Ch2. Basic Concepts

$$0 \le P[A] \le 1$$
$$P[S] = 1 \qquad P[\varnothing] = 0$$

If $A \cap B = \varnothing$, then $P[A \cup B] = P[A] + P[B]$
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Sampling

(Separate) $\prod_k n_k$

(Replacement and Ordering) n^k

(Ordering only) $\prod_{x=0}^{k-1}(n-x)$

(Permutations) $\prod_{x=0}^{n-1}(n-x) = n!$

(Neither) $\frac{n!}{k!(n-k)!} = \binom{n}{k} = \binom{n}{n-k}$

(Replacement only) $\binom{n-1+k}{k} = \binom{n-1+k}{n-1}$

Conditional Probability

($P[A]$ given B) $P[A|B] = \frac{P[A \cap B]}{P[B]}$, $P[B] > 0$

(Total Probability) $P[A] = \sum P[A|B_i]P[B_i] \forall B_i$

(Bayes' Rule) $P[B_j|A] = \frac{P[A|B_j]P[B_j]}{\sum P[A|B_i]P[B_i] \forall B_i}$

Independence of Events

(A and B independent) $P[A \cap B] = P[A]P[B]$

Sequential Experiments

(All independent) $P[A_1 \cap A_2 \cap \ldots] = P[A_1]P[A_2]\ldots$

Bernoulli trials

(n trials, k succeed) $\binom{n}{k} p^k (1-p)^{n-k}$ for $0 \le k \le n$

(k first success) $(1-p)^{k-1}p$

(first success past k) $(1-p)^k$

Markov chain

Markov chains are used for experiments where the probability of the next outcome depends on the results so far.

$$P[s_0, s_1, \ldots, s_n] = P[s_n|s_{n-1}]P[s_{n-1}|s_{n-2}]\ldots P[s_1|s_0]P[s_0]$$

Ch3. Discrete Random Variables

$$\textit{(pmf)} \quad p_X(x) = P[X = x]$$

$$\textit{(mean/expected val.)} \quad m_X = E[X] = \sum_{x \in S_X} x p_X(x)$$

$$\textit{(func mean)} \quad E[g(x)] = \sum_{x \in S_X} g(x) p_X(x)$$

$$\textit{(variance)} \quad \sigma_X^2 = \text{VAR}[X] = E[X^2] - E[X]^2$$

$$\textit{(std.dev)} \quad \sigma_X = \text{STD}[X] = \sqrt{\text{VAR}[X]}$$

$$\textit{(}n^{th}\textit{ moment)} \quad E[X^n] = \sum_{x \in S_X} x^n p_X(x)$$

$$\textit{(cond. pmf)} \quad p_X(x|C) = \frac{P[\{X = x\} \cap C]}{P[C]}$$

$$\textit{(cond. func. mean)} \quad E[g(x)|C] = \sum_{x \in S_X} g(x) p_X(x|C)$$

$$\textit{(cdf)} \quad F_X(x) = P[X \leq x] = \sum_{k \leq x, k \in S_X} p_X(k)$$

$$\textit{(pdf)} \quad f_X(x) = \sum_{k \in S_X} p_X(k) \delta(x - x_k)$$

Important Discrete Random Variables

Bernoulli

$X = 1$ when event A with $P[A] = p$ occurs, $X = 0$ else.

$$S_X = \{0, 1\} \qquad p_0 = 1 - p \qquad p_1 = p$$

$$E[X] = p \qquad \text{VAR}[X] = p(1 - p) \qquad G_X(z) = (q + pz)$$

Binomial

$X =$ number of successes in n Bernoulli trials.

$$S_X = \{0, 1, \ldots, n\} \qquad p_k = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np \qquad \text{VAR}[X] = np(1 - p) \qquad G_X(z) = (q + pz)^n$$

Geometric (type 1)

$X =$ number of Bernoulli trial failures before succeeding.

$$S_X = \{0, 1, \ldots\} \qquad p_k = p(1 - p)^k$$

$$E[X] = \frac{1 - p}{p} \qquad \text{VAR}[X] = \frac{1 - p}{p^2} \qquad G_X(z) = \frac{p}{1 - qz}$$

Geometric (type 2)

$X =$ number of first Bernoulli trial to succeed.

$$S_X = \{1, 2, \ldots\} \qquad p_k = p(1 - p)^{k-1}$$

$$E[X] = \frac{1}{p} \qquad \text{VAR}[X] = \frac{1 - p}{p^2} \qquad G_X(z) = \frac{pz}{1 - qz}$$

Negative Binomial

$X =$ number of r^{th} Bernoulli trial to succeed.

$$S_X = \{r, r + 1, \ldots\} \qquad p_k = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$E[X] = \frac{r}{p} \qquad \text{VAR}[X] = \frac{r(1 - p)}{p^2} \qquad G_X(z) = \left(\frac{pz}{1 - qz}\right)^r$$

Poisson

$X =$ number of events in one time unit when time between events is exponentially distributed with mean $\frac{1}{\alpha}$.

