# Flutter Prediction for Flexible Wings

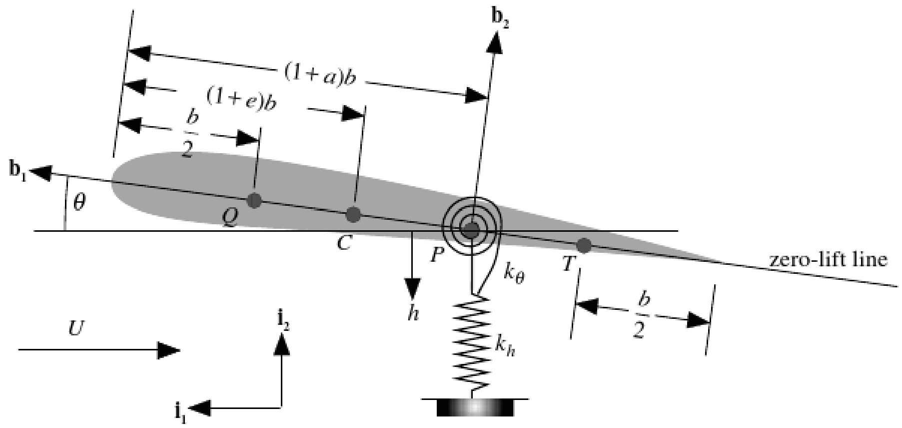


Figure 1: The general wing section setup for the problem. However, here we allow the wing to be elastic instead of idealizing the elasticity at a point.

We must first write down the equations of motion. We will model the wing as a beam undergoing both bending and torsion. We denote the deflection of the elastic axis as $w(x,t)$ and the twist about the elastic axis as $\theta(y,t)$. To find the equations of motion, we will use LaGrange's Equations. We must first find the LaGrangian:

$$L = K - P \tag{1}$$

Where $K$ is the kinetic energy and $P$ is the potential energy. Now, we find expressions for these two. We will start with the potential energy. This can be found by summing up all the contributions to the elastic strain energy. We will have one contribution from bending and one from torsion. We obtain:

$$P = \frac{1}{2}\int_0^l \left[ EI\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + GJ\left(\frac{\partial \theta}{\partial y}\right)^2 \right] dy \tag{2}$$

Now we must construct the kinetic energy term. In general, the kinetic energy is given by:

$$K = \frac{1}{2}\int_0^l \int_A \rho ||\mathbf{V}||^2 dA dy$$

From here, we will consider the density, $\rho$, to be constant. Now, to make this equation useful. $\mathbf{V}$ denotes the velocity of a given point on the beam. We will call this point $G$. We must find an expression for $\mathbf{V}$ in terms of $w$ and $\theta$. From dynamics, the absolute velocity of the point $G$ on the beam with respect to the given coordinate system is:

$$\mathbf{V}_G(x,y,z,t) = \mathbf{V}_P + \mathbf{V}_{G/P} = \mathbf{V}_P + \boldsymbol{\omega} \times \mathbf{r}_{PG}$$

Now, we recognize that $\boldsymbol{\omega} = \frac{\partial \theta}{\partial t}\hat{\mathbf{j}}$ is the rate of twist of the beam and $\mathbf{r}_{PG} = x\hat{\mathbf{i}} + z\hat{\mathbf{k}}$. The cross product of these two gives $\mathbf{V}_{G/P} = z\frac{\partial \theta}{\partial t}\hat{\mathbf{i}} - x\frac{\partial \theta}{\partial t}\hat{\mathbf{k}}$. Also, since the elastic axis can only move up and down, we recognize that $\mathbf{V}_P = \frac{\partial w}{\partial t}\hat{\mathbf{k}}$. Now, we clearly see that:

$$\mathbf{V} = z\frac{\partial \theta}{\partial t}\hat{\mathbf{i}} + \left(\frac{\partial w}{\partial t} - x\frac{\partial \theta}{\partial t}\right)\hat{\mathbf{k}}$$

Now, we plug the magnitude of this vector into the expression for the kinetic energy:

$$K = \frac{1}{2}\int_0^l \int_A \rho \left[ \left(\frac{\partial w}{\partial t}\right)^2 + (x^2+z^2)\left(\frac{\partial \theta}{\partial t}\right)^2 - 2x\frac{\partial w}{\partial t}\frac{\partial \theta}{\partial t} \right] dA dy$$

We can rewrite this in individual integrals and factor out an $A$:

$$K = \frac{1}{2}\left[m\int_0^l \left(\frac{\partial w}{\partial t}\right)^2 dy + \frac{m}{A}\int_A (x^2+z^2)dA\int_0^l \left(\frac{\partial \theta}{\partial t}\right)^2 dy - \frac{2m}{A}\int_A xdA\int_0^l \frac{\partial w}{\partial t}\frac{\partial \theta}{\partial t}dy\right]$$

