According to New Revised Credit System Syllabus (2019 Pattern) **SPPU**

Second Year (S.E.) Degree Course In
COMPUTER ENGG. / INFORMATION TECHNOLOGY (Semester - II)

ENGINEERING MATHEMATICS – III

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ENGINEERING MATHEMATICS – III

(With Large Number of Solved Examples and MCQ's)

FOR

SECOND YEAR DEGREE COURSES IN

COMPUTER ENGINEERING AND INFORMATION TECHNOLOGY

ACCORDING TO NEW REVISED CREDIT SYSTEM SYLLABUS (2019 PATTERN)
OF SAVITRIBAI PHULE PUNE UNIVERSITY, PUNE
(EFFECTIVE FROM ACADEMIC YEAR – 2020, SEMESTER - II)

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PREFACE

Our text books on **"Engineering Mathematics-III"** have occupied place of pride among engineering student's community for more than **twenty-five year** now. All the teachers of this group of authors have been teaching mathematics in engineering colleges for the past several years. Difficulties of engineering students are well understood by the authors and that is reflected in the text material.

As per the policy of the University, Engineering Syllabi is revised every five years. Last revision was in the year 2015. New revision is coming little earlier, as university has introduced Choice Based Credit System from Academic Year 2019-2020.

As per the Choice Based Credit System, the theory examination shall be conducted by university in two phase. Phase-I will be In-Semester Examination of 30 marks theory examination based on first and second units. Phase-II will be End-Semester Examination of 70 marks theory examination based on third, fourth, fifth and sixths units.

New text book on Engineering Mathematics III for second year **Computer Engineering and Information Technology** is written, taking in to account all the new features that have been introduced. All the entrants to the engineering field will definitely find this book, complete in all respect. Students will find the subject matter presentation quite lucid. There are large number of illustrative examples and well graded exercises. Multiple Choice Questions based on the syllabus are appended at the end of the book and will be useful across universities/colleges and various competitive exams.

We take this opportunity to express our sincere thanks to Shri. Dineshbhai Furia of Nirali Prakashan, Pioneer in all fields of education. Thanks are also due to Shri. Jignesh Furia, whose dynamic leadership is helpful to all the authors of Nirali Prakashan.

We specially appreciate the effects of Late M. P. Munde and entire team of Nirali Prakashan namely Mrs. Anagha Kawre (Co-Ordinator and Proof Reader), Late Mr. Santosh Bare, who really have taken keen interest and untiring efforts in publishing this text.

We have no doubt that like our earlier texts, student's community will respond favourably to this new venture.

The advice and suggestions of our esteemed readers to improve the text are most welcomed, and will be highly appreciated.

26th January 2021

Pune **Authors**

SYLLABUS

Unit I : Linear Differential Equations (LDE) **(08 Hrs.)**

LDE of n^{th} order with constant coefficients, Complementary function, Particular integral, General method, Short methods, Method of variation of parameters, Cauchy's and Legendre's DE, Simultaneous and Symmetric simultaneous DE.

Unit II : Transforms **(08 Hrs.)**

Fourier Transform (FT): Complex exponential form of Fourier series, Fourier integral theorem, Fourier transform, Fourier sine and cosine integrals, Fourier transforms, Fourier Sine and Cosine transforms and their inverses, Discrete Fourier Transform.

Z-Transform (ZT): Introduction, Definition, Standard properties, ZT of standard sequences and their inverses. Solution of difference equations.

Unit III : Statistics **(07 Hrs.)**

Measures of central tendency, Measure of dispersion, Coefficient of variation, Moments, Skewness and Kurtosis, Curve fitting: fitting of straight line, parabola and related curves, Correlation and Regression, Reliability of Regression Estimates.

Unit IV : Probability and Probability Distributions **(07 Hrs.)**

Probability, Theorems on Probability, Bayes Theorem, Random variables, Mathematical Expectation, Probability density function, Probability distributions : Binomial, Poisson, Normal and Hypergeometric, Sampling distributions, Test of Hypothesis : Chi-square test, t-test.

Unit V : Numerical Methods **(08 Hrs.)**

Numerical solution of Algebraic and Transcendental equations: Bisection, Secant, Regula-Falsi, Newton-Raphson and Successive Approximation Methods, Convergence and Stability.

Numerical Solution of System of linear equations: Gauss elimination, LU Decomposition, Cholesky, Jacobi and Gauss-Seidel Methods.

Unit VI : Numerical Methods **(08 Hrs.)**

Interpolation: Finite Differences, Newton's and Lagrange's Interpolation formulae, Numerical Differentiation. Numerical Integration: Trapezoidal and Simpson's rules, Bound of truncation error.

Numerical Solution of Ordinary differential equations: Euler's, Modified Euler's, Runge-Kutta 4^{th} order methods and Predictor-Corrector methods.

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UNIT I : LINEAR DIFFERENTIAL EQUATIONS (LDE)

CHAPTER-1

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1.1 INTRODUCTION

Differential equations are widely used in fields of Engineering and Applied Sciences. Mathematical formulations of most of the physical problems are in the forms of differential equations. Use of differential equations is most prominent in subjects like Circuit Analysis, Theory of Structures, Vibrations, Heat Transfer, Fluid Mechanics etc. Differential equations are of two types : Ordinary and Partial Differential Equations. In ordinary equations, there is one dependent variable depending for its value on one independent variable. Partial differential equations will have more than one independent variables.

In what follows, we shall discuss ordinary and partial differential equations, which are of common occurrence in engineering fields. Applications to some areas will also be dealt.

1.2 PRELIMINARIES

I. Second Degree Polynomials and Their Factorization

(a) (i) $D^2 - 2D - 3 = (D + 1)\ (D - 3)$ (ii) $D^2 + 5D + 6 = (D + 2)\ (D + 3)$

(iii) $D^2 + 2D + 1 = (D + 1)^2$ (iv) $D^2 - 5D + 6 = (D - 2)\ (D - 3)$

(v) $D^2 + 3D + 2 = (D + 2)\ (D + 1)$ (vi) $D^2 - D - 2 = (D - 2)\ (D + 1)$

(vii) $D^2 - 4D + 4 = (D - 2)^2$ (viii) $D^2 - a^2 = (D - a)\ (D + a)$

(ix) $D^2 + a^2 = (D + ia)\ (D - ia)$

(b) The roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, these roots are imaginary if $b^2 - 4ac < 0$.

(i) $D^2 + 2D + 2 = 0 \quad \Rightarrow \quad D = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

(ii) $D^2 + D + 1 = 0 \quad \Rightarrow \quad D = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{3}}{2}\, i$

If $D = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2} = \alpha \pm i\beta$ then $\alpha = -\dfrac{1}{2}$, $\beta = \dfrac{\sqrt{3}}{2}$, β is always positive; α may be positive, negative or zero.

(iii) $D^2 + 1 = 0 \quad \Rightarrow \quad D^2 = -1$ i.e. $D = \pm i \quad \therefore \quad \alpha = 0,\ \beta = 1.$

(iv) $D^2 + 4 = 0 \quad \Rightarrow \quad D^2 = -4$ i.e. $D = \pm 2i \quad \therefore \quad \alpha = 0,\ \beta = 2.$

II. Third Degree Polynomials and Their Factorization

(a) (i) $D^3 - a^3 = (D - a)\ (D^2 + aD + a^2)$ (iii) $D^3 + a^3 = (D + a)\ (D^2 - aD + a^2)$

(ii) $D^3 + 3D^2 + 3D + 1 = (D + 1)^3$ (iv) $D^3 - 3D^2 + 3D - 1 = (D - 1)^3$

(b) Use of Synthetic Division :

(i) $f(D) = D^3 - 7D - 6 = 0$; for $D = -1$, $f(-1) = 0$

$\therefore$ $(D + 1)$ is one of the factors.

$\therefore$ $D^3 - 7D - 6 = 0 \Rightarrow (D + 1)\ (D^2 - D - 6) = 0$

$(D + 1)\ (D - 3)\ (D + 2) = 0 \Rightarrow D = -1, -2, 3.$

– 1	1	0	– 7	– 6
		– 1	1	6
	1	– 1	– 6	0

(ii) For $D^3 - 2D + 4 = 0$; $D = -2$

$\therefore$ $f(-2) = 0$ $\therefore$ $(D + 2)$ is one of the factors.

$\therefore$ $D^3 - 2D + 4 = 0 \Rightarrow (D + 2)\ (D^2 - 2D + 2) = 0$

$D = -2$ and $D = 1 \pm i$, $\alpha = 1$, $\beta = 1$.

– 2	1	0	– 2	4
		– 2	4	– 4
	1	– 2	2	0

III. Fourth Degree Polynomials and Their Factorization

(a) $D^4 - a^4 = (D^2 - a^2)(D^2 + a^2) = (D - a)(D + a)(D + ia)(D - ia)$

(b) Making a Perfect Square by Introducing a Middle Term :

(i) For $D^4 + a^4 = 0$; consider $(D^2 + a^2)^2 = D^4 + 2a^2 D^2 + a^4$

$$D^4 + a^4 = (D^4 + 2a^2 D^2 + a^4) - (2a^2 D^2) = (D^2 + a^2)^2 - \left(\sqrt{2}\, a\, D\right)^2$$

$$D^4 + a^4 = \left(D^2 - \sqrt{2}\, a\, D + a^2\right)\left(D^2 + \sqrt{2}\, a\, D + a^2\right)$$

(ii) For
$$D^4 + 1 = D^4 + 2D^2 + 1 - 2D^2 = (D^2 + 1)^2 - \left(\sqrt{2}\, D\right)^2$$

$$D^4 + 1 = \left(D^2 - \sqrt{2}\, D + 1\right)\left(D^2 + \sqrt{2}\, D + 1\right)$$

(c)
$$D^4 + 8D^2 + 16 = (D^2 + 4)^2,\; D^4 + 2D^2 + 1 = (D^2 + 1)^2 = (D + i)^2 (D - i)^2$$

$$D^4 + 10D^2 + 9 = (D^2 + 9)(D^2 + 1) = (D + 3i)(D - 3i)(D + i)(D - i)$$

(d) (i) $f(D) = D^4 - 2D^3 - 3D^2 + 4D + 4 = 0$,

for $D = -1$, $f(-1) = 0$

$\therefore$ Factors are $(D + 1)^2 (D - 2)^2 = 0$.

−1	1	−2	−3	4	4
		−1	3	0	−4
−1	1	−3	0	4	0
		−1	4	−4	
2	1	−4	4	0	
		2	−4		
	1	−2	0		

On a similar line,

(ii) $D^4 - D^3 - 9D^2 - 11D - 4 = (D + 1)^3 (D - 4)$

(e) Perfect square of the type $(a + b + c)^2$

(i)
$$D^4 + 2D^3 + 3D^2 + 2D + 1 = (D^2)^2 + 2 \cdot D^2 \cdot D + D^2 + 2D^2 + 2D + 1 = (D^2 + D)^2 + 2(D^2 + D) + 1$$
$$= [(D^2 + D) + 1]^2 = (D^2 + D + 1)^2$$

(ii)
$$D^4 - 4D^3 + 8D^2 - 8D + 4 = (D^2)^2 - 2D^2 \cdot 2D + (2D)^2 + 4D^2 - 8D + 4$$
$$= (D^2 - 2D)^2 + 4(D^2 - 2D) + 4$$
$$= [(D^2 - 2D) + 2]^2 = (D^2 - 2D + 2)^2$$

IV. Fifth Degree Polynomials and Their Factorization

(i)
$$D^5 - D^4 + 2D^3 - 2D^2 + D - 1 = D^4 (D - 1) + 2D^2 (D - 1) + 1(D - 1) = (D^4 + 2D^2 + 1)(D - 1) = (D - 1)(D^2 + 1)^2$$
$$= (D - 1)(D + i)^2 (D - i)^2$$

1.3 THE n^{th} ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

A differential equation which contains the differential coefficients and the dependent variable in the first degree, does not involve the product of a derivative with another derivative or with dependent variable, and in which the coefficients are constants is called a *linear differential equation with constant coefficients*.

The general form of such a differential equation of order "n" is

$$\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)} \quad \ldots (1)$$

Here $a_0, a_1, a_2 \ldots$ are constants. Equation (1) is a n^{th} order linear differential equation with constant coefficients.

e.g. Put n = 3 in equation (1), we get

$$a_0 \frac{d^3y}{dx^3} + a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x)$$ which is a 3rd order linear differential equation with constant coefficients.

Using the differential operator D to stand for $\frac{d}{dx}$ i.e. $Dy = \frac{dy}{dx}$; $D^2y = \frac{d^2y}{dx^2}$, … $D^ny = \frac{d^ny}{dx}$, the equation (1) will take the form

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots + a_{n-1} Dy + a_n y = f(x)$$

OR $$\mathbf{(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)\, y = f(x)} \qquad \ldots (2)$$

in which each term in the parenthesis is operating on y and the results are added.

Let $\phi(D) \equiv \mathbf{a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n}$, $\phi(D)$ is called as n^{th} order polynomial in D.

$\therefore$ Equation (2) can be written as $$\boxed{\phi(D)\, y = f(x)} \qquad \ldots (3)$$

Note : In equation (1), if $a_0, a_1, \ldots a_n$ are functions of x then it is called n^{th} order linear differential equation.

1.4 THE NATURE OF DIFFERENTIAL OPERATOR "D"

It is convenient to introduce the symbol D to represent the operation of differentiation with respect to x i.e. $D \equiv \frac{d}{dx}$, so that

$$\frac{dy}{dx} = Dy; \quad \frac{d^2y}{dx^2} = D^2y; \quad \frac{d^3y}{dx^3} = D^3y; \ldots\ldots; \frac{d^ny}{dx^n} = D^ny \quad \text{and} \quad \frac{dy}{dx} + ay = (D + a)\, y$$

The differential operator D or (D^n) obeys the laws of Algebra.

Properties of the Operator D :

If y_1 and y_2 are differentiable functions of x and "a" is a constant and m, n are positive integer then

(i) $D^m (D^n)\, y = D^n (D^m)\, y = D^{m+n} y$

(ii) $(D - m_1)(D - m_2)\, y = (D - m_2)(D - m_1)\, y$

(iii) $(D - m_1)(D - m_2)\, y = [D^2 - (m_1 + m_2) D + m_1 m_2]\, y$

(iv) $D(au) = a \cdot D(u)$; $D^n (au) = a \cdot D^n (u)$

(v) $D(y_1 + y_2) = D(y_1) + D(y_2)$; $D^n (y_1 + y_2) = D^n (y_1) + D^n (y_2)$.

1.5 LINEAR DIFFERENTIAL EQUATION ϕ (D) y = 0

Consider, $$(D)\, y = 0 \qquad \ldots (4)$$

where, $\phi(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + a_3 D^{n-3} + \ldots + a_{n-1} D + a_n$ is n^{th} order polynomial in D and D obeys the laws of algebra, we can in general factorise $\phi(D)$ in n linear factors as $\phi(D) = (D - m_1)(D - m_2)(D - m_3) \ldots (D - m_n)$ where $m_1, m_2, m_3, \ldots m_n$ are the roots of the algebraic equation $\phi(D) = 0$

$\therefore$ Equation (4) can be written as;

$$\phi(D)\, y = (D - m_1)(D - m_2)(D - m_3) \ldots (D - m_n)\, y = 0 \qquad \ldots (5)$$

Note : These factors can be taken in any sequence.

1.6 AUXILIARY EQUATION (A.E.)

The equation $\phi(D) = 0$ is called as an *auxiliary equation* (A.E.) for equations (3), (4).

e.g. $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

By using operator D for $\frac{d}{dx}$, we have $(D^2 - 5D + 6)\, y = 0$

$\phi(D) = D^2 - 5D + 6 = 0$ is the A.E.

$\therefore$ $(D^2 - 5D + 6)\, y = (D - 3)(D - 2)\, y = (D - 2)(D - 3)\, y$.

1.7 SOLUTION OF ϕ(D) y = 0

Being n^{th} order DE, equation (4) or (5) will have exactly n arbitrary constants in its general solution.

The equation (5) will be satisfied by the solution of the equation $(D - m_n)\, y = 0$

i.e. $$\frac{dy}{dx} - m_n y = 0$$

On solving this 1st order 1st degree DE by separting variables, we get $y = c_n e^{m_n x}$, where, c_n is an arbitrary constant.

Similarly, since the factors in equation (5) can be taken in any order, the equation will be satisfied by the solution of each of the equations $(D - m_1)\, y = 0$, $(D - m_2)\, y = 0$ … etc., that is by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ ………… etc.

It can, therefore, easily be proved that the sum of these individual solutions, i.e.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \qquad \ldots (6)$$

also satisfies the equation (5) and as it contains n arbitrary constants, and the equation (4) is of the n^{th} order, (6) constitutes the general solution of the equation (4).

$\therefore$ **The general solution of the equation $\phi(D)\, y = 0$ is**

$$\mathbf{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}}$$

where $m_1, m_2, \ldots m_n$ are the roots of the auxiliary equation $\phi(D) = 0$.

Ex. 1 : *Solve* $\dfrac{d^3 y}{dx^3} - 6\dfrac{d^2 y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0.$

Sol. : Let D stand for $\dfrac{d}{dx}$ and the given equation can be written as

$$(D^3 - 6D^2 + 11D - 6)\, y = 0.$$

Here auxiliary equation is

$$D^3 - 6D^2 + 11D - 6 = 0$$

i.e. $(D - 1)(D - 2)(D - 3) = 0 \Rightarrow m_1 = 1, m_2 = 2, m_3 = 3$, are roots of AE.

$\therefore$ The general solution is $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$.

Ex. 2 : *For* $(4D^2 - 8D + 1)\, y = 0.$

Sol. : Here auxiliary equation is $4D^2 - 8D + 1 = 0$ i.e. $D = 1 \pm \dfrac{\sqrt{3}}{2}$

$$y = c_1 e^{\left(1 + \frac{\sqrt{3}}{2}\right) x} + c_2 e^{\left(1 - \frac{\sqrt{3}}{2}\right) x}.$$

1.8 DIFFERENT CASES DEPENDING UPON THE NATURE OF ROOTS OF THE AUXILIARY EQUATION $\phi(D) = 0$

A. The Case of Real and Different Roots

If roots of $\phi(D) = 0$ be $m_1, m_2, m_3 \ldots m_n$, all are real and different, then the solution of $\phi(D)\, y = 0$ will be

$$\boxed{\mathbf{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}}$$

B. The Case of Real and Repeated Roots (The Case of Multiple Roots)

Let $m_1 = m_2, m_3, m_4 \ldots m_n$ be the roots of $\phi(D) = 0$, then the part of solution corresponding to m_1 and m_2 will look like

$$c_1 e^{m_1 x} + c_2 e^{m_1 x}\ (m_1 = m_2) = (c_1 + c_2)\, e^{m_1 x} = c' e^{m_1 x}$$

But this means that number of arbitrary constants now in the solution will be n – 1 instead of n. Hence it is no longer the general solution. The anomaly can be rectified as under.

Pertaining to $m_1 = m_2$, the part of the equation will be $(D - m_1)(D - m_1)\, y = 0$

Put $(D - m_1)\, y = z$, temporarily, then we have $(D - m_1)\, z = 0 \quad \therefore\ z = c_1 e^{m_1 x}$

Hence putting value of z in $(D - m_1)\, y = z$, we have

$$(D - m_1)\, y = c_1 e^{m_1 x} \quad \text{or} \quad \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

which is a linear differential equation. Its I.F. $= e^{-\int m_1 dx} = e^{-m_1 x}$ and hence solution is

$$y\left(e^{-m_1 x}\right) = \int c_1 e^{m_1 x} \cdot e^{-m_1 x}\, dx + c_2 = c_1 x + c_2$$

$$\therefore \qquad y = (c_1 x + c_2)\, e^{m_1 x}$$

If $m_1 = m_2$ are real, and the remaining roots $m_3, m_4, m_5, \ldots, m_n$ are real and different then solution of $\phi(D)\, y = 0$ is

$$\boxed{\mathbf{y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}}}$$

Similarly, when three roots are repeated. i.e. if $m_1 = m_2 = m_3$ are real, and the remaining roots $m_4, m_5, \ldots m_n$ are real and different then solution of $\phi(D)\,y = 0$ is

$$\boxed{y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}}$$

If $m_1 = m_2 = m_3 = \ldots = m_n$ i.e. n roots are real and equal then solution of $\phi(D)\,y = 0$ is

$$\boxed{y = (c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n)\, e^{m_1 x}}$$

Ex. 1. For $(D^2 - 6D + 9)\,y = 0$ A.E. $= (D-3)^2 = 0$ and solution is $y = (c_1 x + c_2)\, e^{3x}$

2. For $(D-1)^3 (D+1)\,y = 0$, solution is $y = (c_1 x^2 + c_2 x + c_3)\, e^x + c_4 e^{-x}$

3. For $(D-1)^2 (D+1)^2\, y = 0$, solution is $y = (c_1 x + c_2)\, e^x + (c_3 x + c_4)\, e^{-x}$.

C. The Case of Imaginary (Complex) Roots

For practical problems in engineering, this case has special importance. Since the coefficients of the auxiliary equation are real, the imaginary roots (if exists) will occur in conjugate pairs. Let $\alpha \pm i\beta$ be one such pair. Therefore $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

The corresponding part of the solution of the equation $\phi(D)\,y = 0$, then takes the form

$$y = A e^{(\alpha + i\beta) x} + B e^{(\alpha - i\beta) x}$$
$$y = e^{\alpha x} [A e^{i\beta x} + B e^{-i\beta x}]$$
$$y = e^{\alpha x} [A (\cos \beta x + i \sin \beta x) + B (\cos \beta x - i \sin \beta x)]$$
$$y = e^{\alpha x} [(A + B) \cos \beta x + i (A - B) \sin \beta x]$$
$$\boxed{y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]}$$

where, $c_1 = A + B$ and $c_2 = i(A - B)$ are arbitrary constants.

Using $c_1 = C \cos \theta$, $c_2 = -\sin \theta$, this can also be put sometimes into the form as given below (recall SHM).

$$\boxed{y = C e^{\alpha x} \cos (\beta x + \theta) \text{ where } C, \theta \text{ are arbitrary constants.}}$$

ILLUSTRATIONS

Ex. 1 : *Solve $(D^2 + 2D + 5)\, y = 0$.*

Sol. : The auxiliary equation is $D^2 + 2D + 5 = 0$ whose roots are $D = -1 \pm 2i$ which are both imaginary. Here $\alpha = -1$, $\beta = 2$. Hence the solution is

$$y = e^{-x} [A \cos 2x + B \sin 2x]$$

Ex. 2 : *Solve $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 28y = 0$.*

Sol. : The auxiliary equation is $D^4 - 5D^2 + 12D + 28 = 0$ having roots $D = -2, -2, 2 \pm \sqrt{3}\, i$. (Here $\alpha = 2$, $\beta = \sqrt{3}$).

Hence the solution is

$$y = (c_1 x + c_2)\, e^{-2x} + e^{2x} \left[A \cos \sqrt{3}\, x + B \sin \sqrt{3}\, x\right]$$

Ex. 3 : *For $(D^2 + 4)y = 0$.*

Sol. : The auxiliary equation is $D^2 + 4 = 0$ having roots $D = 0 \pm 2i$ (Here $\alpha = 0$, $\beta = 2$).

Hence the solution is

$$y = A \cos 2x + B \sin 2x.$$

D. The Case of Repeated Imaginary Roots

If the imaginary roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ occur twice, then the part of solution of $\phi(D)\,y = 0$ will be

$$y = (Ax + B)\, e^{m_1 x} + (Cx + D)\, e^{m_2 x} \qquad \text{… (by using case B)}$$
$$= (Ax + B)\, e^{(\alpha + i\beta) x} + (Cx + D)\, e^{(\alpha - i\beta) x}$$
$$= e^{\alpha x} \left[(Ax + B)\, e^{i\beta x} + (Cx + D)\, e^{-i\beta x}\right]$$
$$= e^{\alpha x} [(Ax + B) \{\cos \beta x + i \sin \beta x\} + (Cx + D) \{\cos \beta x - i \sin \beta x\}]$$
$$= e^{\alpha x} [(Ax + B + Cx + D) \cos \beta x + i (Ax + B - Cx - D) \sin \beta x]$$

$$\therefore \quad \boxed{y = e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]}$$

with proper changes in the constants c_1, c_2, c_3 and c_4.

ILLUSTRATIONS

Ex. 1 : *Solve* $\dfrac{d^6y}{dx^6} + 6\dfrac{d^4y}{dx^4} + 9\dfrac{d^2y}{dx^2} = 0.$

Sol. : The auxiliary equation $D^6 + 6D^4 + 9D^2 = 0$ has roots $D = 0, 0, \pm i\sqrt{3}, \pm i\sqrt{3}$ where the imaginary roots $\pm i\sqrt{3}$ are repeated. Hence the solution is

$$y = c_1 x + c_2 + (c_3 x + c_4)\cos\sqrt{3}\,x + (c_5 x + c_6)\sin\sqrt{3}\,x$$

Ex. 2 : $(D^4 + 2D^2 + 1)\, y = 0.$

Sol. : The auxiliary equation $D^4 + 2D^2 + 1 = 0$ has roots $D = \pm i, \pm i$, repeated imaginary roots.

Hence the solution is

$$y = (c_1x + c_2)\cos x + (c_3x + c_4)\sin x$$

Now we will summarise the four cases for ready reference.

Case 1 : Real & Distinct Roots : A.E. $\Rightarrow (D - m_1)(D - m_2)(D - m_3)\ldots(D - m_n) = 0$

Solution is $y = c_1 e^{m_1x} + c_2 e^{m_2x} + c_3 e^{m_3x} + \ldots + c_n e^{mnx}$

Case 2 : Repeated Real Roots

(i) For $m_1 = m_2 \Rightarrow$ A.E. $\Rightarrow (D - m_1)(D - m_1)(D - m_3)\ldots\ldots(D - m_n) = 0$

Solution is $y = (c_1 x + c_2)\, e^{m_1x} + c_3 e^{m_3x} + \ldots + c_n e^{mnx}$

(ii) For $m_1 = m_2 = m_3 \Rightarrow$ A.E. $\Rightarrow (D - m_1)(D - m_1)(D - m_1)(D - m_4)\ldots(D - m_n) = 0$

Solution is $y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1x} + c_4 e^{m_4x} + \ldots + c_n e^{m_nx}$

Case 3 : Imaginary Roots : For $D = \alpha \pm i\beta$

Solution is $y = e^{\alpha x}\,[c_1 \cos\beta x + c_2 \sin\beta x]$

Case 4 : Repeated Imaginary Roots : For $D = \alpha \pm i\beta$ be repeated twice

Solution is $y = e^{\alpha x}\,[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x]$

ILLUSTRATIONS

Ex.1 : Solve $\dfrac{d^2x}{dt^2} + 4x = 0.$

Sol. : Let D stand for $\dfrac{d}{dt}$.

$\therefore$ A.E. : $D^2 + 4 = 0 \Rightarrow D = 0 \pm 2i$

$\therefore$ The solution is $x = c_1\cos 2t + c_2 \sin 2t.$

Ex. 2 : Solve $\dfrac{d^4y}{dz^2} - 16y = 0.$

Sol. : Let D stand for $\dfrac{d}{dz}$.

$\therefore$ A.E. : $D^4 - 16 = 0,\ (D - 2)(D + 2)(D^2 + 4) = 0.$

$\therefore$ The solution is $y = c_1 e^{2z} + c_2 e^{-2z} + c_3\cos 2z + c_4 \sin 2z.$

Special Case : If the two real roots of $\phi(D)\, y = 0$ be m and – m [e.g. $D^2 - m^2 = 0$], then the corresponding part of the solution is

$$y = A e^{mx} + B e^{-mx}$$

OR $$y = A(\cosh mx + \sinh mx) + B(\cosh mx - \sinh mx)$$

OR $$y = (A + B)\cosh mx + (A - B)\sinh mx$$

$$\therefore \quad \boxed{y = c_1 \cosh mx + c_2 \sinh mx}$$

We note here that (in some particular cases) solution of $D^2 - m^2 = 0$ can be written as

$$y = c_1 e^{mx} + c_2 e^{-mx} \quad \text{or} \quad y = c_1 \cosh mx + c_2 \sinh mx.$$

e.g. 1. $(D^2 - 1)\, y = 0 \Rightarrow y = c_1\cosh x + c_2 \sinh x.$

2. $(D^2 - 4)\, y = 0 \Rightarrow y = c_1\cosh 2x + c_2 \sinh 2x.$

EXERCISE 1.1

Solve the following Differential Equations

1. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 6y = 0.$

Ans. $y = c_1 e^{-x} + c_2 e^{6x}$

2. $2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 10y = 0.$

Ans. $y = c_1 e^{-2x} + c_2 e^{(5/2)\,x}$

3. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

Ans. $y = c_1 + e^{-x}(c_2 x + c_3)$

4. $(D^4 - 2D^3 + D^2)\, y = 0.$

Ans. $y = c_1 x + c_2 + (c_3 x + c_4)\, e^x$

5. $(D^6 - 6D^5 + 12D^4 - 6D^3 - 9D^2 + 12D - 4)\, y = 0.$

Ans. $y = (c_1x^2 + c_2x + c_3)\, e^x + (c_4x + c_5)\, e^{2x} + c_6e^{-x}$

6. $(D^3 + 6D^2 + 11D + 6)\, y = 0.$

Ans. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

7. $4y'' - 8y' + 7y = 0.$

Ans. $y = e^x\left[A\cos\left(\dfrac{\sqrt{3}}{2}x\right) + B\sin\left(\dfrac{\sqrt{3}}{2}x\right)\right]$

8. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 0,\ x(0) = 2,\ x'(0) = 0.$

Ans. $x = e^{-t}(2\cos 2t + \sin 2t)$

9. $\dfrac{d^2s}{dt^2} = -16\dfrac{ds}{dt} - 64\,s,\ s = 0,\ \dfrac{ds}{dt} = -4$ when $t = 0.$

Ans. $s = -4e^{8t}\,t$

10. $(D^3 + D^2 - 2D + 12)\, y = 0.$

Ans. $y = c_1 e^{-3x} + e^x\,[A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x]$

11. $(D^2 + 1)^3 (D^2 + D + 1)^2\, y = 0.$

Ans. $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x + e^{-x/2}\left[(c^7 + c^8 x)\cos\left(\dfrac{\sqrt{3}}{2}x\right) + (c^9 + c^{10} x)\sin\left(\dfrac{\sqrt{3}}{2}x\right)\right]$

12. $\dfrac{d^4y}{dx^4} + m^4y = 0.$

Ans. $y = e^{(mx/\sqrt{2})}\left[A\cos\left(\dfrac{mx}{\sqrt{2}}\right) + B\sin\left(\dfrac{mx}{\sqrt{2}}\right)\right] + e^{-(mx/\sqrt{2})}\left[C\cos\left(\dfrac{mx}{\sqrt{2}}\right) + D\sin\left(\dfrac{mx}{\sqrt{2}}\right)\right]$

13. $4\dfrac{d^2s}{dt^2} = -9s.$ **Ans.** $s = c_1\sin\dfrac{3t}{2} + c_2\cos\dfrac{3t}{2}$

14. The equation for the bending of a strut is $EI\dfrac{d^2y}{dx^2} + Py = 0$. If $y = 0$ when $x = 0$ and $y = a$ when $x = \dfrac{l}{2}$, find y. **Ans.** $y = \dfrac{a\sin\sqrt{\dfrac{P}{EI}}\,x}{\sin\sqrt{\dfrac{P}{EI}}\cdot\dfrac{l}{2}}$

1.9 THE GENERAL SOLUTION OF THE LINEAR DIFFERENTIAL EQUATION $\phi(D)\, y = f(x)$

The general solution of the equation $\phi(D)y = f(x)$ can be written as $\boxed{y = y_c + y_p}$ where,

1. y_c is the solution of the given equation with $f(x) = 0$, that is of equation $\phi(D)\, y = 0$ (which is known as Associated equation or Reduced equation) and is called the *complimentary function* (C.F.). It involves n arbitrary constants and is denoted by C.F. then $\boxed{\phi(D)\, y_c = 0}$.

2. y_p is any function of x, which satisfies the equation $\phi(D)\, y = f(x)$, so that $\boxed{\phi(D)\, y_p = f(x)}$ y_p is called the particular integral and is denoted by P.I. It does not contain any arbitrary constant.

   Thus, on substituting $y = y_c + y_p$ in $\phi(D)\, y$,

   $$\phi(D)\,[y_c + y_p] = \phi(D)\, y_c + \phi(D)\, y_p = 0 + f(x) = f(x)$$

$\therefore$ $y = y_c + y_p$ satisfies the equation $\phi(D)\, y = f(x)$ and as it contains exactly n arbitrary constants, is the general (or complete) solution of the equation.

Note : 1. The complete solution of $\phi(D)\, y = f(x)$ is $y = C.F. + P.I. = y_c + y_p$.

2. The general solution of $\phi(D)\, y = f(x)$ has *arbitrary constants equal in number to the order of the differential equation.*

1.10 THE INVERSE OPERATOR $\frac{1}{\phi(D)}$ AND THE SYMBOLIC EXPRESSION FOR THE PARTICULAR INTEGRAL

We define $\frac{1}{\phi(D)}$ f(x) as that function of x which, when acted upon by the differential operator $\phi(D)$ gives f(x).

Thus by this definition, $\phi(D) \left\{\frac{1}{\phi(D)} f(x)\right\} = f(x)$ and so $\left\{\frac{1}{\phi(D)} f(x)\right\}$ satisfies the equation $\phi(D)\, y = f(x)$ and so is the P.I. of the equation $\phi(D)\, y = f(x)$.

Thus the P. I. of the equation $\phi(D)\, y = f(x)$ is symbolically given by;

$$\boxed{\text{P.I.} = y_p = \frac{1}{\phi(D)} f(x)}$$

e.g. 1. $(D^2 - 1)\, y = x^2 \;\therefore\; y_p = \frac{1}{D^2 - 1} x^2$

2. $(D^2 - 3D + 2)\, y = \sin e^x \;\therefore\; y_p = \frac{1}{D^2 - 3D + 2} \sin e^x$.

1.11 METHODS OF OBTAINING PARTICULAR INTEGRAL

There are three methods to evaluate the particular integral $y_p = \frac{1}{\phi(D)} f(x)$.

(A) General method

(B) Short-cut methods

(C) Method of variation of parameters.

Now we will discuss these methods in detail.

(A) General Method

This method is useful when the short-cut methods given in (B) are not applicable. This method involves integration.

(i) $\frac{1}{D - m}$ f(x) : By definition of the P. I., $\frac{1}{D - m} f(x)$ will be the P.I. of the equation $(D - m)\, y = f(x)$ i.e. the part in the solution of this equation which does not contain the arbitrary constant. We have, $\frac{dy}{dx} - my = f(x)$ (linear)

$$\text{I.F.} = e^{-mx} \text{ and the general solution is}$$

$$y\, e^{-mx} = \int f(x) \cdot e^{-mx}\, dx + c_1$$

$$\therefore \quad y = (c_1 e^{mx}) + \left(e^{mx} \int e^{-mx} f(x) \cdot dx\right)$$

$$\text{i.e.} \quad y = y_c + y_p$$

Here $c_1 e^{mx}$ is the C.F. and $e^{mx} \int e^{-mx} f(x)\, dx$ must be the P.I.

$$\therefore \quad \boxed{y_p = \text{P. I.} = \frac{1}{D - m} f(x) = e^{mx} \int e^{-mx} f(x)\, dx}$$

Similarly,

$$\boxed{y_p = \text{P. I.} = \frac{1}{D + m} f(x) = e^{-mx} \int e^{mx} f(x)\, dx}$$

Put m = 0

$$\boxed{y_p = \frac{1}{D} f(x) = \int f(x)\, dx}$$

Also, $$y_p = \frac{1}{D^2} f(x) = \frac{1}{D}\left[\frac{1}{D} f(x)\right]$$

$$= \frac{1}{D}\left[\int f(x)\,dx\right] = \int\left[\int f(x)\,dx\right] dx$$

$\therefore$ $$\boxed{y_p = \frac{1}{D^2} f(x) = \int\left[\int f(x)\,dx\right]dx}$$

Similarly, $$\boxed{y_p = \frac{1}{D^3} f(x) = \int\left\{\int\left[\int f(x)\,dx\right]dx\right\}dx}\ \text{... and so on.}$$

(ii) $\frac{1}{(D-m_1)(D-m_2)}$ f(x) :

$$y_p = \frac{1}{(D-m_1)(D-m_2)} f(x) = \frac{1}{(D-m_1)} e^{m_2x}\int e^{-m_2x} f(x)\,dx$$

$$y_p = e^{m_1x}\int e^{-m_1x}\left[e^{m_2x}\int e^{-m_2x} f(x)\,dx\right] dx$$

(iii) Use of Partial Fraction :

$$y_p = \frac{1}{(D-m_1)(D-m_2)} f(x)$$

$$= \frac{1}{(m_1-m_2)}\left[\frac{1}{D-m_1}-\frac{1}{D-m_2}\right] f(x)$$

$$= \frac{1}{m_1-m_2}\left\{\frac{1}{D-m_1} f(x) - \frac{1}{D-m_2} f(x)\right\}$$

$$y_p = \frac{1}{m_1-m_2}\left\{e^{m_1x}\int e^{-m_1x} f(x)\,dx - e^{m_2x}\int e^{-m_2x} f(x)\,dx\right\}$$

ILLUSTRATIONS ON GENERAL METHOD

Ex. 1 : *Solve* $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ ***(Dec. 2010, 2016, 2017; May 2011)***

Sol. : For C.F., A.E. is $D^2 + 3D + 2 = 0 \Rightarrow (D+2)(D+1) = 0$. Hence $D = -1, -2$

and $$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

Here $$\text{P. I.} = y_p = \frac{1}{(D+2)(D+1)} (e^{e^x})$$

$$= \frac{1}{D+2}\left[\frac{1}{D+1} e^{e^x}\right]$$

$$= \frac{1}{D+2}\left[e^{-x}\int e^x e^{e^x} dx\right]$$ [put $e^x = t \ \therefore e^x dx = dt$]

$$= \frac{1}{D+2}\left[e^{-x}\int e^t dt\right]$$

$$= \frac{1}{D+2}\left[e^{-x} e^{e^x}\right] = e^{-2x}\int e^{2x} e^{-x} e^{e^x} dx$$

$$\text{P. I.} = e^{-2x}\int e^x e^{e^x} dx = e^{-2x} e^{e^x}$$

Hence the complete solution will be

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

Ex. 2 : *Solve* $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{1+e^x}$. ***(Dec. 2008, 2010, 2012; May 2012)***

Sol. : We have $(D^2 + D)\, y = \frac{1}{1+e^x}$, here $D \equiv \frac{d}{dx}$

For C.F., A.E. $\Rightarrow D(D+1) = 0 \quad \therefore \quad D = 0, -1.$

$$\therefore \quad \text{C.F.} = y_c = c_1 + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D(D+1)} \left(\frac{1}{1+e^x}\right)$$

$$= \left(\frac{1}{D} - \frac{1}{D+1}\right)\left(\frac{1}{1+e^x}\right) \qquad \text{by partial fraction}$$

$$= \frac{1}{D}\left(\frac{1}{1+e^x}\right) - \frac{1}{D+1}\left(\frac{1}{1+e^x}\right)$$

$$= \int \frac{1}{1+e^x}\, dx - e^{-x}\int e^x \frac{dx}{1+e^x}$$

$$= \int \frac{e^x\, dx}{e^x(1+e^x)} - e^{-x}\int e^x \frac{dx}{1+e^x} \qquad \begin{bmatrix}\text{put } 1 + e^x = t \\ e^x\, dx = dt\end{bmatrix}$$

$$= \int \frac{dt}{t(t-1)} - e^{-x}\int \frac{dt}{t}$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt - e^{-x}\log(e^x + 1)$$

$$= \log(t-1) - \log t - e^{-x}\log(e^x+1)$$

$$= \log(e^x) - \log(1+e^x) - e^{-x}\log(e^x+1)$$

$$= x - \log(1+e^x) - e^{-x}\log(e^x+1)$$

Hence the complete solution is

$$y = c_1 + c_2 e^{-x} + x - \log(1+e^x) - e^{-x}\log(1+e^x)$$

Ex. 3 : *Solve* $(D^2 + 5D + 6)\, y = e^{-2x}\sec^2 x\,(1 + 2\tan x)$ ***(Dec. 2005, May 2008, 2011)***

Sol. : A.E. is $D^2 + 5D + 6 = 0$ gives $(D+2)(D+3) = 0 \Rightarrow D = -2, -3$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{P.I.} = \frac{1}{(D+3)(D+2)}\left[e^{-2x}\sec^2 x\,(1 + 2\tan x)\right]$$

$$= \frac{1}{D+3}\left[e^{-2x}\int e^{2x}\cdot e^{-2x}\sec^2 x\,(1+2\tan x)\, dx\right]$$

$$= \frac{1}{D+3}\left[e^{-2x}\int \sec^2 x\,(1+2\tan x)\, dx\right] \qquad \text{put } \tan x = t,\ \sec^2 x\, dx = dt$$

$$= \frac{1}{D+3}\left[e^{-2x}\int (1+2t)\, dt\right] = \frac{1}{D+3}\left[e^{-2x}(t + t^2)\right]$$

$$= \frac{1}{D+3}\left[e^{-2x}(\tan x + \tan^2 x)\right]$$

$$= e^{-3x}\int e^{3x}\cdot e^{-2x}\left[(\tan x - 1) + \sec^2 x\right] dx$$

$$= e^{-3x}\int e^{x}\left[(\tan x - 1) + \sec^2 x\right] dx$$

$$= e^{-3x}\left[e^x(\tan x - 1)\right] \qquad \because \int e^x[f(x) + f'(x)]\, dx = e^x f(x)$$

$$= e^{-2x}(\tan x - 1)$$

Hence the complete solution is

$$y = c_2 e^{-3x} + e^{-2x}[c_1 + \tan x - 1]$$

$$= c_2 e^{-3x} + e^{-2x}[c_3 + \tan x]$$

Ex. 4 : *Solve* $\frac{d^2y}{dx^2} + 9y = \sec 3x$ ***(May 2008)***

Sol. : A.E. is $D^2 + 9 = 0$, or $D = \pm 3i$

$$\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x \text{ and}$$

$$\text{P.I.} = \frac{1}{D^2+9}(\sec 3x) = \frac{1}{(D+3i)(D-3i)} \sec 3x$$

$$= \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right] \sec 3x$$

$$= \frac{1}{6i}\frac{1}{D-3i} \sec 3x - \frac{1}{6i}\frac{1}{D+3i} \sec 3x \quad \dots (1)$$

Now, $$\frac{1}{D-3i} \sec 3x = e^{3ix}\int e^{-3ix} \sec 3x \, dx = e^{3ix}\int \frac{\cos 3x - i \sin 3x}{\cos 3x} dx$$

$$= e^{3ix}\int [1 - i \tan 3x] \, dx$$

$$= e^{3ix}\left[x + \frac{i}{3}\log(\cos 3x)\right]$$

Changing i to $-i$ in this, we have

$$\frac{1}{D+3i}(\sec 3x) = e^{-3ix}\left[x - \frac{i}{3}\log(\cos 3x)\right]$$

Putting values in (1), we have

$$\text{P.I.} = \frac{1}{6i}\left[e^{3ix}\left\{x + \frac{i}{3}\log(\cos 3x)\right\} - e^{-3ix}\left\{x - \frac{i}{3}\log(\cos 3x)\right\}\right]$$

$$= \frac{x}{6i}\cdot e^{3ix} + \frac{e^{3ix}\log(\cos 3x)}{18} - \frac{x\,e^{-3ix}}{6i} + \frac{e^{-3ix}\log(\cos 3x)}{18}$$

Combining the like terms, we get

$$= \frac{x}{3}\left[\frac{e^{3ix} - e^{-3ix}}{2i}\right] + \frac{1}{9}\left[\frac{e^{3ix} + e^{-3ix}}{2}\right]\log(\cos 3x)$$

$$\text{P.I.} = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x \log(\cos 3x)$$

Hence the general solution will be

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x \cdot \log(\cos 3x)$$

Ex. 5 : *Solve* $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$ ***(May 2006, Dec. 2012)***

Sol. : $(D^2 - D - 2)\, y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

A.E. : $D^2 - D - 2 = 0 \quad \therefore \quad (D-2)(D+1) = 0$

$$\therefore \quad y_c = c_1 e^{2x} + c_2 e^{-x}$$

$$y_p = \frac{1}{(D-2)(D+1)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{D-2}\left[e^{-x}\int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) dx\right]$$

$$= \frac{1}{D-2}\left[e^{-x}\int e^{x}\left\{2\log x + \frac{2}{x} - \frac{1}{x} + \frac{1}{x^2}\right\} dx\right]$$

$$= \frac{1}{D-2}\left\{e^{-x}\int e^{x}\left[\left(2\log x-\frac{1}{x}\right)+\left(\frac{2}{x}+\frac{1}{x^2}\right)\right]dx\right\}$$

$$= \frac{1}{D-2}\; e^{-x}\cdot e^{x}\left(2\log x-\frac{1}{x}\right) = \frac{1}{D-2}\left(2\log x-\frac{1}{x}\right)$$

$$= e^{2x}\int e^{-2x}\left(2\log x-\frac{1}{x}\right)dx$$

$$= e^{2x}\left\{\int 2\log x\cdot e^{-2x}\,dx-\int e^{-2x}\frac{1}{x}\,dx\right\}$$

$$= e^{2x}\left\{2\log x\left(\frac{e^{-2x}}{-2}\right)-\int\frac{2}{x}\cdot\left(\frac{e^{-2x}}{-2}\right)dx-\int e^{-2x}\cdot\frac{1}{x}\,dx\right\}$$

$$= e^{2x}\left\{-\log x\cdot e^{-2x}+\int e^{-2x}\frac{1}{x}\,dx-\int e^{-2x}\frac{1}{x}\,dx\right\}$$

$$= e^{2x}\{-\log x\, e^{-2x}\} = -\log x$$

$\therefore$ $y = \text{C.F.} + \text{P.I.} = y_c + y_p$

$$y = c_1 e^{2x} + c_2 e^{-x} - \log x$$

Ex. 6 : *Solve* $(D^2-1)\,y = e^{-x}\sin e^{-x}+\cos e^{-x}$.

Sol. : AE : $D^2-1 = 0$ or $(D-1)(D+1) = 0 \Rightarrow D = -1, +1$

$\therefore$ $y_c = c_1 e^{x} + c_2 e^{-x}$

$$y_p = \frac{1}{(D-1)(D+1)}\,(e^{-x}\sin e^{-x}+\cos e^{-x})$$

$$= \frac{1}{D-1}\left\{e^{-x}\int e^{x}(\cos e^{-x}+e^{-x}\sin e^{-x})\,dx\right\} \qquad \left\{\text{Use}\int e^{x}[f+f']\,dx = e^{x}\cdot f\right\}$$

$$= \frac{1}{D-1}\{e^{-x}\cdot e^{x}\cos e^{-x}\} = \frac{1}{D-1}\cdot\cos e^{-x}$$

$$= e^{x}\int e^{-x}\cos e^{-x}\,dx = -e^{x}\int\cos e^{-x}(-e^{-x}\,dx) \qquad \{\text{Use } e^{-x} = t\}$$

$$= -e^{x}\sin e^{-x}$$

$\therefore$ $y = c_1 e^{x} + c_2 e^{-x} - e^{x}\sin e^{-x}$.

Ex. 7 : *Solve* $(D^2-1)\;y = (1+e^{-x})^{-2}$.

Sol. : A.E. : $D^2-1 = 0$

$$\text{C.F.} = c_1 e^{x} + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)(D-1)}\;(1+e^{-x})^{-2}$$

$$= \frac{1}{D+1}\; e^{x}\int e^{-x}(1+e^{-x})^{-2}\,dx$$

$$= \frac{1}{D+1}\,(-e^{x})\int(1+e^{-x})^{-2}(-e^{-x}\,dx) = \frac{-1}{D+1}\;[-e^{x}(1+e^{-x})^{-1}]$$

$$= e^{-x}\int\frac{e^{x}\cdot e^{x}}{1+e^{-x}}\,dx$$

$$= e^{-x}\int\frac{e^{2x}\cdot(e^{x}\,dx)}{1+e^{x}} \quad (1+e^{x} = t)$$

$$= e^{-x}\int\frac{(t-1)^2}{t}\,dt = e^{-x}\left[\frac{t^2}{2}-2t+\log t\right]$$

$$= e^{-x}\left[\frac{(1+e^{x})^2}{2}-2(1+e^{x})+\log(1+e^{x})\right]$$

$$\therefore \quad y = c_1 e^{x} + c_2 e^{-x} + \frac{e^{-x}}{2}(1+e^{x})^2 + e^{-x}\log(1+e^{x}) - 2e^{-x} - 2$$

$$\text{or} \quad y = A e^{x} + B e^{-x} + e^{-x}\left[\frac{(1+e^{x})^2}{2} + \log(1+e^{x})\right] - 2$$

Ex. 8 :** Solve $(D^2 + 3D + 2)\, y = e^{e^x} + \cos e^x$.* ***(Dec. 2007)

Sol. : A.E. : $D^2 + 3D + 2 = (D+2)(D+1) = 0$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+2)(D+1)}\left(e^{e^x} + \cos e^x\right)$$

$$= \frac{1}{D+2}\, e^{-x}\int e^{x}\left(e^{e^x} + \cos e^x\right) dx$$

$$= \frac{1}{D+2}\, e^{-x}\left(e^{e^x} + \sin e^x\right)$$

$$= e^{-2x}\int e^{2x} e^{-x}\left(e^{e^x} + \sin e^x\right) dx$$

$$= e^{-2x}\int e^{x}\left(e^{e^x} + \sin e^x\right) dx$$

$$= e^{-2x}\left(e^{e^x} - \cos e^x\right)$$

$$\therefore \quad y = c_1 e^{-2x} + c_2 e^{-x} + e^{-2x}\left(e^{e^x} - \cos e^x\right)$$

Ex. 9 :** Solve $(D^2 + 3D + 2)\, y = \sin e^x$.* ***(May 2012, Dec. 2012)

Sol. : A.E. : $D^2 + 3D + 2 = (D+2)(D+1) = 0 \Rightarrow D = -2, -1$.

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+2)(D+1)}\sin e^x = \frac{1}{D+2}\, e^{-x}\int e^{x}\sin e^x\, dx$$

$$= \frac{1}{D+2}\, e^{-x}(-\cos e^x) = -e^{-2x}\int e^{x}\cos e^x\, dx$$

$$= -e^{-2x}\sin e^x$$

$$\therefore \quad y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x}\sin e^x$$

***Ex. 10 :** Solve $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$.*

Sol. : A.E. : $D^2 + 1 = (D+i)(D-i) = 0 \Rightarrow D = \pm i$.

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1}\operatorname{cosec} x = \frac{1}{2i}\left(\frac{1}{D-i} - \frac{1}{D+i}\right)\operatorname{cosec} x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i}\operatorname{cosec} x - \frac{1}{D+i}\operatorname{cosec} x\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int e^{-ix}\operatorname{cosec} x\, dx - e^{-ix}\int e^{ix}\operatorname{cosec} x\, dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int(\cos x - i\sin x)\operatorname{cosec} x\, dx - e^{-ix}\int(\cos x + i\sin x)\operatorname{cosec} x\, dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int(\cot x - i)\, dx - e^{-ix}\int(\cot x + i)\, dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}(\log\sin x - ix) - e^{-ix}(\log\sin x + ix)\right]$$

$$= \frac{1}{2i}\left[(\log\sin x)(e^{ix} - e^{-ix}) - ix(e^{ix} + e^{-ix})\right]$$

$$= \sin x\log\sin x - x\cos x$$

$$\therefore \quad y = c_1\cos x + c_2\sin x + \sin x\log\sin x - x\cos x$$

(B) Short-cut Methods for Finding P.I. in Certain Standard Cases

Although the general method (A) discussed in the previous article will always work in the theory, it many a times leads to laborious and difficult integration. To avoid this, short methods of finding P.I. without actual integration are developed depending upon the particular form of function f(x).

Case I : P.I. when $f(x) = e^{ax}$, a is any constant.

To obtain $y_p = \dfrac{1}{\phi(D)} e^{ax}$, we have $D e^{ax} = a e^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$ ...... $D^n e^{an} = a^n e^{ax}$

$$\therefore \quad (a_0 D^n + a_1 D^{n-1} + \ldots\ldots + a_n) e^{ax} = (a_0 a^n + a_1 a^{n-1} + \ldots\ldots + a_n) e^{ax}$$

$$\text{or} \quad \phi(D) e^{ax} = \phi(a) e^{ax}$$

Operating on both sides by $\dfrac{1}{\phi(D)}$, we have

$$\frac{1}{\phi(D)} [\phi(D) e^{ax}] = \frac{1}{\phi(D)} [\phi(a) e^{ax}]$$

$$\text{or} \quad e^{ax} = \phi(a) \frac{1}{\phi(D)} (e^{ax}), \qquad \left(\because \frac{1}{\phi(D)} \text{ is a linear operator}\right)$$

Dividing by $\phi(a)$, we have the formula

$$\boxed{\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax} \text{ provided } \phi(a) \neq 0} \qquad \ldots (A)$$

Case of Failure : If $\phi(a) = 0$, above rule fails and we proceed as under.

Since $\phi(a) = 0$, $D - a$ must be a factor of $\phi(D)$ (by Factor Theorem).

Let $\phi(D) = (D - a) \psi(D)$ where, $\psi(a) \neq 0$. Then

$$\frac{1}{\phi(D)} (e^{ax}) = \frac{1}{D-a} \frac{1}{\psi(D)} e^{ax}$$

$$= \frac{1}{D-a} \frac{e^{ax}}{\psi(a)} \qquad \ldots \text{ from (A)}$$

$$= \frac{1}{\psi(a)} \frac{1}{D-a} e^{ax}$$

$$= \frac{1}{\psi(a)} e^{ax} \int e^{-ax} e^{ax} dx \qquad \ldots \text{(refer 1.11-A (i))}$$

$$= \frac{1}{\psi(a)} e^{ax} \int dx$$

$$= x \cdot \frac{1}{\psi(a)} e^{ax}, \quad \text{where } \psi(a) = \phi'(a) \neq 0.$$

$$\text{i.e.} \quad \boxed{\frac{1}{\phi(D)} e^{ax} = x \cdot \frac{1}{\phi'(a)} e^{ax} \text{ provided } \phi'(a) \neq 0} \qquad \ldots (B)$$

If $\phi'(a) = 0$ then we shall apply (B) again to get

$$\boxed{\frac{1}{\phi(D)} (e^{ax}) = x^2 \frac{1}{\phi''(a)} e^{ax}, \text{ provided } \phi''(a) \neq 0}, \text{ and so on.}$$

Remark 1 : Since $\phi(D) = (D - a) \psi(D)$

$$\phi'(D) = (D - a) \psi'(D) + \psi(D)$$

$$\therefore \quad \phi'(a) = 0 + \psi(a)$$

$$\text{or} \quad \phi'(a) = \psi(a)$$

Remark 2 : It can also be established that

$$\frac{1}{(D-a)^r \psi(D)} e^{ax} = \frac{1}{\psi(a)} \frac{x^r}{r!} e^{ax}, \text{ provided } \psi(a) \neq 0.$$

Remark 3 : Any constant k can be expressed as $k = k \cdot e^{0x}$

$$\therefore \quad y_p = \frac{1}{\phi(D)} (k) = \frac{1}{\phi(D)} k \cdot e^{0x} = k \cdot \frac{1}{\phi(D)} e^{0x}$$

$$= k \cdot \frac{1}{\phi(0)}, \ \phi(0) \neq 0$$

Remark 4 : If $f(x) = a^x$ then we use $a^x = e^{x \log a}$

$$\therefore \quad y_p = \frac{1}{\phi(D)} a^x = \frac{1}{\phi(D)} e^{x \log a}$$

$$= \frac{1}{\phi(\log a)} a^x$$

Replace D with log a.

If $f(x) = a^{-x}$ then we use $a^{-x} = e^{x \log 1/a} = e^{x(-\log a)}$

$$\therefore \quad y_p = \frac{1}{\phi(D)} a^{-x} = \frac{1}{\phi(D)} e^{x(-\log a)}$$

$$= \frac{1}{\phi(-\log a)} a^{-x}.$$

Replace D with – log a.

Formulae for Ready Reference :

1. $\frac{1}{D-a} e^{ax} = x \cdot e^{ax}$
2. $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$
3. $\frac{1}{(D-a)^3} e^{ax} = \frac{x^3}{3!} e^{ax}$
4. $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$
5. $\frac{1}{(D-a)^r \psi(D)} e^{ax} = \frac{1}{\psi(a)} \frac{1}{(D-a)^r} e^{ax} = \frac{1}{\psi(a)} \frac{x^r}{r!} e^{ax}, \quad \psi(a) \neq 0$

ILLUSTRATIONS

Ex. 1 : *Find the Particular Integral of $(D^2 - 5D + 6)\, y = 3\, e^{5x}$.*

Sol. :
$$\text{P.I.} = \frac{3}{D^2 - 5D + 6} (e^{5x}) = \frac{3\, e^{5x}}{5^2 - 5.5 + 6} = \frac{e^{5x}}{2}$$

Ex. 2 : *Find the Particular Integral of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$*

Sol. : Here
$$\text{P.I.} = \frac{1}{D^2 + 4D + 3} (e^{-3x}), \ \phi(D) = D^2 + 4D + 3$$

But $\phi(-3) = 9 - 12 + 3 = 0$ hence $\phi(-3) = 0$ and case I fails.

$$\therefore \quad \text{P.I.} = \frac{x\, e^{-3x}}{\phi'(a)} = x \cdot \frac{1}{2D + 4} \cdot e^{-3x}, \ D \to a = -3$$

$$= \frac{x\, e^{-3x}}{2(-3) + 4} = \frac{x\, e^{-3x}}{-2}$$

Ex. 3 : *Find the Particular Integral of $(D-1)^3 y = e^x + 2^x - \frac{3}{2}$.*

Sol. :
$$y_p = \frac{1}{(D-1)^3} e^x + \frac{1}{(D-1)^3} 2^x - \frac{3}{2} \frac{1}{(D-1)^3} e^{0x}$$

$$= \frac{x^3}{3!} e^x + \frac{1}{(\log 2 - 1)3} 2^x - \frac{3}{2} \frac{1}{(0-1)^3}$$

$$= \frac{x^3}{6} e^x + \frac{1}{(\log 2 - 1)^3} 2x + \frac{3}{2}$$

Ex. 4 : *Find the Particular Integral of* $(D-2)^2 (D+1)\, y = e^{2x} + 2^{-x}$

Sol. :

$$y_p = \frac{1}{(D-2)^2 (D+1)}\, e^{2x} + \frac{1}{(D-2)^2 (D+1)}\, 2^{-x}$$

$$= \frac{1}{(D-2)^2} \cdot \frac{1}{(2+1)}\, e^{2x} + \frac{2^{-x}}{(-\log 2 - 2)^2 (-\log 2 + 1)}$$

$$= \frac{1}{3} \cdot \frac{x^2}{2!}\, e^{2x} + \frac{2^{-x}}{(-\log 2 - 2)^2 (-\log 2 + 1)}$$

Case II : P.I. when f(x) = sin (ax + b) or cos (ax + b).

To obtain $y_p = \frac{1}{\phi(D^2)} \sin (ax + b)$ or $\frac{1}{\phi(D^2)} \cos (ax + b)$, we have

$$D \sin (ax+b) = a \cos (ax+b)$$

$$D^2 \sin (ax+b) = -a^2 \sin (ax+b)$$

$$D^3 \sin (ax+b) = -a^3 \cos (ax+b)$$

$$D^4 \sin (ax+b) = a^4 \sin (ax+b)$$

or $$(D^2)^2 \sin (ax+b) = (-a^2)^2 \sin (ax+b)$$

Similarly $$(D^2)^p \sin (ax+b) = (-a^2)^p \sin (ax+b)$$

and we may generalise that

$$\phi(D^2) \sin (ax+b) = \phi(-a^2) \sin (ax+b)$$

Operating on both sides by $\frac{1}{\phi(D^2)}$, we have

$$\frac{1}{\phi(D^2)} [\phi(D^2) \sin (ax+b)] = \frac{1}{\phi(D^2)} [\phi(-a^2) \sin (ax+b)]$$

$$\sin (ax+b) = \phi(-a^2) \frac{1}{\phi(D^2)} \sin (ax+b)$$

Dividing now by $\phi(-a^2)$, we have

$$\boxed{\frac{1}{\phi(D^2)} \sin (ax+b) = \frac{1}{\phi(-a^2)} \sin (ax+b), \text{ provided } \phi(-a^2) \neq 0}$$

Case of Failure : But if $\phi(-a^2) = 0$, above rule fails and we proceed as under :

We know by Euler's Theorem that $\cos (ax+b) + i \sin (ax+b) = e^{i(ax+b)}$ hence

$$\frac{1}{\phi(D^2)} \sin (ax+b) = \text{Imag. Part of } \frac{1}{\phi(D^2)} e^{i(ax+b)}$$

$$= \text{I.P. of } \frac{1}{\phi(D^2)} e^{i(ax+b)}$$

$$= \text{I.P. of } x \frac{1}{\phi'(D^2)} e^{i(ax+b)}, \quad (D^2 = -a^2)$$

Hence $$\boxed{\frac{1}{\phi(D^2)} \sin (ax+b) = x \frac{1}{\phi'(-a^2)} \sin (ax+b) \text{ provided } \phi'(-a^2) \neq 0}$$

Add if $\phi'(-a^2) \neq 0$, we have

$$\boxed{\frac{1}{\phi(D^2)} \sin (ax+b) = x^2 \frac{1}{\phi''(-a^2)} \sin (ax+b), \text{ provided } \phi''(-a^2) \neq 0}$$

Similarly formulae for cos (ax + b) viz.

$$\boxed{\frac{1}{\phi(D^2)} \cos (ax+b) = \frac{1}{\phi(-a^2)} \cos (ax+b), \text{ provided } \phi(-a^2) \neq 0}$$

But if $\phi(-a^2) = 0$, we have

$$\boxed{\frac{1}{\phi(D^2)} \cos(ax + b) = x \frac{1}{\phi'(-a^2)} \cos(ax + b), \text{ provided } \phi'(-a^2) \neq 0}$$

And if $\phi'(-a^2) = 0$, we have

$$\boxed{\frac{1}{\phi(D^2)} \cos(ax + b) = x^2 \frac{1}{\phi''(-a^2)} \cos(ax + b), \text{ provided } \phi''(-a^2) \neq 0}$$

and so on and so forth.

Additional Results :

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax, \quad \phi(-a^2) \neq 0 \quad (\text{Replace } D^2 \text{ with } -a^2)$$

$$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax, \quad \phi(-a^2) \neq 0, \ (\text{Replace } D^2 \text{ with } -a^2)$$

For the case of failure, it can also be established that

$$\boxed{\frac{1}{D^2 + a^2} \sin(ax + b) = -\frac{x}{2a} \cos(ax + b)}$$

$$\boxed{\frac{1}{D^2 + a^2} \cos(ax + b) = \frac{x}{2a} \sin(ax + b)}$$

$$\boxed{\frac{1}{(D^2 + a^2)^r} \sin(ax + b) = \left(\frac{-x}{2a}\right)^r \frac{1}{r!} \sin\left(ax + b + \frac{r\pi}{2}\right)}$$

$$\boxed{\frac{1}{(D^2 + a^2)^r} \cos(ax + b) = \left(\frac{-x}{2a}\right)^r \frac{1}{r!} \cos\left(ax + b + \frac{r\pi}{2}\right)}$$

Useful Formulae :

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{e^{ox}}{2} - \frac{\cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} = \frac{e^{ox}}{2} + \frac{\cos 2x}{2}$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$\sin x = 2 \sin\frac{x}{2} \cos\frac{x}{2}$$

$$\sin 2x = 2 \sin x \cos x.$$

Note : Write $D^3 = D^2 \cdot D$; $D^4 = (D^2)^2$; $D^5 = (D^2)^2 \cdot D$. Always replace D^2 by $-a^2$ and *keep D as it is*. To get D^2 in the denominator, rationalise the denominator and then replace D^2 by $-a^2$. Now numerator will contain an operator in D, therefore open the bracket.

ILLUSTRATIONS

Ex. 1 : *Solve* $(D^2 + 2D + 1)\, y = 4 \sin 2x$.

Sol. : A.E. is $D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1$.

$$\text{C.F.} = (c_1 x + c_2)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1}\,(4 \sin 2x)$$

$$= \frac{1}{-4 + 2D + 1}\,(4 \sin 2x) \qquad \text{(putting } D^2 = -2^2 = -4\text{)}$$

$$= \frac{4}{2D - 3}\,(\sin 2x)$$

$$= \frac{4\,(2D + 3)}{4D^2 - 9}\,(\sin 2x) \qquad \text{[(Multiply numerator and denominator by (2D + 3)]}$$

$$= \frac{4\,(2D + 3)}{4\,(-4) - 9}\,(\sin 2x) \qquad \text{(replace } D^2 \text{ with } -4\text{)}$$

$$= -\frac{4}{25}\,(2D + 3)\,(\sin 2x)$$

$$= -\frac{4}{25}\,[4 \cos 2x + 3 \sin 2x]$$

$\therefore$ General solution is

$$y = (c_1 x + c_2)\, e^{-x} - \frac{4}{25}\,[4 \cos 2x + 3 \sin 2x]$$

Ex. 2 : *Solve* $\dfrac{d^3y}{dx^3} + 4\,\dfrac{dy}{dx} = \sin 2x$. ***(Dec. 2010)***

Sol. : A.E. will be $D^3 + 4D = 0 \Rightarrow D\,(D^2 + 4) = 0$. Hence $D = 0$ and $D = \pm 2i$.

$$\text{C.F.} = \text{Complementary Function} = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

$$\text{P.I.} = \frac{1}{D\,(D^2 + 4)}\,(\sin 2x) \qquad [\because D^2 + 4 = 0, \text{ for } D^2 = -2^2 = -4\,]$$

$$= x \cdot \frac{1}{3D^2 + 4}\,(\sin 2x), \left[\frac{d}{dD}\,(D^3 + 4D) = 3D^2 + 4, \text{ then put } D^2 = -4\right]$$

$$= x \cdot \frac{1}{3\,(-4) + 4}\,(\sin 2x)$$

$$= -\frac{x}{8}\,\sin 2x.$$

Hence the solution is $\quad y = c_1 + c_2 \cos 2x + c_3 \sin 2x - \dfrac{x \sin 2x}{8}$

Ex. 3 : *Solve* $(D^2 + 1)\, y = \sin x \sin 2x$. ***(Dec. 2008)***

Sol. : A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore \qquad \text{C.F.} = c_1 \cos x + c_2 \sin x$

We have

$$\text{P.I.} = \frac{1}{D^2 + 1}\,(\sin x \sin 2x)$$

$$= \frac{1}{D^2 + 1}\left[\frac{1}{2}\,(\cos x - \cos 3x)\right]$$

$$= \frac{1}{2}\,\frac{1}{D^2 + 1}\,\cos x - \frac{1}{2}\,\frac{1}{D^2 + 1}\,\cos 3x$$

($D^2 \to -9$ in 2^{nd} term, case fails for 1^{st} term)

$$= \frac{1}{2} x \cdot \frac{1}{2D} \cos x - \frac{1}{2} \frac{1}{-9+1} \cos 3x$$

$$= x \cdot \frac{1}{4} \frac{D}{D^2} \cos x + \frac{1}{16} \cos 3x, \quad (D^2 \to -1)$$

$$= \frac{1}{4} x \cdot \frac{D(\cos x)}{-1} + \frac{1}{16} \cos 3x$$

$$= \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x$$

Hence the solution is $\quad y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x$

Case III : P.I when f(x) = cosh (ax + b) or sinh (ax + b).

To find $y_p = \dfrac{1}{\phi(D^2)} \cosh(ax+b)$ or $\dfrac{1}{\phi(D^2)} \sinh(ax+b)$

As earlier on the similar line, we can prove that

$$\boxed{\begin{array}{ll} & \dfrac{1}{\phi(D^2)} \cosh(ax+b) = \dfrac{1}{\phi(a^2)} \cosh(ax+b), \ \phi(a^2) \neq 0 \\ \text{and} & \dfrac{1}{\phi(D^2)} \sinh(ax+b) = \dfrac{1}{\phi(a^2)} \sinh(ax+b), \ \phi(a^2) \neq 0 \end{array}}$$

ILLUSTRATION

Ex. 1 : *Solve* $\dfrac{d^3y}{dx^3} - 4\dfrac{dy}{dx} = 2 \cosh 2x$

Sol. : A.E. is $D^3 - 4D = 0 \Rightarrow D(D^2 - 4) = 0, \ D = 0, \pm 2$

Hence $\quad$ C.F. $= c_1 + c_2 e^{2x} + c_3 e^{-2x}$

$$\text{P.I.} = \frac{1}{(D^2-4)} \left[\frac{1}{D} (2 \cosh 2x)\right]$$

$$= \frac{1}{D^2-4} \int 2 \cosh 2x \, dx$$

$$= \frac{2}{D^2-4} \left(\frac{\sinh 2x}{2}\right)$$

$$= \frac{1}{D^2-4} (\sinh 2x)$$

[case of failure, hence differentiate ϕ (D)]

$$= \frac{x(\sinh 2x)}{2D} = \frac{x D(\sinh 2x)}{2D^2}$$

$$= \frac{x}{2} \frac{D(\sinh 2x)}{(4)} = \frac{x}{8} D(\sinh 2x)$$

$$= \frac{x}{4} \cosh 2x$$

$\therefore$ Solution is $\quad y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \dfrac{x}{4} \cosh 2x$

Case IV : P.I. when f(x) = x^m

To find $y_p = \dfrac{1}{\phi(D)} x^m$, we write $\dfrac{1}{\phi(D)} (x^m) = [\phi(D)]^{-1} x^m$.

We shall now expand $[\phi(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term-by-term. Since $(m+1)^{th}$ and higher derivatives of x^m will be zero, we need not consider terms beyond D^m.

Important Formulae :

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots\ldots\ldots$$

Also note that $D^n (x^n) = n!$ and $D^{n+1} (x^n) = 0$.

Note : To find $y_p = \frac{1}{\phi(D)} x^m$

(i) We always take constant term common from the denominator and use the formulae $(1+x)^{-1}$, $(1-x)^{-1}$, $(1+x)^n$, $(1-x)^n$.

(ii) If constant term is absent in the denominator then the minimum power of D is taken common from the denominator.

for example.

$$\frac{1}{D^2 - 3D - 2} x^m = \frac{1}{-2\left[1 - \left(\frac{D^2 - 3D}{2}\right)\right]} x^m$$

$$\frac{1}{D^2 - 3D + 3} x^m = \frac{1}{3\left[1 + \left(\frac{D^2 - 3D}{3}\right)\right]} x^m$$

$$\frac{1}{D^3 - 3D^2 + 2D} x^m = \frac{1}{2D\left[1 + \left(\frac{D^2 - 3D}{2}\right)\right]} x^m$$

ILLUSTRATION

Ex. 1 : *Find the particular solution of* $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$.

Sol. : It can be put as $(D^2 - D + 1)\, y = x^3 - 3x^2 + 1$.

$$\text{P.I.} = \frac{1}{(1 - D + D^2)} (x^3 - 3x^2 + 1)$$

$$= [1 - (D - D^2)]^{-1} (x^3 - 3x^2 + 1)$$

Expanding by Binomial theorem upto D^3 terms

$$= [1 + (D - D^2) + (D - D^2)^2 + (D - D^2)^3 + \ldots] (x^3 - 3x^2 + 1)$$

$$= [1 + D - D^2 + D^2 - 2D^3 + \ldots + D^3 + \ldots] (x^3 - 3x^2 + 1)$$

$$= (1 + D - D^3) (x^3 - 3x^2 + 1)$$

$$= x^3 - 6x - 5$$

Hence $\quad \text{P.I.} = x^3 - 6x - 5.$

Case V : P.I. when $f(x) = e^{ax} V$, where V is any function of x.

To find $y_p = \frac{1}{f(D)} e^{ax} V$, we have

$$D (e^{ax} V) = e^{ax} DV + ae^{ax} V = e^{ax} (D + a) V$$

and

$$D^2 (e^{ax} V) = e^{ax} D^2 V + 2a\, e^{ax} DV + a^2 e^{ax} V$$

$$= e^{ax} (D + a)^2 V$$

and proceeding similarly, we may have in general

$$D^n (e^{ax} V) = e^{ax} (D + a)^n V$$

Hence

$$\phi(D) (e^{ax} V) = e^{ax} \phi(D + a) V \qquad \ldots \text{(I)}$$

Now, let

$$\phi(D + a) V = V_1 \Rightarrow V = \frac{1}{\phi(D + a)} V_1$$

If we put value of V in (I), we have

$$\phi(D)\left[e^{ax}\frac{1}{\phi(D+a)}V_1\right] = e^{ax}V_1$$

Operating on both sides by $\dfrac{1}{\phi(D)}$ now, we get

$$e^{ax}\frac{1}{\phi(D+a)}V_1 = \frac{1}{\phi(D)}(e^{ax}V_1)$$

Here V_1 is any function of x, and hence, we have the formula

$$\boxed{\frac{1}{\phi(D)}(e^{ax}V) = e^{ax}\frac{1}{\phi(D+a)}(V)}$$

ILLUSTRATION

Ex. 1 : *Solve* $(D^2 - 4D + 3)\,y = x^3 e^{2x}$.

Sol. : A.E. $= D^2 - 4D + 3 = (D-1)(D-3) \Rightarrow D = 1, 3.$

Hence $\quad$ C.F. $= c_1 e^x + c_2 e^{3x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3}(x^3 e^{2x})$$

$$= e^{2x}\frac{1}{(D+2)2 - 4(D+2) + 3}(x^3) \qquad \ldots (D \to D+2)$$

$$= e^{2x}\frac{1}{D^2 + 4D + 4 - 4D - 8 + 3}(x^3)$$

$$= e^{2x}\frac{1}{D^2 - 1}(x^3) = -e^{2x}(1 - D^2)^{-1}(x^3) \qquad \text{(by case IV)}$$

$$= -e^{2x}[1 + D^2 + D^4 + \ldots\ldots](x^3) = -e^{2x}[x^3 + 6x]$$

Hence solution is $\quad y = c_1 e^x + c_2 e^{3x} - e^{2x}(x^3 + 6x)$

Case VI : P.I. when $f(x) = x^m \sin ax$, or $x^m \cos ax$.

To find $y_p = \dfrac{1}{f(D)} x^m \sin ax$ or $\dfrac{1}{f(D)} x^m \cos ax$, we have

$$\frac{1}{\phi(D)} x^m[\cos ax + i \sin ax] = \frac{1}{\phi(D)} x^m e^{iax}$$

$$= e^{iax}\frac{1}{\phi(D + ia)} x^m$$

Now $\dfrac{1}{\phi(D + ia)} x^m$, can be evaluated by method of case IV and equating the Real and Imaginary parts, we get the required results.

ILLUSTRATION

Ex. 1 : *Solve* $(D^4 + 2D^2 + 1)\,y = x^2 \cos x$.

Sol. : A.E. is $(D^2 + 1)^2 = 0 \Rightarrow D = \pm i, \pm i$.

$\therefore \quad$ C.F. $= (c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x$

For Particular Integral, we have

$$\frac{1}{(D^2+1)^2}[x^2(\cos x + i \sin x)] = \frac{1}{(D^2+1)^2} x^2 \cdot e^{ix}$$

$$= e^{ix}\frac{1}{[(D+i)^2 + 1]^2}(x^2) = e^{ix}\frac{1}{(D^2 + 2iD)^2}(x^2)$$

$$= e^{ix}\frac{1}{-4D^2\left(1 - \dfrac{iD}{2}\right)^2}(x^2) \qquad \left(\because \frac{1}{i} = -i\right)$$

$$= -\frac{e^{ix}}{4}\ \frac{1}{D^2}\ \left(1 - \frac{iD}{2}\right)^{-2}\ (x^2)$$

$$= -\frac{e^{ix}}{4}\ \frac{1}{D^2}\ \left(1 + iD - \frac{3}{4}\ D^2 + \ldots\right)(x^2)$$

$$= -\frac{e^{ix}}{4}\ \frac{1}{D^2}\left[x^2 + 2ix - \frac{3}{2}\right]$$

$$= -\frac{e^{ix}}{4}\left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right] \text{ Integrating twice}$$

$$= -\frac{1}{4}\ [\cos x + i \sin x]\ \left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right]$$

Equating the real parts on both sides,

$$\frac{1}{(D^2 + 1)^2}\ (x^2 \cos x)\ =\ -\frac{1}{4}\left(\frac{x^4}{12} - \frac{3}{4}x^2\right)\cos x + \frac{1}{12}\ x^3 \sin x$$

Hence the general solution is

$$y\ =\ (c_1 x + c_2) \cos x + (c_3 x + c_4) \sin x + \frac{x^3 \sin x}{12} - \frac{(x^4 - 9x^2)}{48} \cos x$$

Case VII : P.I. when f(x) = x V, V being any function of x.

To find $\frac{1}{f(D)}\,xV$ we have by successive differentiations

$$D\,(xV)\ =\ x\,DV + V$$

$$D^2\,(xV)\ =\ x\,D^2V + 2DV$$

$$D^3\,(xV)\ =\ x\,D^3V + 3D^2V$$

and so on, we may have

$$D^n\,(xV)\ =\ x\,D^n V + n\,D^{n-1}\,(V)$$

Or

$$D^n\,(xV)\ =\ x\,D^n V + \frac{d}{dD}\ (D^n)\ V \qquad \ldots (A)$$

Since $\phi(D)$ is a polynomial in D, we may write in general from (A) using $\phi'(D) = \frac{d}{dD}\ \phi\,(D)$

$$\phi\,(D)\,(xV)\ =\ x\,\phi\,(D)\,V + \phi'\,(D)\,V \qquad \ldots (B)$$

Now, put $\phi\,(D)\,V = V_1$ so that $V = \frac{1}{\phi\,(D)}\ V_1$ in equation (B), we have

$$\phi\,(D)\left[x\,\frac{1}{\phi\,(D)}\,V_1\right]\ =\ x\,V_1 + \phi'\,(D)\,\frac{1}{\phi\,(D)}\ V_1$$

Operating on both sides by $\frac{1}{\phi\,(D)}$, we get

$$x \cdot \frac{1}{\phi\,(D)}\ V_1\ =\ \frac{1}{\phi\,(D)}\ [xV_1] + \frac{1}{\phi\,(D)}\ \phi'\,(D)\,\frac{1}{\phi\,(D)}\,V_1$$

and if we adjust the terms on both sides, we get

$$\frac{1}{\phi\,(D)}\ [xV_1]\ =\ \left[x - \frac{1}{\phi\,(D)}\,\phi'\,(D)\right]\frac{1}{\phi\,(D)}\ V_1$$

But here V_1 is any function of x, hence we have the formula

$$\boxed{\frac{1}{\phi\,(D)}\ [xV]\ =\ \left[x - \frac{1}{\phi\,(D)}\,\phi'\,(D)\right]\frac{1}{\phi\,(D)}\,V}$$

Remark :

1. The rule xV is applied if
   (i) power of x is one
   (ii) $\frac{1}{\phi(D)}$ V is not a case of failure.
2. If power of x is one and $\frac{1}{\phi(D)}$ V is a case of failure then do not apply xV rule. In this case, apply rule given by case (VI).

   For example, $y_p = \frac{1}{D^2+1}\ x \sin x$

   Here $\frac{1}{D^2+1}$ sin x is a case of failure. Therefore use case (VI) method.

ILLUSTRATIONS

Ex. 1 : *Solve* $\frac{d^2y}{dx^2} + 4y = x \sin x$.

Sol. : A.E. : $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$,

and

$$\text{P.I.} = \frac{1}{D^2+4}\ (x \sin x)$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{D^2+4}\ (\sin x) \qquad \text{[by case (VII)]}$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{-1+4}\ (\sin x)$$

$$= \frac{1}{3}\left[x - \frac{2D}{D^2+4}\right] \sin x = \frac{1}{3}\left[x \sin x - \frac{2D}{D^2+4} \sin x\right]$$

$$= \frac{1}{3}\left[x \sin x - \frac{2D\ (\sin x)}{-1+4}\right] = \frac{1}{3}\left[x \sin x - \frac{2}{3}(\cos x)\right]$$

$$= \frac{1}{3}\ x \sin x - \frac{2}{9}\ \cos x.$$

Hence the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x \sin x}{3} - \frac{2}{9} \cos x$$

Ex. 2 : *Solve* $(D^2 - 2D + 1)\ y = x\ e^x \sin x$ ***(Dec. 2004)***

Sol. : A.E. : $D^2 - 2D + 1 = 0$

$$\Rightarrow (D-1)^2 = 0,\ D = 1, 1.$$

Hence C.F. $= (c_1 x + c_2)\ e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2}\ [x\ e^x \sin x]$$

$$= e^x \frac{1}{(D+1-1)^2}\ (x \sin x) \qquad \text{(by case V)}$$

$$= e^x \frac{1}{2}\ (x \sin x)$$

$$= e^x\left[x - \frac{2D}{D^2}\right] \frac{1}{D^2}\ (\sin x) \qquad \text{(by case VII)}$$

$$= e^x\left[x - \frac{2}{D}\right] (-\sin x) = -e^x\left[x \sin x - \frac{2}{D} \sin x\right]$$

$$= -e^x\ [x \sin x + 2 \cos x]$$

Hence the complete solution is

$$y = (c_1 x + c_2)\ e^x - e^x\ [x \sin x + 2 \cos x]$$

Now, we will summarise the short-cut methods of P.I. and the corresponding formulae :

Case I : $\frac{1}{\phi(D)} e^{ax} = \frac{e^{ax}}{\phi(a)}, \ \phi(a) \neq 0$

Case of failure : If $\phi(a) = 0, \ \frac{1}{\phi(D)} e^{ax} = x \cdot \frac{1}{\phi'(a)} e^{ax}, \ \phi'(a) \neq 0$

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}; \ \frac{1}{\phi(D)} (k) = k \cdot \frac{1}{\phi(0)}, \ \phi(0) \neq 0$$

$$\frac{1}{\phi(D)} a^x = \frac{a^x}{\phi(\log a)}$$

Case II : $\frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b), \ \phi(-a^2) \neq 0$

$$\frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b), \ \phi(-a^2) \neq 0$$

Case of failure : If $\phi(a^2) = 0, \ \frac{1}{\phi(D^2)} \sin(ax+b) = x \cdot \frac{1}{\phi'(-a)^2} \sin(ax+b), \ \phi'(-a)^2 \neq 0$

If $\phi(a^2) = 0, \ \frac{1}{\phi(D^2)} \cos(ax+b) = x \cdot \frac{1}{\phi'(-a^2)} \cos(ax+b), \ \phi'(-a^2) \neq 0$

Case of failure formulae :

$$\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax; \ \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

$$\frac{1}{(D^2+a^2)^r} \sin(ax+b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!} \sin\left(ax+b+r\frac{\pi}{2}\right) \text{ and}$$

$$\frac{1}{(D^2+a^2)^r} \cos(ax+b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!} \cos\left(ax+b+r\frac{\pi}{2}\right)$$

Case III : $\frac{1}{\phi(D^2)} \sinh ax = \frac{1}{\phi(a^2)} \sinh ax, \ \phi(a^2) \neq 0$ and

$$\frac{1}{\phi(D^2)} \cosh ax = \frac{1}{\phi(a^2)} \cosh ax, \ \phi(a^2) \neq 0$$

Case IV : $\frac{1}{\phi(D)} x^m = [\phi(D)]^{-1} x^m$, expand by using Binomial theorem.

Case V : $\frac{1}{\phi(D)} e^{ax} V = e^{ax} \frac{1}{\phi(D+a)} V$

Case VI : $\frac{1}{\phi(D)} x^m \sin ax = \text{I.P. of } \frac{1}{\phi(D)} x^m e^{iax} = \text{I.P. of } e^{iax} \frac{1}{\phi(D+ia)} x^m$

$$\frac{1}{\phi(D)} x^m \cos ax = \text{R.P. of } \frac{1}{\phi(D)} x^m e^{iax} = \text{R.P. of } e^{iax} \frac{1}{\phi(D+ia)} x^m$$

Case VII : $\frac{1}{\phi(D)} xV = \left[x - \frac{\phi'(D)}{\phi(D)}\right] \frac{1}{\phi(D)} V$

ILLUSTRATIONS ON SHORT-CUT METHODS

Ex. 1 : *Solve* $(D^2 + 2D + 1) y = 2 \cos x + 3x + 2 + 3e^x$.

Sol. : Here AE is $(D+1)^2 = 0 \Rightarrow D = -1, -1$

$\therefore$ C.F. $= (c_1 x + c_2) e^{-x}$

and

$$\text{P.I.} = 2 \frac{1}{D^2+2D+1} \cos x + \frac{1}{[1+(2D+D^2)]} (3x+2) + 3 \frac{1}{(D+1)^2} e^x$$

$$= \frac{1}{-1+2D+1} 2 \cos x + [1+(2D+D^2)]^{-1} (3x+2) + \frac{3e^x}{4}$$

$$= \int \cos x\,dx + [1 - 2D - D^2 + \ldots](3x + 2) + \frac{3}{4}e^x$$

$$= \sin x + 3x + 2 - 6 + \frac{3e^x}{4}$$

$$= \frac{3e^x}{4} + \sin x + 3x - 4$$

Hence the complete solution is

$$y = (c_1 x + c_2)\,e^{-x} + \frac{3e^x}{4} + \sin x + 3x - 4$$

Ex. 2 : *Solve* $\frac{d^2y}{dx^2} + a^2 y = \frac{a^2 R}{p}(l - x)$ *where a, R, p and l are constants, subject to the conditions* $y = 0,\ \frac{dy}{dx} = 0$ *at* $x = 0$.

Sol. : Given equation is

$$(D^2 + a^2)\,y = \frac{a^2 R}{p}(l - x)$$

A.E. = $D^2 + a^2 = 0$ or $D = \pm ia$

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2}\,\frac{a^2 R}{p}(l - x) = \frac{a^2 R}{p}\cdot\frac{1}{a^2}\,\frac{1}{\left(1 + \frac{D^2}{a^2}\right)}(l - x)$$

$$= \frac{R}{p}\left(1 + \frac{D^2}{a^2}\right)^{-1}(l - x) = \frac{R}{p}\left[1 - \frac{D^2}{a^2}\right](l - x) = \frac{R}{p}(l - x)$$

Hence the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{R}{p}(l - x) \qquad \ldots (1)$$

For initial conditions, now put y = 0 when x = 0 in (1), we get

$$0 = c_1 + \frac{R}{p}\,l \ \Rightarrow\ c_1 = \frac{-Rl}{p}$$

If we differentiate equation (1),

$$\frac{dy}{dx} = -ac_1 \sin ax + ac_2 \cos ax - \frac{R}{p}$$

Putting x = 0 and $\frac{dy}{dx} = 0$ in this, we get

$$0 = ac_2 - \frac{R}{p},\ \text{hence } c_2 = \frac{R}{ap}$$

Now put values of c_1 and c_2 in A, then the required particular solution is

$$y = \frac{R}{p}\left[\frac{\sin ax}{a} - l\cos ax + l - x\right]$$

Ex. 3 : *Solve* $(D^3 - 1)\,y = (1 + e^x)^2$.

Sol. : A.E. : $D^3 - 1 = 0$ or $(D - 1)(D^2 + D + 1) = 0 \ \therefore\ D = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$y_c = c_1 e^x + e^{(-1/2)x}[c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x]$$

$$y_p = \frac{1}{D^3 - 1}(1 + e^x)^2 = \frac{1}{D^3 - 1}(1 + 2e^x + e^{2x})$$

$$= \frac{1}{D^3 - 1}e^{ox} + 2\frac{1}{D^3 - 1}e^x + \frac{1}{D^3 - 1}e^{2x}$$

$$= -1 + \frac{2}{3}xe^x + \frac{1}{7}e^{2x}$$

$$\therefore \quad y = c_1 e^x + e^{(-1/2)x}[c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x] - 1 + \frac{2}{3}xe^x + \frac{1}{7}e^{2x}$$

Ex. 4 : *Solve* $(D-1)^2 (D^2+1)^2 y = \sin^2 \frac{x}{2}$.

Sol. : A.E. : $(D-1)^2 (D^2+1)^2 = 0$, $D = 1, 1, \pm i, \pm i$.

$$y_c = (c_1x + c_2) e^x + (c_3x + c_4) \cos x + (c_5x + c_6) \sin x.$$

$$y_p = \frac{1}{(D-1)^2 (D^2+1)^2} \sin^2 \frac{x}{2} = \frac{1}{(D-1)^2 (D^2+1)^2} \left(\frac{1-\cos x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)^2 (D^2+1)^2} e^{ox} - \frac{1}{(D-1)^2 (D^2+1)^2} \cos x\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{(D^2+1)^2 (-1-2D+1)} \cos x\right] = \frac{1}{2}\left[1 + \frac{1}{2} \frac{1}{(D^2+1)^2} \sin x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} x^2 \frac{1}{-8} \sin x\right] \left\{\frac{d^2}{dD^2} (D^2+1)^2 = \frac{d}{dD} 2 (D^2+1) 2D = 4 (3D^2+1), \text{ then put } D^2 = -1\right\}$$

$$= \frac{1}{2} - \frac{1}{32} x^2 \sin x$$

$$\therefore \quad y = (c_1x + c_2) e^x + (c_3x + c_4) \cos x + (c_5x + c_6) \sin x + \frac{1}{2} - \frac{1}{32} x^2 \sin x$$

Ex. 5 : *Solve* $(D^4 - 2D^3 - 3D^2 + 4D + 4) y = x^2 e^x$.

Sol. : A.E. : $(D-2)^2 (D+1)^2 = 0$

$$y_c = (c_1 x + c_2) e^{2x} + (c_3 x + c_4) e^{-x}$$

$$y_p = \frac{1}{(D^2 - D - 2)^2} e^x \cdot x^2 = e^x \frac{1}{[(D+1)^2 - (D+1) - 2]^2} x^2$$

$$= e^x \frac{1}{(D^2 + D - 2)^2} x^2 = \frac{e^x}{4} \frac{1}{\left[1 - \left(\frac{D^2 + D}{2}\right)\right]^2} x^2$$

$$= \frac{e^x}{4} \left[1 + (D^2 + D) + \frac{3}{4} (D^2 + D)^2 + \dots\right] x^2$$

$$= \frac{e^x}{4} \left[1 + D + \frac{7}{4} D^2 + \dots\right] x^2 = \frac{e^x}{4} \left[x^2 + 2x + \frac{7}{2}\right]$$

$$\therefore \quad y = (c_1 x + c_2) e^{2x} + (c_3 x + c_4) e^{-x} + \frac{e^x}{4} \left(x^2 + 2x + \frac{7}{2}\right).$$

Ex. 6 : *Solve* $(D^4 - 1) y = \cos x \cosh x$ ***(Dec. 2006)***

Sol. : A.E. : $(D-1)(D+1)(D+i)(D-i) = 0$

$$y_c = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x.$$

$$y_p = \frac{1}{D^4 - 1} \cos x \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2} \frac{1}{D^4 - 1} e^x \cos x + \frac{1}{2} \cdot \frac{1}{D^4 - 1} e^{-x} \cos x$$

$$= \frac{e^x}{2} \frac{1}{(D+1)^4 - 1} \cos x + \frac{e^{-x}}{2} \frac{1}{(D-1)^4 - 1} \cos x$$

$$= \frac{e^x}{2} \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 1} \cos x + \frac{e^{-x}}{2} \frac{1}{D^4 - 4D^3 + 6D^2 - 4D + 1 - 1} \cos x$$

$$= \frac{e^x}{2} \frac{1}{1 - 4D - 6 + 4D} \cos x + \frac{e^{-x}}{2} \frac{1}{1 + 4D - 6 - 4D}$$

$$= \frac{e^x}{2} \cdot \frac{\cos x}{-5} + \frac{e^{-x}}{2} \frac{\cos x}{-5} = \frac{\cos x}{-5} \cosh x$$

$$\therefore \quad y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{\cos x \cosh x}{5}$$

Ex. 7 : *Solve $(D^4 + 1)\, y = 2 \sinh x \sin x$.* ***(Dec. 2004)***

Sol. : A.E. : $D^4 + 1 = 0 \quad \therefore \quad D^4 + 2D^2 + 1 - 2D^2 = 0 \quad$ or $(D^2 + 1)^2 - (\sqrt{2}\, D)^2 = 0$

or $(D^2 - \sqrt{2}\, D + 1)\,(D^2 + \sqrt{2}\, D + 1) = 0$

$$\therefore \qquad D = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\, i, \qquad D = \frac{-1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\, i$$

$$y_c = e^{x/\sqrt{2}} \left[c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}} \left[c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right]$$

$$y_p = \frac{1}{D^4 + 1}\; 2 \sinh x \sin x$$

$$= \frac{1}{D^4 + 1}\,(e^x - e^{-x}) \sin x$$

$$= \frac{1}{D^4 + 1}\; e^x \sin x - \frac{1}{D^4 + 1}\; e^{-x} \sin x$$

$$= e^x \frac{1}{(D + 1)^4 + 1} \sin x - e^{-x} \frac{1}{(D - 1)^4 + 1} \sin x$$

$$= e^x \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 2} \sin x - e^{-x} \frac{1}{D^4 - 4D^3 + 6D^2 - 4D + 2} \sin x$$

$$= e^x \frac{1}{(-1)^2 + 4D\,(-1) + 6\,(-1) + 4D + 2} \sin x - e^{-x} \frac{1}{(-1)^2 - 4D\,(-1) + 6\,(-1) - 4D + 2} \sin x$$

$$= e^x \left(\frac{\sin x}{-3}\right) - e^{-x} \left(\frac{\sin x}{-3}\right)$$

$$= \frac{-2}{3} \sin x \cdot \left(\frac{e^x - e^{-x}}{2}\right) = -\frac{2}{3} \sin x \sinh x$$

$$\therefore \qquad y = \text{C.F.} + \text{P.I.}$$

$$y = e^{x/\sqrt{2}} \left[c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}} \left[c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right] - \frac{2}{3} \sin x \sinh x$$

Ex. 8 : *Solve $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}\,(1 + x)$.* ***(Dec. 2004)***

Sol. : Given D.E. is written as

$$(D^3 - 7D - 6)\, y = e^{2x}\,(1 + x) \quad \text{where } D \equiv \frac{d}{dx}$$

A.E. : $D^3 - 7D - 6 = 0 \quad \therefore \quad (D + 1)\,(D + 2)\,(D - 3) = 0$

$$\therefore \qquad \text{C.F.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 7D - 6}\; e^{2x}\,(1 + x)$$

$$= e^{2x} \frac{1}{(D + 2)^3 - 7\,(D + 2) - 6}\,(1 + x), \qquad \text{by } D \to D + 2$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 5D - 12}\,(1 + x)$$

$$= \frac{-e^{2x}}{12} \left[1 - \frac{D^3 + 6D^2 + 5D}{12}\right]^{-1} (1 + x) = \frac{-e^{2x}}{12} \left[1 + \frac{5D}{12} + \ldots\right] (1 + x)$$

$$= \frac{-e^{2x}}{12} \left(1 + x + \frac{5}{12}\right) = \frac{-e^{2x}}{12} \left(x + \frac{17}{12}\right)$$

$$\therefore \qquad y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{e^{2x}}{12} \left(x + \frac{17}{12}\right)$$

Ex. 9 : *Solve $(D^2 - 1)\, y = x \sin x + (1 + x^2)\, e^x$.* ***(Dec. 2010)***

Sol. : A.E. : $D^2 - 1 = 0$ or $(D - 1)(D + 1) = 0$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 - 1}\, x \sin x + \frac{1}{D^2 - 1}\, e^x (1 + x^2)$$

$$= x \frac{1}{D^2 - 1} \sin x - \frac{2D}{(D^2 - 1)^2} \sin x + e^x \frac{1}{(D + 1)^2 - 1} (1 + x^2)$$

$$= \frac{x \sin x}{-2} - \frac{2D}{4} \sin x + e^x \frac{1}{D^2 + 2D} (1 + x^2)$$

$$= -\frac{x}{2} \sin x - \frac{\cos x}{2} + e^x \frac{1}{2D} \left(1 - \frac{D}{2} + \frac{D^2}{4} + \ldots\right)(1 + x^2)$$

$$= -\frac{x}{2} \sin x - \frac{\cos x}{2} + \frac{e^x}{2} \frac{1}{D} \left(1 + x^2 - x + \frac{1}{2}\right)$$

$$= -\frac{x}{2} \sin x - \frac{\cos x}{2} + \frac{e^x}{2} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right)$$

$$\therefore \quad y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} (x \sin x + \cos x) + \frac{e^x}{12} (2x^3 - 3x^2 + 9x)$$

Ex. 10 : *$(D^2 - 4D + 4)\, y = e^x \cos^2 x$.* **(Dec. 2015)**

Sol. : A.E. is $D^2 - 4D + 4 = 0 \quad \therefore \quad D = 2, 2$

$$y_c = (c_1 x + c_2)\, e^{2x}$$

$$y_p = \frac{1}{(D - 2)^2} e^x \cos^2 x = e^x \frac{1}{(D - 1)^2} \cos^2 x$$

$$= e^x \frac{1}{(D - 1)^2} \left(\frac{1 + \cos 2x}{2}\right) = 0$$

$$= \frac{e^x}{2} \left[\frac{1}{(D - 1)^2} e^{ox} + \frac{1}{D^2 - 2D + 1} \cos 2x\right]$$

$$= \frac{e^x}{2} \left[1 - \frac{1}{(2D + 3)} \cos 2x\right] = \frac{e^x}{2} \left[1 - \frac{(2D - 3)}{4D^2 - 9} \cos 2x\right]$$

$$= \frac{e^x}{2} \left[1 + \frac{1}{25} (2D - 3) \cos 2x\right] = \frac{e^x}{2} \left[1 - \frac{1}{25} (4 \sin 2x + 3 \cos 2x)\right]$$

$$\therefore \quad y = (c_1 x + c_2)\, e^{2x} + \frac{e^x}{2} \left[1 - \frac{1}{25} (4 \sin 2x + 3 \cos 2x)\right]$$

Ex. 11 : *Solve $(D^2 + 1)\, y = x^2 \sin 2x$.*

Sol. : A.E. is $D^2 + 1 = 0 \;\therefore\; D = \pm i$.

