $y = g(x)$ and $y = h(x)$ where P_0 lays on both graphs. If

  $$\lim_{x \to x_0} f(x, g(x)) \neq \lim_{x \to x_0} f(x, h(x))$$

  then $\lim_{P \to P_0} f(x,y)$ DNE.

  - Or choose $x = g(y)$ and $x = h(y)$ where P_0 lays on both graphs. If

  $$\lim_{y \to y_0} f(g(y), y) \neq \lim_{y \to y_0} f(h(y), y)$$

  then $\lim_{P \to P_0} f(x,y)$ DNE.

- Limit Laws

$$\lim_{P \to P_0} (f(P) \pm g(P)) = \lim_{P \to P_0} f(P) \pm \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (f(P) \cdot g(P)) = \lim_{P \to P_0} f(P) \cdot \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (kf(P)) = k \lim_{P \to P_0} f(P)$$

$$\lim_{P \to P_0} \frac{f(P)}{g(P)} = \frac{\lim_{P \to P_0} f(P)}{\lim_{P \to P_0} g(P)}$$

$$\lim_{P \to P_0} (f(P))^{r/s} = \left(\lim_{P \to P_0} f(P) \right)^{r/s}$$

- Computing Limits

  - Variables not involved in a limit may be eliminated:

  $$\lim_{P \to P_0} f(x) = \lim_{x \to x_0} f(x)$$

  - Due to the Limit Laws, many limits follow the "just plug it in" rule.
  - If plugging in results in a zero in a denominator, use factoring, perhaps with conjugates.
  - L'Hopital's Rule does not apply for multiple variable limits.

- Continuity

  - A function $f(P)$ is **continuous** if $\lim_{P \to P_0} f(P) = f(P_0)$ for all points P_0 in its domain.
  - If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

14.3 Partial Derivatives

- Partial Derivatives

    - For a function f of two variables (x, y):

    $$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

    $$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

    - To compute partial derivatives with respect to a variable, treat all other variables as constants and differentiate as normal.

    - Functions of more than two variables behave similarly. For $T(x, y, z)$:

    $$\frac{\partial T}{\partial z} = T_z(x, y, z) = \lim_{h \to 0} \frac{T(x, y, z + h) - T(x, y, z)}{h}$$

- Higher Order Partial Derivatives

    $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = (f_y)_x = f_{yx}$$

    $$\frac{\partial^2 g}{\partial z^2} = \frac{\partial}{\partial z} \left[\frac{\partial g}{\partial z} \right] = (g_z)_z = g_{zz}$$

- Mixed Derivative Theorem

    - For many naturally occuring functions:

    $$f_{xy} = f_{yx}$$

14.4 Tangent Planes and Linear Approximations

- Tangent Plane to $z = f(x, y)$ at $(a, b, f(a, b))$

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Linearization of $f(x, y)$ at (a, b)

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Differentiability and a Sufficient Condition

    - A multi-variable function f is **differentiable** at a point if its linearizaration approximates the value of the function near that point.
    - If f_x, f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

- Linear Approximation
  If f is differentiable at (a, b), then

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

14.5 The Chain Rule

- Gradient Vector Function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Nested Functions

  - If f is a function of $\mathbf{r} = \langle x, y, z \rangle$, and x, y, z are functions of t, then we say x, y, z are itermediate variables and may consider the following composed function of t:
    $$f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

  - If f is a function of $\mathbf{r} = \langle x, y, z \rangle$, and x, y, z are functions of $\mathbf{s} = \langle t, u, v \rangle$, then we say x, y, z are itermediate variables and may consider the following composed function of t, u, v:

    $$f(\mathbf{r}(\mathbf{s})) = f(x(t, u, v), y(t, u, v), z(t, u, v))$$

- Chain Rule

  - For functions of the form $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$:

    $$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

  - For functions of the form $f(\mathbf{r}(\mathbf{s})) = f(x(t, u, v), y(t, u, v), z(t, u, v))$:

    $$\frac{\partial f}{\partial t} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

- Differentiation by Substitution

  - The multi-variable Chain Rule can be avoided by "plugging in" functions and using single-variable calculus.

- Total Derivative

    - If f is a function of x, y, z, and y, z are also functions of x, then

    $$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

- Implicit Differentiation

    - If $f(x, y) = c$ defines y as a function of x, then

    $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}$$

- Tree Diagram for the Chain Rule

    - The tree diagram for the chain rule can be used to generate the chain rule.
    - It also holds for multiple levels of intermediate variables.