$$S_X = \{0, 1, \ldots\} \qquad p_k = \frac{a^k}{k!} e^{-\alpha}$$

$$E[X] = \alpha \qquad \text{VAR}[X] = \alpha \qquad G_X(z) = e^{\alpha(z-1)}$$

Uniform

$X =$ any of L equally likely outcomes.

$$S_X = \{1, 2, \ldots, L\} \qquad p_k = \frac{1}{L}$$

$$E[X] = -1 + \frac{L + 1}{2} \qquad \text{VAR}[X] = \frac{L^2 - 1}{12} \qquad G_X(z) = \frac{z}{L} \frac{1 - z^L}{1 - z}$$

Zipf

$X =$ any of L outcomes, where few are frequent and most rare.

$$S_X = \{1, 2, \ldots, L\} \qquad p_k = \frac{1}{k c_L} \qquad c_L = \sum_{j=1}^{L} \frac{1}{j}$$

$$E[X] = \frac{L}{c_L} \qquad \text{VAR}[X] = \frac{L(L + 1)}{2 c_L} - \frac{L^2}{{c_L}^2}$$

Ch4. One Random Variable

$$\textit{(pdf)} \quad f_X(x) = \frac{d}{dx}F_X(x)$$

$$P[a \le X \le b] = \int_a^b f_X(x)dx$$

$$\textit{(cdf)} \quad F_X(x) = P[X \le x] = \int_{-\infty}^x p_X(t)dt$$

$$F_X(\infty) = \int -\infty^\infty p_X(t)dt = 1$$

$$\textit{(cond. cdf)} \quad F_X(x|C) = \frac{P[\{X \le x\} \cap C]}{P[C]}$$

$$\textit{(cond. pdf)} \quad f_X(x|C) = \frac{d}{dx}F_X(x|C)$$

$$\textit{(mean/expected val.)} \quad m_X = E[X] = \int_{-\infty}^\infty x f_X(x)dx$$

$$\textit{(func mean)} \quad E[g(x)] = \int_{-\infty}^\infty g(x)f_X(x)dx$$

$$\textit{(variance)} \quad \sigma_X^2 = \mathrm{VAR}[X] = E[X^2] - E[X]^2$$

$$\textit{(std.dev)} \quad \sigma_X = \mathrm{STD}[X] = \sqrt{\mathrm{VAR}[X]}$$

$$\textit{(}n^{th}\textit{ moment)} \quad E[X^n] = \int_{-\infty}^\infty x^n f_X(x)$$

$$\textit{(linear func.)} \quad Y = aX + b \qquad f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$$

$$\textit{(Markov)} \quad P[X \ge a] \le \frac{E[X]}{a} \quad X \ge 0$$

$$\textit{(Chebyshev)} \quad P[|X - m_X| \ge a] \le \frac{\sigma^2}{a^2}$$

Important Continuous Random Variables

Uniform

$X =$ uniformly distributed in interval $[a, b]$.

$$S_X = [a, b] \qquad f_X = \frac{1}{b-a} \qquad E[X] = \frac{a+b}{2}$$

$$\mathrm{VAR}[X] = \frac{(b-a)^2}{12} \qquad \Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$$

Exponential

$X =$ memoryless ($P[X > t + h | X > t] = P[X > h]$).

$$S_X = [0, \infty) \qquad f_X = \lambda e^{-\lambda x} \qquad E[X] = \frac{1}{\lambda}$$

$$\mathrm{VAR}[X] = \frac{1}{\lambda} \qquad \Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$$

Gamma

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad \Gamma(z+1) = z\Gamma(z) \qquad \Gamma(m+1) = m!$$

$$S_X = (0, \infty) \qquad f_X = \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} \qquad E[X] = \frac{\alpha}{\lambda}$$

$$\mathrm{VAR}[X] = \frac{\alpha}{\lambda^2} \qquad \Phi_X(\omega) = \left(\frac{1}{1 - j\omega/\lambda}\right)^\alpha$$

For $\alpha = m$, get an $m-1$ Erlang variable (sum of m exponential variables with parameter λ). For $\alpha = k/2$ and $\lambda = 1/2$ get chi-square variable with k degrees of freedom (sum of k independent Gaussian variables with $m = 0$ and $\sigma = 1$).

Gaussian (Normal)

$X \approx$ sum of a large number of independent random variables.

$$S_X = (-\infty, \infty) \qquad f_X = \frac{e^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \qquad E[X] = m$$

$$\mathrm{VAR}[X] = \sigma^2 \qquad \Phi_X(\omega) = e^{jm\omega - \sigma^2\omega^2/2}$$

Laplacian

$$S_X = (-\infty, \infty) \qquad f_X = \frac{\alpha}{2}e^{-\alpha|x|} \qquad E[X] = 0$$

$$\mathrm{VAR}[X] = \frac{2}{\alpha^2} \qquad \Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$$

