① Tapered wing of length L

$$EI = E_0 I_0 \left(1 - \frac{3}{4}\left(\frac{x}{L}\right)^2\right)$$
$$m = m_0\left(1 - \frac{3}{4}\left(\frac{x}{L}\right)\right)$$

1) Given $K = \frac{1}{2}\int_0^L m\left(\frac{\partial v}{\partial t}\right)^2 dx$ and $P = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$

Determine the expressions for the mass and stiffness matrices for this structure for arbitrary basis functions ψ_is.

$$K = \frac{1}{2}\int_0^L m_0\left(1-\frac{3}{4}\left(\frac{x}{L}\right)\right)\left(\frac{\partial v}{\partial t}\right)^2 dx$$

$$v(x,t) = \sum_{i=1}^{N} \xi_i(t)\psi_i(x)$$
$$\frac{\partial v}{\partial t} = \sum_{i=1}^{N} \dot{\xi}_i(t)\psi_i(x)$$
$$\frac{\partial^2 v}{\partial x^2} = \sum_{i=1}^{n} \xi_i(t)\psi_i''(x)$$

$$K = \sum_{i=1}^{n} \dot{\xi}_i\dot{\xi}_j \cdot \frac{1}{2}\underbrace{\int_0^L m_0\left(1-\frac{3}{4}\frac{x}{L}\right)\psi_i\psi_j\,dx}_{M_{ij}}$$

$$P = \frac{1}{2}\sum_{i=1}^{n}\xi_i\xi_j \underbrace{\int_0^L E_0 I_0\, \psi_i''\psi_j''\,dx}_{K_{ij}}$$

Mass
$$\boxed{M_{ij} = m_0\int_0^L \left(1-\frac{3}{4}\left(\frac{x}{L}\right)\right)\psi_i\psi_j\,dx}$$

Stiffness
$$\boxed{K_{ij} = E_0 I_0\int_0^L \left(1-\frac{3}{4}\left(\frac{x}{L}\right)^2\right)\psi_i''\psi_j''\,dx}$$

2) $X(0)=0$ $\quad x''(L)=0$
$X'(0)=0$ $\quad x'''(L)=0$

$$\boxed{\psi_i(x) = \left(\frac{x}{L}\right)^{i+2} \qquad i=1,\dots,\infty}$$

This set of functions is admissible.

$$\psi_i' = (i+2)\left(\frac{x}{L}\right)^{i+1}\cdot\frac{1}{L} \qquad \psi_i(0)=0$$
$$\psi_i'' = (i+2)(i+1)\left(\frac{x}{L}\right)^{i}\cdot\frac{1}{L^2} \qquad \psi_i'(0)=0$$

3) $$M_{ij} = m_0\int_0^L\left(1-\frac{3}{4}\left(\frac{x}{L}\right)\right)\psi_i\psi_j\,dx = m_0\int_0^L\left(1-\frac{3}{4}\left(\frac{x}{L}\right)\right)\left(\frac{x}{L}\right)^{i+2}\left(\frac{x}{L}\right)^{j+2}dx$$

set $y = \frac{x}{L}$, $\quad \frac{dy}{dx} = \frac{1}{L}$

$$= m_0\int_0^L\left(1-\frac{3}{4}y\right)y^{i+j+4}\,dx = m_0\int_0^L y^{i+j+4}\,dx - \frac{3m_0}{4}\int_0^L y^{i+j+5}\,dx$$

$$= \frac{m_0}{i+j+5}\,y^{i+j+5}\cdot\frac{1}{dy/dx}\Bigg|_0^L - \frac{\frac{3}{4}m_0}{i+j+6}\,y^{i+j+6}\cdot\frac{1}{dy/dx}\Bigg|_0^L$$

$$= \frac{m_0 L}{i+j+5}\left(\frac{x}{L}\right)^{i+j+5}\Bigg|_0^L - \frac{\frac{3}{4}m_0 L}{i+j+6}\left(\frac{x}{L}\right)^{i+j+6}\Bigg|_0^L = \frac{m_0 L}{i+j+5}(1-0)^{i+j+5} - \frac{\frac{3}{4}m_0 L}{i+j+6}(1-0)^{i+j+6}$$

