MANE 4900 Aeroelasticity & Structural Vibrations	Homework 6 solution	April 15, 2011

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_i}\right) - \frac{\partial L}{\partial \xi_i} = \mathbb{F}_i \qquad \text{where } L = K - P \text{ and } \mathbb{F}_i \text{ is generalized force}$$

$K = 0$

$$P = \frac{EI}{2}\int_0^l (w'')^2 d\bar{y} + \frac{GJ}{2}\int_0^l (\bar{\theta}')^2 d\bar{y}$$

w - deflection of wing

$\bar{\theta}$ - twist about elastic axis

$$\delta W = \int_0^l (L'\delta w + M'\delta\bar{\theta})\, d\bar{y}$$

$$L' = qca\cos\Lambda(\alpha_R + \theta)$$

$$M' = qeca\cos\Lambda(\alpha_R + \theta)$$

$$\theta = \bar{\theta}\cos\Lambda - w'\sin\Lambda$$

change in A.O.A due to aeroelastic effects

(1) $\displaystyle w(\bar{y}) = \sum_{i=1}^{N_w} \xi_i \phi_i(\bar{y})$ solve for $-\frac{\partial L}{\partial \xi_i}$ and $\mathbb{F}_i$

$$\bar{\theta}(\bar{y}) = \sum_{i=N_w+1}^{N_w+N_\theta} \xi_i \phi_i(\bar{y})$$

→ solving for $-\frac{\partial L}{\partial \xi_i}$

$$L = \cancel{K}^{0} - P = -P = -\frac{EI}{2}\int_0^l (w'')^2 d\bar{y} - \frac{GJ}{2}\int_0^l (\bar{\theta}')^2 d\bar{y}$$

$$w''(\bar{y}) = \sum_{i=1}^{N_w} \xi_i \phi_i'' \quad \text{and} \quad \bar{\theta}'(\bar{y}) = \sum_{i=N_w+1}^{N_w+N_\theta} \xi_i \phi_i'$$

$$L = -\frac{EI}{2}\int_0^l \left(\sum_{i=1}^{N_w} \xi_i\phi_i''\right)^2 d\bar{y} - \frac{GJ}{2}\int_0^l \left(\sum_{i=N_w+1}^{N_w+N_\theta} \xi_i\phi_i'\right)^2 d\bar{y}$$

utilizing Eqn 2.93, we got

$$L = -\frac{EI}{2}\sum_{i=1}^{N_w}\sum_{j=1}^{N_w} \xi_i\xi_j \int_0^l \phi_i''\phi_j''\, d\bar{y} - \frac{GJ}{2}\sum_{i=N_w+1}^{N_w+N_\theta}\sum_{j=N_w+1}^{N_w+N_\theta} \xi_i\xi_j \int_0^l \phi_i'\phi_j'\, d\bar{y}$$

now we need $-\frac{\partial L}{\partial \xi_i}$

$$\frac{\partial}{\partial \xi_i}\frac{EI}{2}\int_0^l \left(\sum_{m=1}^{N}\xi_m\phi_m''\right)\left(\sum_{n=1}^{N}\xi_n\phi_n''\right) d\bar{y} = \frac{EI}{2}\sum_m\sum_n \left(\int_0^l \phi_m''\phi_n''\, d\bar{y}\right)\frac{\partial}{\partial \xi_i}(\xi_m\xi_n)$$

$$= \frac{EI}{2}\sum_m\sum_n \left(\int_0^l \phi_m''\phi_n''\, d\bar{y}\right)(\xi_m\delta_{in} + \xi_n\delta_{im}) \qquad \delta_{in} = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}$$

$$= \frac{EI}{2}\sum_m \left(\int_0^l \phi_m''\phi_i''\, d\bar{y}\right)\xi_m + \frac{EI}{2}\sum_n \left(\int_0^l \phi_i''\phi_n''\, d\bar{y}\right)\xi_n$$

n and m are dummy indices so they can be equal, $n = m = j$

$$= EI\sum_{j=1}^{N}\left(\int_0^l \phi_i''\phi_j''\, d\bar{y}\right)\xi_j$$

$$\boxed{-\frac{\partial L}{\partial \xi_i} = \begin{cases} \displaystyle\sum_{j=1}^{N_w} EI\left(\int_0^l \phi_i''\phi_j''\, d\bar{y}\right)\xi_j & i \leq N_w \\ \displaystyle\sum_{j=N_w+1}^{N_w+N_\theta} GJ\left(\int_0^l \phi_i'\phi_j'\, d\bar{y}\right)\xi_j & i > N_w \end{cases}}$$