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_p = \frac{1}{D^2 + 1} x^2 \sin 2x = \text{I.P. of } \frac{1}{D^2 + 1} e^{i2x} x^2$$

$$= \text{I.P. of } e^{i2x} \frac{1}{(D + 2i)^2 + 1} x^2 = \text{I.P. of } e^{i2x} \frac{1}{D^2 + 4iD - 4 + 1} x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{(-3)} \frac{1}{\left[1 - \frac{1}{3} (4iD + D^2)\right]} x^2 = \text{I.P. of } \frac{e^{i2x}}{(-3)} \left[1 - \frac{1}{3} (4iD + D^2)\right]^{-1} x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{(-3)} \left[1 + \frac{1}{3} (4iD + D^2) + \frac{1}{9} (-16D^2 + 8iD^3 + D^4) + \ldots\right] x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{(-3)}\left[1+\frac{4}{3}iD-\frac{13}{9}D^2+\ldots\right]x^2$$

$$= \text{I.P. of } \frac{(\cos 2x + i\sin 2x)}{(-3)}\left[\left(x^2-\frac{26}{9}\right)+i\frac{8}{3}x\right]$$

$$= -\frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x-\frac{8}{9}x\cos 2x$$

$$\therefore \quad y = c_1\cos x + c_2\sin x - \frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x - \frac{8}{9}x\cos 2x$$

Ex. 12 : *$(D^2 + D + 1)\, y = x \sin x$.* ***(May 2011)***

Sol. : A.E. is $D^2 + D + 1 = 0 \quad \therefore \; D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

$$y_c = e^{(-1/2)x}\left[c_1\cos(\sqrt{3}/2)x + c_2\sin(\sqrt{3}/2)x\right]$$

$$y_p = \frac{1}{D^2+D+1}x\sin x = \left[x-\frac{2D+1}{D^2+D+1}\right]\frac{1}{D^2+D+1}\sin x$$

$$= \left[x-\frac{2D+1}{D^2+D+1}\right]\frac{1}{D}\sin x = \left[x-\frac{2D+1}{D^2+D+1}\right](-\cos x)$$

$$= -x\cos x + (2D+1)\frac{1}{D}\cos x$$

$$= -x\cos x + (2D+1)\sin x$$

$$= -x\cos x + 2\cos x + \sin x$$

$$\therefore \quad y = e^{(-1/2)x}\left[c_1\cos(\sqrt{3}/2)x + c_2\sin(\sqrt{3}/2)x\right] - x\cos x + 2\cos x + \sin x$$

Ex. 13 : *$(D^2 + 2D + 1)\, y = x\, e^{-x} \cos x$.*

Sol. : A.E. is $D^2 + 2D + 1 = 0 \quad \therefore \; D = -1, 1$.

$$y_c = (c_1x + c_2)\,e^{-x}$$

$$y_p = \frac{1}{(D+1)^2}e^{-x}x\cos x = e^{-x}\frac{1}{D^2}x\cos x$$

$$= e^{-x}\left[x-\frac{2D}{D^2}\right]\frac{1}{D^2}\cos x = e^{-x}\left[x-\frac{2}{D}\right](-\cos x)$$

$$= e^{-x}(-x\cos x + 2\sin x)$$

$$\therefore \quad y = (c_1x+c_2)\,e^{-x} + e^{-x}(-x\cos x + 2\sin x)$$

Ex. 14 : *Solve $(D^2 + 4)\, y = x \sin^2 x$.*

Sol. : A.E. : $D^2 + 4 = 0 \quad \therefore \; D = \pm 2i$

$$y_c = c_1\cos 2x + c_2\sin 2x$$

$$y_p = \frac{1}{D^2+4}x\cdot\left(\frac{1-\cos 2x}{2}\right) = \frac{1}{2}\cdot\frac{1}{D^2+4}x - \frac{1}{2}\,\frac{1}{D^2+4}x\cos 2x.$$

$$= y_{p_1} + y_{p_2}$$

$$y_{p_1} = \frac{1}{2}\cdot\frac{1}{D^2+4}x = \frac{1}{8}\,\frac{1}{1+\frac{D^2}{4}}x$$

$$= \frac{1}{8}\left(1-\frac{D^2}{4}+\ldots\ldots\right)x = \frac{x}{8}$$

$$y_{p_2} = -\frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x.$$

Here we can not apply "xV" rule (case VII) because $\frac{1}{D^2+4}$ cos 2x is a case of failure.

$$\therefore \quad \frac{1}{D^2+4}\, x\, e^{i2x} = e^{i2x} \frac{1}{(D+2i)2+4}\, x$$

$$= e^{i2x} \frac{1}{D2+4iD}\, x = \frac{e^{i2x}}{4\,i\,D}\left(\frac{1}{1-\frac{Di}{4}}\right) x$$

$$= -\frac{e^{i2x}\, i}{4D}\left(1+\frac{iD}{4}\right) x = -\frac{e^{i2x}\, i}{4D}\left(x+\frac{i}{4}\right)$$

$$= -\frac{e^{i2x}\, i}{4}\left(\frac{x^2}{2}+\frac{ix}{4}\right)$$

$$= -\frac{1}{16}(\cos 2x + i \sin 2x)(-x + i\, 2x^2)$$

Taking real parts on both sides,

$$\frac{1}{D^2+4}\, x \cos 2x = -\frac{1}{16}(-x\cos 2x - 2x^2 \sin 2x)$$

$$\therefore \quad y_{p_2} = -\frac{1}{2}\left[\frac{1}{16}(x\cos 2x + 2x^2\sin 2x)\right]$$

$$y_p = \frac{x}{8} - \frac{1}{32}(x\cos 2x + 2x^2 \sin 2x)$$

$$\therefore \quad y = c_1\cos 2x + c_2 \sin 2x + \frac{x}{8} - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x)$$

EXERCISE 1.2

Solve the following Differential Equations :

(A) On General Method

1. $(D^2 + 5D + 6)\, y = e^{e^x}$.

Ans. $y = c_1 e^{-2x} + c_2 e^{-3x} + (e^{-2x} - 2e^{-3x})\, e^{e^x}$.

2. $\frac{d^2y}{dx^2} + a^2 y = \tan ax$

Ans. $y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log[\sec ax + \tan ax]$

3. $(D^2 - 3D + 2)\, y = \frac{1}{e^{e^{-x}}} + \cos\left(\frac{1}{e^x}\right)$ **(Dec. 2006, 2008, 2010)**

Ans. $y = c_1 e^{2x} + c_2 e^x + e^{2x}\left[e^{-e^{-x}} - \cos(e^{-x})\right]$

4. $(D^2 - 9D + 18)\, y = e^{e^{-3x}}$

Ans. $y = c_1 e^{6x} + c_2 e^{3x} + \frac{e^{6x}}{9} e^{e^{-3x}}$

5. $(D^2 - 2D - 3)\, y = 3e^{-3x}\sin(e^{-3x}) + \cos(e^{-3x})$ **(Nov. 2015)**

Ans. $y = c_1 e^{3x} + c_2 e^{-x} - \frac{e^{3x}}{3}\sin e^{-3x}$

(B) On Short Methods :

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$.

Ans. $y = c_1 e^{6x} + c_2 e^x - \frac{e^{2x}}{4}$

2. $\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2 + 3$

Ans. $y = c_1 e^{2x} + c_2 e^{-2x} - 1 - \frac{2}{3}e^x + \frac{xe^{2x}}{4}$

3. $(D^3 - 5D^2 + 8D - 4)\, y = e^{2x} + 2e^x + 3e^{-x} + 2$

Ans. $y = c_1 e^x + (c_2 + c_3 x)\, e^{2x} + \frac{e^{2x}x^2}{2} + 2xe^x - \frac{e^{-x}}{6} - \frac{1}{2}$

4. $(D^4 - 4D^3 + 6D^2 - 4D + 1)\, y = e^x + 2^x + \frac{1}{3}$. **(Dec. 2005, May 2007)**

Ans. $y = (c_1 x^3 + c_2 x^2 + c_3 x + c_4)\, e^x + \frac{x^4}{24} e^x + \frac{1}{(\log 2 - 1)^4} 2^x + \frac{1}{3}$

5. $\frac{d^2y}{dx^2} + 4y = \cos x \cdot \cos 2x \cdot \cos 3x$ **(May 2014)**

Ans. $y = A\cos 2x + B\sin 2x + \frac{1}{16} + \frac{x \sin 2x}{16} - \frac{1}{48}\cos 4x - \frac{1}{128}\cos 6x$

6. $(D^5 - D^4 + 2D^3 - 2D^2 + D - 1)\, y = \cos x$ **(May 2007, 2008)**

Ans. $y = c_1 e^x + (c_2 x + c_3)\cos x + (c_4 x + c_5)\sin x + \frac{1}{16}\,[(x^2 + 2x)\cos x - x^2 \sin x]$

7. $(D^4 - m^4)\, y = \sin mx$ **Ans.** $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3}\cos mx$

8. $(D^3 + D)\, y = \cos x$ **(Dec. 2008)** **Ans.** $c_1 + c_2 \cos x + c_3 \sin x - \frac{x\cos x}{2}$

9. $\operatorname{cosec} x\, \frac{d^4y}{dx^4} + y \operatorname{cosec} x = \sin 2x$

Ans. $y = e^{\frac{x}{\sqrt{2}}}\left[c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right] + e^{-\frac{x}{\sqrt{2}}}\left[c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right] + \frac{1}{2}\left(\frac{\cos x}{2} - \frac{\cos 3x}{82}\right)$

10. $\frac{d^2x}{dt^2} + 9x = 4\cos\left(\frac{\pi}{3} + t\right)$, given that $x = 0$ at $t = 0$ and $x = 2$ at $t = \frac{\pi}{6}$. **Ans.** $x = \frac{1}{4}\cos 3t + 2\sin 3t + \frac{1}{2}\cos\left(\frac{\pi}{3} + t\right)$

11. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \sin^2 t$ **Ans.** $y = e^{-t}[A\cos 2t + b\sin 2t] + \frac{1}{10} - \frac{1}{34}\,[4\sin 2t + \cos 2t]$

12. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin 2x - 2\cos 2x$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. **Ans.** $y = e^{-x}\sin x - \frac{1}{2}\sin 2x$

13. $\frac{d^2y}{dx^2} + n^2 y = h \sin px$, where h, p and n are constants satisfying the condition $y = 0$, $\frac{dy}{dx} = b$ for $x = 0$.

Ans. $y = a\cos nx + \left[\frac{b}{n} - \frac{ph}{n\,(n^2 - p^2)}\right]\sin nx + \frac{h\sin px}{(n^2 - p^2)}$

14. $(D^3 + 1)\, y = \cos(2x - 1) - \cos^2 \frac{x}{2}$

Ans. $y = c_1 e^{-x} + e^{x/2}\left[c^2 \cos \frac{\sqrt{3}}{2} x + c^3 \sin \frac{\sqrt{3}}{2} x\right] + \frac{1}{65}\,[\cos(2x - 1) - 8\sin(2x - 1)] - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$

15. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$. **Ans.** $y = e^x(A\cos x + B\sin x) + 2\sin x + \cos x$

16. $(D^4 + 10D^2 + 9)\, y = 96\sin 2x\cos x$. Given that at $x = 0$, $y = 0$, $y' = -2$, $y'' = -8$, $y''' = -18$.

Ans. $y = \cos 3x - \cos x + x(\cos 3x - 3\cos x)$

17. $(D^4 + 6D^2 + 8)\, y = \sin^2 x \cos 2x$

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos\sqrt{2}\, x + c_4 \sin\sqrt{2}\, x - \frac{x\sin 2x}{16} - \frac{1}{32} - \frac{\cos 4x}{672}$

18. $(D^3 + 3D)\, y = \cosh 2x \sinh 3x$. **Ans.** $y = c_1 + \left(c_2 \cos\sqrt{3}\, x + c_3 \sin\sqrt{3}\, x\right) + \frac{\cosh 5x}{280} + \frac{\cosh x}{8}$

19. $(D^3 - 25D)\, y = \cosh 2x \sinh 3x$. **Ans.** $y = c_1 + c_2 e^{5x} + c_3 e^{-5x} + \frac{x}{100}\sinh 5x - \frac{1}{48}\cosh x$

20. $(D^4 - 1)\, y = \cosh x \sinh x$ **Ans.** $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{1}{30}\sinh 2x$

21. $(D^2 + 13D + 36)\, y = e^{-4x} + \sinh x$. **Ans.** $y = c_1 e^{-9x} + c_2 e^{-4x} + \frac{x}{5}e^{-4x} - \frac{1}{1200}(13\cosh x - 37\sinh x)$

22. $(D^3 + 1)\, y = \sin(2x + 3) + e^{-x} + 2^x$.

Ans. $y = c_1 e^{-x} + e^{(1/2)x}[c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x] + \frac{1}{65}[\sin(2x + 3) + 8\cos(2x + 3)] + \frac{x}{3}e^{-x} + \frac{2^x}{(\log 2)^3 + 1}$

23. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 50x$ with $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$ **Ans.** $y = 5x - 3 + e^{-3x}(3\cos x + 5\sin x)$

24. $(D^2 - 2D + 5)\, y = 25x^2$. **Ans.** $y = e^x\, [c_1 \cos 2x + c_2 \sin 2x] + 5x^2 + 4x - \frac{2}{5}$

25. $(D^4 + D^2 + 1)\, y = 53x^2 + 17$ **(Dec. 2008)**

Ans. $y = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right] + e^{x/2}\left[c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x\right] + 53x^2 - 89$

26. $(D^2 + 5D + 4)\, y = x^2 + 7x + 9$. **(May 2006)** **Ans.** $y = c_1 e^{-4x} + c_2 e^{-x} + \frac{1}{4}\left(x^2 + \frac{9x}{2} + \frac{23}{8}\right)$

27. $(D^4 + 6D^2 + 25)\, y = x^4 + x^2 + 1$.

Ans. $y = e^x\, [c_1 \cos 2x + c_2 \sin 2x] + e^{-x}\, [c_3 \cos 2x + c_4 \sin 2x] + \frac{1}{25}\left[x^4 - \frac{47}{25}x^2 + \frac{589}{625}\right]$

28. $(D^2 - D + 1)\, y = x^3 - 3x^2 + 1$ **Ans.** $y = e^{x/2}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right] + x^3 - 6x - 5$

29. $(D^3 - 3D^2 + 3D - 1)\, y = 2x^3 - 3x^2 + 1$. **Ans.** $y = (c_1 x^2 + c_2 x + c_3)\, e^x - (2x^3 + 15x^2 + 54x + 85)$

30. $(D^3 - 2D + 4)\, y = 3x^2 - 5x + 2$. **(Dec. 2012)** **Ans.** $c_1 e^{-2x} + e^x\, (c_2 \cos x + c_3 \sin x) + \frac{1}{4}(3x^2 - 2x + 1)$

31. $\frac{d^3y}{dx^3} + 8y = x^4 + 2x + 1$. **Ans.** $y = c_1 e^{-2x} + e^x\, \left[A \cos \sqrt{3}\, x + B \sin \sqrt{3}\, x\right] + \frac{1}{8}\, (x^4 - x + 1)$

32. $(D^2 - 3D + 2)\, y = x^2 + \sin x$. **Ans.** $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}(x^2 + 3x + \frac{7}{2} + \frac{1}{10}\sin x + \frac{3}{10}\cos x$

33. $(D^3 + 3D^2 - 4)\, y = 6e^{-2x} + 4x^2$. **Ans.** $y = c_1 e^x + (c_2 x + c_3)\, e^{-2x} - x^2 e^{-2x} - x^2 - \frac{3}{2}$

34. $(D^3 + 6D^2 + 12D + 8)\, y = e^{-2x} + x^2 + 3^x + \cos 2x$.

Ans. $y = (c_1 x + c_2 x + c_3)\, e^{-2x} + \frac{x^3}{6} e^{-2x} + \frac{1}{8}(x^2 - 3x + 3) + \frac{1}{(\log 3 + 2)^3} 3^x + \frac{1}{32}(\sin 2x - \cos 2x)$

35. $(D^2 - 4D + 4)\, y = 8\, (e^{2x} + \sin 2x + x^2)$. **(May 2006)** **Ans.** $y = (c_1 x + c_2)\, e^{2x} + 4x^2 e^{2x} + \cos 2x + 2\left(x^2 + 2x + \frac{3}{2}\right)$

36. $(D^5 - D)\, y = 12\, e^x + 8 \sin x - 2x$ **Ans.** $y = c_1 + c_2 e^{-x} + c_3 e^x + A \cos x + B \sin x + 3x\, e^x + 2x \sin x + x^2$

37. $(D^2 - 1)\, y = e^x + x^3$. **(May 2017)** **Ans.** $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x - x^3 - 6x$

38. $(D^2 - 4D + 4)\, y = e^{2x} + x^3 + \cos 2x$ **(Dec. 2010)** **Ans.** $y = (c_1 + c_2 x)\, e^{2x} + \frac{1}{2}\, x^2 e^{2x} - \frac{1}{8}\, \sin 2x + \frac{1}{8}\, [2x^3 + 6x^2 + 9x + 6]$

39. $(D^5 - D)\, y = 12e^x + 85mx + 2^x$ **(May 2018)**

Ans. $y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + 3x\, e^x - 35\, m \frac{x^2}{2} + \frac{2^x}{(\log 2)^5 - \log 2}$

40. $(D^2 - 4)\, y = e^{3x} x^2$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{3x}}{125}\, (125x^2 - 60x + 62)$

41. $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}\, (1 + x^2)$ **Ans.** $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{e^{2x}}{12}\, \left[\frac{169}{72} + x^2 + \frac{5x}{6}\right]$

42. $(D^3 - 3D^2 + 3D - 1)\, y = \sqrt{x}\, e^x$. **Ans.** $y = (c_1 x^2 + c_2 x + c_3)\, e^x + \frac{8e^x x^{7/2}}{105}$

43. $(D^2 - 4D + 4)\, y = e^{2x} \sin 3x$ **Ans.** $y = (c_1 + c_2 x)\, e^{2x} - \frac{1}{9}\, e^{2x} \sin 3x$

44. $(D^3 - D^2 + 3D + 5)\, y = e^x \cos 3x$ **Ans.** $y = c_1 e^{-x} + e^x\, (c_2 \cos 2x + c_3 \sin 2x) - \frac{e^x}{65}\, (3 \sin 3x + 2 \cos 3x)$

45. $(D^2 + 2D + 1)\, y = \dfrac{e^{-x}}{x + 2}$ **(Dec. 2007)** **Ans.** $y = (c_1 + c_2 x)\, e^{-x} - e^{-x}[x \log (x + 2) + 2 \log (x + 2) - x]$

46. $(D^2 + 6D + 9)\, y = \dfrac{1}{x^3}\, e^{-3x}$ **(Dec. 2014, 2017, May 2018)** **Ans.** $y = (c_1 x + c_2)\, e^{-3x} + \dfrac{e^{-3x}}{2x}$

47. $(D^4 - 3D^3 - 2D^2 + 4D + 4)\, y = x^2 e^x.$ **Ans.** $y = (c_1x + c_2)\, e^{-x} + (c_3x + c_4)\, e^{2x} + \dfrac{e^x}{4}(x^2 + 2x + \dfrac{7}{2})$

48. $(D^3 - 3D - 2)\, y = 540\, x^3 e^{-x}.$ **Ans.** $y = (c_1x + c_2)\, e^{-x} + c_3\, e^{2x} - 180e^{-x}\left(\dfrac{x^5}{20} + \dfrac{x^4}{12} + \dfrac{x^3}{9} + \dfrac{x^2}{9}\right)$

49. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = e^{-x}\sec^3 x$ **Ans.** $y = e^{-x}\left[c_1 \cos x + c_2 \sin x + \dfrac{\sin x}{2}\tan x\right]$

50. $(D^2 + 2D + 1)\; y = e^{-x} \log x.$ **(May 2015)** **Ans.** $y = (c_1x + c_2)\, e^{-x} + \dfrac{e^{-x}x^2}{4}\;(2 \log x - 3x^2)$

51. $(D^4 + D^2 + 1)\, y = e^{-x/2} \cos\left(\dfrac{\sqrt{3}}{2}x\right)$

Ans. $y = e^{x/2}\left[c_1 \cos\dfrac{\sqrt{3}}{2}x + c_2 \sin\dfrac{\sqrt{3}}{2}x\right] + e^{-x/2}\left[c_3 \cos\dfrac{\sqrt{3}}{2}x + c_4 \sin\dfrac{\sqrt{3}}{2}x\right] + \dfrac{1}{4\sqrt{3}}\; x\, e^{-x/2}\left[\sin x\dfrac{\sqrt{3}}{2} + \sqrt{3}\cos x\dfrac{\sqrt{3}}{2}\right]$

52. $(D^3 - D^2 - D + 1)\, y = \cosh x \sin x.$ **Ans.** $y = (c_1x + c_2)\, e^x + c_3\, e^{-x} + \dfrac{e^x}{10}(\cos x - 2 \sin x) - \dfrac{e^{-x}}{50}(3 \cos x - 4 \sin x)$

53. $\dfrac{d^2y}{dx^2} - y = \cosh x \cos x$ **(Dec. 2016)** **Ans.** $y = c_1 e^x + c_2 e^{-x} + \dfrac{1}{5}\;(2 \sinh x \sin x - \cosh x \cos x)$

54. $(D^2 + 40D + 8)\, y = 12e^{-2x} \sin x \sin 3x.$ **(Dec. 2004)** **Ans.** $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \dfrac{3}{2}x\, e^{-2x} \sin 2x + \dfrac{1}{2}e^{-2x}\cos 4x$

55. $(D^3 - 6D^2 + 11D - 6)\, y = e^x x + \sin x + \cos x.$ **Ans.** $y = c_1e^x + c_2e^{2x} + c_3e^{3x} + \dfrac{e^x}{2}\left(\dfrac{x^2}{3} + \dfrac{3}{2}x\right) - \dfrac{1}{10}\cos x + \dfrac{1}{10}\sin x$

56. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-2x} \sin 2x + 4x^2 e^x x$ **(May 2011)**

Ans. $y = c_1 e^{-2x} + c_2 e^{-3x} - \dfrac{e^{-2x}}{10}\;(\cos 2x + 2 \sin 2x) + \dfrac{e^x}{3}\left(x^2 - \dfrac{7}{6}x + \dfrac{37}{72}\right)$

57. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} = 3x + x\, e^x.$ **Ans.** $y = c_1 + c_2 x + c_3 e^x - 2x\, e^x + \dfrac{x^2 e^x}{2} - \dfrac{x^3}{2} - \dfrac{3x^2}{2}$

58. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = x\, e^{3x} + \sin 2x.$ **Ans.** $y = c_2 e^x + c_1 e^{2x} + e^{3x}\left(\dfrac{x}{2} - \dfrac{3}{4}\right) + \dfrac{1}{20}\;(3 \cos 2x - \sin 2x)$

59. $(D^2 - 6D + 13)\, y = 8\, e^{3x} \sin 4x + 2^x$ **Ans.** $y = e^{3x}(A \cos 2x + B \sin 2x) - \dfrac{2e^{3x}\sin 4x}{3} + \dfrac{2^x}{(\log 2)^2 - 6 \log 2 + 13}$

60. $(D^4 + D^2 + 1)\, y = ax^2 + be^{-x} \sin 2x.$ **Ans.** $y = e^{(-1/2)\,x}[c_1 \cos (\sqrt{3}/2)\, x + c_2 \sin (\sqrt{3}/2)\, x] + e^{(1/2)\,x}[c_3 \cos (\sqrt{3}/2)\, x$

$+ c_4 \sin (\sqrt{3/2}\, x] + a\,(x^2 - 2) - \dfrac{b}{481}e^{-x}(20 \cos 2x + 9 \sin 2x)$

61. $(D^2 - 4)\, y = x \sinh x$ **(May 2006)** **Ans.** $y = c_1e^{2x} + c_2\, e^{-2x} - \dfrac{1}{3}[x \sinh x + \dfrac{2}{3}\cosh x]$

62. $(D^2 - 2D + 1)\, y = x^2 e^x \sin x.$ **Ans.** $y = (c_1x + c_2)\, e^x - e^x\,[4x \cos x + (x^2 - 6) \sin x]$

63. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8x^2 \cdot e^{2x} \sin 2x$ **(Dec. 2004)** **Ans.** $y = e^{2x}[c_1 + c_2 x + 3 \sin 2x - 2x^2 \sin 2x - 4x \cos 2x]$

64. $(D^2 + 2D + 1)\, y = x \cos x$ **Ans.** $y = (c_1 x + c_2)\, e^{-x} + \dfrac{1}{2}\;(x \sin x + \cos x - \sin x)$

65. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x \sin 2x$ **Ans.** $y = c_1 e^{-2x} + c_2 e^{-x} + \left(\frac{7-30x}{200}\right) \cos 2x + \left(\frac{12-5x}{100}\right) \sin 2x$

66. $(D^4 + 2D^2 + 1)\, y = x \cos x$. **(May 2012, Dec. 2012)** **Ans.** $y = (c_1x + c_2) \cos x + (c_3x + c_4) \sin x - \frac{x^3}{24} \cos x + \frac{x^2}{2} \sin x$

67. $(D^2 + 1)^2\, y = 24x \cos x$. **Ans.** $y = (c_1x + c_2) \cos x + (c_3x + c_4) \sin x - x^3 \cos x + 3x^2 \sin x$

68. $(D^2 + 2D + 5)^2\, y = x\, e^{-x} \cos 2x$. **(Dec. 2018)**

Ans. $y = e^{-x} [(c_1x + c_2) \cos 2x + (c_3x + c_4) \sin 2x] - \frac{e^{-x}}{32}\left[(x^3 - x^2) \cos 2x - \frac{2}{3} x^3 \sin 2x\right]$

69. $(D^2 - 2D + 4)^2 y = xe^x \cos\left[\sqrt{3}\, x + \alpha\right]$ **(Dec. 2006; May 2007, 2008)**

Ans. $y = e^x \left[(c_1 + c_2 x) \cos \sqrt{3}\, x + (c_3 + c_4 x) \sin \sqrt{3}\, x\right] - \frac{e^x}{12}\left[\frac{x^3}{6} \cos\left(\sqrt{3}\, x + \alpha\right) + \frac{x^2}{2\sqrt{3}} \sin\left(\sqrt{3}\, x + \alpha\right)\right]$

70. $(D^2 - 4D + 4)\, y = x\, e^{2x} \sin 2x$. **(Dec. 2005, 2008)** **Ans.** $y = (c_1x + c_2)\, e^{2x} - \frac{e^{2x}}{4} [x \sin 2x + \cos 2x]$

(C) Method of Variation of Parameters

When the short-cut methods (Art. 1.13) fail to determine the particular integral then one has to make use of general method. But this method involves laborious integration and in such cases other methods are available. One such method is the method of variation of parameters. This method is due to a great Mathematician named Lagrange. To explain the rigours of this method, let us start with a simple differential equation

$$\frac{d^2y}{dx^2} + y = \tan x \qquad \text{... (1)}$$

Here C.F. is very simple but the P.I. will be difficult to obtain even by general method because it is not of those special cases discussed before.

The complementary function is;

$$A \cos x + B \sin x \qquad \text{... (2)}$$

where, A and B are Arbitrary constants. Here Lagrange has shown his ingenuity by evolving the Particular Integral from this C.F. only by assuming that (temporarily) the constants A and B are some functions of x say A (x) and B (x) (of course it looks ridiculous).

Since the method assumes that the quantities A and B vary, this method is called *The Method of Variation of Parameters or Variation of Constants.*

Since two functions A (x) and B (x) are to be determined, they must satisfy two conditions. First is that the assumed solution (P.I.)

$$y = A(x) \cos x + B(x) \sin x \qquad \text{... (3)}$$

must satisfy the differential equation. When determined, (3) actually will deliver to us the Particular Integral. The second condition is at our disposal and we shall choose it at proper time so as to evaluate A (x) and B (x) and thereby solving the equation.

If we differentiate equation (3), we get

$$y' = -A(x) \sin x + B(x) \cos x + A'(x) \cos x + B'(x) \sin x \qquad \text{... (4)}$$

Since further differentiation will involve higher differentials of unknown functions A(x) and B(x), we apply our choice of second condition here only and that is what we assume

$$A'(x) \cos x + B'(x) \sin x = 0 \qquad \text{... (5)}$$

and then (4) becomes simpler as

$$y' = -A(x) \sin x + B(x) \cos x \qquad \text{... (6)}$$

One further differentiation will give

$$y'' = -A(x) \cos x - B(x) \sin x - A'(x) \sin x + B'(x) \cos x \qquad \text{... (7)}$$

Substituting from (3) and (7) in the given differential equation, we find that

$$-A'(x) \sin x + B'(x) \cos x = \tan x \qquad \text{... (8)}$$

Now, if we solve equations (5) and (8) simultaneously, we get

$$A'(x) = -\frac{\sin^2 x}{\cos x} \quad \text{and} \quad B'(x) = \sin x$$

and hence by integration, we get

$$A(x) = \int \frac{\cos^2 x - 1}{\cos x}\, dx = \int (\cos x - \sec x)\, dx$$

$$= \sin x - \log(\sec x + \tan x)$$

and $$B(x) = -\cos x$$

We are not using here constants of integration because it is P.I. part.

Now we frame our P.I. as follows :

$$y = A(x)\cos x + B(x)\sin x$$

$$\text{P.I.} = \cos x\,[\sin x - \log(\sec x + \tan x)] - \sin x \cos x = -[\log(\sec x + \tan x)]\cos x$$

Hence the complete solution is

$$y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$$

Note : Lagrange's method may be extended to higher order linear differential equations too, as may be seen by further exercises.

Second Method of Variation of Parameters

When we have to solve equation of the type $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X$

where, a, b, c are constants and X, any function of x, we also have an alternative method of variation of parameters.

Let the complementary function = $Ay_1 + By_2$ then the particular integral = $uy_1 + vy_2$

where $$u = \int \frac{-y_2 X}{W}\, dx, \quad v = \int \frac{y_1 X}{W}\, dx$$

where $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \text{called "WRONSKIAN"} = (y_1 y_2' - y_1' y_2)$$

ILLUSTRATIONS ON METHOD OF VARIATION OF PARAMETERS

Ex. 1 : *Solve the equation $(D^2 + 4)\, y = \sec 2x$ by the method of variation of parameters.* ***(Dec. 2012)***

Sol. : $$\text{C.F.} = A\cos 2x + B\sin 2x \qquad \dots (1)$$

Let $$\text{P.I.} = y = A(x)\cos 2x + B(x)\sin 2x \qquad \dots (2)$$

Differentiating (2), we have

$$y' = -2A(x)\sin 2x + 2B(x)\cos 2x + A'(x)\cos 2x + B'(x)\sin 2x \qquad \dots (3)$$

Assume here that

$$A'(x)\cos 2x + B'(x)\sin 2x = 0 \qquad \dots (4)$$

Then equation (3) will become

$$y' = -2A(x)\sin 2x + 2B(x)\cos 2x \qquad \dots (5)$$

If we differentiate (5) again, we get

$$y'' = -4A\cos 2x - 4B\sin 2x - 2A'\sin 2x + 2B'\cos 2x \qquad \dots (6)$$

[Briefly A (x) = A, A' = A' (x), B (x) = B, B' = B' (x)]

Putting values of y, y' and y" in the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sec 2x,$$

We have

$$(-4A\cos 2x - 4B\sin 2x - 2A'\sin 2x + 2B'\cos 2x) + (4A\cos 2x + 4B\sin 2x) = \sec 2x$$

$$\Rightarrow -2A'\sin 2x + 2B'\cos 2x = \sec 2x \qquad \dots (7)$$

Solving (4) and (7) simultaneously, we have

$$A' \cos 2x + B' \sin 2x = 0$$

$$-A' \sin 2x + B' \cos 2x = \frac{1}{2} \sec 2x$$

$$B' = \frac{1}{2} \Rightarrow B = \frac{1}{2} x \text{ and}$$

$$A' = \frac{-1}{2} \tan 2x \Rightarrow A = \frac{1}{4} \log (\cos 2x)$$

Hence

$$\text{P.I.} = A \cos 2x + B \sin 2x$$

$$= \frac{1}{4} \cos 2x \log (\cos 2x) + \frac{x}{2} \sin 2x$$

Hence the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \log \cos 2x$$

Alternative Method :

$$(D^2 + 4)\, y = \sec 2x$$

$$\text{C.F.} = A \cos 2x + B \sin 2x = Ay_1 + By_2$$

Here $y_1 = \cos 2x$ and $y_2 = \sin 2x$

Let

$$\text{P. I.} = u(x)\, y_1 + v(x)\, y_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2 (\cos^2 2x + \sin^2 2x) = 2$$

$$u = \int \frac{-y_2 X}{W} dx = \int \frac{-\sin 2x \sec 2x}{2} = -\frac{1}{2} \int \tan 2x \, dx$$

$$= \frac{1}{4} \log (\cos 2x)$$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{\cos 2x \sec 2x}{2} dx = \frac{1}{2} \int dx = \frac{1}{2} x$$

$\therefore$

$$\text{P.I.} = \left\{\frac{1}{4} \log (\cos 2x)\right\} \cos 2x + \left\{\frac{1}{2} x\right\} \sin 2x$$

Hence the general solution is

$$y = A \cos 2x + B \sin 2x + \frac{1}{4} \cos 2x \log (\cos 2x) + \frac{1}{2} x \sin 2x$$

Ex. 2 : *Solve by method of variation of parameters* $\frac{d^2y}{dx^2} + y = \text{cosec } x$. ***(Dec. 2004)***

Sol. : A.E. is $D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$$\text{C.F.} = A \cos x + B \sin x$$

$$= Ay_1 + By_2$$

Here $y_1 = \cos x$ and $y_2 = \sin x$

Let

$$\text{P.I.} = uy_1 + vy_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u = \int \frac{-y_2 X}{W} dx = \int \frac{-\sin x \,\text{cosec}\, x}{1} dx = \int -dx$$

$$= -x$$

and $$v = \int \frac{y_1 X}{D}\, dx = \int \frac{\cos x \operatorname{cosec} x}{1}\, dx = \int \cot x\, dx$$

$$= \log(\sin x)$$

$$\therefore \quad \text{P. I.} = (-x)\cos x + \{\log(\sin x)\}\sin x$$

Hence the general solution is

$$y = A\cos x + B\sin x - x\cos x + \sin x \log(\sin x)$$

Ex. 3 : *Solve by method of variation of parameters* $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$. ***(Dec. 2005, 2006, May 2017)***

Sol. : A.E. is $D^2 - 1 = 0 \therefore D = \pm 1$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$= c_1 y_1 + c_2 y_2$$

Here $y_1 = e^x$ and $y_2 = e^{-x}$, then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$u = \int \frac{-y_2 X}{W}\, dx = -\int \frac{y_2 X}{-2} = -\int \frac{e^{-x}}{-2}\left(\frac{2}{1+e^x}\right) dx$$

$$= \int \frac{e^{-x}}{1+e^x}\, dx = \int \frac{dx}{e^x(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$

$$u = \int e^{-x}\, dx - \int \frac{e^{-x}\, dx}{e^{-x}+1} = -e^{-x} + \log(1+e^{-x})$$

$$v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^x}{-2}\left(\frac{2}{1+e^x}\right) dx$$

$$= -\int \frac{e^x\, dx}{1+e^x} = -\log(1+e^x)$$

$$\therefore \quad \text{P.I.} = u\, y_1 + v\, y_2 = [-e^{-x} + \log(1+e^{-x})]\, e^x - \{\log(1+e^x)\}\, e^{-x}$$

$$= -1 + e^x \log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Hence the general solution is

$$y = c_1 e^x + c_2 e^{-x} - 1 + e^x \log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Ex. 4 : *Solve by method of variation of parameters* $(D^2 - 6D + 9)\, y = \frac{e^{3x}}{x^2}$. ***(May 2009, 2014, Dec. 2010, Nov. 2019)***

Sol. : A.E. is $D^2 - 6D + 9 = 0 \quad \therefore \quad D = 3, 3$

$$\text{C.F.} = (c_1 x + c_2)\, e^{3x}$$

$$= c_1 y_1 + c_2 y_2$$

Here $y_1 = x\, e^{3x}$ and $y_2 = e^{3x}$

Let $\text{P.I.} = uy_1 + vy_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x\, e^{3x} & e^{3x} \\ (3x+1)\, e^{3x} & 3e^{3x} \end{vmatrix} = -e^{6x}$$

$$u = \int \frac{-y_2 X}{W}\, dx = \int \frac{-e^{3x}(e^{3x}/x^2)}{-e^{6x}}\, dx = \int \frac{1}{x^2}\, dx$$

$$= -\frac{1}{x}$$

and
$$v = \int \frac{y_1 X}{W}\, dx = \int \frac{xe^{3x}\,(e^{3x}/x^2)}{-e^{6x}}\, dx = \int -\frac{1}{x}\, dx$$
$$= -\log x$$

$\therefore$
$$\text{P.I.} = -\frac{1}{x}(x\, e^{3x}) - \log x\,(e^{3x}) = -e^{3x}(1 + \log x)$$

Hence the general solution is
$$y = (c_1 x + c_2)\, e^{3x} - e^{3x}(1 + \log x)$$

Ex. 5 : *Use method of variation of parameters to solve* $(D^2 - 2D + 2)\, y = e^x \tan x$ ***(May 2007, 2008, Nov. 2014, 2015)***

Sol. : A.E. is $D^2 - 2D + 2 = 0 \quad \therefore \quad D = 1 \pm i$.
$$\text{C.F.} = e^x (c_1 \cos x + c_2 \sin x)$$
$$= c_1 y_1 + c_2 y_2$$

Here, $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$

Let $\text{P.I.} = u y_1 + v y_2$
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x (\cos x - \sin x) & e^x (\sin x + \cos x) \end{vmatrix} = e^{2x}$$
$$u = \int \frac{-y_2 X}{W}\, dx = \int \frac{-e^x \sin x\, e^x \tan x}{e^{2x}}\, dx$$
$$= \int \frac{-\sin^2 x}{\cos x}\, dx = -\int \frac{(1 - \cos^2 x)}{\cos x}\, dx$$
$$= -\log(\sec x + \tan x) + \sin x$$
$$v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^x \cos x\, e^x \tan x}{e^{2x}}\, dx$$
$$= \int \sin x\, dx = -\cos x$$

$\therefore$
$$\text{P.I.} = (-\log \sec x + \tan x + \sin x)\, e^x \cos x + (-\cos x)\, e^x \sin x$$

Hence the general solution is
$$y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$$

Ex. 6 : *Solve by method of variation of parameters* $(D^2 + 9)\, y = \dfrac{1}{1 + \sin 3x}$. ***(Dec. 2008)***

Sol. : A.E. is $D^2 + 9 = 0 \quad \therefore \quad D = \pm i3$.
$$\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x$$
$$= c_1 y_1 + c_2 y_2$$

Here, $y_1 = \cos 3x$ and $y_2 = \sin 3x$

Let $\text{P.I.} = u y_1 + v y_2$
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} = 3$$
$$u = \int \frac{-y_2 X}{W}\, dx = \int \frac{-\sin 3x\,(1/1 + \sin 3x)}{3}\, dx$$
$$= -\frac{1}{3} \int \frac{\sin 3x\,(1 - \sin 3x)}{(1 + \sin 3x)(1 - \sin 3x)}\, dx$$
$$= -\frac{1}{3} \int \frac{\sin 3x - \sin^2 3x}{\cos^2 3x}\, dx = -\frac{1}{3} \int (\sec 3x \tan 3x - \tan^2 3x)\, dx$$
$$= -\frac{1}{3} \int (\sec 3x \tan 3x - \sec^2 3x + 1)\, dx$$
$$= \frac{1}{3}\left(-\frac{1}{3} \sec 3x + \frac{1}{3} \tan 3x - x\right)$$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{\cos 3x\,(1/1 + \sin 3x)}{3} dx = \frac{1}{3}\int \frac{\cos 3x}{1 + \sin 3x} dx$$

$$= \frac{1}{9} \log (1 + \sin 3x)$$

$$\therefore \qquad \text{P.I.} = \left\{\frac{1}{9}(-\sec 3x + \tan 3x - 3x)\right\} \cos 3x + \left\{\frac{1}{9} \log (1 + \sin 3x)\right\} \sin 3x$$

Hence the general solution is

$$y = (c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{9}(-1 + \sin 3x - 3x \cos 3x) + \frac{1}{9} \sin 3x \log (1 + \sin 3x)$$

Ex. 7 : *Solve by method of variation of parameters* $\dfrac{d^2y}{dx^2} - y = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$ ... *(1)*

Sol. : C.F. $= A e^x + B e^{-x}$

Let P. I. $= y = A(x) e^x + B(x) e^{-x}$ ... (2)

A (x) and B (x) are functions to be determined.

Differentiating (2), we have

$$y' = A e^x - B e^{-x} + A' e^x + B' e^{-x} \qquad \text{... (3)}$$

Put $A' e^x + B' e^{-x} = 0$, ... (4)

then (3) will become $y' = A e^x - B e^{-x}$

Differentiating again

$$y'' = A e^x + B e^{-x} + A' e^x - B' e^{-x} \qquad \text{... (5)}$$

Putting values of y" and y in (1), we have

$A e^x + B e^{-x} + A' e^x - B' e^{-x} - Ae^x - Be^{-x} = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$

$$\therefore \qquad A' e^x - B' e^{-x} = e^{-x} \sin (e^{-x}) + \cos (e^{-x}) \qquad \text{... (6)}$$

Solving (4) and (6) simultaneously for A', B',

$$A' e^x + B' e^{-x} = 0$$

$$A' e^x - B' e^{-x} = e^{-x} \sin (e^{-x}) + \cos (e^{-x}) \qquad \text{... (7)}$$

Adding the equations in (7), we have

$$2 A' e^x = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$$

$$\therefore \qquad A' = \frac{1}{2} e^{-x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})] \qquad \text{... (8)}$$

and similarly, $B' = -\dfrac{1}{2} e^{x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]$ ... (9)

Integrating (8)

$$A = \frac{1}{2} \int e^{-x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx \qquad [\text{put } e^{-x} = t,\ -e^{-x} dx = dt]$$

$$A = -\frac{1}{2} \int [t \sin t + \cos t]\, dt = -\frac{1}{2} [- t \cos t + \sin t + \sin t]$$

$$= \frac{1}{2} t \cos t - \sin t$$

Hence $A(x) = \dfrac{1}{2} e^{-x} \cos (e^{-x}) - \sin (e^{-x})$ ... (10)

If we integrate (9),

$$B = -\frac{1}{2}\int e^x (e^{-x}\sin e^{-x} + \cos e^{-x})\,dx = -\frac{1}{2}\, e^x \cdot \cos e^{-x}$$

$$B(x) = -\frac{1}{2}\, e^x \cos e^{-x}, \text{ hence P.I. will be given by}$$

$$y = e^x\left[\frac{1}{2}\, e^{-x}\cos e^{-x} - \sin e^{-x}\right] - \frac{1}{2}\, e^x \cos(e^{-x})\cdot e^{-x}$$

$$= \frac{1}{2}\cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2}\cos e^{-x} = -e^x\sin(e^{-x})$$

Hence the complete solution is

$$y = A\, e^x + B\, e^{-x} - e^x \sin(e^{-x})$$

Ex. 8 : *By the method of variation of parameters, solve*

$$(D^3 + D)\,y = \operatorname{cosec} x \qquad \ldots (I)$$

Sol. : $\quad$ C.F. $= A + B\cos x + C\sin x$

Let the $\quad$ P.I. $= y_p = A(x) + B(x)\cos x + C(x)\sin x \qquad \ldots$ (II)

where A (x), B (x) and C (x) are the parameters to be determined. For brevity, take A (x) = A, B (x) = B, C (x) = C.

Hence $\quad$ P.I. $= y = A + B\cos x + C\sin x$

$$y' = A' + (B'\cos x - B\sin x) + (C'\sin x + C\cos x)$$

Put $\quad A' + B'\cos x + C'\sin x = 0 \qquad \ldots$ (III)

So that the new value of y' becomes

$$y' = -B\sin x + C\cos x$$

$$\therefore \quad y'' = -B'\sin x - B\cos x + C'\cos x - C\sin x$$

Choose B' and C' such that

$$-B'\sin x + C'\cos x = 0 \qquad \ldots (IV)$$

hence $\quad y'' = -B\cos x - C\sin x$

and $\quad y''' = -B'\cos x + B\sin x - C'\sin x - C\cos x$

Substituting in (I) values of y, y' and y''', we get

$$-B'\cos x - C'\sin x = \operatorname{cosec} x \qquad \ldots (V)$$

Solving simultaneously (III), (IV) and (V), we get

$$A' = \operatorname{cosec} x,\ B' = -\cot x \text{ and } C' = -1$$

and integration yields $\quad A = \log[\operatorname{cosec} x - \cot x]$

$$B = -\log\sin x$$

$$C = -x$$

$$\therefore \quad \text{P.I.} = \log(\operatorname{cosec} x - \cot x) - \cos x \log\sin x - x\sin x$$

Hence the complete solution is

$$y = A + B\cos x + C\sin x + \log[\operatorname{cosec} x - \cot x] - \cos x\, \log(\sin x) - x\sin x$$

EXERCISE 1.3

Solve the following differential equations by the method of variation of parameters.

1. $y'' + y = \sec x$ **(May 2007, 2019)** $\qquad$ **Ans.** $y = A\cos x + B\sin x + x\sin x + \cos x\log\cos x$
2. $\dfrac{d^2y}{dx^2} + y = x\sin x$. **(Dec, 2010)** $\qquad$ **Ans.** $y = A\cos x + B\sin x + \dfrac{x}{2}\sin x - \dfrac{x^2}{4}\cos x$
3. $(D^2 + 3D + 2)\,y = \sin e^x$ $\qquad$ **Ans.** $y = c_1e^{-x} + c_2e^{-2x} - e^{-2x}\sin e^x$

4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$ **Ans.** $y = A + B e^{2x} - \frac{e^x}{2} \sin x$

5. $(D^2 + 4) y = 4 \sec^2 2x$. **(May 2005)** **Ans.** $y = A \cos 2x + B \sin 2x - 1 + \sin 2x \log (\sec 2x + \tan 2x)$

6. $(D^2 + D) y = (1 + e^x)^{-1}$ **(May 2006, 2008)** **Ans.** $y = c_1 + c_2 e^{-x} + x - \log (1 + e^x) - e^{-x} \log (1 + e^x)$

7. $(D^2 - 1) y = (1 + e^{-x})^{-2}$ **Ans.** $y = A e^x + B e^{-x} - 1 + e^{-x} \log (1 + e^x)$

8. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ **(May 2005, 2006, May 2018)** **Ans.** $y = Ae^{-x} + B e^{-2x} + e^{-2x} e^{e^x}$

9. $(D^2 - 4D + 4) y = e^{2x} \sec^2 x$ **(May 2010)** **Ans.** $y = [c_1 + c_2 x + \log (\sec x)] e^{2x}$

10. $\frac{d^2y}{dx^2} + y = \tan x$. **(Dec. 2004)** **Ans.** $y = A \cos x + B \sin x - \cos x \log (\sec x + \tan x)$

11. $\frac{d^2y}{dx^2} + 4y = \tan 2x$ **Ans.** $y = A \cos 2x + B \sin 2x - \frac{1}{4} \cos 2x \log (\sec 2x + \tan 2x)$

12. $\frac{d^2y}{dx^2} + y = \sec x \tan x$ **Ans.** $y = A \cos x + B \sin x + x \cos x - \sin x + \sin x \log (\sec x)$

13. $y'' + y = \sec x$ **(Dec. 2004)** **Ans.** $y = A \cos x + B \sin x + x \sin x + \cos x \log \cos x$

14. $(D^2 + 1) y = 3x - 8 \cot x$. **Ans.** $y = c_1 \cos x + c_2 \sin x + 3x - 8 \sin x \log (\operatorname{cosec} x - \cot x)$

15. $(D^2 + 4) y = \frac{1}{1 + \cos 2x}$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} (\cos 2x) \log (1 + \cos 2x) + \frac{1}{2}\left(x - \frac{1}{2} \tan x\right) \sin 2x$

1.12 EQUATIONS REDUCIBLE TO LINEAR WITH CONSTANT COEFFICIENTS

We shall now study two types of linear differential equations with *variable coefficients* which can be reduced to the case of linear differential equation with constant coefficients by suitable transformations of variables.

1.13 CAUCHY'S OR EULER'S HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

An equation of the type

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \ldots + a_{n-1} x D + a_n) y = F (x)$$

Where, a_0, a_1, a_2 …… a_n are constants is called Cauchy's Homogeneous Equation. It is sometimes attributed to Euler also. It may also be written as

$$a_0 x^n \frac{d^ny}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = F (x) \quad \ldots (1)$$

It can be reduced to linear differential equation with constant coefficients by putting

$$x = e^z \text{ or } z = \log x \quad \ldots (2)$$

Now $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

or $$x\frac{dy}{dx} = \frac{dy}{dz} = Dy, \text{ here we took } D \equiv \frac{d}{dz}$$

Also, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{d^2y}{dz^2}\right)\frac{1}{x} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Hence $$x^2 \frac{d^2y}{dx^2} = -Dy + D^2y = D(D-1)y$$

Similarly, we can show that

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y \text{ and so on.}$$

....

....

$$x^r \frac{d^ry}{dx^r} = D(D-1)(D-2) \,\ldots\ldots\ldots\, (D-r+1)y \qquad \ldots (3)$$

Making these substitutions in (1) it can be reduced to linear differential equation with constant coefficients. The following examples can clarify further.

ILLUSTRATIONS

Ex. 1 : *Solve* $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$ ***(Dec. 2004, May 2005)***

Sol. : Given equation is Cauchy's homogeneous linear differential equation. We use substitution $z = \log x$ or $x = e^z$ and let $D \equiv \frac{d}{dz}$.

Then we note from article (1.18),

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y, \quad x\frac{dy}{dx} = Dy, \text{ where } D \equiv \frac{d}{dz}$$

and equation is transformed into

$$D(D-1)y - Dy + 4y = \cos(z) + e^z \sin z$$

or $$(D^2 - D - D + 4)y = \cos(z) + e^z \sin z$$

or $$(D^2 - 2D + 4)y = \cos(z) + e^z \sin z$$

which is linear with constant coefficients in y and z. Now

A.E. is $D^2 - 2D + 4 = 0 \Rightarrow D = 1 \pm i\sqrt{3}$

Hence $$\text{C.F.} = e^z \left[A \cos\sqrt{3}\,z + B \sin\sqrt{3}\,z\right]$$

and

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 2D + 4}\cos z + \frac{1}{D^2 - 2D + 4} e^z \sin z \\
&= \frac{1}{-1 - 2D + 4}\cos z + e^z \frac{1}{(D+1)^2 - 2(D+1) + 4} \sin z \\
&= \frac{1}{3 - 2D}\cos z + e^z \frac{1}{D^2 + 3} \sin z \\
&= -\frac{2D + 3}{4D^2 - 9}\cos z + e^z \frac{1}{-1 + 3} (\sin z) \\
&= -\frac{(2D + 3)\cos z}{-4 - 9} + e^z \frac{1}{2} \sin z \\
&= \frac{1}{13}[-2\sin z + 3\cos z] + \frac{1}{2} e^z \sin z
\end{aligned}$$

Hence the general solution in terms of y and z is

$$y = e^z \left[A\cos\left(\sqrt{3}\,z\right) + B\sin\left(\sqrt{3}\,z\right)\right] + \frac{1}{13}[3\cos z - 2\sin z] + \frac{1}{2} e^z \sin z$$

Changing to y and x, we have

$$y = x\left[A\cos\sqrt{3}(\log x) + B\sin\sqrt{3}(\log x)\right] + \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{1}{2}x\sin(\log x)$$

Ex. 2 : *Find the equation of the curve, which satisfies the differential equation*

$$4x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + y = 0 \text{ and crosses the x-axis at an angle of } 60^\circ \text{ at } x = 1.$$

Sol. : Given equation is Cauchy's homogeneous linear differential equation. The solution will be the equation of the curve.

Put $x = e^z \Rightarrow z = \log x$, and $\frac{d}{dz} \equiv D$, then the given equation is transformed into

$$[4 D (D - 1) - 4 D + 1] \; y = 0$$

A.E. is $4D^2 - 8D + 1 = 0 \quad \therefore \; D = 1 \pm \frac{\sqrt{3}}{2}$

$$\text{C.F.} = c_1 e^{\left(1 + \frac{\sqrt{3}}{2}\right) z} + c_2 e^{\left(1 - \frac{\sqrt{3}}{2}\right) z} \text{ and solution is}$$

$$y = c_1 x^{\left(1 + \frac{\sqrt{3}}{2}\right)} + c_2 x^{\left(1 - \frac{\sqrt{3}}{2}\right)} \qquad \dots (1)$$

But initially when $x = 1$, $y = 0$ and $\frac{dy}{dx} = \sqrt{3}$

$\therefore \quad 0 = c_1 + c_2 \Rightarrow c_1 = - c_2 \qquad \dots (2)$

Differentiating (1) w.r.t. x

$$\frac{dy}{dx} = \left(1 + \frac{\sqrt{3}}{2}\right) c_1 x^{\frac{\sqrt{3}}{2}} + \left(1 - \frac{\sqrt{3}}{2}\right) c_2 x^{-\frac{\sqrt{3}}{2}}$$

Put $x = 1$ and $\frac{dy}{dx} = \sqrt{3}$ in this

$$\sqrt{3} = \left(1 + \frac{\sqrt{3}}{2}\right) c_1 + \left(1 - \frac{\sqrt{3}}{2}\right) c_2$$

Solving with (2), we get $c_1 = 1$, $c_2 = - 1$

$\therefore$ Solution or the equation of the curve will be

$$y = x^{\left(1 + \frac{\sqrt{3}}{2}\right)} - x^{\left(1 - \frac{\sqrt{3}}{2}\right)}$$

Ex. 3 : *Solve* $x^3 \cdot \frac{d^3y}{dx^3} + 2x^2 \cdot \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ ***(May 2012, 2014, Dec. 2012)***

Sol. : The given equation is Cauchy's homogeneous linear differential equation.

Put $x = e^z$, $\Rightarrow z = \log x$ and $\frac{d}{dz} \equiv D$ then equation is transformed into

$[D (D - 1) (D - 2) + 2 D (D - 1) + 2] \; y = 10 (e^z + e^{-z})$

A.E. is $D^3 - D^2 + 2 = 0 \quad \therefore \quad D = - 1, \; 1 \pm i$

$$\text{C.F.} = c_1 e^{-z} + e^z [c_2 \cos z + c_3 \sin z]$$

$$= \frac{c_1}{x} + x [c_2 \cos (\log x) + c_3 \sin (\log x)]$$

$$\text{P. I.} = 10 \frac{1}{D^3 - D^2 + 2} (e^z + e^{-z})$$

$$= 10 \left[\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} e^{-z}\right]$$

$$= 10 \left[\frac{1}{1 - 1 + 2} e^z + z \frac{1}{3D^2 - 2D} e^{-z}\right] = 10 \left[\frac{e^z}{2} + \frac{1}{5} z e^{-z}\right]$$

$$= 5 e^z + 2 z e^{-z} = 5x + \frac{2}{x} \log x$$

Hence the general solution will be

$$y = \frac{c_1}{x} + x [c_2 \cos (\log x) + c_3 \sin (\log x)] + 5x + \frac{2}{x} \log x$$

Ex. 4 : *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = x^2 \sin(\log x)$. ***(May 2009, 2011, Dec. 2014, 2017, Nov. 2019)***

Sol. : Given equation is Cauchy's homogeneous linear differential equation.

Put $z = \log x$ or $x = e^z$ and $\dfrac{d}{dz} \equiv D$, then equation is transformed into

$$[D(D-1) - 3D + 5]\ y = e^{2z} \sin z$$

$$(D^2 - 4D + 5)\ y = e^{2z} \sin z$$

A.E. is $D^2 - 4D + 5 = 0 \quad \therefore \quad D = 2 \pm i$.

$$\text{C.F.} = e^{2z}(c_1 \cos z + c_2 \sin z)$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 5}\ e^{2z} \sin z = e^{2z} \frac{1}{(D+2)^2 - 4(D+2) + 5} \sin z$$

$$= e^{2z} \frac{1}{D^2 + 1} \sin z = -e^{2z} \frac{z}{2} \cos z$$

$$= -\frac{1}{2} e^{2z} z \cos z$$

General solution in terms of y and z is

$$y = e^{2z}(c_1 \cos z + c_2 \sin z) - \frac{1}{2} e^{2z} z \cos z$$

General solution in terms of y and x is

$$y = x^2 [c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{1}{2} x^2 (\log x) \cos(\log x)$$

Ex. 5 : *Solve* $u = r \dfrac{d}{dr}\left(r \dfrac{du}{dr}\right) + r^3$.

Sol. : Given equation is $u = r \left\{r \dfrac{d^2u}{dr^2} + \dfrac{du}{dr}\right\} + r^3$ or $r^2 \dfrac{d^2u}{dr^2} + r \dfrac{du}{dr} - u = -r^3$

which is a homogeneous equation.

Put $z = \log r$ or $r = e^z$ and using D for $\dfrac{d}{dz}$, equation is transformed into

$$[D(D-1) + D - 1]\ u = -e^{3z} \quad \text{or} \quad (D^2 - 1)\ u = -e^{3z}.$$

A.E. is $D^2 - 1 = 0 \quad \therefore \quad D = \pm 1$

$$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

$$\text{P. I.} = \frac{1}{D^2 - 1}(-e^{3z}) = -\frac{1}{8} e^{3z}$$

$$\therefore \qquad u = c_1 e^z + c_2 e^{-z} - \frac{1}{8} e^{3z}$$

The general solution in u and r is

$$u = c_1 r + \frac{c_2}{r} - \frac{r^3}{8}$$

1.14 LEGENDRE'S LINEAR EQUATION

An equation of the type

$$a_0 (ax + b)^n \frac{d^n y}{dx^n} + a_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = F(x)$$

where, $a_0, a_1, a_2 \ldots \ldots a_n$ are constants is called *Legendre's Linear Equation*.

In case of such equations, we put $ax + b = e^z$ to reduce it to linear with constant coefficients.

If we put $\quad ax + b = e^z \Rightarrow z = \log (ax + b)$

then $\quad \dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \left(\dfrac{a}{ax + b}\right) \dfrac{dy}{dz}$

$$\Rightarrow \quad (ax + b) \frac{dy}{dx} = a \frac{dy}{dz} = a\, Dy \qquad \left[\because \frac{d}{dz} = D\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{a}{ax + b} \cdot \frac{dy}{dz}\right)$$

$$= \frac{-a^2}{(ax + b)^2} \frac{dy}{dz} + \frac{a}{ax + b} \frac{d}{dz} \left(\frac{dy}{dz}\right) \frac{dz}{dx}$$

$$= -\frac{a^2}{(ax + b)^2} \frac{dy}{dz} + \frac{a^2}{(ax + b)^2} \frac{d^2y}{dz^2}$$

$$= \frac{a^2}{(ax + b)^2} \left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$$\Rightarrow \quad (ax + b)^2 \frac{d^2y}{dx^2} = a^2 [D^2 - D]\, y = a^2 D (D - 1)\, y$$

Similarly, we shall get

$$(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D (D - 1) (D - 2)\, y \text{ and so on.}$$

If we make these substitutions in the differential equation (Legendre's), we shall see that it has been transformed into one with constant coefficients.

ILLUSTRATIONS

Ex. 6 : *Solve* $(2x + 1)^2 \dfrac{d^2y}{dx^2} - 2 (2x + 1) \dfrac{dy}{dx} - 12y = 6x$ ***(Dec. 2004, 2008; May 2011)***

Sol. : Put $\quad 2x + 1 = e^z \Rightarrow z = \log (2x + 1), \dfrac{dz}{dx} = \dfrac{2}{2x + 1}, \dfrac{d}{dz} \equiv D.$

Then we shall have

$$(2x + 1)^2 \frac{d^2y}{dx^2} = 4 \cdot D (D - 1)\, y, \quad (2x + 1) \frac{dy}{dx} = 2\, Dy$$

and the equation is transformed into

$$4D (D - 1)\, y - 4 (Dy) - 12y = 6 \left(\frac{e^z - 1}{2}\right)$$

$$\Rightarrow \quad [4 (D^2 - D) - 4D - 12]\, y = 3\, e^z - 3$$

$$\Rightarrow \quad (4D^2 - 8D - 12)\, y = 3\, e^z - 3$$

$$\Rightarrow \quad (D^2 - 2D - 3)\, y = \frac{3}{4} (e^z - 1)$$

which is now linear with constant coefficient in y, z.

$$\text{A.E.} \; : \; D^2 - 2D - 3 = 0 \Rightarrow D = 3, -1$$

$$\text{C.F.} = c_1 e^{3z} + c_2 e^{-z}$$

$$\text{P. I.} = \frac{1}{D^2 - 2D - 3}\,\frac{3}{4}\,(e^z - e^{oz})$$

$$\text{P. I.} = \frac{3}{4}\left[\frac{1}{D^2 - 2D - 3}\,e^z - \frac{1}{D^2 - 2D - 3}\,e^{oz}\right]$$

$$= \frac{3}{4}\left[\frac{e^z}{1 - 2 - 3} - \frac{e^{oz}}{0 - 0 - 3}\right] = \frac{3}{4}\left[\frac{e^z}{-4} + \frac{1}{3}\right]$$

$$= \frac{3e^z}{-16} + \frac{1}{4}$$

Hence the complete solution in terms of y and z is

$$y = c_1 e^{3z} + c_2 e^{-z} - 3\,\frac{e^z}{16} + \frac{1}{4}$$

Changing back to y and x, we have

$$y = c_1 (2x + 1)^3 + c_2 (2x + 1)^{-1} - \frac{3}{16}\,(2x + 1) + \frac{1}{4}$$

Ex. 7 : *Solve* $(1 + x)^2 \dfrac{d^2y}{dx^2} + (1 + x)\dfrac{dy}{dx} + y = 2 \sin [\log (1 + x)]$ ***(Dec. 2007)***

Sol. : Put $(1 + x) = e^z \Rightarrow z = \log (1 + x)$, $\dfrac{d}{dz} \equiv D$

Then the equation will become

$$D (D - 1)\, y + Dy + y = 2 \sin z$$

$$\Rightarrow \quad (D^2 + 1)\, y = 2 \sin z$$

Here A.E. : $D^2 + 1 = 0$, $D = \pm i$, hence

$$\text{C.F.} = A \cos z + B \sin z$$

$$\text{P.I.} = \frac{2 \sin z}{D^2 + 1} = \frac{2 \sin z}{-1 + 1} \quad \text{(case of failure)}$$

$$\therefore \quad \text{P.I.} = z\,\frac{1}{2D}\,2 \sin z = z \int \sin z\, dz = -z \cos z$$

General solution in terms of y and z is

$$y = A \cos z + B \sin z - z \cos z$$

$$\therefore \quad y = A \cos [\log (1 + x)] + B \sin [\log (1 + x)] - \log (1 + x) \cos [\log (1 + x)]$$

Ex. 8 : *Solve* $(3x + 2)^2 \dfrac{d^2y}{dx^2} + 3\,(3x + 2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$ ***(May 2007, 2011, 2012, Dec. 2012)***

Sol. : Given equation is Legendre's linear differential equation.

Put $z = \log (3x + 2)$ or $(3x + 2) = e^z$ and let $\dfrac{d}{dz} \equiv D$ then the equation is transformed into

$$[9D (D - 1) + 3.3D - 36]\, y = \frac{1}{3}(e^{2z} - 1) \text{ or } (D^2 - 4)\, y = \frac{1}{27}(e^{2z} - 1)$$

A.E. is $D^2 - 4 = 0 \quad \therefore \quad D = \pm 2.$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-2z}$$

$$\text{P.I.} = \frac{1}{27}\,\frac{1}{D^2 - 4}\,(e^{2z} - 1) = \frac{1}{27}\left[\frac{1}{D^2 - 4}\,e^{2z} - \frac{1}{D^2 - 4}\,e^{oz}\right]$$

$$= \frac{1}{27}\left[\frac{z\,e^{2z}}{4} + \frac{1}{4}\right] = \frac{1}{108}\,[ze^{2z} + 1]$$

The general solution in y and z is

$$y = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108}[ze^{2z} + 1]$$

The general solution in y and x is

$$y = c_1 (3x+2)^2 + c_2 (3x+2)^{-2} + \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

EXERCISE 1.4

Solve following Differential Equations with Variable Coefficients.

1. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$ **(Dec. 2008)** **Ans.** $y = c_1 x^2 + c_2 x^3 + \frac{x^5}{6}$

2. $x^2 \frac{d2y}{dx2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x$ **(Dec. 2007, Nov. 2015)** **Ans.** $y = c_1 x^4 + \frac{c_2}{x} - \frac{x^2}{6} - \frac{1}{2} \log x + \frac{3}{8}$

3. $x^2 \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = \log x$ **Ans.** $y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos(\sqrt{3}/2) \log x + c_3 \sin(\sqrt{3}/2) \log x\right] + \frac{x}{2}\left(\log x - \frac{3}{2}\right)$

4. $x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - 2y = x^2 + x^{-3}$. **(May 2008)** **Ans.** $y = c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) + \frac{x^2}{5} \log x - \frac{1}{50} x^{-3}$

5. $(x^3 D^3 + x^2 D^2 - 2)\, y = x + x^{-3}$ **Ans.** $y = c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{x}{2} - \frac{1}{50} x^{-3}$

6. $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = A + B \log x$ **Ans.** $y = (c_1 + c_2 \log x) + \frac{A}{4} x^2 + \frac{B}{4} x^2 (\log x - 1)$

7. $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = 0$ **Ans.** $y = c_1 x^4 + c_2 x^2 + c_3 x + \frac{c_4}{x}$

8. $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = x^2$ **Ans.** $y = c_3 x^2 + \frac{c_4}{x} + c_5 x^4 + c_6 x + \frac{x^6}{280}$

9. $(x^2 D^2 - xD + 1)\, y = x \log x$ **(Dec. 2010)** **Ans.** $y = x [A \log x + B] + \frac{x}{6} (\log x)^3$

10. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \log x$. **(Dec. 2005)** **Ans.** $y = x^2 [c_1 \cos(\log x) + c_2 \sin(\log x)] + x^2 \log x$

11. $x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy = \sin(\log x)$ **(May 2008, Dec. 2008, 2012)** **Ans.** $y = \frac{1}{x}\{c_1 + c_2 \log x - \sin(\log x)\}$

12. The radial displacement 'u' in a rotating disc at a distance 'r' from axis is given by $\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} + kr = 0$.

Find the displacement if u = 0 for r = 0, r = a **Ans.** $u = \frac{kr}{8} (a^2 - r^2)$

13. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = \frac{x^3}{1+x^2}$ **Ans.** $y = Ax + \frac{B}{x} + \frac{x}{4} \log(1+x^2) - \frac{x}{4} + \frac{1}{4x} \log(x^2+1)$

14. $u = r \frac{d}{dr}\left[r \frac{du}{dr}\right] + ar^3$ **Ans.** $u = Ar + \frac{B}{r} - \frac{a}{8} r^3$

15. $x \frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0$ [**Hint :** Multiply by x] **Ans.** $y = A + B \log x - \frac{x^2}{4}$

16. $(x^3 D^3 + 2x^2 D^2 + 3x D - 3)\, y = x^2 + x$ **Ans.** $y = c_1 x + c_2 \cos(\log x) + c_3 \sin(\log x) + \frac{x}{7} + \frac{x}{4} \log x$

17. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}$ **Ans.** $y = \frac{1}{x}\left[c_1 \log x + c_2 + \log\left(\frac{x}{x-1}\right)\right]$

18. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \sin(\log x^2)$ **(Dec. 2004)** **Ans.** $y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{3}\sin(\log x^2)$

19. $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$.

Ans. $y = c_1 x^{-2} + x\,(c_2 \cos \sqrt{(3)} \log x + c_3 \sin (\sqrt{3}) \log x) - \sin(\log x) + 8 \cos(\log x)$

20. $(x^2 D^2 + 5xD + 3)\, y = \left(1 + \frac{1}{x}\right)^2 \log x$

21. $(x^2 D^2 - 3xD + 1)\, y = \log x + \left[\frac{\sin(\log x) + 1}{x}\right]$

22. $\left(D^3 - \frac{4}{x} D^2 + \frac{5}{x^2} D - \frac{2}{x^3}\right) y = 1$ **Ans.** $y = c_1 x^2 + c_2\, x^{\left(\frac{5-\sqrt{21}}{2}\right)} + c_3\, x^{\left(\frac{5-\sqrt{21}}{2}\right)} - \frac{x^3}{5}$

23. $(x^2 D^2 - 4xD + 6)\, y = -x^4 \sin x$ **Ans.** $y = c_1 x^2 + c_2 x^3 + x^2 \sin x$

24. $(2x+3)^2 \frac{d2y}{dx2} - 2\,(2x+3) \frac{dy}{dx} - 12y = 6x$ **(May 2009, Dec. 2016)** **Ans.** $y = c_1 (2x+3)^3 + c_2 (2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4}$

25. $(x+a)^2 \frac{d^2y}{dx^2} - 4\,(x+a) \frac{dy}{dx} + 6y = x$ **(Dec. 2008)** **Ans.** $y = A\,(x+a)^3 + B\,(x+a)^2 + \frac{3x+2a}{6}$

26. $7\,(2+x)^2 \frac{d2y}{dx2} + 8\,(2+x) \frac{dy}{dx} + y = 4 \cos[\log(2+x)]$

27. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos[\log(1+x)]$ **Ans.** $y = c_1 \cos[\log(x+1)] + c_2 \sin[\log(1+x)]$

28. $(x+2)^2 \frac{d^2y}{dx^2} - (x+2) \frac{dy}{dx} + y = 3x + 4$ **Ans.** $y = (x+2)\,[c_1 + c_2 \log(x+2)] + \frac{3}{2}(x+2)\,[\log(x+2)]^2 - 2$

29. $(x+2)^2 \frac{d^2y}{dx^2} + 3\,(x+2) \frac{dy}{dx} + y = 4 \sin[\log(x+2)]$ **(May 2010)**

Ans. $y = [\{c_1 + c_2 \log(x+2)\}\,(x+2)^{-1} - 2 \cos[\log(x+2)]$

30. $(2x+1)^2 \frac{d^2y}{dx^2} - 6\,(2x+1) \frac{dy}{dx} + 16y = 8\,(2x+1)^2$. **(May 2015)**

Ans. $y = [c_1 + c_2 \log(2x+1)]\,(2x+1)^2 + (2x+1)^2\,[\log(2x+1)]^2$

31. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} = (2x+3)(2x+4)$. **Ans.** $y = c_1 + c_2 \log(x+1) + (x+1)^2 + 6\,(x+1) + [\log(x+1)]^2$

32. $(4x+1)^2 \frac{d^2y}{dx^2} + 2\,(4x+1) \frac{dy}{dx} + y = 2x + 1$. **Ans.** $y = [c_1 + c_2 \log(4x+1)]\,(4x+1)^{1/4} + \frac{1}{18}(4x+1) + \frac{1}{2}$

33. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} - y = 2 \log(x+1) + x - 1$.

Ans. $y = c_1 (x+1) + c_2 (x+1)^{-1} - 2 \log(x+1) + \frac{1}{2}(x+1) \log(x+1) + 2$

34. $(x-1)^3 \frac{d^3y}{dx^3} + 2\,(x-1)^2 \frac{d^2y}{dx^2} - 4\,(x-1) \frac{dy}{dx} + 4y = 4 \log(x-1)$ **Ans.** $y = c_1 + c_2 (x-1)^2 + c_3 (x-1)^{-2} - \frac{4}{3}(x-1) \log(x-1)$

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CHAPTER-2

SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS, SYMMETRIC SIMULTANEOUS DIFFERENTIAL EQUATIONS

2.1 INTRODUCTION

Sometimes in applications we come across equations, containing one independent but two or more dependent variables. For example :

$$\frac{dx}{dt} + 3\frac{dy}{dt} + y = t$$

$$\frac{dy}{dt} - x - y = t^2$$

Here, t is single independent and x and y are the two dependent variables. Such equations are called *Simultaneous Linear Differential Equations*. The number of equations is the same as the number of dependent variables.

2.2 METHOD OF SOLUTION

Method of solution is analogous to that of solving two linear simultaneous equations in algebra; either by Elimination or by Substitution. The equations of the system are so combined as to get a simple equation containing only one of the dependent variables and its derivatives. Then by integration, a relation between this dependent and the independent variable is found. Then either in a similar way or by substitution, a relation between the second dependent variable and the independent variable can be easily obtained. Examples will explain more.

Illustrations on Simultaneous Linear Differential Equations

Ex. 1 :** Solve* $\frac{dx}{dt} + 2x - 3y = t$ ***(May 2012, Dec. 2012)

$$\frac{dy}{dt} - 3x + 2y = e^{2t}$$

Sol. : Writing in terms of operator $D = \frac{d}{dt}$, we have

$$Dx + 2x - 3y = t \quad \text{or} \quad (D + 2)\,x - 3y = t \qquad \ldots (1)$$

$$Dy - 3x + 2y = e^{2t} \quad \text{or} \quad (D + 2)\,y - 3x = e^{2t} \qquad \ldots (2)$$

Solving for x (i.e. eliminating y) :

Operating (1) by (D + 2), we have

$$(D + 2)^2 x - 3\,(D + 2)\,y = (D + 2)\,t$$

or

$$(D + 2)^2 x - 3\,(D + 2)\,y = 1 + 2t \qquad \ldots (3)$$

Multiplying (2) by 3, we have

$$3\,(D + 2)\,y - 9x = 3e^{2t} \qquad \ldots (4)$$

Adding (3) and (4), we have

$$(D^2 + 4D - 5)\,x = 1 + 2t + 3e^{2t} \qquad \ldots (5)$$

This is a linear differential equation with constant coefficients.

AE : $D^2 + 4D - 5 = 0$ gives $D = -5, 1$

$$\text{C.F.} = c_1\, e^{-5t} + c_2\, e^{t}$$

$$\text{P.I.} = \frac{1}{D^2 + 4D - 5}(1+2t) + \frac{1}{D^2+4D-5}3e^{2t}$$

$$= -\frac{1}{5}\left[1 - \frac{4D+D^2}{5}\right]^{-1}(1+2t) + 3\frac{1}{4+8-5}e^{2t}$$

$$= -\frac{1}{5}\left(1+\frac{4D}{5}\right)(1+2t) + \frac{3}{7}e^{2t}$$

$$= -\frac{1}{5}\left(\frac{13}{5}+2t\right) + \frac{3e^{2t}}{7}$$

Hence the general solution for x is

$$x = c_1 e^{-5t} + c_2 e^t - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7} \qquad \text{... (6)}$$

Next, the general solution for y :

Differentiating (6) with respect to t,

$$\frac{dx}{dt} = -5c_1 e^{-5t} + c_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t}$$

Putting values of x and $\frac{dx}{dt}$ in equation (1), we have

$$y = \frac{1}{3}\left[\frac{dx}{dt} + 2x - t\right]$$

$$= \frac{1}{3}\left[-5c_1 e^{-5t} + c_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t} + 2c_1 e^{-5t} + 2c_2 e^t - \frac{26}{25} - \frac{4t}{5} + \frac{6e^{2t}}{7} - t\right]$$

Simplifying, we get

$$y = -c_1 e^{-5t} + c_2 e^t - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7} \qquad \text{... (7)}$$

Hence (6) and (7) together constitute the general solution.

Ex. 2 : *Solve the simultaneous linear differential equations with given conditions.*

$$\frac{du}{dx} + v = \sin x$$

$$\frac{dv}{dx} + u = \cos x$$

Given that when $x = 0$, then $u = 1$ and $v = 0$. ***(Dec. 2006)***

Sol. : In terms of operator $D \equiv \frac{d}{dx}$, the equations become :

$$Du + v = \sin x \qquad \text{... (1)}$$

$$Dv + u = \cos x \qquad \text{... (2)}$$

On differentiating (1), we get

$$D^2 u + Dv = \cos x \qquad \text{... (3)}$$

Now subtracting (2) from (3), we get

$$D^2 u - u = 0 \Rightarrow (D^2 - 1)u = 0,$$

whose solution is $u = c_1 e^x + c_2 e^{-x}$ ... (4)

and $\frac{du}{dx} = c_1 e^x - c_2 e^{-x}$ ... (5)

Now if we put value of $\frac{du}{dx}$ from (5) in (1), we get v, as

$$v = \sin x - c_1 e^x + c_2 e^{-x} \qquad \text{... (6)}$$

Hence (4) and (6) together constitute the general solution.

To find c_1, c_2, we apply initial condition at x = 0, u = 1 and v = 0, hence

$$c_1 + c_2 = 1 \text{ and } c_2 - c_1 = 0$$

Solving for c_1 and c_2, we obtain $c_1 = c_2 = \frac{1}{2}$

Hence $\quad u = \frac{1}{2}(e^x + e^{-x})$ and $v = \sin x - \frac{1}{2}(e^x - e^{-x})$ … (7)

or $\quad u = \cosh x$ and $v = \sin x - \sinh x$ … (8)

Hence the solution of equations are given by equation (7) and (8).

Ex. 3 : *Solve simultaneously* ***(Dec. 2008)***

$$\frac{dx}{dt} - 3x - 6y = t^2 \quad \text{... (1)}$$

$$\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t \quad \text{... (2)}$$

Sol. : Using $D \equiv \frac{d}{dx}$, equations (1) and (2) can be written as

$$(D-3)\,x - 6y = t^2 \quad \text{... (3)}$$

$$Dx + (D-3)\,y = e^t \quad \text{... (4)}$$

To eliminate x from (3) and (4), operating (3) by D and (4) by (D – 3), we get

$$D(D-3)\,x - 6Dy = 2t \quad \text{... (5)}$$

$$D(D-3)\,x + (D-3)^2\,y = (D-3)\,e^t = e^t - 3e^t = -2e^t \quad \text{... (6)}$$

Subtracting (5) from (6), we have

$$(D^2 + 9)\,y = -2e^t - 2t$$

whose general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t - \frac{e^t}{5} - \frac{2t}{9} \quad \text{... (7)}$$

To eliminate y from (3) and (4), operate (3) by (D – 3) and multiply (4) by 6 and subtract, we obtain

$$(D^2 + 9)\,x = 6e^t - 3t^2 + 2t$$

whose general solution is

$$x = c_3 \cos 3t + c_4 \sin 3t + \frac{3e^t}{5} - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} \quad \text{... (8)}$$

We have too many constants. To deal with this problem (i.e. to obtain relation between c_1, c_2 and c_3, c_4), we put values of x and y in (3) to find

$$(\sin 3t)(-3c_3 - 3c_4 - 6c_2) + \cos 3t\,(3c_4 - 3c_3 - 6c_1) + t^2 = t^2$$

This must be identity, hence

$$-3c_3 - 3c_4 - 6c_2 = 0 \quad \text{and} \quad 3c_4 - 3c_3 - 6c_1 = 0$$

Solving and simplifying these, we get

$$c_1 = \frac{c_4}{2} - \frac{c_3}{2} \quad \text{and} \quad c_2 = -\frac{c_3}{2} - \frac{c_4}{2}$$

Hence on substituting c_1, c_2 in (8) the required solutions are

$$x = c_3 \cos 3t + c_4 \sin 3t + \frac{3}{5}e^t - \frac{1}{3}t^2 + \frac{2t}{9} + \frac{2}{27}$$

and

$$y = \left(\frac{c_4}{2} - \frac{c_3}{2}\right)\cos 3t + \left(-\frac{c_3}{2} - \frac{c_4}{2}\right)\sin 3t - \frac{e^t}{5} - \frac{2t}{9}$$

Remark : Alternatively, we note that in equation (1), coefficient of y is constant. Hence, we can solve the system for x first and then using this solution we can obtain y. Thus we can avoid obtaining a relation between constants c_1, c_2 and c_3, c_4 in the solutions for x and y.

Ex. 4 : *The currents x and y in the coupled circuits are given by* ***(Dec. 2005, Nov. 2014)***

$$L\frac{dx}{dt} + Rx + R(x-y) = E$$

$$L\frac{dy}{dt} + Ry - R(x-y) = 0$$

Find x and y in terms of t, given that x = y = 0 at t = 0.

Sol. : In terms of operator $D \equiv \frac{d}{dt}$, the equations are :

$$(LD + 2R)\ x - Ry = E \qquad \text{... (1)}$$

$$(LD + 2R)\ y - Rx = 0 \qquad \text{... (2)}$$

To eliminate y, operating (1) by (LD + 2R), we have

$$(LD + 2R)^2 x - R\,(LD + 2R)\,y = (LD + 2R)\,E = 0 + 2\,RE$$

or $\quad (LD + 2R)^2 x - R\,(Rx) = 2RE$ (since from (2), we have $(LD + 2R)\,y = Rx$)

or $\quad (LD + 2R)^2 x - R^2 x = 2RE$

i.e. $\quad L^2 \frac{d^2x}{dt^2} + 4RL\frac{dx}{dt} + 3R^2 x = 2RE$

A.E. is $\quad L^2 D^2 + 4RLD + 3R^2 = 0 \quad$ or $\quad (LD + 3R)(LD + R) = 0$

$$\therefore \quad D = -\frac{3R}{L}, -\frac{R}{L}$$

$$\text{C.F.} = c_1\, e^{-(Rt/L)} + c_2\, e^{-(3Rt/L)}$$

$$\text{P.I.} = 2RE\ \frac{1}{L^2 D^2 + 4RLD + 3R^2}\, e^{ot}$$

$$= 2RE\ \frac{1}{0 + 0 + 3R^2}\, e^{ot} = \frac{2E}{3R}$$

Hence the general solution for x is

$$x = c_1\, e^{-(Rt/L)} + c_2\, e^{-(3Rt/L)} + \frac{2E}{3R} \qquad \text{...(3)}$$

To find y, from (1), we have $y = \frac{1}{R}[(LD + 2R)\,x - E]$ ... (4)

Now, $\quad (LD + 2R)\,x = L\,.\frac{dx}{dt} + 2\,Rx$

$$= L\left[-\frac{c_1 R}{L}\, e^{-(Rt/L)} - \frac{3R}{L}\, c_2\, e^{-(3Rt/L)}\right] + 2R\left[c_1\, e^{-(Rt/L)} + c_2\, e^{-(3Rt/L)} + \frac{2E}{3R}\right]$$

$$= -\,c_1 R\, e^{-(Rt/L)} - 3R\, c_2\, e^{-(3Rt/L)} + 2R\, c_1\, e^{-(Rt/L)} + 2R\, c_2\, e^{-(3Rt/L)} + \frac{4E}{3}$$

$$\therefore \quad (LD + 2R)\,x = c_1 R e^{-(Rt/L)} - c_2\, R\, e^{-(3Rt/L)} + \frac{4E}{3} \qquad \text{... (5)}$$

Putting value of (LD + 2R) x from (5) in (4), we obtain

$$y = \frac{1}{R}\left[c_1 R e^{-(Rt/L)} - c_2 R e^{-(3Rt/L)} + \frac{4E}{3} - E\right]$$

$$\therefore \quad y = c_1 e^{-(Rt/L)} - c_2 e^{-(3Rt/L)} + \frac{E}{3R} \qquad \text{... (6)}$$

Initially, at t = 0, x = 0 and y = 0, hence

$$0 = c_1 + c_2 + \frac{2E}{3R} \qquad \text{and} \qquad 0 = c_1 - c_2 + \frac{E}{3R}$$

Solving for c_1 and c_2, we get;

$$c_1 = -\frac{E}{2R} \quad \text{and} \quad c_2 = -\frac{E}{6R}$$

Putting these values of c_1 and c_2 in (3) and (6) for x and y,

$$x = \frac{E}{R}\left[\frac{2}{3} - \frac{1}{2}\, e^{-(Rt/L)} - \frac{1}{6}\, e^{-(3Rt/L)}\right]$$

$$y = \frac{E}{R}\left[\frac{1}{3} - \frac{1}{2}\, e^{-(Rt/L)} + \frac{1}{6}\, e^{-(3Rt/L)}\right]$$

Ex. 5 : *The equations of motion of an electron under certain conditions are :*

$$m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE$$

$$m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0 \qquad \text{... (1)}$$

with condition $x = \frac{dx}{dt} = y = \frac{dy}{dt} = 0$ *when* $t = 0$, *find the path of the electron.*

Sol. : Multiply the second equation of (1) by an arbitrary constant k and add to the first of equation (1).

$$m\frac{d^2}{dt^2}(x + ky) + eH\frac{d}{dt}(y - kx) = eE$$

or
$$m\frac{d^2}{dt^2}(x + ky) - eHk\frac{d}{dt}\left(x - \frac{1}{k}y\right) = eE \qquad \text{... (2)}$$

Here in choose k such that :

$$x - \frac{1}{k}y = x + ky \Rightarrow k = -\frac{1}{k}$$

or
$$k^2 + 1 = 0, \text{ hence } k = \pm i \qquad \text{... (3)}$$

Now put $x + ky = u$, then from (2),

$$m\frac{d^2u}{dt^2} - eHk\frac{du}{dt} = eE$$

$$\Rightarrow \quad \frac{d^2u}{dt^2} - wk\frac{du}{dt} = \frac{eE}{m} \qquad \text{where,} \quad w = \frac{eH}{m} \qquad \text{... (4)}$$

If we solve equation (4) as linear with constant coefficients, we get

$$u = x + ky = A + Be^{wkt} - \frac{Et}{Hk} \qquad \text{... (5)}$$

Also,
$$\frac{du}{dt} = -\frac{E}{Hk}\,Bwk\,e^{wkt} \qquad \text{... (6)}$$

$$= Bwk\,e^{wkt}$$

But initially $x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$ at $t = 0$ and $x + ky = u$, we can easily get

$$u = \frac{du}{dt} = 0 \text{ at } t = 0 \text{ and from (5) and (6), we have at } t = 0$$

$$A + B = 0 \quad \text{and} \quad wkB = \frac{E}{Hk}$$

Solving these two, we get

$$A = -\frac{E}{Hwk^2} \quad \text{and } B = \frac{E}{Hwk^2}$$

Putting for A and B in (5),

$$u = x + ky = -\frac{E}{Hwk^2} + \frac{E}{Hwk^2}\cdot e^{wkt} - \frac{Et}{Hk} \qquad \text{... (7)}$$

But $k = i$ and $-i$.

when $(k = i)$,
$$x + iy = \frac{E}{Hw} - \frac{E}{Hw}e^{iwt} + \frac{iEt}{H} \qquad \text{... (8)}$$

and $(k = -i)$,
$$x - iy = \frac{E}{Hw} - \frac{E}{Hw}e^{-iwt} - \frac{iEt}{H} \qquad \text{... (9)}$$

If we add and subtract (7) and (9), we can easily get

$$x = \frac{E}{Hw}(1 - \cos wt) \qquad \left[w = \frac{eH}{m}\right]$$

$$y = \frac{E}{Hw}(wt - \sin wt)$$

Alternative Method :

System (I) can also be written as

$$\frac{d^2x}{dt^2} + a\frac{dy}{dt} = b \quad \ldots (1)$$

$$\frac{d^2y}{dt^2} - a\frac{dx}{dt} = 0 \quad \text{where, } a = \frac{eH}{m} \text{ and } b = \frac{eE}{m} \quad \ldots (2)$$

Integrating (2) with respect to t, we get

$$\frac{dy}{dt} - ax = c_1$$

Initially, $x = \frac{dy}{dt} = 0$ at $t = 0$ $\therefore$ $c_1 = 0$

$$\therefore \quad \frac{dy}{dt} - ax = 0 \quad \ldots (3)$$

Next, integrating (1) with respect to t, we get

$$\frac{dx}{dt} + ay = bt + c_2$$

Initially, $\frac{dx}{dt} = y = 0$ at $t = 0$ $\therefore$ $c_2 = 0$

$$\therefore \quad \frac{dx}{dt} + ay = bt \quad \ldots (4)$$

From (3), substituting $x = \frac{1}{a}\frac{dy}{dt}$ in (4), we get

$$\frac{1}{a}\frac{d^2y}{dt^2} + ay = bt$$

or

$$\frac{d^2y}{dt^2} + a^2y = abt \quad \ldots (5)$$

which is a linear differential equation.

A.E. is $D^2 + a^2 = 0$ $\therefore$ $D = \pm ia$

$$\text{C.F.} = c_1 \cos at + c_2 \sin at$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} abt = \frac{ab}{a^2}\left[1 + \frac{D^2}{a^2}\right]^{-1} t = \frac{b}{a}\left(1 - \frac{D^2}{a^2}\right) t = \frac{b}{a} t$$

$$\therefore \quad y = c_3 \cos at + c_4 \sin at + \frac{b}{a} t \quad \ldots (6)$$

Again initially, $y = 0$, $t = 0$ $\therefore$ $c_3 = 0$

$$\therefore \quad y = c_4 \sin at + \frac{b}{a} t$$

and

$$\frac{dy}{dt} = a\, c_4 \cos at + \frac{b}{a}$$

Also, given $\frac{dy}{dt} = 0$, $t = 0$ $\therefore$ $c_4 = -\frac{b}{a^2}$

$$\therefore \quad y = -\frac{b}{a^2} \sin at + \frac{b}{a} t = \frac{b}{a^2}(at - \sin at) \quad \ldots (7)$$

From (3),

$$x = \frac{1}{a}\frac{dy}{dt}$$

$$x = \frac{1}{a}\left[\frac{b}{a^2}(a - a\cos at)\right] = \frac{b}{a^2}(1 - \cos at) \quad \ldots (8)$$

where, $a = \frac{eH}{m}$, $b = \frac{eE}{m}$

Hence, (7) and (8) constitute the solution.

Ex. 6 : Solve
$$t\,dx = (t-2x)\,dt$$
$$t\,dy = (tx+ty+2x-t)\,dt \qquad \ldots (1)$$

Sol. : From first of equation (I), we have
$$\frac{dx}{dt}+\frac{2}{t}x = 1 \text{ which is linear in x.}$$
its solution is
$$x = \frac{t}{3}+\frac{c_1}{t^2} \qquad \ldots (2)$$
If we add the two equations in (I), we get,
$$t\,(dx+dy) = [t-2x+tx+ty+2x-t]\,dt = t\,(x+y)\,dt$$
or
$$\frac{dx+dy}{x+y} = dt$$
Integrating,
$$\int \frac{dx+dy}{x+y} = \int dt$$
we get,
$$\log(x+y) = t+c_2'$$
or
$$x+y = c_2\,e^t \qquad (c_2' = \log c_2)$$
$\therefore$
$$y = c_2\,e^t - x$$
or
$$y = c_2\,e^t - \frac{t}{3}-\frac{c_1}{t^2}$$
Hence the general solution is
$$x = \frac{t}{3}+\frac{c_1}{t^2}$$
$$y = c_2\,e^t - \frac{t}{3}-\frac{c_1}{t^2} \qquad \textbf{... Ans.}$$

EXERCISE 2.1

Solve the following Simultaneous Equations :

1. $\frac{dx}{dt}+y = e^t$

$\frac{dy}{dt} - x = e^{-t}$

Ans. $x = c_1\cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-t})$

$y = c_1 \sin t - c_2 \cos t + \frac{1}{2}(e^t - e^{-t})$

2. $(D+2)\,x + (D+1)\,y = t$

$5x + (D+3)\,y = t^2$

Ans. $x = \left(\frac{c_1-3c_2}{5}\right)\sin t - \left(\frac{3c_1+c_2}{5}\right)\cos t - t^2 + t + 3$

$y = c_1 \cos t + c_2 \sin t + 2t^2 - 3t - 4$

3. $\frac{dx}{dt}+5x - 2y = t$

$\frac{dy}{dt} + 2x + y = 0$

having been given that $x = y = 0$ at $t = 0$. **(Dec. 2010)**

Ans. $x = -\frac{1}{27}(1+6t)\,e^{-3t} + \frac{1}{27}(1+3t)$

$y = -\frac{2}{27}(2+3t)\,e^{-3t} + \frac{2}{27}(2-3t)$

4. If $\frac{dx}{dt} - wy = a\cos pt$ **(May 2007, 2010, Dec. 2012)**

and $\frac{dy}{dt} + wx = a \sin pt$

Show that
$$x = A\cos wt + B\sin wt + \frac{a\sin pt}{p+w}$$
$$y = B\cos wt - A\sin wt - \frac{a\cos pt}{p+w}$$

5. In a heat exchange, the temperatures u and v of two liquids, satisfy the equations
$$4\frac{du}{dx} = v-u = 2\frac{dv}{dx} \quad \textbf{(May 2009)}$$
Solve the equations for u and v, given that $u = 20$ and $v = 100$ when $x = 0$.

Ans. $u = -60 + 80\,e^{x/4}$

$v = -60 + 160\,e^{x/4}$

6. The equations of motion of a particle are given by

$$\frac{dx}{dt} + wy = 0, \quad \frac{dy}{dt} - wx = 0$$

Find the path of the particle.

Ans. $x = A\cos wt + B\sin wt$, $y = A\sin wt - B\cos wt$

7. Solve the simultaneous equations for r and θ.

$$\frac{dr}{dt} - 2r - \theta = 0$$

$$\frac{d\theta}{dt} + r - 4\theta = 0$$

given that $\theta(0) = 0$ and $r'(0) = 6$

Ans. $r = 3(e^{3t} - te^{3t})$, $\theta = -3t\,e^{3t}$

8. Solve the simultaneous equations

$$2\frac{dx}{dt} - x + 3y = \sin t,$$

$$2\frac{dy}{dt} + 3x - y = \cos t$$

and obtain x and y if $x = \frac{1}{4}$ and $y = -\frac{1}{20}$ at $t = 0$.

Ans. $x = \frac{1}{10}\left[e^{2t} + e^{-t}\right] + \frac{1}{20}[\cos t + 2\sin t]$

$y = -\frac{1}{10}e^{2t} + \frac{1}{10}e^{-t} + \frac{2}{5}\sin t - \frac{1}{20}\cos t$

9. Solve $(D + 5)x + (D + 7)y = 2$

$(2D + 1)x + (3D + 1)y = \sin t$

under conditions $x = y = 0$, when $t = 0$

Ans. $x = -\frac{4}{3}e^{t} + \frac{4}{3}e^{-2t} - 1 + \cos t + 2\sin t$

$y = -e^{t} + \frac{4}{5}e^{-2t} + 1 - \frac{4}{5}\cos t - \frac{7}{5}\sin t$

10. $(D - 2)x + (D - 1)y = e^t$

$(D + 3)x + y = 0$

Ans. $x = c_1\cos t + c_2\sin t - \frac{1}{2}e^t$

$y = (c_1 - 3c_2)\sin t - (3c_1 + c_2)\cos t + 2e^t$

11. $(D - 1)x + Dy = t$

$3x + (D + 4)y = t^2$ **(May 2011)**

Ans. $x = -2c_1 e^{2t} - \frac{2}{3}c_2 e^{-2t} - \frac{1}{4} - \frac{1}{2}t$

12. $(5D + 4)y - (2D + 1)z = e^x$

$(D + 8)y - 3z = 5e^{-x}$

Ans. $y = c_1e^x + c_2e^{-2x} - \frac{1}{2}xe^x + \frac{5}{4}e^{-x}$

$z = 3c_1e^x + 2c_2e^{-2x} - \frac{1}{6}e^x - \frac{3}{2}xe^x + \frac{5}{4}e^{-x}$

13. $\frac{dx}{dt} + x - y = te^t$

$2y - \frac{dx}{dt} + \frac{dy}{dt} = e^t$

given that $x = y = 0$ when $t = 0$.

Ans. $x = (A\cos t + B\sin t)e^{-t} + \frac{1}{25}(15t - 2)e^t$

$y = (B\cos t - A\sin t)e^{-t} + \frac{1}{25}(5t + 11)e^t$

14. $4\frac{dx}{dt} + 9\frac{dy}{dt} + 44x + 49y = t$

$3\frac{dx}{dt} + 7\frac{dy}{dt} + 34x + 38y = e^t$

Ans. $x = Ae^{-t} + Be^{-6t} + \frac{19}{3}t - \frac{56}{9} - \frac{29}{7}e^t$

$y = -Ae^{-t} + 4Be^{-6t} - \frac{17}{3}t + \frac{55}{9} + \frac{24}{7}e^t$

15. $\frac{d^2x}{dt^2} + 4x + 5y = {}^2$

$\frac{d^2y}{dt^2} + 5x + 4y = t + 1$

Ans. $x = c_1 e^t + c_2 e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$

$y = -c_1 e^t - c_2 e^{-t} + c_3\cos 3t + c_4\sin 3t + \frac{1}{9}\left(5t^2 - 4t + \frac{44}{9}\right)$

16. A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$$

$$2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$$

Obtain the expressions for x and y in terms of t, given x, y, $\frac{dx}{dt'}$ $\frac{dy}{dt}$ all vanish at t = 0.

Ans. $x = \frac{8}{9}\left(1 - \cos\frac{3t}{2}\right)$, $y = \frac{t}{3} - \frac{8}{9}\sin\frac{3t}{2}$

17. $\frac{d^2x}{dt^2} - y = 0$ **Ans.** $x = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x - 1$

$\frac{d^2y}{dt^2} - x - 1 = 0$ $y = c_1e^x + c_2e^{-x} - c_3 \cos x - c_4 \sin x$

18. The small oscillations of a certain system with two degrees of freedom are given by two simultaneous equations

$D^2x + 3x - 2y = 0$ **Ans.** $x = \frac{11}{4} \sin \frac{1}{2} + \frac{1}{12} \sin \frac{3}{2}$

$D^2x + D^2y - 3x + 5y = 0$ $y = \frac{11}{4} \sin \frac{1}{2} - \frac{1}{4} \sin \frac{3}{2}$

If $x = 0 = y$ and $Dx = 3$, $Dy = 2$ when $t = 0$, find x and y when $t = \frac{1}{2}$.