14.6 Directional Derivatives and the Gradient Vector

- Directional Derivative

  - The **directional derivative** of f for the unit vector $\mathbf{u}$ is

  $$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

  - The maximum value of $D_{\mathbf{u}}f$ at a fixed point P_0 is $|\nabla f(P_0)|$, which occurs when $\mathbf{u} = \frac{\nabla f(P_0)}{|\nabla f(P_0)|}$.

- Normal Vector to Level Curves and Surfaces

  - The gradient vectors ∇f are normal vectors to the level curves $f(x, y) = k$ for every (x, y) in the domain of f.

  - The gradient vectors ∇f are normal vectors to the level surfaces $f(x, y, z) = k$ for every (x, y, z) in the domain of f.

14.7 Maximum and Minimum Values

- Local Maximum and Minimum Values

    - Let f be a function of many variables defined near the point P_0.
        * f has a **local maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value of f near P_0
        * f has a **local minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value of f near P_0

- Critical Points

    - If P_0 is a point in the domain of f and

    $$\nabla f(P_0) = 0 \text{ or } \nabla f(P_0) \text{ DNE}$$

    then P_0 is called a **critical point**.
    - Critical points occur when the tangent plane is horizontal or DNE.
    - The local maximum and minimum values of a function always occur at critical points.

- Saddle Points

    - Not every critical point gives a local extreme value.
    - The **saddle points** of f are the critical points which don't yield local extreme values.

- Discriminant Function

    - The **discriminant** of f with variables x, y is the function

    $$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second Derivative Test for Local Extreme Values of $f(x, y)$
  Let (a, b) be a critical point of of f where ∇f is defined.

    - If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
    - If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
    - If $f_D(a, b) < 0$, then f has a saddle point at (a, b).
    - If $f_D(a, b) = 0$, then the test is inconclusive.

- Absolute Maximum and Minimum Values

  - Let f be a function of many variables.
    * f has an **absolute maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value in the range of f
    * f has an **absolute minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value in the range of f
  - Every continuous function of many variables with a closed and bounded domain has an absolute maximum and minimum value.

- Finding Absolute Max/Min of $f(x, y)$ on a Closed and Bounded Region D

  - The following points are candidates for giving the absolute extrema:
    * Critical points of f within D.
    * Critical points for a function which gives part of the boundary of D.
    * Corners of D.
  - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.

14.8 Lagrange Multipliers

- The Method of Lagrange Multipliers

  - The **Method of Lagrange Multipliers** says that if f is a function of many variables which has an absolute max/min value on the restriction $g(P) = k$ where $\nabla g \neq 0$, then the absolute max/min occurs at a point P where

    $$\nabla f(P) = \lambda \nabla g(P) \text{ and } g(P) = k$$

    for some real number λ.

  - If two constraints $g(P) = k$ and $h(P) = l$ are given, then we find where

    $$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P) \text{ and } g(P) = k \text{ and } h(P) = l$$

    for some real numbers λ, μ.

15.1 Double Integrals over Rectangles

- Double Integral

  - We define the **double integral** of a function $f(x, y)$ over a region R to be

    $$\iint\limits_R f(x, y) \, dA = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i}) \Delta A_{n,i}$$

    where for each positive integer n we've defined a way to partition R into n pieces

    $$\Delta R_{n,1}, \Delta R_{n,2}, \ldots, \Delta R_{n,n}$$

    where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

    $$\lim_{n \to \infty} \max(\Delta A_{n,i}) = 0$$

  - Since for $f(x, y) \geq 0$,

    $$\sum_{i=1}^{n} f(x_{n,i}, y_{n,i}) \Delta A_{n,i}$$

    is an approximation of the volume under $z = f(x, y)$ and over R, the double integral is used to define the precise volume.

  - If f is not always positive, then the double integral represents **net volume**: volume above the xy-plane minus volume below the xy-plane.

- Midpoint Rule for Approximating Rectangular Double Integrals

  - For the rectangle
    $$R : a \leq x \leq b, c \leq y \leq d$$

    we may approximate the double integral by partitioning the rectangle into a grid of $m \times n$ rectangular pieces all with area ΔA and evaluating:

    $$\iint\limits_R f(x, y) \, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j}) \Delta A$$

    where $(\overline{x_i}, \overline{y_j})$ is the midpoint of the $i \times j$ rectangle.