→ solving for $\mathbb{F}_i$

$$\delta W = \int_0^l (L'\delta w + M'\delta\bar{\theta})\, d\bar{y}$$

Eqn 2.108 gives $\delta w = \sum_{i=1}^{N_w} \delta\xi_i \phi_i$ and $\delta\bar{\theta} = \sum_{i=N_w+1}^{N_w+N_\theta} \delta\xi_i \phi_i$

$$\delta W = \int_0^l L' \sum_{i=1}^{N_w} \delta\xi_i \phi_i\, d\bar{y} + \int_0^l M' \sum_{i=N_w+1}^{N_w+N_\theta} \delta\xi_i \phi_i\, d\bar{y}$$

$$\delta W = \sum_{i=1}^{N_w} \delta\xi_i \underbrace{\int_0^l L'\phi_i\, d\bar{y}}_{\mathbb{F}_i} + \sum_{i=N_w+1}^{N_w+N_\theta} \delta\xi_i \underbrace{\int_0^l M'\phi_i\, d\bar{y}}_{\mathbb{F}_i}$$

$$\mathbb{F}_i = \begin{cases} \int_0^l L'\phi_i\, d\bar{y} & i \le N_w \\ \int_0^l M'\phi_i\, d\bar{y} & i > N_w \end{cases}$$

$\underline{i \le N_w}$ $\quad \mathbb{F}_i = \int_0^l L'\phi_i d\bar{y} = \int_0^l qca\cos\Lambda(\alpha_R + \theta)\phi_i d\bar{y} = qca\cos\Lambda \int_0^l (\alpha_R + \bar{\theta}\cos\Lambda + w'\sin\Lambda)\phi_i d\bar{y}$

$$\mathbb{F}_i = qca\cos\Lambda\left[\alpha_R \int_0^l \phi_i d\bar{y} + \cos\Lambda \int_0^l \bar{\theta}\phi_i d\bar{y} + \sin\Lambda \int_0^l w'\phi_i d\bar{y}\right]$$

plug in $\bar{\theta} = \sum_{j=N_w+1}^{N_w+N_\theta} \xi_j \phi_j$ and $w' = \sum_{j=1}^{N_w} \xi_i \phi_j'$

$$\mathbb{F}_i = qca\cos\Lambda\left(\alpha_R \int_0^l \phi_i d\bar{y} + \cos\Lambda \sum_{j=N_w+1}^{N_w+N_\theta} \xi_j \left(\int_0^l \phi_i\phi_j d\bar{y}\right) - \sin\Lambda \sum_{j=1}^{N_w} \xi_j \left(\int_0^l \phi_i\phi_j' d\bar{y}\right)\right)$$

$\underline{i > N_w}$ $\quad \mathbb{F}_i = \int_0^l M'\phi_i d\bar{y} = \int_0^l qeca\cos\Lambda(\alpha_R + \theta)\phi_i d\bar{y} = qeca\cos\Lambda \int_0^l (\alpha_R + \bar{\theta}\cos\Lambda + w'\sin\Lambda)\phi_i d\bar{y}$

$$\mathbb{F}_i = qeca\cos\Lambda\left[\alpha_R \int_0^l \phi_i d\bar{y} + \cos\Lambda \sum_{j=N_w+1}^{N_w+N_\theta} \xi_j \left(\int_0^l \phi_i\phi_j d\bar{y}\right) - \sin\Lambda \sum_{j=1}^{N_w} \xi_j \left(\int_0^l \phi_i\phi_j' d\bar{y}\right)\right]$$