$$= \frac{m_0 L(i+j+6) - \frac{3}{4}m_0 L(i+j+5)}{(i+j+5)(i+j+6)} = \boxed{\frac{m_0 L\left[(i+j+6) - \frac{3}{4}(i+j+5)\right]}{(i+j+5)(i+j+6)}} = M_{ij}$$

$$K_{ij} = E_0 I_0 \int_0^L \left(1 - \frac{3}{4}\left(\frac{x}{L}\right)^2\right) \psi_i'' \psi_j'' \, dx$$

$$= \frac{E_0 I_0}{L^4} \int_0^L \left(1 - \frac{3}{4}\left(\frac{x}{L}\right)^2\right) \left[\frac{(i+2)(i+1)}{L^2}\left(\frac{x}{L}\right)^i\right]\left[\frac{(j+2)(j+1)}{L^2}\left(\frac{x}{L}\right)^j\right] dx \qquad y = \frac{x}{L} \quad \frac{dy}{dx} = \frac{1}{L}$$

$$= \frac{E_0 I_0}{L^4} \underbrace{(i+2)(i+1)(j+2)(j+1)}_{C} \int_0^L \left(1 - \frac{3}{4}y^2\right) y^{i+j} \, dx$$

$$= \frac{E_0 I_0 C}{L^4} \left\{ \int_0^L y^{i+j} \, dx - \frac{3}{4}\int_0^L y^{i+j+2} \, dx \right\}$$

$$= \frac{E_0 I_0 C}{L^4} \left\{ \frac{L}{i+j+1} y^{i+j+1} \Big|_0^L - \frac{\frac{3}{4}L}{i+j+3} y^{i+j+3} \Big|_0^L \right\}$$

$$= \frac{E_0 I_0 C}{L^4} \left\{ \frac{L}{i+j+1} \left(\frac{x}{L}\right)^{i+j+1} \Big|_0^L - \frac{\frac{3}{4}L}{i+j+3} \left(\frac{x}{L}\right)^{i+j+3} \Big|_0^L \right\}$$

$$= \frac{E_0 I_0 C}{L^4} \left\{ \frac{L}{i+j+1} - \frac{\frac{3}{4}L}{i+j+3} \right\} = \frac{E_0 I_0 C}{L^3} \left\{ \frac{(i+j+3) - \frac{3}{4}(i+j+1)}{(i+j+1)(i+j+3)} \right\}$$

$$\boxed{K_{ij} = \frac{E_0 I_0}{L^3} (i+2)(i+1)(j+2)(j+1) \left\{ \frac{(i+j+3) - \frac{3}{4}(i+j+1)}{(i+j+1)(i+j+3)} \right\}}$$

4) According to eqn. 2.299 $\left[[K] - \omega^2 [M]\right]\{\xi\} = 0$ where $\bar{\xi}$ is the amplitude of each mode

$$\omega_i = \sqrt{\frac{K_{ij}}{M_{ij}}}$$

$$C_{ki} = (i+2)(i+1)(j+2)(j+1)\left\{\frac{(i+j+3) - \frac{3}{4}(i+j+1)}{(i+j+1)(i+j+3)}\right\}$$

$$C_{Mi} = \frac{(i+j+6) - \frac{3}{4}(i+j+5)}{(i+j+5)(i+j+6)}$$

$$\omega_i = \sqrt{\frac{K_{ij}}{M_{ij}}} = \sqrt{\frac{E_0 I_0 C_{ki}}{m_0 L^4 C_{Mi}}}$$

$$\omega_i = \sqrt{\frac{C_{ki}}{C_{mi}}}\sqrt{\frac{E_0 I_0}{m_0 L^4}} \qquad \bar{\omega}_i = \sqrt{\frac{C_{ki}}{C_{Mi}}}$$

$$\boxed{\omega_i = \bar{\omega}_i \sqrt{\frac{E_0 I_0}{m_0}} \cdot \frac{1}{L^2}}$$