15.2 Iterated Integrals

- Volume as Integral of Area

  - If $A(x)$ is the area of a solid's cross-section, then the solid's volume is

    $$V = \int_a^b A(x)\,dx$$

- Iterated Integrals over Rectangles

  - A double integral over a rectangle

    $$R : a \leq x \leq b, c \leq y \leq d$$

  can be expressed as the **iterated integrals**:

  $$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

15.3 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions

  - For **Type I** regions which may be expressed as

$$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

  a double integral over R may be expressed as the iterated integral:

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  - For **Type II** regions which may be expressed as

$$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

  a double integral over R may be expressed as the iterated integral:

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Finding Limits of Integration

  1. Sketch the region and label bounding curves
  2. Determine if the region is Type I or Type II by identifying (I) bottom/top curves $y = g_1(x), y = g_2(x)$ or (II) left/right curves $x = h_1(y), x = h_2(y)$.

  For Type I:

  3. Use the leftmost and rightmost x-values in the region a, b to complete the iterated integral:

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  For Type II:

  3. Use the bottommost and topmost y-values in the region c, d to complete the iterated integral:

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Swapping Variables of Integration

  – You can only swap the order of integration of an iterated integral by drawing the region and reinterpreting it as a region of the opposite Type.

- Additivity

  If R can be split into two regions R_1, R_2, then

$$\iint\limits_{R} f(x,y)\, dA = \iint\limits_{R_1} f(x,y)\, dA + \iint\limits_{R_2} f(x,y)\, dA$$

- Average Value of Two-Variable Functions

  – The average value of a two-variable function f over a region R is defined to be

$$\frac{1}{\text{Area of } R} \iint\limits_{R} f(x,y)\, dA$$

- Area as a Double Integral

  – The area of a region R in the plane is

$$A = \iint\limits_{R} dA = \iint\limits_{R} 1\, dA$$

15.7 Triple Integrals

- Triple Integral

  - We define the **triple integral** of a function $f(x, y, z)$ over a solid D to be

$$\iiint\limits_{D} f(x, y, z)\, dV = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i}, z_{n,i}) \Delta V_{n,i}$$

  where for each positive integer n we've defined a way to partition D into n pieces

$$\Delta D_{n,1}, \Delta D_{n,2}, \ldots, \Delta D_{n,n}$$

  where $\Delta D_{n,i}$ has volume $\Delta V_{n,i}$, contains the point $(x_{n,i}, y_{n,i}, z_{n,i})$, and

$$\lim_{n \to \infty} \max(\Delta V_{n,i}) = 0$$

- Iterated Integral for Rectangular Boxes

  - The triple integral over the rectangular box

$$D : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$$

  can be expressed as the **iterated integrals**:

$$\iiint\limits_{D} f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dy\, dx$$

$$= \int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dz\, dy = \int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dz\, dx = \ldots$$

- Iterated Integral for Generated Solids

  - If the solid D is determined by the bottom/top surfaces

$$h_1(x, y) \le z \le h_2(x, y)$$

  and has shadow R in the xy-plane, then a triple integral over D can be expressed as:

$$\iiint\limits_{D} f(x, y, z)\, dV = \iint\limits_{R} \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz \right] dA$$

– In general:

$$\iiint\limits_{D} f(x, y, z) \, dV = \iint\limits_{R} \left[\int_{\text{bottom surface}}^{\text{top surface}} f(x, y, z) \, d\square \right] dA$$

where $\square$ is chosen from x, y, z to be the "up" orientation.

- Additivity

  If D can be split into two regions D_1, D_2, then

$$\iint\limits_{D} f(x, y, z) \, dV = \iint\limits_{D_1} f(x, y, z) \, dV + \iint\limits_{D_2} f(x, y, z) \, dV$$

- Average Value of Three-Variable Functions

  – The average value of a three-variable function f over a solid D is defined to be
$$\frac{1}{\text{Volume of } D} \iiint\limits_{R} f(x, y, z) \, dV$$

- Volume as a Triple Integral

  – The volume of a solid D in space is
$$V = \iiint\limits_{D} dV = \iiint\limits_{D} 1 \, dV$$

15.10 Change of Variables in Multiple Integrals

- Transformations

  - Two similar regions in 2D space can be transformed by a "nice" pair of functions

  $$\mathbf{r}(u, v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}) \rangle = \langle x(u, v), y(u, v) \rangle$$

  that map points in a uv plane to the xy plane.