19. The acceleration components of a particle moving in a plane are given by

$$\frac{d^2x}{dt^2} = b \frac{dy}{dt}$$ **Ans.** $b^2x = a (bt - \sin bt)$

and $$\frac{d^2y}{dt^2} = a - b \frac{dx}{dt}$$ $b^2y = a (1 - \cos bt)$

where, a and b are constants, if the particle is initially at rest at the origin then show that the path of the particle is the cycloid.

2.3 SYMMETRICAL SIMULTANEOUS DIFFERENTIAL EQUATIONS

Definition : Equations of the type : $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ...(1)

where P, Q, R are the functions of x, y and z, are said to be *symmetrical simultaneous differential equations*.

There are mainly two methods of solving such equations. The solutions of such a system consist of two independent relations of the type :

$$F_1 (x, y, z) = c_1 \text{ and } F_2 (x, y, z) = c_2$$

(A) Method of Combination or Grouping

If we can observe that z is missing from first group $\frac{dx}{P} = \frac{dy}{Q}$ or, may be cancelled from this equation, then it becomes a differential equation in x and y only. Solution of this will give one relation in the solution of simultaneous equations. Then we consider the second group $\frac{dy}{Q} = \frac{dz}{R}$. If it does not contain x, it is most ideal otherwise we cancel x (if possible) and if not try to eliminate x with the help of first relation just reached. It will then be a differential equation in y and z only and after integration yields the second relation in the solution of the system of simultaneous equations. Following examples will illustrate this method.

Illustrations on Symmetrical Simultaneous Differential Equations

Ex. 1 : *Solve* $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2\ y^2\ z^2}$

Sol. : Consider $$\frac{dx}{y^2} = \frac{dy}{x^2}$$

or $$x^2\, dx = y^2\, dy$$

On integration, $$x^3 = y^3 + c_1$$

$\Rightarrow$ $$x^3 - y^3 = c_1 \quad \text{... (1)}$$

which is the first solution.

Now consider, $$\frac{dy}{x^2} = \frac{dz}{x^2\, y^2\ z^2}$$

Cancelling the common factors, we have

$$\frac{dy}{1} = \frac{dz}{y^2\, z^2} \Rightarrow y^2\, dy = \frac{dz}{z^2}$$

On integration, $\frac{1}{3}y^3 = -\frac{1}{z} + c'_2$ or $y^3 = -\frac{3}{z} + c_2$

$$\Rightarrow \quad y^3 + \frac{3}{z} = c_2 \qquad \dots (2)$$

Equations (1) and (2) taken together constitute the answer.

Note : Here in this question, we could have considered $\frac{dx}{y^2} = \frac{dz}{x^2 y^2 z^2}$ either and after cancelling y^2, got the equation $\frac{dx}{1} = \frac{dz}{x^2 z^2}$ which would have yielded the solution $x^3 + \frac{3}{z} = c_2$ ... (3)

But (2) and (3) are actually the same in the light of solution (1).

***Ex. 2 :** Solve* $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$.

Sol. : Consider first two terms,

$$\frac{dx}{y^2} = \frac{dy}{-xy} \quad \text{or} \quad x\,dx + y\,dy = 0$$

On integrating, $x^2 + y^2 = c_1$ … (1)

Next, consider second and third terms,

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \text{or} \quad z\,dy + y\,dz - 2y\,dy = 0$$

On integrating, $yz - y^2 = c_2$ … (2)

Hence, (1) and (2) together constitute the solution.

Ex. 3 :** Solve* $\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{4xy^2 - 2z}$. ***(Dec. 2005)

Sol. : Consider first two terms together.

$$\frac{dx}{2x} = \frac{dy}{-y} \quad \text{or} \quad \frac{dx}{x} + 2\frac{dy}{y} = 0$$

On integrating,

$$\log x + 2\log y = \log c_1$$

$$xy^2 = c_1 \qquad \dots (1)$$

Next, consider first and last terms together.

$$\frac{dx}{2x} = \frac{dz}{4xy^2 - 2z}$$

Using the solution (1), we remove y from this equation and obtain

$$\frac{dx}{2x} = \frac{dz}{4c_1 - 2z} \quad \text{or} \quad \frac{dx}{x} - \frac{dz}{2c_1 - z} = 0$$

On integrating,

$$\log x + \log(2c_1 - z) = \log c_2$$

$$x(2c_1 - z) = c_2$$

Putting back the expression for $c_1 = xy^2$, we have

$$x(2xy^2 - z) = c_2 \qquad \dots (2)$$

Hence, (1) and (2) constitute the solution of given symmetrical equations.

***Ex. 4 :** Solve* $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}}$

Sol. : First group of equations gives :

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \quad \log x = \log y + \log c_1$$

$$\Rightarrow \quad \log\left(\frac{x}{y}\right) = \log c_1 \Rightarrow x = c_1 y \Rightarrow y = c_2 x$$

We shall put value of y in third ratio and eliminate it from 1st and 3rd ratios, to yield

$$\frac{dx}{x} = \frac{dz}{z - a\sqrt{x^2 + z^2 + c_2^2 x^2}}$$

$$\Rightarrow \qquad \frac{dz}{dx} = \frac{z - a\sqrt{x^2\left(1 + c_2^2\right) + z^2}}{x}$$

which is homogeneous, we now put z = vx and $\frac{dz}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \qquad v + x\frac{dv}{dx} = \frac{vx - a\sqrt{x^2\left(1 + c_2^2\right) + v^2x^2}}{x}$$

$$= v - a\sqrt{v^2 + \left(1 + c_2^2\right)}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = -a\sqrt{v^2 + \left(1 + c_2^2\right)}$$

$$\Rightarrow \qquad \int \frac{dv}{\sqrt{\left(1 + c_2^2 + v^2\right)}} = -a\int \frac{dx}{x} \text{ (variable separable)}$$

$$\Rightarrow \qquad \log\left[v + \sqrt{1 + c_2^2 + v^2}\right] + a \log x = \log c_3$$

$$\Rightarrow \qquad \left(v + \sqrt{1 + v^2 + c_2^2}\right)(x^a) = c_3$$

Now, put $v = \frac{z}{x}$ and $c_2 = \frac{y}{x}$

$$\Rightarrow \qquad z + \sqrt{x^2 + y^2 + z^2} = c_3\, x^{1-a}$$

Hence the required solution is given by :

$$y = c_2 x \text{ and } z + \sqrt{x^2 + y^2 + z^2} - c_3 x^{1-a} = 0$$

(B) Method of Multipliers

Sometimes we select one or two sets of multipliers say *l*, m, n or *l*', m', n', not necessarily constants to find a fourth ratio by which we come to solutions viz. if the equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

then choose multipliers *l*, m, n such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \qquad \ldots (1)$$

Now, suppose the choice of *l*, m, n such that

$$lP + mQ + nR = 0 \text{ then } l\,dx + m\,dy + n\,dz = 0$$

and if it is exact we may find its integral as

$$F_1(x, y, z) = c_1 \qquad \ldots (2)$$

which is the first solution of the system.

If it is further possible to find the other set of multipliers say *l*', m', n' such that :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

and also if $l'P + m'Q + n'R = 0$, then $l'dx + m'dy + n'dz = 0$

and solving we get another solution like

$$F_2(x, y, z) = c_2 \qquad \ldots (3)$$

Thus (2) and (3) constitute the solution of the given set of symmetrical equations.

ILLUSTRATIONS

Ex. 5 : *Solve* $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$

Sol. : If we take the first set of multipliers as 1, 1, 1 we have

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx + dy + dz}{1\,(y-z) + 1\,(z-x) + 1\,(x-y)} = \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

and by integration we get

$$x + y + z = c_1 \qquad \dots (1)$$

as first solution.

Next the question itself suggests that even x, y, z may be a set of multipliers, then

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy + z\,dz}{x\,(y-z) + y\,(z-x) + z\,(x-y)}$$

$$\Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$$

On integration, it yields

$$x^2 + y^2 + z^2 = c_3 \qquad \dots (2)$$

Thus the equations (1) and (2) together constitute the required solutions of the set.

Ex. 6 : *Solve* $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$

Sol. : The equation suggests that (x, y, z) may be the first set of multipliers, hence

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x\,(mz-ny) + y\,(nx-lz) + z\,(ly-mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$$

On integrating,

$$x^2 + y^2 + z^2 = c_1 \qquad \dots (1)$$

Let l, m, n be second set of multipliers, then each ratio equals

$$= \frac{l\,dx + m\,dy + n\,dz}{l\,(mz-ny) + m\,(nx-lz) + n\,(ly-mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\Rightarrow \qquad l\,dx + m\,dy + n\,dz = 0$$

On integration, it gives

$$lx + my + nz = c_2 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution.

Note : In some cases, the 4th term $\dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ turns out to be a more convenient ratio than the previous three ratios and in such a case numerator often turns out to be differential of the denominator. By this fact and by opting one or two given ratios, we are able to solve the system. Examples will strengthen this method further.

Ex. 7 : *Solve* $\dfrac{dx}{3z-4y} = \dfrac{dy}{4x-2z} = \dfrac{dz}{2y-3x}$ ***(May 2015)***

Sol. : Let us choose multipliers x, y, z then each ratio equals $\dfrac{x\,dx + y\,dy + z\,dz}{0}$

$$\Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0, \text{ on integration, we get}$$

$$x^2 + y^2 + z^2 = c_1 \qquad \dots (1)$$

Second set of multipliers may be conveniently chosen as 2, 3, 4 and then each ratio equals $\frac{2\,dx + 3\,dy + 4\,dz}{0}$, from where

$$2\,dx + 3\,dy + 4\,dz = 0$$

$$\Rightarrow \quad 2x + 3y + 4z = c_2 \quad \dots (2)$$

Equations (1) and (2) constitute the answer.

Ex. 8 : *Solve* $\frac{dx}{x\,(2y^4 - z^4)} = \frac{dy}{y\,(z^4 - 2x^4)} = \frac{dz}{z\,(x^4 - y^4)}$ ***(May 2006, 2007, 2008, Dec. 2010)***

Sol. : First set of multipliers may be x^3, y^3, z^3 which will equal each ratio to

$$\frac{x^3\,dx + y^3\,dy + z^3\,dz}{0}$$

$$\Rightarrow \quad x^3\,dx + y^3\,dy + z^3\,dz = 0$$

On integration, this yields

$$x^4 + y^4 + z^4 = c_1 \quad \dots (1)$$

If we choose conveniently the second set of multipliers as $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$, then each ratio will be equal to

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{2\,.\,dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} + \frac{2\,.\,dz}{z} = 0$$

On integration, we have

$$\log x + \log y + 2 \log z = \log c_2$$

$$\Rightarrow \quad x\,y\,z^2 = c_2 \quad \dots (2)$$

Here equations (1) and (2) constitute the answer.

Ex. 9 : *Solve* $\frac{a\,dx}{(b - c)\,yz} = \frac{b\,dy}{(c - a)\,xz} = \frac{c\,dz}{(a - b)\,xy}$

Sol. : Use multipliers x, y, z, then each ratio equals

$$\frac{ax\,dx + by\,dy + cz\,dz}{0} \Rightarrow ax\,dx + by\,dy + cz\,dz = 0$$

Integration yields, $\quad ax^2 + by^2 + cz^2 = c_1 \quad \dots (1)$

Next we shall use multipliers ax, by, cz then each ratio equals

$$\frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{0} \Rightarrow a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$$

Integration will yield,

$$a^2\,x^2 + b^2\,y^2 + c^2\,z^2 = c_2 \quad \dots (2)$$

Equations (1) and (2) constitute the answer.

Ex. 10 : *Solve* $\frac{dx}{y + zx} = \frac{dy}{-x - yz} = \frac{dz}{x^2 - y^2}$.

Sol. : Using the first set of multipliers y, x, 1,

$$\text{each term} = \frac{y\,dx + x\,dy + dz}{0}$$

$$\therefore \quad y\,dx + x\,dy + dz = 0$$

On integration, $\quad xy + z = c_1 \quad \dots (1)$

Again using second set of multipliers x, y, – z,

$$\text{each term} = \frac{x\,dx + y\,dy - z\,dz}{0}$$

$$\therefore \quad x\,dx + y\,dy - z\,dz = 0$$

On integration, $$x^2 + y^2 - z^2 = c^2 \qquad \ldots (2)$$

Thus (1) and (2) are solutions of the given equations.

Ex. 11 : *Solve* $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{(1 + 2xy + 3x^2\, y^2)\,(x + y)\,z}$

Sol. : From the first two ratios, we have

$$\frac{dx}{1} = \frac{dy}{1}$$

On integration, $$x - y = c_1 \qquad \ldots (1)$$

Also, $$\text{each ratio} = \frac{y\,dx + x\,dy}{y + x} \text{ and hence}$$

$$\frac{y\,dx + x\,dy}{y + x} = \frac{dz}{(1 + 2xy + 3x^2\,y^2)\,(x + y)\,z}$$

$$\Rightarrow \quad \frac{y\,dx + x\,dy}{1} = \frac{dz}{(1 + 2xy + 3x^2\,y^2)\,z}$$

$$\Rightarrow \quad \frac{d\,(xy)}{1} = \frac{dz}{(1 + 2xy + 3x^2\,y^2)\,z}$$

$$\Rightarrow \quad (1 + 2xy + 3x^2\,y^2)\,d\,(xy) = \frac{dz}{z}$$

For convenience, put $$xy = v$$

$$\Rightarrow \quad x\,dy + y\,dx = dv$$

$$\Rightarrow \quad d\,(xy) = dv$$

$$\Rightarrow \quad (1 + 2v + 3v^2)\,dv = \frac{dz}{z} \text{ (from where variables are separable)}$$

$$\Rightarrow \quad v + v^2 + v^3 = \log z + c_2$$

$$\Rightarrow \quad xy + (xy)^2 + (xy)^3 - \log z = c_2 \qquad \ldots (2)$$

Hence equations (1) and (2) together represent the solution set of the system.

Ex. 12 : *Solve* $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{(x + y)\,[e^{xy} + \sin xy + x^2\,y^2]}$

Sol. : Consider the first group $\dfrac{dx}{1} = \dfrac{dy}{1}$, which yields

$$\Rightarrow \quad x - y = c_1 \qquad \ldots (1)$$

Now each ratio equals $\dfrac{y\,dx + x\,dy}{y + x}$, hence

$$\frac{y\,dx + x\,dy}{y + x} = \frac{dz}{(x + y)\,[e^{xy} + \sin xy + (xy)^2]}$$

$$\Rightarrow \quad [e^{xy} + \sin xy + (xy)^2]\,(y\,dx + x\,dy) = dz$$

Put $xy = v$, then $x\,dy + y\,dx = dv$

$$\Rightarrow \quad (e^v + \sin v + v^2)\,dv = dz$$

On integration, we get $$e^v - \cos v + \frac{v^3}{3} = z + c_2$$

$$\Rightarrow \quad 3e^v - 3\cos v + v^3 = 3z + c_3$$

$$\Rightarrow \quad 3e^{xy} - 3\cos(xy) + (xy)^3 - 3z = c_3 \qquad \ldots (2)$$

Equations (1) and (2) constitute the answer.

Ex. 13 : *Solve* $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$ ... (1)

Sol. : Each ratio equals

$$\frac{dx - dy}{(x + y + z)(x - y)} = \frac{dy - dz}{(x + y + z)(y - z)} = \frac{dz - dx}{(x + y + z)(z - x)}$$

$$\Rightarrow \quad \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \qquad \text{... (2)}$$

Consider the first two ratios in (2)

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

Each being exact, we may integrate to get

$$\log (x - y) = \log (y - z) + \log c_1$$

$$\Rightarrow \quad \frac{x - y}{y - z} = c_1 \qquad \text{... (3)}$$

Next we shall select two sets of multipliers say x, y, z and (1, 1, 1) and obtain by their help the second relation. Each ratio in (1) equals

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} \text{ as well as } \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

Equating these two, we have

$$\frac{x\,dx + y\,dy + z\,dz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

Cancelling factor $x^2 + y^2 + z^2 - xy - yz - zx$, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x + y + z} = \frac{dx + dy + dz}{1}$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = (x + y + z)(dx + dy + dz)$$

Integration yields

$$x^2 + y^2 + z^2 = (x + y + z)^2 + c_2 \qquad \text{... (4)}$$

$\therefore$ Equations (3) and (4) constitute the solution of the system.

Ex. 14 : *Solve* $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2\,xz}$ ... (1)

Sol. : Here it is convenient to consider the later two ratios to yield $\dfrac{dy}{y} = \dfrac{dz}{z}$ which on integration gives $\log y = \log z + \log c_1$.

$$\Rightarrow \quad \frac{y}{z} = c_1 \Rightarrow y = c_1 z \qquad \text{... (2)}$$

Next we shall use multipliers x, y, z then each ratio in (1) equals

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x[x^2 + y^2 + z^2]}$$

If we consider this with the second ratio in (1)

$$\frac{dy}{2\,xy} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2 + y^2 + z^2}$$

Hence its integration yields

$$\log y = \log (x^2 + y^2 + z^2) + \log c_2$$

$$\Rightarrow \quad \frac{y}{x^2 + y^2 + z^2} = c_2 \qquad \text{... (3)}$$

Equations (2) and (3) constitute the answer.

EXERCISE 2.2

Solve the following system of symmetrical simultaneous equations :

1. $\dfrac{dx}{y^2 z} = \dfrac{dy}{x^2 z} = \dfrac{dz}{y^2 x}$ **(May 2010)**

Ans. $x^3 - y^3 = c_1$ and $x^2 - z^2 = c_2$

2. $\dfrac{x\,dx}{y^3 z} = \dfrac{dy}{x^2 z} = \dfrac{dz}{y^3}$ **(Dec. 2012)**

Ans. $x^4 - y^4 = c_1,\ x^2 - z^2 = c_2$

3. $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{x\,e^{x^2+y^2}}$ **(Dec. 2004)**

Ans. $x^2 + y^2 = c_1,\ y\,e^{x^2+y^2} + z = c_2$

4. $\dfrac{dx}{x(z-2y^2)} = \dfrac{dy}{y(z-y^2-2x^3)} = \dfrac{dz}{z(z-y^2-2x^3)}$

Ans. $\dfrac{y}{z} = c_1,\ \dfrac{z}{x} - \dfrac{y^2}{x} + x^2 = c_2$

Hint : Use solution $y_1 = c_1 z$ to find second solution.

5. $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{-(x+z)}$.

Hint : Use solution $x = c_1 y$ to find second solution.

Ans. $x = c_1 y,\ \dfrac{1}{2}xy + yz = c_2$

6. $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z + \tan(y-3x)}$. **(Nov. 2015)**

Ans. $y - 3x = c_1,\ \ 5x = \log[5z + \tan(y-3x)] + c_2$

7. $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{z(x^2-y^2)}$

Hint : Use the multipliers $1/x, 1/y, 1/z$ and $x, y, -1$.

Ans. $xyz = c_1,\ x^2 + y^2 - 2z = c_2$

8. $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$.

Hint : Use the multipliers $1/x^2, 1/y^2, 1/z^2$ and $1/x, 1/y, 1/z$.

Ans. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = c_1,\ xyz = c_2$

9. $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{2x-3y}$

Ans. $x^2 + y^2 = c_1,\ 3x + 2y + z = c_2$

Hint : Use the multipliers 3, 2, 1.

10. $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$

Ans. $x^2 - y^2 - 2xy = c_1,\ x^2 - y^2 - z^2 = c_2$

11. $\dfrac{dx}{y^3x - 2x^4} = \dfrac{dy}{2y^4 - x^3y} = \dfrac{dz}{9z\,(x^3 - y^3)}$

Hint : Use the multipliers $1/x, 1/y, 1/3z$ and then x^2, y^2 for the first two terms.

Ans. $x^3 y^3 z = c_1,\ (x^3 + y^3)\,z^2 = c_2$

12. $\dfrac{x\,dx}{z^2 - 2yz - y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$ **(Dec. 2010)**

Ans. $x^2 + y^2 + z^2 = c_1,\ y^2 - 2yz - z^2 = c_2$

Hint : Use the multipliers 1, y, z and then consider last two terms.

13. $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{-y(z^2+x^2)} = \dfrac{dz}{z(x^2+y^2)}$

Ans. $\dfrac{yz}{x} = c_2,\ x^2 + y^2 + z^2 = c_1$

14. $\dfrac{x^2\,dx}{y^3} = \dfrac{y^2\,dy}{x^3} = \dfrac{dz}{z}$ **(May 2004, 2012, May 2011)**

Hint : each ratio $= \dfrac{x^2\,dx + y^2\,dy}{y^3 + x^3}$ etc.

Ans. $x^6 - y^6 = c_1,\ x^3 + y^3 = c_2 z^3$

15. $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dx}{(x+y)\,e^{xy} + \sin xy + x^2y^2}$

Ans. $x - y = c_1,\ 3e^{xy} - 3\cos xy + (xy)^3 - 3z = c_2$

16. $\dfrac{dx}{x^2+y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{(x+y)^3 z}$ **(Dec. 2013)**

Ans. $(x+y)^2 - 2\log z = c_1,\ c_2 y = x^2 - y^2$

Hint: $\dfrac{dx+dy}{(x+y)^2} = \dfrac{dz}{(x+y)^3 z}$ and $\dfrac{dx+dy}{(x+y)^2} = \dfrac{dx-dy}{(x-y)^2}$

17. $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$

Ans. $\dfrac{x-y}{y-z} = c_1,\ (x-y)^2\,(x+y+z) = c_2$

Hint : $\dfrac{dx-dy}{x-y} = \dfrac{dy-dz}{y-z}$ and $\dfrac{dx-dy}{x-y} = \dfrac{dx+dy+dz}{2(x+y+z)}$

18. $dx + dy + (x+y)\,dz = 0$,

$z\,(dx+dy) + (x+y)\,dz = 0$

Ans. $x + y = c_1\,e^{-z},\ x + y = c_2/z$

19. $(x-z)\,dx + 2(x+z)\,dy + (z-x)\,dz = 0$

$x(z-x)\,dx + 4y(x+z)\,dy - z(z-x)\,dz = 0$

Ans. $x + z = c_1\,(2y+z) = c_2\,(x-2y)$

UNIT II : TRANSFORMS

CHAPTER-3
FOURIER TRANSFORM

3.1 INTRODUCTION

Transformation is an operation which converts a mathematical expression into a different form with the help of which problems are either solved easily or methods of solution become simple. For example, Logarithmic transformation reduces multiplication, division and one expression raised to power another expression into addition, subtraction and simple multiplication. Transfer of origin and/or axes convert equations of curves and surfaces in simple or standard forms from which useful informations and important properties can be obtained. Similarly, elementary transformations in matrices are useful in solving various problems by simple methods.

Fourier series are powerful tools in treating various problems involving periodic functions. However, in many practical problems, the impressed force or voltage is non-periodic rather than periodic, a single unrepeated pulse, for instance. A suitable representation for non-periodic functions can be obtained by considering the limiting form of Fourier series when the fundamental period is made infinite. We shall find that in such a case, the Fourier series becomes a Fourier integral. Using symmetry, Fourier integral can conveniently be expressed in terms of Fourier transform which transforms a non-periodic function, say f(t) in time domain, into a function F(λ) in frequency domain.

The Fourier integrals and transforms are useful in solving boundary value problems arising in science and engineering for example, Conduction of Heat, Wave Propagation, Theory of Communication etc.

3.2 COMPLEX EXPONENTIAL FORM OF FOURIER SERIES

If f(x) is a periodic function of period 2L, defined in the interval $-L < x < L$, and satisfies Dirichlet's conditions then f(x) can be represented by Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right) \qquad \dots \text{(i)}$$

where,

$$\left.\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(u) \cos \frac{n\pi u}{L}\, du \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(u) \sin \frac{n\pi u}{L}\, du \end{aligned}\right\} \qquad \dots \text{(ii)}$$

Using exponential equivalent of cosine and sine terms

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{-i}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

result (i) can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) + b_n\left(-\frac{i}{2}\right)\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - i\,b_n}{2}\right) e^{in\pi x/L} + \left(\frac{a_n + i\,b_n}{2}\right) e^{-in\pi x/L}\right]$$

If we now define $c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - i\,b_n}{2}, \quad c_{-n} = \frac{a_n + i\,b_n}{2}$

then the above series can be written in more symmetric form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L} \qquad \text{... (iii)}$$

where $c_0 = \frac{a_0}{2}$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{n\pi u}{L}\,du - i\,\frac{1}{L}\int_{-L}^{L} f(u)\sin\frac{n\pi u}{L}\,du\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f(u)\left[\cos\frac{n\pi u}{L} - i\sin\frac{n\pi u}{L}\right]du = \frac{1}{2L}\int_{-L}^{L} f(u)\;e^{-in\pi u/L}\;du.$$

Similarly, $$c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(u)\,e^{in\pi u/L}\,du.$$

Clearly, the index n is positive, negative or zero, c_n is correctly given by the single formula

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(u)\;e^{-in\pi u/L}\;du \qquad \text{... (iv)}$$

Thus the complex exponential form of a Fourier series is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L} \\ \text{where,} \quad c_n &= \frac{1}{2L}\int_{-L}^{L} f(u)\,e^{-in\pi u/L}\,du \end{aligned}} \qquad \text{... (1)}$$

3.3 FOURIER INTEGRAL

We shall now consider the limiting form of Fourier series for periodic function of period 2 L, when L → ∞.

For convenience, we start with the complex exponential form of a Fourier series [result (1)]

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L} \qquad \text{... (i)}$$

where, $$c_n = \frac{1}{2L}\int_{-L}^{L} f(u)\,e^{-in\pi u/L}\,du \qquad \text{... (ii)}$$

Substituting for c_n from (ii) in (i), we obtain

$$f(x) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2L}\int_{-L}^{L} f(u)\,e^{-in\pi u/L}\,du\right] e^{in\pi x/L}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-L}^{L} f(u)\,e^{-in\pi(u-x)/L}\,du\right]\left(\frac{\pi}{L}\right) \qquad \text{... (iii)}$$

Now, let us denote $\lambda = \frac{n\pi}{L}$ (frequency of general term)

$$\therefore \qquad \Delta\lambda = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \qquad \begin{pmatrix}\text{difference in frequency between} \\ \text{successive terms}\end{pmatrix}$$

Then f(x) can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} f(u)\, e^{-i\lambda(u-x)}\, du \right] \Delta\lambda \qquad \ldots \text{(iv)}$$

Now if $L \to \infty$, then $\Delta\lambda \to 0$ and the expression (iv) gives

$$f(x) = \lim_{L \to \infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} f(u)\, e^{-i\lambda(u-x)}\, du \right] \Delta\lambda$$

$$= \lim_{\Delta\lambda \to 0} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du \right] \Delta\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du \right] d\lambda \qquad \text{[By definition of integral as limit of sum]}$$

Thus the Fourier integral representation of f(x), where $-\infty < x < \infty$ is given by,

$$\boxed{f(x) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=\infty} \int_{u=-\infty}^{u=\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda} \qquad \ldots (2)$$

The result (2) is also known as **Fourier Integral Theorem.**

The limitations on f(x) for validity of the result (2) are as follows :

(i) In every finite interval, f(x) satisfies the Dirichlet's conditions.

(ii) The integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.

Note that the above conditions are sufficient but not necessary.

Remark 1 : The result (2) holds if x is a point of continuity of f(x). At a point of discontinuity $x = x_0$, the value of the Fourier integral equals the average value of the left-hand and right-hand limit of f(x) at $x = x_0$ i.e. at $f(x_0) = \frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$ as in the case of Fourier series.

Remark 2 : The result (2) can also be written as

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x}\, d\lambda \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du} \qquad \ldots \text{(2 a)}$$

Remark 3 : Example of a periodic function f(x) of period T and the limiting non-periodic function whose period becomes infinite.

Consider the function

$$f_T(x) = \begin{cases} 0, & -T/2 < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < T/2 \end{cases}$$

having period T > 2. For $T \to \infty$, we obtain a function which is no longer periodic [See Fig. 3.1]. Non-periodic function could be assumed to have infinite period.

$$f(x) = \lim_{T \to \infty} f_T(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

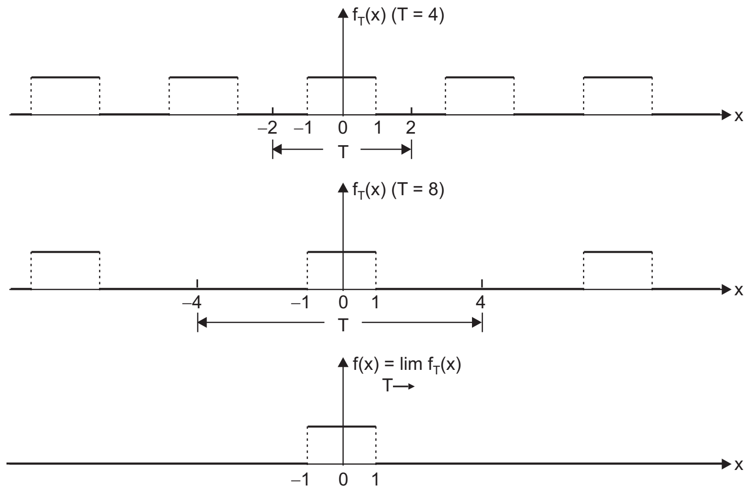

Fig. 3.1 : The non-Periodic Limit of Sequence of Periodic Function whose Period becomes Infinite

Remark 4 : We note that when we extend a function into Fourier series in certain range then the function is defined by the series outside this range in a periodic manner. However, by the Fourier integral, we obtain analytical expression for functions that represent the function throughout the infinite range $-\infty < x < \infty$.

3.4 EQUIVALENT FORMS OF FOURIER INTEGRAL

The Fourier integral can be written in various forms :

From result (2), we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda \qquad \text{... (i)}$$

Replacing the exponential by its trigonometric equivalent, we get

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, [\cos \lambda (u-x) - i \sin \lambda (u-x)]\, du\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \lambda (u-x)\, du\, d\lambda - i\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \sin \lambda (u-x)\, du\, d\lambda \qquad \text{... (ii)}$$

Since $\sin \lambda (u - x)$ is an odd function of λ in $-\infty < \lambda < \infty$, we have $\int_{-\infty}^{\infty} \sin \lambda (u-x)\, d\lambda = 0$ and second integral is always zero.

Then the expression (ii) gives equivalent form of result (2) as

$$\boxed{\mathbf{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \lambda (u-x)\, du\, d\lambda}} \qquad \text{... (3)}$$

Again since the cos λ (u – x) of result (3) is even function of λ in $-\infty < \lambda < \infty$, we have

$$\int_{-\infty}^{\infty} \cos \lambda (u - x)\, d\lambda = 2 \int_{0}^{\infty} \cos \lambda (u - x)\, d\lambda$$

and we get modified form of the result (3) as

$$\boxed{f(x) = \frac{1}{\pi} \int_{\lambda = 0}^{\lambda = \infty} \int_{u = -\infty}^{u = \infty} f(u) \cos \lambda (u - x)\, du\, d\lambda} \qquad \ldots (4)$$

Expanding the factor cos λ (u – x) in the integrand of the result (4), we obtain

$$f(x) = \frac{1}{\pi} \int_{\lambda = 0}^{\lambda = \infty} \int_{u = -\infty}^{u = \infty} f(u) [\cos \lambda u \cos \lambda x + \sin \lambda u \sin \lambda x]\, du\, d\lambda$$

Hence another equivalent form of the result (4) is

$$\boxed{\begin{aligned} f(x) &= \int_{0}^{\infty} [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x]\, d\lambda \\ \text{where} \quad A(\lambda) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \lambda u\, du \\ B(\lambda) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \lambda u\, du \end{aligned}} \qquad \ldots (5)$$

3.5 SINE AND COSINE INTEGRALS

If a function defined in the interval $-\infty < x < \infty$ is either an even function or an odd function, then the Fourier integral representation becomes simpler than in the case of arbitrary function.

Case 1 : When f(x) is an even function, then in the result (5), we have

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \lambda u\, du = 0 \qquad \text{[product } f(u) \sin \lambda u \text{ is odd]}$$

and

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \lambda u\, du = \frac{2}{\pi} \int_{0}^{\infty} f(u) \cos \lambda u\, du$$

[product f(u) cos λu is even]

Hence result (5) reduces to the following simpler form

$$\boxed{f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \cos \lambda u \cos \lambda x\, du\, d\lambda} \qquad \ldots (6)$$

The result (6) is called the **Fourier Cosine Integral** of f(x).

Case 2 : When f(x) is an odd function, then in the result (5), we have

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \lambda u \, du = 0 \qquad \text{[product } f(u) \cos \lambda u \text{ is odd]}$$

and

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin \lambda u \, du = \frac{2}{\pi} \int_{0}^{\infty} f(u) \sin \lambda u \, du \qquad \text{[product } f(u) \sin \lambda u \text{ is even]}$$

Hence result (5) reduces to the following simpler form

$$\boxed{\mathbf{f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \sin \lambda u \sin \lambda x \, du \, d\lambda}} \qquad \text{... (7)}$$

The result (7) is called the **Fourier Sine Integral** of f(x).

Note : If a function f(x) is defined in the interval $0 < x < \infty$, then considering f(x) to be either an even or an odd function of x in $-\infty < x < \infty$, we can express f(x) as a Fourier cosine integral or Fourier sine integral respectively.

These simplifications are quite similar to half range cosine and half range sine expansions of even and odd periodic functions respectively.

3.6 FOURIER TRANSFORMS

From result (2), we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \, e^{-i\lambda(u-x)} \, du \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u) \, e^{-i\lambda u} \, du \right] e^{i\lambda x} \, d\lambda \qquad \text{... (i)}$$

If we write

$$F(\lambda) = \int_{-\infty}^{\infty} f(u) \, e^{-i\lambda u} \, du \qquad \text{... (ii)}$$

then from (i), we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) \, e^{i\lambda x} \, d\lambda \qquad \text{... (iii)}$$

The function F(λ) is called the *Fourier transform* of f(x) (and is written as F(λ) = F [f(x)]), while the function f(x) is the *inverse Fourier transform* of F(λ).

Hence Fourier transform of f(x) is defined as

$$\boxed{\mathbf{F(\lambda) = \int_{-\infty}^{\infty} f(u) \, e^{-i\lambda u} \, du}} \qquad \text{... (8)}$$

and Inverse Fourier transform is given by

$$\boxed{\mathbf{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) \, e^{i\lambda x} \, d\lambda}} \qquad \text{... (9)}$$

Note 1 : It is sometimes convenient to associate the factor $\frac{1}{2\pi}$ with the integral for F(λ) instead with the integral for f(x). It is also possible to achieve more symmetric form by associating the factor $1/\sqrt{2\pi}$ with each of the integrals.

Hence the results (8) and (9) can be written as :

$$\boxed{F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du} \qquad \text{... (8 a)}$$

and

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda} \qquad \text{... (9 a)}$$

Note 2 : To find Fourier integral representation of a function f(x), first find F(λ) from result (8) and then substitute this value of F(λ) in (9).

Note 3 : Symmetrical expressions f(x) and its corresponding function F(λ) constitute a *Fourier Transform pair*.

3.7 FOURIER SINE AND COSINE TRANSFORMS

1. Fourier Cosine Transform :

If a function f(x) defined in the interval $-\infty < x < \infty$ is an *even function*, then from Fourier cosine integral [result (6)], we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(u) \cos \lambda u \cos \lambda x \, du \, d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} \left[\int_0^{\infty} f(u) \cos \lambda u \, du \right] \cos \lambda x \, d\lambda \qquad \text{... (i)}$$

If we write

$$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du \qquad \text{... (ii)}$$

then from (i), it follows that

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(\lambda) \cos \lambda x \, d\lambda \qquad \text{... (iii)}$$

We call $F_C(\lambda)$ the Fourier cosine transform of f(x), while f(x) is the *Inverse Fourier cosine transform* of $F_C(\lambda)$.

Hence the Fourier cosine transform of f(x) is defined as

$$\boxed{F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du} \qquad \text{... (10)}$$

and the Inverse Fourier cosine transform of $F_C(\lambda)$ is given by

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(\lambda) \cos \lambda x \, d\lambda} \qquad \text{... (11)}$$

2. Fourier Sine Transform :

If a function f(x) defined in the interval $-\infty < x < \infty$ is an *odd function*, then from Fourier sine integral [result (7)], we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(u) \sin \lambda u \sin \lambda x \, du \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(u) \sin \lambda u \, du \right] \sin \lambda x \, d\lambda \qquad \text{... (i)}$$

If we write $$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du \qquad \text{... (ii)}$$

then from (i), it follows that

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda \qquad \text{... (iii)}$$

We call $F_S(\lambda)$ the *Fourier sine transform* of f(x), while f(x) is the *Inverse Fourier sine transform* of $F_S(\lambda)$.

Hence the Fourier sine transform of f(x) is defined as

$$\boxed{F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du} \qquad \text{... (12)}$$

and the Inverse Fourier sine transform of $F_S(\lambda)$ is given by

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda} \qquad \text{... (13)}$$

Note :

1. If a function f(x) is defined in the interval $0 < x < \infty$, then we can extend f(x) in the interval $-\infty < x < 0$ so that f(x) becomes an even function in the interval $-\infty < x < \infty$. Thus for even function defined in $-\infty < x < \infty$, Fourier cosine transform and Inverse Fourier cosine transform are given by results (10) and (11) respectively.
2. If a function f(x) is defined in the interval $0 < x < \infty$, then we can also extend f(x) in the interval $-\infty < x < 0$, so that f(x) becomes an odd function in the interval $-\infty < x < \infty$. Thus for odd function defined in $-\infty < x < \infty$, Fourier sine transform and Inverse Fourier sine transform are given by results (12) and (13) respectively.
3. These simplifications are quite similar to those in the case of Fourier series.
4. Results (10) and (11) of Fourier cosine transform and Inverse Fourier cosine transform can be written in more symmetric forms as

$$\boxed{F_C(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \lambda u \, du} \qquad \text{... (10 (a))}$$

and $$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(\lambda) \cos \lambda x \, d\lambda} \qquad \text{... (11 (a))}$$

Similarly, results (12) and (13) of Fourier sine transform and Inverse Fourier sine transform can be written in more symmetrical forms as

$$\boxed{F_S(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u \, du} \quad \text{... (12 (a))}$$

and

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_S(\lambda) \sin \lambda x \, d\lambda} \quad \text{... (13 (a))}$$

In the following table, we have listed Fourier transform pairs for ready reference.

Table 3.1 : Table of Fourier Transforms and Inverse Transforms

Sr. No.	Name of the transform	Interval	Expression for the Transform	Inverse Transform
1.	Fourier	$-\infty < x < \infty$	$F(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} du$	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2.	Fourier cosine (for even function)	$-\infty < x < \infty$	$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(\lambda) \cos \lambda x \, d\lambda$
3.	Fourier sine (for odd function)	$-\infty < x < \infty$	$F_S(\lambda) = \int_0^{\infty} f(u) \sin \lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_S(\lambda) \sin \lambda x \, d\lambda$
4.	Fourier cosine	$0 < x < \infty$	$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(\lambda) \cos \lambda x \, d\lambda$
5.	Fourier sine	$0 < x < \infty$	$F_S(\lambda) = \int_0^{\infty} f(u) \sin \lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_S(\lambda) \sin \lambda x \, d\lambda$

3.8 USEFUL RESULTS FOR EVALUATING THE INTEGRALS IN FOURIER TRANSFORMS

The following results are quite useful in evaluating the integrals :

1. $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$
2. $|x| \le a \Rightarrow -a \le x \le a$ and $|x| \ge a \Rightarrow x \ge a$ and $x \le -a$
3. $\int uv \, dx = u \int v \, dx - \int \left(\frac{\partial u}{\partial x} \int v \, dx\right)$
4. $\int uv \, dx = uv_1 - u'v_2 + u''v_3 \; \ldots\ldots$

6. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$

   $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

7. $\int_0^{\infty} \frac{\sin ax}{x} dx = \begin{cases} \pi/2 & \text{if a is positive} \\ -\pi/2 & \text{if a is negative} \end{cases}$

8. $B(m, n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma (m+n)}$

9. $\Gamma(n+1) = n\,\Gamma n$, $\Gamma(n+1) = n!$ if n is positive integer, $\Gamma(1/2) = \sqrt{\pi}$

10. Rule of differentiation under the integral sign (DUIS) :

If $I(\alpha) = \int_a^b f(x, \alpha)\, dx$, where a and b are constants, then

$$\frac{d\, I(\alpha)}{d\alpha} = \frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx$$

11. $\int_{-\infty}^{\infty} f(x)\, dx = 0$, if f(x) is odd.

12. $\int_{-\infty}^{\infty} f(x)\, dx = 2\int_0^{\infty} f(x)\, dx$, if f(x) is even.

Illustrations on Fourier Integrals and Fourier Transforms

Type 1 : Problems on Fourier Integral Representation

Ex. 1 : *Find the Fourier integral representation of the function*

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$ ***(Dec. 2017)***

and hence

(a) evaluate $\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda$ *(b) deduce the value of* $\int_0^{\infty} \frac{\sin\lambda}{\lambda}\, d\lambda$.

(c) Find the value of above integrals at $|x| = 1$*, which are points of discontinuity of f(x).* ***(Dec. 2005)***

Sol. : Here the given function f(x) is

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases} \quad \text{... (i)}$$

This shows that $f(-x) = f(x)$ i.e. f(x) is an even function in the interval $-\infty < x < \infty$ [See Fig. 3.2].

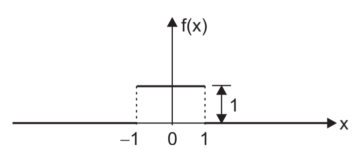

Fig. 3.2

Hence by result (10), the Fourier cosine transform for even function f(x) in the interval $-\infty < x < \infty$ is given by;

$$F_C(\lambda) = \int_0^{\infty} f(u) \cos\lambda u\, du = \int_0^1 \cos\lambda u\, du \quad \text{[from (i)]}$$

$$= \left[\frac{\sin\lambda u}{\lambda}\right]_0^1 = \frac{\sin\lambda}{\lambda} \quad \text{... (ii)}$$

By using inverse transform [result (11)], the Fourier integral representation is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda)\cos\lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos\lambda x\, d\lambda \qquad \text{[substituting } F_C(\lambda) \text{ from (ii)]}$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, dx \qquad \dots \text{(iii)}$$

which is the required Fourier integral representation.

The result (iii) can be expressed as

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x), \text{ where } f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$\therefore \quad \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \qquad \dots \text{(iv)}$$

Now, if we put x = 0 (which lies in $-1 < x < 1$) in (iv), we have

$$\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2} \qquad \dots \text{(v)}$$

At $|x| = 1$, which are points of discontinuity, the value of the Fourier integral equals to average of left-hand and right-hand limit of f(x) at $|x| = 1$.

Thus,

$$\left[\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, dx\right]_{|x| = 1} = \frac{\frac{\pi}{2} + 0}{2} = \frac{\pi}{4}.$$

Note 1 : We note that the integral in (v) is the limit of the so-called *sine integral*

$$S_i(t) = \int_0^t \frac{\sin\lambda}{\lambda}\, d\lambda \quad \text{as} \quad t \to \infty.$$

Note 2 : The Fourier integral representation can also be obtained directly by using result (4) in section 3.4. Thus,

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\lambda=\infty}\int_{u=-\infty}^{u=\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\lambda=\infty}\left[\int_{u=-1}^{u=1}\cos\lambda(u-x)\, du\right] d\lambda \qquad (\because f(u) = 1 \text{ for } -1 < u < 1)$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\lambda=\infty}\left[\frac{\sin\lambda(u-x)}{\lambda}\right]_{-1}^{1} d\lambda$$

$$= \frac{1}{\pi} \int_{\lambda=0}^{\lambda=\infty} \left[\frac{\sin \lambda (1-x) + \sin \lambda (1+x)}{x}\right] d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\lambda=\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda. \qquad \left\{\because \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}\right.$$

Ex. 2 : *Find the Fourier integral for the function*

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \\ 1/2, & x = 0 \end{cases}$$

Sol. : Note that the given function f(x) is neither an even function nor an odd function. Hence the Fourier integral representation can be obtained by first finding Fourier transform F(λ) using result (8) and then substituting this value of F(λ) in result (9).

Thus from result (8), we have

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \int_{-\infty}^{0} f(u)\, e^{-i\lambda u}\, du + \int_{0}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^{0} (0)\, e^{-i\lambda u}\, du + \int_{0}^{\infty} e^{-u}\, e^{-i\lambda u}\, du$$

$$= \int_{0}^{\infty} e^{-(1+i\lambda)u}\, du = \left[\frac{e^{-(1+i\lambda)u}}{-(1+i\lambda)}\right]_0^{\infty}$$

$$= \frac{1}{1+i\lambda} = \frac{1-i\lambda}{1+\lambda^2} \qquad \text{... (i)}$$

By using inverse transform [result (9)], the Fourier integral representation of f(x) is given by,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1-i\lambda}{1+\lambda^2} [\cos \lambda x + i \sin \lambda x]\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} + i\, \frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2}\right] d\lambda \qquad \text{... (ii)}$$

If $\phi_1(\lambda) = \dfrac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}$, then since $\phi_1(-\lambda) = \phi_1(\lambda)$, $\phi_1(\lambda)$ is an even function of λ and hence, we have

$$\int_{-\infty}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda}{1+\lambda^2}\, d\lambda = 2\int_{0}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}\, d\lambda \qquad \text{... (iii)}$$

and if $\phi_2(\lambda) = \dfrac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2}$, then since $\phi_2(-\lambda) = -\phi_2(\lambda)$, $\phi_2(\lambda)$ is an odd function of λ, we have

$$\int_{-\infty}^{\infty} \frac{-\lambda \cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}\, d\lambda = 0 \qquad \text{... (iv)}$$

Substituting (iii) and (iv) in (ii), we get

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}\, d\lambda$$

which is the required Fourier integral representation.

Ex. 3 : *If* $f(x) = \begin{cases} \sin x, & \text{when } 0 < x < \pi \\ 0, & \text{when } x < 0 \text{ or } x > \pi \end{cases}$ ***(May 2009)***

then prove that

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \lambda x + \cos [\lambda (\pi - x)]}{1-\lambda^2}\, d\lambda.$$

Hence deduce that $\displaystyle\int_{0}^{\infty} \frac{\cos \lambda \pi/2}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}$.

Sol. : Here f(x) is defined over the interval $-\infty < x < \infty$ and is neither an even function nor an odd function, hence the Fourier transform of f(x) is given by [result (8)],

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \int_{0}^{\pi} \sin u\, e^{-i\lambda u}\, du$$

$$= \left[\frac{e^{-i\lambda u}}{(-i\lambda)^2 + 1} (-i\lambda \sin u - \cos u)\right]_0^{\pi}$$

$$= \left[\frac{e^{-i\lambda \pi}}{-\lambda^2 + 1} (-\cos \pi) - \frac{1}{-\lambda^2+1} (-\cos 0)\right]$$

$$= \frac{e^{-i\lambda\pi} + 1}{1-\lambda^2} = \frac{(1 + \cos \lambda\pi) - i \sin \lambda\pi}{1-\lambda^2} \qquad \text{... (i)}$$

Using result (9), inverse transform of $F(\lambda)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{(1+\cos \lambda\pi) - i \sin \lambda \pi}{1-\lambda^2}\right] (\cos \lambda x + i \sin \lambda x)\, d\lambda$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} \left[\frac{1+\cos \lambda\pi}{1-\lambda^2}\right] \cos \lambda x\, d\lambda + i \int_{-\infty}^{\infty} \left[\frac{1+\cos\lambda\pi}{1-\lambda^2}\right] \sin \lambda x\, d\lambda \right.$$

$$\left. - i \int_{-\infty}^{\infty} \frac{\sin \lambda\pi \cos \lambda x}{1-\lambda^2}\, d\lambda + \int_{-\infty}^{\infty} \frac{\sin \lambda\pi \sin \lambda x}{1-\lambda^2}\, d\lambda \right]$$

$$= \frac{1}{2\pi}\left[2\int_0^\infty \left(\frac{1+\cos\lambda\pi}{1-\lambda^2}\right)\cos\lambda x\, d\lambda + 2\int_0^\infty \frac{\sin\lambda\pi\sin\lambda x}{1-\lambda^2}\, d\lambda\right]$$

$$\left\{\begin{array}{l} \because \left(\dfrac{1+\cos\lambda\pi}{1-\lambda^2}\right)\cos\lambda x \text{ is an even function of } \lambda,\ \sin\lambda\pi\cos\lambda x \text{ is an odd function of } \lambda \\ \text{and } \left(\dfrac{1+\cos\lambda\pi}{1-\lambda^2}\right)\sin\lambda x \text{ is an odd function of } \lambda,\ \sin\lambda\pi\sin\lambda x \text{ is an even function of } \lambda \end{array}\right.$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x + \cos\lambda\pi\cos\lambda x + \sin\lambda\pi\sin\lambda x}{1-\lambda^2}\, d\lambda$$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x + \cos[\lambda(\pi - x)]}{1-\lambda^2}\, d\lambda \qquad \ldots \text{(ii)}$$

Putting $x = \frac{\pi}{2}$ in (ii), we obtain

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda\,\pi/2 + \cos[\lambda(\pi - \pi/2)]}{1-\lambda^2}\, d\lambda$$

$$\sin\frac{\pi}{2} = \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda\,\pi/2}{1-\lambda^2}\, d\lambda \qquad \therefore \int_0^\infty \frac{\cos\lambda\,\pi/2}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}$$

which is the required deduction.

Ex. 4 : *By considering Fourier sine and cosine integrals of e^{-mx} (m > 0), prove that*

(a) $\displaystyle\int_0^\infty \frac{\lambda\sin\lambda x}{\lambda^2 + m^2}\, d\lambda = \frac{\pi}{2} e^{-mx}, \qquad m > 0,\ x > 0;$ *and*

(b) $\displaystyle\int_0^\infty \frac{\cos\lambda x}{\lambda^2 + m^2}\, d\lambda = \frac{\pi}{2m} e^{-mx}, \qquad m > 0,\ x > 0.$ ***(Nov. 2014, May 2016)***

Sol. : Let $f(x) = e^{-mx}$, m > 0, x > 0, then since f(x) is defined in the half range $0 < x < \infty$, the function f(x) can have either a Fourier sine transform or a Fourier cosine transform.

(a) Taking Fourier sine transform [using result (12)] of $f(x) = e^{-mx}$, we have

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du = \int_0^\infty e^{-mu}\sin\lambda u\, du$$

$$= \left[\frac{e^{-mu}}{m^2+\lambda^2}(-m\sin\lambda u - \lambda\cos\lambda u)\right]_0^\infty = \frac{\lambda}{m^2+\lambda^2} \qquad \ldots \text{(i)}$$

Using result (13), inverse sine transform of $F_S(\lambda)$ is given by;

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{m^2+\lambda^2}\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{m^2+\lambda^2}\, d\lambda \qquad \ldots \text{(ii)}$$

The result (ii) can be expressed as

$$\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2 + m^2}\, d\lambda = \frac{\pi}{2}\, f(x) = \frac{\pi}{2}\, e^{-mx}, \qquad x > 0,\ m > 0 \qquad \text{... (iii)}$$

(b) Taking Fourier cosine transform [using result (10)] of $f(x) = e^{-mx}$, we have

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du = \int_0^\infty e^{-mu} \cos \lambda u\, du$$

$$= \left[\frac{e^{-mu}}{m^2 + \lambda^2}(-\, m \cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty = \frac{m}{m^2 + \lambda^2} \qquad \text{... (iv)}$$

Using result (11), inverse cosine transform of $F_C(\lambda)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{m}{m^2 + \lambda^2} \cos \lambda x\, d\lambda$$

$$= \frac{2m}{\pi}\int_0^\infty \frac{\cos \lambda x}{m^2 + \lambda^2}\, d\lambda \qquad \text{... (v)}$$

The result (v) can be expressed as

$$\int_0^\infty \frac{\cos \lambda x}{\lambda^2 + m^2}\, d\lambda = \frac{\pi}{2m}\, f(x) = \frac{\pi}{2m}\, e^{-mx}.$$

Ex. 5 : *Find the Fourier cosine integral representation for the function*

$$f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. : Using result (10), Fourier cosine transform of f(x) is given by;

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du = \int_0^a u^2 \cos \lambda u\, du$$

$$= \left[u^2\left(\frac{\sin \lambda u}{\lambda}\right) - (2u)\left(-\frac{\cos \lambda u}{\lambda^2}\right) + (2)\left(-\frac{\sin \lambda u}{\lambda^3}\right)\right]_0^a$$

(using generalised rule of integration by parts)

$$= \frac{a^2 \sin \lambda a}{\lambda} + \frac{2a \cos \lambda a}{\lambda^2} - \frac{2 \sin \lambda a}{\lambda^3}$$

and using result (11), corresponding inverse transform is given by;

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{a^2 \sin a\lambda}{\lambda} + \frac{2a \cos a\lambda}{\lambda^2} - \frac{2 \sin a\lambda}{\lambda^3}\right] \cos \lambda x\, d\lambda.$$

Ex. 6 : *Using Fourier integral representation, show that*

(a) $\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x$, where $x > 0$ ***(Dec. 2005, 2008)***

(b) $$\int_0^{\infty} \frac{\cos \frac{\pi\lambda}{2} \cos \lambda x}{1 - \lambda^2}\, d\lambda = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

(c) $$\int_0^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1 + \lambda^2}\, d\lambda = \begin{cases} 0, & x < 0 \\ \frac{\pi}{2}, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$$ ***(Dec. 2004, May 2007)***

(d) $$\int_0^{\infty} \frac{1 - \cos \pi\lambda}{\lambda} \sin \lambda x\, d\lambda = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}.$$

Sol. : (a) To prove the result, consider R.H.S. which defines the function

$$f(x) = \frac{\pi}{2} e^{-x} \cos x, \quad x > 0 \qquad \text{... (i)}$$

Here the function f(x) is defined in the half range $0 < x < \infty$ and since the integral on L.H.S. involves a term $\sin \lambda x$, indicates that, we are required to find the Fourier sine transform of f(x).

Thus from result (12), we have

$$F_S(\lambda) = \int_0^{\infty} f(u) \sin \lambda u\, du = \int_0^{\infty} \frac{\pi}{2} e^{-u} \cos u \sin \lambda u\, du \qquad \text{[from (i)]}$$

$$= \frac{\pi}{4} \int_0^{\infty} e^{-u} [\sin (\lambda + 1) u + \sin (\lambda - 1) u]\, du$$

$$= \frac{\pi}{4} \left[\int_0^{\infty} e^{-u} \sin (\lambda + 1) u\, du + \int_0^{\infty} e^{-u} \sin (\lambda - 1) u\, du \right]$$

$$= \frac{\pi}{4} \left[\frac{e^{-u}}{1 + (\lambda + 1)^2} \{- \sin (\lambda + 1) u - (\lambda + 1) \cos (\lambda + 1) u\} + \frac{e^{-u}}{1 + (\lambda - 1)^2} \{- \sin (\lambda - 1) u - (\lambda - 1) \cos (\lambda - 1) u\} \right]_0^{\infty}$$

$$= \frac{\pi}{4} \left[\frac{\lambda + 1}{\lambda^2 + 2\lambda + 2} + \frac{\lambda - 1}{\lambda^2 - 2\lambda + 2} \right] = \frac{\pi}{4} \left[\frac{2\lambda^3}{(\lambda^2 + 2)^2 - 4\lambda^2} \right]$$

$$= \frac{\pi}{2} \frac{\lambda^3}{\lambda^4 + 4}$$

Now using result (13), inverse sine transform of $F_S(\lambda)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_S(\lambda) \sin \lambda x\, d\lambda = \frac{2}{\pi} \int_0^{\infty} \frac{\pi}{2} \frac{\lambda^3}{\lambda^4 + 4} \sin \lambda x\, d\lambda$$

$$= \int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4}\, d\lambda \qquad \text{... (iii)}$$

The result (iii) can be expressed as

$$\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} = f(x) = \frac{\pi}{2} e^{-x} \cos x \qquad \text{[from (i)]}$$

which is the required result.

(b) To prove the result, consider the function

$$f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases} \qquad \text{... (i)}$$

Here f(x) is an even function of x defined in the interval $-\infty < x < \infty$ and since cosine terms are present in the integral, we find Fourier cosine transform.

Thus from result (10), we have

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du = \int_0^{\pi/2} \frac{\pi}{2} \cos u \cos \lambda u \, du + \int_{\pi/2}^\infty (0) \cos \lambda u \, du$$

$$= \frac{\pi}{4} \int_0^{\pi/2} [\cos (\lambda + 1) u + \cos (\lambda - 1) u] \, du$$

$$= \frac{\pi}{4} \left[\frac{\sin (\lambda + 1) u}{\lambda + 1} + \frac{\sin (\lambda - 1) u}{\lambda - 1} \right]_0^{\pi/2} \qquad \{\because 2 \cos A \cos B = \cos (A + B) + \cos (A - B)\}$$

$$= \frac{\pi}{4} \left[\frac{\sin (\lambda + 1) \pi/2}{\lambda + 1} + \frac{\sin (\lambda - 1) \pi/2}{\lambda - 1} \right]$$

$$= \frac{\pi}{4} \left[\frac{\cos \lambda\pi/2}{\lambda + 1} - \frac{\cos \lambda\pi/2}{\lambda - 1} \right] \qquad \left\{\because \sin \frac{(\lambda + 1)\pi}{2} = \cos \frac{\lambda\pi}{2}\right\}$$

$$= \frac{\pi}{4} \left[\frac{2 \cos \lambda\pi/2}{1 - \lambda^2} \right] = \frac{\pi}{2} \frac{\cos \lambda\pi/2}{1 - \lambda^2} \qquad \text{... (ii)}$$

Using inverse Fourier cosine transform given by result (11), we have

$$f(x) = \frac{2}{\pi} \int_0^\infty F_C(\lambda) \cos \lambda x \, d\lambda = \frac{2}{\pi} \int_0^\infty \frac{\pi}{2} \frac{\cos \lambda \pi/2}{1 - \lambda^2} \cos \lambda x \, d\lambda$$

$$= \int_0^\infty \frac{\cos \frac{\lambda\pi}{2} \cos \lambda x}{1 - \lambda^2} \, d\lambda \qquad \text{... (iii)}$$

The result (iii) can be expressed as

$$\int_0^\infty \frac{\cos \frac{\lambda\pi}{2} \cos \lambda x}{1 - \lambda^2} \, d\lambda = f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

which is the required result.

(c) To prove the result, consider the function

$$f(x) = \begin{cases} 0, & x < 0 \\ \pi e^{-x}, & x > 0 \end{cases} \quad \text{... (i)}$$

This function is defined in $-\infty < x < \infty$ and since the terms $\sin \lambda x$ and $\cos \lambda x$ are present in the integrand, we find general Fourier transform. Also note that f(x) is neither an even function nor an odd function.

Thus from result (8), we have

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \int_{-\infty}^{0} 0\, e^{-i\lambda u}\, du + \int_{0}^{\infty} \pi\, e^{-u}\, e^{-i\lambda u}\, du$$

$$= \pi \int_{0}^{\infty} e^{-(1+i\lambda)u}\, du = \pi \left[\frac{e^{-(1+i\lambda)u}}{-(1+i\lambda)}\right]_0^{\infty}$$

$$= \pi \left[\frac{1}{1+i\lambda}\right] = \pi \left[\frac{1-i\lambda}{1+\lambda^2}\right] \quad \text{... (ii)}$$

Now using result (9), inverse Fourier transform of $F(\lambda)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \left(\frac{1-i\lambda}{1+\lambda^2}\right) [\cos \lambda x + i \sin \lambda x]\, d\lambda$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} + i\, \frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2}\right] d\lambda$$

$$= \frac{1}{2} \left[2 \int_{0}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}\, d\lambda\right] \qquad \left\{ \because \int_{-\infty}^{\infty} \frac{\lambda \cos \lambda x + \sin \lambda}{1+\lambda^2} = 0 \right.$$

since integrand is odd function of λ (refer Ex. 2)

$$= \int_{0}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}\, d\lambda \quad \text{... (iii)}$$

The result (iii) can be expressed as

$$\int_{0}^{\infty} \frac{\cos \lambda x + \lambda\, \sin \lambda x}{1+\lambda^2}\, d\lambda = f(x) = \begin{cases} 0, & x < 0 \\ \pi e^{-x}, & x > 0 \end{cases} \quad \text{... (iv)}$$

To find the value of the integral at x = 0 i.e. f(0), put x = 0 in (iv), we get

$$f(0) = \int_{0}^{\infty} \frac{1}{1+\lambda^2}\, d\lambda = \left[\tan^{-1} \lambda\right]_0^{\infty} = \frac{\pi}{2} \quad \text{... (v)}$$

Hence from (iv) and (v), we get

$$\int_{0}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2}\, d\lambda = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$$

which is the required result.

(d) To prove the result, consider the function

$$f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases} \qquad \text{... (i)}$$

Here the function is defined in $0 < x < \infty$ and since the integral on L.H.S. involves a term $\sin \lambda x$, we find Fourier sine transform of f(x).

Thus from result (12), we have

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du = \int_0^\pi \frac{\pi}{2} \sin \lambda u \, du + \int_\pi^\infty (0) \sin \lambda u \, du$$

$$= \frac{\pi}{2}\left[\frac{-\cos \lambda u}{\lambda}\right]_0^\pi = \frac{\pi}{2}\left[\frac{1-\cos \lambda\pi}{\lambda}\right] \qquad \text{... (ii)}$$

Now using result (13), inverse sine transform of $F_S(\lambda)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\left[\frac{1-\cos \lambda\pi}{\lambda}\right] \sin \lambda x \, d\lambda$$

$$= \int_0^\infty \frac{1-\cos \lambda\pi}{\lambda} \sin \lambda x \, d\lambda \qquad \text{... (iii)}$$

Result (iii) can be expressed as

$$\int_0^\infty \frac{1-\cos \lambda\pi}{\lambda} \sin \lambda x \, d\lambda = f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

which is the required result.

Type 2 : Problems on Fourier Transforms

Ex. 7 : *Find the Fourier transforms of*

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Also graph f(x) and its Fourier transform for a = 3.

Sol. : The Fourier transform of f(x) is [refer result (8)]

$$F(\lambda) = \int_{-\infty}^\infty f(u)\, e^{-i\lambda u} \, du = \int_{-\infty}^a (1)\, e^{-i\lambda u} \, du = \left[\frac{e^{-i\lambda u}}{-i\lambda}\right]_{-a}^a$$

$$= \frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda} = \frac{2 \sin \lambda a}{\lambda}, \ \lambda \neq 0 . \qquad \text{... (i)}$$

For $\lambda = 0$, we obtain $F(\lambda) = 2a$.

The graphs of f(x) and $F(\lambda)$ for a = 3 are shown in Figs. 3.3 and 3.4 respectively.

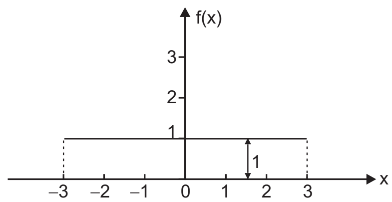

Fig. 3.3

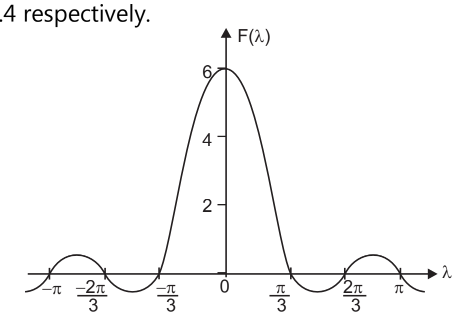

Fig. 3.4

Ex. 8 : *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

and hence evaluate $\displaystyle\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2}\, dx$ ***(Dec. 2004, 2006, 2010, 2012, 2016, May 2008, 2012)***

Sol. : The function f(x) is given by

$$f(x) = \begin{cases} 1 - x^2, & -1 \le x \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{... (i)}$$

This shows that $f(-x) = f(x)$ i.e. f(x) is an even function in the interval $-\infty < x < \infty$. Hence by result (10), the Fourier cosine transform of f(x) is

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du = \int_0^1 (1 - u^2) \cos \lambda u\, du + \int_1^\infty (0) \cos \lambda u\, du \quad \text{[from (i)]}$$

$$= \left[(1 - u^2)\left(\frac{\sin \lambda u}{\lambda}\right) - (-2u)\left(\frac{-\cos \lambda u}{\lambda^2}\right) + (-2)\left(\frac{-\sin \lambda u}{\lambda^3}\right)\right]_0^1$$

$$= 2\left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \quad \text{... (ii)}$$

By using inverse transform [result (11)], the Fourier integral representation is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty 2\left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \cos \lambda x\, d\lambda$$

$$= \frac{4}{\pi}\int_0^\infty \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \cos \lambda x\, d\lambda \quad \text{... (iii)}$$

The result (iii) can be expressed as

$$\int_0^\infty \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \cos \lambda x\, d\lambda = \frac{\pi}{4}\, f(x) = \begin{cases} \frac{\pi}{4}(1 - x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{... (iv)}$$

Putting $x = \frac{1}{2}$, which lies in $-1 \le x \le 1$ i.e. in $|x| \le 1$, we get

$$\int_0^\infty \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \cos \frac{\lambda}{2}\, d\lambda = \frac{\pi}{4}\left(1 - \frac{1}{4}\right) = \frac{3\pi}{16} \quad \text{... (v)}$$

Since variable of integration in definite integral is of no importance, replacing λ by x in (v), we have

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) \cos \frac{x}{2}\, dx = \frac{3\pi}{16}$$

or $$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos \frac{x}{2}\, dx = -\frac{3\pi}{16}.$$

Ex. 9 : *Find the Fourier cosine transform of the function*

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. : Using result (10), cosine transform of f(x) is

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du = \int_0^a \cos u \cos \lambda u\, du + \int_a^\infty (0) \cos \lambda u\, du$$

$$= \frac{1}{2} \int_0^a [\cos(\lambda+1)u + \cos(\lambda-1)u]\, du$$

$$= \frac{1}{2}\left[\frac{\sin(\lambda+1)u}{\lambda+1} + \frac{\sin(\lambda-1)u}{\lambda-1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(\lambda+1)a}{\lambda+1} + \frac{\sin(\lambda-1)a}{\lambda-1}\right]$$

Note : If we use result [10 (a)] for Fourier cosine transform,

$$F_C(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \lambda u\, du$$

we get the result

$$F_C(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2}\left[\frac{\sin(\lambda+1)a}{\lambda+1} + \frac{\sin(\lambda-1)a}{\lambda-1}\right].$$

Ex. 10 : *Show that :*

(a) the Fourier transform of $f(x) = e^{-|x|}$ *is* $\frac{2}{1+\lambda^2}$. ***(Nov. 2015)***

(b) the Fourier cosine transform of $f(x) = e^{-x} + e^{-2x}$ $(x > 0)$ *is* $\frac{6+3\lambda^2}{4+5\lambda^2+\lambda^4}$

Sol. : (a) Fourier transform of $f(x) = e^{-|x|}$ in the interval $-\infty < x < \infty$ is given by

$$F(\lambda) = \int_{-\infty}^\infty f(u)\, e^{-i\lambda u}\, du = \int_{-\infty}^\infty e^{-|u|}\, e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^\infty e^{-|u|} (\cos \lambda u - i \sin \lambda u)\, du$$

$$= \int_{-\infty}^\infty e^{-|u|} \cos \lambda u\, du - i \int_{-\infty}^\infty e^{-|u|} \sin \lambda u\, du$$

Since the integrand in the second integral is odd and hence integral is zero.

$$\therefore \qquad F(\lambda) = 2\int_0^\infty e^{-u}\cos\lambda u\, du = 2\left[\frac{e^{-u}}{1+\lambda^2}(-\cos\lambda u + \lambda\sin\lambda u)\right]_0^\infty$$

$$= \frac{2}{1+\lambda^2}$$

Note : We can also obtain Fourier cosine transform of f(x) directly, since it is an even function.

(d) Here $\quad f(x) = e^{-x} + e^{-2x},\ 0 < x < \infty.$

The Fourier cosine transform of f(x) is

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du = \int_0^\infty (e^{-u} + e^{-2u})\cos\lambda u\, du$$

$$= \left[\frac{e^{-u}}{1+\lambda^2}(-\cos\lambda u + \lambda\sin\lambda u) + \frac{e^{-2u}}{4+\lambda^2}(-2\cos\lambda u + \lambda\sin\lambda u)\right]_0^\infty$$

$$= \frac{1}{1+\lambda^2} + \frac{2}{4+\lambda^2} = \frac{6+3\lambda^2}{4+5\lambda^2+\lambda^4}.$$

Ex. 11 : *Show that :*

(a) the Fourier transform of $f(x) = e^{-x^2/2}$ *is* $e^{-\lambda^2/2}$ ***(Dec. 2008)***

(b) the Fourier cosine transform of $f(x) = e^{-x^2}$ *is* $\frac{1}{\sqrt{2}}e^{-\lambda^2/4}$ ***(May 2008)***

Sol. : (a) Here $f(x) = e^{-x^2/2}$ is an even function of x defined in the interval $-\infty < x < \infty$, hence to obtain the required result, we use formula (10 a) for the Fourier cosine transform.

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\lambda u\, du = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u^2/2}\cos\lambda u\, du \qquad \text{... (i)}$$

Let
$$I(\lambda) = \int_0^\infty e^{-u^2/2}\cos\lambda u\, du \qquad \text{... (ii)}$$

$$\therefore \qquad \frac{dI(\lambda)}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda} e^{-u^2/2}\cos\lambda u\, du \qquad \left[\begin{array}{l}\text{Using rule of differentiation}\\ \text{under the integral sign}\end{array}\right]$$

$$= \int_0^\infty -(u\,e^{-u^2/2})\sin\lambda u\, du$$

Integrating by parts, we have

$$= \left[e^{-u^2/2}\sin\lambda u\right]_0^\infty - \int_0^\infty e^{-u^2/2}\lambda\cos\lambda u\, du \qquad \left\{\begin{array}{l}\because \int -u\,e^{-u^2/2}\,du,\ [u^2/2 = t]\\ = -\int e^{-t}\,dt = +e^{-t} = +e^{-u^2/2}\end{array}\right.$$

$$= 0 - \lambda\int_0^\infty e^{-u^2/2}\cos\lambda u\, du.$$

$$= -\lambda\ I(\lambda), \qquad \text{where, } I(\lambda) = \int_0^\infty e^{-u^2/2}\cos\lambda u\, du \qquad \text{... (iii)}$$

$\therefore \quad \dfrac{dI}{d\lambda} = -\lambda I$ (In variable separable form)

$\therefore \quad I = A e^{-\lambda^2/2}$ ... (iv)

To find constant A, put $\lambda = 0$ in (iv), then

$$[I(\lambda)]_{\lambda=0} = A e^0 = A \qquad \text{... (v)}$$

The value of $[I(\lambda)]$ at $\lambda = 0$ is obtained by putting $\lambda = 0$ in (ii),

$$[I(\lambda)]_{\lambda=0} = \int_0^\infty e^{-u^2/2} du, \text{ putting } u^2 = 2t \text{ or } u = \sqrt{2}\, t^{1/2}$$

$$= \int_0^\infty e^{-t} \frac{1}{\sqrt{2}} t^{-1/2} dt = \frac{1}{\sqrt{2}} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{\sqrt{2}} \Gamma(1/2) = \frac{1}{\sqrt{2}} \sqrt{\pi} = \sqrt{\frac{\pi}{2}}.$$

Thus from (v), we have $A = \sqrt{\dfrac{\pi}{2}}$ and hence from (iv) and (ii), we have

$$I(\lambda) = \int_0^\infty e^{-u^2/2} \cos \lambda u = \sqrt{\frac{\pi}{2}} e^{-\lambda^2/2} \qquad \text{... (vi)}$$

Substituting (vi) in (i), we get

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\pi}{2}} e^{-\lambda^2/2} = e^{-\lambda^2/2}$$

Note : If we use result (10) for finding Fourier cosine transform, we get

$$F(\lambda) = \sqrt{\frac{\pi}{2}} e^{-\lambda^2/2}$$

(b) Here $f(x) = e^{-x^2}$ is defined in the interval $-\infty < x < \infty$. To obtain required result, we use formula (10 a) for Fourier cosine transform.

$$F_C(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \lambda u \, du = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2} \cos \lambda u \, du \qquad \text{... (i)}$$

Let $\quad I(\lambda) = \displaystyle\int_0^\infty e^{-u^2} \cos \lambda u \, du$ ... (ii)

$$I'(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} e^{-u^2} \cos \lambda u \, du = \int_0^\infty (-u\, e^{-u^2}) \sin \lambda u \, du$$

[Using the rule of differentiation under the integral sign]

Integrating by parts, we have

$$= \left[\frac{1}{2} e^{-u^2} \sin \lambda u\right]_0^\infty - \int_0^\infty \frac{1}{2} e^{-u^2} \lambda \cos \lambda u \, du \qquad \left\{ \begin{aligned} &\because \int -u\, e^{-u^2} du \quad [-u^2 = t] \\ &= \int \frac{1}{2} e^t dt = \frac{1}{2} e^{-u^2} \end{aligned} \right.$$

$$= 0 - \frac{\lambda}{2} \int_0^\infty e^{-u^2} \cos \lambda u \, du$$

$$= -\frac{\lambda}{2} \, I(\lambda), \qquad \text{where, } I(\lambda) = \int_0^\infty e^{-u^2} \cos \lambda u \, du \quad \dots \text{(iii)}$$

$$\therefore \qquad \frac{dI}{I} = -\frac{\lambda}{2} \, d\lambda \qquad \text{(In variable separable form)}$$

Integrating, we have

$$I = A\, e^{-\lambda^2/4} \qquad \dots \text{(iv)}$$

To find the constant A, put $\lambda = 0$ in (iv), then

$$[I(\lambda)]_{\lambda = 0} = A \qquad \dots \text{(v)}$$

The value of $[I(\lambda)]_{\lambda = 0}$ is obtained by putting $\lambda = 0$ in (ii),

$$[I(\lambda)]_{\lambda = 0} = \frac{\sqrt{\pi}}{2}, \qquad \text{(refer to similar part of previous example)}$$

Thus from (v), we have $A = \frac{\sqrt{\pi}}{2}$; and hence from (iv) and (ii), we have

$$I(\lambda) = \int_0^\infty e^{-u^2} \cos \lambda u \, du = \frac{\sqrt{\pi}}{2} e^{-\lambda^2/4} \qquad \dots \text{(vi)}$$

Substituting (iv) in (i), we get

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \, \frac{\sqrt{\pi}}{2} \, e^{-\lambda^2/4} = \frac{1}{\sqrt{2}} \, e^{-\lambda^2/4}.$$

Note : If we use result (10) for finding Fourier cosine transform, we get

$$F(\lambda) = \frac{\sqrt{\pi}}{2} \, e^{-\lambda^2/4}$$

Ex. 12 : *Find the Fourier sine transform of* $\frac{e^{-ax}}{x}$ *and hence evaluate* $\int_0^\infty \tan^{-1}\frac{x}{a} \sin x \, dx.$ ***(May 2006)***

Sol. : Using result (12), we have

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du = \int_0^\infty \frac{e^{-au}}{u} \sin \lambda u \, du \qquad \dots \text{(i)}$$

Let

$$I(\lambda) = \int_0^\infty \frac{e^{-au}}{u} \sin \lambda u \, du \qquad \dots \text{(ii)}$$

$$\therefore \quad \frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda} \frac{e^{-au}}{u} \sin \lambda u \, du \qquad \text{[Using DUIS rule]}$$

$$= \int_0^\infty \frac{e^{-au}}{u} (u \cos \lambda u) \, du = \int_0^\infty e^{-au} \cos \lambda u \, du$$

$$= \left[\frac{e^{-au}}{a^2 + \lambda^2} (- a \cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty = \frac{a}{\lambda^2 + a^2}$$

$$\therefore \quad \frac{dI}{d\lambda} = \frac{a}{\lambda^2 + a^2} \qquad \text{... (iii)}$$

Integrating, we have

$$I(\lambda) = \int \frac{a}{\lambda^2 + a^2} \, d\lambda + A = \tan^{-1} \frac{\lambda}{a} + A \qquad \text{... (iv)}$$

To find constant A, we put $\lambda = 0$ in (iv), we get

$$[I(\lambda)]_{\lambda = 0} = 0 + A$$

or
$$\left[\int_0^\infty \frac{e^{-au}}{u} \sin \lambda u \, du\right]_{\lambda = 0} = 0 + A \quad \Rightarrow \quad \boxed{A = 0} \qquad \text{... (by (ii))}$$

$\therefore$ (iv) $\Rightarrow$
$$I(\lambda) = \int_0^\infty \frac{e^{-au}}{u} \sin \lambda u \, du = \tan^{-1} \frac{\lambda}{a} \qquad \text{... (v)}$$

Substituting (v) in (i), we get

$$F_S(\lambda) = \tan^{-1} \frac{\lambda}{a} \qquad \text{... (vi)}$$

Using result (13), inverse sine transform is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda = \frac{2}{\pi} \int_0^\infty \tan^{-1} \frac{\lambda}{a} \sin \lambda x \, d\lambda$$

$$\therefore \quad \int_0^\infty \tan^{-1} \frac{\lambda}{a} \sin \lambda x \, d\lambda = \frac{\pi}{2} f(x) = \frac{\pi}{2} \frac{e^{-ax}}{x} \qquad \text{... (vii)}$$

Putting $x = 1$ in (vii), we get

$$\int_0^\infty \tan^{-1} \frac{\lambda}{a} \sin \lambda \, d\lambda = \frac{\pi}{2} e^{-a}$$

or
$$\int_0^\infty \tan^{-1} \frac{x}{a} \sin x \, dx = \frac{\pi}{2} e^{-a}.$$

Ex. 13 : *Find the Fourier sine and cosine transforms of the function $f(x) = e^{-x}$ and hence show that :*

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m} \text{ and } \int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}.$$ ***(May 2009)***

Sol. : Fourier Cosine Transform : Using result (10), we have

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du = \int_0^\infty e^{-u} \cos \lambda u \, du$$

$$= \left[\frac{e^{-u}}{1+\lambda^2}(-\cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty$$

$$= \frac{1}{1+\lambda^2} \quad \text{... (i)}$$

Now using result (11), inverse cosine transform is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x \, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2} \cos \lambda x \, d\lambda$$

$$\therefore \quad \int_0^\infty \frac{\cos \lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-x} \quad \text{... (ii)}$$

Putting x = m in (ii), we have

$$\int_0^\infty \frac{\cos \lambda m}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} e^{-m} \quad \text{... (iii)}$$

Since variable of integration is immaterial in the definite integral,

$$\text{hence,} \quad \int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}.$$

Fourier Sine Transform : Using result (12), we have

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du = \int_0^\infty e^{-u} \sin \lambda u \, du$$

$$= \left[\frac{e^{-u}}{1+\lambda^2}(-\sin \lambda u - \lambda \cos \lambda u)\right]_0^\infty$$

$$= \frac{\lambda}{1+\lambda^2} \quad \text{... (i)}$$

Now using result (13), inverse sine transform is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2} \sin \lambda x \, d\lambda$$

$$\therefore \quad \int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-x} \quad \text{... (ii)}$$

Putting x = m in (ii), we have

$$\int_0^{\infty} \frac{\lambda \sin \lambda m}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} e^{-m} \qquad \ldots \text{(iii)}$$

Since variable of integration is immaterial in the definite integral,

hence $$\int_0^{\infty} \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}.$$

Ex. 14 : *Find the Fourier sine and cosine transforms of the following function*

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases}$$ ***(May 2011)***

Sol. : Fourier Cosine Transform : Using result (10), we have

$$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u\, du$$

$$= \int_0^{1} (u) \cos \lambda u\, du + \int_1^{2} (2-u) \cos \lambda u\, du + \int_2^{\infty} (0) \cos \lambda u\, du$$

$$= \left[u \frac{\sin \lambda u}{\lambda} + \frac{\cos \lambda u}{\lambda^2}\right]_0^1 + \left[(2-u)\frac{\sin \lambda u}{\lambda} - \frac{\cos \lambda u}{\lambda^2}\right]_1^2$$

$$= \left[\frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2} - \frac{1}{\lambda^2}\right] + \left[-\frac{\cos 2\lambda}{\lambda^2} - \frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2}\right]$$

$$= \frac{2\cos \lambda - (1+\cos 2\lambda)}{\lambda^2} = \frac{2\cos \lambda\,(1-\cos \lambda)}{\lambda^2}.$$

Fourier Sine Transform : Using result (12), we have

$$F_S(\lambda) = \int_0^{\infty} f(u) \sin \lambda u\, du$$

$$= \int_0^{1} (u) \sin \lambda u\, du + \int_1^{2} (2-u) \sin \lambda u\, du + \int_2^{\infty} (0) \sin \lambda u\, du$$

$$= \left[-u \frac{\cos \lambda u}{\lambda} + \frac{\sin \lambda u}{\lambda^2}\right]_0^1 + \left[-(2-u)\frac{\cos \lambda u}{\lambda} - \frac{\sin \lambda u}{\lambda^2}\right]_1^2$$

$$= \left[-\frac{\cos \lambda}{\lambda} + \frac{\sin \lambda}{\lambda^2}\right] + \left[-\frac{\sin 2\lambda}{\lambda^2} + \frac{\cos \lambda}{\lambda} + \frac{\sin \lambda}{\lambda^2}\right]$$

$$= \frac{2\sin \lambda - \sin 2\lambda}{\lambda^2} = \frac{2\sin \lambda\,(1-\cos \lambda)}{\lambda^2}.$$

Ex. 15 : *Find the Fourier sine and cosine transforms of the function f(x) =* x^{m-1}. ***(May 2005, Dec. 2004)***

Sol. : The given function $f(x) = x^{m-1}$ using results (10) and (12), Fourier cosine and sine transforms are given by

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du = \int_0^\infty u^{m-1}\cos\lambda u\, du \quad \text{... (i)}$$

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du = \int_0^\infty u^{m-1}\sin\lambda u\, du \quad \text{... (ii)}$$

Now by definition of Gamma function, we have

$$\overline{|m} = \int_0^\infty e^{-x}\, x^{m-1}\, dx$$

Putting $x = i\lambda u, \quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$, we get

$$\overline{|m} = \int_0^\infty e^{-i\lambda u}(i\lambda u)^{m-1}(i\lambda)\, du = (e^{i\pi/2})^m\, \lambda^m \int_0^\infty u^{m-1} e^{-i\lambda u}\, du$$

$$\therefore \quad \int_0^\infty u^{m-1} e^{-i\lambda u}\, du = \frac{\overline{|m}}{\lambda^m}\, e^{-im\pi/2}$$

$$\therefore \quad \int_0^\infty u^{m-1}(\cos\lambda u - i\sin\lambda u)\, du = \frac{\overline{|m}}{\lambda^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right)$$

Equating real and imaginary parts on both sides, we get

$$F_C(\lambda) = \int_0^\infty u^{m-1}\cos\lambda u\, du = \frac{\overline{|m}}{\lambda^m}\cos\frac{m\pi}{2}$$

and

$$F_S(\lambda) = \int_0^\infty u^{m-1}\sin\lambda u\, du = \frac{\overline{|m}}{\lambda^m}\sin\frac{m\pi}{2}.$$

Ex. 16 : *Find the Fourier sine transform of* $\frac{1}{x}$. ***(Dec. 2010)***

Sol. : Using result (12), we have

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du = \int_0^\infty \frac{1}{u}\sin\lambda u\, du$$

$$= \int_0^\infty \frac{\lambda}{t}(\sin t)\frac{dt}{\lambda}, \qquad \text{putting } \lambda u = t \text{ or } u = \frac{t}{\lambda} \text{ and } du = \frac{dt}{\lambda}.$$

$$= \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

Note : If we use result 12 (a), we would get

$$F_S(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}.$$

***Ex. 17 :** Find the Fourier cosine transform of $f_1(x) = \frac{1}{1 + x^2}$ and hence find the Fourier sine transform of $f_2(x) = \frac{x}{1 + x^2}$.*

Sol. : We know that Fourier cosine transform of $f_1(x) = \frac{1}{1 + x^2}$ is given by [result (10)],

$$F_C(\lambda) = \int_0^\infty f_1(u) \cos \lambda u \, du = \int_0^\infty \frac{1}{1 + u^2} \cos \lambda u \, du \quad \text{... (i)}$$

Let

$$I(\lambda) = \int_0^\infty \frac{1}{1 + u^2} \cos \lambda u \, du \quad \text{... (ii)}$$

Differentiating both sides w.r.t. λ using the rule of DUIS, we get

$$\frac{dI}{d\lambda} = I'(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} \frac{1}{1 + u^2} \cos \lambda u \, du = \int_0^\infty -\frac{u}{1 + u^2} \sin \lambda u \, du$$

$$= \int_0^\infty -\frac{u^2}{u(1 + u^2)} \sin \lambda u \, du = \int_0^\infty \frac{-(u^2 + 1 - 1)}{u(1 + u^2)} \sin \lambda u \, du$$

$$= \int_0^\infty \left(\frac{1}{u(1 + u^2)} - \frac{1}{u}\right) \sin \lambda u \, du = \int_0^\infty \frac{\sin \lambda u}{u(1 + u^2)} \, du - \int_0^\infty \frac{\sin \lambda u}{u} \, du$$

$$= \int_0^\infty \frac{\sin \lambda u}{u(1 + u^2)} \, du - \int_0^\infty \frac{\sin t}{t} \, dt, \quad \left\{\begin{matrix}\text{Putting } \lambda u = t \text{ in the} \\ \text{second integral}\end{matrix}\right\}$$

$$= \int_0^\infty \frac{\sin \lambda u}{u(1 + u^2)} \, du - \frac{\pi}{2}, \quad \left(\because \int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2}\right) \quad \text{... (iii)}$$

Again differentiating both sides w.r.t. λ using the rule of DUIS, we get

$$I''(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} \frac{\sin \lambda u}{u(1 + u^2)} \, du - 0 = \int_0^\infty \frac{u \cos \lambda u}{u(1 + u^2)} \, du$$

$$= \int_0^\infty \frac{\cos \lambda u}{1 + u^2} \, du = I(\lambda) \quad \text{[from (i)]}$$

$$\therefore \quad I''(\lambda) - I(\lambda) = 0 \quad \text{... (iv)}$$

General solution of (iv) is given by

$$I(\lambda) = A e^{\lambda} + B e^{-\lambda} \quad \text{... (v)}$$

Now to evaluate constants A and B, differentiating (v), w.r.t. λ, we get

$$I'(\lambda) = A e^{\lambda} - B e^{-\lambda} \quad \text{... (vi)}$$

Putting λ = 0 in (v) and (vi), we obtain

$$I(0) = \left[\int_0^\infty \frac{\cos \lambda u}{1 + u^2} du\right]_{\lambda = 0} = A + B \quad \text{[from (ii)] ... (vii)}$$

and

$$I'(0) = \left[\int_0^\infty \frac{\sin \lambda u}{u(1 + u^2)} du - \frac{\pi}{2}\right]_{\lambda = 0} = A - B \quad \text{[from (iii)] ... (viii)}$$

From (vii) and (viii), we obtain

$$A + B = \frac{\pi}{2} \qquad \left\{\because \int_0^\infty \frac{1}{1 + u^2} du = \left[\tan^{-1} u\right]_0^\infty = \frac{\pi}{2}\right\}$$

and

$$A - B = -\frac{\pi}{2} \qquad \left\{\because \int_0^\infty \frac{\sin \lambda u}{u(1 + u^2)} du = 0 \text{ at } \lambda = 0\right\}$$

Solving for A and B, we get A = 0 and B = $\frac{\pi}{2}$ and substituting these values of A and B in result (v), we have

$$I(\lambda) = \frac{\pi}{2} e^{-\lambda}$$

$$\therefore \quad F_C(\lambda) = \int_0^\infty f_1(u) \cos \lambda u \, du = \int_0^\infty \frac{1}{1 + u^2} \cos \lambda u \, du = \frac{\pi}{2} e^{-\lambda} \quad \text{... (ix)}$$

Now to find Fourier sine transform of $f_2(x) = \frac{x}{1 + x^2}$, we differentiate result (ix) with respect to λ, we get

$$-\int_0^\infty \frac{u}{1 + u^2} \sin \lambda u \, du = -\frac{\pi}{2} e^{-\lambda}$$

$$\therefore \quad F_S(\lambda) = \int_0^\infty f_2(u) \sin \lambda u \, du$$

$$= \int_0^\infty \frac{u}{1 + u^2} \sin \lambda u \, du = \frac{\pi}{2} e^{-\lambda}$$

Type 3 : Problems on Inverse Fourier Transforms

Ex. 18 : *Using inverse sine transform, find f(x) if*

$$F_S(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$$

(Dec. 2004, 2012, May 2007, 2008)

Sol. : By result (13), inverse sine transform of $F_S(\lambda)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda = \frac{2}{\pi} \int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \sin \lambda x \, d\lambda \quad \text{... (i)}$$

Let

$$I(x) = \int_0^\infty \frac{e^{-a\lambda}}{\lambda} \sin \lambda x \, d\lambda \quad \text{... (ii)}$$

$$\therefore \qquad I'(x) = \int_0^\infty \frac{\partial}{\partial x} \frac{e^{-a\lambda}}{\lambda} \sin \lambda x \, d\lambda \qquad \text{[Using DUIS rule]}$$

$$= \int_0^\infty e^{-a\lambda} \cos \lambda x \, d\lambda = \left[\frac{e^{-a\lambda}}{a^2 + x^2} (-a \cos \lambda x + x \sin \lambda x)\right]_0^\infty$$

$$= \frac{a}{a^2 + x^2} \qquad \text{... (iii)}$$

Integrating, we get

$$I(x) = \int \frac{a}{x^2 + a^2} \, dx + A = \tan^{-1}\frac{x}{a} + A \qquad \text{... (iv)}$$

Putting x = 0, we get

$$[I(x)]_{x=0} = A \qquad \text{... (v)}$$

The value of $[I(x)]_{x=0}$ is obtained from (ii), when x = 0.

$$[I(x)]_{x=0} = \left[\int_0^\infty \frac{e^{-a\lambda}}{\lambda} \sin \lambda x \, d\lambda\right]_{x=0} = 0 \qquad \therefore \; A = 0$$

Hence from (iv), we have

$$I(x) = \tan^{-1}\frac{x}{a} \qquad \text{... (vi)}$$

Thus from (i) and using (vi), we have

$$f(x) = \frac{2}{\pi} I(x) = \frac{2}{\pi} \tan^{-1}\frac{x}{a}.$$

Note : If we use result (13 a), we get $f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{a}$.

Ex. 19 : *What is the function f(x), whose Fourier cosine transform is* $\frac{\sin a\lambda}{\lambda}$ *?*

Sol. : Given that $F_C(\lambda) = \frac{\sin a\lambda}{\lambda}$ and we are required to find f(x). Using result (11), inverse cosine transform is given by;

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x \, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda} \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin (a+x)\lambda + \sin (a-x)\lambda}{\lambda} \, dx$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{\sin (a+x)\lambda}{\lambda} d\lambda + \int_0^\infty \frac{\sin (a-x)\lambda}{\lambda} d\lambda\right]$$

$$= \begin{cases} \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right], & a + x > 0 \text{ and } a - x > 0 \\ 0, & a + x > 0 \text{ and } a - x < 0 \end{cases} \qquad \left[\because \int_0^\infty \frac{\sin ax}{x} dx = \begin{cases} \pi/2 & a > 0 \\ -\pi/2 & a < 0 \end{cases}\right.$$

$$= \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$$

Ex. 20 : *Solve the following integral equations :*

(a) $\int_0^\infty f(x) \sin \lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ ***(May 2009, 2011, 2017, 2018, Nov. 2019)***

(b) $\int_0^\infty f(x) \sin \lambda x \, dx = \begin{cases} 1, & 0 \le \lambda < 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \ge 2 \end{cases}$ ***(Dec. 2010)***

(c) $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}, \lambda > 0.$ ***(May 2005, Dec. 2008, 2018)***

Sol. : (a) Since the term sin λx is present in the integral, using result (12), the Fourier sine transform of f(x) is given by

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases} \qquad \text{... (i)}$$

To find f(x), we obtain inverse Fourier sine transform [by result (13)]. Thus

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda = \frac{2}{\pi} \int_0^1 (1-\lambda) \sin \lambda x \, d\lambda \qquad \text{[from (i)]}$$

$$= \frac{2}{\pi} \left[(1-\lambda) \left(\frac{-\cos \lambda x}{x} \right) - (-1) \left(-\frac{\sin \lambda x}{x^2} \right) \right]_0^1$$

$$= \frac{2}{\pi} \left[-\frac{\sin x}{x^2} + \frac{1}{x} \right] = \frac{2}{\pi} \left(\frac{x - \sin x}{x^2} \right)$$

which is the required result.

Note : If we use (13 a) for inverse sine transform, we would get

$$f(x) = \sqrt{\frac{2}{\pi}} \left(\frac{x - \sin x}{x^2} \right).$$

(b) Since the term sin λx is present in the integral, using result (12), Fourier sine transform is given by

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du = \begin{cases} 1, & 0 \le \lambda < 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \ge 2 \end{cases} \qquad \text{... (i)}$$

Now to find f(x), we use result (13) and obtain Inverse Fourier sine transform. Thus

$$f(x) = \frac{2}{\pi} \int_0^\infty F_S(\lambda) \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \left[\int_0^1 (1) \sin \lambda x \, d\lambda + \int_1^2 (2) \sin \lambda x \, d\lambda + \int_2^\infty (0) \sin \lambda x \, d\lambda \right]$$

$$= \frac{2}{\pi} \left[\left(-\frac{\cos \lambda x}{x} \right)_0^1 + 2 \left(-\frac{\cos \lambda x}{x} \right)_1^2 \right] \qquad \text{[from (i)]}$$

$$= \frac{2}{\pi} \left[\left(\frac{1 - \cos x}{x} \right) + 2 \left(\frac{\cos x - \cos 2x}{x} \right) \right]$$

$$= \frac{2}{\pi} \left(\frac{1 + \cos x - 2 \cos 2x}{x} \right)$$

(c) Presence of cos λx in the integral indicates that, we have to find inverse Fourier cosine transform.

Using result (10), Fourier cosine transform of f(x) is given by

$$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du = e^{-\lambda}, \qquad \text{(given) ... (i)}$$

Hence using result (11), we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(\lambda) \cos \lambda x \, d\lambda = \frac{2}{\pi} \int_0^{\infty} e^{-\lambda} \cos \lambda x \, d\lambda \qquad \text{[from (i)]}$$

$$= \frac{2}{\pi} \left[\frac{e^{-\lambda}}{1 + x^2} (-\cos \lambda x + x \sin \lambda x) \right]_0^{\infty} = \frac{2}{\pi} \left(\frac{1}{1 + x^2} \right).$$

Ex. 21 : *Solve the integral equation*

$$\int_0^{\infty} f(x) \cos \lambda x \, dx = \begin{cases} 1 - \lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$$

and hence show that $\int_0^{\infty} \frac{\sin^2 z}{z^2} dz = \frac{\pi}{2}$ ***(May 2005, 2012, 2014, 2019, Dec. 2006, 2007, 2012)***

Sol. : Since the term cos λx is present in the integral, using result (10), the Fourier cosine transform is given by

$$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du = \begin{cases} 1 - \lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases} \qquad \text{... (i)}$$

To find f(x), we use inverse Fourier cosine transform given by result (11). Thus

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_C(\lambda) \cos \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \left[\int_0^{1} (1 - \lambda) \cos \lambda x \, d\lambda + \int_1^{\infty} (0) \cos \lambda x \, d\lambda \right]$$

$$= \frac{2}{\pi} \left[(1 - \lambda) \left(\frac{\sin \lambda x}{x} \right) - (-1) \left(-\frac{\cos \lambda x}{x^2} \right) \right]_0^{1}$$

$$= \frac{2}{\pi} \left[-\frac{\cos x}{x^2} + \frac{1}{x^2} \right] = \frac{2}{\pi} \left(\frac{1 - \cos x}{x^2} \right) \qquad \text{... (ii)}$$

Now from (i), we have

$$F_C(\lambda) = \int_0^{\infty} f(u) \cos \lambda u \, du = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1 - \cos u}{u^2} \right) \cos \lambda u \, du$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{2 \sin^2 u/2}{u^2} \cos \lambda u \, du \qquad \text{... (iii)}$$

At $\lambda = 0$, we have from result (iii),

$$[F_C(\lambda)]_{\lambda=0} = \frac{2}{\pi}\int_0^{\infty} \frac{2\sin^2 u/2}{u^2}\,(1)\,du \qquad [\because \cos 0 = 1]$$

$$1 = \frac{2}{\pi}\int_0^{\infty} \frac{2\sin^2 u/2}{u^2}\,du \qquad \text{[from (i)]}$$

Putting $u/2 = z$ or $u = 2z$, we have

$$1 = \frac{2}{\pi}\int_0^{\infty} \frac{2\sin^2 z}{(2z)^2}\,2\,dz$$

$$\therefore \quad \int_0^{\infty} \frac{\sin^2 z}{z^2}\,dz = \frac{\pi}{2}$$

which is the required result.

3.9 PROPERTIES AND THEOREMS OF FOURIER TRANSFORMS

(A) Linearity Property : If $F(\lambda)$ and $G(\lambda)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, and if we use notation $F[f(x)] = F(\lambda)$ and $F[g(x)] = G(\lambda)$, then

$$\boxed{F[c_1 f(x) + c_2 g(x)] = c_1 F(\lambda) + c_2 G(\lambda)} \qquad \text{... (I)}$$

where, c_1 and c_2 are constants.