$$\therefore \boxed{\mathbb{F}_i = \begin{cases} qca\cos\Lambda\left[\alpha_R \int_0^l \phi_i d\bar{y} + \cos\Lambda \sum_{j=N_w+1}^{N_w+N_\theta} \xi_j \int_0^l \phi_i\phi_j d\bar{y} - \sin\Lambda \sum_{j=1}^{N_w} \xi_j \int_0^l \phi_i\phi_j' d\bar{y}\right] & i \le N_w \\ qeca\cos\Lambda\left[\alpha_R \int_0^l \phi_i d\bar{y} + \cos\Lambda \sum_{j=N_w+1}^{N_w+N_\theta} \xi_j \int_0^l \phi_i\phi_j d\bar{y} - \sin\Lambda \sum_{j=1}^{N_w} \xi_i \int_0^l \phi_i\phi_j' d\bar{y}\right] & i > N_w \end{cases}}$$

(2) $(\underline{K} + q\underline{A})\vec{\xi} = \vec{f}$

plug in $-\frac{\partial L}{\partial \xi_i}$ and $\mathbb{F}_i$ into Lagrange's Eqns

and note that $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_i}\right) = 0$ because $K=0$

Then rearrange terms to get

Note: The product of a term like $\sum_{j=1}^{N} B_{ij} v_j$ is a vector.

In our case, $(K_{ij} + q A_{ij}) = B_{ij}$ and $\xi_j = v_j$

$$\boxed{f_i = \begin{cases} q c a \cos\Lambda\, \alpha_R \int_0^l \phi_i \, d\bar{y} & i \le N_w \\ q e c a \cos\Lambda\, \alpha_R \int_0^l \phi_i \, d\bar{y} & i > N_w \end{cases}}$$

$$\boxed{K_{ij} = \begin{cases} EI \cdot \int_0^l \phi_i'' \phi_j'' \, d\bar{y} & i \le N_w,\ j \le N_w \\ 0 & i \le N_w,\ j > N_w \\ 0 & i > N_w,\ j \le N_w \\ GJ \cdot \int_0^l \phi_i' \phi_j' \, d\bar{y} & i > N_w,\ j > N_w \end{cases}}$$

$$\boxed{A_{ij} = \begin{cases} c a \cos\Lambda \sin\Lambda \cdot \int_0^l \phi_i \phi_j' \, d\bar{y} & i \le N_w,\ j \le N_w \\ -c a \cos^2\Lambda \cdot \int_0^l \phi_i \phi_j \, d\bar{y} & i \le N_w,\ j > N_w \\ c a e \cos\Lambda \sin\Lambda \cdot \int_0^l \phi_i \phi_j' \, d\bar{y} & i > N_w,\ j \le N_w \\ -c a e \cos^2\Lambda \cdot \int_0^l \phi_i \phi_j \, d\bar{y} & i > N_w,\ j > N_w \end{cases}}$$

(3) $$\phi_i(\bar{y}) = \begin{cases} \left(\frac{\bar{y}}{l}\right)^{i+1} & i \le N_w \\ \left(\frac{\bar{y}}{l}\right)^{i-N_w} & i > N_w \end{cases}$$

Verify that these are admissible → Use page 61

1. satisfy all BCs on displacement and rotation at the root

$$\phi_i(0) = \begin{cases} \left(\frac{0}{l}\right)^{i+1} = 0 & i \le N_w \\ \left(\frac{0}{l}\right)^{i-N_w} = 0 & i > N_w \end{cases} \qquad \text{GOOD}$$

$$\phi'(0) = \begin{cases} \frac{(i+1)}{l^{i+1}}\, 0^{i} = 0 & i \le N_w \\ \frac{(i-N_w)}{l^{i-N_w}}\, 0^{i-N_w+1} = 0 & i > N_w \end{cases} \qquad \text{GOOD}$$