5) find eigenvalues of $[M]^{-1}[K]$

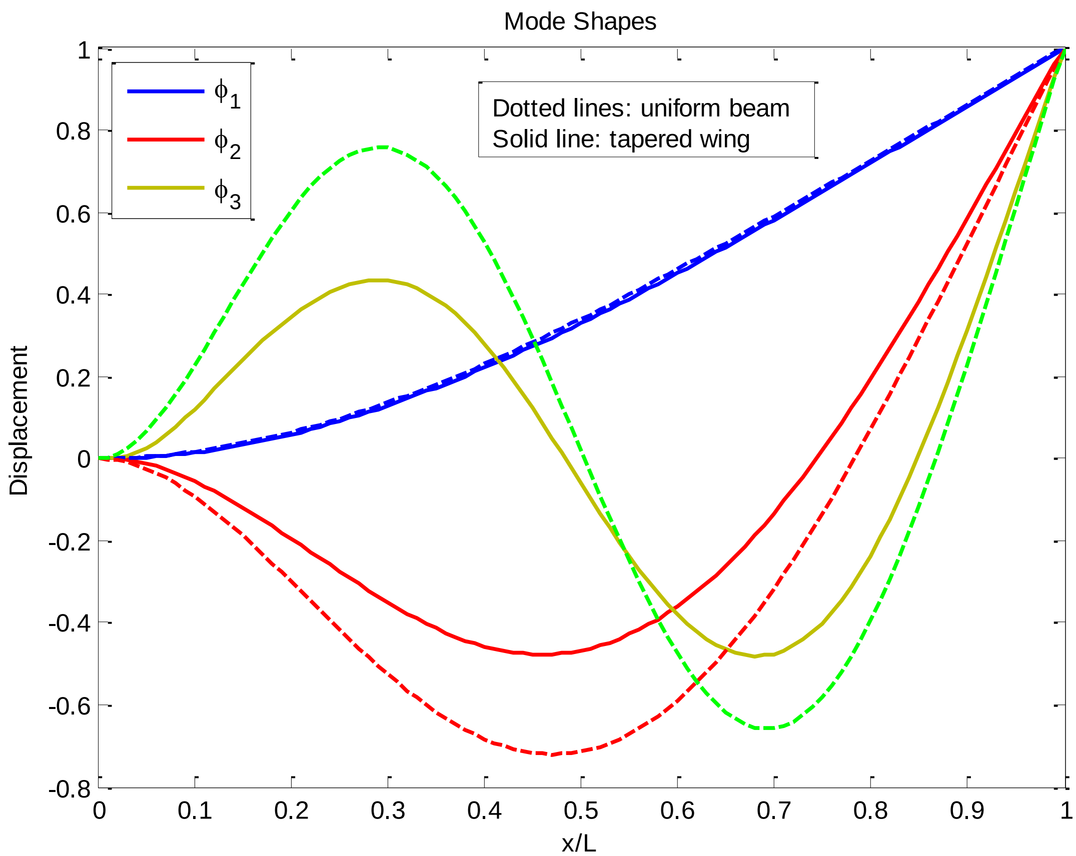
ω is the squareroot of these eigenvalues

$$\boxed{\begin{aligned} \bar{\omega}_1 &= 5.55 \\ \bar{\omega}_2 &= 28.20 \\ \bar{\omega}_3 &= 73.29 \end{aligned}}$$

mode shapes: $\phi_{ij} = \sum_{j=1}^{n} \xi_{ij} \psi_{ij}$ where ξ_{ij} is eigenvector for mode i and ψ_{ij} are basis functions

6) Looking at plot of mode shapes for tapered wing and uniform beam, one can see that the amplitudes are greater for the uniform beam, except for the first mode, where they are nearly identicle. Comparing values of $(\alpha_i l)^2$ for uniform beam case and $\bar{\omega}_i$ for the tapered wing, one sees that the frequencies for tapered wings are higher than those for uniform beams.