Proof : By definition of Fourier transform, we have

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx \quad \text{and} \quad G(\lambda) = \int_{-\infty}^{\infty} g(x)\,e^{-i\lambda x}\,dx$$

$$\therefore \quad F[c_1 f(x) + c_2 g(x)] = \int_{-\infty}^{\infty} [c_1 f(x) + c_2 g(x)]\,e^{-i\lambda x}\,dx$$

$$= c_1 \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx + c_2 \int_{-\infty}^{\infty} g(x)\,e^{-i\lambda x}\,dx$$

$$= c_1 F(\lambda) + c_2 G(\lambda).$$

(B) Change of Scale Property : If $F(\lambda)$ is the complex Fourier transform of $f(x)$, then

$$\boxed{F[f(ax)] = \frac{1}{a} F\left(\frac{\lambda}{a}\right)} \qquad \text{... (II)}$$

Proof : We have, $F[f(x)] = \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx$

$$\therefore \quad F[f(ax)] = \int_{-\infty}^{\infty} f(ax)\,e^{-i\lambda x}\,dx, \quad \text{and put } ax = t$$

$$= \int_{-\infty}^{\infty} f(t)\,e^{-i\lambda t/a}\,\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(t)\,e^{-i(\lambda/a)t}\,dt$$

$$= \frac{1}{a} F\left(\frac{\lambda}{a}\right).$$

(C) Shifting Property : If $F(\lambda)$ is the complex Fourier transform of f(x), then

$$\boxed{\mathbf{F[f(x-a)] = e^{-i\lambda a} F(\lambda)}} \quad \text{... (III)}$$

Proof : We have, $$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

$$\therefore \quad F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)\, e^{-i\lambda x}\, dx, \text{ and put } x = t + a$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda(t+a)}\, dt = e^{-i\lambda a} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t}\, dt$$

$$= e^{-i\lambda a} F(\lambda).$$

(D) Modulation Theorem : If $F(\lambda)$ is the complex Fourier transform of f(x), then

$$\boxed{\mathbf{F[f(x)\cos ax] = \frac{1}{2}[F(\lambda+a) + F(\lambda-a)]}} \quad \text{... (IV)}$$

Proof : We have, $$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

$$\therefore \quad F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x)\cos ax\, e^{-i\lambda x}\, dx = \int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right) e^{-i\lambda x}\, dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\, e^{-i(\lambda+a)x}\, dx + \int_{-\infty}^{\infty} f(x)\, e^{-i(\lambda-a)x}\, dx\right]$$

$$= \frac{1}{2}[F(\lambda+a) + F(\lambda-a)]$$

Note : If $F_s(\lambda)$ and $F_c(\lambda)$ are Fourier sine and cosine transforms of f(x) respectively, then

(i) $$F_S[f(x)\cos ax] = \frac{1}{2}[F_S(\lambda+a) + F_S(\lambda-a)]$$

(ii) $$F_S[f(x)\sin ax] = \frac{1}{2}[F_C(\lambda-a) - F_C(\lambda+a)]$$

(iii) $$F_C[f(x)\cos ax] = \frac{1}{2}[F_C(\lambda+a) + F_C(\lambda-a)]$$

(iv) $$F_C[f(x)\sin ax] = \frac{1}{2}[F_S(\lambda+a) - F_S(\lambda-a)]$$

(E) Convolution Theorem : Definition : The convolution of the functions f(x) and g(x) is denoted by $f(x) * g(x)$ and defined by

$$\boxed{\mathbf{f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du}} \quad \text{... (V)}$$

Theorem : The Fourier transform of the convolution of f(x) and g(x) is equal to the product of the Fourier transforms of f(x) and g(x). In symbols,

$$\boxed{\mathbf{F\,[f(x) * g(x)] = F\,[f(x)]\,F\,[g(x)] = F(\lambda)\,G(\lambda)}} \quad \text{... (VI)}$$

Proof : We have by definition of convolution,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$\therefore \quad F\,[f(x) * g(x)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right] e^{-i\lambda x}\, dx$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(x-u)\, e^{-i\lambda x}\, dx\right] du \qquad \left\{\begin{array}{l}\text{By changing the}\\ \text{order of integration}\end{array}\right.$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t)\, e^{-i\lambda (t+u)}\, dt\right] du, \qquad \text{by } x = t + u$$

$$= \left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du\right]\left[\int_{-\infty}^{\infty} g(t)\, e^{-i\lambda t}\, dt\right] = \left[\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx\right]\left[\int_{-\infty}^{\infty} g(x)\, e^{-i\lambda x}\, dx\right]$$

$$= F\,[f(x)]\;F\,[g(x)] = F(\lambda)\, G(\lambda)$$

Remark : The following important properties of convolution can be proved easily :

1. $f(x) * g(x) = g(x) * f(x)$ 2. $f(x) * \{g(x) * h(x)\} = \{f(x) * g(x)\} * h(x)$
3. $f(x) * \{g(x) + h(x)\} = f(x) * g(x) + f(x) * h(x)$

i.e., the convolution obeys the commutative, associative and distributive laws of algebra.

EXERCISE 3.1

1. (a) Find the Fourier cosine integral representation for the following functions :

(i) $f(x) = \begin{cases} x, & 0 \le x \le a \\ 0, & x > a \end{cases}$

Ans. $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \left(\frac{a \sin a\lambda}{\lambda} + \frac{\cos a\lambda - 1}{\lambda^2}\right) \cos \lambda x\, d\lambda$

(ii) $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Ans. $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \frac{1}{\lambda^3} \{(\lambda^2 - 2) \sin \lambda + 2\lambda \cos \lambda\} \cos \lambda x\, d\lambda$

(iii) $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Ans. $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x\, du\, d\lambda$

(iv) $f(x) = e^{-x} + e^{-2x}$, $x \ge 0$

Ans. $f(x) = \dfrac{6}{\pi} \displaystyle\int_0^{\infty} \frac{\lambda^2 + 2}{\lambda^4 + 5\lambda^2 + 4} \cos \lambda x\, d\lambda$

(v) $f(x) = \dfrac{1}{1 + x^2}$, $x \ge 0$

Ans. $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty}\int_0^{\infty} \frac{\cos \lambda u \cos \lambda x}{1 + u^2}\, du\, d\lambda = \int_0^{\infty} e^{-\lambda} \cos \lambda x\, d\lambda$

(b) Represent the following functions in the Fourier integral form :

(i) $f(x) = \begin{cases} \frac{\pi}{2}\sin x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ **(Dec. 2007)**

Ans. $f(x) = \int_0^\infty \frac{\sin \lambda\pi \sin \lambda x}{1 - \lambda^2}\, d\lambda$

(ii) $f(x) = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$

Ans. $f(x) = \int_0^\infty \frac{\lambda \sin \lambda\pi}{1 - \lambda^2} \cos \lambda x\, d\lambda$

(iii) $f(x) = \begin{cases} 0, & x < -a \\ 1, & -a \le x \le a \\ 0, & x > a \end{cases}$

Ans. $f(x) = \frac{2}{\pi}\int_0^\infty \frac{2 \sin \lambda a \cos \lambda x}{\lambda}\, d\lambda$

(iv) $f(x) = e^{-|x|}, -\infty < x < \infty$

Ans. $f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1 + \lambda^2} \cos \lambda x\, d\lambda$

(v) $f(x) = e^{-x^2/2}, -\infty < x < \infty$

Ans. $f(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda^2/2} \cos \lambda x\, d\lambda$

2. If $f(x) = \begin{cases} 1, & |x| < 1 \\ \frac{1}{2}, & |x| = 1 \\ 0, & |x| > 1 \end{cases}$ then prove that for every x in $-\infty < x < \infty$,

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin [\lambda (1 + x)] + \sin [\lambda (1 - x)]}{\lambda}\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda.$$

3. By applying the Fourier sine integral formula to the function

$$f(x) = \begin{cases} 1, & 0 < x < k \\ \frac{1}{2}, & x = k \\ 0, & x > k \end{cases}$$

obtain the representation

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1 - \cos k\lambda}{\lambda} \sin \lambda x\, d\lambda,\ x > 0.$$

4. Find the Fourier integral for f(x), where

$$f(x) = e^{-kx},\ (x > 0)$$

in the following cases : (i) $f(-x) = f(x)$, (ii) $f(-x) = -f(x)$. **Ans.** (i) $f(x) = \frac{2}{\pi}\int_0^\infty \frac{k \cos \lambda x}{\lambda^2 + k^2}\, d\lambda$, (ii) $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2 + k^2}\, d\lambda$

5. Using the Fourier integral representation, show that :

(i) $\int_0^\infty \frac{\sin \pi\lambda \sin \lambda x}{1 - \lambda^2}\, d\lambda = \begin{cases} \frac{\pi}{2}\sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

(ii) $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \begin{cases} \frac{\pi}{2}, & 0 \le x < 1 \\ \frac{\pi}{4}, & x = 1 \\ 0, & x > 1. \end{cases}$

6. Establish the following representations :

(i) $e^{-x} - e^{-2x} = \dfrac{6}{\pi} \displaystyle\int_0^\infty \dfrac{\lambda \sin \lambda x}{(\lambda^2 + 1)(\lambda^2 + 4)}\, d\lambda,\ x > 0$ **(Dec. 2007)**

(ii) $e^{-x} \sin x = \dfrac{2}{\pi} \displaystyle\int_0^\infty \dfrac{2\lambda \sin \lambda x}{\lambda^4 + 4}\, d\lambda,\ x > 0.$

(iii) $e^{-3x} \sinh x = \dfrac{12}{\pi} \displaystyle\int_0^\infty \dfrac{\lambda \sin \lambda x}{(\lambda^2 + 4)(\lambda^2 + 16)}\, d\lambda$ **(Dec. 2008)**

7. Find the Fourier transforms of the following functions :

(i) $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ **Ans.** $F(\lambda) = 2i \left(\dfrac{\sin a\lambda}{\lambda^2} - \dfrac{a \cos a\lambda}{\lambda}\right),\ F_s(\lambda) = \left(\dfrac{\sin a\lambda}{\lambda^2} - \dfrac{a \cos a\lambda}{\lambda}\right)$

(ii) $f(x) = \begin{cases} x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$ **Ans.** $\dfrac{2}{\lambda^3} \{(a^2 \lambda^2 - 2) \sin a\lambda + 2a\lambda \cos a\lambda\}$

(iii) $f(x) = \begin{cases} \dfrac{\pi}{2} \cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ **Ans.** $\pi \dfrac{\lambda \sin \lambda\pi}{1 - \lambda^2}$

8. Find the Fourier sine transforms of the following functions :

(i) $f(x) = \begin{cases} \sin x, & 0 \le x < a \\ 0, & x > a \end{cases}$ **Ans.** $\dfrac{1}{2}\left[\dfrac{\sin (1 - \lambda) a}{1 - \lambda} - \dfrac{\sin (1 + \lambda) a}{1 + \lambda}\right]$

(ii) $f(x) = \begin{cases} 0, & 0 \le x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$ **Ans.** $\left(\dfrac{a \cos \lambda a - b \cos \lambda b}{\lambda}\right) + \left(\dfrac{\sin \lambda b - \sin \lambda a}{\lambda^2}\right)$

9. Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate $\displaystyle\int_0^\infty \dfrac{x \sin mx}{1 + x^2}\, dx$. **(Dec. 2012)** **Ans.** $\dfrac{\lambda}{1 + \lambda^2},\ \dfrac{\pi}{2}\ e^{-m}$

10. Find the Fourier cosine transforms of the following functions :

(i) $f(x) = 2\, e^{-5x} + 5\, e^{-2x}$ **(Dec. 2005)** **Ans.** $10\left(\dfrac{1}{\lambda^2 + 5} + \dfrac{1}{\lambda^2 + 4}\right)$

(ii) $f(x) = e^{-2x} + 4\, e^{-3x}$ **Ans.** $2\left(\dfrac{1}{\lambda^2 + 4} + \dfrac{6}{\lambda^2 + 9}\right)$

(iii) $f(x) = \begin{cases} x, & 0 < x < \dfrac{1}{2} \\ 1 - x, & 1/2 < x < 1 \\ 0, & x > 1 \end{cases}$ **(Nov. 2013, May 2014)** **Ans.** $\left(\dfrac{-\cos \lambda + 2 \cos \lambda/2 - 1}{\lambda^2}\right)$

11. Find the Fourier sine and cosine transforms of the following functions :

(i) $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$ **Ans.** $\dfrac{1 - \cos\lambda}{\lambda}, \ \dfrac{\sin\lambda}{\lambda}$

(ii) $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & x > 0 \end{cases}$ **Ans.** $\dfrac{1}{\lambda^3}\{2\lambda \sin\lambda - \lambda^2 \cos\lambda + 2(\cos\lambda - 1)\}, \ \dfrac{1}{\lambda^3}\{2\lambda\cos\lambda + \lambda^2 \sin\lambda - 2\sin\lambda\}$

12. Find the Fourier sine transform of $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$ and hence evaluate $\displaystyle\int_0^\infty \frac{\sin^3 x}{x}\,dx$.

Hint : $\displaystyle F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du = \int_0^1 \sin\lambda u\,du = \left[-\frac{\cos\lambda u}{\lambda}\right]_0^1$

$$= \frac{1-\cos\lambda}{\lambda} = \frac{2\sin^2\lambda/2}{\lambda}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda n\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\lambda/2}{2(\lambda/2)}\sin\lambda x\,d\lambda \quad \text{Putting } \lambda/2 = t,$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 t}{t}\sin 2tx\,dt \quad \text{Again putting } x = 1/2,$$

$$f\left(\frac{1}{2}\right) = 1 = \frac{4}{\pi}\int_0^\infty \frac{\sin^3 t}{t}\,dt$$

Ans. $\dfrac{\pi}{4}$

13. Using inverse Fourier cosine transform, find f(x), if

$$F_C(\lambda) = \begin{cases} \sqrt{2/\pi}\left(a - \dfrac{\lambda}{2}\right), & \lambda \le 2a \\ 0, & \lambda > 2a \end{cases}$$

Ans. $\dfrac{2\sin^2 ax}{\pi x^2}$

14. Using inverse Fourier sine transform, find f(x), if

$$F_S(\lambda) = \frac{\lambda}{1+\lambda^2}$$

Ans. e^{-x}

Hint : $\displaystyle f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2}\sin\lambda x\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+1-1}{\lambda(1+\lambda^2)}\sin\lambda x\,d\lambda$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{\sin\lambda x}{\lambda(1+\lambda)^2}\,d\lambda\right] = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda x}{\lambda(1+\lambda^2)}\,d\lambda$$

Using DUIS Rule,

$$f'(x) = 0 - \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial x}\frac{\sin\lambda x}{\lambda(1+\lambda^2)}\,d\lambda = -\frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}\,d\lambda$$

Again using DUIS Rule,

$$f''(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2}\, d\lambda = f(x) \qquad \therefore \quad f''(x) - f(x) = 0$$

$$\text{G.S.} = f(x) = c_1 e^x + c_2 e^{-x} \text{ and } f'(x) = c_1 e^x - c_2 e^{-x}.$$

Show that $c_1 = 0,\ c_2 = 1.$

15. Find the function f(x), satisfying the integral equation

$$\int_0^\infty f(x) \sin \lambda x\, dx = \frac{\lambda}{\lambda^2 + k^2}.$$

Ans. $f(x) = e^{-kx},\ x > 0$

❑❑❑

CHAPTER-4
THE Z-TRANSFORM

4.1 INTRODUCTION

Use of Z-transform is very prominent in the analysis of linear time-invariant systems. Linear time invariant (LTI) systems are characterised either by their Z-transform or by the Fourier transform and their characteristics are related to the location of the poles or zeroes of their system functions. We have already discussed Fourier Transforms in details in Chapter 3.

Number of systems of practical importance such as economical systems, population systems and many other systems occurring in statistical studies are discrete in nature. While the modern technological development has made it possible to consider many systems occurring in Engineering fields as discrete. Number of important types of digital systems including resonators, notch filters, comb filters, all-pass filters and oscillators use Z-transforms for their analysis.

Discrete systems give rise to difference equations, and their solutions as well as their analysis are carried out by using transform techniques. Z-transform plays important role in these aspects. Its role in analysis of discrete systems is same as that of Laplace transform and Fourier transform in continuous systems.

4.2 BASIC PRELIMINARY

I. Sequence

An ordered set of real or complex numbers is called a sequence. It is denoted by $\{f(k)\}$ or $\{f_k\}$. The sequence $\{f(k)\}$ is represented in two ways.

1. The most elementary way is to list all the numbers of the sequence; such as :

(i) $$\{f(k)\} = \{15, 13, 10, \underset{\uparrow}{8}, 5, 2, 0, 3\} \qquad \dots (1)$$

In this representation, a vertical arrow indicates the position corresponding to k = 0.

$\therefore$ $f(0) = 8$, $f(1) = 5$, $f(2) = 2$, $f(3) = 0$, $f(4) = 3$.

$$f(-1) = 10$$
$$f(-2) = 13$$
$$f(-3) = 15$$

(ii) For the sequence

$$\{f(k)\} = \{15, 13, \underset{\uparrow}{10}, 8, 5, 2, 0, 3\} \qquad \dots (2)$$

$f(-2) = 15$, $f(-1) = 13$, $f(0) = 10$, $f(1) = 8$, $f(2) = 5$, $f(3) = 2$, $f(4) = 0$, $f(5) = 3$.

Note : The sequences given in (1) and (2) are having the same listing but they are not treated as identical, since k = a corresponds to different terms in these sequences.

The method of representation, as discussed above, is appropriate only for a sequence with finite number of terms.

When vertical arrow ↑ is not given, then the starting or left hand end term of the sequence denotes the position corresponding to k = 0.

In the sequence :

$$\{f(k)\} = \{9, 7, 5, 3, 1, -2, 0, 2, 4\} \qquad \dots (3)$$

the zeroeth term is 9, the left hand term.

$\therefore$ $f(0) = 9$, $f(1) = 7$, $f(2) = 5$ ... etc.

2. The second way of specifying the sequence is to define the general term of the sequence (if possible) as a function of position i.e. k.

e.g. The sequence {f (k)} where $\{f(k)\} = \dfrac{1}{4^k}$ (k is any integer) represents the sequence

$$\left\{\frac{1}{4^{-8}}, \frac{1}{4^{-7}}, \ldots\ldots, \frac{1}{4^{-1}}, \underset{\uparrow}{1}, \frac{1}{4}, \frac{1}{4^2} \ldots\ldots\right\} \qquad \ldots (4)$$

Here $f(0) = 1$, $f(1) = \dfrac{1}{4}$, $f(2) = \dfrac{1}{4^2}$, etc.

If $f(k) = \dfrac{1}{4^k}$; $-3 \le k \le 5$ then it represents the sequence

$$\left\{4^3, 4^2, 4, \underset{k = 0}{\overset{1}{\uparrow}}, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \frac{1}{4^5}\right\}$$

Hence a sequence {f (k)} can be written as :

$$\{f(k)\} = \{\ldots\ldots f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), f(4), \ldots\}$$

having f (0) as the zeroeth term.

OR $\{f(k)\} = \{f(0), f(1), f(2), \ldots\ldots\}$

OR $\{f(k)\} = \{f(-2), f(-1), f(0), f(1), f(2), f(3)\}$

II. Causal Sequence

Any sequence whose terms corresponding to k < 0 are all zero is called causal sequence.

ILLUSTRATION

Ex. 1 : *What sequence is generated when*

$$f(k) = \begin{cases} 0, & k < 0 \\ \cos\dfrac{k\pi}{3}, & k \ge 0 \end{cases}$$

Sol. : We have

$$\{f(k)\} = \left\{\ldots 0, 0, \underset{k = 0}{\overset{1}{\uparrow}}, \cos\frac{\pi}{3}, \cos\frac{2\pi}{3}, \cos\pi, \ldots\ldots\right\}$$

From application point of view, it is sometimes convenient to consider finite sequences to be of infinite length by appending additional zeroes to each.

e.g. $\{f(k)\} = \{8, 6, 4, 2, 0, 2, 4, 6, 8, 10\}$

$\therefore$ $\{f(k)\} = \{\ldots 0, 0, \ldots, 0, 0, 8, 6, 4, 2, 0, 2, 4, 6, 8, 10, 0, 0, \ldots, 0, 0, \ldots\ldots\}$

III. Basic Operations on Sequences

1. Addition : If {f (k)} and {g (k)} are the two sequences with same number of terms, then the addition of these sequences is a sequence given by {f (k) + g (k)} i.e.

$$\{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$$

2. Scaling : If a is a scalar, then

$$a\{f(k)\} = \{a\, f(k)\}$$

3. Linearity : If a and b are scalars, then

$$\{a\, f(k) + b\, g(k)\} = \{a\, f(k)\} + \{b\, g(k)\} = a\{f(k)\} + b\{g(k)\}$$

ILLUSTRATION

Ex. 1 : *Write the sequence* $\frac{1}{2}\{f(k)\}$, *where* $\{f(k)\}$ *is given by* $f(k) = \frac{1}{2^k}$.

Sol. : $$\frac{1}{2}\{f(k)\} = \left\{\frac{1}{2} f(k)\right\} = \left\{\frac{1}{2}, \frac{1}{2^k}\right\} = \left\{\frac{1}{2^{k+1}}\right\}$$

Ex. 2 : *Write the sequence* $\{f(k) + g(k)\}$, *where* $\{f(k)\}$ *is given by* $f(k) = \frac{1}{2^k}$ *and* $\{g(k)\}$ *is given by* $g(k) = \begin{cases} 0, & k < 0 \\ 3, & k \geq 0 \end{cases}$.

Sol. : $$\{f(k) + g(k)\} = \{f(k)\} + \{g(k)\} = \{h(k)\}$$

where $$h(k) = \begin{cases} \frac{1}{2^k}, & k < 0 \\ \frac{1}{2^k} + 3, & k \geq 0. \end{cases}$$

Ex. 3 : *If* $\{f(k)\}$ *is given by* $f(k) = \begin{matrix} 0, & k < 0 \\ 2, & k \geq 0 \end{matrix}$ *find* $\frac{1}{3}\{f(k)\}$.

Sol. : $$\frac{1}{3}\{f(k)\} = \left\{\frac{1}{3} f(k)\right\} = \left\{\ldots 0,\ 0,\ \ldots\ldots 0,\ \frac{2}{3}, \frac{2}{3}, \ldots\ldots\right\}$$

IV. Useful Results

1. $$\frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \ldots\ldots, \quad |y| < 1.$$

2. $$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \ldots\ldots, \quad |y| < 1.$$

3. $$(1+y)^n = 1 + ny + \frac{n(n-1)}{2!} y^2 + \frac{n(n-1)(n-2)}{3!} y^3 + \ldots, = \sum_{r=0}^{n} {}^nC_r\, y^r, \quad |y| < 1$$

4. $$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\ldots,$$

5. Since S_∞ of the Geometric Progression G.P.

$$a + ar + ar^2 + ar^3 + \ldots\ldots = \frac{a}{1-r} \text{ and it is convergent if } |r| < 1$$

where a = First term and r = Common ratio.

6. If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$

Also, $|z| = 1$ represents

$$\sqrt{x^2 + y^2} = 1 \text{ or } x^2 + y^2 = 1 \text{ a circle. (see Fig. 4.1)}$$

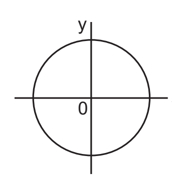

Fig. 4.1

If $z = x + iy$, $|z| > 4$ represents

$$\sqrt{x^2 + y^2} > 4 \text{ or } x^2 + y^2 > 16$$

i.e. the collection of points which lie outside the circle $x^2 + y^2 = 16$ (see Fig. 4.2)

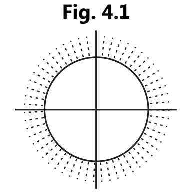
Fig. 4.2

Similarly, $|z| < 1$ represents

$\sqrt{x^2 + y^2} < 1$ or $x^2 + y^2 < 1$ i.e. represents,

the collection of points which lie inside the unit circle

$|z| = 1$ i.e. $x^2 + y^2 = 1$ (see Fig. 4.3).

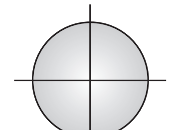
Fig. 4.3

4.3 Z-TRANSFORMS

Definition :

1. The Z-transform of a sequence {f (k)}, symbolically denoted by Z{f(k)} is defined as :

$$\boxed{\mathbf{Z\{f\,(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f\,(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f\,(k)}{z^k}}}$$

where, $z = x + i\,y$ is a complex number. Z is a Z-transform operator and F (z), the Z-transform of {f (k)}.

2. For a finite sequence {f (k)}, $m \le k \le n$, its Z-transform is,

$$Z\,\{f\,(k)\} = F\,(z) = \sum_{k=m}^{n} f\,(k)\, z^{-k}$$

$$Z\,\{f\,(k)\} = f\,(m)\, z^{-m} + f\,(m+1)\, z^{-(m+1)} + \ldots + f\,(n)\, z^{-n}$$

The Z-transform of {f (k)} exists if the sum of the series on R.H.S. exists i.e. the series on R.H.S. converges absolutely.

3. Z-transform of a causal sequence :

$$\{f\,(k)\} = \{0, 0, \ldots 0, 0, f\,(0), f\,(1), \ldots\ldots\}$$

which is defined for positive integers k, is defined as

$$\boxed{\mathbf{Z\,\{f\,(k)\} = F\,(z) = \sum_{k=0}^{\infty} f\,(k)\, z^{-k}}}$$

Note :

1. To obtain Z-transform of a sequence we multiply each term by *negative power of z of the order of that term* and take the sum.

2. Z {f (k)} is a function of a complex variable z and is defined only if the sum is finite i.e. *if the infinite series* $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ *is absolutely convergent.*

ILLUSTRATIONS

Ex. 1 : For {f (k)} if

$$f\,(k) = \{8,\ 6,\ 4,\ 2,\ -1,\ \underset{\uparrow}{0},\ 1,\ 2,\ 3\}$$

$$\{f\,(k)\} = \{f(-5), f(-4), f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3)\}$$

we have

$$F\,(z) = Z\,\{f\,(k)\} = \sum_{k=-5}^{3} f\,(k)\ z^{-k}$$

$$= f(-5)\, z^5 + f(-4)\, z^4 + f(-3)\, z^3 + f(-2)\, z^2 + f(-1)\, z + f(0)\, z^0 + f\,(1)\, z^{-1} + f\,(2)\, z^{-2} + f\,(3)\, z^{-3}$$

$$F\,(z) = Z\,\{f\,(k)\} = 8\, z^5 + 6\, z^4 + 4\, z^3 + 2\, z^2 - 1\,(z) + 0 + 1\,(z^{-1}) + 2\, z^{-2} + 3\, z^{-3}$$

$$F\,(z) = 8\, z^5 + 6\, z^4 + 4\, z^3 + 2\, z^2 - z + 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3}$$

Ex. 2 : For {f (k)} if $f\,(k) = \{4,\ 2,\ \underset{\uparrow}{0},\ -2,\ -4,\ -6\}$

$$F\,(z) = Z\,\{f\,(k)\} = \sum_{k=-3}^{2} f\,(k)\ z^{-k}$$

$$= 4\, z^3 + 2\, z^2 + 0\, z^1 - 2\, z^0 - 4\, z^{-1} - 6\, z^{-2}$$

$$= 4\, z^3 + 2\, z^2 + 0 - 2 - \frac{4}{z} - \frac{6}{z^2}$$

Ex. 3 : For {f (k)}, if f (k) $= \left\{ \ldots\ldots \frac{1}{2^{-2}},\ \frac{1}{2^{-1}},\ \underset{\uparrow}{1},\ \frac{1}{2},\ \frac{1}{2^2},\ \ldots \right\}$

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\ .\ z^{-k}$$

$$= \ldots\ldots + 2^2 \cdot z^2 + 2 \cdot z + 1\,(z^0) + \frac{1}{2} z^{-1} + \frac{1}{2^2} z^{-2} + \ldots\ldots$$

$$F(z) = \ldots\ldots + 2^2 z^2 + 2 \cdot z + 1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \ldots\ldots$$

4.4 INVERSE Z-TRANSFORM

The operation of obtaining the sequence {f (k)} from F (z) is defined as inverse Z-transform and is denoted as :

$$\boxed{\mathbf{Z^{-1}\ [F(z)] = \{f(k)\}}}$$

where, Z^{-1} is inverse Z-transform operator.

4.5 Z-TRANSFORM PAIR

Sequence {f (k)} and its Z-transform F (z) are together termed as Z-transform pair and denoted as {f (k)} $\longleftrightarrow$ F (z).

i.e. $Z\{f(k)\} = F(z)$

and $Z^{-1}[F(z)] = \{f(k)\}$

4.6 UNIQUENESS OF INVERSE Z–TRANSFORM : REGION OF ABSOLUTE CONVERGENCE (ROC)

Consider the two sequences {f (k)} and {g (k)}

where $f(k) = \begin{cases} 0, & k < 0 \\ a^k, & k \geq 0 \end{cases}$; $g(k) = \begin{cases} -b^k, & k < 0 \\ 0, & k \geq 0 \end{cases}$

$\therefore$ Z-transform of the sequence {f (k)} is

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\ z^{-k} = \sum_{k=-\infty}^{-1} f(k)\,z^{-k} + \sum_{k=0}^{\infty} f(k)\ z^{-k}$$

$$= 0 + \sum_{k=0}^{\infty} a^k\ z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(a z^{-1}\right)^k = 1 + \left(a z^{-1}\right) + \left(a z^{-1}\right)^2 + \left(a z^{-1}\right)^3 + \ldots$$

which is an infinite G.P.

$\therefore$ $S_\infty = \frac{a}{1-r}$, $|r| < 1$

Here, a = first term = 1 and r = common ratio $= a z^{-1}$.

$$Z\{f(k)\} = \frac{1}{1 - a z^{-1}} \text{ provided } \left|a z^{-1}\right| < 1$$

$$= \frac{1}{1 - \frac{a}{z}},\ |a| < |z|$$

$$F(z) = \frac{z}{z-a},\ |z| > |a|$$

but $z = x + iy$ $\therefore$ $|z| = \sqrt{x^2 + y^2}$

$$|z| > |a| \Rightarrow \sqrt{x^2 + y^2} > a$$

i.e. $x^2 + y^2 > a^2$, which represents exterior of circle $x^2 + y^2 = a^2$ [refer Fig. 4.4 (a)].

Now consider the Z-transform of the sequence {g (k)},

$$Z\{g(k)\} = G(z) = \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} g(k)\, z^{-k} + \sum_{k=0}^{\infty} g(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} - b^{k}\, z^{-k} + 0$$

Let $\quad k = -r$ when

k = –1	r = 1
k = –∞	r = ∞

and $\sum_{k=-\infty}^{-1} = \sum_{r=\infty}^{1} = \sum_{r=1}^{\infty}$

$$G(z) = -\sum_{r=1}^{\infty} b^{-r} z^{r} = -\sum_{r=1}^{\infty} \left(b^{-1}z\right)^{r}$$

$$= -(b^{-1} z) - \left(b^{-1}z\right)^{2} - \left(b^{-1}z\right)^{3} \text{ ....}$$

$$= -\frac{b^{-1}z}{1-\left(b^{-1}z\right)}, \quad \left|b^{-1}z\right| < 1$$

$$= -\frac{\frac{z}{b}}{1-\frac{z}{b}}, \; |z| < |b| \; = \; \frac{z}{z-b}, \; |z| < |b|$$

but $\quad z = x + iy, \; |z| = \sqrt{x^2+y^2}$ and $|z| < |b| \Rightarrow \sqrt{x^2+y^2} < b$

i.e. $x^2 + y^2 < b^2$ which represents the interior of circle $x^2 + y^2 = b^2$ [Refer Fig. 4.4 (b)]

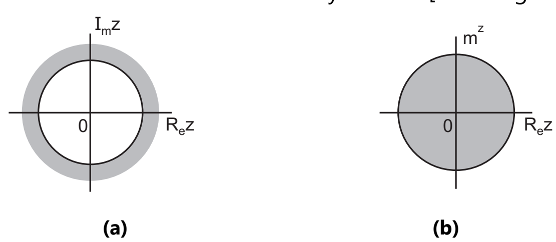

Fig. 4.4

Now if a = b then

$$F(z) = G(z) = \frac{z}{z-a}$$

i.e. for a = b, $\quad Z\{f(k)\} = Z\{g(k)\} = \frac{z}{z-a}$

If a = b, then two sequences {f (k)} and {g (k)} have the same Z-transform, therefore inverse Z-transform of $\frac{z}{z-a}$ will be two different sequences {f(k)} and {g(k)}, indicating that *inverse Z-transform is not unique.*

However, if we specify the region, interior or exterior of circle $x^2 + y^2 = a^2$ known as region of convergence, then we get exactly

$$Z^{-1}\{F(z)\} = \{f(k)\} \quad \text{and} \quad Z^{-1}\{G(z)\} = \{g(k)\}.$$

This implies that *Z – transform and its inverse are uniquely related in the specified region of convergence.*

Note :

1. In case of one-sided sequences (i.e. causal sequences or sequences for which $f(k) = 0$ for $k < 0$), then there is no necessity of specifying the ROC.
2. By the term "region of convergence", we will mean the "region of absolute convergence". This term will be abbreviated to ROC.

Next, consider the sequence {f (k)}, where $f(k) = \begin{cases} -b^k, & k < 0 \\ a^k, & k \geq 1 \end{cases}$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} -b^k\, z^{-k} + \sum_{k=0}^{\infty} a^k\, z^{-k}$$

Put $k = -m \quad \therefore k = -\infty \Rightarrow m = \infty, \quad k = -1 \Rightarrow m = 1$

$$Z\{f(k)\} = -\sum_{m=1}^{\infty} b^{-m}\, z^m + \sum_{k=0}^{\infty} \left(a z^{-1}\right)^k$$

$$= -\sum_{m=1}^{\infty} (b^{-1} z)^m + \sum_{k=0}^{\infty} \left(a z^{-1}\right)^k$$

$$= -\left(\frac{b^{-1} z}{1 - b^{-1} z}\right) + \frac{1}{1 - az^{-1}},$$

provided $\left|b^{-1} z\right| < 1$ and $\left|az^{-1}\right| < 1$

or $|z| < |b|$ and $|a| < |z|$

$$\therefore \quad Z\{f(k)\} = \frac{z}{z-b} + \frac{z}{z-a}, \text{ provided } |a| < |z| < |b|$$

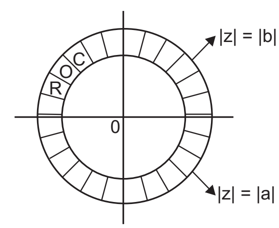

Fig. 4.5

Note that the Z - transform exists only if |b| > |a| and does not exist for |b| = |a| or |b| < |a|.

Note : In general, Z-transform of sum of sequences is the sum of corresponding transforms with region of absolute convergence consisting of those values of z for which all of the individual transforms converge absolutely i.e. the region of absolute convergence of sum of transforms is the intersection of the individual regions of absolute convergence.

Note : For finite sequence, Z-transform exists for all values of z except for $z = 0$ and $z = \infty$.

Note : The region lying between two concentric circles is called an annulus.

e.g. Consider {f (k)}

where $f(k) = 5^k$, for $k < 0$

$= 3^k$, for $k \geq 0$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 5^k\, z^{-k} + \sum_{k=0}^{\infty} 3^k\, z^{-k}$$

Put $k = -r$ when

$k = -\infty$	$r = \infty$
$k = -1$	$r = 1$

$$Z\{f(k)\} = \sum_{r=1}^{\infty} 5^{-r} z^{r} + \sum_{k=0}^{\infty} \left(3z^{-1}\right)^{k}$$

$$= \sum_{r=1}^{\infty} \left(5^{-1} z\right)^{r} + \sum_{k=0}^{\infty} \left(3z^{-1}\right)^{k}$$

$$= \frac{5^{-1} z}{1 - 5^{-1} z} + \frac{1}{1 - 3z^{-1}}, \quad \left|5^{-1} z\right| < \text{and } \left|3z^{-1}\right| < 1$$

$$F(z) = \frac{z}{5 - z} + \frac{z}{z - 3}, \quad |z| < 5 \text{ and } 3 < |z|$$

$\therefore$ F (z) is the sum of two infinite series both of which are G.P. The first series is absolutely (and therefore uniformly) convergent if $\left|5^{-1} z\right| < 1$ and the second one if $\left|3z^{-1}\right| < 1$. Thus, F (z) is defined iff $|z| > 3$ and $|z| < 5$ i.e. Z lies in the annulus $3 < |z| < 5$.

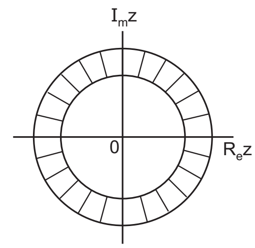

Fig. 4.6

4.7 PROPERTIES OF Z-TRANSFORMS

1. Linearity : If {f(k)} and {g(k)} are such that they can be added and 'a' and 'b' are constants, then

$$\mathbf{Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}}$$

Proof : We have

$$Z\{a\,f(k) + b\,g(k)\} = \sum_{k=-\infty}^{\infty} Z\{a\,f(k) + b\,g(k)\}\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \left[a\,f(k)\,z^{-k} + b\,g(k)\,z^{-k}\right]$$

$$= a \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} + b \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

$$= a\,F(z) + b\,G(z) = a \cdot Z\{f(k)\} + b \cdot Z\{g(k)\}$$

2. If Z {f (k)} = F (z) and Z {g (k)} = G (z) and 'a' and 'b' are constants, then

$$\mathbf{Z^{-1}[a\,F(z) + b\,G(z)] = a\,Z^{-1}[F(z)] + b\,Z^{-1}[G(z)]}$$

Proof : We have,

$$Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

$$= a\,F(z) + b\,G(z)$$

$$\therefore \quad Z^{-1}\{a\,F(z) + b\,G(z)] = \{a\,f(k) + b\,g(k)\} = a\,\{f(k)\} + b\,\{g(k)\}$$

$$= a\,Z^{-1}[F(z)] + b\,Z^{-1}[G(z)]$$

i.e. operator Z^{-1} is a linear operator.

3. Change of scale : If Z {f (k)} = F(z) then Z {$\mathbf{a^k}$ f (k)} = $\mathbf{F\left(\frac{Z}{a}\right)}$

Proof : We have,

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

Replacing z by $\frac{z}{a}$, we get

$$F\left(\frac{z}{a}\right) = \sum_{k=-\infty}^{\infty} f(k) \left(\frac{z}{a}\right)^{-k} = \sum_{k=-\infty}^{\infty} a^{k}\, f(k)\, z^{-k}$$

$$F\left(\frac{z}{a}\right) = Z\left\{a^{k}\, f(k)\right\}$$

$$\therefore \quad Z\left\{a^{k}\, f(k)\right\} = F\left(\frac{z}{a}\right)$$

4. If $Z\{f(k)\} = F(z)$ then $Z\left\{e^{-ak} f(k)\right\} = F(e^a z)$

Proof : We have,

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore \quad Z\left\{e^{-ak} f(k)\right\} = \sum_{-\infty}^{\infty} e^{-ak} \cdot f(k) z^{-k} = \sum_{-\infty}^{\infty} f(k) \ (e^a z)^{-k}$$

$$= F(e^a z)$$

5. Shifting Property :

(a) If $Z\{f(k)\} = F(z)$ then $Z\{f(k+n)\} = z^n F(z)$ and $Z\{f(k-n)\} = z^{-n} F(z)$

Proof : We have,

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k) \ z^{-k}$$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=-\infty}^{\infty} f(k+n) z^{-k} = \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)}$$

For k + n = r if

$k = -\infty$	$r = -\infty$
$k = \infty$	$r = \infty$

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{r=-\infty}^{\infty} f(r) \ z^{-r} = z^n F(z)$$

$$\therefore \quad Z\{f(k+n)\} = z^n F(z)$$

Similarly,

$$Z\{f(k-n)\} = \sum_{k=-\infty}^{\infty} f(k-n) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f(k-n) z^{-(k-n)} . z^{-n}$$

$$= z^{-n} \sum_{k=-\infty}^{\infty} f(k-n) z^{-(k-n)}$$

For k – n = r; if $k = -\infty$, $r = -\infty$ and $k = \infty$, $r = \infty$

$$= z^{-n} \sum_{k=-\infty}^{\infty} f(r) \ z^{-r} = z^{-n} F(z)$$

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z)$$

(b) For One Sided Z-Transform defined as $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}$

(i.e Z - transform for $k \geq 0$) , we have

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$$

and

$$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=-n}^{-1} f(r) z^{-(n+r)}$$

Proof : We have for $k \geq 0$,

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=0}^{\infty} f(k+n) z^{-k} = \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)} \cdot z^n$$

For k + n = r, when

k = 0,	r = n
k = ∞,	r = ∞

$$\therefore \qquad \text{R.H.S.} = z^n \sum_{r=n}^{\infty} f(r)\, z^{-r}$$

Now r = n to ∞ means (r = 0 to ∞) − (r = 0 to n − 1)

$$\therefore \qquad Z\{f(k+n)\} = z^n \sum_{r=0}^{\infty} f(r)\, z^{-r} - z^n \sum_{r=0}^{n-1} f(r)\, z^{-r}$$

$$Z\{f(k+n)\} = z^n\, F(z) - z^n \sum_{r=0}^{n-1} f(r)\, z^{-r}$$

$$= z^n\, F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

Now,

$$Z\{f(k-n)\} = \sum_{k=0}^{\infty} f(k-n)\, z^{-k} = \sum_{k=0}^{\infty} f(k-n)\, z^{-(k-n)} \cdot z^{-n}$$

$$= z^{-n} \sum_{k=0}^{\infty} f(k-n)\, z^{-(k-n)}$$

For k − n = r, when

k = 0	r = − n
k = ∞	r = ∞

For k − n = r, when k = 0, r = − n

k = ∞, r = ∞

$$\therefore \qquad Z\{f(k+n)\} = z^{-n} \sum_{r=-n}^{\infty} f(r)\, z^{-r}$$

Now r = − n to ∞ is (r = − n to − 1) + (r = 0 to ∞)

$$\therefore \qquad Z\{f(k-n)\} = z^{-n} \sum_{r=-n}^{-1} f(r)\, z^{-r} + z^{-n} \sum_{r=0}^{\infty} f(r)\, z^{-r}$$

$$\therefore \qquad Z\{f(k-n)\} = z^{-n}\, F(z) + \sum_{r=-n}^{-1} f(r)\, z^{-(n+r)}$$

Additional Results :

1. If {f (k)} is causal sequence then

$$Z\{f(k-n)\} = z^{-n}\, F(z)$$

because f (– 1) f (– 2) f (– 3) ...... f (– n) are all zero.

2. $$Z\{f(k-n)\} = z^{-n}\, F(z)$$

For n = 1,

$$Z\{f(k-1)\} = z^{-1}\, F(z),\ f(-1) = 0$$

$$Z\{f(k-2)\} = z^{-2}\, F(z),\ f(-1) = 0,\ f(-2) = 0$$

$$Z\{f(k+1)\} = z\, F(z) - z\, f(0)$$

$$Z\{f(k+2)\} = z^2\, F(z) - z^2 f(0) - z\, f(1)$$

Shifting properties are very useful in Z-transforming linear difference equations, from which the solution is obtained by inverse transforming.

6. Multiplication by k :

If $Z\{f(k)\} = F(z)$ then $Z\{k\,f(k)\} = -z\dfrac{d}{dz}\,F(z)$

$\therefore$ **In general** $\quad Z\left\{k^n\ f(k)\right\} = \left(-z\dfrac{d}{dz}\right)^n F(z)$

Proof : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

$$\therefore \quad Z\{k\,f(k)\} = \sum_{k=-\infty}^{\infty} k\,f(k)\, z^{-k}$$

Multiply and divide by (– z) on R.H.S.

$$= \sum_{k=-\infty}^{\infty} -k\,f(k)\, z^{-k-1}\,(-z)$$

$$= -z \sum_{k=-\infty}^{\infty} f(k)\,\left\{-k\,z^{-k-1}\right\}$$

$$= -z \sum_{k=-\infty}^{\infty} f(k)\left(\frac{d}{dz}\,z^{-k}\right) = -z\,\frac{d}{dz}\sum_{k=-\infty}^{\infty} f(k)\,z^{-k}$$

$$\therefore \quad Z\{k\,f(k)\} = -z\frac{d}{dz}\,F(z)$$

$$\therefore \quad Z\{k^2\ f(k)\} = Z\{k\,.\,k\,f(k)\} = \left(-z\frac{d}{dz}\right)\left(-z\,\frac{d}{dz}\,F(z)\right)$$

$$= \left(-z\frac{d}{dz}\right)^2 F(z)$$

$\therefore$ On generalizing, we get

$$Z\{k^n\ f(k)\} = \left(-z\,\frac{d}{dz}\right)^n F(z)$$

Note : $\left(-z\,\dfrac{d}{dz}\right)^2 \neq z^2\dfrac{d^2}{dz^2}$ but it is a repeated operator $\left(-z\,\dfrac{d}{dz}\right)^2 = \left(-z\,\dfrac{d}{dz}\right)\left(-z\,\dfrac{d}{dz}\right)$

Note : Let $k \geq 0$ and let $f(k) = 1$.

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} 1\,.\,z^{-k}$$

$$Z\{1\} = 1 + z^{-1} + z^{-2} + \ldots\ldots$$

$$= \frac{1}{1-z^{-1}} \qquad \left|z^{-1}\right| < 1$$

$$Z\{1\} = \left(1-z^{-1}\right)^{-1} \qquad |z| > 1.$$

$$\therefore \quad Z\{k\} = Z\{k\,.\,1\} = \left(-z\,\frac{d}{dz}\right) F(z)$$

$$= \left(-z\,\frac{d}{dz}\right)\left[(1-z^{-1})^{-1}\right] = -z\left\{-1\,.\left(1-z^{-1}\right)^{-2}\right\}\times z^{-2}$$

$$Z\{k\} = z^{-1}\,(1-z^{-1})^{-2}, \quad |z| > 1$$

Similarly, $\quad Z\{k^n\}\ (k \geq 0) = \left(-z\dfrac{d}{dz}\right)^n \left(1-z^{-1}\right)^{-1}, \ |z| > 1$

7. Division by k :

If $Z\{f(k)\} = F(z)$ **then** $Z\left\{\frac{f(k)}{k}\right\} = -\int^{z} z^{-1} F(z)\, dz.$

Proof : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad Z\left\{\frac{f(k)}{k}\right\} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{k}\, z^{-k}$$

As $$\int z^{-k-1}\, dz = \frac{z^{-k}}{-k}$$

we have,
$$Z\left\{\frac{f(k)}{k}\right\} = -\sum_{k=-\infty}^{\infty} f(k)\, \frac{z^{-k}}{-k}$$

$$= -\sum_{k=-\infty}^{\infty} f(k) \int^{z} z^{-k-1}\, dz$$

$$= -\sum_{k=-\infty}^{\infty} \int^{z} f(k)\, z^{-k}\, z^{-1}\, dz$$

$$= -\int^{z} z^{-1} \left(\sum_{k=-\infty}^{\infty} f(k)\, z^{-k} \right) dz$$

$$Z\left\{\frac{f(k)}{k}\right\} = -\int^{z} z^{-1} F(z)\, dz.$$

Note : For $k > 0$, we use the formula $Z\left\{\frac{f(k)}{k}\right\} = \int_{z}^{\infty} z^{-1} F(z)\, dz.$

8. Initial Value Theorem (One Sided Sequence) :

If $Z\{f(k)\} = F(z)$ **then** $f(0) = \lim_{z \to \infty} F(z).$

Proof : We have,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad F(z) = f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots\ldots$$

$$\therefore \quad \lim_{z \to \infty} F(z) = \lim_{z \to \infty} \left[f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots\ldots \right]$$

As $$\lim_{z \to \infty} z^{-n} = 0$$

$$\therefore \quad \text{R.H.S.} = f(0) + \text{all vanishing terms}$$

$$\therefore \quad f(0) = \lim_{z \to \infty} F(z).$$

9. Final Value Theorem (One Sided Sequence) :

$$\lim_{k \to \infty} \{f(k)\} = \lim_{z \to 1} (z - 1)\; F(z), \text{ if limit exists.}$$

Proof : We have,
$$Z\{[f(k+1) - f(k)]\} = \sum_{k=0}^{\infty} [f(k+1) - f(k)]\, z^{-k}$$

$$\therefore \quad Z\{f(k+1)\} - Z\{f(k)\} = \sum_{k=0}^{\infty} [f(k+1) - f(k)]\, z^{-k} \qquad \text{... (A)}$$

For causal sequence, we have
$$Z\{f(k+n)\} = z^{n} F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$\therefore$ For n = 1, $Z\{f(k+1)\} = z F(z) - f(0)$.

$\therefore$ From equation (A),

$$z F(z) - f(0) - F(z) = \lim_{n \to \infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}$$

$$\therefore \quad \lim_{z \to 1} (z - 1)\; F(z) = f(0) + \lim_{z \to 1} \lim_{n \to \infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}$$

$$= f(0) + \lim_{n \to \infty} \sum_{k=0}^{n} [f(k+1) - f(k)] \lim_{z \to 1} z^{-k}$$

$$= f(0) + \lim_{n \to \infty} [f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \ldots + f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} [f(0) + f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \ldots + f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} f(n+1) = \lim_{k \to \infty} f(k)$$

For k = n + 1, when $n \to \infty$, $k \to \infty$

$$\therefore \quad \lim_{z \to 1} (z - 1)\; F(z) = \lim_{k \to \infty} f(k).$$

10. Partial Sum :

$$\textbf{If } Z\{f(k)\} = F(z) \textbf{ then } Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \frac{F(z)}{1 - z^{-1}}$$

Proof : Form {g (k)} such that :
$$g(k) = \sum_{m=-\infty}^{k} f(m).$$

Hence we have to obtain Z [{g (k)}].

We have,
$$g(k) - g(k-1) = \sum_{m=-\infty}^{k} f(m) - \sum_{m=-\infty}^{k-1} f(m) = f(k)$$

$$\therefore \quad Z[\{g(k) - g(k-1)\}] = Z[\{f(k)\}] = F(z)$$

$$\therefore \quad Z[\{g(k)\}] - Z[\{g(k-1)\}] = F(z)$$

$$\therefore \quad G(z) - z^{-1} G(z) = F(z) \;\Rightarrow\; (1 - z^{-1})\; G(z) = F(z)$$

$$\therefore \quad \sum_{m=-\infty}^{k} f(m) = G(z) = \frac{F(z)}{1 - z^{-1}}$$

Alternative :

$$Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{k} f(m)\right] z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} [\ldots + f(k-3)\, z^{-k} + f(k-2)\, z^{-k} + f(k-1)\, z^{-k} + f(k)\, z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty} [\ldots + f(k-3)\, z^{-(k-3)} z^{-3} + f(k-2)\, z^{-(k-2)} . z^{-2} + f(k-1)\, z^{-(k-1)} z^{-1} + f(k)\, z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{r=0}^{\infty} f(k-r)\, z^{-(k-r)} z^{-r}$$

$$= \sum_{r=0}^{\infty} z^{-r} \sum_{k=-\infty}^{\infty} f(k-r)\, z^{-(k-r)}, \quad (\text{let } k - r = p)$$

$$= \sum_{r=0}^{\infty} z^{-r} \sum_{p=-\infty}^{\infty} f(p)\, z^{-p} = \sum_{r=0}^{\infty} F(z)\, z^{-r}$$

$$= F(z) \sum_{r=0}^{\infty} z^{-r}$$

$$= F(z) \left(1 + z^{-1} + z^{-2} + \ldots\right)$$

$$= F(z) \frac{1}{1 - z^{-1}}, \quad \left|z^{-1}\right| < 1$$

$$Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \frac{F(z)}{1 - z^{-1}}, \quad |z| > 1.$$

Remark : $\lim\limits_{k \to \infty} g(k) = \lim\limits_{k \to \infty} \sum\limits_{m=-\infty}^{k} f(m) = \sum\limits_{m=-\infty}^{k} f(m).$

By final value theorem,

$$\lim_{k \to \infty} g(k) = \lim_{z \to 1} (z-1) \left(\frac{F(z)}{1 - z^{-1}}\right) \text{ (by using property 10) .}$$

$$= \lim_{z \to 1} (z-1) \frac{F(z)}{z-1} \cdot z = F(1).$$

$$\therefore \qquad \boxed{\sum_{m=-\infty}^{\infty} f(m) = F(1)}$$

11. Convolution :

I. General Case

Convolution of two sequences {f (k)} and {g (k)} denoted as {f (k)} * {g (k)}, is defined as :

$$\{h(k)\} = \{f(k)\} * \{g(k)\}$$

where $$h(k) = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$ (Replacing dummy index m by k – m)

$$= \sum_{m=-\infty}^{\infty} g(m)\, f(k-m)$$

$$= \{g(k)\} * \{f(k)\}$$

Taking Z-transform of both sides, we get

$$Z[\{h(k)\}] = \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} f(m)\; g(k-m) \right] z^{-k}$$

Since the power series converges absolutely, it converges uniformly also within the ROC, this allows us to interchange the order of summation, we get

$$Z[\{h(k)\}] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{k} f(m)\; g(k-m)\, z^{-k}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\; z^{-m} \sum_{k=-\infty}^{\infty} g(k-m)\; z^{-(k-m)}$$

$$= \left[\sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \right] G(z)$$

$$H(z) = F(z)\; G(z)$$

ROC of H (z) is common region of convergence of F (z) and G (z).

We have $\{f(k)\} * \{g(k)\} \leftrightarrow F(z)\; G(z)$.

II. Convolution of Causal Sequences

In this case, f (k) and g (k) are zero for negative values of k, due to this

$$h(k) = \sum_{m=-\infty}^{\infty} f(m)\; g(k-m) \text{ becomes}$$

$$= \sum_{m=0}^{k} f(m)\, g(k-m)$$

Because for negative values of m, f (m) is zero and for values of m > k, g (k – m) becomes zero.

The Z-transform of

$$\{h(k)\} = Z[\{f(k)\} * \{g(k)\}]$$
$$= F(z) \cdot G(z)$$

remains unchanged.

4.8 Z-TRANSFORM OF SOME STANDARD SEQUENCES

1. Unit Impulse :

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$\therefore \quad Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} (0+0+0 \ldots + 1 + 0 + 0 \;\ldots)\, z^{-k}$$

$$\therefore \quad Z\{\delta(k)\} = 1 \text{ as } z^{-k} = z^{0} = 1 \quad \text{for } k = 0.$$

2. Discrete Unit Step :

$$U(k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$$

$$\therefore \quad Z\{U(k)\} = \sum_{k=-\infty}^{\infty} U(k) \cdot z^{-k} = \sum_{k=-\infty}^{-1} 0 \,.\, z^{-k} + \sum_{k=0}^{\infty} 1\,(z^{-k})$$

$$= \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \;\ldots\ldots\right) \text{ + an infinite G.P.}$$

$$= \frac{1}{1-\frac{1}{z}} \qquad \text{for } \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1} \qquad \text{for } 1 < |z|$$

$$\therefore \qquad Z\{U(k)\} = \frac{z}{z-1} \qquad \text{for } |z| > 1$$

$$\therefore \qquad Z^{-1}\left\{\frac{z}{z-1}\right\} = \{U(k)\} \qquad \text{for } |z| > 1.$$

3. $\mathbf{f(k) = a^k,\ k \geq 0}$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k} = \sum_{k=0}^{\infty} a^k\, z^{-k} = \sum_{k=0}^{\infty} (a\,z^{-1})^k$$

$$= 1 + a\,z^{-1} + (a\,z^{-1})^2 + \ldots \text{ an infinite G.P.}$$

$$= \frac{1}{1-az^{-1}} \qquad \text{provided } |a\,z^{-1}| < 1$$

$$= \frac{z}{z-a} \qquad \text{if } |a| < |Z|$$

$$\therefore \qquad Z\{a^k\} = \frac{z}{z-a} \qquad \text{for } |z| > |a|$$

$$\therefore \qquad Z^{-1}\left[\frac{z}{z-a}\right] = a^k, \qquad k \geq 0 \text{ provided } |z| > |a|$$

4. $\mathbf{f(k) = a^k,\ k < 0}$

$$Z\{f(k)\} = \sum_{-\infty}^{-1} f(k)\, z^{-k} = \sum_{-\infty}^{-1} a^k\, z^{-k}$$

Replacing $k \to -k,\ -\infty \leq k \leq -1 \Rightarrow \infty \geq -k \geq 1$

$$\therefore \qquad Z\{f(k)\} = \sum_{1}^{\infty} a^{-k}\, z^k = \sum_{1}^{\infty} (a^{-1}\, z)^k$$

$$= a^{-1}z + (a^{-1}z)^2 + (a^{-1}z)^3 + \ldots\ldots \text{ an infinite G.P.}$$

$$= \frac{a^{-1}z}{1-a^{-1}z} \qquad \text{provided } |a^{-1}\,z| < 1$$

$$\underset{k<0}{Z\{a^k\}} = \frac{z}{a-z} \qquad \text{for } |z| < |a|$$

$$\therefore \qquad Z^{-1}\left\{\frac{z}{a-z}\right\} = a^k \qquad \text{for } k < 0 \text{ if } |z| < |a|$$

5. $\mathbf{f(k) = \left\{a^{|k|}\right\}}$ **for all k**

$$\therefore \qquad Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)\, z^{-k} = \sum_{-\infty}^{-1} f(k)\, z^{-k} + \sum_{0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{-\infty}^{-1} a^{|k|}\, z^{-k} + \sum_{0}^{\infty} a^{|k|}\, z^{-k}$$

$$= \sum_{1}^{\infty} a^{|-k|}\, z^{+k} + \sum_{0}^{\infty} a^k\, z^{-k} = \sum_{1}^{\infty} (a\,z)^k + \sum_{0}^{\infty} (a\,z^{-1})^k$$

$$= \underbrace{\left[az + (az)^2 + (az)^3 + \ldots\right]}_{\text{infinite G.P.}} + \underbrace{\left[1 + (az^{-1}) + (az^{-1})^2 + \ldots\right]}_{\text{infinite G.P.}}$$

$$= \frac{az}{1-az} + \frac{1}{1-az^{-1}}, \quad |az| < 1 \text{ and } |a\,z^{-1}| < 1 \text{ or } |z| < \frac{1}{|a|} \text{ and } |a| < |z|$$

$$\therefore \qquad Z\left\{a^{|k|}\right\} = F(z) = \left(\frac{az}{1-az} + \frac{z}{z-a}\right) \text{ for } |a| < |z| < \frac{1}{|a|}$$

6. $f(k) = \cos \alpha k$, $(k \geq 0)$

We have, $\cos \alpha k = \dfrac{e^{i\alpha k} + e^{-i\alpha k}}{2}$ (by Euler's formula)

$$\therefore \quad Z\{\cos \alpha k\} = \sum_{k=0}^{\infty} \frac{\left(e^{i\alpha k} + e^{-i\alpha k}\right)}{2} z^{-k}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} + \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \left(e^{i\alpha} z^{-1}\right)^k + \sum_{k=0}^{\infty} \left(e^{-i\alpha} z^{-1}\right)^k\right]$$

Both are infinite G.P. $\therefore \quad S_\infty = \dfrac{a}{1-r}$

$$= \frac{1}{2}\left[\frac{1}{1-e^{i\alpha} z^{-1}} + \frac{1}{1-e^{-i\alpha} z^{-1}}\right], \left|e^{i\alpha} z^{-1}\right| < 1 \text{ and } \left|e^{-i\alpha} z^{-1}\right| < 1$$

$$= \frac{1}{2}\left[\frac{1-e^{-i\alpha} z^{-1} + 1 - e^{i\alpha} z^{-1}}{1-e^{i\alpha} z^{-1} - e^{-i\alpha} z^{-1} + z^{-2}}\right] = \frac{1}{2}\left[\frac{2-\left(e^{i\alpha} + e^{-i\alpha}\right) z^{-1}}{1-\left(e^{i\alpha} + e^{-i\alpha}\right) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2-(2\cos\alpha) z^{-1}}{1-(2\cos\alpha) z^{-1} + z^{-2}}\right] = \frac{(z-\cos\alpha)/z}{(z^2 - 2z\cos\alpha + 1)/z^2}$$

$$\therefore \quad F(z) = Z\{\cos \alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1.$$

Note : $e^{i\alpha} = \cos\alpha + i\sin\alpha, \; e^{-i\alpha} = \cos\alpha - i\sin\alpha$

$$\therefore \quad \left|e^{i\alpha}\right| = \left|\cos\alpha + i\sin\alpha\right| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1.$$

and $\quad \left|e^{-i\alpha}\right| = |\cos\alpha - i\sin\alpha| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

$$\therefore \quad \left|e^{i\alpha} z^{-1}\right| < 1 \quad \text{i.e.} \quad \left|z^{-1}\right| < 1 \quad \text{i.e.} \quad |z| > 1$$

Also, $\quad \left|e^{-i\alpha} z^{-1}\right| < 1 \;\Rightarrow\; |z| > 1.$

7. $\{f(k)\} = \{\sin \alpha k\}$, $k \geq 0$

We have $\quad \sin \alpha k = \dfrac{e^{i\alpha k} - e^{-i\alpha k}}{2i}$

$$\therefore \quad Z\{\sin \alpha k\} = \sum_{k=0}^{\infty} \frac{\left(e^{i\alpha k} - e^{-i\alpha k}\right)}{2i} \cdot z^{-k}$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} - \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} \left(e^{i\alpha} z^{-1}\right)^k - \sum_{k=0}^{\infty} \left(e^{-i\alpha} z^{-1}\right)^k\right]$$

Both are infinite G.P., $S_\infty = \dfrac{a}{1-r}$

$$= \frac{1}{2i}\left[\frac{1}{1-e^{i\alpha}z^{-1}} - \frac{1}{1-e^{-i\alpha}z^{-1}}\right], \quad \left|e^{i\alpha}z^{-1}\right| < 1 \text{ and } \left|e^{-i\alpha}z^{-1}\right| < 1$$

$$= \frac{1}{2i}\left[\frac{1-e^{-i\alpha}z^{-1} - 1 + e^{i\alpha}z^{-1}}{1-\left(e^{i\alpha}+e^{-i\alpha}\right)z^{-1} + z^{-2}}\right], \quad |z| > 1$$

$$= \frac{1}{2i}\left[\frac{\dfrac{(e^{i\alpha}-e^{-i\alpha})}{z}}{\dfrac{z^2-(e^{i\alpha}+e^{-i\alpha})z+1}{z^2}}\right] = \frac{1}{2i}\;\frac{z\,(2i\sin\alpha)}{z^2-2z\cos\alpha+1}$$

$$\therefore \quad F(z) = z\,\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, \quad |z| > 1.$$

8. $\{f(k)\} = \{\cosh\,\alpha k\}$, $k \geq 0$

We have, $\cosh\alpha k = \dfrac{e^{\alpha k} + e^{-\alpha k}}{2}$

$$\therefore \quad Z\{\cosh k\} = \sum_{k=0}^{\infty}\frac{(e^{\alpha k}+e^{-\alpha k})}{2}z^{-k} = \frac{1}{2}\left[\sum_{k=0}^{\infty}e^{\alpha k}z^{-k} + \sum_{k=0}^{\infty}e^{-\alpha k}z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty}\left(e^{\alpha}z^{-1}\right)^k + \sum_{k=0}^{\infty}\left(e^{-\alpha}z^{-1}\right)^k\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{\alpha}z^{-1}} + \frac{1}{1-e^{-\alpha}z^{-1}}\right], \quad \left|e^{\alpha}z^{-1}\right| < 1 \text{ and } \left|e^{-\alpha}z^{-1}\right| < 1$$

$$= \frac{1}{2}\left[\frac{1-e^{-\alpha}z^{-1}+1-e^{\alpha}z^{-1}}{1-(e^{\alpha}+e^{-\alpha})z^{-1}+z^{-2}}\right], \quad |z| > \max.\left(\left|e^{\alpha}\right| \text{ or } \left|e^{-\alpha}\right|\right)$$

$$= \frac{1}{2}\left[\frac{2-\left(e^{\alpha}+e^{-\alpha}\right)z^{-1}}{1-\left(e^{\alpha}+e^{-\alpha}\right)z^{-1}+z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{\dfrac{2z-2\cosh\alpha}{z}}{\dfrac{z^2-2z\cosh\alpha+1}{z^2}}\right] \qquad (\because\ e^{\alpha}+e^{-\alpha} = 2\cosh\alpha)$$

$$Z\{\cosh\alpha k\} = \frac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}, \quad |z| > \max.\left(\left|e^{\alpha}\right| \text{ or } \left|e^{-\alpha}\right|\right)$$

9. $\{f(k)\} = \{\sinh\alpha k\}$, $k \geq 0$

We have, $\sinh\alpha k = \dfrac{e^{\alpha k} - e^{-\alpha k}}{2}$

$$\therefore \quad Z\{\sinh\alpha k\} = \sum_{0}^{\infty}\frac{\left(e^{\alpha k}-e^{-\alpha k}\right)}{2}z^{-k}$$

$$= \frac{1}{2}\left[\sum_{0}^{\infty}\left(e^{\alpha}z^{-1}\right)^k - \sum_{0}^{\infty}\left(e^{-\alpha}z^{-1}\right)^k\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{\alpha}z^{-1}} + \frac{1}{1-e^{-\alpha}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-\alpha}z^{-1}-1+e^{\alpha}z^{-1}}{1-(e^{\alpha}+e^{-\alpha})z^{-1}+z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{\left(e^{\alpha} - e^{-\alpha}\right) z^{-1}}{1 - \left(e^{\alpha} + e^{-\alpha}\right) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{2 \sinh \alpha}{z}}{\frac{z^2 - 2z \cosh \alpha + 1}{z^2}}\right] \qquad (\because\ e^{\alpha} - e^{-\alpha} = 2 \sinh \alpha)$$

$$\therefore \quad F(z) = Z\{\sinh \alpha k\} = \frac{z \sinh \alpha}{z^2 - 2z \cosh \alpha + 1}, \quad |z| > \max.\left(\left|e^{\alpha}\right| \text{ or } \left|e^{-\alpha}\right|\right)$$

10. $\{f(k)\} = \left\{{}^{n}C_{k}\right\}, \; (0 \le k \le n)$

Since ${}^{n}C_{k} = 0$, if $k > n$ i.e. $0 \le k \le n$

$$\therefore \quad Z\left\{{}^{n}C_{k}\right\} = \sum_{k=0}^{\infty} {}^{n}C_{k}\, z^{-k} = {}^{n}C_{0} + {}^{n}C_{1}\, z^{-1} + {}^{n}C_{2}\, z^{-2} + \ldots = \left(1 + z^{-1}\right)^{n}$$

$$\therefore \quad Z\left\{{}^{n}C_{k}\right\} = \left(1 + z^{-1}\right)^{n}, \quad |z| > 0$$

11. $\{f(k)\} = \left\{{}^{k}C_{n}\right\} \; (k \ge n)$

Since ${}^{k}C_{n} = 0$ if $k < n$

$$\therefore \quad Z\left\{{}^{k}C_{n}\right\} = \sum_{k=n}^{\infty} {}^{k}C_{n}\, z^{-k}$$

Put $k = n + r$, when

$k = n$	$r = \infty$
$k = \infty$	$r = \infty$

$$\therefore \quad Z\left\{{}^{k}C_{n}\right\} = \sum_{r=0}^{\infty} {}^{n+r}C_{n}\, z^{-(n+r)}$$

As ${}^{n}C_{r} = {}^{n}C_{n-r}$, it follows that ${}^{n+r}C_{n} = {}^{n+r}C_{r}$

$$\therefore \quad Z\left\{{}^{k}C_{n}\right\} = \sum_{r=0}^{\infty} {}^{n+r}C_{r}\, z^{-(n+r)} = \sum_{r=0}^{\infty} {}^{n+r}C_{r} \cdot z^{-r} \cdot z^{-n}$$

$$= z^{-n}\left[{}^{n}C_{0} + {}^{n+1}C_{1}\, z^{-1} + {}^{n+1}C_{2}\, z^{-2} \ldots \ldots\right]$$

$$Z\left\{{}^{k}C_{n}\right\} = z^{-n}\left(1 - z^{-1}\right)^{-(n+1)}, \quad |z| > 1.$$

12. $\{f(k)\} = \left\{{}^{(k+n)}C_{n}\right\}$

$$Z\left\{{}^{(k+n)}C_{n}\right\} = \sum_{k=-\infty}^{\infty} {}^{(k+n)}C_{n}\, z^{-k}$$

$${}^{(k+n)}C_{n} = 0 \text{ if } k + n < n \text{ i.e. if } k < 0$$

$$\therefore \quad Z\left[\left\{{}^{(k+n)}C_{n}\right\}\right] = \sum_{k=0}^{\infty} {}^{(k+n)}C_{n}\, z^{-k} = \sum_{k=0}^{\infty} {}^{(k+n)}C_{k}\, z^{-k}$$

$$= \left[1 + \frac{n+1}{1} z^{-1} + \frac{(n+2)(n+1)}{1 \,.\, 2} z^{-2} + \ldots\right]$$

$$= \left(1 - z^{-1}\right)^{-(n+1)}, \quad |z| > 1$$

$$\therefore \quad Z\left[\left\{{}^{(k+n)}C_{n}\right\}\right] (k \ge 0) \leftrightarrow \left(1 - z^{-1}\right)^{-(n+1)}, \quad |z| > 1.$$

13. $\{f(k)\} = \left\{{}^{k+n}C_n\, a^k\right\}$

$$Z\left[\left\{{}^{k+n}C_n\, a^k\right\}\right] = \sum_{k=0}^{\infty} {}^{k+n}C_n \left(az^{-1}\right)^k$$

$$= \left(1 - az^{-1}\right)^{-(n+1)}, \qquad |z| > |a|$$

$\therefore$ By putting n = 1, we have

$$Z\left[\left\{(k+1)\,a^k\right\}\right] = \left(1 - az^{-1}\right)^{-2} = \frac{z^2}{(z-a)^2}, \quad |z| > |a|$$

By putting n = 2, we have

$$Z\left[\left\{\frac{(k+1)\;(k+2)}{2!}\;a^k\right\}\right] = Z\left[\left\{{}^{k+2}C_2\;a^k\right\}\right] = \left(1 - az^{-1}\right)^{-3}$$

$$= \frac{z^3}{(z-a)^3} \quad |z| > |a|$$

By putting n – 1 in place of n, we have

$$Z\left[\left\{\frac{(k+1)\,\ldots\,(k+n-1)}{(n-1)\,!}\right\}\right] = \left(1 - az^{-1}\right)^{-n}, \quad |z| > |a|$$

$$= \frac{z^n}{(z-a)^n}, \qquad |z| > |a|$$

$$\therefore \qquad \left[(k+1)\,a^k\right] \leftrightarrow \frac{z^2}{(z-a)^2}, \qquad |z| > |a|$$

$$\left\{\frac{(k+1)\,(k+2)}{2!}\,a^k\right\} \leftrightarrow \frac{z^3}{(z-a)^3}, \qquad |z| > |a|$$

$$\left\{\frac{(k+1)\,(k+2)\,\ldots\left(k+\overline{n-1}\right)}{(n-1)\,!}\right\} \leftrightarrow \frac{z^n}{(z-a)^n}, \qquad |z| > |a|$$

These results are very useful in obtaining inverse Z-transform.

14. $\{f(k)\} = \left\{\dfrac{a^k}{k!}\right\}$ **, k ≥ 0**

$$Z\{f(k)\} = Z\left\{\frac{a^k}{k\,!}\right\} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\,z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(a\,z^{-1})^k}{k\,!}$$

$$= 1 + \frac{\left(az^{-1}\right)}{1\,!} + \frac{\left(az^{-1}\right)^2}{2\,!} + \frac{\left(az^{-1}\right)^3}{3\,!} + \ldots\ldots\ldots\ldots$$

$$= e^{(az^{-1})} = e^{a/z}$$

$$\therefore \qquad Z\left\{\frac{a^k}{k\,!}\right\} = e^{a/z} = e^{az^{-1}} \qquad \left(\text{Since, } e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots\ldots\ldots\right)$$

15. $\{f(k)\} = \left\{c^k \cos \alpha k\right\}, \; k \geq 0$

We have

$$Z\{\cos \alpha k\} = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1}, \quad |z| > 1$$
$$= F(z)$$

By using change of scale property,

$$Z\{f(k)\} = F(z) \text{ then } Z\left\{c^k f(k)\right\} = F\left(\frac{z}{c}\right)$$

$$\therefore \quad Z\left\{c^k \cos \alpha k\right\} = \frac{\frac{z}{c}\left(\frac{z}{c} - \cos \alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right) \cos \alpha + 1}, \text{ provided } \left|\frac{z}{c}\right| > 1$$

$$= \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}, \text{ provided } |z| > |c|$$

16. $\{f(k)\} = \left\{c^k \sin \alpha k\right\}, \; k \geq 0$

$$Z\{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}, \quad |z| > 1$$

$$\therefore \quad Z\left\{c^k \sin \alpha k\right\} = \frac{\left(\frac{z}{c}\right) \sin \alpha}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right) \cos \alpha + 1}, \quad \left|\frac{z}{c}\right| > 1$$

$$= \frac{c z \sin \alpha}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c|$$

17. $\{f(k)\} = \left\{c^k \cosh \alpha k\right\}, \; k \geq 0$

$$\because \quad Z\{\cosh \alpha k\} = \frac{z(z - \cosh \alpha)}{z^2 - 2z \cosh \alpha + 1}$$

provided $|z| > \max.\left(\left|e^{\alpha}\right| \text{ or } \left|e^{-\alpha}\right|\right)$

$$Z\left\{c^k \cosh \alpha k\right\} = \frac{\frac{z}{c}\left(\frac{z}{c} - \cosh \alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right) \cosh \alpha + 1}$$

provided $|z| > \max.\left(\left|c\,e^{\alpha}\right| \text{ or } \left|c\,e^{-\alpha}\right|\right)$

$$= \frac{z(z - c \cosh \alpha)}{z^2 - 2cz \cosh \alpha + c^2}$$

18. $\{f(k)\} = \left\{c^k \sinh \alpha k\right\}, \; k \geq 0$

Proceeding in the same manner as $c^k \cosh \alpha k$

$$Z\left\{c^k \sinh \alpha k\right\} = \frac{cz \sinh \alpha}{z^2 - 2cz \cosh \alpha + c^2}, \quad |z| > \max.\left(\left|c\,e^{\alpha}\right| \text{ or } \left|c\,e^{-\alpha}\right|\right)$$

4.9 TABLE OF PROPERTIES OF Z-TRANSFORMS

1.	**Definition**	$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$
2.	**Inverse**	$Z^{-1}\{F(z)\} = \{f(k)\}$
3.	**Linearity**	$Z\left\{a f(k) + b g(k)\right\} = a F(z) + b G(z)$
4.	**Change of Scale**	$Z\left\{a^k f(k)\right\} = F\left(\frac{z}{a}\right)$

5. Shifting

(a) Both sided sequence

$$Z\left\{f(k \pm n)\right\} = z^{\pm n}\, F(z)$$

(b) One sided sequence, $k \geq 0$

$$Z\{f(k+n)\} = z^{n}\, F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

and for $k < 0$

$$Z\{f(k-n)\} = z^{-n}\, F(z) + \sum_{r=-n}^{-1} f(r)\, z^{-(n+r)}$$

(c) Causal sequence

$$Z\{f(k+n)\} = z^{n}\, F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$$Z\{f(k-n)\} = z^{-n}\, F(z)$$

6. Multiplication by k

$$Z\{k\, f(k)\} = \left(-z\,\frac{d}{dz}\right) F(z)$$

$$Z\{k^{n}\, f(k)\} = \left(-z\,\frac{d}{dz}\right)^{n} F(z)$$

7. Division by k

$$Z\left\{\frac{f(k)}{k}\right\} = -\int^{z} z^{-1}\, F(z)\, dz$$

8. Initial Value Theorem

$$f(0) = \lim_{z \to \infty} F(z)$$

if $\{f(k)\}$ is one sided sequence i.e. $k \geq 0$

9. Final Value Theorem

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1)\, F(z)$$

if $\{f(k)\}$ is one sided sequence ($k \geq 0$)

10. Partial Sum

$$Z\left\{\sum_{m=-\infty}^{k} f(m)\right\} = \frac{F(z)}{1 - z^{-1}}$$

$$\sum_{m=-\infty}^{\infty} = F(1)$$

11. Convolution

$$Z\left\{\{f(k)\} * \{g(k)\}\right\} = F(z) \cdot G(z)$$

$$h(k) = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$

If causal then $$h(k) = \sum_{m=0}^{k} f(m)\, g(k-m)$$

where $$h(k) = \{f(k)\} * \{g(k)\}$$

12. $$Z\left\{e^{-ak}\, f(k)\right\} = F\left(e^{a}\, z\right)$$

4.10 TABLE OF Z-TRANSFORM OF SOME STANDARD SEQUENCES

1.	$Z\{\delta(k)\} = 1,$	for all z
2.	$Z\{U(k)\} = \dfrac{z}{z-1},$	$\lvert z\rvert > 1$
3.	$Z\{1\} = \dfrac{z}{z-1},$	$\lvert z\rvert > 1$
4.	$Z\{a^k\} = \dfrac{z}{z-a},\ k \geq 0$	$\lvert z\rvert > \lvert a\rvert$
5.	$Z\{a^k\} = \dfrac{z}{a-z},\ k < 0,$	$\lvert z\rvert < \lvert a\rvert$
6.	$Z\left\{a^{\lvert k\rvert}\right\} = \dfrac{az}{1-az} + \dfrac{z}{z-a},$	$\lvert a\rvert < \lvert z\rvert < \dfrac{1}{\lvert a\rvert}$
7.	$\underset{k \geq 0}{Z\{\cos\alpha k\}} = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1},$	$\lvert z\rvert > 1$
8.	$\underset{k \geq 0}{Z\{\sin\alpha k\}} = \dfrac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1},$	$\lvert z\rvert > 1$
9.	$\underset{k \geq 0}{Z\{\cosh\alpha k\}} = \dfrac{z(z-\cosh\alpha)}{z^2 - 2z\cosh\alpha + 1},$	$\lvert z\rvert > \max.\left(\lvert e^{\alpha}\rvert \text{ or } \lvert e^{-\alpha}\rvert\right)$
10.	$\underset{k \geq 0}{Z\{\sinh\alpha k\}} = \dfrac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1},$	$\lvert z\rvert > \max.\left(\lvert e^{\alpha}\rvert \text{ or } \lvert e^{-\alpha}\rvert\right)$
11.	$\underset{k \geq 0}{Z\{c^k\cos\alpha k\}} = \dfrac{z(z-c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2},$	$\lvert z\rvert > \lvert c\rvert$
12.	$\underset{k \geq 0}{Z\{c^k\sin\alpha k\}} = \dfrac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2},$	$\lvert z\rvert > \lvert c\rvert$
13.	$\underset{k \geq 0}{Z\{c^k\cosh\alpha k\}} = \dfrac{z(z-c\cosh\alpha)}{z^2 - 2cz\cosh\alpha + c^2},$	$\lvert z\rvert > \max.\left(\lvert ce^{\alpha}\rvert \text{ or } \lvert ce^{-\alpha}\rvert\right)$
14.	$\underset{k \geq 0}{Z\{c^k\sinh\alpha k\}} = \dfrac{cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c^2},$	$\lvert z\rvert > \max.\left(\lvert ce^{\alpha}\rvert \text{ or } \lvert ce^{-\alpha}\rvert\right)$
15.	$Z\left\{{}^nC_k\right\} = \left(1 + z^{-1}\right)^n,\quad 0 \leq k \leq n,$	$\lvert z\rvert > 0$
16.	$Z\left\{{}^kC_n\right\}, (k \geq n) = z^{-n}\left(1 - z^{-1}\right)^{-(n+1)},$	$\lvert z\rvert > 1$
17.	$Z\left\{{}^{k+n}C_n\right\}, k \geq 0 = \left(1 - z^{-1}\right)^{-(n+1)},$	$\lvert z\rvert > 1$
18.	$Z\left\{{}^{k+n}C_n\, a^k\right\} = \left(1 - az^{-1}\right)^{-(n+1)},$	$\lvert z\rvert > \lvert a\rvert$
19.	$Z\{(k+1)\,a^k\} = \dfrac{z^2}{(z-a)^2},$	$\lvert z\rvert > \lvert a\rvert$
20.	$Z\left\{\dfrac{(k+1)(k+2)}{2!}a^k\right\} = \dfrac{z^3}{(z-a)^3},$	$\lvert z\rvert > \lvert a\rvert$
21.	$Z\left\{\dfrac{(k+1)(k+2)\ \ldots\ (k+(n-1))}{(n-1)!}a^k\right\} = \dfrac{z^n}{(z-a)^n},$	$\lvert z\rvert > \lvert a\rvert$
22.	$Z\left\{\dfrac{a^k}{k!}\right\} = e^{a/z},\ k \geq 0$	$\forall z$

ILLUSTRATIONS ON Z-TRANSFORMS

Ex. 1 : *Find the Z – transform and its ROC of*

(i) 2^k, $k \geq 0$, ***(Dec, 2017)*** *(ii)* 3^k, $k < 0$ ***(Dec. 2017)***

(iii) $\left(\frac{1}{3}\right)^k$, $k \geq 0$ *(iv)* $\left(\frac{1}{5}\right)^k$, $k < 0$

Sol. : (i)

$$f(k) = 2^k, \; k \geq 0$$

$$Z\{2^k\} = \sum_{k=0}^{\infty} 2^k z^{-k} = \sum_{k=0}^{\infty} \left(2z^{-1}\right)^k$$

$$= 1 + \left(2z^{-1}\right) + \left(2z^{-1}\right)^2 + \ldots\ldots\ldots$$

$$= \frac{1}{1 - 2z^{-1}}, \quad \text{if } \left|2z^{-1}\right| < 1$$

$$Z\{2^k\} = \frac{z}{z-2}, \quad |z| > 2$$

$$\{2^k\} \leftrightarrow \frac{z}{z-2}, \quad k \geq 0$$

(ii) $f(k) = 3^k$, $k < 0$

$$Z\{3^k\} = \sum_{k=-\infty}^{-1} 3^k z^{-k} \quad \text{Put } k = -r,$$

$k = -\infty$	$r = \infty$
$k = -1$	$r = 1$

$$= \sum_{r=1}^{\infty} 3^{-r} z^r = \sum_{r=1}^{\infty} (3^{-1} z)^r = (3^{-1} z) + (3^{-1} z)^2 + \ldots\ldots$$

$$= \frac{3^{-1} z}{1 - 3^{-1} z}, \quad \text{if } \left|3^{-1} z\right| < 1$$

$$= \frac{z}{3-z}, \quad \text{if } |z| < 3$$

$$\underset{(k<0)}{\{3^k\}} \leftrightarrow \frac{z}{3-z}$$

(iii)

$$f(k) = \left(\frac{1}{3}\right)^k, \; k \geq 0$$

$$Z\left\{(1/3)^k\right\} = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^k$$

$$= 1 + \left(\frac{1}{3} z^{-1}\right) + \left(\frac{1}{3} z^{-1}\right)^2 + \ldots\ldots$$

$$= \frac{1}{1 - \frac{1}{3} z^{-1}}, \text{ if } \left|\frac{1}{3} z^{-1}\right| < 1$$

$$= \frac{z}{z - \frac{1}{3}}, \quad \text{if } |z| > \frac{1}{3}$$

$$\left(\frac{1}{3}\right)^k \leftrightarrow \frac{z}{z - \frac{1}{3}}, \quad k \geq 0$$

(iv) $f(k) = \left(\frac{1}{5}\right)^k, \; k < 0$

$$Z\left\{\left(\frac{1}{5}\right)^k\right\} = \sum_{k=-\infty}^{-1} \left(\frac{1}{5}\right)^k z^{-k} \quad \text{Put } k = -r$$

$k = -\infty$	$r = \infty$
$k = -1$	$r = 1$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{5}\right)^{-r} z^r = \sum_{r=1}^{\infty} \left[\left(\frac{1}{5}\right)^{-1} z\right]^r$$

$$= \sum_{r=1}^{\infty} (5z)^r = 5z + (5z)^2 + \ldots\ldots$$

$$= \frac{5z}{1-5z}, \quad |5z| < 1$$

$$\underset{(k<0)}{\left(\frac{1}{5}\right)^k} \leftrightarrow \frac{5z}{1-5z}, \quad |z| < \frac{1}{5}$$

Ex. 2 : *Find Z {f (k)}*

where $f(k) = 3^k, \quad k < 0$ ***(Dec. 2010)***

$\qquad = 2^k, \quad k \geq 0$

Sol. :

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 3^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= \sum_{r=1}^{\infty} 3^{-r} z^r + \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= \sum_{r=1}^{\infty} (3^{-1} z)^r + \sum_{k=0}^{\infty} \left(2z^{-1}\right)^k$$

$$= \frac{3^{-1} z}{1 - 3^{-1} z} + \frac{1}{1 - 2z^{-1}} \quad \text{provided } |3^{-1} z| < 1 \text{ and } |2z^{-1}| < 1$$

$$F(z) = \frac{z}{3-z} + \frac{z}{z-2}, \quad |z| < 3 \text{ and } 2 < |z|$$

$$= \frac{z}{(3-z)(z-2)} \quad \text{if } 2 < |z| < 3$$

Ex. 3 : *Find Z {f (k)} if* $f(x) = \left(\frac{1}{4}\right)^{|k|}$ *for all k.* ***(May 2006, 2010)***

Sol. :

$$Z\left\{\left(\frac{1}{4}\right)^{|k|}\right\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{4}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k z^{-k}$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{4}\right)^r z^r + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k z^{-k}$$

$$= \frac{\frac{1}{4}z}{1-\frac{1}{4}z} + \frac{1}{1-\frac{1}{4z}} \quad \text{provided} \quad \left|\frac{1}{4}z\right| < 1 \quad \text{and} \quad \left|\frac{1}{4z}\right| < 1$$

$$F(z) = \frac{1}{4}\frac{z}{1-\frac{z}{4}} + \frac{z}{z-\frac{1}{4}}, \quad \frac{1}{4} < |z| < 4$$

Ex. 4 : *Find Z {f (k)} if (i)* $f(k) = \frac{1}{k}$, $k \geq 1$, *(ii)* $f(k) = \frac{a^k}{k}$, $k \geq 1$.

Sol. : (i) $f(k) = \frac{1}{k}, \quad k \geq 1$

Assuming $f(k) = 0$ for $k \leq 0$

$$Z\{f(k)\} = Z\left\{\frac{1}{k}\right\} = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} = z^{-1} + \frac{(z^{-1})^2}{2} + \frac{(z^{-1})^3}{3} + \ldots\ldots = -\log\left(1 - z^{-1}\right)$$

Applying D'Alembert's Ratio test, we find that the series is convergent if $|z| > 1$.

Note :
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$
$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots$$

(ii) $f(k) = \frac{a^k}{k}, \quad k \geq 1$

Assuming $f(k) = 0, \; k \leq 0$

$$Z\left\{\frac{a^k}{k}\right\} = \sum_{k=1}^{\infty} \frac{a^k}{k} z^{-k}$$
$$= a z^{-1} + \frac{(a z^{-1})^2}{2} + \frac{(a z^{-1})^3}{3} + \ldots\ldots$$
$$= -\log\left(1 - a z^{-1}\right)$$

Applying D'Alembert's Ratio test, we find that the series is convergent if $|a z^{-1}| < 1$ i.e. $|a| < |z|$ or $|z| > |a|$.

Ex. 5 : *Find Z {f (k)} where (i)* $f(k) = \frac{2^k}{k!}$ $k \geq 0$ ***(May 2011, 2012, Dec. 2012)***

(ii) $f(k) = e^{-ak}$, $k \geq 0$. ***(May 2009, Dec. 2010)***

Sol. : (i)
$$Z\left\{\frac{2^k}{k!}\right\} = \sum_{k=0}^{\infty} \frac{2^k}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{(2z^{-1})^k}{k!}$$
$$= \frac{1}{0!} + \frac{(2z^{-1})}{1!} + \frac{(2z^{-1})^2}{2!} + \ldots\ldots = e^{2z^{-1}} = e^{2/z} \text{ where, ROC is all of Z-plane.}$$

(ii)
$$Z\{e^{-ak}\} = \sum_{k=0}^{\infty} e^{-ak} z^{-k} = \sum_{k=0}^{\infty} \left(e^{-a} z^{-1}\right)^k$$
$$= 1 + \left(e^{-a} z^{-1}\right) + \left(e^{-a} z^{-1}\right)^2 + \ldots\ldots$$
$$= \frac{1}{1 - e^{-a} z^{-1}}, \quad |e^{-a} z^{-1}| < 1$$
$$= \frac{z}{z - e^{-a}}, \quad |z| > |e^{-a}|$$

Ex. 6 : *Find Z {f (k)}, where*

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{2}\right)^k, & k = 0, 2, 4, 6, \ldots\ldots \\ \left(\frac{1}{3}\right)^k, & k = 1, 3, 5, 7, \ldots\ldots \end{cases}$$

(Dec. 2005, 2017)

Sol. :

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{2n} f(k)\, z^{-k} + \sum_{k=1}^{2n-1} f(k)\, z^{-k} \qquad \text{for } n = 1, 2, 3, \ldots\ldots$$

$$= \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{2n} \left(\frac{1}{2}\right)^k z^{-k} + \sum_{k=1}^{2n-1} \left(\frac{1}{3}\right)^k z^{-k}$$

$$= \sum_{r=1}^{\infty} 2^{-r} z^r + \sum_{k=0}^{2n} \left(\frac{1}{2} z^{-1}\right)^k + \sum_{k=1}^{2n-1} \left(\frac{1}{3} z^{-1}\right)^k$$

$$= \frac{2^{-1} z}{1 - 2^{-1} z} + \left[1 + \left(\frac{1}{2} z^{-1}\right)^2 + \left(\frac{1}{2} z^{-1}\right)^4 + \ldots\right] + \left[\frac{1}{3} z^{-1} + \left(\frac{1}{3} z^{-1}\right)^3 + \left(\frac{1}{3} z^{-1}\right)^5 + \ldots\right]$$

$$= \frac{z}{2 - z} + \frac{1}{1 - \left(\frac{1}{2} z^{-1}\right)^2} + \frac{\frac{1}{3} z^{-1}}{1 - \left(\frac{1}{3} z^{-1}\right)^2} = \frac{z}{2 - z} + \frac{1}{1 - \frac{1}{4 z^2}} + \frac{\frac{1}{3 z}}{1 - \frac{1}{9 z^2}}$$

$$F(z) = \frac{z}{2 - z} + \frac{4z^2}{4z^2 - 1} + \frac{3z}{9z^2 - 1}$$

provided $\left|2^{-1} z\right| < 1$; $\left|\left(\frac{1}{2} z^{-1}\right)^2\right| < 1$; $\left|\left(\frac{1}{3} z^{-1}\right)^2\right| < 1$ or $|z| < 2$; $\frac{1}{2} < |z|$; $\frac{1}{3} < |z|$

$\therefore$ ROC is $\frac{1}{2} < |z| < 2$

Ex. 7 : *Find Z {f (k)} where,*

(i) $f(k) = \begin{cases} -\left(-\frac{1}{3}\right)^k, & k < 0 \\ \left(-\frac{1}{4}\right)^k, & k \geq 0 \end{cases}$ *(ii)* $f(k) = 4^k + 5^k,\ k \geq 0.$

Sol. : (i)

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} -\left(-\frac{1}{3}\right)^k z^{-k} + \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k z^{-k}$$

$$= -\sum_{r=1}^{\infty} \left(-\frac{1}{3}\right)^{-r} z^r + \sum_{k=0}^{\infty} \left(-\frac{1}{4} z^{-1}\right)^k$$

$$= -\frac{\left(-\frac{1}{3}\right)^{-1} z}{1 - \left(-\frac{1}{3}\right)^{-1} z} + \frac{1}{1 - \left(-\frac{1}{4} z^{-1}\right)}$$

$$= \frac{3z}{1 + 3z} + \frac{4z}{4z + 1} \text{ provided } \left|\left(-\frac{1}{3}\right)^{-1} z\right| < 1 \text{ and } \left|-\frac{1}{4} z^{-1}\right| < 1 \text{ or } |z| < \frac{1}{3} \text{ and } \frac{1}{4} < |z|$$

$\therefore$ ROC is $\frac{1}{4} < |z| < \frac{1}{3}$

(ii) $$Z\{4^k + 5^k\} = Z\{4^k\} + Z\{5^k\}$$

$$= \sum_{k=0}^{\infty} 4^k z^{-k} + \sum_{k=0}^{\infty} 5^k z^{-k}$$

$$= \frac{1}{1-4z^{-1}} + \frac{1}{1-5z^{-1}}, \quad |4z^{-1}| < 1 \text{ and } |5z^{-1}| < 1$$

$$= \frac{z}{z-4} + \frac{z}{z-5}, \quad 4 < |z| \text{ and } 5 < |z|$$

$\therefore$ ROC is $|z| > 5$.

Ex. 8 : *Find Z {f (k)} if*

(i) $f_k = \left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1}, \; k \geq 0$ *(ii)* $f_k = \begin{cases} \left(\frac{1}{3}\right)^k, & k \geq 0 \\ 2^k, & k < 0 \end{cases}$ ***(Dec. 2007)***

Sol. : (i) $$f(k) = \left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1} = \left(-\frac{1}{2}\right)^k \left(-\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{2}\right)^k + \frac{3}{2}\left(\frac{1}{2}\right)^k.$$

$$Z\{f(k)\} = Z\left\{-\frac{1}{2}\left(-\frac{1}{2}\right)^k + \frac{3}{2}\left(\frac{1}{2}\right)^k\right\} = -\frac{1}{2} Z\left\{\left(-\frac{1}{2}\right)^k\right\} + \frac{3}{2} \cdot Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$= -\frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k z^{-k} + \frac{3}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^{-k} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^k + \frac{3}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^k$$

$$= -\frac{1}{2}\frac{1}{1-\left(-\frac{1}{2}z^{-1}\right)} + \frac{3}{2}\frac{1}{1-\frac{1}{2}z^{-1}} \text{ provided } \left|-\frac{1}{2}z^{-1}\right| < 1 \text{ and } \left|\frac{1}{2}z^{-1}\right| < 1$$

$$= -\frac{1}{2}\cdot\left(\frac{z}{z+\frac{1}{2}}\right) + \frac{3}{2}\left(\frac{z}{z-\frac{1}{2}}\right), \qquad \frac{1}{2} < |z| \text{ or } |z| > \frac{1}{2}.$$

$\therefore$ ROC is $|z| > \frac{1}{2}$.

(ii) $$f(k) = \begin{cases} \left(\frac{1}{3}\right)^k, & k \geq 0 \\ 2^k, & k < 0 \end{cases}$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k z^{-k}$$

$$= \sum_{r=1}^{\infty} 2^{-r} z^r + \sum_{k=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^k$$

$$= \frac{2^{-1}z}{1-2^{-1}z} + \frac{1}{1-1/3\,z^{-1}} \text{ provided } |(2)^{-1} z \,|\, z < 1 \text{ and } \left|\frac{1}{3}z^{-1}\right| < 1$$

$$= \frac{z}{2-z} + \frac{z}{z-1/3} \qquad \text{provided } |z| < 2 \text{ and } \frac{1}{3} < |z|$$

$\therefore$ ROC is $\frac{1}{3} < |z| < 2$

Ex. 9 : *Find Z {f (k)} if* $f(k) = \left(\frac{1}{2}\right)^{|k|}$ *for all k.* ***(May 2005)***

Sol. :

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{|k|} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k}$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^{r} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^{k}$$

$$= \frac{\frac{1}{2} z}{1 - \frac{1}{2} z} + \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \text{provided } \left|\frac{1}{2} z\right| < 1 \text{ and } \left|\frac{1}{2} z^{-1}\right| < 1$$

$$= \frac{z}{2 - z} + \frac{z}{z - \frac{1}{2}} \quad \text{provided } |z| < 2 \quad \text{and} \quad \frac{1}{2} < |z|$$

$\therefore$ ROC is $\frac{1}{2} < |z| < 2$.

Ex. 10 : *Find Z {f (k)} if* $f(k) = a \cos k\alpha + b \sin k\alpha,\ k \geq 0$.