  - Two similar solids in 3D space can be transformed by a "nice" triple of functions

  $$\mathbf{r}(u, v, w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}) \rangle = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$$

  that map points in a uvw space to the xyz space.

- The Jacobian

  - The Jacobian of a 2D transformation given by $\mathbf{r}(u, v)$ is the determinant

  $$\mathbf{r}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

  - The Jacobian of a 3D transformation given by $\mathbf{r}(u, v, w)$ is the determinant

  $$\mathbf{r}_J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- 2D Substitution

  - Suppose that the region R in the xy-plane is the result of applying the transformation $\mathbf{r}(u, v)$ to the region G in the uv-plane.

  - Then it follows that

  $$\iint\limits_R f(x, y)\, dx\, dy = \iint\limits_G f(x(u, v), y(u, v)) |\mathbf{r}_J(u, v)|\, du\, dv$$

- 3D Substitution

    – Suppose that the solid D in xyz space is the result of applying the transformation $\mathbf{r}(u, v, w)$ to the region H in uvw space.

    – Then it follows that

    $$\iiint\limits_{D} f(x, y, z) \, dx \, dy \, dz$$

    $$= \iiint\limits_{H} f(x(u, v, w), y(u, v, w), z(u, v, w)) |\mathbf{r}_J(u, v, w)| \, du \, dv \, dw$$

15.4 Double Integrals in Polar Coordinates

- Integrating over Regions expressed using Polar Coordinates

    - The polar coordinate transformation

    $$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta \rangle$$

    from polar G into Cartesian R yields

    $$\iint\limits_R f(x, y)\, dA = \iint\limits_G f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$$

15.8 Triple Integrals in Cylindrical Coordinates

- Cylindrical Coordinates

    - The cylindrical coordinate transformation

    $$\mathbf{r}(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle$$

    from cylindrical H into Cartesian D yields

    $$\iiint\limits_D f(x, y, z)\, dV = \iiint\limits_H f(r \cos \theta, r \sin \theta, z)\, r\, dr\, d\theta\, dz$$

15.9 Triple Integrals in Spherical Coordinates

- Spherical Coordinates

    - The spherical coordinate transformation

    $$\mathbf{r}(\rho, \phi, \theta) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$$

    from spherical H into Cartesian D yields

    $$\iiint\limits_D f(x, y, z)\, dV = \iiint\limits_H f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$$

14.1 Line Integrals

- Line Integrals with Respect to Arclength

    - The area of the ribbon with base along the curve C in xyz space and height given by $f(x, y, z)$ is given by the **line integral of $f(x, y, z)$ over C with respect to arclength** s:

    $$\int_C f(x, y, z)\, ds$$

    - Arclength line integrals can be evaulated by finding a smooth parametrization $\mathbf{r}(s)$ of the curve C with respect to arclength s for $a \leq s \leq b$:

    $$\int_C f(x, y, z)\, ds = \int_{s=a}^{s=b} f(x(s), y(s), z(s))\, ds$$

    - If $\mathbf{r}(t)$ is an arbitrary parametrization of C for $a \leq t \leq b$, then

    $$\int_C f(x, y, z)\, ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t))|\mathbf{v}(t)|\, dt$$

- Additivity

$$\int_{C_1+C_2} f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$$

- Reversing Arclength Line Integrals

$$\int_C f\, ds = \int_{-C} f\, ds$$

- **Suggested Exercises for 14.1:**

    - Identifying vector equations for graphs: 1-8
    - Evaluating line integrals: 9-22

14.2 Vector Fields, Work, Circulation, and Flux

- Line Integrals with Respect to Variables

  - The net projected area of the ribbon with base curve C and height $f(x, y, z)$ with respect to the x-axis is given by the **line integral of $f(x, y, z)$ over C with respect to x**:

$$\int_C f(x, y, z)\, dx$$

  (similar for y, z)

  - Line integrals with respect to variables can be evaluated by finding a parametrization $\mathbf{r}(t)$ for the curve C:

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\frac{dx}{dt}\, dt$$

  - Such integrals have the property

$$\int_{-C} f\, dx = -\int_C f\, dx$$

- Vector Fields

  - A **vector field** is a function

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$$

  ($\mathbf{F} = \langle M, N, P \rangle$ for short) which assigns a vector to each point in its domain.