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2. Continuous and p times differentiable
for our case $p=2$ for $i \le N_w$ and $p=1$ $i > N_w$

$$\phi_i'(\bar{y}) = \begin{cases} \dfrac{(i+1)}{l^{i+1}} \bar{y}^{i} & i \le N_w \\ \dfrac{(i-N_w)}{l^{i-N_w}} (\bar{y})^{i-N_w-1} & i > N_w \end{cases}$$

$$\phi_i''(\bar{y}) = \begin{cases} \dfrac{(i^2+i)}{l^{i+1}} \bar{y}^{i-1} & i \le N_w \\ \dfrac{(i-N_w)(i-N_w-1)}{l^{i-N_w}} \bar{y}^{i-N_w-2} & i > N_w \end{cases}$$

GOOD

3. Set of functions is complete.
4. Set of functions is linearly independent.

$\therefore$ These functions are admissible.

Now evaluate the integrals from previous part.

① $i \le N_w$ $$\int_0^l \phi_i \, d\bar{y} = \int_0^l \left(\frac{\bar{y}}{l}\right)^{i+1} d\bar{y} = \frac{l}{i+2}\left(\frac{\bar{y}}{l}\right)^{i+2}\Bigg|_0^l = \frac{l}{i+2}$$

② $i > N_w$ $$\int_0^l \phi_i \, d\bar{y} = \int_0^l \left(\frac{\bar{y}}{l}\right)^{i-N_w} d\bar{y} = \frac{l}{i-N_w+1}\left(\frac{\bar{y}}{l}\right)^{i-N_w+1}\Bigg|_0^l = \frac{l}{i-N_w+1}$$

③ $i \le N_w, j \le N_w$ $$\int_0^l \phi_i'' \phi_j'' \, d\bar{y} = \frac{(i^2+i)(j^2+j)}{l^{i+j+2}} \int_0^l \bar{y}^{i+j-2} d\bar{y} = \frac{(i^2+i)(j^2+j)}{l^{i+j+2}} \frac{\bar{y}^{i+j-1}}{i+j-1}\Bigg|_0^l$$
$$= \frac{(i^2+i)(j^2+j)}{l^3(i+j-1)}$$

④ $i > N_w, j > N_w$ $$\int_0^l \phi_i' \phi_j' \, d\bar{y} = \frac{(i-N_w)(j-N_w)}{l^{i+j-2N_w}} \int_0^l \bar{y}^{i+j-2N_w-2} d\bar{y} = \frac{(i-N_w)(j-N_w)}{l^{i+j-2N_w}} \frac{\bar{y}^{i+j-2N_w-1}}{i+j-2N_w-1}\Bigg|_0^l$$
$$= \frac{(i-N_w)(j-N_w)}{l(i+j-2N_w-1)}$$

⑤ $i \le N_w, j \le N_w$ $$\int_0^l \phi_i \phi_j' \, d\bar{y} = \frac{(j+1)}{l^{i+j+2}} \int_0^l \bar{y}^{i+j+1} d\bar{y} = \frac{(j+1)}{l^{i+j+2}} \frac{y^{i+j+2}}{i+j+2}\Bigg|_0^l = \frac{(j+1)}{i+j+2}$$

⑥ $i \le N_w, j > N_w$ $$\int_0^l \phi_i \phi_j \, d\bar{y} = \int_0^l \left(\frac{\bar{y}}{l}\right)^{i+1}\left(\frac{\bar{y}}{l}\right)^{j-N_w} d\bar{y} = \int_0^l \left(\frac{\bar{y}}{l}\right)^{i+j-N_w+1} d\bar{y} = \frac{l}{i+j-N_w+2}\left(\frac{\bar{y}}{l}\right)^{i+j-N_w+2}\Bigg|_0^l$$
$$= \frac{l}{i+j-N_w+2}$$

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⑦ $i > N_w,\ j \le N_w$

$$\int_0^l \phi_i \phi_j' \, dy = \frac{(j+1)}{l^{i+j-N_w+1}} \int_0^l \bar{y}^{\,i-N_w+j} \, d\bar{y} = \frac{(j+1)}{l^{i+j-N_w+1}} \left. \frac{\bar{y}^{\,i+j-N_w+1}}{i+j-N_w+1} \right|_0^l = \frac{(j+1)}{i+j-N_w+1}$$