Sol. : $Z\{a \cos \alpha k + b \sin \alpha k\} = a\, Z\{\cos \alpha k\} + b Z\{\sin \alpha k\}$ (by using linearity property)

$$= a \cdot \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1} + b \cdot \frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

$$= \frac{a z^2 + z(b \sin\alpha - a\cos\alpha)}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

Ex. 11 : *Find Z {f (k)} if*

(i) $f(k) = \frac{\sin ak}{k},\ k > 0$ ***(May 2005, 2012, 2015; Dec. 2010, 2012)*** *(ii)* $f(k) = \frac{2^k}{k},\ k \geq 1$ ***(Dec. 2006)***

Sol. : (i)

$$f(k) = \frac{\sin ak}{k}, \quad k > 0$$

$$Z\{\sin ak\} = \frac{z \sin a}{z^2 - 2z\cos a + 1}$$

$$Z\left\{\frac{\sin ak}{k}\right\} = \int_z^{\infty} \frac{1}{z} \frac{z \sin a}{z^2 - 2z\cos a + 1}\, dz = \int_z^{\infty} \frac{\sin a}{z^2 - 2z\cos a + 1}\, dz$$

$$= \sin a \int_z^{\infty} \frac{dz}{(z - \cos a)^2 + \sin^2 a}$$

$$= \sin a \left[\frac{1}{\sin a} \tan^{-1}\left(\frac{z - \cos a}{\sin a}\right)\right]_z^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z - \cos a}{\sin a}\right) = \cot^{-1}\left(\frac{z - \cos a}{\sin a}\right).$$

(ii)
$$f(k) = \frac{2^k}{k}, \quad k \geq 1$$

$$Z\{2^k\} = \sum_{k=1}^{\infty} 2^k z^{-k} = \frac{2z^{-1}}{1-2z^{-1}}, \quad |z| > 2$$

$$= \frac{2}{z-2}, \quad |z| > 2$$

$$Z\left\{\frac{2^k}{k}\right\} = \int_z^{\infty} z^{-1} \frac{2}{z-2} dz = 2\int_z^{\infty} \frac{1}{z(z-2)} dz$$

$$= 2\int_z^{\infty} \left(-\frac{1/2}{z} + \frac{1/2}{z-2}\right) dz = \int_z^{\infty} \left(-\frac{1}{z} + \frac{1}{z-2}\right) dz$$

$$= \left[-\log z + \log(z-2)\right]_z^{\infty} = -\log \frac{z-2}{z}$$

$$= -\log\left(1-2z^{-1}\right), \quad |z| > 2$$

Ex. 12 : *Find Z {f (k)} where*

(i) $f(k) = \sin\left(\frac{k\pi}{4} + \alpha\right), \quad k \geq 0$ *(ii)* $f(k) = \cos\left(\frac{k\pi}{4} + \alpha\right), \quad k \geq 0$ ***(Dec. 2005)***

Sol. : (i)
$$\sin\left(\frac{k\pi}{4} + \alpha\right) = \sin\frac{k\pi}{4} \cdot \cos\alpha + \cos\frac{k\pi}{4} \cdot \sin\alpha$$

$$Z\left\{\sin\left(\frac{k\pi}{4} + \alpha\right)\right\} = \cos\alpha \cdot Z\left\{\sin\left(\frac{k\pi}{4}\right)\right\} + \sin\alpha \cdot Z\left\{\cos\left(\frac{k\pi}{4}\right)\right\}$$

$$= \cos\alpha \cdot \frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1} + \sin\alpha \cdot \frac{z\left(z - \cos\frac{\pi}{4}\right)}{z^2 - 2z\cos\frac{\pi}{4} + 1}$$

$$= \frac{\cos\alpha \, \frac{z}{\sqrt{2}}}{z^2 - \frac{2z}{\sqrt{2}} + 1} + \sin\alpha \, \frac{z\left(z - \frac{1}{\sqrt{2}}\right)}{z^2 - \frac{2z}{\sqrt{2}} + 1}$$

$$= \frac{z}{\sqrt{2}}\left[\frac{\cos\alpha + \sin\alpha\left(\sqrt{2}z - 1\right)}{z^2 - \sqrt{2}z + 1}\right], \quad |z| > 1$$

(ii)
$$\cos\left(\frac{k\pi}{4} + \alpha\right) = \cos\frac{k\pi}{4}\cos\alpha - \sin\frac{k\pi}{4}\sin\alpha.$$

$$Z\left\{\cos\left(\frac{k\pi}{4} + \alpha\right)\right\} = \cos\alpha \cdot Z\left\{\cos\frac{k\pi}{4}\right\} + \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$= \cos\alpha \cdot \frac{z\left(z - \cos\frac{\pi}{4}\right)}{z^2 - 2z\cos\frac{\pi}{4} + 1} - \sin\alpha \cdot \frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}$$

$$= \frac{\cos\alpha z\left(z - \frac{1}{\sqrt{2}}\right)}{z^2 - \frac{2z}{\sqrt{2}} + 1} - \frac{\sin\alpha \, . \left(z/\sqrt{2}\right)}{z^2 - \frac{2z}{\sqrt{2}} + 1} = \frac{z}{\sqrt{2}}\left[\frac{\cos\alpha\left(\sqrt{2}z - 1\right) - \sin\alpha}{z^2 - \sqrt{2}z + 1}\right]$$

***Ex. 13 :** Find Z {f (k)} if*

(i) $f(k) = e^{-ak} \cos bk, \quad k \geq 0$

(ii) $f(k) = e^{-ak} \sin bk, \quad k \geq 0$ ***(May 2011, 2014)***

(iii) $f(k) = e^{-3k} \cos 4k, \quad k \geq 0$

Sol. : Here we will make use of property No. 4 i.e.

if $Z\{f(k)\} = F(z)$ then $Z\left\{e^{-ak} f(k)\right\} = F\left(e^a z\right)$ i.e. replace z by e^a z.

(i)
$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1}$$

$$Z\left\{e^{-ak} \cos bk\right\} = \frac{(e^a z)(e^a z - \cos b)}{(e^a z)^2 - 2e^a z\cos b + 1} = \frac{z\left(z - e^{-a}\cos b\right)}{z^2 - (2e^{-a}\cos b)z + e^{-2a}}$$

(ii)
$$Z\{\sin bk\} = \frac{z\sin b}{z^2 - 2z\cos b + 1}$$

$$Z\left\{e^{-ak} \sin bk\right\} = \frac{\left(e^a z\right)\sin b}{\left(e^a z\right)^2 - 2(e^a z)\cos b + 1} = \frac{z e^{-a}\sin b}{z^2 - 2e^{-a}\cos bz + e^{-2a}}$$

(iii) Left as an exercise [refer part (i)].

***Ex. 14 :** Find Z {f (k)} if*

(i) $f(k) = 2^k \cos(3k + 2), \quad k \geq 0$

(ii) $f(k) = 4^k \sin(2k + 3), \quad k \geq 0$ ***(May 2006)***

(iii) $f(k) = 3^k \sinh \alpha k, \quad k \geq 0$ ***(May 2010)***

(iv) $f(k) = 2^k \cosh \alpha k, \quad k \geq 0$ ***(Dec. 2009, May 2009)***

Sol. : Here we will use property No. 3 (change of scale) i.e.

if $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

(i)
$$\cos(3k + 2) = \cos 3k \cos 2 - \sin 3k \sin 2$$

$$Z\{\cos(3k + 2)\} = \cos 2 Z\{\cos 3k\} - \sin 2 Z\{\sin 3k\}$$

$$= \cos 2 \frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1} - \frac{\sin 2(z\sin 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\left[z \cdot \cos 2 - (\cos 3 \cdot \cos 2 + \sin 3 \cdot \sin 2)\right]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\left[z\cos 2 - \cos(3 - 2)\right]}{z^2 - 2z\cos 3 + 1} = \frac{z(z\cos 2 - \cos 1)}{z^2 - 2z\cos 3 + 1}$$

$$Z\{2^k \cos(3k + 2)\} = \frac{\frac{z}{2}\left(\frac{z}{2}\cos 2 - \cos 1\right)}{\left(\frac{z}{2}\right)^2 - 2 \cdot \frac{z}{2} \cdot \cos 3 + 1} = \frac{z\left(z\cos 2 - 2\cos 1\right)}{z^2 - 4z\cos 3 + 4}$$

(ii)
$$Z\left\{\sin(2k + 3)\right\} = \cos 3 Z\left\{\sin 2k\right\} + \sin 3 Z\left\{\cos 2k\right\}$$

$$= \cos 3 \frac{z\sin 2}{z^2 - 2z\cos 2 + 1} + \sin 3 \cdot \frac{z\left(z - \cos 2\right)}{z^2 - 2z\cos 2 + 1}$$

$$= \frac{z\left[\cos 3\sin 2 - \sin 3\cos 2 + z\sin 3\right]}{z^2 - 2z\cos 2 + 1}$$

$$= \frac{z\left[z\sin 3 - \sin 1\right]}{z^2 - 2z\cos 2 + 1}$$

$$Z\left\{4^k \sin(2k+3)\right\} = \frac{\frac{z}{4}\left(\frac{z}{4}\sin 3 - \sin 1\right)}{\left(\frac{z}{4}\right)^2 - 2\frac{z}{4}\cos 2 + 1} = \frac{z\left(z\sin 3 - 4\sin 1\right)}{z^2 - 8z\cos 2 + 16}$$

(iii) $$Z\{\sinh \alpha k\} = \frac{z \sinh \alpha}{z^2 - 2z\cosh \alpha + 1}$$

$$Z\left\{3^k \sinh \alpha k\right\} = \frac{\frac{z}{3}\sinh \alpha}{\left(\frac{z}{3}\right)^2 - 2\frac{z}{3}\cosh \alpha + 1} = \frac{z\sinh \alpha}{z^2 - 6z\cosh \alpha + 9}$$

(iv) $$Z\{\cosh \alpha k\} = \frac{z(z-\cosh \alpha)}{z^2 - 2z\cosh \alpha + 1}$$

$$Z\{2^k \cosh \alpha k\} = \frac{\frac{z}{2}\left(\frac{z}{2} - \cosh \alpha\right)}{\left(\frac{z}{2}\right)^2 - 2\frac{z}{2}\cosh \alpha + 1} = \frac{z\left(z - 2\cosh \alpha\right)}{z^2 - 4z\cosh \alpha + 4}$$

Ex. 15 : *Find Z {f (k)} if*

(i) $f(k) = k, \quad k \geq 0$ ***(Dec. 2018)***

(ii) $f(k) = k\, 5^k, \quad k \geq 0$

(iii) $f(k) = (k+1)\, a^k, \quad k \geq 0$ ***(Dec. 2007, 2012, May 2011, 2017)***

Sol. : Here we will use property No. 6 (multiplication by k) i.e.

if Z {f (k)} = F (z) then, $Z\{k\, f(k)\} = \left(-z\frac{d}{dz}\right)(f(z))$.

(i) Let $f(k) = 1$

$$Z\{f(k)\} = Z\{1\} = \frac{z}{z-1} = \left(1-z^{-1}\right)^{-1}$$

$$\therefore \quad Z\{k\} = Z\{k \cdot 1\} = -z\frac{d}{dz}\left[(1-z^{-1})^{-1}\right]$$

$$= -z\left\{-\left(1-z^{-1}\right)^{-2} \cdot z^{-2}\right\}$$

$$= \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

(ii) $$Z\{5^k\} = \frac{z}{z-5} = \left(1-5z^{-1}\right)^{-1}$$

$$Z\{k\,5^k\} = -z\frac{d}{dz}\left[\left(1-5z^{-1}\right)^{-1}\right]$$

$$= -z\left\{-\left(1-5z^{-1}\right)^{-2} 5z^{-2}\right\} = \frac{5z^{-1}}{\left(1-5z^{-1}\right)^2} = \frac{5z}{(z-5)^2}$$

(iii) $$Z\{(k+1)\,a^k\} = Z\{k\,a^k + a^k\} = Z\{k\,a^k\} + Z\{a^k\}$$

$$= -z\frac{d}{dz}\left(1-az^{-1}\right)^{-1} + \frac{z}{z-a}$$

$$= -z\left[-\left(1-az^{-1}\right)^{-2} \cdot az^{-2}\right] + \frac{z}{z-a}$$

$$= \frac{a \cdot z^{-1}}{\left(1-az^{-1}\right)^2} + \frac{z}{z-a} = \frac{az}{(z-a)^2} + \frac{z}{z-a}$$

$$= \frac{az + z(z-a)}{(z-a)^2} = \frac{z^2}{(z-a)^2}$$

Ex. 16 : *Find Z {f (k)} if*

(i) $f(k) = k^2 e^{-ak}, \quad k \geq 0$ ***(Dec. 2010, Nov. 2019)***

(ii) $f(k) = k^2 a^{k-1}, \quad k \geq 0$

(iii) $f(k) = k^2 a^{k-1} U(k-1)$ ***(May 2006)***

Sol. : (i)

$$Z\left\{e^{-ak}\right\} = \frac{z}{z-e^{-a}}$$

$$Z\left\{k\,e^{-ak}\right\} = -z\frac{d}{dz}\left[\left(1-e^{-a}\,z^{-1}\right)^{-1}\right]$$

$$= -z\left[-\left(1-e^{-a}\,z^{-1}\right)^{-2}\,e^{-a}\,z^{-2}\right]$$

$$= \frac{e^{-a}\,z^{-1}}{\left(1-e^{-a}\,z^{-1}\right)^2} = \frac{z\,e^{-a}}{\left(z-e^{-a}\right)^2}$$

$$Z\left\{k^2 e^{-ak}\right\} = Z\{k\,.\,k\,e^{-ak}\}$$

$$= \left(-z\,\frac{d}{dz}\right)\cdot\left(\frac{z\,e^{-a}}{\left(z-e^{-a}\right)^2}\right) = -z\,e^{-a}\left\{\frac{d}{dz}\left(\frac{z}{\left(z-e^{-a}\right)^2}\right)\right\}$$

$$= \left(-z\,e^{-a}\right)\left\{\frac{\left(z-e^{-a}\right)^2 - z\,.\,2\left(z-e^{-a}\right)}{\left(z-e^{-a}\right)^4}\right\}$$

$$= \left(-z\,e^{-a}\right)\frac{\left[z-e^{-a}-2z\right]}{\left(z-e^{-a}\right)^3} = \frac{z\,e^{-a}\left(e^{-a}+z\right)}{\left(z-e^{-a}\right)^3}, \quad |z| > \left|e^{-a}\right|$$

(ii) We know that if {f (k)} is causal sequence, then $Z\left\{f(k-1)\right\} = z^{-1}\,F(z)$ and $Z\{f(k-1)\} = z^{-1}\,F(x)$

$$\therefore \qquad Z\left\{a^k\right\} = \frac{z}{z-a}$$

$$Z\left\{a^{k-1}\right\} = z^{-1}\left(\frac{z}{z-a}\right) = \frac{1}{z-a}$$

$$Z\left\{k^2 a^{k-1}\right\} = \left(-z\frac{d}{dz}\right)\left(-z\,\frac{d}{dz}\right)\left(\frac{1}{z-a}\right) = \left(-z\,\frac{d}{dz}\right)\cdot(-z)\left(\frac{-1}{(z-a)^2}\right)$$

$$= -z\,\frac{d}{dz}\left(\frac{z}{(z-a)^2}\right) = -z\,.\left[\frac{(z-a)^2 - z\,.\,2\,(z-a)}{(z-a)^4}\right]$$

$$= \frac{-z\,[z-a-2z]}{(z-a)^3} = \frac{z\,(z+a)}{(z-a)^3}, \quad |z| > |a|$$

(iii)

$$Z\left\{U(k)\right\} = \frac{z}{z-1}$$

$$Z\{a^k\,U(k)\} = \frac{z/a}{z/a-1} = \frac{z}{z-a}$$

$$Z\left\{a^{k-1}\,U(k-1)\right\} = z^{-1}\left(\frac{z}{z-a}\right) = \frac{1}{z-a}$$

$$Z\left\{k^2 a^{k-1} U(k-1)\right\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{1}{z-a}\right) = \frac{z\,(z+a)}{(z-a)^3}, \quad |z| > |a|$$

Ex. 17 : *Find Z {f (k)} if*

(i) $f(k) = (k+1)\,(k+2)\,2^k, \; k \geq 0$ ***(Dec. 2012)***

(ii) $f(k) = \frac{1}{2\,!}\,(k+1)\,(k+2)\,a^k, \; k \geq 0$ ***(Dec. 2006, May 2009)***

Sol. : (i)

$$Z\{2^k\} = \frac{z}{z-2} = \left(1-2z^{-1}\right)^{-1}$$

$$Z\{k\,2^k\} = -z\,\frac{d}{dz}\left[\left(1-2z^{-1}\right)^{-1}\right]$$

$$= -z\left[-\left(1-2z^{-1}\right)^{-2}\left(2z^{-2}\right)\right]$$

$$= \frac{2z^{-1}}{\left(1-2z^{-1}\right)^{2}} = 2z^{-1}\left(1-2z^{-1}\right)^{-2}$$

$$\therefore \quad Z\left\{(k+1)\,2^k\right\} = Z\left\{k\,2^k\right\} + Z\left\{2^k\right\}$$

$$= 2z^{-1}\left(1-2z^{-1}\right)^{-2} + \left(1-2z^{-1}\right)^{-1} = \left(2z^{-1} + 1 - 2z^{-1}\right)\left(1-2z^{-1}\right)^{-2}$$

$$= \left(1-2z^{-1}\right)^{-2}$$

$$Z\left\{k\,(k+1)\,2^k\right\} = -z\,\frac{d}{dz}\left(1-2z^{-1}\right)^{-2}$$

$$= -z\left[-2\left(1-2z^{-1}\right)^{-3}\left(2z^{-2}\right)\right] = 4z^{-1}\left(1-2z^{-1}\right)^{-3}$$

$$Z\left\{(k+1)\,(k+2)\,2^k\right\} = Z\left\{k\,(k+1)\,2^k\right\} + Z\left\{2\,(k+1)\,2^k\right\}$$

$$= 4z^{-1}\left(1-2z^{-1}\right)^{-3} + 2\left(1-2z^{-1}\right)^{-2}$$

$$= \left(1-2z^{-1}\right)^{-3}\left[4z^{-1} + 2\left(1-2z^{-1}\right)\right]$$

$$= 2\left(1-2z^{-1}\right)^{-3}$$

(ii) From (i),

$$Z\left\{(k+1)\,(k+2)\,a^k\right\} = 2\left(1-az^{-1}\right)^{-3}$$

$$\therefore \quad Z\left\{\frac{1}{2\,!}\,(k+1)\,(k+2)\,a^k\right\} = \left(1-a\,z^{-1}\right)^{-3}$$

Ex. 18 : Find $Z\{x_k\}$ if $x_k = \frac{1}{1^k} * \frac{1}{2^k} * \frac{1}{3^k},\ k \geq 0$ ***(Dec. 2008)***

Sol. : Let

$$A(k) = \frac{1}{k}$$

then

$$Z\{A(k)\} = \frac{z}{z-1}, \quad |z| > 1$$

Next,

$$B(k) = \frac{1}{2^k}$$

then

$$Z\{B(k)\} = \sum_{k=0}^{\infty} \frac{1}{2^k}\,z^{-k} = \sum_{k=0}^{\infty}\left(2^{-1}z^{-1}\right)^k = 1 + \left(2^{-1}z^{-1}\right) + \left(2^{-1}z^{-1}\right)^2 + \ldots\ldots\ldots$$

$$= \frac{1}{1-2^{-1}\,z^{-1}}, \quad \left|2^{-1}\,z^{-1}\right| < 1 = \frac{2z}{2z-1}, \quad |z| > \frac{1}{2}$$

Next,

$$C(k) = \frac{1}{3^k}$$

then

$$Z\{C(k)\} = Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}, \quad |z| > \frac{1}{3}$$

By using convolution property,

$$Z\{x_k\} = Z\,\{A(k) * B(k) * C(k)\}$$

$$= Z\,\{A(k)\} * Z\{B(k)\} * Z\{C(k)\}$$

$$= \left(\frac{z}{z-1}\right)\left(\frac{2z}{2z-1}\right)\left(\frac{3z}{3z-1}\right), \quad |z| > 1.$$

***Ex. 19 :** Verify convolution theorem for $f_1(k) = k$ and $f_2(k) = k$.*

Sol. :

$$Z\{k\} = Z\{k \,.\, 1\} = -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$Z\{f_1(k)\} = F_1(z) = \frac{z}{(z-1)^2}$$

$$\therefore \quad Z\{f_2(k)\} = F_2(z) = \frac{z}{(z-1)^2}$$

$$\therefore \quad F_1(z)\, F_2(z) = \frac{z^2}{(z-1)^4} \qquad \text{... (I)}$$

$$\{F_1(k) * F_2(k)\} = \sum_{m=0}^{\infty} f_1(m)\, f_2(k-m) = \sum_{m=0}^{\infty} m(k-m)$$

$$= k \sum_{m=0}^{\infty} m - \sum_{m=0}^{\infty} m^2$$

$$= k \frac{k(k+1)}{2} - \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{k(k+1)}{6}(3k-2k-1) = \frac{k}{6}(k^2-1)$$

$$Z\{f_1(k) * f_2(k)\} = Z\left\{\frac{k(k^2-1)}{6}\right\}$$

$$= \frac{1}{6}[Z\{k^3\} - Z\{k\}]$$

$$= \frac{1}{6}\left[\left(-z\frac{d}{dz}\right)^3 (1-z^{-1})^{-1} \left(-z\frac{d}{dz}\right)(1-z^{-1})^{-1}\right]$$

$$= \left[\frac{1}{6}\frac{z(z^2+4z+1)}{(z-1)^4} - \frac{z}{(z-1)^2}\right]$$

$$= z\left[\frac{z^2+4z+1-z^2+2z-1}{6(z-1)^4}\right]$$

$$= \frac{z^2}{(z-1)^4} \qquad \text{... (II)}$$

From (I) and (II), convolution theorem is verified.

EXERCISE 4.1

For each of the following sequences, evaluate corresponding Z-transforms specifying ROC of the transform.

1. $f(k) = 3^k, \quad k \geq 0$

Ans. $\frac{z}{z-3}, \; |z| > 3$

2. $f(k) = 2, \quad k \geq 0$

Ans. $\frac{2z}{z-1}, \; |z| > 1$

3. $f(k) = \left(\frac{1}{3}\right)^k, \quad k \geq 0$

Ans. $\frac{z}{z-\frac{1}{3}}, \; |z| > \frac{1}{3}$

4. $f(k) = \frac{1}{3^k}, \quad k \geq 0$

Ans. $\frac{3z}{3z-1}, \; |z| > \frac{1}{3}$

5. $f(k) = 4^k, \quad k < 0$

Ans. $\frac{z}{4-z}, \; |z| < 4$

6. $f(k) = \left(\frac{1}{3}\right)^k, \quad k < 0$

Ans. $\frac{3z}{1-3z}, \; |z| < \frac{1}{3}$

7. $f(k) = 3\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{5}\right)^k,\ k \geq 0$

Ans. $\frac{12\,z}{4\,z-1} + \frac{20\,z}{5\,z-1},\ |z| > \frac{1}{4}$

8. $f(k) = 4^k + 5^k,\ k \geq 0$

Ans. $\frac{z}{z-4} + \frac{z}{z-5},\ |z| > 5$

9. $f(k) = 5^k,\ k < 0$

Ans. $\frac{2\,z}{(5-z)(z-3)},\ 3 < |z| < 5$

10. $f(k) = \begin{cases} \frac{5^k}{k}, & k > 1 \\ 3^k, & k \geq 0 \end{cases},$

Ans. $-\log\left(1 - 5\,z^{-1}\right),\ |z| > 5$

11. $f(k) = \left(\frac{1}{2}\right)^{|k|}$ for all k **(May 2012)**

Ans. $\frac{z}{2-z} + \frac{2\,z}{2\,z-1},\ \frac{1}{2} < |z| < 2$

12. $f(k) = 2^k + \left(\frac{1}{2}\right)^k,\ k \geq 0$

Ans. $\frac{z}{z-2} + \frac{z}{z-\frac{1}{2}},\ |z| > 2$

13. $f(k) = \begin{cases} 3^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k = 0, 2, 4, 6, \ldots \\ \left(\frac{1}{2}\right)^k, & k = 1, 3, 5, 7, 9, \ldots \end{cases}$

Ans. $\frac{z}{3-z} + \frac{9\,z^2}{9\,z^2-1} + \frac{2\,z}{4\,z^2-1},\ \frac{1}{2} < |z| < 3$

14. $f(k) = \frac{3^k}{k\,!},\ k \geq 0$

Ans. $e^{3/z}$, ROC – z plane

15. $f(k) = e^{k\alpha},\ k \geq 0$

Ans. $\frac{z}{z-e^{\alpha}},\ |z| > |e^{\alpha}|$

16. $f(k) = \cos\left(\frac{k\pi}{8} + \alpha\right),\ k \geq 0$

Ans. $\frac{z^2 \cos\alpha - z\cos\left(\frac{\pi}{8} - \alpha\right)}{z^2 - 2z\cos\frac{\pi}{8} + 1},\ |z| > 1$

17. $f(k) = \sin 4\,k,\ k \geq 0$

Ans. $\frac{z \sin 4}{z^2 - 2\,z\cos 4 + 1},\ |z| > 1$

18. $f(k) = \sin(3k + 5),\ k \geq 0$ **(May 2008, Dec. 2012)**

Ans. $\frac{z^2 \sin 5 - z\sin 2}{z^2 - 2\,z\cos 3 + 1},\ |z| > 1$

19. $f(k) = \cos(7\,k + 2),\ k \geq 0$ **(May 2008)**

Ans. $\frac{z^2\cos 2 - z\cos 5}{z^2 - 2\,z\cos 7 + 1},\ |z| > 1$

20. $f(k) = \cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right),\ k \geq 0$ **(May 2018)**

Ans. $\frac{z^2 - z}{\sqrt{2}\,(z^2+1)},\ |z| > 1$

21. $f(k) = \sin\left(\frac{k\pi}{2} + \alpha\right),\ k \geq 0$

Ans. $\frac{z^2\sin\alpha + z\cos\alpha}{z^2 + 1},\ |z| > 1$

22. $f(k) = \cosh\left(\frac{k\pi}{2}\right),\ k \geq 0$

Ans. $\frac{z\left(z - \cosh\frac{\pi}{2}\right)}{z^2 - 2z\cosh\frac{\pi}{2} + 1},$

$|z| > \max.\left(|e^{\pi/2}|,\ |e^{-\pi/2}|\right)$

23. $f(k) = \sinh\frac{k\pi}{2},\ k \geq 0$

Ans. $\frac{z\sinh\frac{\pi}{2}}{z^2 - 2z\cosh\frac{\pi}{2} + 1},$

$|z| > \max.\left(|e^{\pi/2}|,\ |e^{-\pi/2}|\right)$

24. $f(k) = \cosh\left(\frac{k\pi}{2} + \alpha\right),\ k \geq 0$

Ans. $\frac{z^2\cosh\alpha - z\cosh\left(\frac{\pi}{2} - \alpha\right)}{z^2 - 2z\cosh\frac{\pi}{2} + 1}$

25. $f(k) = 2^k \cos(3k + 2)$

Ans. $\dfrac{z^2 \cos 2 - 2z \cos 1}{z^2 - 4z \cos 3 + 4}, \quad |z| > 2$

26. $f(k) = \begin{cases} -\left(-\dfrac{1}{4}\right)^k, & k < 0 \\ \left(-\dfrac{1}{5}\right)^k, & k \geq 0 \end{cases}$

Ans. $\dfrac{4z}{4z + 1} + \dfrac{5z}{5z + 1}, \quad \dfrac{1}{5} < |z| < \dfrac{1}{4}$

27. $f(k) = e^{-3k} \sin 4k, \; k \geq 0$

Ans. $\dfrac{z e^{-3} \sin 4}{z^2 - 2e^{-3} z \cos 4 + e^{-6}}, \quad |z| > |e^{-3}|$

28. $f(k) = k e^{-ak}, \quad k \geq 0$ **(May 2012)**

Ans. $\dfrac{z e^{-a}}{(z - e^{-a})^2}, \quad |z| > |e^{-a}|$

29. $f(k) = k^2, \; k \geq 0$ **(Dec. 2018)**

Ans. $\dfrac{z(z + 1)}{(z - 1)^3}, \quad |z| > 1$

30. $f(k) = k^3, \; k \geq 0$

Ans. $\dfrac{z(z^2 + 4z + 1)}{(z - 1)^4}, \quad |z| > 1$

31. $f(k) = k a^{k-1} U(k - 1), \; k \geq 0$

(Dec. 2004, May 2007, 2008)

Ans. $\dfrac{z}{(z - a)^2}$

32. $f(k) = \dfrac{1}{3^k} * \dfrac{1}{4^k}, \quad k \geq 0$

Ans. $\left(\dfrac{3z}{3z - 1}\right)\left(\dfrac{4z}{4z - 1}\right), \quad |z| > \dfrac{1}{4}$

33. $f(k) = * \; 4^k, \; k \geq 0$

Ans. $\dfrac{z^2}{(z - 3)(z - 4)}, \quad |z| > 3.$

Table of Inverse Z-Transforms of the Partial Fraction Terms of F (z)

Partial Fraction Term	Inverse Z-Transform f (k) if | z| > | a |, k > 0	Inverse Z-Transform if | z | < | a |, k < 0
$\dfrac{z}{z-a}$	$a^k U(k)$	$-a^k$
$\dfrac{z^2}{(z-a)^2}$	$(k+1) a^k$	$-(k+1) a^k$
$\dfrac{z^3}{(z-a)^3}$	$\dfrac{1}{2!}(k+1)(k+2) a^k U(k)$	$-\dfrac{1}{2!}(k+1)(k+2) a^k U(-k+2)$
$\dfrac{z^n}{(z-a)^n}$	$\dfrac{1}{(n-1)!}(k+1) \ldots (k+n-1) a^k U(k)$	$-\dfrac{1}{(n-1)!}(k+1)(k+2) \ldots\ldots (k+n-1) a^k$
$\dfrac{1}{z-a}$	$a^{k-1} U(k-1)$	$-a^{k-1} U(-k)$
$\dfrac{1}{(z-a)^2}$	$(k-1) a^{k-2} U(k-2)$	$-(k-1) a^{k-2} U(-k+1)$
$\dfrac{1}{(z-a)^3}$	$\dfrac{1}{2}(k-2)(k-1) a^{k-3} U(k-3)$	$-\dfrac{1}{2}(k-2)(k-1) a^{k-3} U(-k+2)$
$\dfrac{z}{z-1}$	$U(k)$	
$\dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}, \; \|z\| > 1$	$\cos \alpha k$	
$\dfrac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}, \; \|z\| > 1$	$\sin \alpha k$	

4.11 INVERSE Z-TRANSFORM

Our aim is to obtain the sequence {f (k)} from its Z-transform F(z) which we assume to be a rational function of z as given below :

$$F(z) = \frac{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \ldots\ldots + b_m}{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}$$

Here we shall study the three different methods :

(I) Power Series Method. (II) Partial Fraction Method (III) Inversion Integral Method

Now we shall study the methods in detail.

(I) Power Series Method

By Direct Division : Since Z-transform F(z) of {f(k)} absolutely converges within the ROC, for inversion of F(z) uniquely, the knowledge of ROC of F(z) is necessary.

We can express F(z) as a series in powers of z by actual division. The process of division depends on ROC.

Case (i) : $|z| < R$: Here we obtain the power series in z which converges in the same region as F (z) by beginning the division with *the lowest power of z* in the denominator (i.e. with a_n). The coefficient of z^k is identified as f (– k).

Case (ii) : $|z| > R$: We should begin the division with the highest power of z in the donominator. The coefficient of z^{-k} is identified as f (k).

ILLUSTRATION

***Ex. 1 :** Find $Z^{-1}\left(\frac{z}{z-a}\right)$ if (i) $|z| > |a|$, (ii) $|z| < |a|$.*

Sol. : (i) If $|z| > |a|$,

$$F(z) = \frac{z}{z-a}$$

Here we will perform actual division :

$$\begin{array}{r|l}
z-a & z \quad \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\ldots\ldots\right. \\
 & z - a \\
 & - \quad + \\
\hline
 & \quad a \\
 & \quad a - \frac{a^2}{z} \\
 & \quad - \quad + \\
\hline
 & \qquad \frac{a^2}{z} \\
 & \qquad \frac{a^2}{z} - \frac{a^3}{z^2} \\
 & \qquad - \quad + \\
\hline
 & \qquad\quad \frac{a^3}{z^2}
\end{array}$$

$\therefore$
$$F(z) = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\ldots + \frac{a^k}{z^k} + \ldots\ldots\ldots = \sum_{k=0}^{\infty} a^k z^{-k}$$

$\therefore$ Coefficient of z^{-k} is a^k.

Inverting, $\{f(k)\} = \{a^k\}, \quad k \geq 0$

(ii) If $|z| < |a|$ then we write

$$F(z) = \frac{z}{z-a} = \frac{z}{-a+z} \qquad \text{(Note this step)}$$

$$
\begin{array}{r|l}
-a+z & z \quad \left(-\dfrac{z}{a} - \dfrac{z^2}{a^2} - \dfrac{z^3}{a^3} - \cdots\cdots\cdots \right. \\
 & z - \dfrac{z^2}{a} \\
 & \underline{- \quad +} \\
 & \quad \dfrac{z^2}{a} \\
 & \quad \dfrac{z^2}{a} - \dfrac{z^3}{a^2} \\
 & \quad \underline{- \quad\quad +} \\
 & \quad\quad \dfrac{z^3}{a^2} \\
 & \quad\quad \dfrac{z^3}{a^2} - \dfrac{z^4}{a^3} \\
 & \quad\quad \underline{- \quad\quad\quad +} \\
 & \quad\quad\quad \dfrac{z^4}{a^3}
\end{array}
$$

$$F(z) = -\frac{z}{a} - \frac{z^2}{a^2} - \frac{z^3}{a^3} - \cdots\cdots\cdots = -\sum_{k=1}^{\infty} a^{-k} z^k$$

Replacing k by − k,

$$= -\sum_{k=-\infty}^{-1} a^k \; z^{-k} = -\sum_{k=-1}^{-\infty} a^k \; z^{-k}$$

∴ Coefficient of z^{-k} is $-a^k$.

Inverting, $\{f(k)\} = \{-a^k\},\; k < 0.$

Note : Alternatively, we can use binomial expansion method as explained below.

By Binomial Expansion : Here we take a suitable factor common depending upon ROC from the denominator so that the denominator is of the form $(1 \pm r)^n$, where $|r| < 1$ and then expand using Binomial theorem. The coefficient of z^{-k} in the series is defined as f(k).

$$\frac{1}{1+y} = (1+y)^{-1} = 1 - y + y^2 - y^3 + y^4 - \cdots\cdots\cdots, \quad |y| < 1$$

$$\frac{1}{1-y} = (1-y)^{-1} = 1 + y + y^2 + y^3 + y^4 + \cdots\cdots\cdots, \quad |y| < 1$$

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{2!} y^2 + \frac{n(n-1)(n-2)}{3!} y^3 + \ldots, \quad |y| < 1$$

ILLUSTRATIONS

Ex. 1 : *Find* $Z^{-1}\left(\dfrac{z}{z-a}\right)$ *when (i)* $|z| > |a|$, *(ii)* $|z| < |a|$.

Sol. : (i) If $|z| > |a|$, i.e. $\left|\dfrac{z}{a}\right| > 1$ i.e. $\left|\dfrac{a}{z}\right| < 1$

We take "z" outside and write

$$F(z) = \frac{z}{z-a} = \frac{z}{z\left(1 - \dfrac{a}{z}\right)} = \frac{1}{1 - \dfrac{a}{z}}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\ldots + \frac{a^k}{z^k} + \ldots\ldots = \sum_{k=0}^{\infty} a^k a^{-k}$$

$\therefore$ Coefficient of $z^{-k} = a^k, \quad k \geq 0$

$$\therefore \qquad Z^{-1}\left(\frac{z}{z-a}\right) = \{f(k)\} = a^k,\ k \geq 0 \quad |z| > |a|$$

(ii) If $|z| < |a|$ i.e. $\left|\frac{z}{a}\right| < 1$

$\therefore$ We take "a" outside and write

$$F(z) = \frac{z}{z-a} = \frac{z}{a\left(\frac{z}{a}-1\right)} = \frac{-z}{a\left(1-\frac{z}{a}\right)}$$

$$= -\frac{z}{a}\left(1 + \frac{z}{a} + \frac{z^2}{a^2} + \ldots\ldots + \frac{z^k}{a^k} + \ldots\ldots\right)$$

$$= -\left[\frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \ldots\ldots + \frac{z^{k+1}}{a^{k+1}} + \ldots\right] = -\sum_{k=0}^{\infty}\left(\frac{z}{a}\right)^{k+1}$$

Put $k + 1 = -r$ (Note this step) then

$k = 0$	$r = -1$
$k = \infty$	$r = -\infty$

$$F(z) = -\sum_{r=-1}^{-\infty}\left(\frac{z}{a}\right)^{-r} = -\sum_{r=-\infty}^{-1} a^r z^{-r}$$

$\therefore$ Coefficient of $z^{-k} = a^k$, $k < 0$.

$$Z^{-1}\left(\frac{z}{z-a}\right) = -a^k,\ k < 0, \quad |z| < |a|$$

We note that :

$$\boxed{\mathbf{Z^{-1}\left(\frac{z}{z-a}\right) = \begin{cases} a^k, & k \geq 0,\ |z| > |a| \\ -a^k, & k < 0,\ |z| < |a| \end{cases}}}$$

Ex. 2 : *Find* $Z^{-1}\left(\frac{1}{z-a}\right)$ *when (i)* $|z| < |a|$, *(ii)* $|z| > |a|$.

Sol. : (i) If $|z| < |a|$ i.e. $\left|\frac{z}{a}\right| < 1$

$\therefore$ We take "a" outside and write

$$F(z) = \frac{1}{z-a} = \frac{1}{a\left(\frac{z}{a}-1\right)} = -\frac{1}{a}\,\frac{1}{1-\frac{z}{a}} = -\frac{1}{a}\left(1-\frac{z}{a}\right)^{-1}$$

$$= -\frac{1}{a}\left\{1 + \frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \ldots\ldots + \frac{z^k}{a^k} + \ldots\ldots\right\}$$

$$= -\left\{\frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \ldots\ldots + \frac{z^k}{a^{k+1}} + \ldots\ldots\right\}$$

$\therefore$ Coefficient of f $z^k = -a^{-k-1}$, $k \geq 0$

$\therefore$ Coefficient of $z^{-k} = -a^{k-1}$, $k \leq 0$

$$\therefore \qquad Z^{-1}\cdot\left(\frac{1}{z-a}\right) = \{f(k)\} = -a^{k-1},\ k \leq 0$$

(ii) If $|z| > |a|$, $\left|\frac{z}{a}\right| > 1$ i.e. $\left|\frac{a}{z}\right| < 1$

$\therefore$ We take "z" outside and write

$$F(z) = \frac{1}{z-a} = \frac{1}{z\left(1-\frac{a}{z}\right)}$$

$$= \frac{1}{z}\left(1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\ldots\ldots+\frac{a^{k-1}}{z^{k-1}}+\ldots\ldots\right)$$

$$= \frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\ldots\ldots+\frac{a^{k-1}}{z^k}+\ldots\ldots$$

Coefficient of $z^{-k} = a^{k-1}, \quad k \geq 1$

$$\therefore \quad Z^{-1}\left(\frac{1}{z-a}\right) = \{f(k)\} = a^{k-1}, \quad k \geq 1.$$

We note that, $Z^{-1}\left(\frac{1}{z-a}\right) = \begin{cases} -a^{k-1}, & k \leq 0, \ |z| < |a| \\ a^{k-1}, & k \geq 0, \ |z| > |a| \end{cases}$

Ex. 3 : *Find* $Z^{-1}\left(\frac{z}{z-5}\right)$ *if* (i) $|z| > 5$, (ii) $|z| < 5$.

Sol. :
$$F(z) = \frac{z}{z-5}$$

Case (i) : $|z| > 5$, $\left|\frac{z}{5}\right| > 1$, $\left|\frac{5}{z}\right| < 1$

$$F(z) = \frac{z}{z\left(1-\frac{5}{z}\right)} = \frac{1}{1-\frac{5}{z}} = 1+\frac{5}{z}+\left(\frac{5}{z}\right)^2+\ldots+\left(\frac{5}{z}\right)^k+\ldots$$

Coefficient of $z^{-k} = 5^k, \quad k \geq 0$

$$Z^{-1}\left(\frac{z}{z-5}\right) = 5^k, \quad k \geq 0 \text{ if } |z| > 5.$$

Case (ii) : $|z| < 5$, $\left|\frac{z}{5}\right| < 1$

$$F(z) = \frac{z}{z-5} = \frac{z}{-5\left(1-\frac{z}{5}\right)}$$

$$= -\frac{z}{5}\left[1+\frac{z}{5}+\left(\frac{z}{5}\right)^2+\ldots\ldots\right]$$

$$= -\left[\frac{z}{5}+\left(\frac{z}{5}\right)^2+\left(\frac{z}{5}\right)^3+\ldots\ldots+\left(\frac{z}{5}\right)^k+\ldots\ldots\right]$$

$$= -\sum_{k=1}^{\infty}\left(\frac{z}{5}\right)^k \qquad \text{Put } k = -r$$

$$= -\sum_{-1}^{-\infty}\left(\frac{z}{5}\right)^{-r} = -\sum_{-1}^{-\infty} 5^r \ z^{-r}$$

$$\therefore \quad \{f(k)\} = -5^k, \quad k < 0, \quad |z| < 5.$$

Ex. 4 : *Find* $Z^{-1}\ \dfrac{1}{(z-a)^2}$ *if* $|z| < a$.

Sol. : If $|z| < a$, $\left|\dfrac{z}{a}\right| < 1$

$$\therefore \quad F(z) = \frac{1}{(z-a)^2} = \frac{1}{\left[a\left(\frac{z}{a}-1\right)\right]^2}$$

$$= \frac{1}{\left[-a\left(1-\frac{z}{a}\right)\right]^2} = \frac{1}{a^2}\ \frac{1}{\left(1-\frac{z}{a}\right)^2} = \frac{1}{a^2}\left(1-\frac{z}{a}\right)^{-2}$$

$$= \frac{1}{a^2}\left[1 + 2\,\frac{z}{a} + 3\left(\frac{z}{a}\right)^2 + \ldots\ldots + (n+1)\,\frac{z^n}{a^n} + \ldots\ldots\right]$$

$$= \frac{1}{a^2} + 2\,\frac{z}{a^3} + 3\,\frac{z^2}{a^4} + \ldots\ldots + (n+1)\,\frac{z^n}{a^{n+2}} + \ldots\ldots$$

Coefficient of $\quad z^n = \dfrac{n+1}{a^{n+2}}, \quad n \ge 0$

Coefficient of $\quad z^{-k} = \dfrac{-k+1}{a^{-k+2}}, \quad k \le 0$

$$Z^{-1}\left\{\frac{1}{(z-a)^2}\right\} = \{f(k)\} = \frac{-k+1}{a^{-k+2}},\ k \le 0, \quad |z| < a.$$

Ex. 5 : Find $Z^{-1}\ \dfrac{1}{(z-5)^3}$, $|z| > 5$.

Sol. : $\quad |z| > 5 \Rightarrow \left|\dfrac{z}{5}\right| > 1$

$$\Rightarrow \left|\frac{5}{z}\right| < 1.$$

$$F(z) = \frac{1}{\left[z\left(1-\frac{5}{z}\right)\right]^3} = \frac{1}{z^3}\left(1-\frac{5}{z}\right)^{-3}$$

$$= z^{-3}\left[1 + (-3)\left(-\frac{5}{z}\right) + \frac{(-3)(-4)}{2\,!}\left(-\frac{5}{z}\right)^2 + \ldots\ldots\right]$$

$$= z^{-3}\left[1 + 3\cdot5\cdot z^{-1} + 6\cdot5^2\cdot z^{-2} + 10\cdot5^3\cdot z^{-3} + \ldots\ldots + \frac{(n+1)(n+2)}{2}\,5^n\,z^{-n} + \ldots\ldots\right]$$

$$= z^{-3} + 3\cdot5\cdot z^{-4} + 6\cdot5^2\cdot z^{-5} + 10\cdot5^3\cdot z^{-6} + \ldots\ldots + \frac{(n+1)(n+2)}{2}\,5^n\,z^{-n-3} + \ldots\ldots\ldots$$

Coefficient of $\quad z^{-n-3} = \dfrac{(n+1)\ (n+2)}{2}\,5^n, \quad n \ge 0,\ n+3 = k$

Coefficient of $\quad z^{-k} = \dfrac{(k-3+1)(k-3+2)}{2}\ 5^{k-3}, \quad k \ge 3$

$$\{f(k)\} = \frac{(k-2)(k-1)}{2}\ 5^{k-3}, \quad k \ge 3.$$

Ex. 6 : *Find* $Z^{-1}\ \dfrac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)},\ \frac{1}{3} < |z| < \frac{1}{2}$. ***(Dec. 2005, 2010)***

Sol. :

$$F(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$$

$$|z| < \frac{1}{2} \Rightarrow |2z| < 1 \quad \text{and} \quad |z| > \frac{1}{3} \Rightarrow |3z| > 1 \Rightarrow \left|\frac{1}{3z}\right| < 1$$

$$F(z) = \frac{6}{-\frac{1}{2}(1-2z)} - \frac{6}{z\left(1-\frac{1}{3z}\right)}$$

$$= -12\,[1 + (2z) + (2z)^2 + \ldots] - \frac{6}{z}\left[1 + \frac{1}{3z} + \left(\frac{1}{3z}\right)^2 + \ldots\ldots\right]$$

$$= -12\,[1 + (2z) + (2z)^2 + \ldots\ldots] - 6\left[\frac{1}{z} + \frac{1}{3z^2} + \frac{1}{3^2 z^3} + \ldots\ldots\right]$$

$$= -12\sum_{k=0}^{\infty}(2z)^k - 6\sum_{k=1}^{\infty}\frac{1}{3^{k-1}z^k}$$

Coefficient of z^k in first series $= -12\,(2)^k, \quad k \geq 0$

Coefficient of z^{-k} in first series $= -12\,(2)^{-k}, \quad k \leq 0$

Coefficient of z^{-k} in second series $= \dfrac{-6}{3^{k-1}}, \quad k \geq 1$

$$\therefore \qquad \{f(k)\} = \underset{(k\leq 0)}{-12\,(2)^{-k}} - \underset{(k \geq 1)}{\frac{6}{3^{k-1}}},$$

Alternatively, By using the formula :

$$Z^{-1}\left(\frac{1}{z-a}\right) = \begin{cases} -a^{k-1}, & k \leq 0,\ |z| < |a| \\ a^{k-1}, & k \geq 1,\ |z| > |a| \end{cases}$$

$$\therefore \qquad Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right] = Z^{-1}\left(\frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}\right)$$

$$= 6\left[Z^{-1}\left(\frac{1}{z-\frac{1}{2}}\right) - Z^{-1}\left(\frac{1}{z-\frac{1}{3}}\right)\right]$$

$|z| < \dfrac{1}{2}$ and $|z| > \dfrac{1}{3}$.

$$\therefore \qquad \{f(k)\} = 6\underset{(k\leq 0)}{\left\{-\left(\frac{1}{2}\right)^{k-1}\right\}} - 6\underset{(k\geq 1)}{\left\{\left(\frac{1}{3}\right)^{k-1}\right\}}$$

$$= -12\cdot\underset{(k\leq 0)}{\{2^{-k}\}} - 6\underset{(k\geq 1)}{\left\{\frac{1}{3^{k-1}}\right\}}$$

Ex. 7 : *Find* $Z^{-1}\left(\dfrac{1}{(z-3)\,(z-2)}\right),\ 2 < |z| < 3.$ ***(Dec. 2004, 2016; May 2007, 2011)***

Sol. :

$$F(z) = \frac{1}{(z-3)\,(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1 \quad \text{and} \quad |z| > 2 \Rightarrow \left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$$

$$F(z) = \frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\ldots\ldots\right] - \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\ldots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\frac{1}{3^3}z^2+\ldots+\frac{1}{3^{k+1}}z^k+\ldots\right] - \left[\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\ldots\ldots+\frac{2^k}{z^{k+1}}+\ldots\ldots\right]$$

Coefficient of z^k in first series $= -\dfrac{1}{3^{k+1}},\ k \geq 0$

Coefficient of z^{-k} in first series $= -\dfrac{1}{3^{-k+1}},\ k \leq 0$

Coefficient of $z^{-(k+1)}$ in second series $= 2^k,\ k \geq 0$

Coefficient of z^{-k} in second series $= -2^{k-1},\ k \geq 1.$

$$\therefore \quad \{f(k)\} = \underset{(k \leq 0)}{-\frac{1}{3^{-k+1}}} - \underset{(k \geq 1)}{2^{k-1}}$$

$$\{f(k)\} = \underset{(k \leq 0)}{-3^{k-1}} - \underset{(k \geq 1)}{2^{k-1}}$$

Ex. 8 : *Show that* $Z^{-1}\left\{\dfrac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right\} = \{x_k\}$ for $|z| > \frac{1}{2}$ where, $x_k = 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right],\ k \geq 1.$

Sol. : ***(Dec. 2005, 2010)***

$$X(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$$

$$|z| > \frac{1}{2} \Rightarrow |2z| > 1 \Rightarrow \left|\frac{1}{2z}\right| < 1 \Rightarrow \left|\frac{1}{3z}\right| < 1$$

$$X(z) = \frac{6}{z\left(1-\frac{1}{2z}\right)} - \frac{6}{z\left(1-\frac{1}{3z}\right)}$$

$$= \frac{6}{z}\left(1+\frac{1}{2z}+\left(\frac{1}{2z}\right)^2+\ldots\right) - \frac{6}{z}\left(1+\frac{1}{3z}+\left(\frac{1}{3z}\right)^2+\ldots\right)$$

$$= 6\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{2^2z^3}+\ldots\ldots+\frac{1}{2^{k-1}z^k}+\ldots\ldots\right] - 6\left[\frac{1}{z}+\frac{1}{3z^2}+\frac{1}{3^2z^3}+\ldots\ldots+\frac{1}{3^{k-1}z^k}+\ldots\ldots\right]$$

$$\{x_k\} = 6\cdot\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right],\ k \geq 1.$$

Ex. 9 : *Show that* $Z^{-1}\left[\dfrac{1}{(z-2)(z-3)}\right] = \{x_k\}$, *for* $|z| < 2$, *where,* $x_k = 2^{k-1} - 3^{k-1},\ k \leq 0.$

Sol. :

$$X(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$|z| < 2, \quad \left|\frac{z}{2}\right| < 1 \Rightarrow \left|\frac{z}{3}\right| < 1$$

$$X(z) = \frac{1}{-3\left(1-\frac{z}{3}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\ldots\ldots\right) + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\ldots\right]$$

$$= -\left[\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\ldots\ldots+\frac{z^k}{3^{k+1}}+\ldots\ldots\right] + \left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\ldots\ldots+\frac{z^k}{2^{k+1}}+\ldots\ldots\right]$$

$$\text{Coefficient of } z^k \text{ in the first series} = -\frac{1}{3^{k+1}}, \quad k \geq 0$$

$$\text{Coefficient of } z^{-k} \text{ in the first series} = -\frac{1}{3^{-k+1}} = -3^{k-1}, \quad k \leq 0$$

$$\text{Coefficient of } z^{-k} \text{ in the second series} = 2^{k-1}, \quad k \leq 0$$

$$\therefore \qquad \{x_k\} = 2^{k-1} - 3^{k-1}, \quad k \leq 0.$$

II. Partial Fraction Method

To apply this, it is necessary that the degree of numerator is not greater than the degree of the denominator i.e. fraction must be proper fraction.

In case this is not, we carry out actual division till the remainder satisfies the condition given above. In this case,

$$F(z) = P(z) + \frac{Q(z)}{R(z)}$$

The expression $\frac{Q(z)}{R(z)}$ is then considered for expressing it in partial fraction.

Let F (z) satisfy the condition specified.

Then we obtain partial fraction of $\frac{F(z)}{z}$ and not that of F (z).

Linear Non-Repeated Factors :

Suppose
$$\frac{F(z)}{z} = \frac{B_1}{z-\alpha_1} + \frac{B_2}{z-\alpha_2} + \frac{B_3}{z-\alpha_3} + \ldots\ldots\ldots$$

$$Z^{-1}[F(z)] = B_1 \, Z^{-1}\left(\frac{z}{z-\alpha_1}\right) + B_2 \cdot Z^{-1}\left(\frac{z}{z-\alpha_2}\right) + \ldots\ldots\ldots$$

$$\{f(k)\} = B_1 \{\alpha_1\}^k + B_2 \{\alpha_2\}^k + \ldots\ldots\ldots$$

provided $|z| > |\alpha_1|$, $|z| > |\alpha_2|$ and so on. $(k \geq 0)$

Similarly,
$$Z^{-1}[F(z)] = B_1 \, Z^{-1}\left(\frac{z}{z-\alpha_1}\right) + B_2 \, Z^{-1}\left(\frac{z}{z-\alpha_2}\right) + \ldots\ldots$$

$$= B_1 \{-\alpha_1^k\} + B_2 \{-\alpha_2^k\} + \ldots\ldots\ldots$$

provided $|z| < |\alpha_1|$, $|z| < |\alpha_2|$ and so on. $(k < 0)$.

Linear Repeated Factors :

Suppose $F(z) = \frac{B_1}{(z-\alpha_1)^2}$ or $F(z) = \frac{B_1}{(z-\alpha_1)^3}$ or $F(z) = B_1 \cdot \frac{z}{(z-\alpha_1)^2}$

then always use series expansion of z which is obtained by Binomial expansion.

Note : Here we will note some important formulae of inverse Z-transform.

$$Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = \{(k+1)\,a^k\}, \qquad |z| > |a|, \quad k \geq 0$$

$$= -(k+1)\,a^k, \qquad |z| < |a|, \quad k < 0$$

$$Z^{-1}\left\{\frac{z^3}{(z-a)^3}\right\} = \frac{1}{2!}(k+1)(k+2)\,a^k\,U(k), \qquad |z| > |a|, \quad k \geq 0$$

$$= -\frac{1}{2!}(k+1)(k+2)\,a^k\,U(-k+2), \qquad |z| < |a|, \ k < 0$$

$$Z^{-1}\left\{\frac{z^n}{(z-a)^n}\right\} = \frac{1}{(n-1)!}(k+1)(k+2)\ldots\ldots(k+n-1)\,a^k\,U(k), \quad |z| > |a|, \ k \geq 0$$

$$= -\frac{1}{(n-1)!}(k+1)(k+2)\ldots\ldots(k+n-1)\,a^k, \qquad |z| < |a|, \ k < 0.$$

ILLUSTRATIONS

Ex. 1 : *Find* $Z^{-1}\,\dfrac{z}{(z-1)\ (z-2)}$, *if* $|z| \geq 2$. ***(May 2019)***

Sol. :

$$F(z) = \frac{z}{(z-1)(z-2)}$$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{(-1)}{z-1} + \frac{(1)}{z-2}$$

$$\frac{F(z)}{z} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$F(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$$Z^{-1}[F(z)] = Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-1}\right)$$

Given $|z| \geq 2 \Rightarrow |z| \geq 1$

$\therefore$

$$\{f(k)\} = 2^k - 1^k,\ k > 0$$

$$\{f(k)\} = 2^k - 1,\ k > 0.$$

Ex. 2 : *Find* $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right]$, if $\dfrac{1}{3} < |z| < \dfrac{1}{2}$. ***(May 2005)***

Sol. :

$$F(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$\frac{F(z)}{z} = \frac{3}{z-\frac{1}{2}} + \frac{(-2)}{z-\frac{1}{3}}$$

$$F(z) = 3\cdot\left(\frac{z}{z-\frac{1}{2}}\right) - 2\cdot\frac{z}{z-\frac{1}{3}}$$

$$Z^{-1}[F(z)] = 3\,Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - 2\,Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)$$

Given $\dfrac{1}{3} < |z| < \dfrac{1}{2} \Rightarrow |z| < \dfrac{1}{2}$ and $|z| > \dfrac{1}{3}$

$\therefore$

$$f(k) = \underset{(k<0)}{3\left[-\left(\frac{1}{2}\right)^k\right]} - \underset{(k\geq 0)}{2\cdot\left(\frac{1}{3}\right)^k}$$

$$f(k) = \underset{(k<0)}{-3\left(\frac{1}{2}\right)^k} - \underset{(k\geq 0)}{2\left(\frac{1}{3}\right)^k}$$

Ex. 3 :** Find* $Z^{-1}\left[\dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right]$, $|z| > \dfrac{1}{4}$. ***(Nov. 2014)

Sol. :

$$F(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$$

$$\frac{F(z)}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{(20)}{z-\frac{1}{4}} + \frac{(-20)}{z-\frac{1}{5}}$$

$$F(z) = 20\cdot\frac{z}{z-\frac{1}{4}} - 20\cdot\frac{z}{z-\frac{1}{5}}$$

Given $\quad |z| > \dfrac{1}{4} \Rightarrow |z| > \dfrac{1}{5}$

$$\therefore \quad Z^{-1}[F(z)] = 20\cdot Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 20\,Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right)$$

$$f(k) = 20\left(\frac{1}{4}\right)^k - 20\left(\frac{1}{5}\right)^k,\ k \geq 0.$$

Ex. 4 :** Find* $Z^{-1}\ \dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$, $|z| > \dfrac{1}{2}$. ***(Dec. 2006, 2012, May 2006, 2008)

Sol. :

$$F(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{3}{z-\frac{1}{2}} - \frac{2}{z-\frac{1}{3}}$$

$$F(z) = 3\cdot\frac{z}{z-\frac{1}{2}} - 2\cdot\frac{z}{z-\frac{1}{3}}$$

$$\therefore \quad Z^{-1}[F(z)] = 3\cdot Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - 2\cdot Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)$$

Given $\quad |z| > \dfrac{1}{2} \Rightarrow |z| > \dfrac{1}{3}$

$$\therefore \quad f(k) = 3\left(\frac{1}{2}\right)^k - 2\left(\frac{1}{3}\right)^k,\quad k \geq 0$$

Ex. 5 :** Find* $Z^{-1}\left(\dfrac{3z^2+2z}{z^2-3z+2}\right)$, $1 < |z| < 2$. ***(Dec. 2004, 2012; May 2005, 2015)

Sol. :

$$F(z) = \frac{3z^2+2z}{z^2-3z+2}$$

$$\frac{F(z)}{z} = \frac{3z+2}{(z-2)(z-1)} = \frac{(8)}{z-2} + \frac{(-5)}{z-1}$$

$$F(z) = 8\ \frac{z}{z-2} - 5\ \frac{z}{z-1}$$

$$Z^{-1}\{f(z)\} = 8\ Z^{-1}\left(\frac{z}{z-2}\right) - 5\ Z^{-1}\left(\frac{z}{z-1}\right)$$

Given, $1 < |z| < 2 \Rightarrow |z| > 1$ and $|z| < 2.$

$$\therefore \quad f(k) = \underset{(k<0)}{8[-(2)^k]} - \underset{(k \geq 0)}{5(1)^k}$$

$$= \underset{(k<0)}{-8(2)^k} - \underset{(k \geq 0)}{5}$$

Ex. 6 : *Find* $Z^{-1}\left[\dfrac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right],\ |z| > 1.$ ***(Dec. 2004, 2012; May 2005)***

Sol. :

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} + \frac{C}{\left(z-\frac{1}{2}\right)^2}$$

$$z^2 = A\left(z-\frac{1}{2}\right)^2 + B\left(z-\frac{1}{2}\right)(z-1) + C(z-1)$$

$$z = 1 \Rightarrow A = 4;\ z = \frac{1}{2} \Rightarrow C = -\frac{1}{2}$$

$$z = 0 \Rightarrow B = -3$$

$$\frac{F(z)}{z} = \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{\frac{1}{2}}{\left(z-\frac{1}{2}\right)^2}$$

$$F(z) = 4 \cdot \frac{z}{z-1} - 3 \cdot \frac{z}{z-\frac{1}{2}} - \frac{1}{2}\ \frac{z}{\left(z-\frac{1}{2}\right)^2},\quad |z| > 1$$

$$f(k) = 4(1)^k - 3 \cdot \left(\frac{1}{2}\right)^k - \frac{1}{2} \cdot k\left(\frac{1}{2}\right)^{k-1},\quad k \geq 0$$

$$= 4 - 3\left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^k,\ k \geq 0,\ |z| > 1$$

$$f(k) = 4-(k+3)\left(\frac{1}{2}\right)^k,\ k \geq 0,\ |z| > 1.$$

Note : Here we have used the formula $Z^{-1}\left[\dfrac{z}{(z-a)^2}\right] = k\,a^{k-1},\ k \geq 0,\ |z| > |a|.$

Ex. 7 : *Find* $Z^{-1}\left(\frac{z(z+1)}{z^2-2z+1}\right)$, $|z| > 1$. ***(May 2006, 2014)***

Sol. :

$$\frac{F(z)}{z} = \frac{z+1}{z^2-2z+1} = \frac{z+1}{(z-1)^2}$$

$$= \frac{(z-1)+2}{(z-1)^2} = \frac{1}{z-1} + \frac{2}{(z-1)^2}$$

$$F(z) = \frac{z}{z-1} + 2\cdot\frac{z}{(z-1)^2}, \quad |z| > 1$$

$$\therefore \quad f(k) = (1)^k + 2\,.\,k\,(1)^{k-1}, \; k \geq 0, \quad |z| > 1$$

$$f(k) = 1 + 2k, \quad k \geq 0, \; |z| > 1.$$

Ex. 8 : *Find* $Z^{-1}\left(\frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right)$, $|z| > \frac{1}{2}$.

Sol. :

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{1/2}{\left(z-\frac{1}{2}\right)^2}$$

$$F(z) = 4\cdot\frac{z}{z-1} - 3\cdot\frac{z}{z-\frac{1}{2}} - \frac{1}{2}\,\frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad f(k) = 4\,(1)^k - 3\cdot\left(\frac{1}{2}\right)^k - \frac{1}{2}\cdot k\cdot\left(\frac{1}{2}\right)^{k-1}, \; k \geq 0$$

$$= 4 - (3+k)\left(\frac{1}{2}\right)^k, \quad k \geq 0.$$

Ex. 9 : *Show that* $Z^{-1}\left[\frac{z^3}{\left(z-\frac{1}{4}\right)^2(z-1)}\right] = \{x_k\}$, *for* $|z| > 1$ where $x_k = \frac{16}{9} - \frac{4}{9}\left(\frac{1}{4}\right)^k - \frac{1}{3}(k+1)\left(\frac{1}{4}\right)^k$, $k \geq 0$. ***(May 2009)***

Sol. :

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{4}\right)^2}$$

$$= \frac{16/9}{z-1} - \frac{7/9}{z-\frac{1}{4}} - \frac{1/12}{\left(z-\frac{1}{4}\right)^2}$$

$$X(z) = \frac{16}{9}\,\frac{z}{z-1} - \frac{7}{9}\,\frac{z}{z-\frac{1}{4}} - \frac{1}{12}\,\frac{z}{\left(z-\frac{1}{4}\right)^2}$$

Given, $|z| > 1 \Rightarrow |z| > \frac{1}{4}$

$$\therefore \quad x_k = \frac{16}{9}(1) - \frac{7}{9}\left(\frac{1}{4}\right)^k - \frac{1}{12}\; k \,.\left(\frac{1}{4}\right)^{k-1}, \quad k \geq 0, \quad |z| > 1$$

$$= \frac{16}{9} - \frac{4}{9}\left(\frac{1}{4}\right)^k - \frac{3}{9}\left(\frac{1}{4}\right)^k - \frac{1}{12}\; k \left(\frac{1}{4}\right)^{k-1}$$

$$= \frac{16}{9} - \frac{4}{9}\left(\frac{1}{4}\right)^k - \frac{1}{3}\left(\frac{1}{4}\right)^k [1 + k]$$

$$x_k = \frac{16}{9} - \frac{4}{9}\left(\frac{1}{4}\right)^k - \frac{1}{3}(k+1)\left(\frac{1}{4}\right)^k, \quad k \geq 0, \quad |z| > 1.$$

Ex. 10 : *Find* $Z^{-1}\left(\dfrac{2z^2 + 3z}{z^2 + z + 1}\right), \; |z| > 1.$

Sol. :

$$F(z) = \frac{2z^2 + 3z}{z^2 + z + 1}$$

$$\frac{F(z)}{z} = \frac{2z + 3}{z^2 + z + 1}$$

The roots of $z^2 + z + 1$ are $z = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\, i$

$$\frac{F(z)}{z} = \frac{2z + 3}{\left[z - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\right]\left[z - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) i\right]}$$

$$= \frac{\dfrac{2 + \sqrt{3}\, i}{\sqrt{3}\, i}}{z - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)} + \frac{\dfrac{2 - \sqrt{3}\, i}{-\sqrt{3}\, i}}{z - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)} \qquad \text{(by partial fraction)}$$

$$F(z) = \frac{1}{\sqrt{3}\, i}\left\{\frac{(2 + \sqrt{3}\, i)\, z}{z - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)} - \frac{(2 - \sqrt{3}\, i)\, z}{z - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)}\right\}$$

Now, $\left|-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$ and $\left|-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right| \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\therefore \quad |z| > 1.$

Taking inverse Z-transform,

$$\{f(k)\} = \frac{1}{\sqrt{3}\, i}\left\{(2 + \sqrt{3}\, i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^k - \left(2 - \sqrt{3}\, i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^k\right\}, |z| > 1.$$

We know from complex numbers, if $z = x + iy = r\, e^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \dfrac{y}{x}$

$\therefore \quad \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} = 1\,.\, e^{i\pi/3} \qquad \left\{\because \; r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \text{ and } \theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}\sqrt{3} = \pi/3\right\}$

$\therefore \quad -\dfrac{1}{2} + i\,\dfrac{\sqrt{3}}{2} = 1 \cdot e^{i\,2\pi/3}$ and $-\dfrac{1}{2} - i\,\dfrac{\sqrt{3}}{2} = 1 \cdot e^{-i\,2\pi/3}$

$$\therefore \quad \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)^k = \left(e^{i\,2\pi/3}\right)^k = e^{i\frac{2\pi k}{3}} \text{ and } \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\,i\right)^k = \left(e^{-i2\pi/3}\right)^k = e^{-i\frac{2\pi k}{3}} \text{ (by De Moivre's theorem)}$$

$$f(k) = \frac{1}{\sqrt{3}\,i}\left\{\left(2+\sqrt{3}\,i\right)e^{i\frac{2\pi k}{3}} - \left(2-\sqrt{3}\,i\right)e^{-i\frac{2\pi k}{3}}\right\}$$

$$= \frac{1}{\sqrt{3}\,i}\left\{2\left(e^{i\frac{2\pi k}{3}} - e^{-i\frac{2\pi k}{3}}\right) + \sqrt{3}\,i\left(e^{i\frac{2\pi k}{3}} + e^{-i\frac{2\pi k}{3}}\right)\right\}$$

$$= \frac{2}{\sqrt{3}}\,(2)\left(\frac{e^{\frac{i2\pi k}{3}} - e^{\frac{-i2\pi k}{3}}}{2i}\right) + 2\left(\frac{e^{\frac{i2\pi k}{3}} + e^{\frac{-i2\pi k}{3}}}{2}\right)$$

$$f(k) = \frac{4}{\sqrt{3}}\sin\frac{2\pi k}{3} + 2\cos\frac{2\pi k}{3},\ k \geq 0,\ |z| > 1.$$

$$\left[\textbf{Note: } \frac{e^{i\theta} - e^{-i\theta}}{2\,i} = \sin\theta\ ;\ \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta\right]$$

***Ex. 11 :** Find* $Z^{-1}\left(\frac{z\,(z+1)}{(z-1)\,(z^2+z+1)}\right),\ |z| > 1.$

Sol. :

$$F(z) = \frac{z\,(z+1)}{(z-1)\,(z^2+z+1)}$$

$$\frac{F(z)}{z} = \frac{z+1}{(z-1)\,(z^2+z+1)}$$

$$\frac{z+1}{(z-1)\,(z^2+z+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+z+1}$$

Here,

$$z+1 = A\,(z^2+z+1) + (Bz+C)\,(z-1)$$

$$z+1 = Az^2 + Az + A + Bz^2 - Bz + Cz - C$$

$$z+1 = (A+B)\,z^2 + (A-B+C)\,z + (A-C)$$

Equating coefficients $A + B = 0$; $A - B + C = 1$; $A - C = 1$

Solving $A = \frac{2}{3}$; $B = -\frac{2}{3}$; $C = -\frac{1}{3}$

$$\frac{F(z)}{z} = \frac{2}{3}\,\frac{1}{z-1} + \frac{-\frac{2}{3}z - \frac{1}{3}}{z^2+z+1}$$

$$F(z) = \frac{2}{3}\,\frac{z}{z-1} - \frac{1}{3}\,\frac{z\,(2z+1)}{z^2+z+1}$$

$$= \frac{2}{3}\,\frac{z}{z-1} - \frac{2}{3}\left(\frac{z\left(z+\frac{1}{2}\right)}{z^2+2z\left(\frac{1}{2}\right)+1}\right)$$

$$Z^{-1}\left(\frac{z}{z-1}\right) = 1,\ |z| > 1.$$

To find $Z^{-1}\left\{\frac{z\left(z+\frac{1}{2}\right)}{z^2+2z\left(\frac{1}{2}\right)+1}\right\}$ we compare with $Z^{-1}\left(\frac{z\,(z-\cos\alpha)}{z^2-2z\cos\alpha+1}\right) = \{\cos\alpha k\},\ k \geq 0.$

$$\therefore \quad -\cos\alpha = \frac{1}{2} \qquad \therefore \quad \cos\alpha = -\frac{1}{2} \qquad \alpha = \frac{2\pi}{3}.$$

$$f(k) = \frac{2}{3} Z^{-1}\left(\frac{z}{z-1}\right) - \frac{2}{3} Z^{-1}\left(\frac{z\left(z+\frac{1}{2}\right)}{z^2+2z\left(\frac{1}{2}\right)+1}\right)$$

$$= \frac{2}{3}(1) - \frac{2}{3}\cos\frac{2\pi k}{3}, \quad k \geq 0$$

$$f(k) = \frac{2}{3}\left(1-\cos\frac{2\pi k}{3}\right), \quad k \geq 0, \ |z| > 1.$$

Ex. 12 : *Show that* $Z^{-1}\left[\frac{z^2+z}{z^2+z+1}\right] = \{x_k\}$ for $|z| > 1$, *where* $x_k = \cos\frac{2\pi k}{3} + \frac{1}{\sqrt{3}}\sin\frac{2\pi k}{3}, \ k \geq 0.$

Sol. :

$$\frac{X(z)}{z} = \frac{z+1}{z^2+z+1}$$

$$\frac{X(z)}{z} = \frac{z+1}{\left[z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]\left[z-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right]}$$

$$= \frac{\dfrac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\sqrt{3}\,i}}{z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} + \frac{\dfrac{\frac{1}{2}-\frac{\sqrt{3}}{2}i}{-\sqrt{3}\,i}}{z-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}$$

$$X(z) = \frac{1}{2\sqrt{3}\,i}\left\{\frac{\left(1+\sqrt{3}\,i\right)z}{z-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)i} - \frac{\left(1-\sqrt{3}\,i\right)z}{z-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}\right\}$$

Taking inverse Z-transform and noting that

$$\left|-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1 \text{ and } \left|-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1 \text{ and } |z| > 1.$$

$$\therefore \quad \{x_k\} = \frac{1}{2\sqrt{3}\,i}\left\{\left(1+\sqrt{3}\,i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^k - \left(1-\sqrt{3}\,i\right)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^k\right\}$$

But $$-\frac{1}{2}+\frac{\sqrt{3}}{2}i = 1\cdot e^{i\frac{2\pi}{3}} \text{ and } -\frac{1}{2}-\frac{\sqrt{3}}{2}i = 1\cdot e^{-i\frac{2\pi}{3}}$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^k = e^{i\frac{2\pi k}{3}} \text{ and } \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^k = e^{-i\frac{2\pi k}{3}}$$

$$x_k = \frac{1}{2\sqrt{3}\,i}\left\{\left(1+\sqrt{3}\,i\right)e^{i\frac{2\pi k}{3}} - \left(1-\sqrt{3}\,i\right)e^{-i\frac{2\pi k}{3}}\right\}$$

$$= \frac{1}{2\sqrt{3}\,i}\left\{\left(e^{i\frac{2\pi k}{3}} - e^{-i\frac{2\pi k}{3}}\right) + \sqrt{3}\,i\left(e^{i\frac{2\pi k}{3}} + e^{-i\frac{2\pi k}{3}}\right)\right\}$$

$$= \frac{1}{\sqrt{3}}\left(\frac{e^{i\frac{2\pi k}{3}} - e^{-i\frac{2\pi k}{3}}}{2i}\right) + \left(\frac{e^{i\frac{2\pi k}{3}} + e^{-i\frac{2\pi k}{3}}}{2}\right)$$

$$x_k = \frac{1}{\sqrt{3}}\sin\frac{2\pi k}{3} + \cos\frac{2\pi k}{3}, \quad k \geq 0.$$

Ex. 13 : *Show that* $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right] = \{x_k\}$ for $|z| < \dfrac{1}{5}$, *where* $x_k = 4\left(\dfrac{1}{5}\right)^k - 5\left(\dfrac{1}{4}\right)^k,\ k < 0.$

Sol. :

$$\frac{X(z)}{z} = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{5}{z-\frac{1}{4}} - \frac{4}{z-\frac{1}{5}}$$

$$X(z) = 5\cdot\frac{z}{z-\frac{1}{4}} - 4\cdot\frac{z}{z-\frac{1}{5}}$$

$$|z| < \frac{1}{5} \Rightarrow |z| < \frac{1}{4}$$

$\therefore$
$$x_k = 5\left[-\left(\frac{1}{4}\right)^k\right] - 4\left[-\left(\frac{1}{5}\right)^k\right],\ k < 0$$

$$= -5\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{5}\right)^k,\quad k < 0.$$

Ex. 14 : *Show that* $Z^{-1}\left[\dfrac{z+1}{(z-1)^2}\right] = \{x_k\}$ for $|z| > 1$, *where* $x_k \begin{array}{ll} = 2k-1, & k \geq 1 \\ = 0, & k < 1 \end{array}.$

Sol. :

$$X(z) = \frac{z+1}{(z-1)^2} = \frac{(z-1)+2}{(z-1)^2}$$

$$X(z) = \frac{1}{z-1} + \frac{2}{(z-1)^2}$$

Given $\quad |z| > 1, \quad \left|\dfrac{1}{z}\right| < 1$

$$X(z) = \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{2}{z^2\left(1-\frac{1}{z}\right)^2}$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots\right)$$

$$+ \frac{2}{z^2}\left[1 + (-2)\left(-\frac{1}{z}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{1}{z}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{1}{z}\right)^3 + \ldots\ldots\right]$$

$$= \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots + \frac{1}{z^k} + \ldots\right] + \left[\frac{2}{z^2} + \frac{4}{z^3} + \frac{6}{z^4} + \frac{8}{z^5} + \ldots \frac{2k}{z^{k+1}} + \ldots\right]$$

Coefficient of z^{-k} in first series $= 1,\quad k \geq 1$

Coefficient of z^{-k-1} in second series $= 2k,\ k \geq 1$

Coefficient of z^{-k} in second series $= 2(k-1),\ k-1 \geq 1$

$= 2k-2,\quad k \geq 2.$

$$\{x_k\} = \underset{k \geq 1}{(1)} + \underset{k \geq 2}{(2k-2)}$$

$$= 2k-1,\ k \geq 1$$

$$= 0,\quad k < 1.$$

Ex. 15 :** Show that* $Z^{-1}\left\{\dfrac{z^2}{z^2+1}\right\} = \{x_k\}$ *for* $|z| > 1$*, where* $x_k = \cos\dfrac{k\pi}{2},\ k \geq 0.$ ***(May 2006; Dec. 2006)

Sol. :

$$X(z) = \frac{z^2}{z^2+1} = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1} \quad \text{(Note this adjustment)}$$

$$x_k = Z^{-1}\left(\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1}\right)$$

$$\{x_k\} = \left\{\cos\frac{k\pi}{2}\right\},\ k \geq 0.$$

EXERCISE 4.2

Find f(k) if :

1. $\dfrac{1}{z-a}$, $|z| < |a|$, $|z| > |a|$

Ans. : $-a^{k-1}$, $k \leq 0$; a^{k-1}, $k \geq 1$

2. $\dfrac{1}{z+a}$, $|z| > a$

Ans. : $(-a)^{k-1}$, $k \geq 1$

3. $\dfrac{1}{(z-a)^2}$, $|z| < |a|$, $|z| > |a|$

Ans. : $\dfrac{-k+1}{a^{-k+2}}$, $k \leq 0$; $(k-1)\,a^{k-2}$, $k \geq 2$

4. $\dfrac{1}{(z-5)^3}$, $|z| > 5$, $|z| < 5$

Ans. : $\dfrac{(k-2)(k-1)}{2}\ 5^{k-3}$, $k \geq 3$;

$\dfrac{-(-k+1)(-k+2)}{2}\ \dfrac{1}{5^{-k+3}}$, $k \leq 0$

5. $\dfrac{1}{(z-3)(z-2)}$ **(Dec. 2011, May 2016)**

if (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| > 3$

Ans. : (i) $-3^{k-1} + 2^{k-1}$, $k \leq 0$

(ii) $f(k) = \begin{cases} -3^{k-1}, & k \leq 0 \\ -2^{k-1}, & k \geq 1 \end{cases}$

(iii) $f(k) = \begin{cases} 3^{k-1} - 2^{k-1}, & k \geq 1 \\ 0, & k \leq 0 \end{cases}$

6. $\dfrac{z+2}{z^2-2z+1}$, $|z| > 1$ **(Nov. 2015)**

Ans. : $3k-2$, $k \geq 1$

7. $\dfrac{2z^2-10z+13}{(z-3)^2(z-2)}$ $2 \leq |z| < 3$

Ans. : $f(k) = \begin{cases} 2^{k-1} & k \geq 1 \\ \dfrac{-k-2}{3^{-k+2}} \leq & k < 0 \end{cases}$

8. $\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$ **(Dec. 2008, 2012)**

if (i) $\dfrac{1}{5} < |z| < \dfrac{1}{4}$, (ii) $|z| < \dfrac{1}{5}$

Ans. :

(i) $f(k) = \begin{cases} -\dfrac{1}{5}\left(\dfrac{1}{4}\right)^k, & k < 0 \\ -4\left(\dfrac{1}{5}\right)^k, & k \geq 0 \end{cases}$

(ii) $f(k) = \begin{cases} 4\,(5)^{-k}, & k < 0 \\ -5\,(4)^{-k}, & k < 0 \end{cases}$

9. $\dfrac{3z^2+2z}{z^2+3z+2}$, $1 < |z| < 2$

Ans. : $f(k) = \begin{cases} -5, & k \geq 0 \\ -8\,(2)^k, & k < 0 \end{cases}$

10. $\dfrac{z}{(z-2)(z-3)}$,

if (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| < 3$

Ans. : (i) $2^k - 3^k$, $k \leq 0$

(ii) $f(k) = \begin{cases} -2^k, & k > 0 \\ -3^k, & k \leq 0 \end{cases}$

(iii) $3^k - 2^k$, $k \geq 0$

11. $\dfrac{z^3}{(z-1)\ (z-2)^2}$, $|z| > 2$ **(May 2007)**

Ans. : $1 + k \cdot 2^{k+1},\ k \geq 0$

12. $\dfrac{z^3}{(z-3)\ (z-2)^2}$, $|z| > 3$ **(Dec. 2008)**

Ans. : $3^{k+2} - 2^{k+2} - k \cdot 2^{k+1},\ k \geq 0$

13. $\dfrac{z^2}{z^2 + a^2}$, $|z| > |a|$ **(Dec. 2008)**

Ans. : $a^k \cos \dfrac{k\pi}{2}$

14. $\dfrac{z}{(z-1)(z-2)}$, $|z| > 2$

Ans. : $2^k - 1,\ k \geq 0$

15. $\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$, $|z| > \dfrac{1}{4}$ **(Dec. 2007, 2013)**

Ans. : $5\left(\dfrac{1}{4}\right)^k - 4\left(\dfrac{1}{5}\right)^k,\ k \geq 0$

16. $\dfrac{2z^2 + 3z}{z^2 + z + \frac{1}{16}}$, $|z| > 2 + \sqrt{3}$

Ans. : $2\left\{\left(-\dfrac{1}{4}\right)^k \cosh \alpha k - \dfrac{8}{\sqrt{3}}\left(-\dfrac{1}{4}\right)^k \sinh \alpha k\right\},\ k \geq 0$

where $\cosh \alpha = 2$

III. Inversion Integral Method

4.12 SINGULAR POINT, POLE, RESIDUE

We note that the point z_0, where the function f(z) ceases to be analytic is called the **singular point** of the function f(z). If in the small neighbourhood of z_0, say $|z - z_0| < \in$, there is no singular point of f(z) other than z_0, then z_0 is called isolated singular point. In such a case, f(z) can be expanded around $z = z_0$ in a series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\ldots + a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \ldots\ a_{-n}(z-z_0)^{n}$$

called Laurent's series.

This series contains two parts : (i) Series consisting of positive powers of $(z - z_0)$ is called analytic part of the Laurent's series. (ii) Series consisting of negative powers of $(z - z_0)$ is called principal part of the Laurent's series. a_n and a_{-n} are given by the integral

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \quad n = 0, \pm 1, \pm 2 \ldots\ldots$$

'c' is the circle surrounding the singular point z_0.

If the principal part of the Laurent's series contains n terms

$$a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \ldots\ a_{-n}(z-z_0)^{-n}$$

then singular point z_0 is called pole of the order n.

If the principal part of the Laurent's series contains only one term $a_{-1}(z - z_0)^{-1}$ then z_0 is called simple pole. If it contains two terms $a_{-1}(z - z_0)^{-1} + a_{-2}(z - z_0)^{-2}$ then z_0 is a double pole or the pole of the order 2. Like this we can have poles of various orders.

a_{-1} which is the residue of f(z) at isolated singular point z_0 is given by $a_{-1} = \dfrac{1}{2\pi i} \oint_c f(z)\, dz$.

In practice, we calculate residues at the poles by different formulae.

We have,
$$Z\left[\{f(k)\}\right] = F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$F(z) = f(0) + f(1)\ z^{-1} + f(2)\, z^{-2} + \ldots\ldots + f(k)\, z^{-k} + \ldots\ldots$$

Inversion Integral Method :

By multiplying both sides of this last equation by z^{k-1}, we obtain

$$F(z) \cdot z^{k-1} = f(0) \cdot z^{k-1} + f(1)\ z^{k-2} + f(2)\ z^{k-3} + \ldots\ldots + f(k)\, z^{-1} + \ldots$$

Now we integrate both sides of the above equation along a circle such that all the poles (i.e. values of z such that F (z) is infinite) of F (z) lie within a circle C, in anticlockwise direction, we get

$$\oint_C F(z)\, z^{k-1}\, dz = \oint_C f(0)\, z^{k-1}\, dz + \oint_C f(1)\ z^{k-2}\, dz + \ldots + \oint_C f(k)\, z^{-1}\ dz + \ldots$$

Applying Cauchy's theorem of complex integration, we see that all terms of R.H.S. of above equation are zero except the term

$$\oint_C f(k)\, z^{-1}\, dz = (2\pi i)\, f(k)$$

$$\therefore \quad \oint_C F(z)\, z^{k-1}\, dz = \oint_C f(k)\, z^{-1}\, dz = (2\pi i)\, f(k)$$

$$\therefore \quad \boxed{\mathbf{f(k) = \frac{1}{2\pi i} \oint_C F(z)\, z^{k-1}\, dz}} \qquad \text{... (I)}$$

Equation (I) is known as the inversion integral for inverse of Z-transform and equation (I) is equivalent to stating that

$$\boxed{\mathbf{f(k) = \sum [\text{Residues of } F(z)\, z^{k-1} \text{ at the poles of } F(z)]}}$$

We have from the theory of complex variables,

(i) Residue for simple pole z = a is $= \left[(z-a)\, z^{k-1}\, F(z)\right]_{z=a}$

(ii) Residue for r times repeated poles at z = a is $= \dfrac{1}{(r-1)!} \cdot \dfrac{d^{r-1}}{dz^{r-1}} \left[(z-a)^r\, z^{k-1}\, F(z)\right]_{z=a}$

The method of inversion integral is most convenient method than earlier methods, in determining inverse of Z-transform.

Additional Results :

1. **Pole of F (z) :** Pole of F (z) is the value (or values) of z for which F (z) is infinite.

   e.g. $\quad F(z) = \dfrac{z}{(z-a)(z-b)}$

Here z = a and z = b are the poles. These are also called as simple poles of F (z).

2. **Multiple Pole of F (z) :** If a pole is repeated more than once, it is called a multiple pole.

   e.g. $\quad F(z) = \dfrac{z^3}{(z-1)\,(z-2)^2}$

   Here $\quad z = 1$ is a simple pole and $z = 2$ is called a double pole.

The following examples will illustrate the inversion integral method.

ILLUSTRATIONS

Ex. 1 :** Find $Z^{-1}\left[\dfrac{1}{(z-2)\,(z-3)}\right]$ by inversion integral method.* ***(May 2016)

Sol. : Given, $\quad F(z) = \dfrac{1}{(z-2)(z-3)}$. The poles of F (z) are simple poles at z = 2, z = 3.

Consider, $\quad F(z)\, z^{k-1} = \dfrac{z^{k-1}}{(z-2)\,(z-3)}$

Residue of $F(z)\, z^{k-1}$ at z = 2 is

$$= \left[(z-2)\, F(z)\, z^{k-1}\right]_{z=2} = \left[(z-2)\, \frac{z^{k-1}}{(z-2)\,(z-3)}\right]_{z=2}$$

$$= \left[\frac{z^{k-1}}{z-3}\right]_{z=2} = \frac{2^{k-1}}{-1} = -2^{k-1} \qquad \text{... (i)}$$

Residue of $F(z)\, z^{k-1}$ at z = 3 is

$$= \left[(z-3)\, z^{k-1}\, F(z)\right]_{z=3} = \left[(z-3)\, \frac{z^{k-1}}{(z-2)\,(z-3)}\right]_{z=3}$$

$$= \left[\frac{z^{k-1}}{z-2}\right]_{z=3} = \frac{3^{k-1}}{1} = 3^{k-1} \qquad \text{... (ii)}$$

From (i) and (ii),

$$f(k) = \text{algebraic sum of all the residues of } F(z)\, z^{k-1}$$

$$= 3^{k-1} - 2^{k-1}, \quad k \geq 1, \ |z| > 3.$$

Ex. 2 : *Obtain {f (k)} by use of the inversion integral when F (z) is given by*

$$F(z) = \frac{10z}{(z-1)(z-2)}$$

(Dec. 2010, 2012, May 2011, 2018, Nov. 2014, 2015)

Sol. : The poles of F (z) are simple poles at z = 1, z = 2.

Consider $\quad F(z)\, z^{k-1} = \dfrac{10 \cdot z^k}{(z-1)(z-2)}$

Residue of F (z) z^{k-1} at z = 1 is

$$= [z-1]\ (z^{k-1}\ F(z))_{z=1}$$

$$= \left[(z-1)\frac{10z^k}{(z-1)(z-2)}\right]_{z=1}$$

$$= \left[\frac{10z^k}{(z-2)}\right]_{z=1} = \frac{10}{-1} = -10$$

Residue of F (z) z^{k-1} at z = 2 is

$$= \left[(z-2)\, z^{k-1}\ F(z)\right]_{z=2}$$

$$= \left[(z-2)\frac{10z^k}{(z-1)(z-2)}\right]_{z=2}$$

$$= \left[\frac{10z^k}{(z-1)}\right]_{z=2} = \frac{10(2)^k}{1} = 10(2)^k$$

$$f(k) = \text{algebraic sum of all the residues of } z^{k-1} F(z).$$

$$= 10\,[2^k - 1], \quad k \geq 0.$$

Ex. 3 : *Find* $Z^{-1}\left[\dfrac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]$ *by using inversion integral method.*

Sol. : Given, $\quad F(z) = \dfrac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$. The poles of F (z) are simple poles at z = 1 and double pole at $z = \dfrac{1}{2}$.

Consider, $\quad F(z)\ z^{k-1} = \dfrac{z^{k+2}}{(z-1)\left(z-\frac{1}{2}\right)^2}$

Residue of F (z) z^{k-1} at z = 1 is

$$= \left[(z-1)\, z^{k-1}\ F(z)\right]_{z=1}$$

$$= \left[(z-1)\frac{z^{k+2}}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]_{z=1}$$

$$= \left[\frac{z^{k+2}}{\left(z-\frac{1}{2}\right)^2}\right]_{z=1} = \frac{1}{\frac{1}{4}}$$

Residue of F (z) z^{k-1} for 2 times repeated pole at $z = \frac{1}{2}$ is

$$= \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[\left(z-\frac{1}{2}\right)^2 z^{k-1} F(z)\right]_{z=\frac{1}{2}}$$

$$= \frac{1}{1!} \frac{d}{dz} \left[\left(z-\frac{1}{2}\right)^2 \frac{z^{k+2}}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]_{z=\frac{1}{2}}$$

$$= \frac{d}{dz}\left[\frac{z^{k+2}}{(z-1)}\right]_{z=\frac{1}{2}}$$

$$= \left[\frac{(k+2)\, z^{k+1}}{z-1} - \frac{z^{k+2}}{(z-1)^2}\right]_{z=\frac{1}{2}}$$

$$= \frac{(k+2)\left(\frac{1}{2}\right)^{k+1}}{\left(-\frac{1}{2}\right)} - \frac{\left(\frac{1}{2}\right)^{k+2}}{\left(\frac{1}{2}\right)^2}$$

$$= -(k+2)\left(\frac{1}{2}\right)^k - \left(\frac{1}{2}\right)^k = -(k+3)\left(\frac{1}{2}\right)^k$$

$$f(k) = 4 - (k+3)\left(\frac{1}{2}\right)^k, \quad k \geq 0, \ |z| > 1.$$

Ex. 4 : *Find* $Z^{-1}\left(\frac{z^2}{z^2+1}\right)$ *by using inversion integral method.* ***(Dec. 2005, 2012, May 2008, 2016, 2017)***

Sol. : Given, $$F(z) = \frac{z^2}{z^2+1}$$

Consider, $z^{k-1} F(z) = \frac{z^{k+1}}{(z+i)(z-i)}$ which has poles at $z = i, \ z = -i$

Residue of F (z) z^{k-1} at z = i

$$= \left[(z-i)\, z^{k-1} F(z)\right]_{z=i} = \left[\frac{z^{k+1}}{(z+i)}\right]_{z=i} = \frac{(i)^{k+1}}{2i} = \frac{(i)^k}{2}$$

Residue of F (z) z^{k-1} at z = – i is

$$= \left[(z+i)\, z^{k-1} F(z)\right]_{z=-i}$$

$$= \left(\frac{z^{k+1}}{z-i}\right)_{z=-i} = \frac{(-i)^{k+1}}{-2i} = \frac{(-i)^k}{2}$$

$\therefore$ $$f(k) = \frac{(i)^k}{2} + \frac{(-i)^k}{2} = \frac{(i)^k + (-i)^k}{2}$$

But $$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$\therefore$ $$(i)^k = e^{i\frac{k\pi}{2}}$$

Similarly, $$(-i)^k = e^{-i\frac{k\pi}{2}}$$

$$\frac{(i)^k + (-i)^k}{2} = \frac{e^{ik\pi/2} + e^{-ik\pi/2}}{2} = \cos k\frac{\pi}{2}$$

$\therefore$ $$f(k) = \cos k\frac{\pi}{2}, \ k \geq 0, |z| > 1.$$

Ex. 5 : *Use inversion integral to find inverse transforms of*

$$F(z) = \frac{2z^2 - 3z}{(z-1)\left(z^2 - 2z + \frac{1}{4}\right)}.$$

Sol. : The roots of the equation $z^2 - 2z + \frac{1}{4} = 0$ are $Z = \frac{2 \pm \sqrt{4-1}}{2} = 1 \pm \frac{\sqrt{3}}{2}$.

$\therefore$ Consider the function :

$$F(z)\, z^{k-1} = \frac{(2z-3)\, z^k}{(z-1)\left[z - \left(1 + \frac{\sqrt{3}}{2}\right)\right]\left[z - \left(1 - \frac{\sqrt{3}}{2}\right)\right]}$$

which has poles at $z = 1$, $z = 1 + \frac{\sqrt{3}}{2}$ and $z = 1 - \frac{\sqrt{3}}{2}$.

$\therefore$ Residue of $F(z)\, z^{k-1}$ at $z = 1$ is

$$= \left[(z-1)\, z^{k-1}\, F(z)\right]_{z=1}$$

$$= \left[(z-1)\, \frac{(2z-3)\, z^k}{(z-1)\left(z^2 - 2z + \frac{1}{4}\right)}\right]_{z=1} = \frac{4}{3}.$$

Residue of $F(z)\, z^{k-1}$ at $z = 1 + \frac{\sqrt{3}}{2}$ is

$$= \left[\left(z - \left(1 + \frac{\sqrt{3}}{2}\right)\right) z^{k-1}\, F(z)\right]_{z = 1 + \frac{\sqrt{3}}{2}}$$

$$= \left[\frac{(2z-3)\, z^k}{(z-1)\left(z - \left(1 - \frac{\sqrt{3}}{2}\right)\right)}\right]_{z = 1 + \frac{\sqrt{3}}{2}}$$

$$= \frac{2}{3}\left(\sqrt{3} - 1\right)\left(1 + \frac{\sqrt{3}}{2}\right)^k = -\frac{2}{3}\left(1 - \sqrt{3}\right)\left(1 + \frac{\sqrt{3}}{2}\right)^k$$

Residue of $F(z)\, z^{k-1}$ at $z = 1 - \frac{\sqrt{3}}{2}$ is

$$= \left[z^{k-1} \cdot F(z)\left(z - \left(1 - \frac{\sqrt{3}}{2}\right)\right)\right]_{z = 1 - \frac{\sqrt{3}}{2}} = \left[\frac{(2z-3)\, z^k}{(z-1)\left(z - \left(1 + \frac{\sqrt{3}}{2}\right)\right)}\right]_{z = 1 - \frac{\sqrt{3}}{2}}$$

$$= \frac{2}{3}\left(-\sqrt{3} - 1\right)\left(1 - \frac{\sqrt{3}}{2}\right)^k = -\frac{2}{3}\left(1 + \sqrt{3}\right)\left(1 - \frac{\sqrt{3}}{2}\right)^k$$

$\therefore$ $f(k)$ = algebraic sum of all the residues of $z^{k-1}\, F(z)$.

$$f(k) = \frac{4}{3} - \frac{2}{3}\left[\left(1 - \sqrt{3}\right)\left(1 + \frac{\sqrt{3}}{2}\right)^k + \left(1 + \sqrt{3}\right)\left(1 - \frac{\sqrt{3}}{2}\right)^k\right]$$

EXERCISE 4.3

Find inverse Z-transforms by inversion integral method :

1. $\dfrac{z(z+1)}{(z-1)(z^2+z+1)}$

Ans. $\frac{2}{3}\left(1 - \cos\frac{2\pi k}{3}\right)$

2. $\dfrac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$ **(May 2007)**

Ans. $3\left(\frac{1}{2}\right)^k - 2\left(\frac{1}{3}\right)^k$

3. $\dfrac{z^3}{\left(z-\frac{1}{4}\right)^2 (z-1)}$

Ans. $\dfrac{16}{9} - \dfrac{4}{9}\left(\dfrac{1}{4}\right)^k - \dfrac{1}{3}(k+1)\left(\dfrac{1}{4}\right)^k$

4. $\dfrac{2z^2 + 3z}{z^2 + z + 1}$

Ans. $2\cos\dfrac{2\pi k}{3} + \dfrac{4}{\sqrt{3}}\sin\dfrac{2\pi k}{3}$

5. $\dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$ **(Dec. 2010)**

Ans. $20\left[\left(\dfrac{1}{4}\right)^k - \left(\dfrac{1}{5}\right)^k\right]$

6. $\dfrac{z}{(z-2)(z+4)^2}$

Ans. $\dfrac{1}{36}\left[2^k - 6k(-4)^{k-1} - (-4)^k\right]$

7. $\dfrac{1}{(z-a)^3}$ $\left[\textbf{Hint} : \text{Residue} = \dfrac{1}{2}\left[\dfrac{d^2}{dz^2} z^{k-1}\right]_{z=a}\right]$

Ans. $\dfrac{1}{2}(k-1)(k-2)a^{k-3}$

4.12 SOLUTIONS OF DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS USING Z-TRANSFORM

A relation between f (k) and f (k + 1), f (k + 2), f (k + 3), ……… is called *difference equation* and an expression for f (k) in terms of k which satisfies the equation is called its solution.

A Laplace transform, transforms a differential equation to algebraic equation, the Z-transform, transforms a difference equation to algebraic equation in z and initial data is automatically included in algebraic equation. We take Z- transform of the entire equation to solve a difference equation and write F (z). The inverse Z-transform of F (z) gives the required solution.

Useful Standard Results : For a causal sequence {f(k)}, $k \ge 0$,

$$\begin{aligned}
Z\{f(k)\} &= F(z) \\
Z\{f(k+1)\} &= z\,F(z) - z\,f(0) \\
Z\{f(k+2)\} &= z^2 F(z) - z^2 f(0) - z\,f(1) \\
Z\{f(k-1)\} &= z^{-1} F(z) \\
Z\{f(k-2)\} &= z^{-2} F(z).
\end{aligned}$$

Note : f (k) is considered as causal sequence.

ILLUSTRATIONS

Ex. 1 : *Obtain f (k) given that* $f(k+1) + \dfrac{1}{2} f(k) = \left(\dfrac{1}{2}\right)^k$, $k \ge 0$, $f(0) = 0$.