Now, we recognize that $\frac{1}{A}\int_A (x^2+z^2)dA = b^2r^2$ is exactly the radius of gyration of the section. We also recognize that $\frac{1}{A}\int_A xdA = bx_\theta$. Plugging in these identities, we obtain the final expression for the kinetic energy:

$$K = \frac{m}{2}\left[\int_0^l \left[\left(\frac{\partial w}{\partial t}\right)^2 + b^2r^2\left(\frac{\partial \theta}{\partial t}\right)^2 - 2bx_\theta \frac{\partial w}{\partial t}\frac{\partial \theta}{\partial t}\right]dy\right] \tag{3}$$

Now, we need the generalized forces. To do this, we get an expression for the virtual work. This is where we include contributions from the sectional lift and moment about the aerodynamic center. There will be a contribution from bending and a contribution from the twist angle.

$$\delta W = \int_0^l L' dy \delta w + \int_0^l \left(M'_{ac} + L'b\left(\frac{1}{2}+a\right)\right)dy\delta\theta \tag{4}$$

Now we can use LaGrange's Equations which are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi_i}}\right) - \frac{\partial L}{\partial \xi_i} = \mathbb{F}_i \tag{5}$$

Now we need to find the generalized coordinates. To do this, we will assume that both the deflection and twist functions are given by:

$$w(y,t) = \sum_{i=1}^{N_w} \xi_i(t)\phi_i(y) \tag{6}$$

$$\theta(y,t) = \sum_{i=N_w+1}^{N} \xi_i(t)\phi_i(y) \tag{7}$$

Now we evaluate $-\frac{\partial L}{\partial \xi_i}$, for $i \leq N_w$. Since there is no contribution here from the kinetic energy (all of the terms depend on time, and therefore $\dot{\xi}$):

$$-\frac{\partial L}{\partial \xi_i} = \frac{\partial P}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left[\frac{1}{2}\int_0^l \left[EI\left(\sum_{m=1}^{N_w}\xi_m\phi''_m\right)\left(\sum_{j=1}^{N_w}\xi_j\phi''_j\right) + GJ\left(\sum_{m=N_w+1}^{N}\xi_m\phi'_m\right)\left(\sum_{j=N_w+1}^{N}\xi_j\phi'_j\right)\right]dy\right]$$

Since for now, we are only considering $i \leq N_w$, the terms multiplying $GJ$ contribute nothing. We obtain:

$$\begin{aligned}\frac{\partial P}{\partial \xi_i} &= \frac{EI}{2}\sum_{m=1}^{N_w}\sum_{j=1}^{N_w}\frac{\partial}{\partial \xi_i}\left(\xi_m\xi_j\int_0^l \phi''_m\phi''_j dy\right)\\ &= \frac{EI}{2}\sum_{m=1}^{N_2}\sum_{j=1}^{N_w}\left[\frac{\partial \xi_m}{\partial \xi_i}\xi_j + \xi_m\frac{\partial \xi_j}{\partial \xi_i}\right]\int_0^l \phi''_m\phi''_j dy\end{aligned}$$

By orthogonality, we only care about the case where $i = m$. In this case, the derivatives with respect to $\xi_i$ evaluate to one. We can then replace $m$ with $j$ since it is just a dummy variable. As a result, we obtain:

$$\frac{\partial P}{\partial \xi_i} = EI\sum_{j=1}^{N_w}\xi_j\int_0^l \phi''_i\phi''_j dy \tag{8}$$

We now define a dimensionless parameter, $\eta$, as:

$$\eta = \frac{y}{l} \Rightarrow dy = l\ d\eta \tag{9}$$

We can use the chain rule to find a relationship between $\frac{d\phi_i}{dy}$ and $\frac{d\phi_i}{d\eta}$. This gives:

$$\frac{d\phi_i}{dy} = \frac{d\phi_i}{d\eta}\frac{d\eta}{dy} = \frac{1}{l}\frac{d\phi_i}{d\eta}$$

We also now define:

$$(\ )' \equiv \frac{d}{d\eta}$$

Substiuting this into (8):

$$\frac{\partial P}{\partial \xi_i} = \left(\frac{EI}{l^3}\sum_{j=1}^{N_w}\int_0^1 \phi_i''\phi_j''\, d\eta\right)\xi_j \tag{10}$$