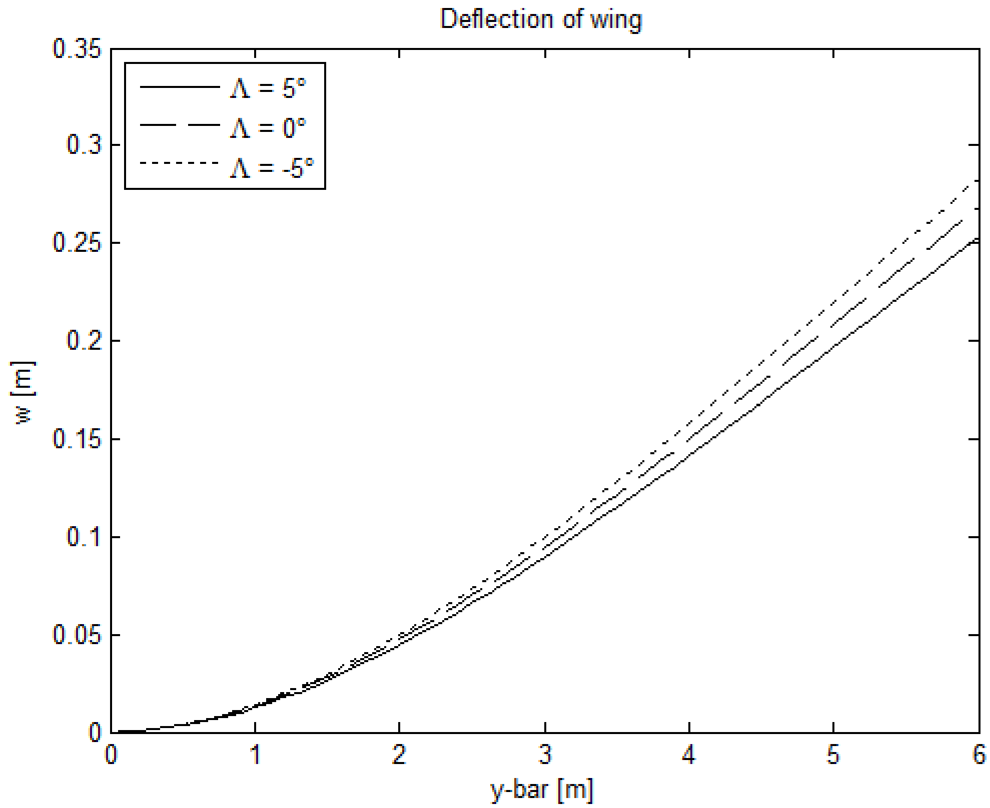
⑧ $i > N_w,\ j > N_w$

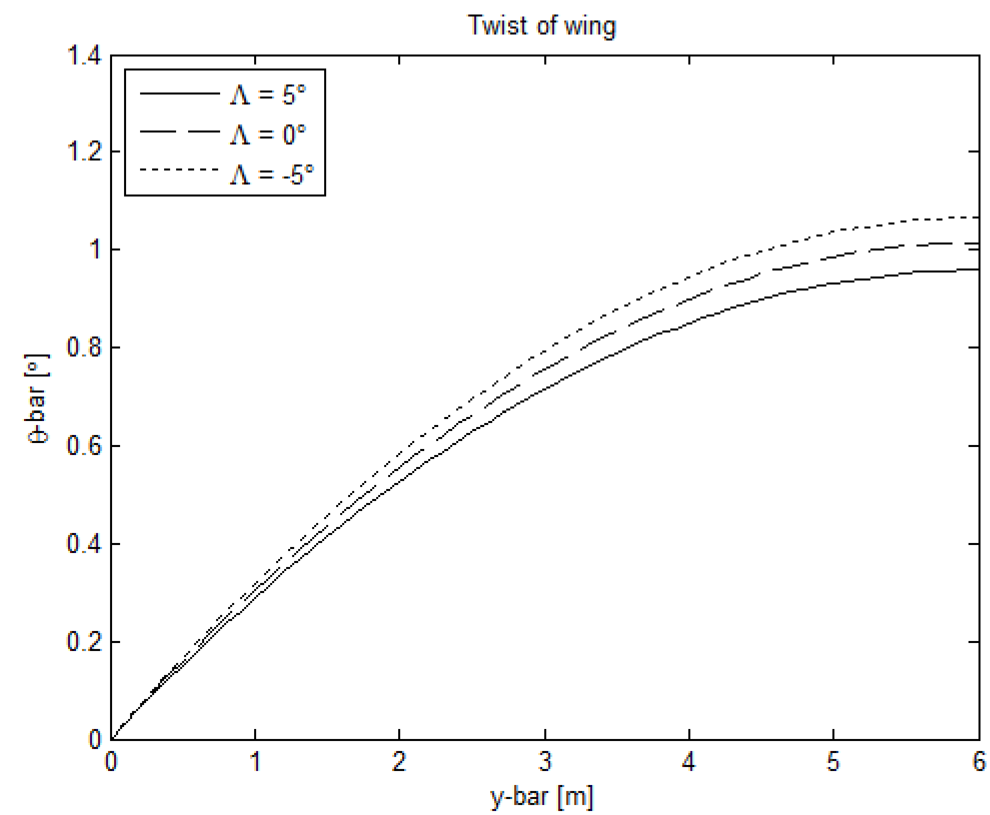
$$\int_0^l \phi_i \phi_j \, d\bar{y} = \int_0^l \left(\frac{\bar{y}}{l}\right)^{i-N_w} \left(\frac{\bar{y}}{l}\right)^{j-N_w} d\bar{y} = \int_0^l \left(\frac{\bar{y}}{l}\right)^{i+j-2N_w} d\bar{y} = \frac{l}{i+j-2N_w+1} \left. \left(\frac{\bar{y}}{l}\right)^{i+j-2N_w+1} \right|_0^l = \frac{l}{i+j-2N_w+1}$$