Plot and Code for Problem 1



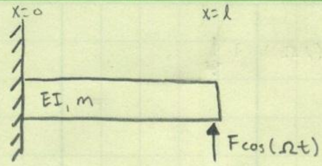

```
clear; clc;
n = 10; % number of basis functions
for i=1:n;
    for j=1:n;
        M(i,j) = ((i+j+6)-0.75*(i+j+5))/((i+j+5)*(i+j+6));
        K(i,j) = (i+2)*(i+1)*(j+2)*(j+1)*((i+j+3)-0.75*(i+j+1))/((i+j+1)*(i+j+3));
    end
end

[e,lam] = eig(K,M);
omega = sqrt(diag(lam));
disp('omega_1='); disp(omega(1));
disp('omega_2='); disp(omega(2));
disp('omega_3='); disp(omega(3));

x=0:.01:1;
N = length(x);
phi = zeros(3,N);
for i=1:n
    for j=1:N
        phi(1,j) = e(i,1)*x(j)^(i+2)+phi(1,j);
        phi(2,j) = e(i,2)*x(j)^(i+2)+phi(2,j);
        phi(3,j) = e(i,3)*x(j)^(i+2)+phi(3,j);
    end
end

alpha = [1.87510 4.69409 7.85476];
for j=1:N
    for i=1:3
       Beta(i) = (cosh(alpha(i))+cos(alpha(i)))/(sinh(alpha(i))+sin(alpha(i)));
       phi_u(i,j) = cosh(alpha(i)*x(j))-cos(alpha(i)*x(j))-Beta(i)*(sinh(alpha(i)*x(j))-sin(alpha(i)*x(j)));
    end
end

plot(x,phi(1,:)/phi(1,N),x,phi(2,:)/phi(2,N),x,phi(3,:)/phi(3,N))
hold on
plot(x,phi_u(1,:)/phi_u(1,N),'--b',x,phi_u(2,:)/phi_u(2,N),'--r',x,phi_u(3,:)/phi_u(3,N),'--g')
```

MANE 4900
Aeroelasticity + Structural Vibrations

Homework 4
Solution

March 4, 2011

② Problem 2.15



BCs

@ $x=0$ $\quad v(0,t) = \frac{\partial v}{\partial x}(0,t) = 0 \rightarrow X(0) = X'(0) = 0$

@ $x=l$ $\quad M(l,t) = V(l,t) = 0 \rightarrow X''(l) = X'''(l) = 0$

a) From Example 9:

$$\phi_i(x) = \cosh(\alpha_i x) - \cos(\alpha_i x) - \beta_i\left[\sinh(\alpha_i x) - \sin(\alpha_i x)\right]$$

$$\beta_i = \frac{\cosh(\alpha_i l) + \cos(\alpha_i l)}{\sinh(\alpha_i l) + \sin(\alpha_i l)}$$

$$\phi(l) = 2(-1)^{i+1}$$

$$\omega_i = (\alpha_i l)^2 \sqrt{\frac{EI}{ml^4}}$$

Generalized Eqn of Motion: $M_i(\ddot{\xi}_i + \omega_i^2 \xi_i) = \Xi_i$

$$\Xi_i = \int_0^l F(x,t)\phi_i(x)\,dx = \int_0^l F\cos(\Omega t)\,\delta(x-l)\,\phi_i(x)\,dx = F\cos(\Omega t)\cdot\phi_i(l) = 2(-1)^{i+1}F\cos(\Omega t)$$

for point load @ l

$$M_i = m\int_0^l \phi_i^2\,dx = ml$$

Note: $\int_0^l \phi_i^2\,dx = l$ according to example 9, eqn 2.253

substituting in gives

$$ml\left(\ddot{\xi}_i + \omega_i^2\xi_i\right) = 2(-1)^{i+1}F\cos(\Omega t)$$

$$\boxed{\ddot{\xi}_i + \omega_i^2\xi_i = 2(-1)^{i+1}\frac{F}{ml}\cos(\Omega t)}$$

b) $v(x,t) = \sum_{i=1}^{\infty} \xi_i(t)\phi_i(x)$

Assume $\xi_i(t)$ of the form $\quad \xi_i(t) = A_i\sin(\omega_i t) + B_i\cos(\omega_i t) + C_i\cos(\Omega t)$

