ECSE-2410 SIGNALS AND SYSTEMS FALL 2010
Rensselaer Polytechnic Institute

EXAM #1 (2 hours) October 05, 2010

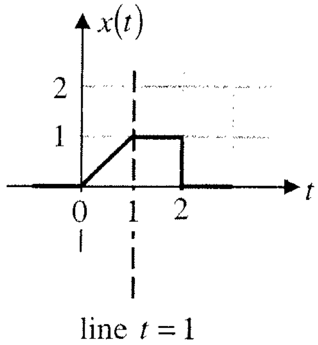
NAME: SOLUTIONS SECTION 8:30 am____
10:00am____

Do all work on these sheets.
NO crib notes.
NO Calculators.
Label and Scale axes on all sketches and indicate all key values
Show all work for full credit

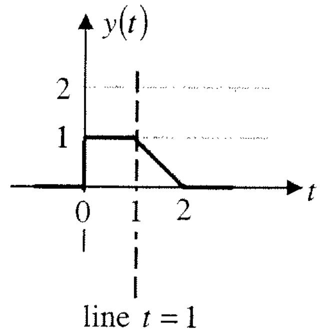
Total Grade for Exam #1:

Problem	Points	Grade
1	6	
2	5	
3	5	
4	15	
5	6	
6	8	
7	8	
8	16	
9	5	
10	5	
11	11	
12	10	
TOTAL	100	

1(6). The signal

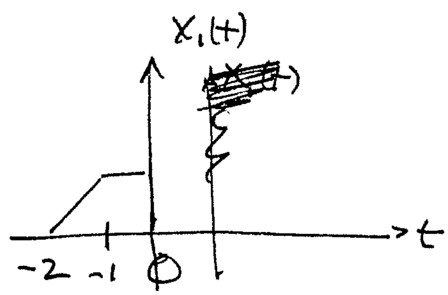


When flipped about the vertical line $t = 1$ results in

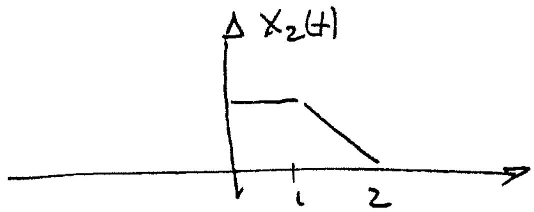


where $y(t) = x(at + b)$. Find a and b.

Let $x_1(t) = x(t+2)$



$x_2(t) = x_1(-t) = y(t)$

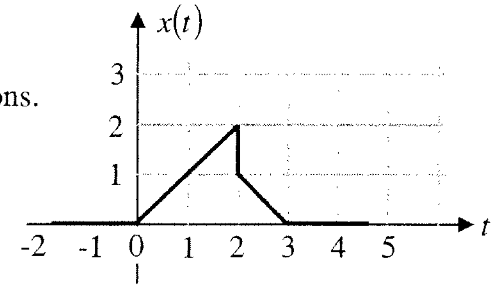


$\therefore$

$y(t) = x_1(-t) = x(-t+2)$

$$\boxed{\begin{aligned} a &= -1 \\ b &= 2 \end{aligned}}$$

2(5). Express the signal shown in terms of step functions.



$$x(t) = t\,u(t) - 2(t-2)\,u(t-2) - u(t-2) + (t-3)\,u(t-3)$$

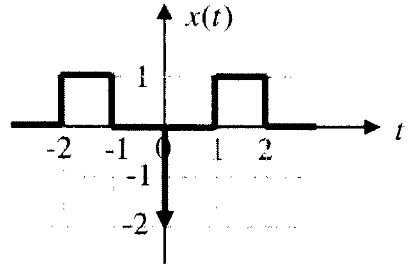
OR

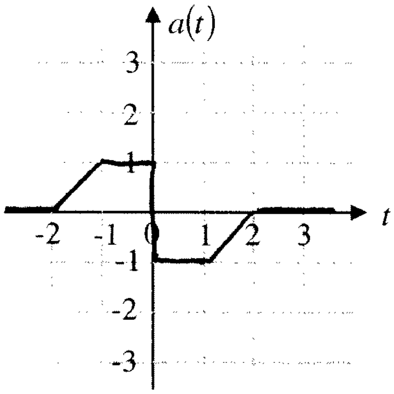
$$x(t) = t\big(u(t) - u(t-2)\big) \quad -(t-3)\big(u(t-2) - u(t-3)\big)$$