  - Gradient functions $\nabla f = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ and transformations $\langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$ are examples of vector fields.

- Line Integrals of Vector Fields

  - The **line integral of $\mathbf{F} = \langle M, N, P \rangle$ over C** is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dx + P\, dz$$

  gives the sum of the line integrals of each component of $\mathbf{F}$ with respect to each variable x, y, z.

– These line integrals can be calculated by using parametrizations of C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dx + P\,dz = \int_a^b \left(M\frac{dx}{dt} + N\frac{dx}{dt} + P\frac{dz}{dt} \right) dt$$

$$= \int_a^b \mathbf{F} \cdot \mathbf{v}\,dt = \int_a^b \mathbf{F} \cdot \mathbf{T}\,ds$$

– It follows that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

- Work over a Smooth Curve

    – Work is given by the product of force and displacement:

    $$W = \mathbf{F} \cdot \mathbf{D}$$

    – So work over a smooth curve can be approximated by the Riemann sum:

    $$W \approx \sum_{i=1}^{n} \mathbf{F}(x_i, y_i, z_i) \cdot \Delta \mathbf{r_i}$$

    – We limit this sum to infinity to define work over a smooth curve:

    $$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Flow

    – The **flow** of a fluid along a curve C is defined to be the line integral

    $$\text{Flow} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

    – If C is closed (its starting point and ending point are the same), then the flow is also known as the **circulation**.

- Flux (2D)

  - The two-dimenstional **flux** of $\mathbf{F}$ across C is

  $$\int\limits_C \mathbf{F} \cdot \mathbf{n} \, ds$$

  where $\mathbf{n}$ is the outward unit normal vector to C.

  - If C is oriented counter-clockwise, then

  $$\int\limits_C \mathbf{F} \cdot \mathbf{n} \, ds = \int\limits_C M \, dy - N \, dx = \int\limits_a^b \left(M \frac{dy}{dt} - N \frac{dx}{dt} \right) \, dt$$

- **Suggested Exercises for 14.2:**

  - Work over a curve: 7-22
  - Circulation, flow, and flux: 23-28, 37-40

14.3 Path Independence, Potential Functions, and Conservative Fields

- Several Equivalencies for Conservative Fields

  The following are all equivalent for piecewise smooth curves and vector fields with continuous first derivatives:

  - $\mathbf{F} = \langle M, N, P \rangle$ is a **conservative field**.

  - $\mathbf{F} \cdot d\mathbf{r} = M\,dx + N\,dy + P\,dz$ is **exact**.

  - $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent**: the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the endpoints of the curve C.

  - There exists a **potential function** f such that $\nabla f = \mathbf{F}$.

  - (Closed Loop Property of Conservative Fields)

    $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed loop C in D.

  - (Fundamental Theorem of Line Integrals)

    $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$ for every path C connecting A to B.

  - (Component Test for Conservative Fields)
    $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$, $\dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$, and $\dfrac{\partial N}{\partial x} = \dfrac{\partial N}{\partial y}$.

- **Suggested Exercises for 14.3:**

  - Determining if a field is conservative: 1-6

  - Finding potential functions: 7-12

  - Evaluating integrals of differential forms: 13-22

14.4 Green's Theorem in the Plane

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Divergence

  - The **divergence** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by
  $$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \mathbf{F}$$
  In physics, divergence is often called the **flux density**.

- Spin

  - The **spin** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by
  $$\operatorname{spin} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$
  In physics, spin is often called the **circulation density**.
  - Spin is also the **k-component of curl**, defined in a later section.

- Simple Curves

  - A curve which does not cross itself is said to be **simple**.

- Green's Theorem in the Plane

  - Let C be a piecewise smooth, simple closed curve enclosing the region R and oriented counter-clockwise. Let $\mathbf{F} = \langle M, N \rangle$ be a vector field for which M, N have continuous first partial derivatives in an open region containing R. Then:
  $$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \operatorname{div} \mathbf{F} \, dA$$
  $$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R \operatorname{spin} \mathbf{F} \, dA$$

- **Suggested Exercises for 14.4:**

  - Using Green's Theorem to find circulation and flux: 5-14
  - Using Green's Theorem to evaluate line integrals: 17-20

14.5 Surfaces and Area

- Parametrization of Surfaces

  - Vector functions of two variables

  $$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

  may be used to parametrize surfaces in xyz space.