Initial conditions $\quad v(x,0) = \frac{\partial v}{\partial t}(x,0) = 0 \quad$ (beam at rest)

$$\rightarrow \xi_i(0) = \dot{\xi}_i(0) = 0$$

$\xi_i(0) = B_i + C_i = 0 \longrightarrow \underline{B_i = -C_i}$

$\dot{\xi}_i(0) = \underline{A_i = 0}$

$$\xi_i(t) = C_i\left[\cos\Omega t - \cos\omega_i t\right]$$

$$\dot{\xi}_i(t) = C_i\left[-\Omega\sin\Omega t + \omega_i\sin\omega_i t\right]$$

$$\ddot{\xi}_i(t) = C_i\left[-\Omega^2\cos\Omega t + \omega_i^2\cos\omega_i t\right] = -C_i\left[\Omega^2\cos\Omega t - \omega_i^2\cos\omega_i t\right]$$

Plug into generalized eqn. of motion from part A

$$-C_i\left[\Omega^2\cancel{\cos\Omega t} - \cancel{\omega_i^2\cos\omega_i t}\right] + C_i\left[\omega_i^2\cancel{\cos\Omega t} - \cancel{\omega_i^2\cos\omega_i t}\right] = 2(-1)^{i+1}\frac{F}{ml}\cancel{\cos\Omega t}$$

$$C_i(\omega_i^2 - \Omega^2) = 2(-1)^{i+1}\frac{F}{ml}$$

$$C_i = \frac{2F}{ml}\frac{(-1)^{i+1}}{\omega_i^2 - \Omega^2} \qquad \therefore\ \xi_i(t) = \frac{2F}{ml}\frac{(-1)^{i+1}}{(\omega_i^2 - \Omega^2)}\left[\cos\Omega t - \cos\omega_i t\right]$$

$$\boxed{v(x,t) = \frac{2F}{ml}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{(\omega_i^2 - \Omega^2)}\left[\cos\Omega t - \cos\omega_i t\right]\phi_i(x)}$$

where $\phi_i(x)$ and ω_i are defined in Part A.

c) $\Omega = 0$ determine tip displacement, so @ $x = l$

$$v(x,t) = \frac{2F}{ml}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{\omega_i^2}\left[1-\cos(\omega_i t)\right]\phi_i(x)$$

$$v(l,t) = \frac{2F}{ml}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{\omega_i^2}\left[1-\cos\omega_i t\right]\phi_i(l)$$

$$\frac{1}{\omega_i^2} = \frac{ml^4}{EI}\cdot\frac{1}{(\alpha_i l)^4} \qquad \phi_i(l) = 2(-1)^{i+1}$$

$$v(l,t) = \frac{2Fl^3}{EI}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{(\alpha_i l)^4}\cdot 2(-1)^{i+1}$$

$\cos\omega_i t = 0$ because we are ignoring time-dependent terms

also, $(-1)^{i+1}(-1)^{i+1} = 1$

$$v(l,t) = \frac{4Fl^3}{EI}\underbrace{\left[\frac{1}{(\alpha_1 l)^4}+\frac{1}{(\alpha_2 l)^4}+\frac{1}{(\alpha_3 l)^4}+\frac{1}{(\alpha_4 l)^4}+\frac{1}{(\alpha_5 l)^4}\right]}_{\simeq 0.0833}$$

$\downarrow$

$4 \times 0.0833 = \frac{1}{3}$

$$\boxed{v(l,t) \simeq \frac{Fl^3}{3EI}}$$

Example 9:

$\alpha_1 l = 1.87510$
$\alpha_2 l = 4.69409$
$\alpha_3 l = 7.85476$
$\alpha_4 l = 10.9955$
$\alpha_5 l = 14.1372$

Use MATLAB to plot tip deflection using varying number of terms

One can see that the result converges to $\frac{Fl^3}{3EI}$ quickly.

Code and Plot for 2.15

```
clear; clc;

al=[1.87510; 4.69409; 7.85476; 10.9955; 14.1372];

for i=1:5;
    vi(i,1) = 4/(al(i)^4);
    vstatic(i,1) = 1/3;
end

v = cumsum(vi);
m=1:5;

plot(m,v,'b',m,vstatic,'r');
legend('Modal Representation for \Omega=0','Elementary Beam Theory','Location','East')
xlabel 'Number of Mode Shapes'
ylabel 'v(x,t) / (4Fl^3/EI)'
title 'Static Tip Deflection of Uniform Cantilever Beam'
```