(May 2005, 2008, 2011, 2017, Dec. 2010, Nov. 2015, 2019)

Sol. : Taking Z-transform of both sides, we get

$$Z\{f(k+1)\} + \frac{1}{2} Z\{f(k)\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$[z\,F(z) - z\,f(0)] + \frac{1}{2} F(z) = \frac{z}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$\left(z+\frac{1}{2}\right) F(z) = \frac{z}{z-\frac{1}{2}}$$

$$F(z) = \frac{z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$$

$$\frac{F(z)}{z} = \frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} = \frac{1}{z-\frac{1}{2}} - \frac{1}{z+\frac{1}{2}}$$

$$F(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z+\frac{1}{2}}$$

$$\therefore \qquad f(k) = \left(\frac{1}{2}\right)^k - \left(-\frac{1}{2}\right)^k, \quad k \geq 0.$$

Ex. 2 : *Obtain f (k), given that f (k + 2) − 4f (k) = 0, f(0) = 0, f(1) = 2.*

Sol. : Taking Z-transform of both sides,

We get
$$Z\{f(k+2)\} - 4Z\{f(k)\} = 0$$
$$[z^2F(z) - z^2f(0) - z\,f(1)] - 4[F(z)] = 0$$
$$[z^2F(z) - z.2)] - 4[F(z)] = 0$$
$$(z^2 - 4)\,F(z) = 2z$$
$$f(z) = \frac{2z}{z^2-4} \quad \text{(By Partial fractions}$$
$$F(z) = \frac{1}{2}\left[\frac{1}{z-2} - \frac{1}{z+2}\right]$$
$$\therefore \qquad F(k) = \frac{1}{2}[2^k - (-2)^k], \quad k \geq 0$$

Ex. 3 : *Obtain f (k), given that 12 f (k + 2) – 7 f (k + 1) + f (k) = 0, k ≥ 0, f (0) = 0, f (1) = 3.*

(Dec. 2006, 2007, 2012, 2016, 2017; May 2019)

Sol. : Taking Z-transform of both sides, we get

$$12\,Z\{f(k+2)\} - 7\,Z\{f(k+1)\} + Z\ f(k) = 0$$
$$12\left[z^2 F(z) - z^2 f(0) - z\,f(1)\right] - 7\left[z\,F(z) - z\,f(0)\right] + F(z) = 0$$
$$12\left[z^2 F(z) - 3z\right] - 7z\ F(z) + F(z) = 0$$
$$\left(12z^2 - 7z + 1\right) F(z) = 36z$$
$$F(z) = \frac{36z}{(4z-1)\ (3z-1)}$$
$$\frac{F(z)}{z} = \frac{36}{(4z-1)(3z-1)} = 36\left\{-\frac{4}{4z-1} + \frac{3}{3z-1}\right\} \text{ (By partial fractions)}$$
$$F(z) = 36\left[\frac{3z}{3z-1} - \frac{4z}{4z-1}\right] = 36\left[\frac{z}{z-\frac{1}{3}} - \frac{z}{z-\frac{1}{4}}\right]$$
$$\{f(k)\} = 36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right], \quad k \geq 0.$$

Ex. 4 : *Obtain the output of the system, where input is U_k and the system is given by $y_k - 4\,y_{k-2} = U_k$, $k \geq 0$*

$$\text{where} \quad U_k = \left(\frac{1}{2}\right)^k, \qquad k \geq 0$$
$$= 0, \qquad k < 0$$

(May 2006)

Sol. : We have,
$$Z\{y_k\} = Y(z)$$
$$Z\{y_{k-2}\} = z^{-2}\ Y(z)$$

Since $\{y_k\}$ is considered as causal sequence. $\therefore$ y_{-1}, y_{-2} are zero.

$$\therefore \qquad Z\{y_k\} - 4Z\{y_{k-2}\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$
$$Y(z) - 4z^{-2}\ Y(z) = \frac{z}{z-\frac{1}{2}}$$

$$\left(\frac{z^2 - 4}{z^2}\right) Y(z) = \frac{z}{\left(z - \frac{1}{2}\right)}$$

$$Y(z) = \frac{z^3}{\left(z - \frac{1}{2}\right)(z^2 - 4)}$$

$$\frac{Y(z)}{z} = \frac{z^2}{(z-2)(z+2)\left(z - \frac{1}{2}\right)}$$

$$Y(z) = \frac{2}{3}\frac{z}{z-2} + \frac{2}{5}\frac{z}{z+2} - \frac{1}{15}\frac{z}{z - \frac{1}{2}}$$

$$\{y_k\} = \frac{2}{3} 2^k + \frac{2}{5}(-2)^k - \frac{1}{15}\left(\frac{1}{2}\right)^k, \; k \geq 0.$$

[**Note :** $Z\{y_{k-1}\} = z^{-2} Y(z) + y_{-1} z^{-1} + y_{-2} z^0$]

Ex. 5 :** Solve* $y_k - \frac{5}{6} y_{k-1} + \frac{1}{6} y_{k-2} = \left(\frac{1}{2}\right)^k, \; k \geq 0.$ ***(Dec. 2004, May 2018)

Sol. :

$$Z\{y_k\} - \frac{5}{6} Z\{y_{k-1}\} + \frac{1}{6} Z\{y_{k-2}\} = Z\left(\frac{1}{2}\right)^k$$

y_{-1}, y_{-2} are zero, since y_k is considered as causal sequence.

$$Y(z) - \frac{5}{6} \cdot z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = \frac{z}{z - \frac{1}{2}}$$

$$\left(1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right) Y(z) = \frac{z}{z - \frac{1}{2}}$$

$$\left(\frac{z^2 - \frac{5}{6} z + \frac{1}{6}}{z^2}\right) Y(z) = \frac{z}{z - \frac{1}{2}}$$

$$Y(z) = \frac{z^3}{\left(z - \frac{1}{2}\right)\left(z^2 - \frac{5}{6} z + \frac{1}{6}\right)}$$

$$\frac{Y(z)}{z} = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

$$= \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)^2} = \frac{4}{z - \frac{1}{3}} - \frac{3}{z - \frac{1}{2}} + \frac{\frac{3}{2}}{\left(z - \frac{1}{2}\right)^2}$$

$$Y(z) = 4 \cdot \frac{z}{z - \frac{1}{3}} - 3 \cdot \frac{z}{z - \frac{1}{2}} + \frac{3}{2} \cdot \frac{z}{\left(z - \frac{1}{2}\right)^2}$$

$$\{y_k\} = 4 \cdot \left(\frac{1}{3}\right)^k - 3 \cdot \left(\frac{1}{2}\right)^k + \frac{3}{2} \cdot k \cdot \left(\frac{1}{2}\right)^{k-1}$$

$$= 4 \cdot \left(\frac{1}{3}\right)^k - 3 \cdot \left(\frac{1}{2}\right)^k + 3 \cdot k \cdot \left(\frac{1}{2}\right)^k, \; k \geq 0.$$

[**Note :** $Z\{y_{k-1}\} = z^{-2} Y(z) + y_{-1} z^{-1} + y_{-2} z^0$]

Ex. 6 : *From the equation $y_k - 3y_{k-1} + 2y_{k-2} = 1$, $k \geq 0$ and $y_{-1} = y_{-2} = 2$, show that the unilateral transform Y(z) of the sequence $\{y_k\}$, using the given initial conditions, is* $\dfrac{z\left(3z^2 - 6z + 4\right)}{(z-1)^2 \ (z-2)}$.

Sol. Taking Z-transform of both sides, we get

$$Z\{y_k\} - 3Z\{y_{k-1}\} + 2 \, . \, Z\{y_{k-2}\} = Z\{1\}$$

$$Y(z) - 3 \cdot \left[z^{-1} Y(z) + y_{-1} z^0\right] + 2\left[z^{-2} Y(z) + y_{-1} z^{-1} + y_{-2} z^0\right] = \frac{z}{z-1}$$

$$Y(z) - 3 \, . \left[z^{-1} Y(z) + 2\right] + 2\left[z^{-2} Y(z) + 2z^{-1} + 2\right] = \frac{z}{z-1}.$$

$$(1 - 3z^{-1} + 2z^{-2}) \, Y(z) = \frac{z}{z-1} + 2 - \frac{4}{z}$$

$$\left(\frac{z^2 - 3z + 2}{z^2}\right) Y(z) = \frac{z^2 + 2z^2 - 2z - 4z + 4}{z(z-1)}$$

$$Y(z) = \frac{(3z^2 - 6z + 4)\, z}{(z-1)(z^2 - 3z + 2)} = \frac{z(3z^2 - 6z + 4)}{(z-1)(z-1)(z-2)}$$

$$Y(z) = \frac{z(3z^2 - 6z + 4)}{(z-1)^2 (z-2)}$$

[**Note :** $Z\{y_{k-1}\} = z^{-2} Y(z) + y_{-1} z^{-1} + y_{-2} z^0$]

EXERCISE 4.4

Solve the following difference equations :

1. $f(k+2) + 3f(k+1) + 2f(k) = 0$, $f(0) = 0$, $f(1) = 1$

(Dec. 2005; May 2007, 2009, 2015, 2016, Nov. 2014, Dec. 2018)

Hint : $F(z) = \dfrac{z}{z^2 + 3z + 2} = \dfrac{z}{z+1} - \dfrac{z}{z+2}$ **Ans. :** $f(k) = (-1)^k - (-2)^k$, $k \geq 0$, $|z| > 2$

2. Obtain f(k), given that $f(k+1) + \dfrac{1}{4} f(k) = \left(\dfrac{1}{4}\right)^k$, $k \geq 0$, $f(0) = 0$.

Hint : $F(z) = \dfrac{z}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{4}\right)} = \dfrac{\frac{1}{2}}{z - \frac{1}{4}} - \dfrac{\frac{1}{2}}{z + \frac{1}{4}}$ **Ans. :** $\dfrac{1}{2}\left[\left(\dfrac{1}{4}\right)^k - \left(-\dfrac{1}{4}\right)^k\right]$, $k \geq 0$

3. $f(k+2) - 3f(k+1) + 2f(k) = U(k)$, where $f(k) = 0$ for $k \leq 0$ i.e. $k \geq 1$.

$U(k) = 1$ for $k = 0$ and $U(k) = 0$ for $k < 0$ and $k > 0$. **Ans. :** $f(k) = 2^{k-1} - 1$, $k > 0$, $|z| > 2$

4. $u_{n+2} + u_{n+1} + u_n = 0$, $u_0 = 1$, $u_1 = 1$ **Ans. :** $u_n = \cos \dfrac{2n\pi}{3} + \sqrt{3} \sin \dfrac{2n\pi}{3}$, $n \geq 0$

5. $4f(k) + f(k-2) = 4\left(\dfrac{1}{2}\right)^k \sin \dfrac{k\pi}{2}$, $k \geq 0$ **Ans. :** $f(k) = (k+1)\left(\dfrac{1}{2}\right)^k \sin \dfrac{k\pi}{2}$, $k \geq 0$, $|z| > \dfrac{1}{2}$

4.13 RELATIONSHIP OF Z-TRANSFORM WITH FOURIER TRANSFORM

We have already defined the Z-transform of sequence {f (k)} as

$$Z[\{f(k)\}] = F(z) = \sum_{k=-\infty}^{\infty} f(k) \, z^{-k}$$

Then $F(e^{i\theta})$ is known as the discrete time Fourier transform of the sequence {f (k)}.

$$\therefore \quad F\left(e^{i\theta}\right) = \sum_{k=-\infty}^{\infty} f(k) \, e^{-ik\theta}$$

$$\therefore \quad F\left(e^{i\theta}\right) e^{in\theta} = \sum_{k=-\infty}^{\infty} f(k)\, e^{i(n-k)\theta}$$

$$\int_{-\pi}^{\pi} F\left(e^{i\theta}\right) e^{in\theta} d\theta = \int_{-\pi}^{\pi} \left[\sum_{k=-\infty}^{\infty} f(k)\, e^{i(n-k)\theta} d\theta\right]$$

$$= \sum_{k=-\infty}^{\infty} \left[\int_{-\pi}^{\pi} f(k)\, e^{i(n-k)\theta} d\theta\right]$$

$$= f(n) \,.\, 2\pi$$

For $n \neq k$,
$$\int_{-\pi}^{\pi} f(k)\, e^{i(n-k)\theta} d\theta = 0$$

$$\Rightarrow \quad \boxed{\mathbf{f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left(e^{i\theta}\right) e^{in\theta}\, d\theta}}$$

ILLUSTRATION

Ex. 1 : *Find the sequence f if $F\left(e^{i\theta}\right) = \cos 3\theta$*

Sol. :
$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left(e^{i\theta}\right) e^{in\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos 3\theta \; e^{in\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{3i\theta} + e^{-3i\theta}}{2}\right) e^{in\theta}\, d\theta$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[e^{i(n+3)\theta} + e^{i(n-3)\theta}\right] d\theta$$

$$= 0 \quad \text{if} \quad n \neq 3, -3 = \frac{1}{4\pi}(2\pi) \quad \text{if} \quad n = 3, -3$$

$$f(n) = \frac{1}{2} \quad \text{if} \quad n = 3, -3$$

$$= 0, \quad \text{otherwise.}$$

i.e.
$$f = \left\{ \ldots\ldots \frac{1}{2},\ 0,\ 0,\ \underset{k=0}{0},\ 0,\ 0,\ \frac{1}{2},\ \ldots\ldots \right\}$$

Here $f(-3) = \frac{1}{2}$, $f(-2) = f(-1) = f(0) = f(1) = f(2) = 0$, $f(3) = \frac{1}{2}$ and so on.

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UNIT III : STATISTICS

CHAPTER - 5
STATISTICS, CORRELATION AND REGRESSION

5.1 INTRODUCTION

In recent decades, the growth of statistics has made its place in almost every major phase of human activity, particularly so in the field of Engineering and Science. Everything dealing with the collection, presentation, processing, analysis and interpretation of numerical data belongs to the field of statistics. Collection and processing of data is usually referred to as statistical survey. Before any major project work is undertaken, the statistical survey is a must. Only when statistical survey gives green signal, actual start of the work is undertaken.

For example, if a Dam is to be constructed on a river, many aspects have to be taken into account. Foremost is the selection of dam site. For making a proper choice, it may be necessary to consider average rainfull in the catchment area for the past say 100 years, the extent of the area which may be submerged, the population which is going to be benefitted, the availability of labour and many other aspects. Good statistical survey should be able to answer all these questions. All such considerations and statistical survey have to be made whenever a new industry is to be started. The success of such project depends to a great extent upon sound statistical survey. Apart from these basic considerations, modern statistical techniques are widely used in the fields of statistical work, Quality control, reliability needs of the highly complex products of space technology and operation research.

Aim of this work is to introduce to the readers, the simple aspects of collection, classification and enumeration of numerical data, which are very essential for development of modern statistical techniques, used in engineering fields.

5.2 COLLECTION AND CLASSIFICATION OF DATA

The data according to the method of collection are of two types viz. (a) Primary data, (b) Secondary data. ***Primary data*** also, called ***raw data*** may be a result of a survey or investigation through questionaries and data taken from sources which are already collected by some other agency viz. media reports, office record, bulletins, magazine, website etc. is called ***secondary data***.

Data collected in a statistical survey as a result of some kind of experimentation is usually large in size and is in the form which is not very useful for arriving at any specific conclusions. The first task is to present this data in a proper form. As a first step, this data which is generally in the form of numerical observations, is arranged either in the ascending or descending order. For example, the set of observations 45, 35, 0, 10, 0, 51, 81, 71, 95, 17, 97, 21, 26, 86, 100, 55, 46, 56, 37, 92 (which are in all 20) is rearranged in ascending order as 0, 0, 10, 17, 21, 26, 35, 37, 45, 51, 55, 56, 71, 81, 86, 92, 95, 97, 100.

This way of presentation immediately reveals that the minimum value of the observation is 0 and maximum is 100. It also indicates that observations are well spread out in the interval (0, 100). In different experiments, these observations could carry different meanings. In some experiments, these figures may indicate the number of syntax errors committed by a group of 20 students in their first attempt to write a computer program. In yet another experiment, these figures may indicate marks obtained out of 100 by a group of 20 students in the paper of numerical computational methods. In an altogether different context, these figures may indicate Rainfall in centimeters in a certain catchment area for the past 20 year. For development of statistical techniques it is unimportant, what is exactly represented by these observations. In presentation of data, these observations are represented by symbol x, called in statistical language, a **variable (variety)**.

After arranging the data in ascending or descending order, to make it more compact (or further classified), it is presented in a tabular form consisting of columns headed by symbols x and f. The column headed by x consists of various observations recorded out of experimentation, arranged in proper order, and column headed by f contains entries which indicate number of times particular value of x occur.

Consider the Table 5.1, which shows various values of x and f. It shows that the value of x = 1 is recorded twice, x = 4 occurs six times, x = 8 occurs four times, etc.

The total numbers of observations being $\sum f = 45$. In statistical language, this table means x = 1 has frequency 2, x = 4 has frequency 6 and so on. This way of arrangement of data is called ***frequency distribution***. In the above example, the range of variety is from x = 1 to x = 10. When the range is wide and the total number of observations is very large, the data can be expressed in still more compact form by dividing the range in class intervals.

Table 5.1

x	**f**
1	2
2	3
3	5
4	6
5	10
6	6
7	4
8	4
9	3
10	2
–	$\sum f = 45$

Next, consider the table 5.2. Here the range of variety (0, 100) is divided into 10 class intervals each of width 10. The class interval 0 – 10 has width 10, the lower limit 0 and the upper limit 10. Here $\frac{10 + 0}{2} = 5$ is the middle value of the class interval and 16 is the frequency corresponding to this class interval. The middle value x = 5 represents the class interval (0 – 10) of f = 16 is taken as frequency of variety x. This way of representing the data is called ***Grouped frequency distribution***. In such type of presentation, the class intervals must be well defined. One such way of defining the class interval is that, all the values of x = 0 and above but less than 10 are included in the class interval 0 – 10. The total frequency of all such observations is 16 and is the frequency of class interval 0 – 10 or is the frequency of variable (variety) x = 5.

Table 5.2

C.I. (Class interval)	**Mid-value x**	**Frequency f**
0 – 10	5	16
10 – 20	15	18
20 – 30	25	20
30 – 40	35	22
40 – 50	45	40
50 – 60	55	45
60 – 70	65	35
70 – 80	75	20
80 – 90	85	19
90 – 100	95	15
Total	**–**	$\sum f = 250$

Similarly, all the observations having the value x = 10 and above but less than 20 are included in the class interval 10 – 20 and so on. Slight change in the definition of last class interval is made. Here all the values of x = 90 and above and less than or equal to 100 are included in the class interval 90 – 100. $\sum f = 250$ gives the total frequency which is sometimes denoted by N.

In presenting the data in Grouped frequency distribution form, the following points must be noted :

(i) The class interval must be well defined that is there must not be any ambiguity about the inclusion of value of *x* in one or the other class interval. In the Table 5.2, the way of defining class interval enables us to put x = 10 in the class interval 10 – 20 while x = 100 is put in the interval 90 – 100.

(ii) The class intervals must be exhaustive that is no observation should escape classification. For this, the entire range of observations should be divided into well defined class intervals.

(iii) The width of the class interval should be uniform as far as possible.

(iv) The number of class intervals should neither be too large nor too small. Depending upon the range of variate x and the total frequency of observations, the total number of class intervals is divided into about 10 to 25 class intervals.

Sometimes the additional column of *cumulative frequency (c.f.)* supplements the grouped frequency distribution or frequency distribution table.

In the Table 5.3, the number 76 against x = 35 shows the total frequency upto and including the observation x = 35 which is the middle value of the interval (30 – 40).

Table 5.3

C.I.	Mid-value (x)	Frequency (f)	Cumulative Frequency (c. f.)
0 – 10	5	16	16
10 – 20	15	18	34
20 – 30	25	20	54
30 – 40	35	22	76
40 – 50	45	40	116
50 – 60	55	45	161
60 – 70	65	35	196
70 – 80	75	20	216
80 – 90	85	19	235
90 – 100	95	15	250
Total	**–**	**$\sum f$ = 250**	**N = 250**

This type of cumulative frequencies also called ***less than cumulative frequency***. If we reverse the process, that is computing cumulative sum of frequencies from highest class to lowest class, than this type of cumulative frequencies is called ***more than cumulative frequency***.

5.3 GRAPHICAL REPRESENTATION OF DATA

To observe the data at a glance, it is exhibited by following graphical methods :

1. Histogram : A Histogram is drawn by constructing rectangles over the class intervals, such that the areas of rectangles are proportional to the class frequencies.

If the class intervals are of equal width, the heights of the rectangles will be proportional to the class frequencies themselves, otherwise these would be proportional to the ratios of the frequencies to the width of the classes (See Fig. 5.1).

2. Frequency Polygon : Consider the set of points (x, f), where x is the middle value of the class interval and f is the corresponding frequency. If these set of points are joined by straight lines, they form a frequency polygon. It is shown by dotted lines in Fig. 5.1.

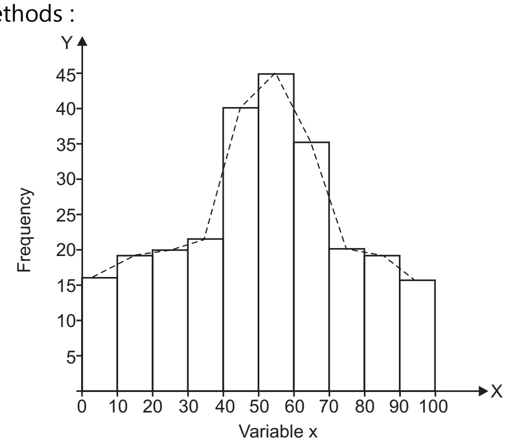


Fig. 5.1

3. Cumulative Frequency Curve or The Ogive : Taking upper limit of classes of x co-ordinate and corresponding cumulative frequency as y co-ordinate, if the points are plotted and then joined by free hand curve, it gives what is called as ogive (See Fig. 5.2).

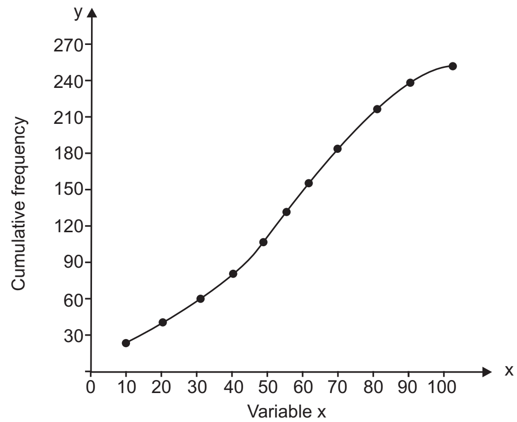


Fig. 5.2

5.4 LOCATION OF CENTRAL TENDENCY

After collecting the data and arranging it in the proper order in the form of frequency distribution or grouped frequency distribution, next task is to study this data carefully and to draw valid conclusions. If data collected relates to marks obtained by the students in Mathematics paper, it should be able to reveal the general performance of the students. Whether the class contains large number of good students or the overall calibre of students is medicore, all this must be inferred from the data. If the numerical data collected relates to the industrial project, the whole success of the project will depend upon the appropriate conclusions drawn from the study of this data. The first step in this direction is the location of central tendency. It means what is represented by data by and large. Whether the data is favourable to a particular project or not will depend upon the criterion that is decided upon. But overall picture must be exhibited. This overall picture or central tendency of the data is known by obtaining what we call the ***Mean or Average***. There are various methods to calculate the mean or the average. Depending upon the project under study, the particular method is selected. Various types of measures of central tendency are as given below :

(1) Arithmetic mean (2) Geometric mean (3) Harmonic mean

(4) Median (5) Mode.

Among the above stated, arithmetic mean, geometric mean and harmonic mean are called as mathematical averages and median and mode are called positional averages.

Out of these, Arithmetic mean is of greater importance and serves the purpose in many cases. Now, we see how these measures are calculated.

5.4.1 Arithmetic Mean

Consider the variate x which takes *n* values x_1, x_2, x_3 ...... x_n, (set of n observations) then the Arithmetic mean (A.M.) is a sum of the observations divided by number of observations, denoted by $\bar{x}$ and is given by,

$$\text{A.M.} = \bar{x} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n} = \frac{\sum x}{n}$$

If the data is presented in the form of frequency distribution

x	x_1	x_2	x_3	......	x_n
f	f_1	f_2	f_3	......	f_n

then the sum of observations is $f_1 x_1 + f_2 x_2 + \ldots + f_n x_n$ and arithmetic mean $\bar{x}$ is given by

$$\text{A.M.} = \bar{x} = \frac{f_1 x_1 + f_2 x_2 + \ldots + f_n x_n}{f_1 + f_2 + f_3 + \ldots + f_n} = \frac{\sum fx}{\sum f} = \frac{\sum fx}{N}$$

where, $N = \sum f = f_1 + f_2 + \ldots\ldots + f_n$ is the total frequency.

ILLUSTRATIONS

Ex. 1 : *Find the Arithmetic mean for the following distribution :*

x	0	1	2	3	4	5	6	7	8	9	10
f	4	5	12	12	13	16	15	13	12	5	6

Sol. : Writing the tabulated values as :

x	f	x × f
0	4	0
1	5	5
2	12	24
3	12	36
4	13	52
5	16	80
6	15	90
7	13	91
8	12	96
9	5	45
10	6	60
Total	**$\sum f = 113$**	**$\sum fx = 579$**

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{579}{113} = 5.12 \text{ (approximately)}$$

To reduce the calculations, we consider the variable d = x – A. Where, A is middle value or value near to it in the range of variable x, A is sometimes called assumed mean.

Now we can write

$$f \times d = f \times x - f \times A$$

or $$\sum fd = \sum fx - \sum fA$$

Dividing by $\sum f$ throughout

$$\frac{\sum fd}{\sum f} = \frac{\sum fx}{\sum f} - \frac{\sum fA}{\sum f}$$

i.e. $$\frac{\sum fd}{\sum f} = \bar{x} - A\frac{\sum f}{\sum f}$$

(A being constant is taken outside the $\sum$ notation)

$$\frac{\sum fd}{\sum f} = \bar{x} - A$$

or $$\bar{x} = A + \frac{\sum fd}{\sum f} = A + \bar{d}$$ [$\bar{d}$ is the mean of the variable d]

fd and $\sum$ fd are smaller numbers as compared to fx and $\sum$ fx, which result in the reduction of the calculations.

Further, for reduction in calculations for grouped frequencies distribution can be achieved by taking

$$u = \frac{x - A}{h} \quad \text{or} \quad u = \frac{d}{h}$$

that gives $$hu = x - A$$

Then proceeding as before, we get

$$h\frac{\sum fu}{\sum f} = \bar{x} - A$$

or $$\bar{x} = A + h\frac{\sum fu}{\sum f}$$

This formula is mostly used in grouped frequency distribution, where, h is chosen to be equal to the width of the class interval.

Ex. 2 : *Calculate arithmetic mean for the following frequency distribution :*

Observations (x)	*103*	*110*	*112*	*118*	*95*
Frequency (f)	*4*	*6*	*10*	*12*	*3*

Sol. : We solve the problem by direct methods.

x	f	fx
103	4	$103 \times 4 = 412$
110	6	$110 \times 6 = 660$
112	10	$112 \times 10 = 1120$
118	12	$118 \times 12 = 1416$
95	3	$95 \times 3 = 285$
Total	**N = 35**	**$\sum f x = 3893$**

$$\therefore \qquad \bar{x} = \frac{\sum fx}{\sum f} = \frac{3893}{35} = 111.2286$$

Ex. 3 : *Marks obtained in a paper of statistics are given in the following table.*

Marks Obtained	***No. of Students***
0 – 10	*8*
10 – 20	*20*
20 – 30	*14*
30 – 40	*16*
40 – 50	*20*
50 – 60	*25*
60 – 70	*13*
70 – 80	*10*
80 – 90	*5*
90 – 100	*2*

Find the Arithmetic mean of the distribution.

Sol. : Preparing the table as : A = 45, h = 10.

C.I.	**Mid-value x**	**f**	**$u = \frac{x-45}{10}$**	**$f \times u$**
0 – 10	5	8	– 4	– 32
10 – 20	15	20	– 3	– 60
20 – 30	25	14	– 2	– 28
30 – 40	35	16	– 1	– 16
40 – 50	45	20	0	0
50 – 60	55	25	1	25
60 – 70	65	13	2	26
70 – 80	75	10	3	30
80 – 90	85	5	4	20
90 – 100	95	2	5	10
Total	**–**	**$\sum f = 133$**	**–**	**$\sum fu = -25$**

$$\bar{x} = A + h \frac{\sum fu}{\sum f} = 45 + 10 \left(\frac{-25}{133}\right)$$

$$= 45 + 10 \left(\frac{-25}{133}\right) = 45 - \frac{250}{133} = 43.12$$

Combined Arithmetic Mean (Mean of composite series)

Consider two sets of data

1. $x_1, x_2, \ldots, x_{n_1}$ containing n_1 items
2. $y_1, y_2, \ldots, y_{n_2}$ containing n_2 items

$\therefore$ $\bar{x}$, the mean of first set is given by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n_1}$$

$$\therefore \quad n_1 \bar{x} = x_1 + x_2 + \ldots + x_{n_1}$$

and the mean of second set is given by

$$\bar{y} = \frac{y_1 + y_2 + y_3 + \ldots + y_{n_2}}{n_2}$$

$$\therefore \quad n_2 \bar{y} = y_1 + y_2 + \ldots + y_{n_2}$$

Hence, by definition the joint arithmetic mean $\bar{z}$ is given by

$$\bar{z} = \frac{(x_1 + x_2 + x_3 + \ldots + x_{n_1}) + (y_1 + y_2 + y_3 + \ldots + y_{n_2})}{n_1 + n_2}$$

$$\therefore \quad \bar{z} = \frac{n_1 \bar{x} + n_2 \bar{y}}{n_1 + n_2} \quad \ldots (A)$$

The above result can be generalized to k ($k \geq 2$) groups.

(A) gives combined Arithmetic Mean (A.M.) of the composite series.

Same type of formula holds good for sets of data presented in frequency distribution form. Consider two sets of data :

Set 1	
x	**f**
x_1	f_1
x_2	f_2
x_3	f_3
…	…
…	…
x_{n_1}	f_{n_1}
	$\boldsymbol{\sum f = N_1}$

Set 2	
y	**f**
y_1	f_1
y_2	f_2
y_3	f_3
…	…
…	…
y_{n_1}	f_{n_1}
	$\boldsymbol{\sum f = N_2}$

Means $\bar{x}$, $\bar{y}$ for two sets are given by

$$\bar{x} = \frac{\sum fx}{N_1} \qquad \therefore \quad N_1 \bar{x} = \sum fx$$

$$\bar{y} = \frac{\sum fy}{N_2} \qquad \therefore \quad N_2 \bar{y} = \sum fy$$

Hence, $\bar{z}$ the combined mean is given by

$$\bar{z} = \frac{\sum fx + \sum fy}{N_1 + N_2} = \frac{N_1 \bar{x} + N_2 \bar{y}}{N_1 + N_2} \quad \ldots (A)$$

Ex. 4 : *Marks obtained in paper of Applied Mechanics by a group of Computer and Electronics students are as given in following tables :*

Group (A) of Computer students :

Marks Obtained	***No. of Students***
0 – 10	*5*
10 – 20	*6*
20 – 30	*15*
30 – 40	*15*
40 – 50	*9*
	$\Sigma f = 50$

Group (B) of Electronics students :

Marks Obtained	***No. of Students***
0 – 10	*8*
10 – 20	*15*
20 – 30	*18*
30 – 40	*13*
40 – 50	*6*
	$\Sigma f = 60$

Find the Combined mean of the two groups.

Sol. : For group (A) :

C.I.	**Mid-value x**	**f**	**f × x**
0 – 10	5	5	25
10 – 20	15	6	90
20 – 30	25	15	375
30 – 40	35	15	525
40 – 50	45	9	405
Total	–	$N_1 = \sum f = 50$	$\sum fx = 1420$

For group (B) :

C.I.	**Mid-value x**	**f**	**f × x**
0 – 10	5	8	40
10 – 20	15	15	225
20 – 30	25	18	450
30 – 40	35	13	455
40 – 50	45	6	270
Total	–	$N_2 = \sum f = 60$	$\sum fy = 1440$

Mean $\bar{x}$ of group (A) is given by,

$$\bar{x} = \frac{\sum fx}{\sum f} \Rightarrow \bar{x} \sum f = N_1 \bar{x} = \sum fx = 1420$$

Mean $\bar{y}$ of group (B) is given by,

$$\bar{y} = \frac{\sum fy}{\sum f} \Rightarrow \bar{y} \sum f = N_2 \bar{y} = \sum fy = 1440$$

Combined mean $\bar{z}$ is given by,

$$\bar{z} = \frac{N_1 \bar{x} + N_2 \bar{y}}{N_1 + N_2} = \frac{1420 + 1440}{50 + 60}$$

$$= \frac{2860}{110} = 26$$

Ex. 5 : *Arithmetic mean of weight of 100 boys is 50 kg and the arithmetic mean of 50 girls is 45 kg. Calculate the arithmetic mean of combined group of boys and girls.*

Sol. : Let $\bar{X}_1$ and N_1 be the mean and size of group of boys and $\bar{Y}$ and N_2 be the mean and size of group of girls. So $N_1 = 100$, $\bar{X} = 50$, $N_2 = 50$, $\bar{Y} = 45$. Hence, combined mean is

$$Z = \frac{N_1\bar{X} + N_2\bar{Y}}{N_1 + N_2} = \frac{(100 \times 50) + (50 \times 45)}{100 + 50} = \frac{7250}{150} = 48.3333$$

Ex. 6 : *The mean weekly salary paid to 300 employees of a firm is ₹1,470. There are 200 male employees and the remaining are females. If mean salary of males is ₹1,505, obtain the mean salary of females.*

Sol. : Suppose $\bar{X}$ and N_1 are mean and group size of males. $\bar{Y}$ and N_2 are mean and size of group of females, $\bar{x}_C$ mean of all the employees considered together.

Now, $$Z = \frac{N_1\bar{X} + N_2\bar{Y}}{N_1 + N_2}$$

$$\therefore \quad 1470 = \frac{(200 \times 1505) + (100 \times \bar{Y})}{200 + 100}$$

$$\therefore \quad 1470 = \frac{301000 + 100\bar{Y}}{300}$$

$$\therefore \quad 441000 = 301000 + 100Y$$

$$\therefore \quad 4410 = 3010 + Y$$

$$\therefore \quad \bar{Y} = ₹\ 1{,}400$$

5.4.2 Geometric Mean

Geometric mean of a set of an observations $x_1, x_2, \ldots, x_n$ is given by n^{th} root of their product.

Thus Geometric Mean (G.M.) is given by,

$$\text{G.M.} = (x_1 \cdot x_2 \cdot x_3 \ldots x_n)^{1/n}$$

In case of frequency distribution

x	x_1	x_2	x_3	……… x_n
f	f_1	f_2	f_3	……… f_n

$$\text{G.M.} = G = \left(x_1^{f_1} \cdot x_2^{f_2} \cdot x_3^{f_3} \ldots x_n^{f_n}\right)^{1/N} \quad \text{where,} \quad N = \Sigma f$$

To calculate it, denoting G.M. by G and taking logarithms of both sides

$$\log G = \frac{1}{N}\left[\log \left(x_1^{f_1} \cdot x_2^{f_2} \cdot x_3^{f_3} \ldots x_n^{f_n}\right)^{1/N}\right]$$

$$= \frac{1}{N}\,[f_1 \log x_1 + f_2 \log x_2 \ldots + f_n \log x_n]$$

$$= \frac{1}{N}\,\Sigma f \log x$$

or $$G = \text{antilog}\left(\frac{1}{N}\,\Sigma f \log x\right)$$

It is seen that logarithm of G is the arithmetic mean of the logarithms of the given values. In case of grouped frequency distribution, x is taken as mid-value of the class interval.

For two sets of observations $(x_1, x_2, \ldots, x_{n_1})$, $(y_1, y_2, \ldots, y_{n_2})$ with geometric means G_1, G_2, it can be established that

$$\log G = \frac{n_1 \log G_1 + n_2 \log G_2}{n_1 + n_2}$$

where, G is the ***combined or common geometric mean*** of the two series.

It may be noted here that if one of the observations is zero, geometric mean becomes zero and if one of the observations is negative, geometric mean becomes imaginary. Naturally, calculation of geometric mean becomes meaningless in such cases.

5.4.3 Harmonic Mean

(H.M.) Harmonic mean of set of observations ($x_1, x_2, \ldots\ldots, x_n$) is the reciprocal of the arithmetic mean of the reciprocals of the given values. Thus, H.M. or H is given by,

$$\text{H.M.} = H = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}\right)}$$

In case of frequency distribution (x, f),

$$H = \frac{1}{\frac{1}{N}\Sigma\left(\frac{f}{x}\right)}$$

where, $N = \sum f$

5.4.4 Median

Median of a distribution is the value of the variable (or variate) which divides it into two equal parts. It is the value such that the number of observations above it is equal to the number of observations below it. Sometimes, Median is called positional average.

In case of ungrouped data, if the number of observations n is odd, then the median is the middle value which is $\left(\frac{n+1}{2}\right)^{th}$ observations of the set of observations after they are arranged in ascending or descending order. For even number of observations, it is the arithmetic mean of the two middle terms given by,

$$\frac{1}{2}\left[\text{The value of } \frac{n}{2}^{th} \text{ observation} + \text{The value of } \left(\frac{n}{2}+1\right)^{th} \text{ observations}\right]$$

For a data presented in the form of frequency distribution :

x	x_1	x_2	x_3	……… x_n
f	f_1	f_2	f_3	……… f_n

where, $\Sigma f = N$

We prepare the cumulative frequency column. Then consider cumulative frequency (c.f.) equal to $\frac{N}{2}$ or just greater than $\frac{N}{2}$, the corresponding value of x is the median.

ILLUSTRATIONS

Ex. 7 : For the ordered arrangement of n = 8 observations x = 1, 5, 9, $\boxed{11, 21}$, 24, 27, 30, the middle terms are 11 and 21 and median $= \frac{11+21}{2}$ = 16 and for the ordered arrangement of n = 7 observations x = 35, 35. 36, $\boxed{37}$, 38, 39, 40, the middle term is 37 and Median = 37 (4^{th} observation).

Ex. 8 : *Obtain the median of the distribution :*

x	1	3	5	7	9	11	13	15	17
f	3	6	8	12	16	16	15	10	5

Sol. : Preparing the table as :

x	f	c.f.
1	3	3
3	6	9
5	8	17
7	12	29
9	16	45
11	16	61
13	15	76
15	10	86
17	5	91
Total	$\sum f = 91$	–

Here the total frequency N = 91; $\frac{N}{2} = 45.5$.

The value of c.f. just greater than 45.5 is 61, the corresponding value of x is 11 and thus, median is 11.

In case of grouped frequency distribution, the class corresponding to the c.f. just greater than $\frac{N}{2}$ is called the median class and the value of median is obtained by the formula :

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where, l is the lower limit of the median class.

f is the frequency of the median class.

h is the width of the median class.

c is the c.f. of the class preceding the median class.

Ex. 9 : *Wages earned in Rupees per day by the labourers are given by the table :*

Wages in ₹	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
No. of Labourers	5	8	13	10	8

Find the median of the distribution.

Sol. :

Wages in ₹ C.I.	No. of Labourers f	(c.f.)
10 – 20	5	5
20 – 30	8	13
30 – 40	13	26
40 – 50	10	36
50 – 60	8	44
Total	$\sum f = N = 44$	–

Here $\frac{N}{2} = \frac{44}{2} = 22$

Cumulative frequency (c.f.) just greater than 22 is 26 and the corresponding class is 30 – 40.

Using formula to calculate median, with

$$l = 30,\ f = 13,\ h = 10,\ \frac{N}{2} = 22,\ c = 13$$

$$\text{Median} = 30 + \frac{10}{13}(22 - 13)$$

$$= 30 + \frac{10}{13}(9) = 30 + \frac{90}{13} = 36.923$$

5.4.5 Mode

It is the value of the variate which occurs most frequently in a set of observations, or is the value of variate corresponding to maximum frequency.

We note that general nature of frequency curve is bell shaped in majority of situations. Thus, initially frequency is small, it increases and reaches the maximum and then it declines. The value on x-axis at which the maxima or peak of the frequency curve appear is a ***mode***.

For the mode of the data 35, 38, 40, 39, 35, 36, 37, it can be clearly seen the observation 35 has maximum frequency, hence it is mode.

For the mode of the following frequency distirubution

x	10	11	12	13	14	15
f	2	5	10	21	12	13

Since maximum frequency is associated with observation 13, the mode is 13.

In case of grouped frequency distribution, Mode is given by the formula :

$$\text{Mode} = l + h \times \frac{(f_1 - f_0)}{(f_1 - f_0) - (f_2 - f_1)}$$

$$= l + h \times \frac{(f_1 - f_0)}{2f_1 - f_0 - f_2}$$

Here, l is the lower limit of the modal class

h is the width of the modal class

f_1 is the frequency of the modal class

f_0 is the frequency of the class preceding to the modal class

f_2 is the frequency of the class succeeding to the modal class.

ILLUSTRATION

Ex. 10 : *Find the Mode for the following distribution :*

C.I.	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
f	4	7	8	12	25	18	10

Sol. : Here C.I. 40 – 50 corresponding to which f = 25 is maximum, is the modal class.

$$l = 40,\ h = 10,\ f_1 = 25,\ f_0 = 12,\ f_2 = 18$$

$$\text{Mode} = 40 + \frac{10(25 - 12)}{(2 \times 25 - 12 - 18)}$$

$$= 40 + \frac{130}{20} = 40 + 6.5 = 46.5$$

So far we have considered various ways in which average can be calculated. It is clear that no single average is suitable for all types of data. Arithmetic mean, Geometric mean and Harmonic mean are rigidly defined and are based on all the observations, they are suitable for further mathematical treatment. They are not much affected by fluctuations of sampling. In fact among all the averages, Arithmetic mean is least affected by fluctuations. Geometric mean becomes zero if any one of the observations is zero. Geometric and Harmonic means are not easy to understand and are difficult to compute. They give greater importance to small items and are useful when small items have to be given a very high weightage. Median and Mode are not amenable to algebraic treatment. Their main advantage is that they are not affected by extreme values, but compared to Arithmetic mean they are affected much by fluctuations of sampling. All the averages have merits and demerits, but Arithmetic mean because of its simplicity and its stability is much more familiar to a lyman. It has wide applications in statistical theory and is considered as best among all the averages.

5.5 MEASURES OF DISPERSION

After calculation of the average using any of the five methods discussed in previous section, question arises whether the average calculated gives correct information about the central tendency of the data, the purpose for which it is calculated. Main point to be discussed is whether the average is true representative of the data or not. As an illustration, consider the two sets of observations.

(i) 5, 10, 15, 20, 25.

(ii) 13, 14, 15, 16, 17.

The Arithmetic mean of both these sets is 15. It is obvious that 15 is better average for second than the first, because the observations in the second set are much closer to the value 15 as compared to the first set. In the second set, the values of the variate are much less scattered or dispersed from the mean as compared to the first. We note that average remains good representative if dispersion is less (i.e. observations are closed to it). Thus, dispersion decides the reliability of average.

There are two widely accepted ways of measuring the degree of scatteredness from the mean. These are :

(i) Mean deviation

(ii) Standard deviation.

These are the measures of dispersion, which decide whether the average truely represents the given data or not. Besides these two standard measures, there are other measures of dispersion such as ***range*** and ***quartile deviation*** or ***semi-interquartile range***. But these are not as much of consequence as it uses only two extreme items or depends upon only two portion values.

We shall now discuss about the two measures of dispersion mentioned earlier.

(i) Mean Deviation : It is defined as the arithmetic mean of absolute deviations from any average is called as mean deviation about the respective average. For the variate x which takes n values $x_1, x_2, \ldots x_n$, mean deviation from the average A (usually, Arithmetic mean $\bar{x}$ or at most median or mode) is given by

$$\text{Mean deviation} = \frac{1}{n} \sum |x - A|$$

For a frequency distribution (x, f) i.e. variate which takes n values $x_1, x_2, \ldots, x_n$ and corresponding frequencies $f_1, f_2, \ldots, f_n$.

$$\text{Mean deviation} = \frac{1}{N} \sum f\ |x - A|$$

where, $N = \sum f$ is the total frequency, $|x - A|$ represents the modulus or the absolute value of the deviation $(x - A)$ ignoring the – ve sign. It can be broadly stated that when deviation is a small number, the average is good.

ILLUSTRATION

Ex. 1 : *Calculate Arithmetic mean and Mean deviation of the following frequency distribution :*

x	*1*	*2*	*3*	*4*	*5*	*6*
f	*3*	*4*	*8*	*6*	*4*	*2*

Sol. : Preparing the table :

x	**f**	**x × f**	**x – A**	**|x – A|**	**f × |x – A|**
1	3	3	– 2.37	2.37	7.11
2	4	8	– 1.37	1.37	5.48
3	8	24	– 0.37	0.37	2.96
4	6	24	0.63	0.63	3.78
5	4	20	1.63	1.63	6.52
6	2	12	2.63	2.63	5.26
Total	**∑f = 27**	**∑ f x = 91**	**–**	**–**	**∑f × |x – A| = 31.11**

$$\text{A.M.} = A = \bar{x} = \frac{\sum fx}{\sum f} = \frac{91}{27} = 3.37 \quad \text{(approximately)}$$

$$\text{Mean deviation} = \frac{\sum f \times |x - A|}{\sum f} = \frac{31.11}{27} = 1.152 \quad \text{(approximately).}$$

(ii) Standard Deviation : ***Standard deviation*** *is defined as the positive square root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean.* It is denoted by the symbol σ.

For the variate x which takes n values $x_1, x_2, \ldots x_n$,

$$\text{S.D.} = \sigma = \sqrt{\frac{1}{n}\Sigma (x - \bar{x})^2}$$

For a frequency distribution (x, f), i.e. for variate x which takes n values $x_1, x_2, \ldots x_n$ and corresponding frequencies $f_1, f_2, \ldots, f_n$.

$$\text{S.D.} = \sigma = \sqrt{\frac{1}{N}\Sigma f \left(x - \bar{x}\right)^2}$$

where, $\bar{x}$ is A.M. of the distribution and $N = \Sigma f$.

(iii) Variances : The square of the standard deviation is called ***variance***, denoted by Var (x).

For the variate x which takes the values $x_1, x_2, \ldots, x_n$,

$$\text{Variance} = \text{Var}(x) = \sigma^2 = \frac{1}{n}\Sigma \left(x - \bar{x}\right)^2$$

For a frequency distribution (x, f), i.e. for variate x which taken n values $x_1, x_2, \ldots, x_n$ and corresponding frequencies $f_1, f_2, \ldots, f_n$,

$$\text{Variance} = \text{Var}(x) = \sigma^2 = \frac{1}{N}\Sigma f \left(x - \bar{x}\right)^2$$

The step of squaring the deviations $\left(x - \bar{x}\right)$ overcomes the drawback of ignoring the signs in Mean deviation. Standard deviation is also suitable for further mathematical treatment. Moreover among all the measures of dispersion, standard deviation is affected by least fluctuations of sampling, hence it is considered as most reliable measure of dispersion.

(iv) Root mean square deviation is given by

$$\text{R.M.S.} = S = \sqrt{\frac{1}{N}\Sigma f\ (x - A)^2}$$

where, A is any arbitrary number (reference number).

Also, square of the root mean square deviation is called **Mean square deviation** denoted by S^2.

$$\text{M.S.D.} = S^2 = \frac{1}{N}\Sigma f(x - A)^2$$

When $A = \bar{x}$, the Arithmetic mean, Root mean square deviation becomes equal to the standard deviation.

(v) For computation purpose the above formulae can be simplified as follows :

Method of Calculating σ :

For variate x which takes n values $x_1, x_2, \ldots, x_n$

$$\sigma^2 = \frac{1}{n}\Sigma (x - \bar{x})^2$$

$$= \frac{1}{n}\Sigma (x^2 - 2x\bar{x} + \bar{x}^2)$$

$$= \frac{1}{n}\Sigma x^2 - 2\bar{x}^2 + \bar{x}^2$$

$$\therefore \quad \boxed{\sigma^2 = \frac{1}{n}\Sigma x^2 - \left(\frac{1}{n}\Sigma x\right)^2} \quad \ldots (1)$$

For frequency distribution (x, f),

$$\sigma^2 = \frac{1}{N}\Sigma f \left(x - \bar{x}\right)^2$$

$$= \frac{1}{N}\Sigma f \left(x^2 - 2x\bar{x} + \bar{x}^2\right)$$

$$= \frac{1}{N}\Sigma f x^2 - \frac{2\bar{x}}{N}\Sigma fx + \bar{x}^2 \cdot \frac{\Sigma f}{N}$$

$$= \frac{1}{N}\Sigma f x^2 - 2\bar{x}^2 + \bar{x}^2 \qquad \left[\because \frac{\Sigma fx}{N} = \bar{x},\ \frac{\Sigma f}{N} = 1\right]$$

$$= \frac{1}{N} \sum f x^2 - \bar{x}^2$$

$$\boxed{\sigma^2 = \frac{1}{N} \sum f x^2 - \left(\frac{1}{N} \sum f x\right)^2} \quad \ldots (2)$$

Usually, product terms fx and fx^2 are large, hence to reduce the volume of calculations further, we proceed as follows :

$$\sigma^2 = \frac{1}{N} \sum f \left(x - \bar{x}\right)^2$$

$$= \frac{1}{N} \sum f \left(x - A + A - \bar{x}\right)^2, \quad \text{(where, A is arbitrary number)}$$

$$= \frac{1}{N} \sum f \left[(x - A)^2 + 2(x - A)\left(A - \bar{x}\right) + \left(A - \bar{x}\right)^2\right]$$

$$= \frac{1}{N} \sum f (x - A)^2 + \frac{2}{N} \sum \left(A - \bar{x}\right) \sum f (x - A) + \left(A - \bar{x}\right)^2 \frac{\sum f}{N}$$

Let $d = x - A$ then using $\bar{x} = A + \frac{1}{N} \sum fd$

$$\therefore \quad \sigma^2 = \frac{1}{N} \sum fd^2 + \frac{2}{N}\left[A - A - \frac{1}{N}\sum fd\right] \sum fd + \left[A - A - \frac{1}{N} \sum f d\right]^2 \cdot 1$$

$$= \frac{1}{N} \sum fd^2 - \frac{2}{N^2} (\sum fd)^2 + \frac{1}{N^2} (\sum fd)^2$$

$$= \frac{1}{N} \sum fd^2 - \frac{1}{N^2} (\sum fd)^2$$

$$\boxed{\sigma^2 = \frac{1}{N} \sum fd^2 - \left(\frac{\sum fd}{N}\right)^2} \quad \ldots (3)$$

or

$$\boxed{\sigma = \sqrt{\frac{1}{N} \sum f d^2 - \left(\frac{\sum fd}{N}\right)^2}} \quad \ldots (4)$$

Terms fd, fd^2 are numerically smaller as compared to fx, fx^2 and use of formula (4) reduces the calculations considerably in obtaining σ.

In dealing with data presented in grouped frequency distribution form and to reduce the calculations further, we put $u = \frac{x - A}{h}$, where h is generally taken as width of class interval.

Thus $u = \frac{d}{h}$ or $d = hu$ putting $d = hu$ in formula (3),

$$\sigma = \sqrt{\frac{1}{N} \sum f h^2 u^2 - \left(\frac{\sum f hu}{N}\right)^2}$$

$$\boxed{\sigma = h \sqrt{\frac{1}{N} \sum f u^2 - \left(\frac{\sum f u}{N}\right)^2}} \quad \ldots (5)$$

Formula (5) is quite useful for data presented in grouped frequency distribution form.

(vi) Relation Between σ and S : By definition, we have

$$S^2 = \frac{1}{N} \sum f (x - A)^2$$

$$= \frac{1}{N} \sum f \left(x - \bar{x} + \bar{x} - A\right)^2$$

$$= \frac{1}{N} \sum f \left[\left(x - \bar{x}\right)^2 + 2\left(x - \bar{x}\right)\left(\bar{x} - A\right) + \left(\bar{x} - A\right)^2\right]$$

$$= \frac{1}{N} \sum f \left(x - \bar{x}\right)^2 + 2\left(\bar{x} - A\right) \frac{1}{N} \sum f \left(x - \bar{x}\right) + \left(\bar{x} - A\right)^2 \frac{\sum f}{N}$$

Note that $(\bar{x} - A)$ being constant, is taken outside the summation.

Now since $\frac{1}{N} \Sigma f (x - \bar{x}) = \frac{1}{N} \Sigma fx - \bar{x} . \frac{1}{N} \Sigma f = \bar{x} - \bar{x} = 0$

$\therefore \quad S^2 = \frac{1}{N} \Sigma f (x - \bar{x})^2 + (\bar{x} - A)^2$, as $\sum f = N$

Thus, $\boxed{S^2 = \sigma^2 + d^2}$, where $d = \bar{x} - A$ ... (6)

If $\bar{x} = A$, thus S^2 would be least as d = 0. Thus, Mean square deviation (S^2) and consequently Root mean square (S) deviation are least when deviations are taken from $A = \bar{x}$.

(vii) Coefficient of Variation : The relative measure of standard deviation is called coefficient of variation and is given by

$$\boxed{\text{C.V.} = \frac{\sigma}{\text{A.M.}} \times 100} \quad \text{... (7)}$$

For comparing the variability of two series, we calculate coefficient of variation for each series.

ILLUSTRATION

Ex. 1 : *Runs scored in 10 matches of current IPL season by two batsmen A and B are tabulated as under*

Batsman A	*46*	*34*	*52*	*78*	*65*	*81*	*26*	*46*	*19*	*47*
Batsman B	*59*	*25*	*81*	*47*	*73*	*78*	*42*	*35*	*42*	*10*

Decide who is better batsman and who is more consistent.

Sol.: For Batsman A :

x	**d = x – 46**	**d^2**
46	0	0
34	–12	144
52	6	36
78	32	1024
65	19	361
81	35	1225
26	–20	400
46	0	0
19	–27	729
47	1	1
	Σ d = 34	**Σd^2 = 3920**

$$\bar{x}_A = 46 + \frac{\Sigma d}{10} = 46 + 3.4 = 49.4$$

$$\sigma_A = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2} = \sqrt{392 - 11.56} = 19.50 \quad (N = 10)$$

Coefficient of variation for A $= \frac{\sigma_A}{\text{A.M.}} \times 100 = \frac{19.50}{49.4} \times 100 = 39.47$

For Batsman B

x	d = x – 42	d^2
59	17	289
25	–17	289
81	39	1521
47	5	25
73	31	961
78	36	1296
42	0	0
35	–7	49
42	0	0
10	–32	1024
	$\sum d = 72$	**$\sum d^2 = 5454$**

$$\bar{x}_B = 42 + \frac{\sum d}{10} = 42 + 7.2 = 49.2$$

$$\sigma_B = \sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2} = \sqrt{545.4 - 51.84} - \sqrt{493.56} = 22.22$$

$$\text{Coefficient of variation for B} = \frac{\sigma_B}{\text{A.M.}} \times 100 = \frac{22.22}{49.2} \times 100 = 45.16$$

Conclusion : A.M. for A is slightly higher than B A.M. for B so A is slightly better and coefficient of variation for A is less than that of B.

$\therefore$ A is more consistent.

Ex. 2 : *Fluctuations in the Aggregate of marks obtained by two groups of students are given below. Find out which of the two shows greater variability.* ***(Dec. 2012)***

Group A	518	519	530	530	544	542	518	550	527	527	531	550	550	529	528
Group B	825	830	830	819	814	814	844	842	842	826	832	835	835	840	840

Sol. : To solve this problem, we have to determine coefficient of variation $\frac{\sigma}{\text{A.M.}} \times 100$ in each case. First we present the data in frequency distribution form.

For Group A :

x	f	d = x – 530	d^2	fd	fd^2
518	2	– 12	144	– 24	288
519	1	– 11	121	– 11	121
527	2	– 3	9	– 6	18
528	1	– 2	4	– 2	4
529	1	– 1	1	– 1	1
530	2	0	0	0	0
531	1	1	1	1	1
542	1	12	144	12	144
544	1	14	196	14	196
550	3	20	400	60	1200
Total	**$\sum f = 15$**	**–**	**–**	**$\sum fd = 43$**	**1973**

$$\text{A.M.} = \bar{x}_A = 530 + \frac{\sum fd}{\sum f} = 530 + \frac{43}{15} = 532.866$$

$$\sigma_A = \sqrt{\frac{1}{N}\sum fd^2 - \left(\frac{\sum fd}{N}\right)^2} = \sqrt{\frac{1973}{15} - \left(\frac{43}{15}\right)^2} = \sqrt{131.533 - 8.218}$$

$$= 11.105$$

$$\text{Coefficient of variation} = \frac{\sigma_A}{\text{A.M.}} \times 100 = \frac{11.105}{532.866} \times 100 = 2.0840$$

For Group B :

x	f	d = x – 830	d^2	fd	fd^2
814	2	– 16	256	– 32	512
819	1	– 11	121	– 11	121
825	1	– 5	25	– 5	25
826	1	– 4	16	– 4	16
830	2	0	0	0	0
832	1	2	4	2	4
835	2	5	25	10	50
840	2	10	100	20	200
842	2	12	144	24	288
844	1	14	196	14	196
Total	**$\sum f = 15$**	**–**	**–**	**$\sum fd = 18$**	**$\sum fd^2 = 1412$**

$$\text{A.M.} = \bar{x}_B = 830 + \frac{18}{15} = 831.2$$

$$\sigma_B = \sqrt{\frac{1412}{15} - \left(\frac{18}{15}\right)^2} = \sqrt{94.133 - 1.44} = 9.628$$

$$\text{Coefficient of variation} = \frac{\sigma_B}{\text{A.M.}} \times 100 = \frac{9.628}{831.2} \times 100 = 1.158$$

Coefficient of variation of group A is greater than that of group B.

$\therefore$ Group A has greater variability, or Group B is more consistent.

Ex. 3 : *Calculate standard deviation for the following frequency distribution. Decide whether A.M. is good average.*

Wages in Rupees earned per day	*0 – 10*	*10 – 20*	*20 – 30*	*30 – 40*	*40 – 50*	*50 – 60*
No. of labourers	*5*	*9*	*15*	*12*	*10*	*3*

Sol. : Preparing the table for the purpose of calculations.

Wages Earned C.I.	Mid-value x	Frequency f	$u = \frac{x - 25}{10}$	fu	fu^2
0 – 10	5	5	– 2	– 10	20
10 – 20	15	9	– 1	– 9	9
20 – 30	25	15	0	0	0
30 – 40	35	12	1	12	12
40 – 50	45	10	2	20	40
50 – 60	55	3	3	9	27
Total	**–**	**$\sum f = 54$**	**–**	**$\sum fu = 22$**	**$\sum fu^2 = 108$**

Using formula (4),

$$\sigma = 10\sqrt{\frac{1}{54} \times 108 - \left(\frac{22}{54}\right)^2}$$

$$= 10\sqrt{2 - 0.166} = 13.54 \text{ approximately}$$

In this problem,

$$\text{A.M.} = 25 + h\frac{\sum fu}{N} = 25 + 10\ (0.4074) = 29.074$$

Conclusion : σ = 13.54 is quite a large value and Arithmetic mean 29.074 is not a good average.

Ex. 4 : *Arithmetic mean and standard deviation of 30 items are 20 and 3 respectively out of these 30 items, item 22 and 15 are dropped. Find new A.M. and S.D.. Calculate A.M. and S.D. if item 22 is replaced by 8 and 15 is replaced by 17.*

Sol. : $\bar{x} = \dfrac{\Sigma x}{30}$, $\bar{x} = 20$ $\therefore$ $\Sigma x = 20 \times 30 = 600$

Since items 22 and 15 are dropped. Now $\Sigma x = 600 - (22 + 15)$

Now $\Sigma x = 563$, and total item are 28

Now $\bar{x} = \dfrac{563}{28} = 20.107$

Again $V = \sigma^2 = \dfrac{\Sigma x^2}{N} - \left(\dfrac{\Sigma x}{N}\right)^2$ where N = No. of items = 30

$$\sigma^2 = \frac{\Sigma x^2}{30} - 400$$

$$\therefore \quad 9 = \frac{\Sigma x^2}{30} - 400$$

$$\therefore \quad \Sigma x^2 = 30 \times 9 + 30 \times 400 = 12270.$$

Items 22 and 15 are dropped.

Now $\Sigma x^2 = 12270 - (22)^2 - (15)^2 = 12270 - 709 = 11561$

Now $\text{S.D.} = \sqrt{\dfrac{11561}{28} - (20.107)^2} = 2.933$

In the second case if items are replaced

Now $\Sigma x = 600 - (22 - 8) + (17 - 15) = 600 - 14 + 2 = 588$, $N = 30$

$\therefore$ Now $\bar{x} = \dfrac{588}{30} = 19.6$

Now $\Sigma x^2 = 12270 - (14)^2 + 4 = 12078$

Now $\text{S.D.} = \sqrt{\dfrac{12078}{30} - (19.6)^2} = 4.294$

Ex. 5 : *Prove that for any discrete distribution standard deviation, σ is greater than or equal to Mean deviation from the mean.*

$$\sigma^2 = \frac{1}{N} \Sigma f \left(x - \bar{x}\right)^2, \quad M.D. = \frac{1}{N} \Sigma f \left|x - \bar{x}\right|$$

Sol. : Required result implies

$$\frac{1}{N} \Sigma f \left(x - \bar{x}\right)^2 \geq \left(\frac{1}{N} f \left|x - \bar{x}\right|\right)^2$$

Putting $\left|x - \bar{x}\right| = z$

which means $\left(x - \bar{x}\right)^2 = z^2$

We have to prove that $\dfrac{1}{N} \Sigma f z^2 \geq \left(\dfrac{1}{N} \Sigma f z\right)^2$

i.e. $\dfrac{1}{N} \Sigma f z^2 - \left(\dfrac{1}{N} \Sigma f z\right)^2 \geq 0$

i.e. $\dfrac{1}{N} \Sigma f \left(z - \bar{z}\right)^2 \geq 0$ [Refer article 5.4 (i)]

which is always true.

Hence the required results.

Ex. 6 : *Two sets containing n_1 and n_2 items have means m_1 and m_2 and standard deviations σ_1 and σ_2 respectively. Show that combined group has variance given by :*

$$\sigma^2 = \frac{n_1 \sigma_1^2 + n_2 \sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)} (m_1 - m_2)^2$$

Sol. : Let the variates in two series be denoted by x and y respectively. The first series contains n_1 values of variate x and the second series contains n_2 values of variate y.

By definition, $$m_1 = \frac{\sum x}{n_1}, \quad m_2 = \frac{\sum y}{n_2}$$

$$\sigma_1^2 = \frac{1}{n_1}\sum (x - m_1)^2, \quad \sigma_2^2 = \frac{1}{n_2}\sum (y - m_2)^2$$

By formula (A) of article 5.3.

The A.M. $\bar{z}$ of combined series is given by,

$$\bar{z} = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}$$

The variance σ^2 of combined series is given by

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[\sum \left(x - \bar{z}\right)^2 + \sum \left(y - \bar{z}\right)^2\right]$$

Now, $$\sum \left(x - \bar{z}\right)^2 = \sum \left(x - m_1 + m_1 - \bar{z}\right)^2$$

$$= \sum (x - m_1)^2 + 2(x - m_1)\left(m_1 - \bar{z}\right) + \left(m_1 - \bar{z}\right)^2$$

$$= \sum (x - m_1)^2 + 2\left(m_1 - \bar{z}\right)\sum (x - m_1) + \sum\left(m_1 - \bar{z}\right)^2$$

$$\sum (x - m_1) = \sum x - \sum m_1 = n_1 m_1 - n_1 m_1 = 0$$

Let $$d_1 = m_1 - \bar{z}$$

$$\sum \left(x - \bar{z}\right)^2 = \sum (x - m_1)^2 + n_1 d_1^2 = n_1 \sigma_1^2 + n_1 d_1^2$$

Similarly, we can show that

$$\sum \left(y - \bar{z}\right)^2 = n_2 \sigma_2^2 + n_2 d_2^2 \text{ where, } d_2 = m_2 - \bar{z}$$

Thus, $$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\sigma_1^2 + n_1 d_1^2 + n_2\sigma_2^2 + n_2 d_2^2\right]$$

$$= \frac{1}{n_1 + n_2}\left[n_1\sigma_1^2 + n_1 d_1^2 + n_2\sigma_2^2 + n_2 d_2^2\right]$$

To express it in required form :

$$d_1 = m_1 - \bar{z} = m_1 - \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2} = \frac{n_2 (m_1 - m_2)}{n_1 + n_2}$$

$$d_2 = m_2 - \bar{z} = m_2 - \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2} = \frac{n_1 (m_1 - m_2)}{n_1 + n_2}$$

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2\right]$$

$$= \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{1}{n_1 + n_2}\left[\frac{n_1 n_2^2 (m_1 - m_2)^2}{(n_1 + n_2)^2} + \frac{n_2 n_1^2 (m_1 - m_2)^2}{(n_1 + n_2)^2}\right]$$

$$= \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2 (m_1 - m_2)^2}{(n_1 + n_2)^3}(n_2 + n_1)$$

$$= \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2 (m_1 - m_2)^2}{(n_1 + n_2)^2}$$

which is the required result.

5.6 MOMENTS, SKEWNESS AND KURTOSIS

So far we have studied tow aspects of frequency distribution viz. average and dispersion. In order to study few more aspects such as symmetry, shape of the frequency distribution (or frequency curve), more general type of descriptive measure called moments are useful.

5.6.1 Moments

The r^{th} moment about the mean $\bar{x}$ (or central moments) of a frequency distribution is denoted by μ_r and is given by,

$$\mu_r = \frac{1}{N} \Sigma f \left(x - \bar{x}\right)^r, \quad \text{where, } N = \Sigma f \text{ and } \bar{x} = \text{A.M. of the distribution.}$$

Putting r = 0, 1, 2, 3, 4 etc., we get

$$\mu_0 = \frac{1}{N} \Sigma f = 1,$$

$$\mu_1 = \frac{1}{N} \Sigma f \left(x - \bar{x}\right) = \bar{x} - \bar{x} = 0 \text{ gives first moment of distribution about the mean.}$$

$$\mu_2 = \frac{1}{N} \Sigma f \left(x - \bar{x}\right)^2 = \sigma^2 = \text{Variance} = \text{Var(x) gives second moment of the distribution about the mean.}$$

$$\mu_3 = \frac{1}{N} \Sigma f \left(x - \bar{x}\right)^3 \text{ gives } 3^{rd} \text{ moment of the distribution about the mean.}$$

$$\mu_4 = \frac{1}{N} \Sigma f \left(x - \bar{x}\right)^4 \text{ gives } 4^{th} \text{ moment of the distribution about the mean and so on}$$

Since actual evaluation of r^{th} moment μ_r about the mean $\bar{x}$ is numerically complicated, we find r^{th} moments viz. μ_r' about arbitrary convenient number A of the distribution with much less calculation (Refer Art. 5.4). The r^{th} moment μ_r' about arbitrary number (assumed mean) is given by

$$\mu_r' = \frac{1}{N} \Sigma f (x - A)^r, \text{ it can be seen on putting r = 0, 1, 2, …etc. that}$$

$$\mu_0' = 1$$

$$\mu_1' = \frac{1}{N} \Sigma f (x - A) = \frac{1}{N} \Sigma f x - \frac{\Sigma f}{N} A = \bar{x} - A$$

$$\mu_2' = \frac{1}{N} \Sigma f (x - A)^2 = S^2 \text{ the mean square deviation}$$

$$\mu_3' = \frac{1}{N} \Sigma f (x - A)^3 \text{ etc.}$$

5.6.2 Relation Between μ_r and μ_r'

We know by definition of μ_r,

$$\mu_r = \frac{1}{N} \Sigma f \left(x - \bar{x}\right)^r$$

$$= \frac{1}{N} \Sigma f \left(x - A + A - \bar{x}\right)^r$$

Let $d = x - A$ and hence $\bar{d} = \dfrac{\Sigma fd}{N} = \dfrac{\Sigma fx}{N} - \dfrac{A \cdot \Sigma f}{N}$ or $\bar{d} = \bar{x} - A = \mu'_1$

Thus, $$\mu_r = \frac{1}{N} \Sigma f \left(d - \bar{d}\right)^r$$

On expanding $\left(d - \bar{d}\right)^r$ binomially, we obtain

$$\mu_r = \frac{1}{N} \Sigma f \left(d^r - {}^rC_1 \, d^{r-1} \, \bar{d} + {}^rC_2 \, d^{r-2} \, \bar{d}^2 + \ldots + (-1)^r \, \bar{d}^r\right)$$

where $\quad {}^rC_1 = r,\ {}^rC_2 = \dfrac{r(r-1)}{2!},\quad {}^rC_3 = \dfrac{r(r-1)(r-2)}{3!}$ etc.

$$\therefore \quad \mu_r = \frac{1}{N}\sum fd^r - {}^rC_1 \frac{1}{N}\sum fd^{r-1}\,\bar{d} + {}^rC_2 \frac{1}{N}\sum fd^{r-2}\,\bar{d}^{\,2} - {}^rC_3 \frac{1}{N}\sum fd^{r-3}\,\bar{d}^3 \ldots (-1)^r \frac{1}{N}\sum f\,\bar{d}^r$$

Using, $\dfrac{1}{N}\sum f\,d^r = \mu_r'$ and $\bar{d} = \mu_1'$, relation between μ_r and μ_r' is

$$\therefore \quad \mu_r = \mu_r' - {}^rC_1\ \mu'_{r-1}\ \mu_1' + {}^rC_2\ \mu'_{r-2}\ \mu_1' + \ldots + (-1)^r\ (\mu_r')^r$$

We have already seen that $\mu_0 = 1,\ \mu_1 = 0$.

Putting r = 2, 3, 4 etc., we get

$$\begin{aligned}
\mu_2 &= \mu_2' - {}^2C_1\ \mu_1'\ \mu_1' + {}^2C_2\ \mu_0'\,\mu_1' \\
&= \mu_2' - 2\,\mu_1'^2 + \mu_1'^2 \\
&= \mu_1' - \mu_1'^2 && \ldots (A) \\
\mu_3 &= \mu_3' - {}^2C_1\ \mu_2'\ \mu_1' + {}^3C_2\ \mu_1'\ \mu_1'^2 - \mu_1'^3 \\
&= \mu_3' - {}^3C_1\ \mu_2' + {}^3C_2\ \mu_1'\ \mu_1'^2 - \mu_1'^3 \\
&= \mu_3' - 3\,\mu_2'\ \mu_1' + 2\,\mu_1'^3 && \ldots (B) \\
\mu_4 &= \mu_4' - {}^4C_1\,\mu_3'\ \mu_1' + {}^4C_2\ \mu_2'\ \mu_1'^2 - {}^4C_2\ \mu_1'\ \mu_1'^3 + {}^4C_4\ \mu_1'^4 \\
&= \mu_4' - 4\,\mu_3'\ \mu_1' + 6\,\mu_2'\ \mu_1'^2 - 4\,\mu_1'^4 + \mu_1'^4 \\
&= \mu_4' - 4\,\mu_3'\ \mu_1' + 6\,\mu_2'\ \mu_1'^2 - 3\,\mu_1'^4 && \ldots (C)
\end{aligned}$$

The moments of higher order μ_5, μ_6 etc. can be similarly expressed.

Note : From the above relations (A), (B) and (C) we note the following :

(i) Sum of the coefficients on R.H.S. of each of the relations is zero.

(ii) First term in the expression is positive and alternative terms are negative.

(iii) The last term in the expression of μ_r is $(\mu_1')^r$.

While dealing with data presented in group frequency distribution, to reduce the calculation of μ_r' further, we use the following procedure.

Put $u = \dfrac{x-A}{h}$ where, h is taken generally width of class interval, then the expressions for the moments μ_r' about any arbitrary point A (assumed or convenient mean) are given by

$$\mu_r' = \frac{1}{N}\sum f\,(x-A)^r, \qquad N = \sum f$$

$$= \frac{1}{N}\sum f\,(hu)^r$$

$$\therefore \quad \mu_r' = h^r \frac{1}{N}\sum f\,u^r, \quad r = 1, 2, 3, \ldots \qquad \ldots (D)$$

We know that first moment about mean $\bar{x}$ is $\mu_1 = 0$. The second, third and fourth moments about the mean $\bar{x}$ are obtained using relations (A), (B) and (C).

Note : r^{th} moment about the mean $\bar{x}$ (or central moment) of individual observation $(x_1, x_2, \ldots, x_n)$ denoted by μ_r and is given by

$$\mu_r = \frac{1}{n}\sum (x-\bar{x})^r, \qquad r = 0, 1, 2, 3, 4$$

r^{th} moment about the arbitrary mean A of individual observation $(x_1, x_2, \ldots, x_n)$ is denoted by μ'_r and is given by

$$\mu_r' = \frac{1}{n}\sum (x-A)^r \qquad \ldots (E)$$

Remark :

(i) Change of Origin Property : The rth moment about the mean $\bar{x}$ (central moments) are invariant to the change of origin. If $u = x - A$ then (μ_r of u) = (μ_r of x).

(ii) Change of Origin and Scale : If $u = \frac{x - A}{h}$ then $(\mu_r \text{ of } u) = \frac{1}{h^r} (\mu_r \text{ of } x)$.

Sheppard's Correction for Moments : In case of grouped frequency distribution, we take mid-values of class intervals to represent the class interval. This involves some error in calculation of moments. W.F. Sheppard suggested some corrective formulae :

$$\mu_2 \text{ (corrected)} = \mu_2 - \frac{1}{12} h^2$$

$$\mu_3 = \mu_3$$

$$\mu_4 \text{ (corrected)} = \mu_4 - \frac{1}{2} h^2 \mu_1 + \frac{7}{240} h^4,$$ where, h is the width of class interval.

5.6.3 Skewness

A frequency distribution is symmetric above a value 'a' (say); if the corresponding frequency curve is symmetric about a. For symmetric frequency curve, the point 'a' turns out to be arithmetic mean, mode as well as median refer Fig. 5.3 (a).

Skewness signifies departure from symmetry (or a lacks of symmetry). We study skewness to have an idea about the shape of the curve which we draw with the given data.

If the frequency curve stretches to the right as in Fig. 5.3 (b) i.e. the mean is to the right of the mode then the distribution is right skewed or is said to have positive skewness. If the curve stretches to left or mode is to the right of the mean then the distribution is said to have negative skewness refer Fig. 5.3 (c).

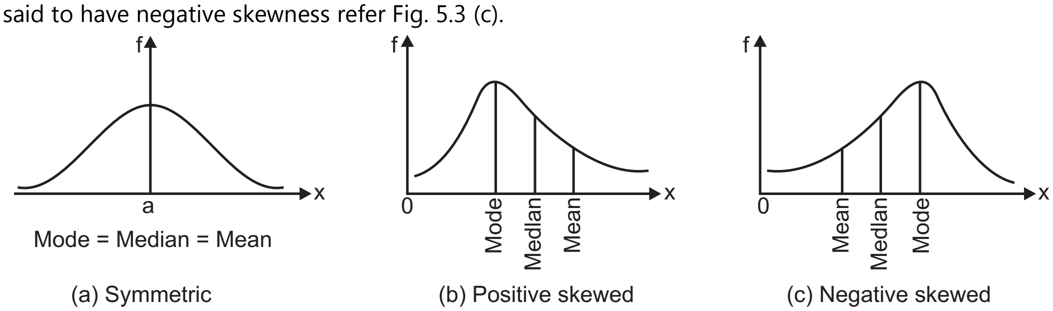


Fig. 5.3

The different measures of skewness are :

(i) $$\text{Skewness} = \frac{3 \text{ (Mean – Median)}}{\text{Standard deviation}}$$

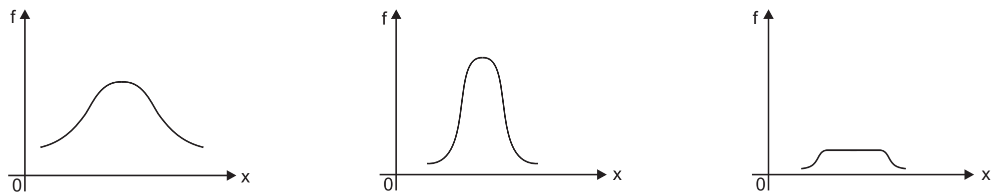
(ii) $$\text{Coefficient of skewness, } \beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

The distribution is positively skewed if skewness or coefficient of skewness β_1 is positive. If coefficient of skewness is negative the distribution is negatively skewed. It is also clear from (i). Now to decide the sign of β_1, We introduce the parameter $\gamma_1 = \pm \sqrt{\beta_1}$. Now $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ where both numerator and denominator are positive. To decide the sign of γ_1 we associate with sign of μ_3. It μ_3 is negative, we take γ_1 as negative and μ_3 is positive, we take γ_1 as positive. In short distribution is positively skewed if μ_3 is positive and it is negatively skewed if μ_3 is negative.

5.6.4 Kurtosis

To get complete idea of the distribution in addition to the knowledge of mean, dispersion and skewness, we should have an idea of the flatness or peakedness of the curve. It is measured by the coefficient β_2 given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad \text{and} \quad \gamma_2 = \beta_2 - 3$$



(a) Mesokurtic (b) Leptokurtic (c) Platykurtic

Fig. 5.4

The curve of Fig. 5.4 (a) which is neither flat nor peaked is called the normal curve or Mesokurtic curve. $\gamma_2 = \beta_2 - 3$ gives the excess of kurtosis. For a normal distribution, $\beta_2 = 3$ and the excess is zero. The curve of Fig. 5.4 (c) which is flatter than the normal curve is called Platykurtic and that of Fig. 5.4 (b) which is more peaked is called Leptokurtic. For Platykurtic curves $\beta_2 < 3$, for Leptokurtic curves $\beta_2 > 3$.

ILLUSTRATIONS

Ex. 7 : *Calculate the first four moments about the mean of the given distribution. Also find β_1 and β_2.*

(Dec. 2005, 2008, 2011; May 2006, 2007)

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f	4	36	60	90	70	40	10

Sol. : Taking A = 3.5, h = 0.5 and $u = \dfrac{x - 3.5}{0.5}$

We prepare the table for calculating μ_1', μ_2', μ_3' and μ_4'.

x	f	$u = \dfrac{x-3.5}{0.5}$	fu	fu^2	fu^3	fu^4
2.0	4	– 3	– 12	36	– 108	342
2.5	36	– 2	– 72	144	– 288	576
3.0	60	– 1	– 60	60	– 60	60
3.5	90	0	0	0	0	0
4.0	70	1	70	70	70	70
4.5	40	2	80	160	320	640
5.0	10	3	30	90	270	810
Total	$\sum f = 310$		$\sum fu = 36$	$\sum fu^2 = 560$	$\sum fu^3 = 204$	$\sum fu^4 = 2480$

For moments about arbitrary mean A = 3.5, we use formula (D).

$$\mu_r' = h^r \frac{\sum fu^r}{\sum f}$$

$$\mu_1' = h \frac{\sum fu}{\sum f} = (0.5) \frac{36}{310} = 0.058064$$

$$\mu_2' = h^2 \frac{\sum fu^2}{\sum f} = (0.5)^2 \frac{560}{310} = 0.451612$$

$$\mu_3' = h^3 \frac{\sum fu^3}{\sum f} = (0.5)^3 \frac{204}{310} = 0.082259$$

$$\mu_4' = h^4 \frac{\sum fu^4}{\sum f} = (0.5)^4 \frac{2480}{310} = 0.5$$

Using relations A, B, C of section 5.5 central moments are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_2')^2 = (0.451612) - (0.058064)^2 = 0.44824$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,(\mu_1')^3$$

$$= (0.082259) - 3\,(0.451612)\,(0.058064) + 2\,(0.058064)^3$$

$$= 0.082259 - 0.078668 + 0.0003916$$

$$= 3.9826 \times 10^{-3} = 0.0039826$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\,\mu_4'\,(\mu_1')^2 - 3\,(\mu_1')^4$$

$$= 0.5 - 0.01911 + 0.009136 - 0.0000341$$

$$= 0.48999$$

$$\beta_1 = \frac{\mu_3{}^2}{\mu_2{}^3} = \frac{0.0000159}{0.0900599} = 1.76549$$

$$\beta_2 = \frac{\mu_4}{\mu_2{}^2} = \frac{0.48999}{0.20092} = 2.43874$$

Ex. 8 : *For the following distribution, find (i) first 4 moments about the mean, (ii) β_1 and β_2, (iii) arithmetic mean, (iv) standard deviation.* ***(Dec. 2011)***

x	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*5*
f	*5*	*38*	*65*	*92*	*70*	*40*	*10*

Sol. : Let $u = \frac{x - 3.5}{0.5}$, A = 3.5, h = 0.5.

x	**f**	**u**	**fu**	**fu²**	**fu³**	**fu⁴**
2	5	– 3	– 15	45	– 135	405
2.5	38	– 2	– 76	152	– 304	608
3	65	– 1	– 65	65	– 65	65
3.5	92	0	0	0	0	0
4	70	1	70	70	70	70
4.5	40	2	80	160	320	640
5	10	3	30	90	270	810
Total	**320**	**–**	**24**	**582**	**156**	**2598**

(i) Moment about the mean M.

When assumed mean is A = 3.5 and using (D), we have

$$\mu_1' = h\,\frac{\sum fu}{\sum f} = 0.5\left(\frac{24}{320}\right) = 0.0375$$

$$\mu_2' = h^2\,\frac{\sum fu^2}{\sum f} = (0.5)^2\left(\frac{582}{320}\right) = 0.4546$$

$$\mu_3' = h^3 \frac{\sum fu^3}{\sum f} = (0.5)^3 \left(\frac{156}{320}\right) = 0.0609$$

$$\mu_4' = h^4 \frac{\sum fu^4}{\sum f} = (0.5)^4 \left(\frac{2598}{320}\right) = 0.5074$$

Using results (A), (B), (C), we have four moments about the mean M

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = (0.4546) - (0.0375)^2 = 0.453$$

$$\begin{aligned} \mu_3 &= \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,(\mu_1')^3 \\ &= (0.0609) - 3\,(0.4546)\,(0.0375) + 2\,(0.0375)^3 \\ &= 0.0600 \end{aligned}$$

$$\begin{aligned} \mu_4 &= \mu_4' - 4\,\mu_3'\,\mu_2' + 6\,(\mu_1')^2\,\mu_2' - 3\,(\mu_1')^4 \\ &= (0.5074) - 4\,(0.0609)\,(0.0375) + 6\,(0.0375)^2\,(0.4546) - 3\,(0.0375)^4 \\ &= 0.502 \end{aligned}$$

(ii) By definition of β_1 and β_2, we have

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.0600)^2}{(0.453)^3} = 0.0387$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{0.502}{(0.453)^2} = 2.4463$$

Since $\beta_2 < 3$, the distribution is platy kurtic i.e. it is flatter than the normal distribution.

(iii) Arithmetic Mean : Using result (B), we have

$$A = \frac{\sum fu}{\sum f} = \frac{24}{320} = 0.075$$

(iv) Standard Deviation :

$$\sigma^2 = h^2 \left\{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2\right\}$$

$$= (0.5)^2 \left\{\frac{582}{320} - \left(\frac{24}{320}\right)^2\right\} = 0.453$$

$$\sigma = 0.673$$

Ex. 9 : *Calculate the first four moments about the mean of the given distribution. Find β_1, β_2 and comment on skewness and kurtosis.*

x	*5*	*7*	*13*	*24*	*29*	*36*	*40*	*45*	*50*
f	*4*	*6*	*17*	*25*	*18*	*12*	*9*	*3*	*2*

Sol.: Taking A = 24, d = x – 24

x	f	d = x – 24	fd	fd^2	fd^3	fd^4
5	4	–19	–76	1444	–27436	521284
7	6	–17	–102	1734	–29478	501126
13	17	–11	–187	2057	–22627	248897
24	25	0	0	0	0	0
29	18	5	90	450	2250	11250
36	12	12	144	1728	20736	248832
40	9	16	144	2304	36864	589924
45	3	21	63	1323	27783	583443
50	2	26	52	1352	35152	913952
Total	**$\sum f = 96$**	**–**	**$\sum fd = 128$**	**$\sum fd^2 = 12392$**	**$\sum fd^3 = 43244$**	**$\sum fd^4 = 3618608$**

$$\mu_1' = \frac{\sum fd}{\sum f} = 1.33$$

$$\mu_2' = \frac{\sum fd^2}{\sum f} = 129.08$$

$$\mu_3' = \frac{\sum fd^3}{\sum f} = 450.46$$

$$\mu_4' = \frac{\sum fd^4}{\sum f} = 37693.83$$

$$\mu_1 = 0,$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 127.31$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -59.86$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 36657.97$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = 0.001736$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 2.262$$

β_1 is very small, so the curve is symmetrical, skewness is negative.

$\beta_2 = 2.262 < 3$, curve is playticurtic type.

Ex. 10 : *Compute the first four central moments for the following frequencies :* ***(Dec. 2010, May 2011)***

No. of Jobs Completed	*0-10*	*10-20*	*20-30*	*30-40*	*40-50*
No. of Workers	*6*	*26*	*47*	*15*	*6*

Sol. :

Class	**Mid-Pts. (x)**	**Freq. (f)**	$u = \frac{x-25}{10}$	**fu**	fu^2	fu^3	fu^4
0-10	5	6	– 2	– 12	24	– 48	96
10-20	15	26	– 1	– 26	26	– 26	26
20-30	25	47	0	0	0	0	0
30-40	35	15	1	15	15	15	15
40-50	45	6	2	12	24	48	96
Total	–	**100**	–	**– 11**	**89**	**– 11**	**233**

For moments about arbitrary mean A = 25 we use the formula (D)

$$\mu_4' = h^r \frac{\sum fu^r}{\sum f}$$

$$\therefore \quad \mu_1' = h\frac{\sum fu}{\sum f} = 10\left(\frac{-11}{100}\right) = 10(-0.11) = -1.1$$

$$\mu_2' = h^2\frac{\sum fu^2}{\sum f} = (10)^2\left(\frac{89}{100}\right) = 100(0.89) = 89$$

$$\mu_3' = h^3\frac{\sum fu^3}{\sum f} = (10)^3\left(\frac{-11}{100}\right) = (1000)(-0.11) = -110$$

$$\mu_4' = h^4 \frac{\sum fu^4}{\sum f} = (10)^4 \left(\frac{233}{100}\right) = (10)^4 (2.33) = 23300$$

Using relations A, B, C of article 5.5 central moments are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 89 - (-1.1)^2 = 89 - (1.21) = 87.79$$

$$\begin{aligned}
\mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 \\
&= -110 - 3(89)(-1.1) + 2(-1.1)^3 \\
&= -110 + 293.7 + 2(-1.331) \\
&= -110 + 293.7 - 2.662 \\
&= 181.038
\end{aligned}$$

$$\begin{aligned}
\mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \\
&= 23300 - 4(-110)(-1.1) + 6(89)(-1.1)^2 - 3(-1.1)^4 \\
&= 23300 - 484 + 646.14 - 4.3923 \\
&= 23457.7477
\end{aligned}$$

Ex. 11 : *If $\sum f = 27$, $\sum fx = 91$, $\sum fx^2 = 359$, $\sum fx^3 = 1567$, $\sum fx^4 = 7343$. Find first four moments about origin. Find A.M., S.D., μ_3 and μ_4. Find coefficients of skewness and kurtosis. Comment on skewness and kurtosis.*

Sol.:

$$\mu_1' = \frac{\sum fx}{\sum f} = \frac{91}{27} = 3.37$$

$$\mu_2' = \frac{\sum fx^2}{\sum f} = \frac{359}{27} = 13.297$$

$$\mu_3' = \frac{\sum fx^3}{\sum f} = \frac{1567}{27} = 58.04$$

$$\mu_4' = \frac{\sum fx^4}{\sum f} = \frac{7343}{27} = 271.963$$

$$\text{A.M.} = \mu_1' = 3.37$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 13.297 - (3.37)^2 = 1.94$$

$$\text{S.D.} = \sqrt{\mu_2} = 1.3928$$

$$\begin{aligned}
\mu_3 &= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 \\
&= 58.04 - 3 \times 3.37 \times 13.297 + 2 \times (3.37)^3 \\
&= 58.04 - 134.43267 + 38.2727 \times 2 \\
&= 58.04 - 134.43267 + 76.5455 = 0.15283
\end{aligned}$$

$$\begin{aligned}
\mu_4 &= \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4 \\
&= 271.963 - 4 \times 58.04 \times 3.37 + 6(13.37)^2 \times 13.297 - 3 \times (3.37)^4 \\
&= 8.7311
\end{aligned}$$

$$\beta_1 = \text{coefficient of skewness} = \frac{\mu_3^2}{\mu_2^3} = 0.003197$$

Skewness is very small and curve is symmetrical

$$\beta_2 = \text{coefficient of kurtosis} = \frac{\mu_4}{\mu_2^2} = 2.3198$$

$$\gamma_2 = \beta_2 - 3 \text{ given excess of kurtosis, which is small}$$

Since $\beta_2 < 3$, it is playtikurtic type curve.

Ex. 12 : *The first four moments about the working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25. Calculate the first four moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.*

(Dec. 2005, 2006, May 2010, Nov. 2019)

Sol. : The first four moments about the arbitrary origin 30.2 are

$$\mu_1' = 0.255, \quad \mu_2' = 6.222, \quad \mu_3' = 30.211, \quad \mu_4' = 400.25$$

$$\therefore \qquad \mu_1' = \frac{1}{N}\sum f_i (x_i - 30.2) = \frac{1}{N}\sum f_i x_i - 30.2 = \bar{x} - 30.2 = 0.255$$

$$\text{or} \qquad \bar{x} = 30.455$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 6.222 - (0.255)^2 = 6.15698$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = 30.211 - 3(6.222)(0.255) + 2(0.255)^3$$

$$= 30.211 - 4.75983 + 0.03316275$$

$$\mu_3 = 25.48433$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 440.25 - 4(30.211)(0.255) + 6(6.222)(0.255) - 3(0.255)^4$$

$$\mu_4 = 378.9418$$

$$\therefore \qquad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(25.48433)^2}{(6.15698)^3} = 2.78255$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{378.9418}{(6.15698)^2}$$

$$\beta_2 = 9.99625$$

$$\therefore \qquad \gamma_1 = \sqrt{\beta_1} = \sqrt{2.78255} = 1.6681$$

which indicates considerable positive skewness of the distribution.

$$\gamma_2 = \beta_2 - 3 = 9.99625 - 3 = 6.99625$$

which shows that the distribution is leptokurtic.

Ex. 13 : *The first four moments of a distribution about the value 5 are 2, 20, 40 and 50. From the given information obtain the first four central moments, mean, standard deviation and coefficient of skewness and kurtosis.* ***(Dec. 2007; May 2015, 2019)***

Sol. : $A = 5$, $\mu_1' = 2$, $\mu_2' = 20$, $\mu_3' = 40$ and $\mu_4' = 50$.

On the basis of given information we can calculate the various central moments, mean, standard deviation and coefficient of skewness and kurtosis.

The first moment about zero gives the value of the distribution.

$$\therefore \qquad \text{Mean} = \bar{x} = A + \mu_1' = 5 + 2 = 7$$

Now we calculate central moments.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 20 - (2)^2 = 16$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= 40 - 3(2)(20) + 2(2)^3$$

$$= 40 - 120 + 16$$

$$= -64$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

$$= 50 - 4(2)(40) + 6(2)^2(20) - 3(2)^4$$

$$= 50 - 320 + 480 - 48$$

$$= 162$$

The second central moment gives the value of variance.

$\therefore$ Variance $= \mu_2 = 16$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = \sqrt{16} = 4$

Coefficient of skewness is given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-64)^2}{(16)^3} = 1$$

Since μ_3 is negative, the distribution is negatively skewed. Coefficient of kurtosis is given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{162}{(16)^2} = 0.63$$

Since the value of β_2 is less than 3, hence the distribution is platykurtic.

Ex. 14 : *The first four central moments of distribution are 0, 2.5, 0.7 and 18.75. Comment on the skewness and kurtosis of the distribution.* ***(May 2009)***

Sol. : Testing of Skewness : $\mu_1 = 0$, $\mu_2 = 2.5$, $\mu_3 = 0.7$ and $\mu_4 = 18.75$

Coefficient of skewness is given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.7)^2}{(2.5)^3} = 0.0314$$

Since, μ_3 is positive, the distribution is positively skewed slightly.

Testing of Kurtosis : Coefficient of kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{18.75}{(2.5)^2} = 3$$

Since, β_2 is exactly three, the distribution is mesokurtic.

EXERCISE 5.1

1. Find the Arithmetic Mean, Median and Standard deviation for the following frequency distribution.

x	5	9	12	15	20	24	30	35	42	49
f	3	6	8	8	9	10	8	7	6	2

Ans. $\bar{x} = 22.9851$, $M = 20$, $\sigma = 11.3538$

2. Age distribution of 150 life insurance policy-holders is as follows :

Age as on Nearest Birthday	Number
15 – 19.5	10
20 – 24.5	20
25 – 29.5	14
30 – 34.5	30
35 – 39.5	32
40 – 44.5	14
45 – 49.5	15
50 – 54.5	10
55 – 59.5	5

Calculate mean deviation from median age. **Ans.** M.D. = 8.4284

3. The Mean and Standard deviation of 25 items is found to be 11 and 3 respectively. It was observed that one item 9 was incorrect. Calculate the Mean and Standard deviation if :

(i) The wrong item is omitted.

(ii) It is replaced by 13. **(May 2012)** **Ans.** (i) $\bar{x} = 11.08, \sigma = 3.345$, (ii) $\bar{x} = 11.16, \sigma = 2.9915$

4. Following table gives the Marks obtained in a paper of statistics out of 50, by the students of two divisions :

C.I.	**0 – 5**	**5 – 10**	**10 – 15**	**15 – 20**	**20 – 25**	**25 – 30**	**30 – 35**	**35 – 40**	**40 – 45**	**45 – 50**
Div. A(f)	2	6	8	8	15	18	12	11	9	4
Div. B(f)	3	5	7	9	12	16	11	5	6	2

Find out which of the two divisions show greater variability. Also find the common mean and standard deviation.

Ans. B has greater variability , $\bar{x} = 26.1458$, $\sigma = 11.1267$

5. Calculate the first four moments about the mean of the following distribution. Find the coefficient of Skewness and Kurtosis.

x	1	2	3	4	5	6	7	8	9	10
f	6	15	23	42	62	60	40	24	13	5

Ans. $\mu_1 = 0$, $\mu_2 = 3.703$, $\mu_3 = 0.04256$, $\mu_4 = 37.5$, $\beta_1 = 0.00005572$, $\beta_2 = 2.8411$

6. The first four moments of a distribution about the mean value 4 are −1.5, 17, −30 and 108. Find the moments about the mean and β_1 and β_2. **Ans.** $\mu_1 = 0$, $\mu_2 = 14.75$, $\mu_3 = 39.75$, $\mu_4 = 142.31$; $\beta_1 = 0.4926$, $\beta_2 = 0.6543$.

5.7 CURVE FITTING

5.7.1 Least Square Approximation

As a result of certain experiment suppose the values of the variables (x_i, y_i) are recorded for i = 1, 2, 3, … n.

If these points are plotted, usually it is observed that a smooth curve passes through most of these points, while some the points are slightly away from this curve. The curve passing through these points may be a first degree curve i.e. a straight line say $y = ax + b$ or a second degree parabola such as $y = ax^2 + bx + c$ or in general an n^{th} degree curve.

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} \dots a_n$$

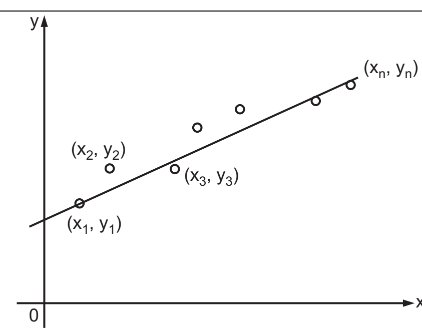


Fig. 5.5

To determine the equation of the curve which very nearly passes through the set of points, we assume some form of relation between x and y may be a straight line or a parabola of second degree, third degree and so on, which we expect to be the best fit.

In Fig. 5.5, we observe that a straight line very nearly passes through the set of points. We may assume the equation of the straight line as

$$y = ax + b \quad \dots (1)$$

If point (x_i, y_i) is assumed to lie on (1) then y co-ordinate of the point can be calculated as,

$$y'_i = ax_i + b$$

If point actually lies on (1) then,

$$y_i = y'_i$$

otherwise $y_i - y'_i$ will represent the deviation of observed value y_i from the calculated value of y'_i using the formula (1).

In method of least squares we take the sum of the squares of these deviations and minimize this sum using the principle of maxima or minima. Values of a, b in (1) are calculated using this criteria. This is called least square criteria. Curve (1) can be of any degree using least square criteria we can find its equation.

In what follows we shall discuss fitting of straight line and second degree parabola to a given set of points.

5.7.2 Fitting Straight Line

Let (x_i, y_i); i = 1, 2, 3, … n be the observed values of (x, y).

To fit the straight line,

$$y = ax + b$$

using least square criteria

$(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ are the observed values.

$$y'_1 = ax_1 + b,\ y'_2 = ax_2 + b \ldots y'_n = ax_n + b$$

are the calculated values of y co-ordinates under the presumption that the points lie on the straight line y = ax + b.

Let $$s = (y'_1 - y_1)^2 + (y'_2 - y_2)^2 + (y'_3 - y_3)^2 \ldots (y'_n - y_n)^2$$

i.e. $$s = (ax_1 + b - y_1)^2 + (ax_2 + b - y_2)^2 \ldots (ax_n + b - y_n)^2$$

For minimum s, $\frac{\partial s}{\partial b} = 0,\ \frac{\partial s}{\partial a} = 0.$

$\frac{\partial s}{\partial b} = 0$ gives,

$$2(ax_1 + b - y_1) + 2(ax_2 + b - y_2) \ldots 2(ax_n + b - y_n) = 0$$

or $$a(x_1 + x_2 \ldots x_n) + nb - (y_1 + y_2 \ldots y_n) = 0$$

$$\therefore \quad a\sum x + nb = \sum y \qquad \ldots (1)$$

$\frac{\partial s}{\partial a} = 0$ gives

$$2(ax_1 + b - y_1)x_1 + 2(ax_2 + b - y_2)x_2 \ldots 2(ax_n + b - y_n)x_n = 0$$

or $$a(x_1^2 + x_2^2 + \ldots + x_n^2) + b(x_1 + x_2 \ldots x_n) - (x_1y_1 + x_2y_2 \ldots x_ny_n) = 0$$

$$\therefore \quad a\sum x^2 + b\sum x = \sum xy \qquad \ldots (2)$$

Solving (1) and (2), we determine the values of a and b which gives the straight line y = ax + b, best fit for the given data.

Some times to reduce the volume of calculations we shift the origin by taking

$$x = X + h \quad \text{or} \quad X = x - h$$

where, h is generally the centrally located value in the set of observed values of x.

We find the straight line

$y = aX + b$ and then replace X by x – h.