$$= t\,u(t) + \underbrace{(-t - t+3)}_{-2t+4-1}\,u(t-2) + (t-3)\,u(t-3)$$

$$= t\,u(t) - 2(t-2)\,u(t-2) - u(t-2) + (t-3)\,u(t-3)$$

3(5). Sketch the running integral $a(t) = \int_{-\infty}^{t} x(\tau)d\tau$ of the signal





$$x(t) = u(t+2) - u(t+1) - 2\delta(t) + u(t-1) - u(t-2)$$

$$a(t) = \int_{-\infty}^{t} \left[u(\tau+2) - u(\tau+1) - 2\delta(\tau) + u(\tau-1) - u(\tau-2) \right] d\tau$$

$$= \int_{-\infty}^{t} u(\tau+2)d\tau - \int_{-\infty}^{t} u(\tau+1)d\tau - 2\int_{-\infty}^{t} \delta(\tau)d\tau + \int_{-\infty}^{t} u(\tau-1)d\tau - \int_{-\infty}^{t} u(\tau-2)d\tau$$

$$= \int_{-2}^{t} d\tau - \int_{-1}^{t} d\tau - 2u(t) + \int_{1}^{t} d\tau - \int_{2}^{t} d\tau$$

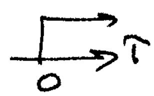
$$= (t+2)u(t+2) - (t+1)u(t+1) - 2u(t) + (t-1)u(t-1) - (t-2)u(t-2)$$

4(15). Evaluate and simplify the following:

(a) $y(t)=\int_{-\infty}^{t}\sin\left(\frac{\pi}{2}\tau\right)\delta(\tau-1)\,d\tau \;=\; \int_{-\infty}^{t}\overset{1}{\cancel{\sin(\pi/2)}}\,\delta(\tau-1)\,d\tau \;=\; \boxed{u(t-1)}$

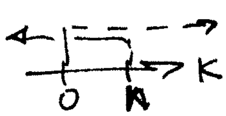
(b) $y(t)=\int_{-\infty}^{\infty}\tau\,\delta(\tau-1)\,d\tau \;=\; \int_{-\infty}^{\infty}1\cdot\delta(\tau-1)\,d\tau = \boxed{1}$

(c) $y(t)=\int_{-\infty}^{t}e^{-\tau}u(\tau)\,d\tau \;=\; \int_{0}^{t}e^{-\tau}\,d\tau = -e^{-\tau}\Big|_{0}^{t} = -\left(e^{-t}-1\right) = \boxed{\left(1-e^{-t}\right)u(t)}$



$t>0$ $\quad$ $t>0$

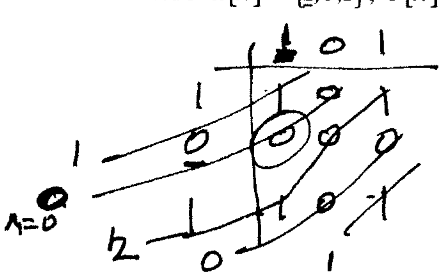
(d) $y[n]=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2}\right)^{k}u[k]u[n-k] \;=\; \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \boxed{2\left(1-\left(\frac{1}{2}\right)^{n+1}\right)u[n]}$



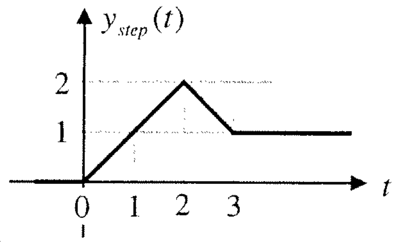
$n\geq 0$ $\quad$ $n\geq 0$

(e) $y[n]=\sum_{k=-\infty}^{\infty}a[k]b[n-k] \;=\; \boxed{\{1, \underline{0}, 2, 0, 1\}}$

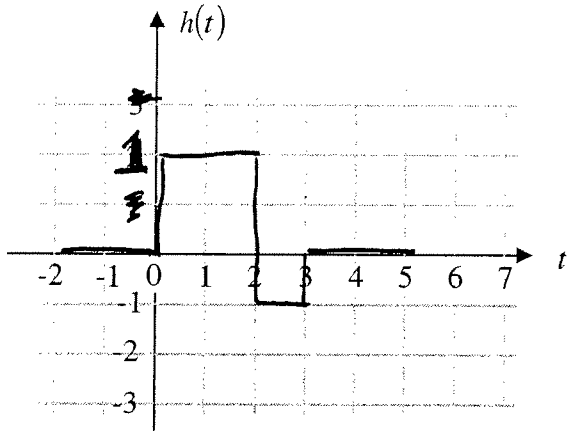
where $a[n]=\{\underline{1},0,1\}$, $b[n]=\{1,\underline{0},1\}$