Now we wish to find $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_i}\right) = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\xi}_i}\right)$. We plug (3) into this definition:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\xi}_i}\right) = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\xi}_i}\left(\frac{m}{2}\int_0^l\left[\left(\sum_{m=1}^{N_w}\dot{\xi}_m\phi_m\right)\left(\sum_{j=1}^{N_w}\dot{\xi}_j\phi_j\right) + b^2r^2\left(\sum_{m=N_w+1}^{N}\dot{\xi}_m\phi_m\right)\left(\sum_{j=N_w+1}^{N}\dot{\xi}_j\phi_j\right)\right.\right.\right.$$
$$\left.\left.\left.-2bx_\theta\left(\sum_{m=N_w+1}^{N}\dot{\xi}_m\phi_m\right)\left(\sum_{j=N_w+1}^{N}\dot{\xi}_j\phi_j\right)\right]\, dy\right)\right\}$$

$$= \frac{m}{2}\frac{d}{dt}\frac{\partial}{\partial \dot{\xi}_i}\left[\sum_{m=1}^{N_w}\sum_{j=1}^{N_w}\dot{\xi}_m\dot{\xi}_j\int_0^l \phi_m\phi_j\, dy + b^2r^2\sum_{m=N_w+1}^{N}\sum_{j=N_w+1}^{N}\dot{\xi}_m\dot{\xi}_j\int_0^l \phi_m\phi_j\, dy - 2bx_\theta\sum_{m=1}^{N_w}\sum_{j=N_w+1}^{N}\dot{\xi}_m\dot{\xi}_j\int_0^l \phi_m\phi_j\, dy\right]$$

Similarly to what we did before, we take the derivative using the product rule, so we only care about when $m = i$ and when $j = i$. Also, the $b^2r^2$ term sums from $N_w + 1$ to $N$, so it does not contribute to this expression. As a result, we have:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\xi}_i}\right) = m\frac{d}{dt}\left[\sum_{j=1}^{N_w}\left(\int_0^l \phi_i\phi_j\, dy\right)\dot{\xi}_j - bx_\theta\sum_{j=N_w+1}^{N}\left(\int_0^l \phi_i\phi_j\, dy\right)\dot{\xi}_j\right] \tag{11}$$

Differentiating and non-dimensionalizing:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\xi}_i}\right) = ml\left[\sum_{j=1}^{N_w}\left(\int_0^1 \phi_i\phi_j\, d\eta\right)\ddot{\xi}_j - bx_\theta\sum_{j=N_w+1}^{N}\left(\int_0^1 \phi_i\phi_j\, d\eta\right)\ddot{\xi}_j\right] \tag{12}$$

Now, we want to know what the generalized forces are. We consider (4), and plug in the definitions $\delta w = \delta\theta = \delta\xi_i\phi_i$:

$$\sum_{i=1}^{N_w}\delta\xi_i\int_0^l L'\phi_i\, dy + \sum_{i=N_w+1}^{N}\delta\xi_i\int_0^l M'\phi_i\, dy$$

and from here we can see that:

$$\mathbb{F}_i = \begin{cases}\int_0^l L'\phi_i\, dy, & i \le N_w\\ \int_0^l M'\phi_i\, dy, & i > N_w\end{cases} \tag{13}$$

Summarizing our results, we have:

$$ml\left[\sum_{j=1}^{N_w}\left(\int_0^1 \phi_i\phi_j\, d\eta\right)\ddot{\xi}_j - bx_\theta\sum_{j=N_w+1}^{N}\left(\int_0^1 \phi_i\phi_j\, d\eta\right)\ddot{\xi}_j\right] + \frac{EI}{l^3}\sum_{j=1}^{N_w}\left(\int_0^1 \phi_i''\phi_j''\, d\eta\right)\ddot{\xi}_j = l\int_0^1 L'\phi_i\, d\eta \tag{14}$$

Similarly, we consider all of the same quatities but for $i > N_w$. First we consider the potential energy term:

$$\frac{\partial L}{\partial \xi_i} = \frac{\partial P}{\partial \xi_i} = GJ \sum_{j=N_w+1}^{N} \left( \int_0^l \phi_i' \phi_j' \, dy \right) = GJl \sum_{j=N_w+1}^{N} \left( \int_0^1 \phi_i' \phi_j' \, d\eta \right) \tag{15}$$

Now, considering the kinetic energy term:

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{\xi}_i} \right) = ml \left[ -bx_\theta \sum_{j=1}^{N_w} \left( \int_0^1 \phi_i \phi_j \, d\eta \right) \ddot{\xi}_j + b^2 r^2 \sum_{j=N_w+1}^{N} \left( \int_0^1 \phi_i \phi_j \, d\eta \right) \ddot{\xi}_j \right] \tag{16}$$

At this point, we should be able to recognize that this can be written as a matrix system. If we let