- Smooth Vector Functions

  - A surface parametrized by $\mathbf{r}(u,v)$ is called **smooth** if

  $$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \ \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

  are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ on the interior of the surface.

- Surface Area of a Parametrized Surface

  - The area of a smooth surface with parametrizing vector function $\mathbf{r}(u,v)$ for a region R in the uv plane is given by

  $$A = \iint\limits_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

- Implicit Surface

  - Level surfaces $F(x,y,z) = c$ are sometimes called **implicit surfaces**.
  - If $\mathbf{p}$ is a unit vector normal a coordinate plane, then the surface area defined by $F(x,y,z)$ bounded by the cylinder given by a region R in that coordinate plane is

  $$\iint\limits_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA$$

- Surface Area Differential

  - The integral $\iint\limits_S d\sigma$ is used to represent surface area, and $d\sigma$ is known as the surface area differential.

  $$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} \, dA$$

- **Suggested Exercises for 14.5:**

  - Finding parametrizations of surfaces: 1-16
  - Finding surface area: 17-26

14.6 Surface Integrals and Flux

- Surface Integrals

  - The **surface integral** of a function $G(x, y, z)$ over a surface S is given by

  $$\iint\limits_{S} G(x, y, z)\, d\sigma = \iint\limits_{R} G(x(u,v), y(u,v), z(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

  $$= \iint\limits_{R} G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA$$

- Orientable Surfaces

  - A surface is said to be **orientable** if it is "two-sided" - there exists a continuous normal unit vector field $\mathbf{n}$ to the surface.

- Flux in Three Dimensions

  - The flux of a three dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is given by the surface integral

  $$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

- **Suggested Exercises for 14.6:**

  - Evaluating surface integrals: 1-14
  - Three-dimensional flux: 15-24

14.7 Stokes' Theorem

- Curl

  - The **curl** of a vector field $\mathbf{F}$ is defined as

    $$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \operatorname{curl}\mathbf{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

  - The counterclockwise spin with respect to a unit vector $\mathbf{u}$ is given by

    $$\operatorname{spin}_{\mathbf{u}}\mathbf{F} = \operatorname{curl}\mathbf{F} \cdot \mathbf{u} = \nabla \times \mathbf{F} \cdot \mathbf{u}$$

  - In particular, in 2D:

    $$\operatorname{spin}\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \operatorname{spin}_{\mathbf{k}}\mathbf{F}$$

    and in 3D:

    $$\operatorname{curl}\mathbf{F} = \left\langle \operatorname{spin}_{\mathbf{i}}\mathbf{F}, \operatorname{spin}_{\mathbf{j}}\mathbf{F}, \operatorname{spin}_{\mathbf{k}}\mathbf{F} \right\rangle$$

- Stokes' Theorem

  - If a curve C in $\mathbb{R}^3$ is the boundary of a surface S, and we want to compute the counter-clockwise circulation with respect to unit normal vectors $\mathbf{n}$ on the surface, we may use

    $$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \operatorname{spin}_{\mathbf{n}}\mathbf{F}\, d\sigma = \iint_S (\operatorname{curl}\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

- Identities and Properties

  - Due to the Mixed Derivative Theorem,

    $$\operatorname{curl}\nabla f = \nabla \times \nabla f = \mathbf{0}$$

  - If $\nabla \times \mathbf{F} = \mathbf{0}$ for every point in a region D, then

    $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$$

    for every curve C and surface S within D.

- **Suggested Exercises for 14.7:**

  - Using Stokes' Theorem: 1-10

14.8 Divergence Theorem and a Unified Theory

- Divergence Theorem

  - Divergence in $\mathbb{R}^3$ is defined as

  $$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$

  - The Divergence Theorem lets us measure the flux on a closed surface S by integrating over the divergence within its bounded region D:

  $$\text{Flux} = \iint\limits_{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint\limits_{D} \operatorname{div} \mathbf{F} \, dV = \iiint\limits_{D} \nabla \cdot \mathbf{F} \, dV$$

- **Suggested Exercises for 14.8:**

  - Using the Divergence Theorem: 5-16