5(6). The step response of an LTI system is



Sketch the system impulse response $h(t)$.



$$h(t) = \frac{d\,y_{step}(t)}{dt}$$

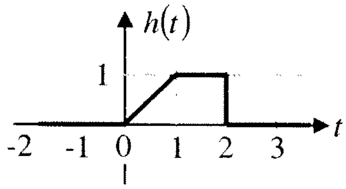
OR

$$y_{step}(t) = t\,u(t) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

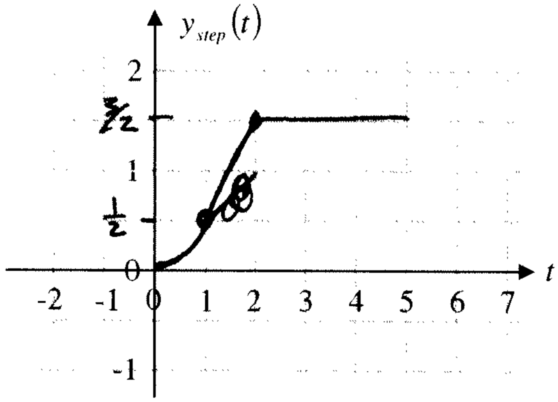
$$\frac{d\,y_{step}(t)}{dt} = \underbrace{\cancel{t\,\delta(t)}}_{0} + u(t) - 2\left[\underbrace{\cancel{(t-2)\,\delta(t-2)}}_{0} + u(t-2)\right] + \left[\underbrace{\cancel{(t-3)\,\delta(t-3)}}_{0} + u(t-3)\right]$$

$$= u(t) - 2u(t-2) + u(t-3)$$

6(8). The impulse response of a LTI system is



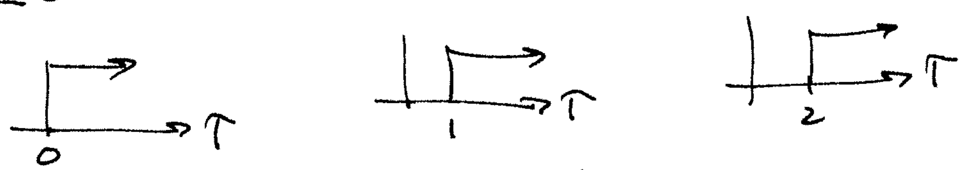
Find and sketch the step response.



$$y_{step}(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$h(t) = t\,u(t) - (t-1)\,u(t-1) - u(t-2)$$

$$\therefore\ y_{step}(t) = \int_{-\infty}^{t} \tau\,u(\tau)\,d\tau - \int_{-\infty}^{t} (\tau-1)\,u(\tau-1)\,d\tau - \int_{-\infty}^{t} u(\tau-2)\,d\tau$$



$$= \int_{0}^{t} \tau\,d\tau - \int_{1}^{t} (\tau-1)\,d\tau - \int_{2}^{t} d\tau$$

$(t>0)$ $(t>1)$ $(t>2)$

$u(t-2)$

$$= \frac{\tau^2}{2}\Big|_0^t - \frac{(\tau-1)^2}{2}\Big|_1^t - \tau\Big|_2^t = \frac{t^2}{2}u(t) - \left[\frac{(t-1)^2}{2}\right]u(t-1) - (t-2)$$

For $0<t<1$, $y_{step}(t) = \frac{t^2}{2}$

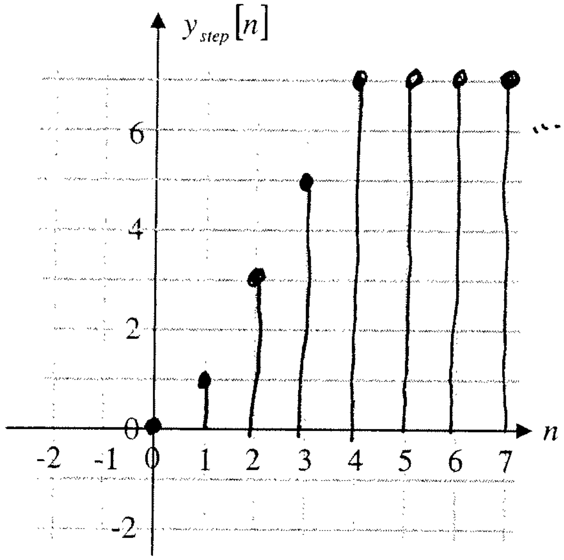
t^2-2t+1

For $1<t<2$, $y_{step}(t) = \frac{t^2}{2} - \frac{(t-1)^2}{2} = t-\frac{1}{2}$

For $t>2$, $y_{step} = \left(t-\frac{1}{2}\right) - (t-2) = 3/2$

7(8). The impulse response of a LTI system is $h[n] = \{\underline{0}, 1, 2, 2, 2, 0, \ldots\}$.