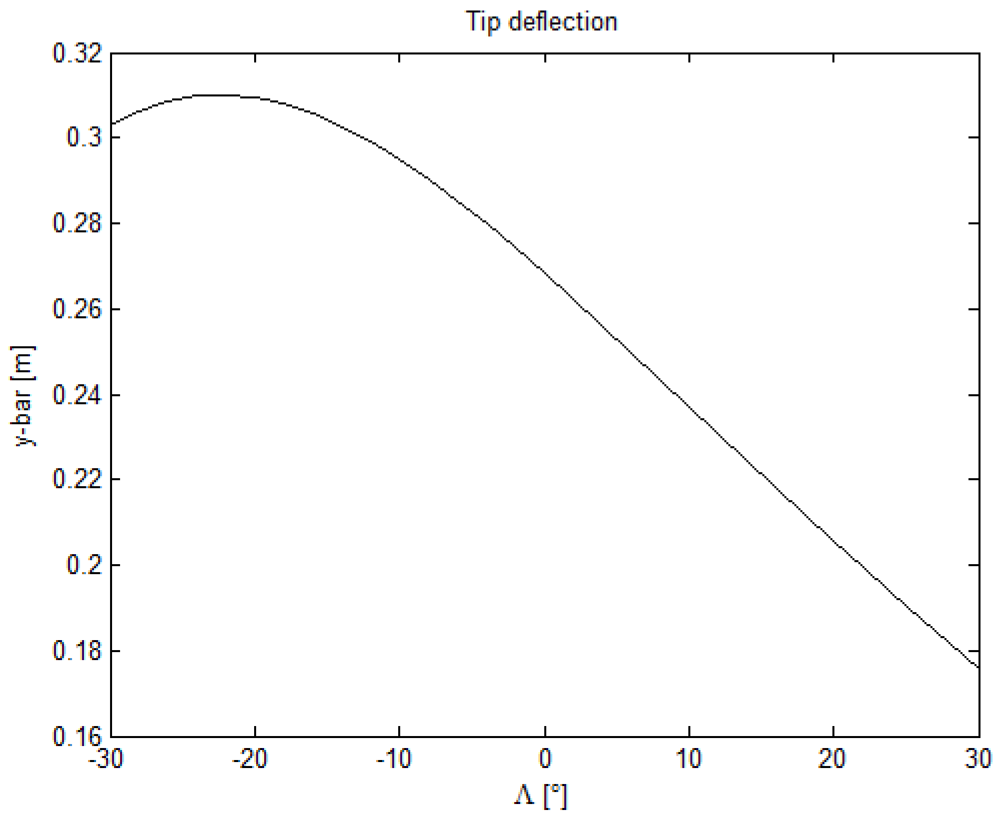
Now plug into f_i

$$\boxed{\begin{aligned}
f_i &= \begin{cases} q c a \cos\Lambda \, \alpha_R \left(\dfrac{l}{i+2}\right) & i \le N_w \\[2ex] q e c a \cos\Lambda \, \alpha_R \left(\dfrac{l}{i-N_w+1}\right) & i > N_w \end{cases} \\[3ex]
K_{ij} &= \begin{cases} EI \cdot \left(\dfrac{(i^2+i)(j^2+j)}{l^3 (i+j-1)}\right) & i \le N_w,\ j \le N_w \\[2ex] 0 & i \le N_w,\ j > N_w \\[2ex] 0 & i > N_w,\ j \le N_w \\[2ex] GJ \cdot \left(\dfrac{(i-N_w)(j-N_w)}{l(i+j-2N_w-1)}\right) & i > N_w,\ j > N_w \end{cases} \\[3ex]
A_{ij} &= \begin{cases} c a \cos\Lambda \sin\Lambda \cdot \left(\dfrac{j+1}{i+j+2}\right) & i \le N_w,\ j \le N_w \\[2ex] -c a \cos^2\Lambda \cdot \left(\dfrac{l}{i+j-N_w+2}\right) & i \le N_w,\ j > N_w \\[2ex] c a e \cos\Lambda \sin\Lambda \cdot \left(\dfrac{j+1}{i+j-N_w+1}\right) & i > N_w,\ j \le N_w \\[2ex] -c a e \cos^2\Lambda \cdot \left(\dfrac{l}{i+j-2N_w+1}\right) & i > N_w,\ j > N_w \end{cases}
\end{aligned}}$$

4.


Deflection of wing
Λ = 5°
Λ = 0°
Λ = -5°
w [m]
y-bar [m]
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0
1
2
3
4
5
6


Twist of wing
Λ = 5°
Λ = 0°
Λ = -5°
θ-bar [°]
y-bar [m]
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
1
2
3
4
5
6