5.7.3 Fitting Second Degree Parabola

We now explain the method of fitting a parabola of the form,

$$y = ax^2 + bx + c$$

to the set of observed values

$$(x_i, y_i)\ ; \quad i = 1, 2, 3, \ldots n.$$

As before calculated values of y co-ordinates under the assumption that points satisfy the equation of parabola are given by :

$$y'_1 = ax_1^2 + bx_1 + c,\ y'_2 = ax_2^2 + bx_2 + c''' \ldots y'_n = ax_n^2 + bx_n + c \qquad \ldots (1)$$

Let $$s = (y'_1 - y_1)^2 + (y'_2 - y_2)^2 \dots (y'_n - y_n)^2$$

or $$s = (ax_1^2 + bx_1 + c - y_1)^2 + (ax_2^2 + bx_2 + c - y_2)^2 \dots (ax_n^2 + bx_n + c - y_n)^2$$

For minimum s, $\frac{\partial s}{\partial c} = 0, \quad \frac{\partial s}{\partial b} = 0, \quad \frac{\partial s}{\partial a} = 0,$

$\frac{\partial s}{\partial c} = 0$ gives,

$$2(ax_1^2 + bx_1 + c - y_1) + 2(ax_2^2 + bx_2 + c - y_2) \dots 2(ax_n^2 + bx_n + c - y_n) = 0$$

or $$a(x_1^2 + x_2^2 + \dots x_n^2) + b(x_1 + x_2 \dots x_n) + nc - (y_1 + y_2 \dots y_n) = 0$$

$$\therefore \quad a\sum x^2 + b\sum x + nc = \sum y \qquad \dots (2)$$

$\frac{\partial s}{\partial b} = 0$ gives,

$$2(ax_1^2 + bx_1 + c - y_1)x_1 + 2(ax_2^2 + bx_2 + c - y_2)x_2 \dots 2(ax_n^2 + bx_n + c - y_n)x_n = 0$$

or $$a(x_1^3 + x_2^3 + \dots x_n^3) + b(x_1^2 + x_2^2 \dots x_n^2) + c(x_1 + x_2 \dots x_n) - x_1y_1 - x_2y_2 \dots - x_ny_n = 0$$

$$\therefore \quad a\sum x^3 + b\sum x^2 + c\sum x = \sum xy \qquad \dots (3)$$

Similarly, $\frac{\partial s}{\partial a} = 0$, gives,

$$2(ax_1^2 + bx_1 + c - y_1)x_1^2 + 2(ax_2^2 + bx_2 + c - y_2)x_2^2 \dots 2(ax_n^2 + bx_n + c - y_n)x_n^2 = 0$$

$$\therefore \quad a(x_1^4 + x_2^4 \dots x_n^4) + b(x_1^3 + x_2^3 \dots x_n^3) + c(x_1^2 + x_2^2 \dots x_n^2) = y_1x_1^2 + y_2x_2^2 \dots y_nx_n^2$$

$$\therefore \quad a\sum x^4 + b\sum x^3 + c\sum x^2 = \sum x^2y \qquad \dots (4)$$

(2), (3), (4) are three simultaneous equations in three unknowns a, b, c. Solving these equations we determine a, b, c, which gives best fit parabola for the given data :

As before, to reduce the calculations, we shift the origin by taking

$$x = X + h \quad \text{or} \quad X = x - h$$

Equations (2), (3), (4) take the form

$$a\sum X^2 + b\sum X + nc = \sum y \qquad \dots (5)$$

$$a\sum X^3 + b\sum X^2 + c\sum X = \sum Xy \qquad \dots (6)$$

$$a\sum X^4 + b\sum X^3 + c\sum X^2 = \sum X^2y \qquad \dots (7)$$

Solving (5), (6) and (7), we find a, b, c and determine the parabola.

$$y = aX^2 + bX + c$$

then we replace X by x – h and get the parabola in terms of x and y.

Illustrations

Ex. 1 : *Fit a straight line of the form $y = mx + c$ to the following data, by using the method of least squares.*

x	0	1	2	3	4	5	6	7
y	– 5	– 3	– 1	1	3	5	7	9

Sol. : Preparing the table as

x	**y**	**xy**	**x^2**
0	–5	0	0
1	–3	–3	1
2	–1	–2	4
3	1	3	9
4	3	12	16
5	5	25	25
6	7	42	36
7	9	63	49
$\sum x = 28$	$\sum y = 16$	$\sum xy = 140$	$\sum x^2 = 140$

n = 8 (Total number of points)

Substituting in (1) and (2) of (5.6.2) after replacing a by m and b by c we get,

$$28m + 8c = 16 \qquad 140m + 28c = 140$$

or $\quad 7m + 2c = 4 \quad \ldots (1) \quad$ or $\quad 5m + c = 5 \quad \ldots(2)$

Solving (1) and (2), we get m = 2, c = –5.

Hence the equation of the straight line is

$$y = 2x - 5$$

Ex. 2 : *Fit a parabola of the form $y = ax^2 + bx + c$ to the following data using least square criteria.*

x	1	2	3	4	5	6	7
y	–5	–2	5	16	31	50	73

Sol. :

x	y	X = x – 4	X^2	X^3	X^4	Xy	X^2y
1	–5	–3	9	–27	81	15	–45
2	–2	–2	4	–8	16	4	–8
3	5	–1	1	–1	1	–5	5
4	16	0	0	0	0	0	0
5	31	1	1	1	1	31	31
6	50	2	4	8	16	100	200
7	73	3	9	27	81	219	657
	$\sum y = 168$	$\sum X = 0$	$\sum X^2 = 28$	$\sum X^3 = 0$	$\sum X^4 = 196$	$\sum Xy = 364$	$\sum X^2y = 840$

n = 7.

Substituting in equations (5), (6) and (7) of (5.6.3)

$$28a + 0b + 7c = 168 \qquad \ldots (1)$$

$$a \cdot 0 + 28b + c0 = 364 \qquad \ldots (2)$$

$$196a + 0 \cdot b + 28c = 840 \qquad \ldots (3)$$

(1), (2), (3) can be written as

$$4a + 0 \cdot b + c = 24 \qquad \ldots (4)$$

$$a \cdot 0 + b + c \cdot 0 = 13 \qquad \ldots (5)$$

$$7a + 0 \cdot b + c = 30 \qquad \ldots (6)$$

From (5) b = 13 and from (4) and (5) we get,

$$a = 2, \; c = 16$$

Equation of parabola in terms of variable X is

$$y = 2X^2 + 13X + 16$$

Putting X = x – 4 $\qquad y = 2(x-4)^2 + 14(x-4) + 16$

$$y = 2x^2 - 3x - 4$$

is the required fit for the data.

Ex. 3 : *A simply supported beam carries a concentrated load P(kg) its middle point. Corresponding to various values of P, the maximum deflection y cms is tabulated as :*

P	100	120	140	160	180	200
Y	0.90	1.10	1.20	1.40	1.60	1.70

Find a law of the form $y = aP + b$ by using least square criteria.

Sol. : Preparing the table as

P(x)	y	X = P – 140	X^2	Xy
100	0.90	– 40	1600	–36
120	1.10	– 20	400	–22
140	1.20	0	0	0
160	1.40	20	400	28
180	1.60	40	1600	64
200	1.70	60	3600	102
	$\sum y = 7.9$	$\sum X = 60$	$\sum X^2 = 7600$	$\sum xy = 136$

n = 6 (No. of points)

From (1) and (2) of (5.6.2)

$$60a + 6b = 7.9 \quad \dots (1)$$

$$7600a + 60b = 136 \quad \dots (2)$$

Solving (1) and (2) we get,

$$a = 0.008143 \qquad b = 1.2352$$

$\therefore$ $$y = 0.008143\, X + 1.2352$$

but $$X = P - 140$$

$$y = 0.008143\, (P - 140) + 1.2352$$

$$y = 0.008143\, P + 0.9518$$

is the required result.

Ex. 4 : Values of x and y are tabulated as under :

x	1	1.5	2.0	2.5
y	25	56.2	100	156

Find the law of the form $x = ay^n$ to satisfy the given by data

Sol. : Taking logarithms, we get,

$$\log x = \log a + n \log y$$

which can be written as,

$$X = nY + c$$

where X = log x; Y = log y.

x	y	X	Y	Y^2	XY
1.0	25	0.0	1.3979	1.9541	0
1.5	56.2	0.1761	1.7497	3.0615	0.3081
2.0	100	0.301	2.0	4.0	0.602
2.5	156	0.3979	2.1931	4.8097	0.8726
		0.875	7.3407	13.8253	1.7827

Substituting in (1) and (2) of (5.6.1) where x is replaced by Y and y by X, a by n, b by log a = c. n in (1) of (6.1) = 4 (No. of points)

$$7.3407n + 4c = 0.875 \quad \dots (1)$$

$$13.8253n + 7.3407c = 1.7827 \quad \dots (2)$$

Solving (1) and (2) we get,

$$n = 0.5, \quad c = \log a = -0.6988375 \quad \therefore \quad a = 0.2$$

Hence required law of the form $x = ay^n$ is $x = 0.2\, y^{0.5}$.

Ex. 5 : *Following is the data given for values of X and Y. Fit a second degree polynomial of the type $ax^2 + bx + c$ where a, b, c are constants.*

X	–3	–2	–1	0	1	2	3
Y	12	4	1	2	7	15	30

Sol. : For best fitted parabola a, b and c should satisfy equation.

$$a \sum x^4 + b \sum x^3 + c \sum x^2 = \sum x^2 y$$
$$a \sum x^3 + b \sum x^2 + c \sum x = \sum xy$$
$$a \sum x^2 + b \sum x + cn = \sum y$$

where, n = number of point = 7

x	y	xy	x^2	x^2y	x^3	x^4
–3	12	–36	9	108	–27	81
–2	4	–8	4	16	–8	16
–1	1	–1	1	1	–1	1
0	2	0	0	0	0	0
1	7	7	1	7	1	1
2	15	30	4	60	8	16
3	30	90	9	270	27	81
$\sum x$	$\sum y$	$\sum xy$	$\sum x^2$	$\sum x^2y^2$	$\sum x^3$	$\sum x^4$
0	71	82	28	462	0	196

Substituting we get,

$$196a + 0b + 28c = 462 \qquad \dots \text{(i)}$$
$$0a + 28b + 0c = 82 \qquad \dots \text{(ii)}$$
$$28a + 0b + 7c = 71 \qquad \dots \text{(iii)}$$

Solving (i) and (iii) simultaneously,

$$196a + 0b + 28c = 462 \qquad \dots \text{(i)}$$
$$28a + 0b + 7c = 71 \qquad \dots \text{(ii)}$$
$$a = 2.119 \approx 2.12$$
$$c = 1.66$$

From equation (ii), we get b = 2.92.

Now solving equations (i) and (iii) simultaneously,

$$196a + 0b + 28c = 362$$

and we get, $a = 2.119$

$c = 1.00$

Hence, a = 2.12, b = 2.92, c = 1.66.

Ex. 6 : *Fit a curve $y = ax^b$ using the following data :*

x	2000	3000	4000	5000	6000
y	15	15.5	16	17	18

Find out values of a and b.

Sol. : Given curve : $y = ax^b$

Taking logarithms we get,

$$\log y = \log a + b \log x$$

which can be written as, $Y = bX + C$

where, $Y = \log y; \quad X = \log x$

By least square regression we get,

$$\sum Y = b \sum X + Cn \quad \dots (1)$$

$$\sum XY = b \sum X^2 + C \sum X \quad \dots (2)$$

∴ From given data :

x	y	X = log x	Y = log y	X^2	XY
2000	15	3.30103	1.17609	10.8967	3.88498
3000	15.5	3.47712	1.19033	12.09037	4.13892
4000	16	3.60206	1.20412	12.97483	4.33731
5000	17	3.69897	1.23044	13.682379	4.55139
6000	18	3.77815	1.2552	14.27442	4.7426
		$\sum X = 17.85$	$\sum Y = 6.056$	$\sum X^2 = 63.9188$	$\sum XY = 21.652$

Putting these values in equation (1) and (2)

$$6.056 = b * 17.85 + 5 * C \quad \dots (3)$$

$$21.625 = b * 63.9188 + 17.85 * C \quad \dots (4)$$

Multiplying (3) by 17.85 and (4) and 5

$$108.0996 = b * 318.6225 + 89.25 * C$$

$$108.125 = b * 319.594 + 89.25 * C$$

∴ Solving these equation we get,

$$b = 0.026, \; C = 1.117 \text{ but } C = \log a$$

∴ $$a = 13.09$$

Ex. 7 : *Given the table of points :*

x	0	2	4	6	8	12	20
y	10	12	18	22	20	30	30

Use least square method to fit a straight line to the data and find the value of y (22).

Sol. : We fit straight line y = ax + b for n = 7 points.

The various summations are as follow :

x	y	x^2	xy
0	10	0	0
2	12	4	24
4	18	16	72
6	22	36	132
8	20	64	160
12	30	144	360
20	30	400	600
$\sum x = 52$	$\sum y = 142$	$\sum x^2 = 664$	$\sum xy = 1348$

From (1) and (2) of 5.6.2 we obtain

$$52a + 7b = 142 \quad \dots(1)$$

$$664a + 52\,b = 1348 \quad \dots(2)$$

Solving (1) and (2),

$$a = 1.0555 \quad \text{and} \quad b = 12.4485$$

Therefore the straight line equation y = ax + b is

$$y = 1.0555x + 12.4485$$

Alternatively, we can also find the values of a and b in y = ax + b on solving (1) and (2 of 5.6.2).

$$a = \frac{\sum x \sum y - n \sum xy}{(\sum x)^2 - n\sum x^2} \text{ and } b = \frac{\sum y}{n} - a\frac{\sum x}{n}$$

Thus,
$$a = \frac{(52) \times (142) - 7\ (1348)}{(52)^2 - 7\ (664)} = 1.0555$$

and
$$b = \frac{142}{7} - 1.0555\,\frac{52}{7} = 12.4485$$

Ex. 8 : *For the tabulated values of x and y given below fit a linear curve of the type y = mx + c.*

x	1.0	3.0	5.0	7.0	9.0
y	1.5	2.8	4.0	4.7	6.0

Sol. : Here n = 5 . The various summations are as follows :

x	**y**	**x^2**	**xy**
1.0	1.5	1	1.5
3.0	2.8	9.	8.4
5.0	4.0	25	20.0
7.0	4.7	49	32.9
9.0	6.0	81	54.0
$\sum x = 25$	$\sum y = 19$	$\sum x^2 = 165$	$\sum xy = 116.8$

From (1) and (2) of 5.6.2 ,we obtain

$$25a + 5b = 19 \qquad \ldots(1)$$
$$165a + 25b = 116.8 \qquad \ldots(2)$$

Solving (1) and (2),

$$a = 0.545 \text{ and } b = 1.075$$

Therefore the equation of straight line y = ax + b is

$$y = 0.545\,x + 1.075$$

Alternatively,
$$a = \frac{\sum x \sum y - n\sum xy}{(\sum x)^2 - n\sum x^2} = 0.545$$
$$b = \frac{\sum y}{n} - a\,\frac{\sum x}{n}$$
$$b = 1.075$$

Ex. 9 : *Following data refers to the load lifted and corresponding force applied in a pulley system. If the load lifted and effort required are related by equation effort = a * (load lifted) b, where a and b are constants. Evaluate a and b by linear curve fitting.*

Load lifted in kN	10	15.0	20.0	25.0	30.0
Effort applied in kN	0.750	0.935	1.100	1.200	1.300

Sol. : Here n = 5 preparing table as,

Sr. No.	**Load x_i**	**Effort y_i**	**x_iy_i**	**x_i^2**
1	10	0.75	7.5	100
2	15	0.935	14.025	225
3	20	1.1	22	400
4	25	1.2	30	625
5	30	1.3	39	900
	$\sum x_i = 90$	$\sum y_i = 5285$	$\sum x_iy_i = 112.525$	$\sum x_i^2 = 2250$

From (1) and (2) of 5.6.2, we obtain

$$90a + 5b = 5.285 \qquad \ldots(1)$$
$$2250a + 90b = 112.525 \qquad \ldots(2)$$

Solving (1) and (2), $a = 0.02761$ and $b = 0.56$

Effort = 0.02761 (load lifted) + 0.56

Ex. 10 : *If the relation between x and y is of the type $y = a\,b^x$. Using following values of x and y, find he values of constants a and b for the best fitting curve.*

x	2.1	2.5	3.1	3.5	4.1
y	5.14	6.788	10.29	13.58	20.578

Sol. : $y = a \cdot b^x$

Taking logarithms, $\log y = \log a + x \log b$

$\therefore \quad Y = C + Bx$

where, $Y = \log y$, $C = \log a$, $B = \log b$

By least square regression we get

$$\sum Y = nC + B \sum x \quad \dots (1)$$

and $$\sum xY = (\sum x)C + B \sum x^2 \quad \dots (2)$$

x	**y**	**Y = log y**	**x^2**	**xY**
2.1	5.14	0.71096	4.41	1.493016
2.5	6.788	0.83174	6.25	2.07935
3.1	10.29	1.01241	9.61	3.138471
3.5	13.58	1.13289	12.25	3.965115
4.1	20.578	1.3134	16.81	5.38494
$\sum x = 15.3$		$\therefore \sum Y = 5.0014$	$\sum x^2 = 49.33$	$\sum xY = 16.060892$

Substituting in equations (1) and (2)

$$5.0014 = 5C + B\,(15.3) \quad \dots (3)$$

$$16.060892 = 15.3C + B\,(49.33) \quad \dots (4)$$

Solving (3) and (4) $B = 0.301197$ and $C = 0.00786157$

$$B = \log b$$

$$\therefore \quad b = 10^B = 10^{0.3011974} = 2$$

$$C = \log a$$

$$\therefore \quad a = 10^C = 10^{0.0786157} = 1.1984386$$

Therefore the required relation $y = ab^x$ is $y = (1.1984386)\,2^x$.

Ex. 11 : *If X and Y are connected by the relation $ax^2 + by^2 = X$, find the value of a and b by rearranging the relation into linear form by using least square criteria for following data :*

x	1	2	3	4	5
y	3.35	5.92	8.43	10.93	13.45

Sol. : Function to be fitted is,

$$ax^2 + by^2 = x$$

Deviation is, $$D = ax^2 + by^2 - x$$

Sum of square of deviations is,

$$D^2 = \sum (ax^2 + by^2 - x)^2$$

Differentiating with respect to a and equate to zero,

$$\frac{\partial D^2}{\partial a} = \sum 2\,(ax^2 + by^2 - x)\,x^2 = 0$$

$\dfrac{\partial D^2}{\partial a} = 0$ gives,

$$\sum ax^4 + b\sum x^2y^2 - \sum x^3 = 0 \quad \ldots(1)$$

Similarly, differentiating with respect to b and equation to zero gives,

$$\frac{\partial D^2}{\partial b} = \sum 2\,(ax^2 + by^2 - x)\,y^2 = 0$$

or $\qquad \sum (ax^2y^2 + by^4 - xy^2) = 0$

$\frac{\partial D^2}{\partial b} = 0$ gives,

$$a\sum x^2y^2 + b\sum y^4 - \sum xy^2 = 0 \quad \ldots(2)$$

Here n = 5. Various summations are as follows :

x	**y**	**x^4**	**x^2y^2**	**x^3**	**y^4**	**xy^2**
1	3.35	1	11.22	1	125.94	11.22
2	5.92	16	140.19	8	1228.25	70.095
3	8.43	81	639.58	27	5050.22	213.193
4	10.93	256	1911.44	64	14271.86	477.86
5	13.45	625	4522.56	125	32725.72	904.512
	$\sum$	$\sum x^4 = 979$	$\sum x^2y^2 = 7224.99$	$\sum x^3 = 224$	$\sum y^4 =$ 53401.99	$\sum xy^2 =$ 1676.88

Using in equations (1) and (2), we obtain

$$979\,(a) + 7224.99\,(b) - 224 = 9$$
$$979a + 7224.99b = 224$$
$$a + 7.37997b = 0.2288 \quad \ldots(3)$$
$$\text{and}\;7224.99a + 53401.99b = 1676.88$$
$$a + 7.391289b = 0.232094 \quad \ldots(4)$$

Solving (3) and (4)

$$0.011319b = 0.0032944$$
$$b = 0.29105$$

Using value of b in equation (3)

$$a + 7.37997 \times 0.29105 = 0.2288$$
$$a = -1.91914$$

$\therefore$ Equation of the required curve is, $-1.91914x^2 + 0.29105y^2 = x$

5.8 CORRELATION

We have already considered distributions involving one variable or what we call as univariate distributions. In many problems of practical nature, we are required to deal with two or more variables. If we consider the marks obtained by a group of students in two or more subjects, the distribution will involve two or more variables. Distributions using two variables are called *Bivariate distributions*. In such distributions, we are often interested in knowing whether there exists some kind of relationship between the two variables involved. In language of statistics, this means whether there is correlation or co-variance between the two variables. If the change in one variable affects the change in the other variable, the variables are said to be ***correlated***. For example, change in rainfall will affect the crop output and thus the variables 'Rainfall recorded' and 'crop output' are correlated. Similarly, for a group of workers, the variables 'income' and 'expenditure' would be correlated. If the increase (or decrease) in one variable causes corresponding increase (or decrease) in the other, the correlation is said to be ***positive*** or ***direct***. On the other hand, if increase in the value of one variable shows a corresponding decrease in the value of the other or vice versa, the correlation is called ***negative*** or ***inverse***. As the income of a worker increases, as a natural course his expenditure also increases, hence the correlation between income and expenditure is positive or direct. Correlation between heights and weights of a group of students will also be positive. If we consider the price and demand of a certain commodity then our experience tells us that as the price of a commodity rises, its demand falls and thus the correlation between these variables is negative or inverse. Several such examples can be given. Correlation can also be classified as linear and non-linear. It is based upon the constancy of the ratio of change between the two variables. As an example, consider the values assumed by variables x and y.

x	5	8	11	15	17	19	20
y	10	16	22	30	34	38	40

Here the ratio $\frac{y}{x}$ is equal to 2 for all the values of x and y.

Correlation in such case is called *linear*.

When the amount of change in one variable is not in a constant ratio to the amount of change in other variable, the correlation is called *non-linear*. In such a case, the relationship between the variables x and y is not of the form y = mx (or of the form y = mx + c). In practical situations, the correlation is generally non-linear, but its analysis is quite complicated. Usually, it is assumed that the relation between x and y is linear and further analysis is made. There are different methods to determine whether the two variables are correlated. Some of these methods such as 'Scatter Diagram' are graphical methods and give rough idea about the correlation. These methods are not suitable if the number of observations is large. There are mathematical methods such as '*Karl Pearson*'s Coefficient of Correlation', 'Concurrent Deviation Method' etc. which are more suitable. We shall discuss '*Karl Pearson*'s Coefficient of Correlation' which is widely used in practice.

5.9 KARL PEARSON'S COEFFICIENT OF CORRELATION

To measure the intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called *correlation coefficient*.

Correlation coefficient between two variables x and y denoted by r (x, y) is defined as

$$r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

In bivariate distribution if (x_i, y_i) take the values (x_1, y_1) (x_2, y_2) ... (x_n, y_n)

$$\text{cov}(x, y) = \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y})$$

where, $\bar{x}$, $\bar{y}$ are arithmetic means for x and y series respectively.

Similarly, $$\sigma_x = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2} \quad \text{and} \quad \sigma_y = \sqrt{\frac{1}{n} \sum (y_i - \bar{y})^2}$$

which are the standard deviations for x and y series.

$$\begin{aligned}
\text{cov}(x, y) &= \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y}) \\
&= \frac{1}{n} \sum (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y}) \\
&= \frac{1}{n} \sum x_i y_i - \bar{y} \frac{1}{n} \sum x_i - \bar{x} \frac{1}{n} \sum y_i + \frac{1}{n} \sum \bar{x}\bar{y} \\
&= \frac{1}{n} \sum x_i y_i - \bar{y}\bar{x} - \bar{x}\bar{y} + \frac{1}{n}(n\bar{x}\bar{y}) \\
&= \frac{1}{n} \sum x_i y_i - \bar{x}\bar{y}
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= \frac{1}{n} \sum (x_i - \bar{x})^2 = \frac{1}{n} \sum x_i^2 - 2x_i \bar{x} + \bar{x}^2 \\
&= \frac{1}{n} \sum x_i^2 - 2\frac{\bar{x}}{n} \sum x_i + \frac{1}{n} \sum \bar{x}^2 \\
&= \frac{1}{n} \sum x_i^2 - 2\bar{x}^2 + \frac{1}{n}(n\bar{x}^2) \\
&= \frac{1}{n} \sum x_i^2 - \bar{x}^2
\end{aligned}$$

Similarly, $\sigma_y^2 = \frac{1}{n} \sum y_i^2 - \bar{y}^2$

$r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$ can then be calculated.

If we put, $u_i = x_i - A$ or $\frac{x_i - A}{h}$ and $v_i = y_i - B$ or $\frac{y_i - B}{k}$

then $\text{cov}(u, v) = \frac{1}{n} \sum u_i v_i - \bar{u}\,\bar{v}$, $\sigma_u^2 = \frac{1}{n} \sum u_i^2 - \bar{u}^2$, $\sigma_v^2 = \frac{1}{n} \sum v_i^2 - \bar{v}^2$

r (u, v) is given by, $r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v}$

It can be established that r (x, y) = r (u, v). Calculation of r (u, v) is simpler as compared to r (x, y).

Correlation coefficient 'r' always lies between –1 and 1 i.e. $-1 \leq r \leq 1$.

If r > 0, the correlation is positive and if r < 0, the correlation is negative. If r = 0, we say the variables are uncorrelated. In general, if |r| > 0.8, we consider high correlation. If |r| is between 0.3 to 0.8, we say that, correlation is considerable. If |r| < 0.3, we say that correlation is negligible. If r = 1, we say that there is perfect positive correlation; where as if r = –1, we say that there is perfect negative correlation.

We also, note that covariance can also be considered as a joint central moment of order (1, 1) of (X, Y). Hence, we denote μ_{11} = cov (X, Y).

ILLUSTRATIONS

Ex. 1 : *Following are the values of import of raw material and export of finished product in suitable units.*

Export	*10*	*11*	*14*	*14*	*20*	*22*	*16*	*12*	*15*	*13*
Import	*12*	*14*	*15*	*16*	*21*	*26*	*21*	*15*	*16*	*14*

Calculate the coefficient of correlation between the import values and export values.

Sol. : Let X : Quantity exported, Y : Quantity imported, Preparing table as follows calculations can be made simple.

x	y	x^2	y^2	xy
10	12	100	144	120
11	14	121	196	154
14	15	196	225	210
14	16	196	256	224
20	21	400	441	420
22	26	484	676	572
16	21	256	441	336
12	15	144	225	180
15	16	225	256	240
13	14	169	196	182
Total = 147	**170**	**2291**	**3056**	**2638**

Here, n = 10, hence $\bar{x} = \frac{\sum x}{N} = \frac{147}{10} = 14.7$

and $\bar{y} = \frac{\sum y}{N} = \frac{170}{10} = 17$

$$r = \frac{\sum x y - n\,\bar{x}\,\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2) \times (\sum y^2 - n\bar{y}^2)}}$$

$$= \frac{2638 - 10 \times 14.7 \times 17}{\sqrt{(2291 - 10 \times 14.7^2)\,(3056 - 10 \times 17^2)}}$$

$$= \frac{139}{\sqrt{130.1 \times 166}} = 0.9458$$

Ex. 2 : *Calculate the correlation coefficient for the following weights (in kg) of husband (x) and wife (y).* ***(Dec. 2012)***

x	65	66	67	67	68	69	70	72
y	55	58	72	55	66	71	70	50

Sol. :

x	y	x^2	y^2	xy
65	55	4225	3025	3575
66	58	4356	3364	3828
67	72	4489	5184	4824
67	55	4489	3025	3685
68	66	4624	4354	4488
69	71	4761	5041	4899
70	70	4900	4900	4900
72	50	5184	2500	3600
544	**497**	**37028**	**31393**	**33799**

$$\bar{x} = \frac{\sum x}{n} = \frac{544}{8} = 68$$

$$\bar{y} = \frac{\sum y}{n} = \frac{497}{8} = 62.125$$

Correlation coefficient between x and y is given by

$$r(x, y) = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{\frac{1}{n}\sum xy - \bar{x}\,\bar{y}}{\sqrt{\left(\frac{1}{n}\sum x^2 - (\bar{x})^2\right)\left(\frac{1}{n}\sum y^2 - (\bar{y})^2\right)}}$$

$$= \frac{\frac{1}{8}(33799) - 68\,(62.125)}{\sqrt{\left(\frac{37028}{8} - (68)^2\right)\left(\frac{31398}{8} - (62.125)^2\right)}}$$

$$= \frac{4224.875 - 4224.5}{\sqrt{(4628.5 - 4624)\,(3924.125 - 3859.52)}}$$

$$= \frac{0.375}{\sqrt{4.5 \times 64.605}} = \frac{0.375}{\sqrt{290.7225}} = \frac{0.375}{17.051}$$

$$r(x, y) = 0.022$$

Ex. 3 : *From a group of 10 students, marks obtained by each in papers of Mathematics and Applied Mechanics are given as :*

x Marks in Maths	23	28	42	17	26	35	29	37	16	46
y Marks in App. Mech.	25	22	38	21	27	39	24	32	18	44

Calculate Karl Pearson's Coefficient of correlation.

Sol. : The data is tabulated as :

x	y	u = x – 35	v = y – 39	u^2	v^2	uv
16	18	– 19	– 21	361	441	399
17	21	– 18	– 18	324	324	324
23	25	– 12	– 14	144	196	168
26	27	– 09	– 12	81	144	108
28	22	– 07	– 17	49	289	119
29	24	– 06	– 15	36	225	90
35	39	– 00	00	00	00	00
37	32	02	– 07	04	49	– 14
42	38	07	– 01	49	01	– 07
46	44	11	05	121	25	55
Total		$\sum u = -51$	$\sum v = -100$	$\sum u^2 = 1169$	$\sum v^2 = 1694$	$\sum uv = 1242$

$$\bar{u} = \frac{-51}{10} = -5.1, \quad \bar{u}^2 = 26.01$$

$$\bar{v} = \frac{-100}{10} = -10, \quad \bar{v}^2 = 100$$

$$\text{cov}(u, v) = \frac{1}{n}\sum u_i v_i - \bar{u}\,\bar{v} = \frac{1}{10}(1242) - 51 = 73.2$$

$$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - \bar{u}^2 = \frac{1169}{10} - 26.01 = 90.89$$

$$\sigma_u = \sqrt{90.89} = 9.534$$

$$\sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{v}^2 = \frac{1694}{10} - 100 = 69.4$$

$$\sigma_v = \sqrt{69.4} = 8.33$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u\,\sigma_v} = \frac{73.2}{9.534 \times 8.33} = 0.9217$$

Ex. 4 : *Compute correlation coefficient between supply and price of commodity using following data.*

Supply	*152*	*158*	*169*	*182*	*160*	*166*	*182*
Price	*198*	*178*	*167*	*152*	*180*	*170*	*162*

Sol. : Let x = Supply, u = x – 150, y = price, v = y – 160

x	y	u	v	u^2	v^2	uv
152	198	2	38	4	1444	76
158	178	8	18	64	324	144
169	167	19	7	361	49	133
182	152	32	– 8	1024	64	– 256
160	180	10	20	100	400	200
166	170	16	10	256	100	160
182	162	32	2	1024	4	64
Total	**–**	**119**	**87**	**2833**	**2385**	**521**

Here, $n = 7, \sum u = 119, \sum v = 87, \sum u^2 = 2833, \sum v^2 = 2385, \sum uv = 521$

$\therefore$ $$\bar{u} = 17, \quad \bar{v} = 12.4286$$

$$r = \frac{\sum uv - n\,\bar{u}\,\bar{v}}{\sqrt{\left(\sum u^2 - n\bar{u}^2\right) \times \left(\sum v^2 - n\bar{v}^2\right)}}$$

$$r = \frac{521 - 7 \times 17 \times 12.4286}{\sqrt{(2833 - 7 \times 17^2)\,(2.385 - 7 \times 12.4286)^2}}$$

$$r = \frac{-958}{\sqrt{810 \times 1303.7142}} = \frac{-958}{1027.6227}$$

$$= -0.9322$$

Interpretation : There is high negative correlation between supply and price.

Ex. 5 : *Obtain correlation coefficient between population density (per square miles) and death rate (per thousand persons) from data related to 5 cities.* ***(Dec. 2010, 2017; May 2010)***

Population Density	*200*	*500*	*400*	*700*	*800*
Death Rate	*12*	*18*	*16*	*21*	*10*

Sol. : Let x = Population density and y = Death rate.

Let, $u = x - a$ and $v = y - b$

$= x - 500$ $= y - 15$

x	y	u = x – 500	v	u^2	v^2	uv
200	12	– 300	– 3	90000	9	900
500	18	0	3	0	9	0
400	16	– 100	1	10000	1	– 100
700	21	200	6	40000	36	1200
800	10	300	– 5	90000	25	– 1500
Total	**–**	**100**	**2**	**230000**	**80**	**500**

Here, $n = 5$, $\sum u = 100$, $\sum v = 2$, $\sum u^2 = 230000$, $\sum v = 80$, $\sum uv = 500$

$$\bar{u} = \frac{\sum u}{n} = \frac{100}{5} = 20$$

$$\bar{v} = \frac{\sum v}{n} = \frac{2}{5} = 0.4$$

$$r(u, v) = \frac{\sum uv - n\,\bar{u}\,\bar{v}}{\sqrt{\left[\sum u^2 - n(\bar{u})^2\right]\left[\sum v^2 - n(\bar{v})^2\right]}}$$

$$= \frac{500 - 5(20)(0.4)}{\sqrt{230000 - 5(20)^2}\sqrt{80 - 5(0.4)^2}}$$

$$= \frac{460}{\sqrt{228000}\sqrt{79.2}}$$

$$= \frac{460}{4249.42} = 0.1082$$

Ex. 6 : *Calculate the coefficient of correlation for the following distribution.* ***(Dec. 2006)***

x	*5*	*9*	*15*	*19*	*24*	*28*	*32*
y	*7*	*9*	*14*	*21*	*23*	*29*	*30*
f	*6*	*9*	*13*	*20*	*16*	*11*	*7*

Sol. : Tabulating the data as

x	y	f	u = x–19	v = y–21	fu	fv	fu^2	fv^2	fuv
5	7	6	– 14	– 14	– 84	– 84	1176	1176	1176
9	9	9	– 10	– 12	– 90	– 108	900	1296	1080
15	14	13	– 4	– 7	– 52	– 91	208	637	364
19	21	20	0	0	0	0	0	0	0
24	23	16	5	2	80	32	400	64	160
28	29	11	9	8	99	88	891	704	792
32	30	7	13	9	91	63	1183	567	819
Total		**$\sum f = 82$**			**$\sum fu = 44$**	**$\sum fv = -100$**	**$\sum fu^2 = 4758$**	**$\sum fv^2 = 4444$**	**$\sum fuv = 4391$**

$$\bar{u} = \frac{\sum fu}{\sum f} = \frac{44}{82} = 0.5366; \quad \bar{u}^2 = 0.288$$

$$\bar{v} = \frac{\sum fv}{\sum f} = \frac{-100}{82} = 1.2195; \quad \bar{v}^2 = 1.4872$$

$$\text{cov}(u, v) = \frac{1}{\sum f} \sum f u_i v_i - \bar{u}\,\bar{v} = \frac{4391}{82} - 0.6544 = 52.89$$

$$\sigma_u^2 = \frac{1}{\sum f} \sum f u_i^2 - \bar{u}^2 = \frac{758}{82} - 0.288 = 57.7364$$

$$\sigma_v^2 = \frac{1}{\sum f} \sum f v_i^2 - \bar{v}^2 = \frac{4444}{82} - 1.4872 = 52.708$$

$$\sigma_u = 7.598$$

$$\sigma_v = 7.26$$

$$r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v} = \frac{52.89}{55.16} = 0.9588$$

$\therefore$ Coefficient of correlation $= r(x, y) = 0.9588$

Ex. 7: *Find correlation coefficient between X and Y, given that, $n = 25, \sum x = 75, \sum y = 100, \sum x^2 = 250, \sum y^2 = 500, \sum xy = 325$.*

Sol. : Here $\bar{x} = \frac{75}{25} = 3$, $\bar{y} = \frac{100}{25} = 4$.

$$\therefore \quad r = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2) \times \sum(y^2 - n\bar{y}^2)}}$$

$$r = \frac{325 - 25 \times 3 \times 4}{\sqrt{(250 - 25 \times 9)\,(500 - 25 \times 16)}} = \frac{25}{\sqrt{25 \times 100}} = \frac{25}{50} = 0.5$$

Ex. 8 : *Calculate the coefficient of correlation from the following information.*

$n = 10$, $\sum x = 40$, $\sum x^2 = 190$, $\sum y^2 = 200$, $\sum xy = 150$, $\sum y = 40$.

Sol. : Here,

$$\bar{x} = \frac{\sum x}{n} = \frac{40}{10} = 4$$

$$\bar{y} = \frac{\sum y}{n} = \frac{40}{10} = 4$$

Coefficient of correlation is given by,

$$r = \frac{\sum xy - n\,\bar{x}\,\bar{y}}{\sqrt{(\sum x^2 - n\,(\bar{x})^2) \times (\sum y^2 - n\,(\bar{y})^2)}}$$

$$r = \frac{150 - 10\,(4)\,(4)}{\sqrt{190 - 10\,(4)^2}\,\sqrt{200 - 10\,(4)^2}}$$

$$= \frac{150 - 160}{\sqrt{30}\,\sqrt{40}} = \frac{-10}{34.6410} = -0.2850$$

Ex. 9 : *Given : $n = 6$, $\sum (x - 18.5) = -3$, $\sum (y - 50) = 20$, $\sum (x - 18.5)^2 = 19$, $\sum (y - 50)^2 = 850$, $\sum (x - 18.5)(y - 50) = -120$. Calculate coefficient of correlation.*

Sol. : Let $u = x - 18.5$ and $v = y - 50$

$$\bar{u} = \frac{-3}{6} = -0.5$$

and

$$\bar{v} = \frac{20}{6} = 3.33$$

From the given data $\sum u = -3$, $\sum v = 20$, $\sum u^2 = 19$, $\sum v^2 = 850$ and $\sum uv = -120$

Coefficient of correlation is given by

$$r = \frac{\sum uv - n\,\bar{u}\,\bar{v}}{\sqrt{[\sum u^2 - n\,(\bar{u})^2] \times [\sum v^2 - n\,(\bar{v})^2]}}$$

$$= \frac{-120 - 6\,(-0.5) \times (3.33)}{\sqrt{[19 - 6\,(-0.5)^2] \times [850 - 6\,(3.33)^2]}}$$

$$= \frac{-120 + 9.99}{\sqrt{(17.5)\ (783.47)}} = \frac{-110.01}{117.0928}$$

$$r = -0.9395$$

Ex. 10 : *Given : r = 0.9, $\sum XY = 70$, $\sigma_y = 3.5$, $\sum X^2 = 100$.*

Find the number of items, if X and Y are deviations from arithmetic mean.

Sol. :

$$\sum X^2 = 100 \qquad \sum XY = 70$$

$$r = 0.9 \qquad \sigma_y = 3.5$$

We have to find the value of n

$$\sigma_x^2 = \frac{1}{n} \sum (x - \bar{x})^2 = \frac{1}{n} \sum X^2 = \frac{100}{n}$$

$$r = \frac{\sum (x - \bar{x})\ (y - \bar{y})}{n\ \sigma_x\ \sigma_y} = \frac{\sum XY}{n\ \sigma_x\ \sigma_y}$$

Squaring we get,

$$r^2 = \frac{(\sum XY)^2}{n^2\ \sigma_x^2\ \sigma_y^2}$$

$$(0.9)^2 = \frac{(70)^2}{n^2 \times \left(\frac{100}{n}\right) \times (3.5)^2}$$

$$0.81 = \frac{4900}{1225 \cdot n}$$

$$0.81 \times 1225\ n = 4900$$

$$992.25\ n = 4900$$

$$n = 4.9383$$

$$n \approx 5$$

Ex. 11 : *Find the coefficient of correlation for distribution in which S.D. of x = 4, and S.D. of y = 1.8. Coefficient of regression of y on x is 0.32.*

Sol. : $\sigma_x = 4$, $\sigma_y = 1.8$ and $b_{yx} = 0.32$

We have,

$$b_{yx} = r \frac{\sigma_y}{\sigma_x}$$

$$0.32 = r \times \frac{1.8}{4}$$

$$\therefore \quad r = \frac{0.32 \times 4}{1.8} = 0.711$$

5.10 REGRESSION

After having established that the two variables are correlated, we are generally interested in estimating the value of one variable for a given value of the other variable.

For example, if we know that rainfall affects the crop output then it is possible to predict the crop output at the end of a rainy season. If the variables in a bivariate distribution are related, the points in scatter diagram cluster round some curve called the curve of regression or the regression curve. If the curve is a straight line, it is called the ***line of regression*** and in such case the regression between two variables is linear. The line of regression gives best estimate for the value of one variable for some specified value of the other variable. If correlation is not perfect (i.e. $r \neq \pm 1$), then several lines can be drawn through given points. Being the line of best fit, the regression line is obtained by using the method of least squares.

Consider the set of values of (x_i, y_i), i = 1, 2, ... n. Let the line of regression of y on x be $y = mx + c$

From the method of least squares, the normal equations for estimating unknown m and c are given by

$$\sum y_i = nc + m \sum x_i \qquad \text{... (1)}$$

$$\sum x_i\ y_i = c \sum x_i + m \sum x_i^2 \qquad \text{... (2)}$$

Dividing (1) by n, we get

$$\frac{1}{n}\sum y_i = c + m\left(\frac{1}{n}\sum x_i\right)$$

i.e.
$$\bar{y} = c + m\bar{x} \qquad \text{...(3)}$$

which shows that the point $(\bar{x}, \bar{y})$ lies on the line of regression.

We know that,
$$\mu_{11} = \text{cov}(x, y) = \frac{1}{n}\sum x_i y_i - \bar{x}\bar{y}$$

$$\therefore \quad \frac{1}{n}\sum x_i y_i = \mu_{11} + \bar{x}\bar{y} \qquad \text{... (4)}$$

Also
$$\sigma_x^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$\therefore \quad \frac{1}{n}\sum x_i^2 = \sigma_x^2 + \bar{x}^2 \qquad \text{...(5)}$$

Dividing (2) by n, we get

$$\frac{1}{n}\sum x_i y_i = c\frac{\sum x_i}{n} + m\frac{\sum x_i^2}{n} \qquad \text{... (6)}$$

Substituting from (4) and (5) in (6),

$$\mu_{11} + \bar{x}\bar{y} = c\bar{x} + m\left(\sigma_x^2 + \bar{x}^2\right) \qquad \text{... (7)}$$

Multiplying (3) by $\bar{x}$ and subtracting from (7), we get

$$\mu_{11} = m\,\sigma_x^2$$

$$m = \frac{\mu_{11}}{\sigma_x^2}$$

Equation of regression line which passes through $(\bar{x}, \bar{y})$ and which has slope $\frac{\mu_{11}}{\sigma_x^2}$ is thus given by the equation

$$y - \bar{y} = \frac{\mu_{11}}{\sigma_x^2}(x - \bar{x}) \qquad \text{... (8)}$$

This equation gives regression line of y on x.

Similarly, if we start with regression line of x on y as x = my + c same procedure will give

$$x - \bar{x} = \frac{\mu_{11}}{\sigma_y^2}(y - \bar{y}) \qquad \text{... (9)}$$

Also, we know that
$$r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{\mu_{11}}{\sigma_x \sigma_y}$$

Putting for μ_{11} in (8) and (9), we have

$$y - \bar{y} = \frac{r\,\sigma_x \sigma_y}{\sigma_x^2}(x - \bar{x}) \quad \text{and} \quad x - \bar{x} = \frac{r\,\sigma_x \sigma_y}{\sigma_x^2}(y - \bar{y})$$

Thus, the regression line of y on x is

$$\boxed{y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) = b_{yx}(x - \bar{x})} \qquad \text{... (10)}$$

Similarly, the regression line of x on y is

$$\boxed{x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) = b_{xy}(y - \bar{y})} \qquad \text{... (11)}$$

The coefficient b_{yx} involved in the equation (10) is known as ***regression coefficient of y on x*** and the coefficient b_{xy} involved in the equation (11) is known as ***regression coefficient of x on y***.

Remark 1 : For obtaining (10) and (11) we have to calculate r = r (x, y) the correlation coefficient, which can be also determined using change of origin and scale property.

Thus, $$r = r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v} = r(u, v)$$

If $$u = \frac{x-a}{h}, \quad v = \frac{y-b}{k}, \quad \text{then} \quad \sigma_x = h\,\sigma_u, \quad \sigma_v = k\,\sigma_v$$

and $$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - \bar{u}^2 \quad \text{and} \quad \sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{v}^2$$

and $$\bar{x} = a + h\bar{u}, \quad \text{and} \quad \bar{y} = b + k\bar{v}$$

In particular, if $u = x - a,\ v = y - b$ then, $h = k = 1$ and $\sigma_x = \sigma_u$ and $\sigma_y = \sigma_v$

and $$\bar{x} = a + \bar{u}, \qquad \bar{y} = b + \bar{v}$$

These results help us to determine (10) and (11).

Remark 2 : Correlation coefficient and regression coefficients have same algebraic signs. If r > 0, then $b_{yx} > 0$ and $b_{xy} > 0$. If r < 0, then $b_{yx} < 0$ and $b_{xx} < 0$.

Remark 3 : Since $b_{yx} \times b_{xy} = r^2$ therefore correlation coefficient = r = $\sqrt{b_{xy} \times b_{yx}}$. i.e. geometric mean of regression coefficients. Choose positive square root, if regression coefficients are positive, otherwise negative.

Remark 4 : The acute angle θ between the regression lines is given by,

$$\theta = \tan^{-1}\left\{\frac{1-r^2}{|r|} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right\}$$

Remark 5 : The point of intersection of two regression line is $(\bar{x}, \bar{y})$.

ILLUSTRATIONS

Ex. 1 : *Obtain regression lines for the following data :* ***(Dec. 2012, 2016)***

x	*6*	*2*	*10*	*4*	*8*
y	*9*	*11*	*5*	*8*	*7*

Sol. : To find regression lines we require to calculate regression coefficient b_{xy} and b_{yx}. These coefficients depend upon ∑ x, ∑ y, ∑ x^2, ∑ y^2 and ∑ xy. So we prepare the following table and simplify the calculations.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32
8	7	64	49	56
$\sum x_i = 30$	**$\sum y_i = 40$**	**$\sum x_i^2 = 220$**	**$\sum y_i^2 = 340$**	**$\sum x_i y_i = 214$**

No. of observations = n = 5

$$\bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6 \quad \text{and} \quad \bar{y} = \frac{\sum y_i}{n} = \frac{40}{5} = 8$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{220}{5} - (6)^2 = 44 - 36 = 8$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2 = \frac{340}{5} - (8)^2 = 68 - 64 = 4$$

$$\text{Cov}(x, y) = \frac{\sum (x_i y_i)}{n} - \bar{x}\,\bar{y} = \frac{214}{5} - 6 \times 8$$

$$\text{Cov}(x, y) = 42.8 - 48 = -5.2$$

$$b_{yx} = \frac{\text{Cov}(x, y)}{\sigma_x^2} = \frac{-5.2}{8} = -0.65$$

$$b_{xy} = \frac{\text{Cov}(x, y)}{\sigma_y^2} = \frac{-5.2}{6} = -1.3$$

Regression line of Y on X is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 8 = -0.65(x - 6)$$

$$y = -0.65x + 3.9 + 8$$

$$y = -0.65x + 11.9$$

Regression line of X on Y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 6 = -1.3(y - 8)$$

$$x - 6 = -1.3y + 10.4$$

$$x = -1.3y + 10.4 + 6$$

$$x = -1.3y + 16.4$$

Ex. 2 : *Obtain regression lines for the following data :*

X	*2*	*3*	*5*	*7*	*9*	*10*	*12*	*15*
Y	*2*	*5*	*8*	*10*	*12*	*14*	*15*	*16*

Find estimate of (i) Y when X = 6 and (ii) X when Y = 20.

Sol. : To find regression lines we require to calculate regression coefficients b_{xy} and b_{yx}. These coefficients depend upon $\sum x$, $\sum y$, $\sum x^2$, $\sum y^2$, $\sum xy$. So we prepare the following table and simplify the calculations :

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
2	2	4	4	4
3	5	9	25	15
5	8	25	64	40
7	10	49	100	70
9	12	81	144	108
10	14	100	196	140
12	15	144	225	180
15	16	225	256	240
Total = 63	**82**	**637**	**1014**	**797**

n = number of pairs of observations = 8

$$\bar{x} = \frac{\sum x_i}{n} = \frac{63}{8} = 7.875$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2$$

$$= \frac{637}{8} - (7.875)^2 = 17.6094$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{82}{8} = 10.25$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2$$

$$= \frac{1014}{8} - (10.25)^2 = 21.6875$$

$$\text{Cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\,\bar{y} = \frac{797}{8} - 7.875 \times 10.25$$

$$= 18.9063$$

$$b_{yx} = \frac{\text{Cov}(x, y)}{\sigma_x^2} = \frac{18.9063}{17.6094} = 1.0736$$

$$b_{xy} = \frac{\text{Cov}(x, y)}{\sigma_y^2} = \frac{18.9063}{21.6875} = 0.8718$$

Regression line of Y on X : $Y - \bar{Y} = b_{yx}(X - \bar{X})$

$$y - 10.25 = 1.0736\,(X - 7.875)$$

$$Y = 1.0736\,X + 1.7954$$

(i) Estimate of y for x = 6 can be obtained by substituting x = 6 in the above regression equation.

$\therefore$ $$Y = 1.0736 \times 6 + 1.7954 = 8.237$$

Regression line of X on Y :

$$X - \bar{X} = b_{xy}(Y - \bar{Y})$$

$$X - 7.875 = 0.8718\,(Y - 10.25)$$

$$X = 0.8718\,Y - 1.06095$$

(ii) Estimate of x can be obtained by substituting y = 20 in the above equation.

$$X = 16.37505$$

Note : For estimation of x and estimation of y, separate equations are to be used.

Ex. 3 : *Find the lines of regression for the following data :*

x	*10*	*14*	*19*	*26*	*30*	*34*	*39*
y	*12*	*16*	*18*	*26*	*29*	*35*	*38*

and estimate y for x = 14.5 and x for y = 29.5. ***(May 2010)***

Sol. : Tabulating the data as :

x	y	u = x – 26	v = y – 26	u^2	v^2	uv
10	12	– 16	– 14	256	196	224
14	16	– 12	– 10	144	100	120
19	18	– 7	– 8	49	64	56
26	26	0	0	0	0	0
30	29	4	3	16	9	12
34	35	8	9	64	81	72
39	38	13	12	169	144	156
Total	**–**	**$\sum u = -10$**	**$\sum v = -8$**	**$\sum u^2 = 698$**	**$\sum v^2 = 594$**	**$\sum uv = 640$**

Here n = 7, $\bar{u} = \frac{-10}{7} = -1.429,$ $\bar{v} = \frac{-8}{7} = -1.143$

$\bar{u}^2 = 2.042,$ $\bar{v}^2 = 1.306$

$$\text{cov}(u, v) = \frac{1}{n} \sum uv - \bar{u}\,\bar{v}$$

$$= \frac{1}{7}(640) - (1.429)(1.143) = 89.795$$

$$\sigma_u^2 = \frac{1}{n} \sum u_i^2 - \bar{u}^2 = \frac{1}{7}(698) - 2.042 = 97.672$$

$$\therefore \quad \sigma_u = 9.883$$

$$\sigma_v^2 = \frac{1}{n} \sum v_i^2 - \bar{v}^2 = \frac{1}{7}(594) - 1.306 = 83.551$$

$$\therefore \quad \sigma_v = 9.14$$

$$r = r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u\,\sigma_v} = \frac{89.795}{9.883 \times 9.14}$$

$$= \frac{89.795}{90.33062} = 0.9941$$

$$r \times \frac{\sigma_y}{\sigma_x} = r \times \frac{\sigma_v}{\sigma_u} = 0.9941 \times \frac{9.14}{9.883} = 0.9194$$

$$r \times \frac{\sigma_x}{\sigma_y} = r \times \frac{\sigma_u}{\sigma_v} = 0.9941 \times \frac{9.883}{9.14} = 1.0749$$

$$\bar{x} = a + \bar{u} = 26 - 1.429 = 24.571$$

$$\bar{y} = b + \bar{v} = 26 - 1.143 = 24.857$$

Regression line of y on x is given by equation (10)

$$y - 24.857 = 0.9194(x - 24.571) \qquad \text{... (i)}$$

Regression line of x on y is given by equation (11)

$$x - 24.571 = 1.0749(y - 24.857) \qquad \text{... (ii)}$$

To estimate y for x = 14.5

put x = 14.5 in (i), $\therefore$ y = 24.857 + 0.9194 (14.5 – 24.571) = 15.5977

Estimate of x for y = 29.5 is obtained from (ii).

$$x = 24.571 + 1.0749(29.5 - 24.857)$$

$$= 29.56176$$

Ex. 4 : *The table below gives the respective heights x and y of a sample of 10 fathers and their sons :*

(i) Find regression line of y on x.

(ii) Find regression line of x on y.

(iii) Estimate son's height if father's height is 65 inches.

(iv) Estimate father's height if son's height is 60 inches.

(v) Compute correlation coefficient between x and y.

(vi) Find the angle between the regression lines.

Height of Father x (inches)	*65*	*63*	*67*	*64*	*68*	*62*	*70*	*66*	*68*	*67*
Height of Son y (inches)	*68*	*66*	*68*	*65*	*69*	*66*	*68*	*65*	*71*	*67*

Sol. : Let u = x – 62, v = y – 65. We prepare the table to simplify the computations.

x	y	u	v	u^2	v^2	uv
65	68	3	3	9	9	9
63	66	1	1	1	1	1
67	68	5	3	25	9	15
64	65	2	0	4	0	0
68	69	6	4	36	16	24
62	66	0	1	0	1	0
70	68	8	3	64	9	24
66	65	4	0	16	0	0
68	71	6	6	36	36	36
67	67	5	2	25	4	10
Total		**40**	**23**	**216**	**85**	**119**

$$n = \text{Number of pairs} = 10$$

$$\bar{u} = \frac{40}{10} = 4, \qquad \sigma_u^2 = \frac{216}{10} - 4^2 = 5.6$$

$$\bar{v} = \frac{23}{10} = 2.3, \qquad \sigma_v^2 = \frac{85}{10} - (2.3)^2 = 3.21$$

$$\text{Cov}(u, v) = \frac{119}{10} - 4 \times 2.3 = 2.7$$

$$\therefore \quad b_{xy} = b_{uv} = \frac{2.7}{3.21} = 0.8411, \text{ and } b_{yx} = b_{vu} = \frac{2.7}{5.6} = 0.4821$$

$$\bar{x} = \bar{u} + 62 = 66, \quad \bar{y} = \bar{v} + 65 = 67.3$$

(i) Regression line of y on x is $Y - \bar{Y} = b_{yx}(X - \bar{X})$

$$\therefore \quad Y - 67.3 = 0.4821 (X - 66)$$

$$\therefore \quad Y = 0.4821X + 35.4814$$

(ii) Regression line of x on y is $X - \bar{X} = b_{xy}(Y - \bar{Y})$

$$\therefore \quad X - 66 = 0.8411 (Y - 67.3)$$

$$\therefore \quad X = 0.8411 Y + 9.3940$$

(iii) Estimate of son's height Y for x = 65

$$Y = 0.4821 \times 65 + 35.4814 = 66.8179 \text{ inches}$$

(iv) Estimate of father's height x for y = 60

$$X = 0.8411 \times 60 + 9.394 = 59.86 \text{ inches}$$

(v) Correlation coefficient,

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{0.8411 \times 0.4821} = 0.63678$$

We choose positive square root because regression coefficients are positive.

(vi) The acute angle between the regression lines is given by

$$\tan\theta = \frac{1 - r^2}{|r|} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} = \frac{1 - (0.63678)^2}{0.63678} \times \frac{\sqrt{5.6 \times 3.21}}{(5.6 + 3.21)}$$

$$= 0.933621 \times \frac{4.2398}{8.81} = 0.4493$$

$$\therefore \quad \theta = \tan^{-1}(0.4493) = 24.19^\circ$$

Ex. 5 : *The following are marks obtained by 10 students in Statistics and Economics.*

No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Marks in Economics	*25*	*28*	*35*	*32*	*31*	*36*	*29*	*38*	*34*	*32*
Marks in Statistics	*43*	*46*	*49*	*41*	*36*	*32*	*31*	*30*	*33*	*39*

Marks are out of 50. Obtain regression equation to estimate marks in Statistics if marks in Economics are 30.

Sol. : n = 10. Let us denote marks in Economics by x and marks in Statistics by y.

Let u = x – 30 and v = y – 35.

x	**y**	**u = x – 30**	**v = y – 35**	$\mathbf{u^2}$	$\mathbf{v^2}$	**uv**
25	43	– 5	8	25	64	– 40
28	46	– 2	11	4	121	– 22
35	49	5	14	25	196	70
32	41	2	6	4	36	12
31	36	1	1	1	1	1
36	32	6	– 3	36	9	– 18
29	31	– 1	– 4	1	16	4
38	30	8	– 5	64	25	– 40
34	33	4	– 2	16	4	– 8
32	39	2	4	4	16	8
–	–	$\sum u = 2$	$\sum v = 30$	$\sum u^2 = 180$	$\sum v^2 = 488$	$\sum uv = -33$

$$\bar{u} = \frac{\sum u}{n} = \frac{20}{10} = 2 \quad \text{and} \quad \bar{v} = \frac{\sum v}{n} = \frac{30}{10} = 3$$

$u = x - 30 \quad \therefore \quad u = \bar{x} - 30$

$\therefore \quad \bar{x} = \bar{u} - 30 = 2 + 30 = 32$

$v = y - 35 \quad \therefore \quad \bar{v} = \bar{y} - 35$

$\therefore \quad \bar{y} = \bar{v} + 35 = 3 + 35 = 38$

$$\sigma_u^2 = \frac{\sum u^2}{n} - (\bar{u})^2 = \frac{180}{10} - (2)^2 = 18 - 4 = 14$$

$$\sigma_v^2 = \frac{\sum v^2}{n} - (\bar{v})^2 = \frac{488}{10} - (3)^2 = 48.8 - 9 = 39.8$$

$\therefore \quad \sigma_u = 3.742 \quad \text{and} \quad \sigma_v = 6.309$

Standard deviation is invariant to the change of origin.

$\therefore \quad \sigma_x = 3.742 \quad \text{and} \quad \sigma_y = 6.309$

$\therefore \quad \sigma_x^2 = 14 \quad \text{and} \quad \sigma_y^2 = 39.8$

$$\text{Cov}(u, v) = \frac{\sum uv}{n} - \bar{u}\,\bar{v} = \frac{-33}{10} - 2\,(3) = -3.3 - 6$$

$\therefore \quad \text{Cov}(u, v) = -9.3$

Covariance is invariant to the change of origin.

$\therefore \quad \text{Cov}(x, y) = \text{Cov}(u, v) = -9.3$

We have to find regression equation of y on x. It is given by

$$y - \bar{y} = b_{yx}\,(x - \bar{x})$$

$$b_{yx} = \frac{\text{Cov}(x, y)}{\sigma_x^2} = \frac{-9.3}{14} = -0.664$$

∴ Regression equation becomes,

$$y - 38 = -0.664\,(x - 32)$$
$$y = -0.664\,x + 21.248 + 38$$
$$y = -0.664\,x + 59.248$$

Now, we have to estimate marks in Statistics if marks in Economics are 30, i.e. we have to find value of y when x = 30.

Substituting x = 30 in above equation, we get

$$y = -0.664 \times 30 + 59.248$$
$$y = 39.328$$

∴ Marks in Economics are 39.328 i.e. approximately 39.

Ex. 6 : *Determine regression line for price, given the supply, hence estimate price when supply is 180 units, from the following information : x = supply, y = Price, n = 7,* $\Sigma(x - 150) = 119$, $\Sigma(y - 160) = 84$, $\Sigma(x - 150)^2 = 2835$, $\Sigma(y - 160)^2 = 2387$, $\Sigma(x - 150)\,\Sigma(y - 160) = 525$. *Also, find correlation coefficient between price and supply.* ***(Dec. 2018)***

Sol. : Let u = x – 150, v = y – 160

$$\therefore \quad \bar{u} = \frac{119}{7} = 17, \quad \bar{v} = \frac{84}{7} = 12$$

$$\sigma_u^2 = \frac{1}{7}(2835) - (17)^2 = 405 - 289 = 116$$

$$\sigma_v^2 = \frac{1}{7}(2835) - (12)^2 = 341 - 144 = 197$$

$$\text{cov}(x, y) = \text{cov}(u, v) = \frac{1}{7}(525) - 17 \times 12 = -129$$

$$\bar{x} = 150 + \bar{u} = 167, \; \bar{y} = 160 + \bar{v} = 172$$

$$b_{xy} = b_{vu} = \frac{\text{cov}\,(u, v)}{\sigma_u^2} = \frac{-129}{116} = -1.1121$$

Equation of regression line y on x is

$$y - \bar{y} = b_{yx}\,(x - \bar{x})$$
$$y - 172 = (-1.1121)\,(x - 167)$$

Correlation coefficient r is obtain as

$$r = \frac{\text{cov}\,(u, v)}{\sigma_u\,\sigma_v} = \frac{-129}{\sqrt{116 \times 197}} = -0.8534$$

Since both the regression coefficients are negative, we take r = – 0.663.

Ex. 7 : *Given x – 4y = 5 and x – 16 y = – 64 are the regression lines, fine regression coefficient of x on y, regression coefficient of y on x and* $\bar{x}$, $\bar{y}$.

Sol.: Here by looking at the equations we cannot decide which of the equation is regression equation of x on y and which is of y on x. We arbitrarily decide one line as regression line of y on x and find regression coefficients. Then we verify whether these values are admissible.

Suppose the equation x – 16 y = – 64 represent regression line of x on y. The equation can be written as

$$x = 16y - 64 \qquad \therefore \; b_{xy} = 16 \qquad \dots (1)$$

Next, let the equation x – 4y = 5 will be regression line of y on x. The equation can be written as

$$y = \frac{1}{4}\,x - \frac{5}{4} \qquad \therefore \; b_{yx} = \frac{1}{4} \qquad \dots (2)$$

From (1) and (2), we have

$$r^2 = b_{xy} \times b_{yx} = 16 \times \frac{1}{4} = 4 > 1$$

Hence, our choice of regression lines is incorrect.

Next, exchanging the choice,

Suppose x – 16 y = – 64 as regression line x on y. The equation can be written as

$$y = \frac{1}{16} x + 4 \qquad \therefore \; b_{yx} = \frac{1}{16} \qquad \ldots (3)$$

and, let x – 4y = 5 as regression line x on y. The equation can be written as

$$x = 4y + 5 \qquad \therefore \; b_{xy} = 4 \qquad \ldots (4)$$

From (3) and (4), we have

$$r^2 = b_{yx} \times b_{xy} = \frac{1}{16} \times 4 = \frac{1}{4} < 1$$

Thus, from (3) and (4) $b_{yx} = \frac{1}{16}$ and $b_{xy} = 4$ are correct regression coefficients.

$$\text{correlation coefficient} = r^2 = b_{yx} \times b_{xy} = \frac{1}{16} \times 4 = \frac{1}{4} < 1 \quad \therefore r = \frac{1}{2} \qquad \ldots (5)$$

We choose positive square root because regression coefficients are positive.

We also, note that since $(\bar{x}, \bar{y})$ is the point of intersection of regression lines. Thus, $(\bar{x}, \bar{y})$ will satisfy both the equations

$$\therefore \qquad \bar{x} - 4\bar{y} = 5 \qquad \ldots (6)$$

$$\text{and} \qquad \bar{x} - 16\bar{y} = -64 \qquad \ldots (7)$$

Solving equations (6) and (7), we get

$$\bar{x} = 28, \; \bar{y} = \frac{23}{4}$$

Ex. 8 : *If the two lines of regression are 9x + y – λ = 0 and 4x + y = μ and the means of x and y are 2 and –3 respectively, find the values of λ, μ and the coefficient of correlation between x and y.* **(May 2009)**

Sol. : $\bar{x} = 2$ and $\bar{y} = -3$.

The lines of regression are 9x + y = λ and 4x + y = μ.

The point of intersection of two regression lines is (x, y) i.e. $(\bar{x}, \bar{y})$ lies on both the regression lines.

$$9\bar{x} + \bar{y} = \lambda \qquad \ldots (1)$$

$$4\bar{x} + \bar{y} = \mu \qquad \ldots (2)$$

Substituting values of $\bar{x}$ and $\bar{y}$, we get

$$9(2) + (-3) = \lambda$$

$$\lambda = 18 - 3 = 15$$

$$\text{and} \qquad 4(2) + (-3) = \mu$$

$$\therefore \qquad \mu = 8 - 3 = 5$$

Thus, the regression lines are,

$$9x + y = 15 \quad \text{and} \quad 4x + y = 5$$

Let $9x + y = 15$ be the regression line of x on y, so it can be written as

$$x = \frac{15}{9} - \frac{y}{9}$$

$$\therefore \quad b_{xy} = -\frac{1}{9} = -0.11$$

Let $4x + y = 5$ be the regression line of y on x. So it can be written as $y = 5 - 4x$.

$$\therefore \quad b_{yx} = -4$$

Correlation coefficient between x and y is given as,

$$r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{(-4) \times (-0.11)} = \sqrt{0.44} = 0.663$$

Since both the regression coefficients are negative, we take $r = -0.663$.

Ex. 9 : *The regression equations are $8x - 10y + 66 = 0$ and $40x - 18y = 214$. The value of variance of x is 9. Find :*

(1) The mean values of x and y.

(2) The correlation x and y and

(3) The standard deviation of y. ***(Nov. 2015, May 2019)***

Sol. : (1) Since both the regression lines pass through the point $(\bar{x}, \bar{y})$, we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \quad \text{and} \quad 40\bar{x} - 18\bar{y} = 214$$

Solving these two equations, we get

$$\bar{x} = 13 \quad \text{and} \quad \bar{y} = 17$$

(2) Let $8x - 10y + 66 = 0$ be the line of regression of y on x and $40x - 18y = 214$ be the line of regression of x on y. These equations can be written in the form

$$y = \frac{8}{10}x + \frac{66}{10} \quad \text{and} \quad x = \frac{18}{40}y + \frac{214}{40}$$

i.e. $$y = 0.8x + 6.6 \quad \text{and} \quad x = 0.45y + 5.35$$

$$\therefore \quad b_{yx} = \text{Regression coefficient of y on x} = 0.8$$

and $$b_{xy} = \text{Regression coefficient of x on y} = 0.45$$

Correlation coefficient between x and y is given by

$$r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{0.45 \times 0.8} = \pm 0.6$$

But since both the regression coefficients are positive, we take

$$r = +0.6$$

(3) Variance of x = 9, i.e. $\sigma_x^2 = 9$

$$\therefore \quad \sigma_x = 3$$

We have, $$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$0.8 = 0.6 \times \frac{\sigma_y}{3}$$

$$\sigma_y = 4$$

Ex. 10 : *Given the following information*

	Variable x	***Variable y***
Arithmetic Mean	*8.2*	*12.4*
Standard Deviation	*6.2*	*20*

Coefficient of correlation between x and y is 0.9. Find the linear regression estimate of x, given y = 10.

Sol. : Given that $\bar{x} = 8.2$, $\bar{y} = 12.4$, $\sigma_x = 6.2$, $\sigma_y = 20$ and $r_{xy} = 0.9$. We want to find x for y = 10.

Line of regression of x on y is

$$x - \bar{x} = b_{xy}\ (y - \bar{y})$$

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y} = 0.9 \times \frac{6.2}{20} = 0.279$$

Substituting value of $\bar{x}$, $\bar{y}$ and b_{xy} in above equation, we get

$$x - 8.2 = 0.279\ (y - 12.4)$$

$$x = 0.279\ y - 3.4596 + 8.2$$

$$x = 0.279\ y + 4.7404$$

Putting y = 10 in equation, we get

$$x = 0.279 \times 10 + 4.7404$$

$$x = 7.5304$$

5.11 RELIABILITY OF REGRESSION ESTIMATES

Standard Error of Regression Estimate :

In order to study the reliability of regression estimate, we require to know its standard error. For further statistical analysis such as testing the significance of regression coefficient, standard error is required.

Suppose $\{(x_i, y_i), i = 1, 2, ..., n\}$ is a sample on the variables X and Y. The sample variances of X and Y are σ_x^2 and σ_y^2 respectively. The sample correlation coefficient between X and Y is r. The regression line of Y on X is given by $y - \bar{y} = b_{yx}\ (x - \bar{x})$. We can write it as $y = b_{yx}\ (x - \bar{x}) + \bar{y}$. Clearly, the error in estimation is (observed value of y) – (regression estimate of y). The positive square root of the mean sum of squares of error is called as ***standard error of regression estimate***. We denote it by S_y.

If y_i is observed value of y; and $\hat{y}_i$ is the regression estimate for given x_i then $y_i = b_{yx}\ (x_i - \bar{x}) + \bar{y}$.

Therefore,

$$S_y^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \frac{1}{n} \sum_{i=1}^{n} [y_i - b_{yx}\ (x_i - \bar{x}) - \bar{y}\ ]^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} [(y_i - \bar{y}) - b_{yx}\ (x_i - \bar{x})]^2$$

$$= \frac{1}{n} \left[\sum (y_i - \bar{y})^2 + b_{yx}^2 \sum (x_i - \bar{x})^2 - 2\ b_{yx} \sum (x_i - \bar{x})\ (y_i - \bar{y})\right]$$

$$= \sigma_y^2 + b_{yx}^2\ \sigma_x^2 - 2\ b_{yx}\ \text{cov}\ (x, y) \qquad \left(\because\ b_{yx} = r\frac{\sigma_y}{\sigma_x},\quad \text{cov}\ (x, y) = r\ \sigma_x\ \sigma_y\right)$$

$$= \sigma_y^2 + r^2\ \sigma_y^2 - 2r^2\ \sigma_y^2 = \sigma_y^2\ (1 - r^2)$$

Hence the standard error of regression estimate of y on x is

$$\boxed{S_y = \sigma_y\ \sqrt{1 - r^2}}$$

Note that larger the value of r^2, smaller is the error. Hence the regression estimates are close to the actual values of y_i for large r^2. If $r = \pm 1$, the correlation is perfect and the standard error is zero, which means observed values and estimated values of y agree.

The standard error of regression estimate of x on y is given by,

$$\boxed{S_x = \sigma_x\ \sqrt{1 - r^2}}$$

Note : The above discussion leads to conclusion that rather than r we should consider r^2 for ***testing reliability of regression estimates***. Therefore, regression analysis claims validity if r^2 is sufficiently large. The quantity r^2 is called as the ***coefficient of determination***.

EXERCISE 5.2

1. Find Karl Pearson's coefficient of correlation for the following data and determine the probable error.

x	20	22	23	25	25	28	29	30	30	34
y	18	20	22	24	21	26	26	25	27	29

[**Hint :** Probable error = $0.6745 \left(\frac{1 - r^2}{\sqrt{N}}\right)$ where, r is the coefficient correlation and N the number of pairs of observations.] **Ans.** 0.952, 0.02.

2. Find the coefficient of correlation for the following table : **(Dec. 2006, 2014; May 2017) Ans.** r = 0.6013

x	10	14	18	22	26	30
y	18	12	24	6	30	36

3. The following marks have been obtained by a group of students in Engineering Mathematics.

Paper I	80	45	55	56	58	60	65	68	70	75	85
Paper II	82	56	50	48	60	62	64	65	70	74	90

Calculate the coefficient of correlation. **Ans.** 9277

4. For the following tabulated data, find the coefficient of correlation.

y \ x	**18**	**19**	**20**	**21**	**Total**
10 – 20	4	2	2	–	8
20 – 30	5	4	6	4	19
30 – 40	6	8	10	11	35
40 – 50	4	4	6	8	22
50 – 60	–	2	4	4	10
60 – 70	–	2	3	1	6
Total	**19**	**22**	**31**	**28**	**100**

Ans. 0.25

5. Coefficient of correlation between two variables X and Y is 0.8. Their covariance is 20. The variance of X is 16. Find the standard deviation of Y series. **Ans.** 1.5625.

6. Determine the equations of regression lines for the following data :

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

and obtain an estimate of y for x = 4.5. **(May 2007)** **Ans.** 0.95 x + 7.25, x = 0.957 – 6.4 = 11.525.

7. Two examiners A and B independently award marks to seven students.

R. No.	1	2	3	4	5	6	7
Marks by A	40	44	28	30	44	38	31
Marks by B	32	39	26	30	38	34	28

Obtain the equations of regression lines. If examiner A awards 36 marks to Roll No. 8, what would be the marks expected to be awarded by examiner B to the same candidate ? **Ans.** y = 11.885 + 0.587 x, 33.017.

8. The two regression equations of the variables x and y are

$$x = 19.13 - 0.87\, y \qquad y = 11.64 - 0.50\, x$$

Find (i) $\bar{x}$, $\bar{y}$, (ii) The correlation coefficient between x and y. **(Dec. 2006)**

Ans. $\bar{x} = 15.935$, $\bar{y} = 3.673$, $r = 0.6595$

9. Determine the reliability of estimates for the data :

x	10	14	19	26	30	34	39
y	12	16	18	26	29	35	38

Ans. $r^2 = 0.988$ high.

10. If θ is the acute angle between the two regression lines in the case of two variables x and y, show that

$$\tan\theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

❒❒❒

UNIT IV : PROBABILITY AND PROBABILITY DISTRIBUTIONS

CHAPTER - 6
PROBABILITY AND PROBABILITY DISTRIBUTIONS

6.1 INTRODUCTION

Theory of Probability had its origin in Mid-eighteenth century studies in games of chance related to dice throw gambling. Galileo was the first man to attempt at a quantitative measure of probability. The first foundation of Modern Mathematical theory of probability was laid down by French Mathematicians B. Pascal and P. Fermat. J. Bernoulli, Demoivre, T. Bayes, P. S. Laplace alongwith many others did considerable work in this field. Almost every human activity involves some kind of chance element and the role for the theory of probability to play. The subject of statistics originated much earlier than probability and dealt mainly with the collection and organisation of data. With the advent of probability, it was realized that statistics could be used in drawing valid conclusions and making reasonable decisions on the basis of analysis of data, such as in sampling theory and prediction of forecasting. As time progressed, probability theory found its way into many applications not only in Engineering and Science but also in the fields like Actuarial Science, Agriculture, Commerce, Medicine and Psychology.

The scope of this work is to introduce to the readers the modern concept of probability and the subject of probability distribution, which forms the basis of the modern theory of probability.

6.2 TERMINOLOGY

Before we take-up the subject matter, we shall define and explain certain terms which are encountered so very often.

(i) Experiment : Students of Science and Engineering are familiar with experiments which when performed repeatedly under the same conditions give identical results. In theory of probability, our interest is centred around the kind of experiment, which though repeated under essentially identical conditions, does not give unique results but may result in any one of the several possible outcomes. Such an experiment is also called a trial and the outcome an event or a case.

For example, the throw of a coin is an experiment or a trial which can result in one of the two outcomes a Head or a Tail. Drawing a card from a well shuffled pack is a trial which may result in any one of 52 outcomes.

(ii) Equally Likely : The outcomes of a trial are said to be equally likely if any one of them cannot be expected to occur in preference to another. In tossing an unbiased coin, the outcomes Head or Tail are equally likely.

(iii) Mutually Exclusive : The outcomes of a trial are said to be mutually exclusive, if the occurrence of one of them precludes the occurrence of all other outcomes. In the experiment of throw of a die, the occurrence of number 1 at uppermost face will exclude automatically the occurrence of numbers 2, 3, 4, 5 and 6.

In tossing a coin, events Head or Tail are mutually exclusive.

(iv) Exhaustive : All possible outcomes of a trial form exhaustive set of cases or events. In throw of a coin, the events Head and Tail constitute exhaustive set of events. In throw of a die, appearance of numbers 1, 2, 3, 4, 5, 6 constitute exhaustive set of events.

(v) Sample Space : A set of all possible outcomes of a trial which are exhaustive is called a sample space. In a throw with two dice, sample space consists of following outcomes :

$$\begin{array}{cccccc}
(1, 1), & (1, 2), & (1, 3), & (1, 4), & (1, 5), & (1, 6) \\
(2, 1), & (2, 2), & (2, 3), & (2, 4), & (2, 5), & (2, 6) \\
(3, 1), & (3, 2), & (3, 3), & (3, 4), & (3, 5), & (3, 6) \\
(4, 1), & (4, 2), & (4, 3), & (4, 4), & (4, 5), & (4, 6) \\
(5, 1), & (5, 2), & (5, 3), & (5, 4), & (5, 5), & (5, 6) \\
(6, 1), & (6, 2), & (6, 3), & (6, 4), & (6, 5), & (6, 6)
\end{array}$$

which forms a sample space.

(vi) Independent : Events A and B are said to be independent if happening of A has nothing to do with the happening of B and vice-a-versa. If a coin is thrown twice, the event 'Occurrence of Head in first throw' has nothing to do with the event '*Occurrence of Head in second throw*'. The two successive throws, or for that matter n successive throws, are considered as independent trials, which result in independent outcomes. Note that independent events are quite different from mutually exclusive events.

If the occurrence of event B is affected by the occurrence of event A, then such events are Dependent events. For example, if two cards are drawn successively from a well shuffled pack of cards *without replacement*, then the event of appearance of king at second draw will certainly depend upon the result of the first draw.

6.3 DEFINITION OF PROBABILITY AND RELATED EXAMPLES

Question of defining Probability is quite delicate. In Modern Axiomatic approach, the term 'Probability' is left undefined. However, we shall give two definitions of probability, which have got their limitations, but serve the purpose in many cases.

(1) Classical or 'a *priori*' Probability : If a trial results in n exhaustive cases which are mutually exclusive and equally likely and out of which m are favourable to the happening of event A, then the probability P of the happening of event A also denoted by P (A) is given by,

$$p = P(A) = \frac{m}{n}$$

If $\bar{A}$ indicates non-happening of A, then the number of cases favourable to this event are obviously n – m and

$$q = P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$\therefore \qquad P(\bar{A}) = 1 - P(A)$$

$$\text{Or} \qquad P(A) + P(\bar{A}) = 1 \quad \text{i.e.} \quad p + q = 1$$

If an event A is certain to happen

$$P(\bar{A}) = 0 \text{ and } P(A) = 1.$$

(2) Statistical or Empirical Definition : If a trial be repeated for a large number of times, say n, under the same conditions and a certain event A occurs on p × n occasions, then, the probability of happening of event A is given by,

$$P(A) = \lim_{n \to \infty} \frac{p \times n}{n} = p$$

The classical definition of probability gives the relative frequency of favourable cases to the total number of cases, which in the empirical definition is the limit of the relative frequency of the happening of the event.

Note that $\qquad 0 \le P(A) \le 1.$

$P(A) = \frac{m}{n}$ is sometimes expressed by saying that the odds in favour of A are m : (n – m) or the odds against A are (n – m) : n.

ILLUSTRATIONS

Ex. 1 : *Find the probability of drawing the king from a well shuffled pack of cards.*

Sol. : Here the total number of outcomes is n = 52 and there are four kings in the pack of cards.

$$\therefore \qquad \text{Required probability} = p = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$$

Ex. 2 : *If 3 of 20 tubes are defective and 4 of them are randomly chosen for inspection (i.e. each tube has the same chance of being selected), then what is the probability that only one of the defective tubes will be included ?* ***(May 2010)***

Sol. : 4 tubes can be selected out of 20 in $^{20}C_4$ ways.

$$\therefore \qquad n = {}^{20}C_4 = \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4} = 4845$$

One defective tube can be chosen in $^{3}C_1$ ways and each of which should be associated with 3 non-defective tubes which can be selected in $^{17}C_3$ ways.

Thus $$m = {}^3C_1 \times {}^{17}C_3 = 3 \times \frac{17 \cdot 16 \cdot 15}{1 \cdot 2 \cdot 3} = 2040$$

$$p = \frac{m}{n} = \frac{2040}{4845} = \frac{8}{19} = 0.42 \text{ approximately.}$$

Ex. 3 : *A throw is made with two dice. Find the probability of getting a score of (i) 10 points; (ii) At least 10 points; (iii) At most 10 points.*

Sol. : All possible outcomes are sample space.

(1, 1),	(1, 2),	(1, 3),	(1, 4),	(1, 5),	(1, 6)
(2, 1),	(2, 2),	(2, 3),	(2, 4),	(2, 5),	(2, 6)
(3, 1),	(3, 2),	(3, 3),	(3, 4),	(3, 5),	(3, 6)
(4, 1),	(4, 2),	(4, 3),	(4, 4),	(4, 5),	(4, 6)
(5, 1),	(5, 2),	(5, 3),	(5, 4),	(5, 5),	(5, 6)
(6, 1),	(6, 2),	(6, 3),	(6, 4),	(6, 5),	(6, 6)

(1) The total number of outcomes are n = 36, the number of cases favourable to the event.

(i) of total of 10 points are (4, 6), (5, 5), (6, 4) i.e. m = 3

$\therefore$ p (score of 10 points) = $\frac{3}{36} = \frac{1}{12}$.

(ii) The number of cases favourable to the event score of at least 10 points are the score with 10, 11 and 12 points. These are (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6) i.e. m = 6

$\therefore$ p (at least 10 points) = $\frac{6}{36} = \frac{1}{6}$.

(iii) The number of cases favourable to the event at most 10 points are the score with less than or equal to 10 points. These are all the first four rows of sample space containing 24 cases, 5 cases from fifth row and 4 cases from fourth row.

$$m = 24 + 5 + 4 = 33$$

$$\therefore \quad \text{p (at most 10 points)} = \frac{33}{36} = \frac{11}{12}$$

Ex. 4 : *What is the probability that a leap year selected at random will contain 53 Mondays ?*

Sol. : Leap year will have 366 days, 52 Mondays will be contained in 364 days. Now the extra two days could be

(1) Sunday, Monday, (2) Monday, Tuesday, (3) Tuesday, Wednesday, (4) Wednesday, Thursday, (5) Thursday, Friday, (6) Friday, Saturday or (7) Saturday, Sunday. Thus there are seven possible ways in which two extra days can occur i.e. n = 7 out of these, first two cases contain Monday.

Thus in m = 2 cases, there will be extra Monday.

Thus required probability,

$$p = \frac{2}{7}$$

Ex. 5 : *From a well shuffled pack of cards, three cards are drawn at random. Find the probability that they form a King, Queen, Jack combination.*

Sol. : 3 cards can be drawn in ${}^{52}C_3$ ways.

$$n = {}^{52}C_3 = \frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3}$$

King, Queen and Jack each can be chosen in ${}^4C_1$ ways.

The combination can be chosen in

$${}^4C_1 \times {}^4C_1 \times {}^4C_1 = 64$$

$$m = 64$$

$$p = p \text{ (KQJ)} = \frac{64}{\frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3}} = \frac{64}{26 \cdot 17 \cdot 50} = \frac{16}{5525}$$

Ex. 6 : *Among six books, there are two volume of one book. These books are arranged in a random order on a shelf. Find the probability that the two volumes are always together.*

Sol. : Six books can be arranged on a shelf in 6! = 720 ways, thus n = 720.

If first two places are occupied by same volume and remaining by other four books, this can happen in $2! \times 4! = 48$ ways.

Two volume can come together if they occupy 1^{st} and 2^{nd}, 2^{nd} and 3^{rd}, 3^{rd} and 4^{th}, 4^{th} and 5^{th} or 5^{th} and 6^{th} positions. Each of which can happen in 48 ways.

Thus $\quad m = 48 \times 5 = 240$

$$\therefore \quad p = \frac{m}{n} = \frac{240}{720} = \frac{1}{3}$$

6.4 ALGEBRA OF SETS

We discuss below elementary ideas about sets, Venn diagrams and operations on sets.

Set : Set can be thought of as a well-defined collection of objects, called members or elements of the set. In order for a set to be well-defined, we must be able to determine whether a particular object does or does not belong to the set. Elements of the set can be enumerated by roster method or by the property method. The set of all vowels in the English alphabet can be defined by the roster method as (a, e, i, o, u). In this method, we actually list all the elements. The set of all natural numbers can be written as {1, 2, 3, ...}. Here we do not list all the elements, but write the elements in such a way that the complete description is evident. Set of vowels can be described as {x | x is a vowel} which reads as, the set of all elements x and that x is a vowel. This is a property method to define a set. If an element a belongs to the set D, we write $a \in D$. If a does not belong to D, we write $a \notin D$.

Subsets : If each element of set A also belongs to a set B then we call A as subset of B, written as $A \subset B$ or $B \supset A$ and 'A is contained in B' or 'B contains A' respectively.

It follows that for all sets A, we have $A \subset A$.

If $A \subset B$ and $B \subset A$ then we call A and B equal and write A = B. In such a case, A and B have exactly the same elements.

If A is not equal to B i.e. if A and B do not have exactly the same elements, then we write $A \neq B$.

If $A \subset B$ but $A \neq B$, then we call A a proper subset of B. For example, if A = {1, 2, 5}, B = {1, 2, 3, 4, 5} then $A \neq B$ and $A \subset B$. Thus A is proper subset of B.

It can be easily established that if $A \subset B$ and $B \subset C$ then $A \subset C$.

Universal Set and Empty Set : For many purposes, we restrict our discussion to subsets of some particular set called the universal set denoted by U.

It is also useful to consider a set having no elements at all. This is called the empty set or null set and it is denoted by ϕ. It is a subset of any set.

In dealing with a problem of throw of a die, the universal set is {1, 2, 3, 4, 5, 6}. The set of outcomes consisting of faces 8 or 10 on a single die is the null set.

In dealing with problems on probability, the sample space S usually constitutes a Universal set.

Venn Diagrams : A universal set U can be represented geometrically by the set of points inside a rectangle. In such a case, subsets of U (such as A and B shown shaded in Fig. 6.1) are represented by sets of points inside circles. Such diagrams called Venn diagrams, often serve to provide geometric intuition regarding possible relationships between sets.

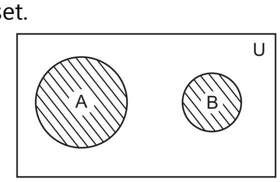


Fig. 6.1

Set Operations :

1. Union : The set of all elements (or points) which belong to either A or B or both A and B is called the union of A and B and is denoted by $A \cup B$, shown by shaded region in Fig. 6.2.

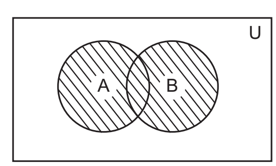


Fig. 6.2

2. Intersection : The set of all elements which belong to both A and B is called the intersection of A and B and is denoted by $A \cap B$, shown by shaded region in Fig. 6.3.

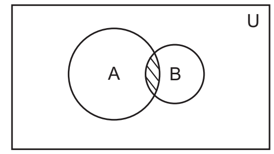


Fig. 6.3

Two sets A and B such that $A \cap B = \phi$ i.e. which have no elements in common are called disjoint sets. In Fig. 6.1, A and B are disjoint.

3. Difference : The set consisting of all the elements of A which do not belong to B is called the difference of A and B, denoted by $A - B$, shown by shaded region in Fig. 6.4.

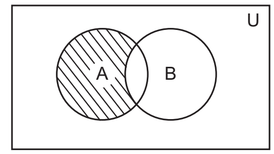


Fig. 6.4

4. Complement : If $B \subset A$ then $A - B$ is called the complement of B relative to A and is denoted by B'_A, shown by shaded region in Fig. 6.5.

If A = U, the universal set, we refer to U – B as simply the complement of B and denote it by B', shown shaded in Fig. 6.6. The complement of $A \cup B$ is denoted by $(A \cup B)'$.

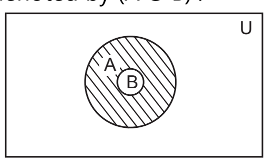


Fig. 6.5

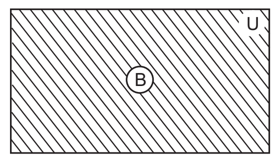


Fig. 6.6

Operations Involving Sets

1. Commutative law for Unions $A \cup B = B \cup A$
2. Associative law for Unions $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
3. Commutative law for intersections $A \cap B = B \cap A$
4. Associative law for intersections $A \cap (A \cap C) = (A \cap B) \cap C = A \cap B \cap C$
5. First Distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. Second Distributive law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
7. $A - B = A \cap B'$
8. If $A \subset B$, then $A' \supset B'$ or $B' \subset A'$
9. $A \cup \phi = A$, $A \cap \phi = \phi$
10. $A \cup U = U$, $A \cap U = A$
11. Demorgan's first law $(A \cap B)' = A' \cup B'$
12. Demorgan's second law $(A \cap B)' = A' \cup B'$
13. For any sets A and B $A = (A \cap B) \cup (A \cap B')$

An event is a subset A of the sample space S, i.e. it is a set of all possible outcomes. If the outcome of an experiment is an element of A, then we say that the event A has occurred.

If A and B are any two events, then

1. $A \cup B$ is the event either A or B or both, sometimes this is denoted by A + B.
2. $A \cap B$ is the event both A and B, alternatively it is denoted by AB.
3. A' or $\bar{A}$ is the event not A.
4. $A - B$ is the event A but not B.

If the sets corresponding to events A and B are disjoint i.e. $A \cap B = \phi$, then we often say that the events are mutually exclusive.

6.5 THEOREMS ON PROBABILITY

1. **Theorem of Total probability :** If A and B are any two events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof : Consider the sample space containing n cases, out of which m_1 are favourable to the event A, m_2 are favourable to the event B and m_3 are favourable to the event $A \cap B$ (or AB).

$n(A) = m_1,\ n(B) = m_2,\ n(A \cap B) = m_3$

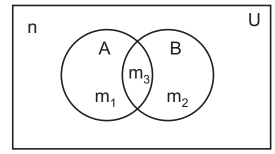


Fig. 6.7

$\therefore$ Total number of cases favourable to the event $A \cup B$ (or A + B) are $m_1 + m_2 - m_3$ (Because addition of m_1 and m_2 cases includes m_3 twice).

$$\therefore \quad P(A \cup B) = \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

$$= P(A) + P(B) - P(A \cap B)$$

If the events A, B, are mutually exclusive then $P(A \cap B) = 0$

$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

Similarly for three events A, B, C, it can be proved that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Generalization of this theorem for n events $A_1, A_2, A_3 \ldots A_n$ can also be made.

In particular if $A_1, A_2, A_3 \ldots A_n$ are mutually exclusive event, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

2. **Theorem of Compound Probability :**

$$P(A \cap B) = P(A)\, P(B|A)$$

where, B|A means occurrence of event B, subject to the condition that A has already occurred.

Proof : Consider the sample space containing n cases, out of which m_1 are favourable to the event A and out of these m_1 cases, m_2 are favourable to the event B $(B \subset A)$.

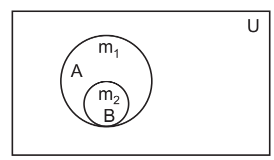


Fig. 6.8

Thus m_2 cases correspond to the event $A \cap B$.

Now we can write, $\quad \dfrac{m_2}{n} = \dfrac{m_2}{m_1} \cdot \dfrac{m_1}{n}$

It is clear that $\quad \dfrac{m_2}{m_1} = P(B|A)$

$\therefore \quad \dfrac{m_2}{n} = P(A \cap B)$

$\quad \dfrac{m_1}{n} = P(A)$

$$\therefore \quad P(A \cap B) = P(B|A) \cdot P(A)$$

Similarly, if $A \subset B$ then

$$P(A \cap B) = P(A|B) \cdot P(B)$$

If A and B are independent events, then

$$P(B|A) = P(B) \text{ or } P(A|B) = P(A)$$

and we have $\quad P(A \cap B) = P(A) \cdot P(B)$

Similarly, it can be proved that for any three events A_1, A_2, A_3

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2)$$

In particular if A_1, A_2, A_3 are independent events, then

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

3. Baye's Theorem : If $A_1, A_2, \ldots, A_n$ are mutually exclusive events whose union is the sample space and A is any event, then

$$P(A_k|A) = \frac{P(A_k)\,P(A|A_k)}{\sum\limits_{k=l}^{n} P(A_k)\,P(A|A_k)}$$

Proof :

$$P(A) = P(A_1)\,P(A|A_1) + P(A_2)\,P(A|A_2) + \ldots + P(A_n)\,P(A|A_n)$$

$$= \sum_{k=l}^{n} P(A_k)\,P(A|A_k)$$

Thus,

$$P(A_k|A) = \frac{P(A_k \cap A)}{P(A)} = \frac{P(A_k)\,P(A|A_k)}{\sum\limits_{k=l}^{n} P(A_k)\,P(A|A_k)}$$

This proves the theorem.

ILLUSTRATIONS

Ex. 1 : *Two cards are drawn from a well shuffled pack of 52 cards. Find the probability that they are both kings if*

(i) the first card drawn is replaced, (ii) first card drawn is not replaced.

Sol. : (i) First card drawn is replaced : Let A_1 be event of king on first draw and A_2 be the event of king on second draw. $A_1 \cap A_2$ is the event of king at both draws.

If first card drawn is replaced before the second draw then A_1, A_2 are independent.

$$\therefore \quad P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$P(A_1) = \frac{4}{52} = \frac{1}{13} \text{ (as there are four kings)}$$

$$P(A_2) = \frac{4}{52} = \frac{1}{13}$$

$$\therefore \quad P(A_1 \cap A_2) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

(ii) First card drawn is not replaced : In this case, the two events are dependent.

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 | A_1)$$

$$\therefore \quad P(A_1) = \frac{4}{52} = \frac{1}{13}$$

If first draw is king then for second draw, 3 kings are left to be chosen out of 51 cards.

$$P(A_2 | A_1) = \frac{3}{51} = \frac{1}{17}$$

$$\therefore \quad P(A_1 \cap A_2) = \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$$

Ex. 2 : *A can hit the target 1 out of 4 times, B can hit the target 2 out of 3 times, C can hit the target 3 out of 4 times. Find the probability of at least two hit the target.*

Sol. : Let A be the event of A hitting the target, B the event of B hitting the target, C the event of C hitting the target.

$$P(A) = \frac{1}{4}, \quad P(\bar{A}) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(B) = \frac{2}{3}, \quad P(\bar{B} = \frac{1}{3}$$

$$P(C) = \frac{3}{4}, \quad P(\bar{C}) = \frac{1}{4}$$

The required event 'at least two hitting the target' can occur in following mutually exclusive cases :

$A \cap B \cap \bar{C}$, $A \cap \bar{B} \cap C$, $\bar{A} \cap B \cap C$, $A \cap B \cap C$

P (at least two hitting the target)

$$= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

Now A, B, $\bar{C}$ etc. are all independent.

$$\therefore \quad P(A \cap B \cap \bar{C}) = P(A)\,P(B)\,P(\bar{C}) = \frac{1}{4} \times \frac{2}{3} \times \frac{1}{4} = \frac{1}{24}$$

$$P(A \cap \bar{B} \cap C) = P(A)\,P(\bar{B})\,P(C) = \frac{1}{4} \times \frac{1}{3} \times \frac{3}{4} = \frac{1}{16}$$

$$P(\bar{A} \cap B \cap C) = P(\bar{A})\,P(B)\,P(C) = \frac{3}{4} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{8}$$

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C) = \frac{1}{4} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{8}$$

P (at least two hit the target)

$$= \frac{1}{24} + \frac{1}{16} + \frac{3}{8} + \frac{1}{8} = \frac{2 + 3 + 18 + 6}{48} = \frac{29}{48}$$

Ex. 3 : *An envelope contains 6 tickets with numbers 1, 2, 3, 5, 6, 7. Another envelope contains 4 tickets with numbers 1, 3, 5, 7. An envelope is chosen at random and ticket is drawn from it. Find the probability that the ticket bears the numbers (i) 2 or 5, (ii) 2.*

Sol. : (i) Required event can happen in the following mutually exclusive ways.

(1) First envelope is chosen and then ticket is drawn.

(2) Second envelope is chosen and then ticket is drawn. Probability of choosing an envelope is $\frac{1}{2}$.

Required probability p is

$$p = P(1) + P(2) = \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{1}{4} = \frac{1}{6} + \frac{1}{8} = \frac{4 + 3}{24} = \frac{7}{24}$$

(ii)

$$p = P(1) + P(2) = \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times 0 = \frac{1}{12}$$

Ex. 4 : *An urn contains 6 white and 8 red balls. Second urn contains 9 white and 10 red balls. One ball is drawn at random from the first urn and put into the second urn without noticing its colour. A ball is then drawn at random from the second urn. What is the probability that it is red ?*

Sol. : There are two mutually exclusive cases.

(i) The white ball is transferred from the first urn to the second and then red ball is drawn from it.

(ii) The red ball is transferred from the first urn to the second and then ball is drawn from it.

Let A be the event of transferring a white ball from the first urn, B be the event of transferring a red ball from the first urn, and C the event of drawing a red ball from the second urn.

$$P(A) = \frac{6}{14} = \frac{3}{7}$$

$$P(B) = \frac{8}{14} = \frac{4}{7}$$

$$P(C|A) = \frac{10}{20} = \frac{1}{2}$$

(In this, second urn will contain 10 white and 10 red)

$$P(C|B) = \frac{11}{20} \text{ (In this case, second urn will contain 9 white and 11 red)}$$

$$\therefore \quad P(i) = P(A \cap C) = P(A)\, P(C|A) = \frac{3}{7} \times \frac{1}{2} = \frac{3}{14}$$

$$\therefore \quad P(ii) = P(B \cap C)$$

$$= P(B) \cdot P(C|B) = \frac{4}{7} \times \frac{11}{20} = \frac{11}{35}$$

Required probability P is given by

$$P = P(i) + P(ii)$$

$$= \frac{3}{14} + \frac{11}{35} = \frac{105 + 154}{490} = \frac{259}{490}$$

Ex. 5 : *A, B play a game of alternate tossing a coin, one who gets head first wins the game. Find the probability of B winning the game if A has a start.*

Sol. : In following mutually exclusive run or trails, B wins the game.

$A \to T, B \to H$ Run TH

$A \to T, B \to T, A \to T, B \to H$ Run TTTH

Similarly, TTTTTH

....

....

$$P(TH) = P(T)\, P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = \frac{1}{2^2}$$

$$P(TTTH) = P(T)\, P(T)\, P(T)\, P(H) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^4}$$

$$P(TTTTTH) = \frac{1}{2^6}$$

Required probability is $P = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots\ldots\ldots$

which is G.P. with common ratio $= \frac{1}{4}$

$$P = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

Ex. 6 : *A box contains 6 red balls, 4 white balls and 5 blue balls. Three balls are drawn successively from the box. Find the probability that they are drawn in the order red, white and blue if each ball is not replaced.*

Sol. : Let A be the event - red on the first draw, B be the event - white on second draw, C be the event - blue on third draw. We want to find $P(A \cap B \cap C)$.

By theorem on Compound probability,

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

$$P(A) = \frac{6}{15} = \frac{2}{5}$$

$$P(B|A) = \frac{4}{14}$$

$$P(C|A \cap B) = \frac{5}{13}$$

$$\therefore \quad P(A \cap B \cap C) = \frac{2}{5} \cdot \frac{4}{14} \cdot \frac{5}{13} = \frac{4}{91}$$

Ex. 7 : *Supposing that out of 12 test matches played between India and Pakistan during last 3 years, 6 are won by India, 4 are won by Pakistan and 2 have ended in a draw. If they agree to play a test series consisting of three matches, find the probability that India wins the test series on the basis of past performance.*

Sol. : India wins test series in following mutually exclusive cases :

(i) one win for India, 2 draws

(ii) two wins for India, 1 draw

(iii) two wins for India, 1 win for Pakistan.