Sketch the step response, $y_{step}[n]$, $0 \le n \le 6$.



$$y_{step}[n] = \sum_{k=-\infty}^{n} h[k]$$

n	$y_{step}[n]$
0	0
1	0+1 = 1
2	0+1+2 = 3
3	0+1+2+2 = 5
4	0+1+2+2+2 = 7
5, 6	0+1+2+2+2 = 7

8a(7). Find the convolution $y_1(t) = x_1(t) * h(t)$, where $x_1(t) = e^{jt}$ and $h(t) = e^{-t}u(t)$.

$$y_1(t) = \int_{-\infty}^{\infty} e^{j\tau} e^{-(t-\tau)} u(t-\tau)\, d\tau = e^{-t}\int_{-\infty}^{t} e^{j\tau} e^{\tau}\, d\tau = e^{-t}\int_{-\infty}^{t} e^{(1+j)\tau}\, d\tau$$

$$= e^{-t}\left[\frac{1}{1+j} e^{(1+j)\tau}\Big|_{-\infty}^{t}\right] = e^{-t}\left[\frac{1}{1+j}\left(e^{t}e^{jt} - \underbrace{e^{-\infty}}_{0} e^{-j\infty}\right)\right]$$

$$= \frac{1}{1+j} e^{jt}$$

8b(5). Find the convolution $y_2(t) = x_2(t) * h(t)$, where $x_2(t) = e^{-jt}$ and $h(t) = e^{-t}u(t)$.

$$y_2(t) = \int_{-\infty}^{\infty} e^{-j\tau} e^{-(t-\tau)} d\tau = y_1^*(t) = \frac{1}{1-j} e^{-jt}$$

8c(4). Using the results from 8a and 8b above, form $x(t) = x_1(t) + x_2(t)$ so that $y(t) = y_1(t) + y_2(t)$, since $h(t)$ is an LTI system. Express $y(t)$ in terms of sines and cosines.

$$y(t) = \frac{1}{1+j} e^{jt} + \frac{1}{1-j} e^{-jt}$$

$$= \frac{(1-j)e^{jt} + (1+j)e^{-jt}}{\underbrace{(1+j)(1-j)}_{1+1}} = \frac{e^{jt} + e^{-jt}}{2} - j\left(\frac{e^{jt} - e^{-jt}}{2}\right)$$

$$= \frac{e^{jt} + e^{-jt}}{2} + \left(\frac{e^{jt} - e^{-jt}}{2j}\right)$$

$$= \cos(t) + \sin(t)$$

9(5). Given $H(\omega)=\dfrac{e^{j\frac{\pi}{4}}}{(j\omega)(2+j\omega)}$, find the value of ω so that $arg(H(\omega))=\angle H(\omega)=-\dfrac{\pi}{2}$.

$$\angle H(\omega) = \frac{\pi}{4} - \cancel{\frac{\pi}{2}} - \tan^{-1}\frac{\omega}{2} = \cancel{-\frac{\pi}{2}}$$

$$\therefore \tan^{-1}\frac{\omega}{2} = \frac{\pi}{4}$$

$$\frac{\omega}{2} = \tan\left(\frac{\pi}{4}\right) = 1$$

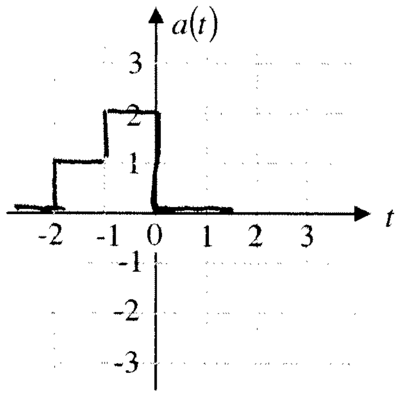
$$\therefore \boxed{\omega = 2}$$

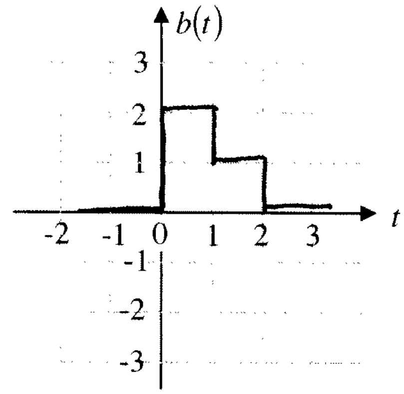
10(5). Express $Y(\omega)=\dfrac{2(1-j\omega)}{(j\omega)(1+\omega^2)}$ in polar form.