Divergence does not occur.

```matlab
%% Code for MANE 4900 Spring 2011, Homework 6
clear; clc;
Nw=10; Nt=10;
GJ = 1.5e6; %N-m^2
EI = 5*GJ; %N-m^2
L = 6; %m
e = 0.02*L; %m
a = 2*pi;
q = 12e3; %Pa
alphaR = 5*(pi/180); %radians
c = 1.6; %m
Lambda = [5 0 -5]; %degrees

%% Deflection and twist
for k=1:length(Lambda)
    Lam = Lambda(k);
    for i=1:Nw+Nt
        if i<=Nw
            f(i,k) = q*c*a*cosd(Lam)*alphaR*L/(i+2);
        end
        if i>Nw
            f(i,k) = q*e*c*a*cosd(Lam)*alphaR*L/(i-Nw+1);
        end

        for j=1:Nw+Nt
            if i<=Nw && j<=Nw
                K(i,j,k) = (EI/L^3)*(i^2+i)*(j^2+j)/(i+j-1);
                A(i,j,k) = c*a*cosd(Lam)*sind(Lam)*(j+1)/(i+j+2);
            end
            if i<=Nw && j>Nw
                K(i,j,k) = 0;
                A(i,j,k) = -c*a*(cosd(Lam)^2)*L/(i+j-Nw+2);
            end
            if i>Nw && j<=Nw
                K(i,j,k) = 0;
                A(i,j,k) = c*a*e*cosd(Lam)*sind(Lam)*(j+1)/(i+j-Nw+1);
            end
            if i>Nw && j>Nw
                K(i,j,k) = (GJ/L)*(i-Nw)*(j-Nw)/(i+j-2*Nw-1);
                A(i,j,k) = -c*e*a*(cosd(Lam)^2)*L/(i+j-2*Nw+1);
            end
        end
    end
end

B = K+q.*A;
for k=1:length(Lambda)
    xi(:,k) = B(:,:,k)\f(:,k);
end

ybar = 0:.1:6;

for k=1:length(Lambda)
    for m=1:length(ybar)
        sumw = 0;
        sumt = 0;
        for i=1:Nw+Nt
            if i<=Nw
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                sumw = sumw + xi(i,k)*(ybar(m)/L)^(i+1);
            end
            if i>Nw
                sumt = sumt + xi(i,k)*(ybar(m)/L)^(i-Nw);
            end
        end
        w(m,k) = sumw;
        tbar(m,k) = sumt*180/pi;
    end
end


figure (1)
plot(ybar,w(:,1),'-black',ybar,w(:,2),'--black',ybar,w(:,3),':black')
title 'Deflection of wing'
xlabel 'y-bar [m]'
ylabel 'w [m]'
legend ('\Lambda = 5\circ', '\Lambda = 0\circ', '\Lambda = -
5\circ','Location','northwest')

figure(2)
plot(ybar,tbar(:,1),'-black',ybar,tbar(:,2),'--
black',ybar,tbar(:,3),':black')
title 'Twist of wing'
xlabel 'y-bar [m]'
ylabel '\theta-bar [\circ]'
legend ('\Lambda = 5\circ', '\Lambda = 0\circ', '\Lambda = -
5\circ','Location','northwest')


%% Tip Deflection
Lambda2 = -30:2:30; %degrees

for k=1:length(Lambda2)
    Lam = Lambda2(k);
    for i=1:Nw+Nt
        if i<=Nw
            f(i,k) = q*c*a*cosd(Lam)*alphaR*L/(i+2);
        end
        if i>Nw
            f(i,k) = q*e*c*a*cosd(Lam)*alphaR*L/(i-Nw+1);
        end

        for j=1:Nw+Nt
            if i<=Nw && j<=Nw
                K(i,j,k) = (EI/L^3)*(i^2+i)*(j^2+j)/(i+j-1);
                A(i,j,k) = c*a*cosd(Lam)*sind(Lam)*(j+1)/(i+j+2);
            end
            if i<=Nw && j>Nw
                K(i,j,k) = 0;
                A(i,j,k) = -c*a*(cosd(Lam)^2)*L/(i+j-Nw+2);
            end
            if i>Nw && j<=Nw
                K(i,j,k) = 0;
                A(i,j,k) = c*a*e*cosd(Lam)*sind(Lam)*(j+1)/(i+j-Nw+1);
            end
            if i>Nw && j>Nw
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                K(i,j,k) = (GJ/L)*(i-Nw)*(j-Nw)/(i+j-2*Nw-1);
                A(i,j,k) = -c*e*a*(cosd(Lam)^2)*L/(i+j-2*Nw+1);
            end
        end
    end
end

B = K+q.*A;
for k=1:length(Lambda2)
    xi(:,k) = B(:,:,k)\f(:,k);
end

ybar = 0:.1:6;

for k=1:length(Lambda2)
    sumw = 0;
    for i=1:Nw
        if i<=Nw
            sumw = sumw + xi(i,k);
        end
    end
    wtip(k) = sumw;
end

figure(3)
plot(Lambda2,wtip,'-black')
title 'Tip deflection'
xlabel '\Lambda [\circ]'
ylabel 'y-bar [m]'
```