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{N_w} \\ \xi_{N_w+1} \\ \vdots \\ \xi_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi}^w \\ \boldsymbol{\xi}^\theta \end{bmatrix}$$

then we can write the system as:

$$ml \begin{bmatrix} M^{ww} & -bx_\theta M^{w\theta} \\ -bx_\theta (M^{w\theta})^T & b^2 r^2 M^{\theta\theta} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}^w \\ \boldsymbol{\xi}^\theta \end{bmatrix} + \begin{bmatrix} \frac{EI}{l^3} K^{ww} & 0 \\ 0 & \frac{GJ}{l} K^{\theta\theta} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}^w \\ \boldsymbol{\xi}^\theta \end{bmatrix} = \begin{bmatrix} \mathbf{F}^w \\ \mathbf{F}^\theta \end{bmatrix} \tag{17}$$

Now, we can perform a classical flutter analysis. To do this, we look for time harmonic solutions. In other words, we look for solutions of the form:

$$w(y,t) = \bar{w}(y) e^{i\omega t} = \left( \sum_{j=1}^{N_w} \bar{\xi}_j \phi_j(y) \right) e^{i\omega t} = \sum_{j=1}^{N_w} (\bar{\xi}_j e^{i\omega t}) \phi_j(y) \tag{18}$$

$$\theta(y,t) = \bar{\theta}(y) e^{i\omega t} = \left( \sum_{j=N_w+1}^{N} \bar{\xi}_j \phi_j(y) \right) e^{i\omega t} = \sum_{j=N_w+1}^{N} (\bar{\xi}_j e^{i\omega t}) \phi_j(y) \tag{19}$$

where $\xi_j = \bar{\xi}_j e^{i\omega t}$ and $i = \sqrt{-1}$. Now we plug these into the equations of motion. First, we apply them to (13). So for $i \leq N_w$ we only need to consider the sectional lift term. For this, we select an unsteady aerodynamic theory and go from there. We continue the analysis for a general aerodynamic theory:

$$L'(y,t) = -\pi \rho_\infty b^3 \omega^2 \left[ -l_h(k) \frac{\bar{w}(y)}{b} + l_\theta(k) \bar{\theta}(y) \right] e^{i\omega t} \tag{20}$$

Finding the generalized force:

$$\mathbb{F}_i = -\pi \rho_\infty b^3 l \omega^2 \left[ -\frac{l_h}{b} \sum_{j=1}^{N_w} \bar{\xi}_j \int_0^1 \phi_i \phi_j \, d\eta + l_\theta \sum_{N_w+1}^{N} \bar{\xi}_j \int_0^1 \phi_i \phi_j \, d\eta \right] e^{i\omega t} \tag{21}$$

Writing this as a vector:

$$\mathbf{F}^w = -\pi \rho_\infty b^3 l \omega^2 \left[ -\frac{l_h}{b} M^{ww} \boldsymbol{\xi}^w + l_\theta M^{w\theta} \boldsymbol{\xi}^\theta \right] e^{i\omega t} \tag{22}$$

For $i > N_w$:

$$M'(y,t) = -\pi \rho_\infty b^4 l \omega^2 \left[ m_h(k) \frac{\bar{w}(y)}{b} + m_\theta \bar{\theta}(y) \right] e^{i\omega t} \tag{23}$$

And similarly, we can write this as a vector:

$$\mathbf{F}^\theta = -\pi\rho_\infty b^4 l\omega^2 \left[-\frac{m_h}{b}(M^{w\theta})^T\boldsymbol{\xi}^w + m_\theta M_{\theta\theta}\boldsymbol{\xi}^\theta\right]e^{i\omega t} \tag{24}$$

Lastly, we combine this all into one matrix equation of motion:

$$-ml\omega^2\begin{bmatrix} b^2M^{ww} & -b^2x_\theta M^{w\theta} \\ -b^2x_\theta(M^{w\theta})^T & b^2r^2M^{\theta\theta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^w/b \\ \boldsymbol{\xi}^\theta\end{bmatrix} + \begin{bmatrix}\frac{b^2EI}{l^3}K^{ww} & 0 \\ 0 & \frac{GJ}{l}K^{\theta\theta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^w/b \\ \boldsymbol{\xi}^\theta\end{bmatrix} \tag{25}$$

$$= -\pi\rho_\infty b^4 l^3\omega^2\begin{bmatrix} -l_hM^{ww} & l_\theta M^{w\theta} \\ -m_h(M^{w\theta})^T & m_\theta M^{\theta\theta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^w/b \\ \boldsymbol{\xi}^\theta\end{bmatrix} \tag{26}$$

Which can be written in the form:

$$(A+B+C)\boldsymbol{\xi} = 0 \tag{27}$$

Since we are only interested in non-trivial solution, we find the determinant of the coefficient matrix and set it equal to zero to solve the system.