$$\underline{Y}(\omega) = \frac{2\cancel{(1-j\omega)}}{j\omega(1+j\omega)\cancel{(1-j\omega)}} = \frac{2}{\omega e^{j\pi/2}\sqrt{1+\omega^2}\, e^{j\tan^{-1}\omega}}$$

$$= \frac{2}{\omega\sqrt{1+\omega^2}}\, e^{-j\left(\frac{\pi}{2}+\tan^{-1}\omega\right)}$$

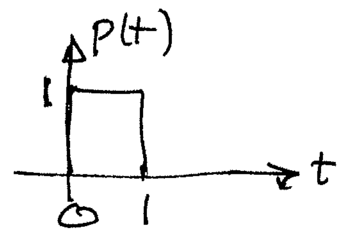
11a(4). Sketch $a(t)=u(t+2)+u(t+1)-2u(t)$, and $b(t)=2u(t)-u(t-1)-u(t-2)$.

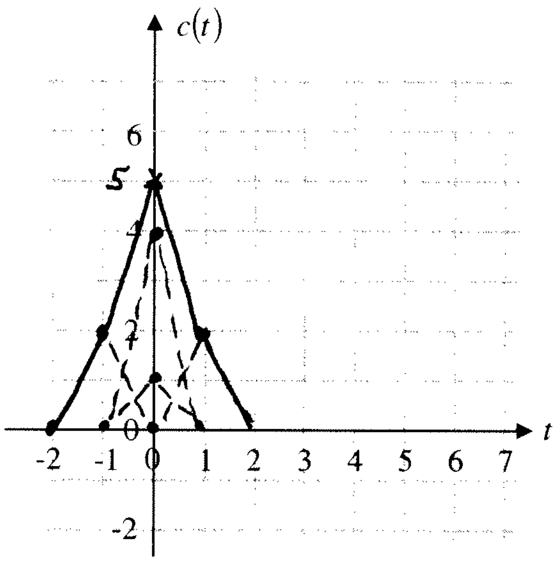




11b(7). Find and sketch $c(t)=a(t)*b(t)$, where

~~a(t)~~ =

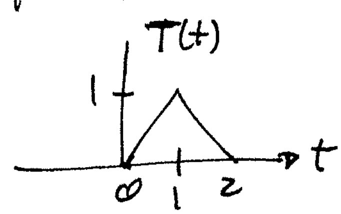




$\therefore\ a(t) = p(t+2) + 2p(t+1)$

$b(t) = 2p(t) + p(t-1)$

But $p(t) * p(t) =$



$$\therefore\ c(t) = a(t) * b(t) = \big(p(t+2) + 2p(t+1)\big) * \big(2p(t) + p(t-1)\big)$$

$$= 2p(t+2) * p(t) + p(t+2) * p(t-1) + 4p(t+1) * p(t) + 2p(t+1) * p(t-1)$$

$$= 2T(t+2) + T(t+1) + 4T(t+1) + 2T(t)$$

12(10). Using classical techniques, solve the differential equation, $\frac{dy}{dt} + y = x(t)$, where the input is $x(t) = 1 + e^{-2t}$, $t \geq 0$, and the initial condition is $y(0) = \alpha$.

$y_h(t) = Ae^{-t}, \quad t \geq 0$

$y_p(t) = K_1 + K_2 e^{-2t}$

Subs in de

$$-2K_2 e^{-2t} + K_1 + K_2 e^{-2t} = 1 + e^{-2t}$$

$$\therefore \quad -2K_2 + K_2 = 1 \implies K_2 = -1$$

$$K_1 = 1$$

$\therefore$

$$y(t) = y_h(t) + y_p(t) = Ae^{-t} + 1 - e^{-2t}$$

IC's

$$\underset{\alpha}{y(0)} = A(1) + \cancel{1} - \cancel{1} \implies A = \alpha$$

$$\boxed{\therefore y(t) = \alpha e^{-t} + 1 - e^{-2t}, \quad t \geq 0}$$