Rayleigh

$$S_X = [0, \infty) \qquad f_X = \frac{x}{\alpha^2}e^{-x^2/2\alpha^2}$$

$$E[X] = \alpha\sqrt{\pi/2} \qquad \mathrm{VAR}[X] = (2 - \pi/2)\alpha^2$$

Cauchy

$$S_X = (-\infty, \infty) \qquad f_X = \frac{\alpha/\pi}{x^2 + \alpha^2} \qquad \Phi_X(\omega) = e^{-\alpha|\omega|}$$

Pareto

Has a long tail like a continuous Zipf random variable.

$$S_X = (x_m > 0, \infty) \qquad E[X] = \frac{\alpha x_m}{\alpha - 1} \quad \alpha > 1$$

$$f_X = \alpha\frac{{x_m}^\alpha}{x^{\alpha+1}} \qquad \mathrm{VAR}[X] = \frac{\alpha {x_m}^2}{(\alpha-2)(\alpha-1)^2} \quad \alpha > 2$$

Beta

$$S_X = (0, 1) \qquad f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

$$E[X] = \frac{\alpha}{\alpha+\beta} \qquad \mathrm{VAR}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Reliability

(Reliability) $R(t) = P[T > t] = 1 - P[T \le t] = 1 - F_T(t)$

(Mean Time to Fail) $E[T] = \int_0^\infty f_T(t)dt = \int_0^\infty R(t)dt$

(Future failure) $f_T(x|T > t) = \frac{f_T(x)}{R(t)} \quad x \ge t$

(Failure rate) $r(t) = f_T(t|T > t) = \frac{-R'(t)}{R(t)}$

(System failure rate) $r_{sys}(t) = \sum_{cpnt} r(t)$

Ch5. Pairs of Random Variables

Discrete

(joint pmf) $p_{X,Y}(x,y) = P[X = x, Y = y]$

(marginal pmf) $p_X(x) = \sum_y p_{X,Y}(x,y)$

Continuous

(joint cdf) $F_{X,Y}(x,y) = P[X \le x, Y \le y]$

(marginal cdf) $F_X(x) = F_{X,Y}(x,\infty)$

(joint pdf) $f_{X,Y}(x,y) = \frac{d^2}{dx\,dy}F_{X,Y}(x,y)$

(marginal pdf) $f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x,y)dy$

(X and Y independent) $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

(X and Y independent) $F_{X,Y}(x,y) = F_X(x)F_Y(y)$

(fn. mean) $E[g(x,y)] = \int_{-\infty}^\infty \int_{-\infty}^\infty g(x,y)f_{X,Y}(x,y)dx\,dy$

(mean of sum) $E[\sum X] = \sum E[X]$

(mean of ind. product) $E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)]$

(joint moment) $E[X^jY^k] = \int_{-\infty}^\infty \int_{-\infty}^\infty x^jy^k f_{X,Y}(x,y)dx\,dy$

(central moment) $E[(X - E[X])^j(Y - E[Y])^k]$

(covariance) $\mathrm{COV}(X,Y) = E[(X - E[X])(Y - E[Y])]$

(0 for independent vars) $\mathrm{COV}(X,Y) = E[XY] - E[X]E[Y]$

(correlation coef.) $\rho_{X,Y} = \frac{\mathrm{COV}(X,Y)}{\sigma_X\sigma_Y} \to [-1,1]$

(X and Y uncorrelated) $\rho_{X,Y} = 0$

Conditional Probability

(discrete cond. pmf) $p_Y(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$

(discrete conditional mean) $E[Y|x] = \sum_y y f_Y(y|x)$

(continuous cond. cdf) $f_Y(y|x) = \frac{\int_{-\infty}^y f_{X,Y}(x,y')dy'}{f_X(x)}$

(continuous cond. pdf) $f_Y(y|x) = \frac{d}{dy}F_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

(continuous conditional mean) $E[Y|x] = \int_{-\infty}^\infty y f_Y(y|x)dy$

Functions of Two Variables

(function cdf) $F_Z(z) = \iint_{g(x,y)=z} f_{X,Y}(x,y)$

(function pdf) $f_Z(z) = \frac{d}{dz}F_Z(z)$

(function pdf) $f_Z(z) = \int_{-\infty}^\infty f_Z(z|y)f_Y(y)dy$

Ch6. Vector Random Variables

$$\textit{(joint cdf)} \quad F_X(\mathbf{x}) = F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n]$$

$$\textit{(joint pmf)} \quad p_X(\mathbf{x}) = p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

- To get a marginal CDF, simply set the appropriate parameters to ∞ in the joint cdf for the whole vector to remove the effect of those variables on the function.
- To get a marginal PMF, simply sum the joint PMF over all values of all variables to be removed from the pmf.
- To get the joint PDF, take the derivative of the CDF once with respect to each variable in the vector. To turn it into a marginal PDF, integrate out any variables to be removed from the vector.
- To get the conditional PDF, take the joint PDF over a partial joint PDF. The same goes for PMF.
- A vector is independent if its joint CDF is equal to the product of each one-dimensional CDF. The same applies to PMF (for discrete vectors) and PDF (for jointly continuous vectors).
- To find the CDF of a function of a vector, find the vector(s) that will produce the value of the function and integrate the joint pdf inside that area. To find its PDF, take the derivative.