(iv) three wins for India.

On the basis of past performance

$$\text{P (Indian win)} = \frac{6}{12} = \frac{1}{2} \quad \text{P (Pakistan win)} = \frac{4}{12} = \frac{1}{3}$$

$$\text{P (Draw)} = \frac{2}{12} = \frac{1}{6}$$

(i) One sequence in which Indian win and 2 draws can occur is Indian win, draw, draw (WDD).

$$\therefore \quad \text{P (WDD)} = \text{P (W) P (D) P (D)} = \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{72}$$

One Indian win and 2 draws can occur in ${}^3C_1$ or 3 mutually exclusive ways.

$$\text{P (i)} = 3 \cdot \frac{1}{72} = \frac{1}{24}$$

(ii) One sequence in which two Indian wins and a draw can occur is

Indian win, Indian win, draw (WWD)

$$\therefore \quad \text{P (WWD)} = \text{P (W) P (W) P (D)} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24}$$

Two Indian wins and a draw can occur in 3 mutually exclusive ways

$$\therefore \quad \text{P (ii)} = 3 \cdot \frac{1}{24} = \frac{1}{8}$$

(iii) One sequence in which two Indian wins and one Pakistan win can occur is Indian win, Indian win, Pakistan win.

$$\text{P (Indian win, Indian win, Pakistan win)} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$$

$$\text{P (iii)} = 3 \cdot \frac{1}{12} = \frac{1}{4}$$

(iv) Three Indian wins can occur in only one way. Indian win, Indian win, Indian win.

$$\text{P (iv)} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Required probability P of Indian winning the series is

$$\begin{aligned} P &= \text{P (i)} + \text{P (ii)} + \text{P (iii)} + \text{P (iv)} \\ &= \frac{1}{24} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1+3+6+3}{24} = \frac{13}{24} \end{aligned}$$

Ex. 8 : *A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even ?*

Sol. : Let probability of an odd number be P, so probability of an even number appearing is 2P.

Now there are 6 outcomes (1, 2, 3, 4, 5, 6)

$$\therefore \quad P(1) = P(3) = P(5) = P, \quad P(2) = P(4) = P(6) = 2P$$

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

$$\therefore \quad 9P = 1 \quad \therefore \quad P = \frac{1}{9}$$

When a die is thrown twice, the sample space is described as,

(1, 1),	(1, 2),	(1, 3),	(1, 4),	(1, 5),	(1, 6)
(2, 1),	(2, 2),	(2, 3),	(2, 4),	(2, 5),	(2, 6)
(3, 1),	(3, 2),	(3, 3),	(3, 4),	(3, 5),	(3, 6)
(4, 1),	(4, 2),	(4, 3),	(4, 4),	(4, 5),	(4, 6)
(5, 1),	(5, 2),	(5, 3),	(5, 4),	(5, 5),	(5, 6)
(6, 1),	(6, 2),	(6, 3),	(6, 4),	(6, 5),	(6, 6)

We want to find the probability of event of sum of the two numbers even.

In first row, cases (1, 1), (1, 3), (1, 5) are favourable to the event with probabilities

$$P(1, 1) = P(1, 3) = P(1, 5) = \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$$

In second row, cases (2, 2), (2, 4), (2, 6) are favourable to the event where

$$P(2, 2) = P(2, 4) = P(2, 6) = \frac{2}{9} \times \frac{2}{9} = \frac{4}{81}$$

Similarly 3rd and 5th row have cases (3, 1), (3, 3), (3, 5) and (5, 1), (5, 3), (5, 5); and 4th and 6th row have cases (4, 2), (4, 4), (4, 6) and (6, 2), (6, 4), (6, 6).

Required probability P is given by,

$$\begin{aligned} P = {} & P(1, 1) + P(1, 3) + P(1, 5) + P(2, 2) + P(2, 4) + P(2, 6) \\ & + P(3, 1) + P(3, 3) + P(3, 5) + P(4, 2) + P(4, 4) + P(4, 6) \\ & + P(5, 1) + P(5, 3) + P(5, 5) + P(6, 2) + P(6, 4) + P(6, 6) \\ = {} & \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{4}{81} + \frac{4}{81} + \frac{4}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{4}{81} + \frac{4}{81} + \frac{4}{81} \\ & + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{4}{81} + \frac{4}{81} + \frac{4}{81} = \frac{45}{81} = \frac{5}{9} \end{aligned}$$

Ex. 9 : *A is one of the eight horses entered for a race and is to be ridden by one of the two jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win, whereas with rider C, A's chance is doubled.*

(1) Find the probability that A wins.

(2) What are odds against A's winning ? ***(Dec. 2008)***

Sol. : (1) A can win in the following two mutually exclusive cases.

(i) B rides A and A wins.

(ii) C rides A and A wins.

$$P(i) = \frac{2}{3} \times \frac{1}{8} = \frac{1}{12} \qquad P(ii) = \frac{1}{3} \times \frac{2}{8} = \frac{1}{12}$$

Probability of A winning

$$= P(i) + P(ii) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

(2) Probability of A's losing $= 1 - \frac{1}{6} = \frac{5}{6}$.

Hence odds against A's winning are $\frac{5}{6} : \frac{1}{6}$, i.e. 5 : 1.

EXERCISE 6.1

1. A throw is made with two dice. Find the probability that (i) the sum is 7 or less, (ii) the sum is a perfect square.

Ans. (i) $\frac{7}{12}$; (ii) $\frac{7}{36}$.

2. Three coins are tossed simultaneously. Find the probability of getting at least 2 Heads. **Ans.** $\left(\frac{1}{2}\right)$

3. From a deck of 52 cards, two cards are drawn at random. Find the probability that (i) Both are Hearts, (i) Both the cards are of different suits. **Ans.** (i) $\frac{39}{633}$; (ii) $\frac{13}{17}$.

4. There are six married couples in a room. If two persons are chosen at random, find the probability that (i) they are of different sex, (ii) they are married to each other. **Ans.** (i) $\frac{6}{11}$; (ii) $\frac{1}{11}$.

5. Find the probability that six people selected at random will have six different birth dates.

[**Hint :** Sample space $(365)^6$] **Ans.** 0.9595.

6. A five figure number is formed by the digits 0, 1, 2, 3, 4, (without repetition). Find the probability that the number formed is divisible by 4. **Ans.** $\frac{3}{10}$.

7. A, B, C throw the coin alternatively in that order. One who gets Tail first wins the game. Find the probability of B winning the game if C has a start. **Ans.** $\frac{1}{7}$.

8. A box contains 5 red and 4 white marbles. 2 marbles are drawn successively from the box without replacement and it is noted that the second one is white. What is the probability that the first is also white ? **Ans.** $\frac{1}{6}$.

9. Box A contains 3 red and 2 blue marbles. The box B contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin shows Head, a marble is chosen from box A, if it shows Tail, a marble is chosen from box B. Find the probability that a red marble is chosen. **Ans.** $\frac{2}{5}$.

10. One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0.5$, $P(E_2) = 0.6$, $P(E_3) = 0.7$ and E_1, E_2, E_3 are independent events, then find the probability that at least two hits are registered. **Ans.** 0.65

11. A problem on computer mathematics is given to the three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved ? **Ans.** $\frac{29}{32}$.

12. Urn I contains 6 white and 4 black balls and urn II contains 4 white and 5 black balls. From urn I, two balls are transferred to urn II without noticing the colour. Sample of size 2 is then drawn without replacement from urn II. What is the probability that the sample contains exactly 1 white ball ? **Ans.** $\frac{4}{5}$.

6.6 PROBABILITY DISTRIBUTION

In Chapter 5, we have seen that statistical data can be presented in the form of frequency distribution, giving tabulated values of variate x and corresponding frequencies. Probability distribution for a variate x can be presented in a similar manner.

6.6.1 Random Variable, Probability Density Function Sample Space

If a trial or an experiment is conducted, the set S of all possible outcomes is called **sample space**.

In an experiment of tossing a fair coin, which results in Head H or Tail T, the sample space S = {H, T}. If a coin is tossed two times successively, all possible outcomes are HH, TT, HT, TH. Sample space in this case is the set S = {HH, TT, HT, TH}.

If a die is thrown two times successively, sample space

$$
\begin{aligned}
S &= \{(1, 1), (1, 2) \dots (1, 6)\} \\
&\quad \dots\dots\dots\dots\dots\dots \\
&\quad \dots\dots\dots\dots\dots\dots \\
&= \{(6, 1)\ (6, 2) \dots (6, 6)\}
\end{aligned}
$$

Random Variable : It is a real valued function defined over the sample space of an experiment. A variable whose value is a number determined by the out come of an experiment, associated with a sample spaces is called **random variable**. It is usually denoted by capital letter X or Y etc. If out comes are x_i $i = 1, 2, 3, \dots$ then $X(x_i)$ stands for the value x at $X = x_i$.

Probability Function : X is random variable with values x_i, $i = 0, 1, 2, \dots n$ and associated probabilities $P(x_i)$. The set p with elements $[x_i, P(x_i)]$ is called the probability function or probability distribution function of X. It can also be called **probability mass function or probability density function of x**.

Illustration

Ex. 1 : *A coin is tossed which results in Head or Tail. Let X be the random variable whose value for any outcome is the number of Heads obtained. Find the probability function of x and contruct a probability distribution table.*

Sol. : Let H denote a head and T a tail

Sample space is $S = \{H, T\}$

$$X(H) = 1, \ X(T) = 1$$

x is number of Heads which takes the values 0 and 1

$$f(x) = P(X = x)$$

$$f(0) = \frac{1}{2},\ f(1) = \frac{1}{2}$$

Probability distribution table is

X(x)	0	1
f(x)	$\frac{1}{2}$	$\frac{1}{2}$

Ex. 2 : *A coin is tossed two times successively X is the random variable whose value for any outcome is the number of heads obtained. Find the probability function of X and construct probability distribution table.*

Sol. : Sample space is $S = \{HH, TT, HT, TH\}$

$$X(x = 0) = 1,\quad X(x = 1) = 2,\quad X(x = 2) = 1$$

x is the number of heads which takes values 0, 1 and 2.

$$P(X = 0) = P(TT) = P(T)\,P(T) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$P(X = 1) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{4}$$

Probability distribution table is,

X (x)	0	1	2
P(X = x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Note that, $\sum P(x) = 1$

Ex. 3 : *For an experiment of simultaneous throw with three coins. The results of three tosses are independent of each other. Find the probability function and construct distribution table.*

Sol.: All the outcomes can be

HHH (all the three tosses giving Heads), HHT, HTH, HTT, THH, THT, TTH and TTT.

There are in all eight outcomes. Probability of any of these outcomes viz. the event HHH is given by;

$$P(HHH) = P(H)\,P(H)\,P(H)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

All these outcomes will have the same probabilities.

Let x denote the number of Heads appearing in each case, which takes values 0, 1, 2 and 3.

$\therefore$ (i) $P(x = 0) = P(TTT) = \frac{1}{8}$

(ii) $P(x = 1) = P(HTT) + P(TTH) + P(THT)$, mutually exclusive events

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(iii) $P(x = 2) = P(HTH) + P(THH) + P(HHT)$, mututally exclusive events

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(iv) $P(x = 3) = P(HHH) = \frac{1}{8}$

Probability distribution table is

x	0	1	2	3
f	1	3	3	1
P (x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Here the variate x is taking the values x = 0, 1, 2, 3 and the corresponding probabilities are $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ respectively.

Thus, in the distribution, if the frequencies are replaced by corresponding relative frequencies, then it is called probabilities in the probability distribution.

Here, $\sum P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$ which is always the case when all possible outcomes are considered. Thus, we note that

(i) $P(x) \geq 0$ for all x (ii) $\sum P(x) = 1$.

Ex. 4 : *The outcomes of a certain experiment are $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. The associated probability function is*

$$P(x) = \begin{cases} k, & x = 1 \\ 2k, & x = 2 \\ 5k, & x = 3 \end{cases}$$

Find P (x < 2), P (x ≤ 2), P (x ≤ 3).

Sol.: $P(x(x_1)) + P(x(x_2)) + P(x(x_3)) = 1$

$$\therefore \quad k + 2k + 5k = 1, \; 8k = 1 \; \therefore \; k = \frac{1}{8}$$

$$P(x < 2) = P(x(x_1)) = \frac{1}{8}$$

$$P(x \leq 2) = P(x = 1) + P(x = 2)$$

$$= \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$P(x \leq 3) = P(x = 1) + P(x = 2) + P(x = 3)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{5}{8} = \frac{8}{8} = 1$$

Ex. 5 : *Given the following probability function*

x	*0*	*1*	*2*	*3*	*4*	*5*
P(x)	*0*	*c*	*2c*	*2c*	*1c*	$7c^2$

(i) Find c (ii) Find P (x ≥ 2) (iii) Find P (x < 3).

Sol.:

(i) $c + 2c + 2c + c + 7c^2 = 1$

$$\therefore \quad 7c^2 + 6c - 1 = 0$$

$$(7c - 1)(c + 1) = 0, \; 7c - 1 = 0 \text{ or } c = \frac{1}{7}$$

c = –1 is rejected as probability cannot be negative.

(ii) $P(x \geq 2) = P(x = 2) + P(x = 3) + P(x = 4) + P(x = 5)$

$$= \frac{2}{7} + \frac{2}{7} + \frac{1}{7} + \frac{7}{49} = \frac{6}{49}$$

(iii) $P(x < 3) = P(x = 0) + P(x = 1) + P(x = 2)$

$$= 0 + \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$$

6.6.2 Mathematical Expectation

If X is random variable with all possible values $x_1, x_2, x_3, \ldots x_n$ and probability functions (probabilities) $p(x_1), p(x_2), p(x_3), \ldots p(x_n)$, then the mathematical expectation of X is denoted by E(x) and is given by;

$$E(x) = \sum_{i=1}^{n} x_i P(x_i)$$

Since, $\sum P(x_i) = 1$.

E (x) is called ***expected value*** or ***mean value*** of X.

Note :

1. **E(x) is the arithmetic mean μ of random variable** X **:**

We note that corresponding to the number $\bar{x}$ which is Arithmetic Mean in frequency distribution and is given by;

$$\bar{x} = \frac{\sum f x}{\sum f} = \frac{f_1}{\sum f} x_1 + \frac{f_2}{\sum f} x_2 + \ldots + \frac{f_n}{\sum f} x_n$$

We have, $$\bar{x} = \sum_{i=1}^{n} P_i x_i = E(x) = \mu$$

where, $P_i = \frac{f_i}{\sum f_i}$, $i = 1, 2, \ldots n$ are the relative frequencies of $x_1, x_2, \ldots, x_n$ respectively.

2. **Variance Var (X) = σ^2 of a random variable X :**

The expected value of X, E(X) provides a measure of central tendency of the probability distribution. However, it does not provide any idea regarding the spread of the distribution. Thus, we define variance of X.

If X be a random variable with probability distributions $[x_i, P_i]$, $i = 1, 2, \ldots n$. Variance of X denoted by;

$$\text{Var}(X) = \sigma^2 = E[X - E(X)]^2$$

The mean of X is generally denoted by μ.

Thus, $$\text{Var}(X) = \sigma^2 = E(X - \mu)^2$$

For computation simplification,

$$\text{Var}(X) = \sigma^2 = E(x - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P_i$$

$$= \sum_{i=1}^{n} x_i^2 P_i - 2\mu \sum_{i=1}^{n} x_i P_i + \mu_2^2 \sum P_i$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$\therefore$ $$\text{Var}(X) = \sigma^2 = E(X^2) - [E(X)]^2$$

3. **Moments of a random variable :**

The mean measures central tendency. While the variance measure spread. In order to get complete information of the probability distribution, we also have to study the shape of the probability distribution function. Thus, we need measures of skewness (lack of symmetry) and kurtosis (peakedness) of probability distribution. Moments of a random variable serves the purpose.

Let $[x_i, P_i]$, $i = 1, 2, 3, \ldots n$ represent a probability distribution of random variable X.

(i) Moments about any arbitrary point a :

$$\mu'_r = \mu'_r(a) = E(X - a)^r = \sum_{i=1}^{n} (x_i - a)^r P_i, \quad r = 1, 2, 3, \ldots$$

In particular, $$\mu'_1(a) = E(X - a) = E(X) - a$$

$$\mu'_2(a) = E(X - a)^2$$

(ii) Moments about the origin (i.e. zero) or Raw moments :

$$\mu'_r = \mu'_r(0) = E(X-0)^r = \sum_{i=1}^{n} x_i^r P_i, \quad r = 1, 2, 3 \ldots$$

In particular, $\mu'_1 = E(X) = \text{mean}$

$$\mu'_2 = E(X^2) = \sum_{i=1}^{n} x_i^2 P_i^2$$

(iii) Moments about the arithmetic mean E(x) or Central moments :

$$\mu_r = \mu'_r(E(X)) = E[X - E(X)]^r$$

$$= \sum_{i=1}^{n} [x_i - E(x)]^r, \quad r = 1, 2, 3$$

In particular, $\mu_1 = E[X - E(X)] = E(x) - E(x) = 0$

$$\mu_2 = E[X - E(X)]^2 = \text{Var}(X) \text{ etc.}$$

Ex. 6 : *A die is tossed once. Random variable x denote the digit that appears. Find the expectation of x.*

Sol.:

x	1	2	3	4	5	6
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(x) = \sum x\, P(x)$$

$$= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= \frac{(1+2+3+4+5+6)}{6} = \frac{21}{6} = \frac{7}{2}$$

Ex. 7 : *A die is thrown twice. X denote the sum of digits in two throws. Find mathematical expectation of x.*

Sol.:

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(x) = \sum x\, P(x) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = \frac{252}{36} = 7$$

$$E(x) = 7$$

Ex. 8 : *There are three envelopes containing ₹ 100, ₹ 400, and ₹ 700 respectively. A player selects an envelop and keep with him, what he gets. Find the expected gain of the player.*

Sol.: Each envelope has probability $\frac{1}{3}$ of getting selected probability table is;

x	100	400	700
P(x)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$E(x) = \sum xP(x) = 100 \times \frac{1}{3} + 400 \times \frac{1}{3} + 700 \times \frac{1}{3}$$

$$= \frac{1200}{3} = 400$$

Expected gain is ₹400.

Ex. 9 : *If random variable X takes the values X = 1, 2, 3 with corresponding probabilities* $\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$. *Find E* (x^2).

Sol.: Probability distribution table is

x	1	2	3
P(x)	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

$$E(x^2) = \sum x^2 \cdot P(x) = (1)^2 \times \frac{1}{6} + (2)^2 \times \frac{2}{3} + (3)^2 \times \frac{1}{6}$$

$$= \frac{1}{6} + \frac{8}{3} + \frac{9}{6} = \frac{1 + 16 + 9}{6} = \frac{26}{6} = \frac{13}{3}$$

$$E(x) = 1 \times \frac{1}{6} + 2 \times \frac{2}{3} + 3 \times \frac{1}{6}$$

$$= \frac{1}{6} + \frac{4}{3} + \frac{1}{2} = \frac{1 + 8 + 3}{6} = \frac{12}{6} = 2$$

$$[E(x)]^2 = 4$$

It is clear that $E(x^2) \neq [E(x)]^2$

Ex. 10 : *Variable x takes the values 0, 1, 2, 3, 4, 5 with probability of each as* $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{3}{15}, \frac{5}{15}, \frac{1}{15}$, *find expectation of x.*

Sol.: Given that

x	0	1	2	3	4	5
P (x)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{3}{15}$	$\frac{5}{15}$	$\frac{1}{15}$

$$E(x) = \sum xP(x) = 0 \times \frac{1}{15} + 1 \times \frac{2}{15} + 2 \times \frac{3}{15} + 3 \times \frac{3}{15} \times 4 \times \frac{5}{15} + 5 \times \frac{1}{15}$$

$$= \frac{2 + 6 + 9 + 20 + 5}{15} = \frac{42}{15}$$

6.6.3 Continuous Random Variables

In our discussion so far random variable X was taking discrete values. X can take the values in the interval $a \leq x \leq b$ or in the range [a, b], for example, temperature, height and weights take all values in the interval. Let X be a random variable which can take all the values in some interval, then x is called continuous random variable.

The ***probability density function f(x)***, called **p.d.f.** satisfies following conditions

(i) $f(x) \geq 0$ for all $x \in R_x$

(ii) $\int_{R_x} f(x)\, dx = 1$

We define probability for any interval $c < x < d$

$$P(c < x < d) = \int_c^d f(x)\, dx$$

Ex. 11 : *Given the density function*

$$f(x) = ke^{-\alpha x}, \quad x \geq 0, \alpha > 0$$
$$= 0, \quad \text{otherwise}$$

Find k.

Sol.: We have $\int_{-\infty}^{\infty} f(x, \alpha)\, dx = \int_{-\infty}^{0} f(x, \alpha)\, dx + \int_{0}^{\infty} f(x, \alpha)\, dx = 1 = 0 + \int_{0}^{\infty} k\, e^{-\alpha x}\, dx = 1$

$$\therefore \quad k\left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = 1 \quad \text{or} \quad k\left[\frac{1}{\alpha}\right] = 1$$

$$\therefore \quad k = \alpha$$

Ex. 12 : *A continuous random variable x has the p.d.f. defined as*

$$P(x) = \frac{1}{2}x, \quad 0 < x \leq 1$$
$$= \frac{1}{4}(3 - x), \quad 1 < x \leq 2$$
$$= \frac{1}{4}, \quad 2 < x \leq 3$$
$$= \frac{1}{4}(4 - x), \quad 3 < x \leq 4$$

(i) Compute P (3 < x ≤ 4) (ii) Compute P (1 < x ≤ 4)

Sol.:

(i) $$P(x \geq 3) = \int_3^4 P(x)\, dx = \int_3^4 \frac{1}{4}(4 - x)dx = \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4 = \frac{1}{4}\left[(16 - 8) - \left(12 - \frac{9}{2}\right)\right]$$

$$= \frac{1}{4}\left[8 - 12 + \frac{9}{2}\right] = \frac{1}{4}\left[-4 + \frac{9}{2}\right] = \frac{1}{8}$$

(ii) $$P(1 < x \leq 4) = \int_1^2 \frac{1}{4}(3 - x)\, dx + \int_2^3 \frac{1}{4}\, dx + \int_3^4 \frac{1}{4}(4 - x)\, dx$$

$$= \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^2 + \frac{1}{4}\, [x]_2^3 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{4}\left[(6 - 2) - \left(3 - \frac{1}{2}\right) + \frac{1}{4}(3 - 2) + \frac{1}{8}\right] = \frac{1}{4}\left[4 - \frac{5}{2} + 1 + \frac{1}{2}\right]$$

$$= \frac{3}{4}$$

Ex. 13 : *The probability density function f(x) of a continuous random variable x is defined by* $f(x) = ke^{-|x|}$, $x \in (-\infty, \infty)$. *Find the value of k.*

Sol.: We have

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} ke^{x}\, dx + \int_{0}^{\infty} ke^{-x}\, dx = 1$$

$$\therefore \quad k\left[(e^x)_{-\infty}^{0} + (-e^{-x})_0^{\infty}\right] = 1$$

$$k[1 + 1] = 1$$

$$\text{or} \quad k = \frac{1}{2}$$

6.7 BINOMIAL PROBABILITY DISTRIBUTIONS

Consider the experiment or a trial which has only two outcomes, a success or failure with p as the probability of success and q as the probability of failure. Since there are only two outcomes, $p + q = 1$.

Let us consider series of n such independent trials each of which either results in success or failure.
To find the probability of r successes in n trials, consider one run of outcomes.

$$\underbrace{SSS \,\ldots\ldots\, S}_{r} \quad \underbrace{FFF \,\ldots\ldots\, F}_{n-r}$$

In which there are r consecutive successes and $n - r$ failures.

Probability of this event P(r success in n trial) is given by;

$$\begin{aligned} P\,(SSS \ldots S\ FFF \ldots F) &= P\,(S)\,P\,(S) \ldots (r \text{ times}) \times P\,(F)\,P\,(F) \ldots \{(n-r) \text{ times}\} \\ &= p\,p \ldots p\ (r \text{ times}) \times q\,q \ldots q\ (n - r \text{ times}) \\ &= p^r\, q^{n-r} \end{aligned}$$

r success and $n - r$ failures can occur in ^nC_r mutually exclusive cases each of which has the probability $p^r\, q^{n-r}$.

$\therefore$ Probability of r success in n trials is $^nC_r \cdot p^r\, q^{n-r}$. This formula gives probability of $r = 0, 1, 2, 3, \ldots n$ success in n trials.

Putting it in tabular form,

r	0	1	2	3	......	n
p (r)	$^nC_0\, p^0\, q^n$	$^nC_1\, p^1\, q^{n-1}$	$^nC_2\, p^2\, q^{n-2}$	$^nC_3\, p^3\, q^{n-3}$	......	$^nC_n\, p^n\, q^{n-n}$

$$^nC_0 = 1, \quad ^nC_n = 1$$

Consider now the Binomial expansion of

$$(q + p)^n = q^n + {^nC_1}\, q^{n-1}\, p + {^nC_2}\, q^{n-2}\, p^2 + \ldots + p^n$$

Terms on R.H.S. of this expansion give probability of $r = 0, 1, 2, \ldots, n$ success. This is the reason for above probability distribution to be called Binomial probability distribution. It is denoted by B (n, p, r).

Thus, $$B\,(n, p, r) = {^nC_r}\, p^r\, q^{n-r}$$

If n independent trials constitute one experiment and the experiment is repeated N times then r successes would be expected to occur $N \times {^nC_r}\, p^r\, q^{n-r}$ times. This is called the expected frequency of r success in N experiments.

6.7.1 Mean and Variance of Binomial Distribution :

We shall first obtain moments of the binomial distribution $X \to B(n, p, r)$ about $r = 0$ (about origin) (refer article 6.6.2).

$$\begin{aligned} \mu_1' &= \mu_1 = E(x - 0) = \text{Mean} = \sum_{r=0}^{n} r\, p(r) \\ &= \sum_{r=0}^{n} r \cdot {^nC_r}\, p^r\, q^{n-r} \\ &= 0 \cdot q^n + 1 \cdot {^nC_1}\, p^1\, q^{n-1} + 2 \cdot {^nC_2}\, p^2\, q^{n-2} + 3 \cdot {^nC_3}\, p^3\, q^{n-3} \ldots np^n \end{aligned}$$

$$= npq^{n-1} + 2 \cdot \frac{n(n-1)}{2!} p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{2!} p^3 q^{n-3} \ldots np^n$$

$$= np \left[q^{n-1} + (n-1) pq^{n-2} + \frac{(n-1)(n-2)}{2!} p^2 q^{n-3} \ldots p^{n-1} \right]$$

$$\mu_1' = np\,[(q+p)^{n-1}] = np$$

$$\therefore \qquad \boxed{\mu_1' = \mu_1 = E(X) = \text{Mean} = np}$$

Hence, Mean of the Binomial distribution (which is also the expectation of variable r) is np.

Next, consider second moment about origin (refer article 6.6.2),

$$\mu_2' = E[X^2] = E[X(X-1)] + E(X)$$

$$\mu_2' = \sum_{r=0}^{n} r^2\, {}^nC_r\, p^r q^{n-r} = \sum_{r=0}^{n} \{r(r-1) + r\}\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + np$$

$$= 1.2\, {}^nC_2\, p^2 q^{n-2} + 2.3\, {}^nC_3\, p^3 q^{n-3} + 3.4\, {}^nC_4\, p^4 q^{n-4} \ldots n(n-1)\, p^n + np$$

$$= 1.2 \frac{n(n-1)}{2!} p^2 q^{n-2} + 2.3 \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + 3.4 \frac{n(n-1)(n-2)(n-3)}{4!} p^4 q^{n-4} \ldots n(n-1)\, p^n + np$$

$$= n(n-1) \times p^2 \left[q^{n-2} + (n-2)\, p\, q^{n-3} + \frac{(n-2)(n-3)}{2!} p^2 q^{n-4} \ldots p^{n-2} \right] + np$$

$$= n(n-1)\, p^2 \{q+p\}^{n-2} + np = n(n-1)\, p^2 + np$$

$$= n^2 p^2 - np^2 + np = n^2 p^2 + np\,\{1-p\}$$

$$= n^2 p^2 + npq$$

$$\therefore \qquad \mu_2 = \text{Variance} = \mu_2' - \mu_1'^2 = E(X^2) - [E(X)]^2$$

$$= n^2 p^2 + npq - (np)^2$$

$$\therefore \qquad \boxed{\text{Var}(X) = \sigma^2 = npq}$$

$$\text{and} \qquad \boxed{\text{S.D.} = \sigma = \sqrt{npq}}$$

ILLUSTRATIONS

Ex. 1 : *An unbiased coin is thrown 10 times. Find the probability of getting exactly 6 Heads, at least 6 Heads.*

Sol. : Here $p = q = \frac{1}{2}$ and $n = 10$. Here occurrence of Head is treated as success.

Probability of getting exactly 6 Heads is

$$P(\text{exactly 6 heads}) = {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4$$

Events of at least six Heads occur when coin shows up Head 6, 7, 8, 9 or 10 times the probabilities for these events are

$$P(7) = {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 \qquad P(8) = {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2$$

$$P(9) = {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 \qquad P(10) = {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^{10}$$

$$\begin{aligned} p\text{ (at least 6 Heads)} &= p(6) + p(7) + p(8) + p(9) + p(10) \\ &= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right] \\ &= \frac{386}{1024} = 0.37695 \end{aligned}$$

[The events of 6, 7, ... etc. Heads are mutually exclusive].

Ex. 2 : *Probability of Man aged 60 years will live for 70 years is $\frac{1}{10}$. Find the probability of 5 mean selected at random 2 will live for 70 years.*

Sol. : Here $p = \frac{1}{10}$, $q = \frac{9}{10}$, $r = 2$, $n = 5$.

$$P\text{ (2 men living for 70 years)} = {}^5C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^3 = 0.0729$$

Ex. 3 : *On an average a box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives ?* ***(May 2016)***

p = Probability of box containing defective articles = $\frac{2}{10} = \frac{1}{5}$

q = Probability of non-defective items = $\frac{4}{5}$

Sol. : Probability of box containing three or less defective articles

$P(r \le 3) = p(r = 0) + p(r = 1) + p(r = 2) + p(r = 3)$ [r denotes the number of defective items.]

$$P(r = 0) = {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} = 0.1074$$

$$P(r = 1) = {}^{10}C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9 = 0.2684$$

$$P(r = 2) = {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8 = 0.302$$

$$P(r = 3) = {}^{10}C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7 = 0.2013$$

$$P(r \le 3) = 0.1074 + 0.2684 + 0.302 + 0.2013 = 0.8791$$

$$\therefore \quad NP(r \le 3) = 100 \times 0.8791 = 87.91$$

88 boxes are expected to contain three or less defectives.

Ex. 4 : *The probability of a man hitting a target is $\frac{1}{3}$. If he fires 5 times, what is the probability of his hitting the target at least twice ?*

Sol. : Probability of hitting target = $p = \frac{1}{3}$

Probability of no hit (failure) = $q = \frac{2}{3}$

r denotes the number of hits (successes) and n = 5.

Probability of man hitting the target at least twice.

$$\begin{aligned} P(r \geq 2) &= P(r = 2) + P(r = 3) + P(r = 4) + P(r = 5) \\ &= 1 - P(r < 2) \\ &= 1 - P(r = 0) - P(r = 1) \\ &= 1 - {}^5C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^5 - {}^5C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4 \\ &= \frac{131}{243} = 0.539 \end{aligned}$$

Ex. 5 : *A coin is so biased that appearance of head is twice likely as that of tail. If a throw is made 6 times, find the probability that atleast 2 heads will appear.*

Sol.: Here $P(H) = \frac{2}{3}, q = P(T) = \frac{1}{3}$

A is the event of appearance of atleast two heads

$$\begin{aligned} P(A \geq 2) &= P(2) + P(3) + P(4) + P(5) + P(6) \\ &= 1 - P(0) - P(1) \\ &= 1 - {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6 - {}^6C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^5 \\ &= 1 - \left\{1 \times \left(\frac{1}{3}\right)^6 + 6 \times \frac{2}{3} \left(\frac{1}{3}\right)^5\right\} \\ &= 0.98213 \end{aligned}$$

Ex. 6 : *Assume that an average one telephone number out of 15 called between 12 p.m. to 3 p.m. on week days is busy. What is the probability that if 6 randomly selected telephone numbers called (i) not more than 3 (ii) at least 3 of them be busy?*

Sol. : The probability that the telephone number, called beween 12 p.m. to 3 p.m., is busy is

$$p = \frac{1}{15} \qquad \therefore \quad q = 1 - \frac{1}{15} = \frac{14}{15} \qquad (\because p + q = 1)$$

Hence, probability that r numbers called out of 6 called (by binomial distribution is)

$$P(r) = {}^6C_r\, p^r\, q^{n-r} = {}^6C_r \left(\frac{1}{15}\right)^r \left(\frac{14}{15}\right)^{6-r} \qquad \ldots (1)$$

(i) For not more than 3 calls be busy,

$$\begin{aligned} P(r \leq 3) &= P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3) \\ &= {}^6C_0 \left(\frac{1}{15}\right)^0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{1}{15}\right)^1 \left(\frac{14}{15}\right)^5 + {}^6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4 + {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3 \\ &= \frac{(14)^3}{(15)^6} \left[(14)^3 + 6(14)^2 + \frac{6 \cdot 5}{2 \cdot 1}(14) + \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}(1)\right] \\ &= (0.002409)\ [2744 + 1176 + 210 + 20] \\ &= 0.997 \end{aligned}$$

(ii) For at least 3 calls to be busy of 6 balls, we have the probability as

$$\begin{aligned} P(r \geq 3) &= 1 - P(r < 3) \\ &= 1 - [P(r = 0) + P(r = 1) + P(r = 2)] \\ &= 1 - \left[{}^6C_0 \left(\frac{1}{15}\right)^0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{1}{15}\right)^1 \left(\frac{14}{15}\right)^5 + {}^6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4\right] \end{aligned}$$

$$= 1 - \frac{(14)^4}{(15)^6}\left[(14)^2 + 6(14) + \frac{6\cdot5}{2\cdot1}(1)\right]$$

$$= 1 - (0.00034)[196 + 84 + 15]$$

$$= 1 - 0.9949$$

$$= 0.0051$$

Ex. 7 : *20% of bolts produced by a machine are defective. Determine the probability that out of 4 bolts chosen at random (i) 1 is defective (ii) zero are defective (iii) at most 2 bolts are defective.*

Sol. : The probability of defective bolts is

$$p = \frac{20}{100} = 0.2 \quad \therefore \quad q = 0.8 \qquad (\because p + q = 1)$$

(i) The probability of having 1 defective bolts out of 4 is

$$P(r = 1) = {}^4C_1 (0.2)^1 (0.8)^3 = 4(0.2)(0.8)^3 = 0.4096$$

(ii) The probability of having zero bolts defective is

$$P(r = 0) = {}^4C_0 (0.2)^0 (0.8)^4 = 4(0.8)^4 = 0.4096$$

(iii) The probability of having at most 2 bolts out of 4 is

$$\begin{aligned} P(r \le 2) &= P(r = 0) + P(r = 1) + P(r = 2) \\ &= {}^4C_0 (0.2)^0 (0.8)^4 + {}^4C_1 (0.2)^1 (0.8)^3 + {}^4C_2 (0.2)^2 (0.8)^2 \\ &= (0.8)^2\left[(0.8)^2 + 4(0.2)(0.8) + \frac{4.3}{2.1}(0.2)^2\right] \\ &= 0.9728 \end{aligned}$$

Ex. 8 : *Out of 2000 families with 4 children each, how many would you expect to have (i) at least a boy (ii) 2 boys, (iii) 1 or 2 girls, (iv) no girls ?*

Sol. :

$$p = \text{probability of having a boy} = \frac{1}{2}$$

$$q = \text{probability of having a girl} = 1 - \frac{1}{2} = \frac{1}{2} \qquad (\because p + q = 1)$$

(i)

$$\begin{aligned} p(\text{at least a body}) &= P(r \ge 1) \\ &= 1 - P(r < 1) = 1 - P(r = 0) \\ &= 1 - {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16} = \frac{15}{16} \end{aligned}$$

Hence, expected number of families having at least a boy,

$$2000\, P(r \ge 1) = 2000 \times \frac{15}{16} = 1875$$

(ii)

$$\begin{aligned} P(\text{having 2 boys}) &= P(r = 2) \\ &= {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{4.3}{2.1} \times \frac{1}{16} = \frac{3}{8} \end{aligned}$$

Expected number of families having 2 boys

$$2000\, P(r = 2) = 2000 \times \frac{3}{8} = 750$$

(iii)

$$\begin{aligned} P(\text{having 1 or 2 girls}) &= P(\text{having 3 boys or 2 boys}) \\ &= P(r = 3) + P(r = 2) \\ &= {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 + {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{(2)^4}\left[4 + \frac{4.3}{2.1}\right] = \frac{10}{16} = \frac{5}{8} \end{aligned}$$

Expected number of families having 1 or 2 girls

$$2000\,[P(r = 3) + p(r = 2)] = 2000 \times \frac{5}{8} = 1250$$

(iv) $$P(\text{having no girls}) = P(\text{having 4 boys}) = P(r = 4)$$

$$= {}^4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 = \frac{1}{16}$$

Expected number of families having no girls

$$2000\,[p(r = 4)] = 2000 \times \frac{1}{16} = 125$$

Ex. 9 : *In a quality control department of a rubber tube manufacturing factory, 10 rubber tubes are randomly selected from each day's production for inspection. If not more than 1 of the 10 tubes is found to be defective, the production lot is approved. Otherwise it is rejected. Find the probability of the rejection of a day's production lot if the true proportion of defectives in the lot is 0.3.*

Sol. : Suppose X denotes the number of defective tubes in the 10 randomly selected tubes.

$\therefore \quad X \to B\,(n = 10,\ p = 0.3)$

The production lot is accepted if not more than one tube (i.e. at the most one tube) is found defective.

$$\therefore \quad P\,(\text{Accepting the lot}) = P\,(X = 0) + P\,(X = 1)$$

$$= q^n + npq^{n-1}$$

$$= (0.7)^{10} + 10\,(0.3)\,(0.7)^9$$

$$= (0.7)^9\,[3.7] = 0.1493$$

$$\therefore \quad P\,(\text{Rejection of the lot}) = 1 - 0.1493 = 0.8507$$

Ex. 10 : *A department in a works has 10 machines which may need adjustment from time to time during the day. Three of these machines are old, each having a probability of $\frac{1}{10}$ of needing adjustment during the day and 7 are new, having corresponding probability $\frac{1}{20}$. Assuming that no machine needs adjustment twice on the same day, determine the probabilities that on a particular day.*

(i) just two old and no new machines need adjustment.

(ii) just two machines need adjustment which are of the same type.

Sol. : Out of 3 old machines, if 2 need adjustment then this combination of 2 needing adjustment and one needing cannot occur in ${}^3C_2$ ways.

$$p_1 = \text{Probability of old machine needing adjustment} = \frac{1}{10}$$

$$q_1 = \text{Probability of old machine not needing an adjustment} = \frac{9}{10}$$

$$p_2 = \text{Probability of new machine needing an adjustment} = \frac{1}{20}$$

$$q_2 = \text{Probability of new machine not needing an adjustment} = \frac{19}{20}$$

We are looking for an event A where 2 old machines needing adjustment alongwith one not needing and remaining 7 new machines not needing an adjustment. Hence

$$P\,(A) = {}^3C_2 \left[\left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)\right]\left(\frac{19}{20}\right)^7 = \frac{3 \times 9 \times (19)^7}{10^3\,(20)^7} = 0.0189$$

Consider the event B in which case, no old machine out of 3 need adjustment and 2 out of 7 new machines need adjustment. Proceeding in the same way

$$P(B) = \left(\frac{9}{10}\right)^3 \left[{}^7C_2 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^5\right] = \frac{7 \cdot 6}{1 \cdot 2} \left(\frac{9}{10}\right)^3 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^5$$

$$= 21 \left(\frac{9}{10}\right)^3 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^5 = 0.0296$$

Event, two machines needing an adjustment which are of the same type A + B.

$$\therefore \quad P(A + B) = P(A) + P(B) \text{ [A, B are mutually exclusive].}$$

$$\text{Required probability} = 0.0189 + 0.0296 = 0.0485$$

Ex. 11 : *A r.v. $X \to B\ (n = 6, p)$. Find p if 9 P (R = 4) = P (R = 2).*

Sol. : $P(R = r) = {}^nC_r\, p^r q^{n-r} \quad r = 0, 1, \ldots n$

Here, $n = 6$

$\therefore \quad 9P(4) = P(2)$

$$\Rightarrow \quad 9 \cdot \binom{6}{4} p^4 q^2 = \binom{6}{2} p^2 q^4$$

$$\therefore \quad 9p^2 = q^2 \qquad \because \binom{6}{4} = \binom{6}{2}$$

$$\therefore \quad 9p^2 = (1 - p)^2 = 1 - 2p + p^2$$

$$\therefore \quad 8p^2 + 2p - 1 = 0$$

$$\therefore \quad (4p - 1)(2p + 1) = 0$$

$$\Rightarrow \quad p = \frac{1}{4} \text{ or } p = -\frac{1}{2}$$

The value $p = -\frac{1}{2}$ is inadmissible. Hence, the answer is $p = \frac{1}{4}$.

Ex. 12 : *Point out the fallacy of the statement 'The Mean of Binomial distribution is 3 and variance 5'.*

Sol. : Given Mean = np = 3, Variance = npq = 5

$$\therefore \quad q = \frac{npq}{np} = \frac{5}{3} > 1$$

which is not possible since probability cannot exceed unity.

Ex. 13 : *The Mean and Variance of Binomial distribution are 4 and 2 respectively. Find p (r $\geq$ 1).* ***(Dec. 2012)***

Sol. : Here r denotes the number of successes in n trials. Given that

$$\text{Mean} = np = 6 \quad \text{and} \quad \text{Variance} = npq = 2$$

$$\therefore \quad q = \frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$np = 6 \quad \therefore \quad n = \frac{6}{2/3} = 9$$

$$P(r \geq 1) = 1 - P(r = 0) = 1 - q^n = 1 - \left(\frac{1}{3}\right)^9 = 0.999949$$

6.8 HYPERGEOMETRIC DISTRIBUTION

We know that binomial distribution is applied whenever we draw a random sample *with* replacement. This is because, in sampling with replacement, the probability of getting 'success' p, remains same at every draw. Also, the successive draws remain independent. Thus, the assumptions of binomial experiment are satisfied. Now, consider the following situation.

A bag contains 4 red and 5 black balls. Suppose 3 balls are drawn at random from this bag *without* replacement and we are interested in the number of red balls drawn. Clearly at the first draw, probability of getting a red ball is $\frac{4}{9}$. Now, suppose a red ball is selected at the first draw. Because, it would be kept aside, the probability of getting a red ball at the second draw would be $\frac{3}{8}$. Thus 'p' does not remain constant. Also, the successive draws are not independent. Probability of getting red balls in the second draw is dependent on which ball you have drawn at the first draw. Thus, in case of sampling *without* replacement, the binomial distribution cannot be applied.

In such situations the hypergeometric distribution is used. Consider the following situation.

Suppose a bag contains N balls of which M are red and N – M are black. A sample of 'n' balls is drawn *without replacement* from the N balls. Let X denote the number of red balls in the *sample*. Hence, the possible values of X are 0, 1, 2, ..., n (assuming $n \leq M$). The p.m.f. is obtained in the following manner.

We want to get P [X = x].

$$\begin{array}{ccc} & N & \\ & \downarrow & \\ \downarrow & & \downarrow \\ M & & N-M \\ \downarrow & & \downarrow \\ x & & n-x \end{array}$$

If the sample of 'n' balls contains 'x' red balls, then it will contain 'n – x' black balls. Hence, number of ways in which x red balls can be selected from M red balls is $\binom{M}{x}$ and number of ways in which n – x black balls can be selected from N – M black balls is $\binom{N-M}{n-x}$. The sample contains both red and black balls. Therefore, the total number of ways in which the above event can occur is $\binom{M}{x}\binom{N-M}{n-x}$. In all 'n' balls are selected from N balls. Therefore, the total number of possible selections is $\binom{N}{n}$. Using the definition of probability of an event, we get,

$$P(x) = P[X = x] = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \quad ; \quad x = 0, 1, \ldots, \min(n, M)$$

$$= 0 \quad ; \quad \text{otherwise}$$

The above P (x) is called as the p.m.f. of hypergeometric distribution with parameters N, M and n.

Notation : $X \rightarrow H(N, M, n)$.

If we don't assume $n \leq M$, then the range X is 0, 1, 2, ..., min (n, M). This is because at the most M red balls can be there in the sample.

Remark : Applicability of Hypergeometric Distribution

Hypergeometric distribution is applied whenever a random sample is taken *without* replacement from a population consisting of two classes. Following are some such situations.

(i) In quality control department, a random sample of items is inspected from a consignment containing defective and non-defective items.

(ii) A lake contains N fish. A sample of fish is taken from the lake, marked and released back in the lake. Next time, another sample of fish is selected and number of marked fish are counted.

(iii) A committee of n persons is to be formed from N persons of whom M are ladies and N – M are gentlemen. The number of ladies on the committee follows hypergeometric distribution.

(iv) In opinion surveys, where the persons have to give answers of 'yes', 'no' type.

The following conditions should be satisfied for the application of hypergeometric distribution.

1. The population is divided into two mutually exclusive categories.
2. The successive outcomes are dependent.
3. The probability of 'success' changes from trial to trial.
4. The number of draws are fixed.

ILLUSTRATIONS

Ex. 1 : *A room has 4 sockets. From a collection of 12 bulbs of which only 5 are good. A person selects 4 bulbs at random (without replacement) and puts them in the sockets. Find the probability that (i) the room is lighted, (ii) exactly one bulb in the selected bulbs is good.*

Sol. : Notice that N = 12, M = 5, n = 4, X = number of good bulbs in the sample.

$\therefore \quad X \to H\,(N = 12, M = 5, n = 4)$

$$\therefore \qquad P(x) = \frac{\binom{5}{x}\binom{7}{4-x}}{\binom{12}{4}}\ ; \qquad x = 0, 1, \ldots, 4$$

(i) The room is lighted even if a single bulb is good. Therefore the required probability is

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{5}{0}\binom{7}{4}}{\binom{12}{4}} = 0.9292$$

(ii)
$$P[X = 1] = \frac{\binom{5}{1}\binom{7}{3}}{\binom{12}{4}} = 0.707$$

Ex. 2 : *Among the 200 employees of a company, 160 are union numbers and the others are non-union. If four employees are to be chosen to serve on the staff welfare committee, find the probability that two of them will be union members and the others non-union, using hypergeometric distribution.*

Sol. : Let X denote number of union members selected in the sample.

$\therefore \quad X \to H\,(N = 200, M = 160, n = 4).$

The required probability is

$$P[X = 2] = \frac{\binom{160}{2}\binom{40}{2}}{\binom{200}{4}} = \frac{\dfrac{160 \times 159}{2} \times \dfrac{40 \times 39}{2}}{\dfrac{200 \times 199 \times 198 \times 197}{4 \times 3 \times 2}} = 0.1534$$

6.9 POISSON DISTRIBUTION

Poisson distribution is the discrete probability distribution of a discrete random variable X.

When 'p' be the probability of success is very small and n the number of trials is very large and np is finite then we get another distribution called Poisson distribution. It is considered as limiting case of Binomial distribution with $n \to \infty$, $p \to 0$ and np remaining finite.

Consider the Binomial distribution

$$B(n, p, r) = {}^nC_r\ p^r\, q^{n-r} = \frac{n(n-1)(n-2)\ \ldots\ (n-(r-1))}{r!}\ p^r\ (1-p)^{n-r}$$

Let $z = np \quad \therefore \quad p = \frac{z}{n}$

Hence, $$B(n, p, r) = \frac{np\,(np - p)\,(np - 2p) \ldots (np - (r-1)\,p)}{r!} \times \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{z\left(z - \frac{z}{n}\right)\left(z - \frac{2z}{n}\right) \ldots \left[z - (r-1)\frac{z}{n}\right]}{r!} \times \frac{\left(1 - \frac{z}{n}\right)^n}{\left(1 - \frac{z}{n}\right)^r}$$

Now taking the limit as $n \to \infty$ and $p \to 0$ in such a way that $np = z\ (> 0)$ remains constant.

$$\lim_{n \to \infty} B(n, p, r) = \frac{e^{-z} z^r}{r!} \qquad \left[\because \lim_{n \to \infty}\left(1 - \frac{z}{n}\right)^n = e^{-z} \text{ and } \lim_{n \to \infty}\left(1 - \frac{z}{n}\right)^r = 1\right]$$

This is called Poisson distribution which may be denoted by p (r).

Thus, the probability of r successes in a series of large number of trials n with p the probability of success at each trial, a small number, probability mass function (p.m.f.) is given by,

$$p(r) = \frac{e^{-z} z^r}{r!}; \quad r = 0, 1, 2, \ldots$$

Here $z > 0$ is called the parameter of the distribution.

We note that (i) $p(r) \geq 0 \ \forall r, \ \because \ e^{-z} > 0$ (ii) $\sum_{r=0}^{\infty} p(r) = \sum_{r=0}^{\infty} \frac{e^{-z} z^r}{r!} = e^{-z}\left(1 + z + \frac{z^2}{3!} + \ldots\right) = e^{-z} e^{z} = 1$. Hence p(r) is a p.m.f.

We experience number of situations where chance of occurrence of an event is a short time interval is very small. However, there are infinitely many opportunities to occur. The number of occurrences of such event follows poisson distribution. Example of such events are :

(i) Number of defectives in a production centre.

(ii) Number of accidents on a highway.

(iii) Number of printing mistakes per page.

(iv) Number of telephone calls during a particular (odd) time.

(v) Number of bad (dishonoured) cheques at a bank.

6.9.1 Mean and Variance of Poisson Distribution

We shall obtain moments of the poisson distribution $X \to P(z)$ about r = 0 (about origin) (refer article 6.6.2).

$$\mu_1' = \mu_1 = E(x) = \text{Mean} = \sum_{r=0}^{\infty} r\,p(r)$$

$$= \sum_{r=1}^{\infty} r\,\frac{e^{-z} z^r}{r!} = z\,e^{-z} \sum_{r=1}^{\infty} \frac{z^{r-1}}{(r-1)\,!} \qquad \text{(Since term corresponding to r = 0 is zero)}$$

$$= z\,e^{-z}\left(1 + z + \frac{z^2}{2!} + \ldots\right)$$

$$= z\,e^{-z} e^{z}$$

$$\boxed{\mu_1 = \mu_1' = E(x) = \text{Mean} = z}$$

Next, consider second moment about origin (refer article 6.6.2),

$$\mu_2' = E(X^2) = E[X(X-1)] + E(X)$$

or $$\mu_2' = \sum_{r=0}^{\infty} r^2\,p(r) = \sum_{r=0}^{\infty} [r(r-1) + r]\left(\frac{e^{-z} z^r}{r!}\right)$$

$$= \sum_{r=0}^{\infty} r(r-1)\,\frac{e^{-z} z^r}{r!} + \sum_{r=0}^{\infty} r\,\frac{e^{-z} z^r}{r!}$$

$$= e^{-z} z^2 \sum_{r=2}^{\infty} \frac{z^{r-2}}{(r-2)!} + E(X)$$

$$= e^{-z} z^2 \left(1 + z + \frac{z^2}{2!} + \dots\right) + z$$

$$= e^{-z} z^2 e^z + z$$

$$= z^2 + z$$

$$\therefore \qquad \mu_2 = \text{Var}(X) = \mu_2' - \mu_1'^2 = E(X^2) - [E(X)]^2$$

$$= z^2 + z - z^2 = z$$

$$\therefore \qquad \boxed{\text{Var}(X) = \sigma^2 = z}$$

and $\boxed{\text{S.D.} = \sigma = \sqrt{z}}$

Thus, we note that variance of Poisson distribution = mean of Poisson distribution.

ILLUSTRATIONS

Ex. 1 : *A manufacturer of cotter pins knows that 2% of his product is defective. If he sells cotter pins in boxes of 100 pins and guarantees that not more than 5 pins will be defective in a box, find the approximate probability that a box will fail to meet the guaranteed quality.* ***(May 2010)***

Sol. : Here, n = 100.

p the probability of defective pins $= \frac{2}{100} = 0.02$

$$z = \text{mean number of defective pins in a box}$$

$$z = np = 100 \times 0.02 = 2$$

Since p is small, we can use Poisson distribution.

$$P(r) = \frac{e^{-z} z^r}{r!} = \frac{e^{-2} 2^r}{r!}$$

Probability that a box will fail to meet the guaranteed quality is

$$P(r > 5) = 1 - P(r \le 5)$$

$$= 1 - \sum_{r=0}^{5} \frac{e^{-2} 2^r}{r!} = 1 - e^{-2} \sum_{r=0}^{5} \frac{2^r}{r!} = 0.0165$$

Ex. 2 : *In a certain factory turning out razor blades, there is a small chance of 1/500 for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective and two defective blades, in a consignment of 10,000 packets.* ***(Dec. 2018, May 2018)***

Sol. : Here p = 0.002, n = 10, z = np = 0.02

$$P(\text{no defective}) = P(r = 0) = \frac{e^{-0.02} (0.02)^0}{0!} = \frac{1}{e^{0.02}}$$

$$P(\text{2 defectives}) = P(r = 2) = \frac{e^{-0.02} (0.02)^2}{2}$$

Number of packets containing no defective blades in a consignment of 10,000 packets

$$= 10{,}000 \times \frac{1}{e^{0.02}} = 9802$$

Number of packets containing 2 defective blades

$$= 10{,}000 \times \frac{(0.02)^2}{2 \times e^{0.02}} = 2$$

Ex. 3 : *The average number of misprints per page of a book is 1.5. Assuming the distribution of number of misprints to be Poisson, find*

(i) The probability that a particular book is free from misprints.

(ii) Number of pages containing more than one misprint if the book contains 900 pages. **(Nov. 2019)**

Sol. : Let X : Number of misprints on a page in the book.

Given : $X \rightarrow P(z = 1.5)$ $\quad E(r) = z = 1.5$

Here the p.m.f. is given by,

$$P(r) = \frac{e^{-z} z^r}{r!}$$

$$P(r) = \frac{e^{-1.5} (1.5)^r}{r!}$$

(i) $$P(r = 0) = \frac{e^{-1.5} (1.5)}{0!} = e^{-1.5} = 0.223130$$

Note : The Poisson probabilities for m = 0.1, 0.2, 0.3 ... 15.0 are also given in the statistical tables.

(ii)
$$\begin{aligned} P(r > 1) &= 1 - P[r \le 1] \\ &= 1 - [P(r = 0) + P(r = 1)] \\ &= 1 - \left[e^{-1.5} + \frac{e^{-1.5} (1.5)^1}{1!}\right] \\ &= 1 - (0.223130 + 0.334695) \\ &= 0.442175 \end{aligned}$$

$\therefore$ Number of pages in the book containing more than one misprint.

$$\begin{aligned} &= (900)\, P[r > 1] = (900)\ (0.442175) \\ &= 397.9575 \approx 398 \end{aligned}$$

Ex. 4 : Number of road accidents on a high way during a month follows a Poisson distribution with mean 5. Find the probability that in a certain month number of accidents on the highway will be *(i) Less than 3 (ii) Between 3 and 5 (iii) More than 3.*

Sol. : Let X : number of road accidents on a highway during a month.

Given : $X \rightarrow P(z = 5)$

$\therefore$ The p.m.f is given by,

$$P(r) = \frac{e^{-z} Z^{-r}}{r!} \ ; \ r = 0, 1, 2 \ldots\ldots$$

$$P(r) = \frac{e^{-5} 5^r}{r!}$$

(i)
$$\begin{aligned} P(r < 3) = P(r \le 2) &= P(r = 0) + P(r = 1) + P(r = 2) \\ &= \frac{e^{-5}\ 5^0}{0!} + \frac{e^{-5}\ 5^1}{1!} + \frac{e^{-5}\ 5^2}{2!} \\ &= 0.006738 + 0.033690 + 0.084224 \\ &= 0.124652 \end{aligned}$$

(ii)
$$\begin{aligned} P(3 \le r \le 5) &= P(r = 3) + P(r = 4) + P(r = 5) \\ &= 0.140374 + 0.175467 + 0.175467 = 0.491308 \end{aligned}$$

(iii)
$$\begin{aligned} P(r > 3) = 1 - P(r \le 3) &= [P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)] \\ &= 0.734974 \end{aligned}$$

Ex. 5 : *A car hire firm has 2 cars which it hires out day-by-day., The number of demands for the car on each day is distributed as Poisson distribution with parameter 1.5. Calculate the probability of days on which neither car is used and for the days on which demand is refused,.*

Sol. : Let r be number demands for each day, where r takes the value 0, 1, 2. The Poisson probability distribution for demand of r cars on each day is given by

$$p(r) = \frac{e^{-z} z^r}{r!} = \frac{e^{-1.5} (1.5)^r}{r!} \qquad (\because z = 1.5)$$

(i) The probability of the days on which neither car is used, is

$$p(r = 0) = \frac{e^{-1.5} (1.5)^0}{0!} = 0.22$$

(ii) When r > 2, the demand will be refused by the firm. Hence the probability of days when the demands is refused is

$$\begin{aligned} p(r > 2) &= 1 - p(r \le 2) \\ &= 1 - [p(r = 0) + p(r = 1) + p(r = 2)] \\ &= 1 - \left[\frac{e^{-1.5}(1.5)^0}{0.!} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right] \\ &= 1 - e^{-1.5}\left[1 + (1.5) + \frac{(1.5)^2}{2}\right] \\ &= 0.2025 \end{aligned}$$

Ex. 6 : *A telephone switch board handles 600 calls on the average during rush hour. The board can make a maximum of 20 connections per minute. Use Poisson distribution to estimate the probability the board will be overtaxed during any given minute.*

Sol. : The mean of calls handled per minute is

$$z = \frac{600}{60} = 10$$

Thus the probability of Poisson distribution for r calls to be handled is given by

$$p(r) = \frac{e^{-10}(10)^r}{r!} \qquad (\because z = 10)$$

The board will be overtaxed during given minute, when calls are more than 20 per minute. Hence, the probability that the board will be over taxed for a given minute is

$$\begin{aligned} p(r > 20) &= 1 - p(r \le 20) \\ &= 1 - \sum_{r=0}^{20} \frac{e^{-10}(10)^r}{r!} \end{aligned}$$

Hence the answer.

Ex. 7 : *The accidents per shift in a factory are given by the table :*

Accidents x per Shift	*0*	*1*	*2*	*3*	*4*	*5*
Frequency f	*142*	*158*	*67*	*27*	*5*	*1*

Fit a Poisson distribution to the above table and calculate theoretical frequencies.

Sol. : To tit a Poisson distribution, we determine the only parameter z (mean) of the distribution from given data.

$$\begin{aligned} z = np &= \text{The mean number of accidents is A.M.} = \frac{\sum fx}{\sum f} \\ &= \frac{0 \times 142 + 1 \times 158 + 2 \times 67 + 3 \times 27 + 4 \times 5 + 5 \times 1}{142 + 158 + 67 + 27 + 5 + 1} \\ &= \frac{158 + 134 + 81 + 20 + 5}{400} = \frac{398}{400} \qquad (\because \textstyle\sum f = 400) \\ &= 0.995 \end{aligned}$$

Thus, the Poisson distribution that fits to the given data is

$$p(r) = \frac{e^{-0.995}(0.995)^r}{r!}$$

$p(0) = e^{-0.995} = 0.3697$ $\quad p(1) = 0.36785$ $\quad p(2) = 0.813$

$p(3) = 0.0607$ $\quad p(4) = 0.0151$ $\quad p(5) = 0.003$

Theoretical (expected) frequencies are = (Total frequency) × (Probabilities)

$$\begin{aligned} 400 \times p(0) &= 400 \times 0.3697 = 148 \\ 400 \times p(1) &= 400 \times 0.36785 = 147 \\ 400 \times p(2) &= 400 \times 0.183 = 73 \\ 400 \times p(3) &= 400 \times 0.0607 = 24 \\ 400 \times p(4) &= 400 \times 0.0151 = 6 \\ 400 \times p(5) &= 400 \times 0.003 = 1 \end{aligned}$$

Theoretical frequencies are compare well with observed frequencies.

Ex. 8 : *Fit a poisson distribution to the following frequency distribution and compare the theortical frequencies with observed frequencies.*

x	*0*	*1*	*2*	*3*	*4*	*5*
f	*150*	*154*	*60*	*35*	*10*	*1*

Sol.: To fit a Poisson distribution, we determine the only parameter z (mean)

$$z = np = \text{mean} = \frac{\sum fx}{\sum f} = \frac{0 \times 150 + 1 \times 154 + 2 \times 60 + 3 \times 35 + 4 \times 10 + 5 \times 1}{150 + 154 + 60 + 35 + 10 + 1}$$

$$= \frac{424}{410} = 1.034 \qquad (\because \sum f = 410)$$

The Poisson distribution that fits to the given data is

$$p(r) = \frac{e^{-1.034}\,(1.034)^r}{r!}$$

$$p(0) = e^{-1.034} = 0.356, \quad p(1) = 0.368, \quad p(2) = 0.190,$$

$$p(3) = 0.0656, \quad p(4) = 0.017, \quad p(5) = 0.0035$$

Theoretical frequencies are = (Total frequency) × (Probability)

$$410 \times p(0) = 410 \times 0.356 = 145.96$$

$$410 \times p(1) = 410 \times 0.368 = 150.88$$

$$410 \times p(2) = 410 \times 0.190 = 77.9$$

$$410 \times p(3) = 410 \times 0.0656 = 26.896$$

$$410 \times p(4) = 410 \times 0.017 = 6.97$$

$$410 \times p(5) = 410 \times 0.0035 = 1.435$$

Remark : Theoretical frequencies compare well with observed frequencies.

Ex. 9 : *In a Poisson distribution if p (r = 1) = 2p (r = 2), find p (r = 3).*

Sol. :

$$p\,(r) = \frac{e^{-z} z^r}{r!}$$

$$p\,(r = 1) = \frac{e^{-z} z}{1}, \quad p\,(r = 2) = \frac{e^{-z} z^2}{2}$$

$$\therefore \quad ze^{-z} = 2 \times \frac{e^{-z} z^2}{2} \quad \text{which gives } z = 1$$

$$p\,(r = 3) = \frac{e^{-1}(1)}{3!} = e^{-1}\frac{1}{6} = \frac{1}{6e} = 0.0613$$

Ex. 10 : *Show that in a Poisson distribution with unit mean, mean deviation about mean is $\frac{2}{e}$ times the standard deviation.*

Sol. : Here z = 1, $\quad p\,(r) = \dfrac{e^{-z} z^r}{r!} = \dfrac{e^{-1}(1)^r}{r!} = \dfrac{e^{-1}}{r!}$

Mean deviation about mean 1 is

$$\sum_{r=0}^{\infty} |r - 1|\, p\,(r) = \sum_{r=0}^{\infty} |r - 1| \frac{e^{-1}}{r!}$$

$$= e^{-1}\left[1 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots\right]$$

We have, $\quad \dfrac{n}{(n+1)!} = \dfrac{(n+1) - 1}{(n+1)!} = \dfrac{1}{n!} - \dfrac{1}{(n+1)!}$

$\therefore$ Mean deviation about mean

$$= e^{-1}\left[1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots\right]$$

$$= e^{-1}(1 + 1) = \frac{2}{e} \times 1$$

But for Poisson distribution, standard deviation $\sqrt{z} = 1$.

$\therefore$ Mean deviation about mean = $\dfrac{2}{e}$ (standard deviation).

6.10 NORMAL DISTRIBUTION

Normal distribution is the probability distribution of a continuous random variable X, known as normal random variable or normal variate.

Normal distribution is obtained as a limiting form of Binomial distribution when n the number of trials is very large and neither p nor q is very small. Most of the modern statistical methods have been based on this distribution.

Normal distribution of a continuous random variable X with parameters μ and σ^2 is denoted $X \to N(\mu, \sigma^2)$ and given by probability density function (p.d.f.).

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The graph (shape) of the normal distribution curve y = f(x) is bell shaped curve with symmetry about the ordinate at $x = \mu$. The mean, median and mode coincide and therefore the normal curve is unimodal (has only one maximum point). Also, normal curve is asymptotitc to both positive x-axis and negative x-axis. (Refer Fig. 6.9).

The total area under the normal curve is unity i.e. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

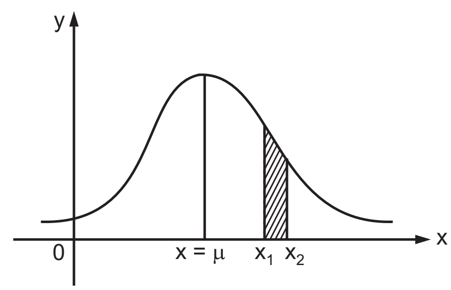


Fig. 6.9

The probability that the continuous random variable x lies between $x = x_1$ and $x = x_2$ is given by the area under the curve y = f(x) bounded by x-axis, $x = x_1$ and $x = x_2$ which is shown by shaded area in the Fig. 6.9.

$$p(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma^2}\right)^2} dx \qquad \dots (2)$$

Introducing $z = \frac{x-\mu}{\sigma}$, integral (2) becomes independent (dimensionless) of two parameters μ and σ. Here z is known as standard variable (variate). With $z = \frac{x-\mu}{\sigma}$, $dx = \sigma\, dz$ and $z_1 = \frac{x_1-\mu}{\sigma}$, $z_2 = \frac{x_2-\mu}{\sigma}$, integral (2) is

$$p(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz \qquad \dots (3)$$

If $\mu = 0$ and $\sigma = 1$, then standard variable $z = \frac{x-\mu}{\sigma}$ is called **standard normal variable** i.e. N(0, 1). The p.d.f. of standard normal random variable is given by

$$y(z) = F(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} \qquad \dots (4)$$

If $z = \frac{x-\mu}{\sigma} \to N(0, 1)$ then the distribution is called **standard normal distribution** and its normal curve as standard normal cure (refer Fig. 6.10).

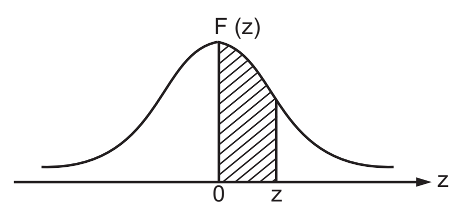


Fig. 6.10

The probability of random variable x lying between $x = \mu$ and any value of $x = x_1$ is given by;

$$p(\mu \le x \le x_1) = \int_{\mu}^{x_1} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma^2}\right)} dx$$

When $\frac{x-\mu}{\sigma} = z$, $dx = \sigma\, dz$. Also, $x = \mu$, $z = 0$ and $x = x_1$, $z = \frac{x_1-\mu}{\sigma} = z_1$ (say)

$$P(0 \le z \le z_1) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z_1} e^{-\frac{1}{2}z^2} dz \qquad \dots (5)$$

If A(z) denotes the area (refer Fig. 6.10) under the normal curve $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, from 0 to z_1, z_1 being any number then from (5), we can write

$$p(z) = A(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}z^2} dx \qquad \dots (6)$$

The definite integral (6) is called normal probability integral (or error function), the value of this integral for different values of z are given in table 6.1. Thus, the entries in the normal table gives (represents) the area under the normal curve between z = 0 to z (shaded in figure 6.10). Hence the determination of normal probabilities (3) reduce to the determination of area as

$$\begin{aligned} p(x_1 \le x \le x_2) &= p(z_1 \le z \le z_2) = p(z_2) - p(z_1) \\ &= (\text{Area under the normal curve from 0 to } z_2) - (\text{Area under the normal curve from 0 to } z_1) \end{aligned}$$

Table 6.1

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5259
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9494	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9871	0.9678	0.9686	0.9693	0.9699	0.9708
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9783	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9634	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.98686	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9809	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9988	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9998	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9898

In each row and each column 0.5 to be subtracted.

6.10.1 The Area under the Normal Curve

Normal distribution curve is given by

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The area under the normal curve is

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\,(2\pi)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = z \quad \therefore\ dx = \sigma\, dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} e^{-\frac{1}{2}(-z)^2} dz\right] \qquad \text{(by definite integral theorem)}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz$$

Put $\frac{z^2}{2} = t,\quad z\,dz = dt$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \left\lceil \frac{1}{2} \right. = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

6.10.2 Mean Deviation from the Mean

$$\text{M.D.} = \int_{-\infty}^{\infty} |x-\mu|\, p(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu|\ e^{-(x-\mu)^2/2\sigma^2}\, dx$$

Put $\frac{x-\mu}{\sigma} = z;\quad dx = \sigma\, dz$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2\, |z|\, e^{-z^2/2}\, dz = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z|\, e^{-z^2/2}\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} |z|\, e^{-z^2/2}\, dz \qquad [f(z) = |z| = e^{-z^2/2} \text{ is even function of } z]$$

$$\text{M.D.} = \sqrt{\frac{2}{\pi}}\ \sigma \int_{0}^{\infty} z e^{-z^2/2}\, dz \quad [|z| = z \text{ for } 0 < z < \infty]$$

$$= \sqrt{\frac{2}{\pi}}\ \sigma \int_{0}^{\infty} \frac{1}{2}\, e^{-z^2/2}\, d(z^2)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sigma}{2} \left[\frac{e^{-z^2/2}}{-1.2}\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\ \sigma\,[1] = 0.8\,\sigma = \frac{8}{10}\sigma = \frac{4}{5}\ \sigma \text{ (approximately)}$$

ILLUSTRATIONS

Ex. 1 : *The mean weight of 500 students is 63 kgs and the standard deviation is 8 kgs. Assuming that the weights are normally distributed, find how many students weigh 52 kgs ? The weights are recorded to the nearest kg.*

Sol. : The frequency curve for the given distribution is

$$y = 500 \frac{1}{8\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-63}{8}\right)^2} \qquad \ldots (1)$$

Since the weights are recorded to the nearest kg, the students weighing 52 kgs have their actual weights between x = 51.5 and 52.5 kg. So the area under the curve (1) from x = 51.5 to x = 52.5 is to be obtained.

Using $z = \frac{x-\mu}{\sigma}$, we have

$$z_1 = \frac{51.5-63}{8} = -1.4375 = -1.44 \text{ (appx)}$$

$$z_2 = \frac{52.5-63}{8} = -1.3125 = -1.31 \text{ (appx)}$$

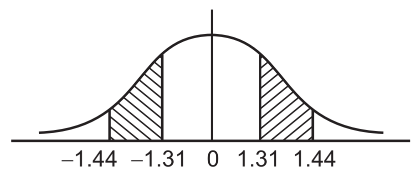


Fig. 6.11

$$P(51.5 \le x \le 52.5) = 500\, P(-1.44 \le z \le -1.31)$$

$$500 \int_{51.5}^{52.5} p(x)\, dx = \frac{500}{\sqrt{2\pi}} \int_{-1.44}^{-1.31} e^{-z^2/2}\, dz$$

The number of students weighing 52 kg

$$= 500\,(A_1 - A_2)$$

$$= 500\,(0.4251 - 0.4049) = 10 \text{ students approximately.}$$

where, $A_1 = 0.4251$ is the area for $z_1 = 1.44$,

and $A_2 = 0.4049$ is the area for $z_2 = 1.31$.

Ex. 2 : *For a normal distribution when mean $\mu = 1$, $\sigma = 3$, find the probabilities for the intervals :*

(i) $3.43 \le x \le 6.19$; (ii) $-1.43 \le x \le 6.19$

Sol. : Using $z = \frac{x-\mu}{\sigma}$, we have

(i) $z_1 = \frac{3.43-1}{3} = 0.81, \quad z_2 = \frac{6.19-1}{3} = 1.73$

$$P(3.43 \le x \le 6.19) = P(0.81 \le z \le 1.73)$$

$\therefore$ Required probability $= A_2 - A_1$

$$= (0.4582 - 0.2910) = 0.1672$$

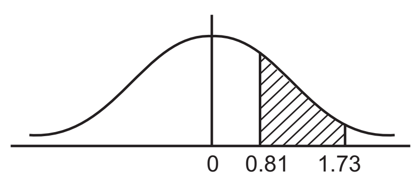


Fig. 6.12

where, $A_2 = 0.4592$ is area corresponding to $z_2 = 1.73$

$A_1 = 0.2910$ is area corresponding to $z_1 = 0.81$

(ii) $z_1 = \frac{-1.43-1}{3} = -0.81, \quad z_2 = \frac{6.19-1}{3} = 1.73$

$$P(-1.43 \le x \le 6.19) = P(-0.81 \le z \le 1.73)$$

$\therefore$ Required probability $= A_1 + A_2$

$$= 0.2910 + 0.4582 = 0.7492$$

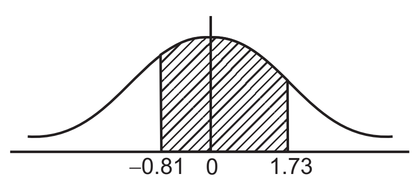


Fig. 6.13

Ex. 3 : *Assuming that the diameters of 1000 brass plugs taken consecutively from machine form a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm. How many of the plugs are likely to be approved if the acceptable diameter is 0.752 ± 0.004 cm ?* ***(May 2018)***

Sol. : Given, $\sigma = 0.0020, \quad \mu = 0.7515$

For $x_1 = 0.752 + 0.004 = 0.756$

and $x_2 = 0.752 - 0.004 = 0.748$

Using $z = \dfrac{x-\mu}{\sigma}$, we have

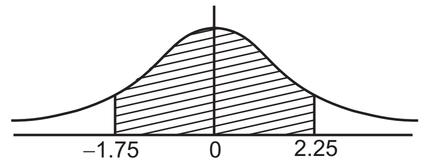


Fig. 6.14

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{0.756 - 0.7515}{0.0020} = 2.25$$

$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{0.748 - 0.7515}{0.0020} = -1.75$$

A_1 corresponding to (z_1 = 2.25) = 0.4878 (Refer table 6.1)

A_2 corresponding to (z_2 = 1.75) = 0.4599

$$p\,(0.748 < x < 0.756) = A_1 + A_2$$

$$= 0.4878 + 0.4599 = 0.9477$$

Number of plugs likely to be approved = 1000×0.9477 = 948 approximately.

Ex. 4 : *In a certain examination test, 2000 students appeared in a subject of statistics. Average marks obtained were 50% with standard deviation 5%. How many students do you expect to obtain more than 60% of marks, supposing that marks are distributed normally ?* ***(May 2010, 2014, Dec. 2012, Nov. 2019)***

Sol. : Given : $\mu = 0.5, \qquad \sigma = 0.05$

$$x_1 = 0.6, \qquad z_1 = \frac{0.6 - 0.5}{0.05} = 2 \qquad \left(\because z = \frac{x-\mu}{\sigma}\right)$$

A corresponding to z = 2 is 0.4772

$\therefore \quad P\,(x \geq 6) = P(z \geq 2) = 0.5 - 0.4772 = 0.0228$

Number of students expected to get more than 60% marks

$$= 0.0228 \times 2000 = 46 \text{ students approximately.}$$

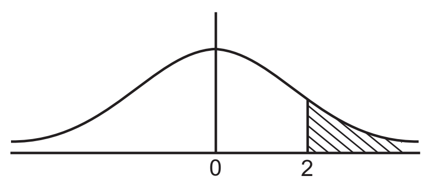


Fig. 6.15

Ex. 5 : *In a certain city 4000 tube lights are installed. If the lamps have average life of 1500 burning hours with standard deviation 100 hours.. Assuming normal distribution*

(i) How many lamps will fail in first 1400 hours

(ii) How many lamps will last beyond 1600 hours

Sol. : $$z = \frac{x-\mu}{\sigma}$$

(i) $$z = \frac{1400 - 1500}{100} = -\frac{100}{100} = 1$$

Corresponding to z = 1, A = 0.3413

No. of tubes failing in first 1400 hours will be

$$= 4000 \times 0.3413$$

$$= 1365.2 \text{ or } 1365$$

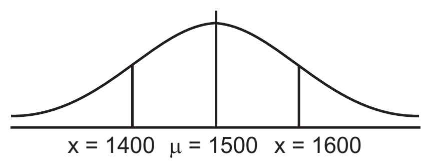


Fig. 6.16

(ii) $$z = \frac{1600 - 1500}{100} = 1$$

Area to the right of x = 1600 = 0.5 – 0.3413 = 0.1587

No. of tubes which will continue to burn after 1600 hours will be 4000×0.1587 or 635.

Ex. 6 : *In a normal distribution 10% of items are under 40 and 5% are over 80. Find the mean and standard deviation of distribution.*

Sol. : $p(x < 40) = 0.1$ and $p(x > 80) = 0.05$

$x = 40$, $x = 80$ are located as shown in Fig. 6.17.

for $x = 40$, $\quad z = \dfrac{40-\mu}{\sigma} = -z_1$

(say – ve sign because $x = 40$ is to the left of $x = \mu$).

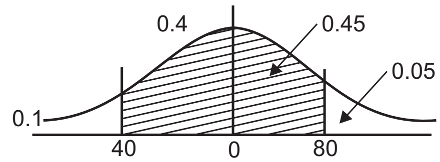


Fig. 6.17

when $x = 80$, $\quad z = \dfrac{80-\mu}{\sigma} = z_2$ (+ ve sign because $x = 80$ is to the right of $x = \mu$)

Corresponding to A = 0.4, z = 1.29

Corresponding to A = 0.45, z = 1.65

$$\frac{40-\mu}{\sigma} = -1.29, \quad \frac{80-\mu}{\sigma} = 1.65$$

$$40-\mu = -1.29\,\sigma, \quad 80-\mu = 1.65\,\sigma$$

$$40 = (1.65 + 1.29)\sigma = 2.94\sigma$$

$$\therefore \quad \sigma = 13.6$$

$$\mu = 40 + 1.29\,\sigma = 57.54$$

Ex. 7 : *Suppose heights of students follows normal distribution with mean 190 cm and variance 80 cm^2. In a school of 1000, students how many would you expect to be above 200 cm tall.*

Sol. : Let, x = Height of students and $x \to N(190, 80)$

Proportion of students having height above 200 cm.

$$= p(x > 200) = p\left(\frac{x-\mu}{\sigma} > \frac{200-190}{\sqrt{80}}\right)$$

$$= p(z > 1.1180) = 0.5 - 0.3686$$

$$= 0.1314$$

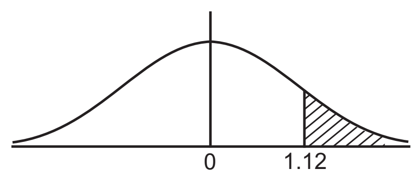


Fig. 6.18

$\therefore$ Number of students out of 1000 having height above 200 cm

= 1000 × Proportion of students having height above 200 cm

= 1000 × 0.1314 = 1.31 student

Ex. 8 : *Let $x \to N(4, 16)$. Find (i) $P(x > 5)$, (ii) $P(x < 2)$, (iii) $P(x > 0)$, (iv) $P(6 < x < 8)$, (v) $P(|x| > 6)$.*

Sol. : Let $x \to N(4, 16) = N(\mu, \sigma^2)$, hence $\mu = 4$, and $\sigma^2 = 16 \Rightarrow \sigma = 4$.

(i) $$P(x > 5) = P\left(z = \frac{x-\mu}{\sigma} > \frac{5-4}{4}\right)$$

$$= P(z > 1/4)$$

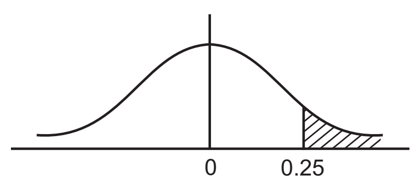


Fig. 6.19

$\therefore$ From normal probability integral table, we get area of shaded region as,

$$p(z > 1/4) = 0.40129$$

(ii) $$p(x < 2) = P\left(\frac{x-\mu}{\sigma} > \frac{2-4}{4}\right)$$

$$= P\left(z < -\frac{2}{4}\right)$$

$$= P(z < -0.5)$$

$$= P(z > 0.5) \qquad \text{(Due to symmetry)}$$

$$= 0.30854 \qquad \text{(From the table)}$$

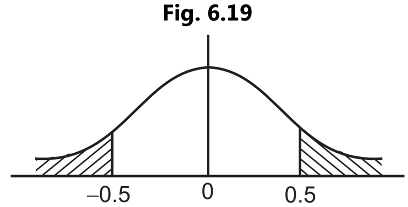


Fig. 6.20

(iii) $$p\,(x > 0) = P\left(\frac{x-\mu}{\sigma} > \frac{0-4}{4}\right)$$

$$= P\,(z > -1) = B$$

Since, only tail area is given in the table, we use the fact that $A + B = 1$.

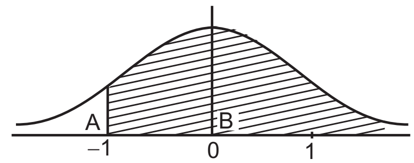


Fig. 6.21

$\therefore$ $$p\,(z > -1) = 1 - A = 1 - p\,(z < -1)$$

$$= 1 - p\,(z > 1) \qquad \text{(Due to symmetry)}$$

$$= 1 - 0.15866$$

$$= 0.84134$$

(iv) $$p\,(6 < x < 8) = P\left(\frac{6-4}{4} < \frac{x-\mu}{\sigma} < \frac{8-4}{\sigma}\right)$$

$$= P\left(\frac{2}{4} < z < 1\right)$$

$$= P\,(0.5 < z < 1) = A$$

$$= (A + B) - B$$

$$= P\,(z > 0.5) - P\,(z > 1)$$

$$= 0.30854 - 0.15866$$

$$= 0.14988$$

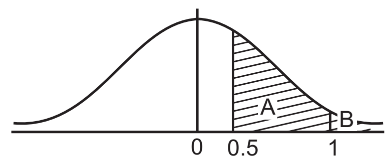


Fig. 6.22

(v) $$p\,(|x| > 6) = p\,(x > 6) + P\,(x < -6)$$

$$= p\left(\frac{x-\mu}{6} > \frac{6-4}{4}\right) + P\left(\frac{x-\mu}{\sigma} > \frac{-6-4}{4}\right)$$

$$= p\,(z > 0.5) + p\,(z < -2.5)$$

$$= p\,(z > 0.5) + p\,(z > 2.5)$$

$$= 0.30854 + 0.0062097)$$

$$= 0.31475$$

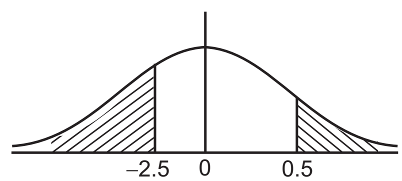


Fig. 6.23

Ex. 9 : *In a distribution, exactly normal, 7% of the items are under 35 and 89% are under 63. Find the mean and standard deviation of the distribution.*

Sol. : From Fig. 6.24, it is clear that 7% of items are under 35 means area under 35 is 0.07. Similarly area for $x > 63$ is 0.11.

$$p\,(x < 35) = 0.07 \quad \text{and} \quad p\,(x > 63) = 0.11$$

$x = 35$, $x = 63$ are located as shown in Fig. 6.24.

When $x = 35$, $z = \dfrac{35-\mu}{\sigma} = -z_1$ (say), (– ve sign because $x = 35$ to the left of $x = \mu$)

When $x = 63$, $z = \dfrac{63-\mu}{\sigma} = z_2$ (say), (+ve sign for $x = 63$ lies to the right of $x = \mu$)

$\therefore$ From Table 6.1, we get

Area $A_1 = p\,(0 < z < z_1) = 0.43$ corresponds to $z_1 = 1.48$ (appx)

& Area $A_2 = p\,(0 < z < z_2) = 0.39$ corresponds to $z_2 = 1.23$ (appx)

Thus, we get two simultaneous equations

$$\frac{35-\mu}{\sigma} = -z_1 = -1.48 \qquad \text{... (1)}$$

$$\frac{63-\mu}{\sigma} = z_2 = 1.23 \qquad \text{... (2)}$$

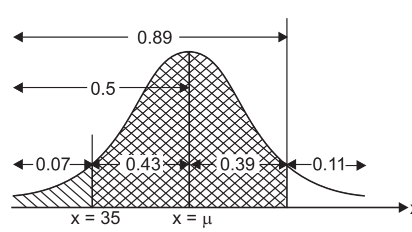


Fig. 6.24

Subtracting (1) from (2),

$$\frac{28}{\sigma} = 2.71 \quad \Rightarrow \quad \sigma = 10.33 \text{ (approximately)}$$

and (2) $\Rightarrow$

$$\mu = 63 - \sigma \times 1.23$$
$$= 63 - 10.33 \times 1.23 = 50.3 \text{ (approximately)}$$

***Ex. 10 :** A fair coin is tossed 600 times. Using normal approximation find the probability of getting (i) number of heads less than 270. (ii) number of heads between 280 to 360.*

Sol. : A fair coin tossing 600 times result into head or tail each with probability p = 0.5.

Let, x = Number of heads in 600 tosses and $x \to B(600, 0.5)$

$$\mu = E(X) = np = 600 \times 0.5 = 300 \text{ and } \sigma^2 = Var(X) = npq = 600 \times 0.5 \times 0.5 = 150$$

(i) p (number of heads less than 270).

$$P(x < 270) = P\left(\frac{x-\mu}{\sigma} < \frac{270-30}{\sqrt{150}}\right) P\left(\frac{x-np}{\sqrt{npq}} < \frac{270-300}{\sqrt{150}}\right)$$

$$= P(z < -2.4495) \qquad \text{(Using normal approximate)}$$
$$= P(z > 2.4495) \qquad \text{(Due to symmetry)}$$
$$= 0.0071428$$

(ii) P (Number of heads are between 280 and 350)

$$P(280 < x < 350) = P\left(\frac{280-300}{\sqrt{150}} < z < \frac{350-300}{\sqrt{150}}\right)$$

Using normal approximate, we get, $z = \frac{x-\mu}{\sigma} \to N(0, 1)$

$$p \approx p(-1.633 < z < 4.0823) = B = 1 - A - C$$
$$= 1 - P(z < -1.633) - P(z > 4.0823) \qquad \text{(Due to symmetry)}$$
$$= 1 - P(z > 1.633) - P(z > 4.0823) \qquad \text{(Using standard table values)}$$
$$= 1 - 0.51551 - 0.000022518 = 0.4845$$

***Ex. 11 :** If x is a random variable with p.d.f.* $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{18}(x-6)^2}$. *Find : (i) p (x > 5), (ii) p (2x + 3 > 10).*

Sol. : We know that $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ and given p.d.f. is $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{18}(x-6)^2}$

Comparing the p.d.f., we get, $\mu = 6$, $\sigma^2 = 9$ and $x \to N(6, 9)$

(i) For x = 5, $z = \frac{x-\mu}{\sigma} = \frac{5-6}{9} = -\frac{1}{9} = 0.33$

$$p(x > 5) = P(z > -0.33)$$
$$= 0.5 + p(z < 0.33) = 0.5 + 0.1293 = 0.6293$$

(ii)

$$2x + 3 \to N(\mu', \sigma^2)$$
$$\mu' = E(2x + 3) = 2\mu + 3$$
$$= 2(6) + 3 = 12 + 3 = 15$$
$$\sigma'^2 = Var(2x + 2) = 4\, Var(x) = 36$$
$$2x + 3 \to N(15, 36)$$

$$\therefore \qquad p\,(2x + 3 > 10) = p\left(\frac{2x + 3 - 15}{6} > \frac{10 - 15}{6}\right)$$

$$= p\,(z > -0.8333)$$

$$= 1 - p\,(z > 0.8333)$$

$$= 1 - 0.20327$$

$$= 0.79673$$

6.11 SAMPLING DISTRIBUTIONS

One of the important tools in statistical inference is testing of hypothesis. In order to draw inference about a certain phenomenon, sampling is a well accepted tool. Entire population cannot be studied due to several reasons. In such a situation sampling is the only alternative. A properly drawn sample is much useful in drawing reliable conclusions. Here, we draw a sample from probability distribution rather than a group of objects. Using simulation technique sample is drawn.

Random Sample from a Continuous Distribution

A random sample from a continuous probability distribution f (x, θ) is nothing but the values of independent and identically distributed random variables with the common probability density function f (x, θ).

Definition : If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, with p.d.f. f (x, θ), then we say that, they form a sample from the population with p.d.f. f (x, θ).

Note :

1. For drawing inference, we use the numerical values of $X_1, X_2, \ldots, X_n$.
2. The joint p.d.f. of $X_1, X_2, \ldots X_n$ is,

$$f\,(x_1, x_2, \ldots, x_n) = f\,(x_1), f\,(x_2), \ldots, f\,(x_n) = \prod_{i=1}^{n} f\,(x_i)$$

6.12 STATISTIC AND PARAMETER

Using the random sample $X_1, X_2, \ldots, X_n$ we draw conclusion about the unknown probability distribution. However probability distribution can be studied if the parameter θ is known. In other words, study of probability distribution reduces to the study of parameter θ. We use sampled observations for this purpose. There are various ways of summarizing the sampled observations. The summarized quantity is called as ***statistic***. A statistic is used to estimate the value of parameter. We define it precisely as follows.

Definition : If $X_1, X_2, \ldots, X_n$ is a random sample from a probability distribution f (x, θ), then $T = T\,(x_1, x_2, \ldots, x_n)$ a function of sample values which does not involve unknown parameter θ is called as a *statistic (or estimator)*.

Some typical statistics are given below :

(i) Sample mean : $\quad T = T\,(x_1, x_2, \ldots, x_n) = \dfrac{\sum x_i}{n}$

$\therefore \qquad T = \bar{X}$ is a statistic

(ii) Sample variance :

$$T = T\,(x_1, x_2, \ldots x_n) = \frac{1}{n-1} \sum (x_i - \bar{x})^2 \quad \text{is a statistic.}$$

(iii) Similiary, it can be show that sample proportion (p) is also a statistic.

6.13 CHI-SQUARE DISTRIBUTION

Introduction :

The Chi-square (pronounced as Ki, sky without 's') distribution is one of the important distributions in Statistics. It is mainly applied in testing of hypothesis for testing the independence of attributes, testing the goodness of fit of a model etc.

The chi-square variable is denoted by χ_n^2. Hence n is the parameter of the distribution, also, called as the 'degrees of freedom' (d.f.). The χ_n^2 variate is defined as sum of squares of n independent standard normal [N (0, 1)] variables.

6.14 DEFINITION

Let $X_1, X_2, ..., X_n$ be n independent N (0, 1) variables, then

$$Y = \sum_{i=1}^{n} X_i^2 \text{ follows chi-square distribution with n degrees of freedom (d.f.).}$$

Notation : $Y \rightarrow \chi_n^2$. $(\because$ n is positive integer)

6.15 ADDITIVE PROPERTY

Statement : If Y_1 and Y_2 are independent χ^2 variates with n_1 and n_2 d.f. respectively, then $Y_1 + Y_2$ has χ^2 distribution with $(n_1 + n_2)$ degrees of freedom.

6.16 TESTING OF HYPOTHESIS

Meaning of Statistical Hypothesis

We are mainly interested in testing certain claims about the population parameters such as mean, variance, proportion. For example, a particular scooter gives average of 50 km per litre, proportion of unemployed persons is same for two different states. Average life of an article produced by company A is larger than that of company B, etc. These claims stated in terms of population parameters or statistical distribution are called hypothesis.

Hypothesis : Definition : It is a statement or assertion about the statistical distribution or unknown parameter of statistical distribution.

In other words, hypothesis is a claim to be tested.

Average life of a bulb is 1000 hours. It can be written as $H_0 : \mu = 1000$ where μ represents population.

6.17 NULL HYPOTHESIS AND ALTERNATIVE HYPOTHESIS

In each problem of test of significance, two hypothesis are to be set. These are set in such a way that if one is rejected, the other is to be accepted. These hypothesis are referred to as null hypothesis and alternative hypothesis.

Null Hypothesis : A hypothesis of "no difference" is called as null hypothesis or according to R. A. Fisher, null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true. Null hypothesis is denoted by H_0.

For Example : $H_0 : \mu = 100$. Here the hypothesis states that there is no difference in population mean and 100. $H_0 : \mu_1 = \mu_2$ or $H_0 : \mu_1 - \mu_2 = 0$. This hypothesis states that there is no difference between two population means. While conducting the test, some difference will be observed in sample value and hypothesized value. Whether this difference is just due to chance element, is decided in testing procedure.

Alternative Hypothesis : It is a hypothesis to be accepted in case null hypothesis is rejected. In other words, a complementary hypothesis to null hypothesis is called as alternative hypothesis. It is denoted by H_1.

For Example : If $H_0 : \mu_1 = \mu_2$ then alternative hypothesis may be $H_1 : \mu_1 \neq \mu_2$ or $H_1 : \mu_1 < \mu_2$ or $H_1 : \mu_1 > \mu_2$.

6.18 ONE AND TWO SIDED HYPOTHESIS

By considering the nature of hypothesis, these are classified as one sided or two sided.

Hypothesis of the type $H_1 : \mu > \mu_0$, $H_1 : P < 0.5$, $H_1 : \mu_1 < \mu_2$, $H_0 : \sigma_1 > \sigma_2$ etc. are called as ***one sided***. On the other hand, the hypothesis of the type $H_1 : P_1 \neq 0.5$, $H_1 : \sigma_1 \neq \sigma_2$, $H_1 : \mu \neq \mu_0$ etc. are called as ***two sided***.

In this text we will consider null hypothesis to be hypothesis of equality and alternative hypothesis to be two sided. Choice of one sided hypothesis as null hypothesis is beyond the scope of the book.

6.19 TYPE I AND TYPE II ERRORS

Since decision of acceptance or rejection of H_0, is based on sampling, it is subject to two kinds of errors. For instance, in the inspection of a lot of manufactured items, the inspector will choose a sample of suitable size and accordingly take decision whether to accept or reject the lot. In this process, two errors are possible viz, rejection of a good lot and acceptance of a bad lot. In testing of hypothesis these errors are called as type I and type II errors.

Type I error : Rejecting H_0 when it is true.

Type II error : Accepting H_0 when it is false.

These errors can be put in tabular form to remember easily.

Actual Situation	Decision	
	Reject H_0	**Accept H_0**
H_0 is true	Type I error	Correct decision
H_0 is false	Correct decision	Type II error

6.20 CRITICAL REGION

Let $x_1, x_2, ..., x_n$ be a random sample taken for testing H_0. The set of values of $(x_1, x_2, ..., x_n)$ for which H_0 is rejected is called as critical region or rejection region. Many a times, critical region is expressed with the help of test statistic e.g. $\bar{x} \geq c$, where c is constant. Critical region is denoted by W. The set of all sample observations can be partitioned into two subsets : critical region (W) and acceptance region (W^c) as shown below.

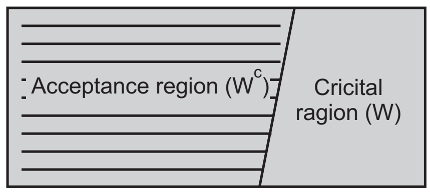


Fig. 6.25

6.21 TEST OF HYPOTHESIS

A rule which leads to the decision of acceptance of H_0 or rejection of H_0 on the basis of observations in a random sample is called test of hypothesis.

Statistical inference is that branch of statistics which concerned with using probability concept to deal with uncertainly in decision making field of statistical inference has a fruitful development since the letter half of the 19^{th} century. It refers to the process of selecting and using a sample statistic to draw inference about a population parameter based on a sub-set of it the sample drawn from the population. Statistical inference treats two different classes of problems.

(i) Hypothesis testing and (ii) Estimation.

Hypothesis testing begins with us assumption called as hypothesis, which is made by the population parameter. A hypothesis is a supposition made as basis for reasoning.

In article 6.14 to 6.21 we discussed the various terms like statistical hypothesis, null hypothesis and alternative hypothesis, critical region, level of significance etc.

Now we will see the method of testing population mean (μ) equal to specified value (μ_0).

Testing Population Mean (μ) Equal to Specified Value (μ_0) : Test statistic is

$$U = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Under H_0,
$$U = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \rightarrow N(0, 1)$$

Critical Region : Value of $|U| > 1.96$ at 5% level of significance.

If calculated value of U is more than 1.96 or less than – 1.96, H_0 is rejected and accepted otherwise.

6.22 ONE SIDED AND TWO SIDED TESTS

The tests used for testing null hypothesis are called as one sided or two sided tests according as the alternative hypothesis are one sided or two sided.

6.23 TEST STATISTIC

A function of sample observations which is used to test null hypothesis H_0 is called a *test statistic*. The distribution of test statistic is completely known under H_0. Hence, it can be used to test H_0.

6.24 LEVEL OF SIGNIFICANCE

Probability of rejecting H_0 when it is true is called as *level of significance*. Thus, it is probability of committing type I error. It is denoted by α.

Level of significance can be interpreted as proportion of cases in which H_0 is rejected though it is true.

If we try to minimize level of significance, the probability of type II error increases. So level of significance cannot be made zero. However, we can fix it in advance as 0.05 (i.e. 5%) or 0.01 (i.e. 1 %). In most of the cases, it is taken as 5%.

6.25 TESTS BASED ON χ^2 DISTRIBUTION

Test for Goodness of Fit of χ^2 Distribution :

For a given data (frequency distribution), we try to fit some probability distribution. Since there are several probability distributions, which distribution will fit properly may be a question of interest. In such cases, we want to test the appropriateness of the fit. Hence we desire to test H_0 : Fitting of the probability distribution to given data is proper (good). The test based on χ^2 distribution used to test this H_0 is called ***χ^2 test of goodness of fit***. The alternative hypothesis is H_1 : The specific distribution does not fit.

In this case, we compare the expected and observed frequencies. Thus, we can take H_0 : There is no significant difference between observed and (theoretical) expected frequencies. The test is carried out as follows :

Suppose $o_1, o_2, \ldots, o_i, \ldots, o_k$ be a set of observed frequencies and $e_1, e_2, \ldots, e_i \ldots, e_k$ be corresponding expected frequencies obtained under H_0.

$$\sum_{i=1}^{k} o_i = N = \sum_{i=1}^{k} e_i$$

Suppose, p = number of parameters estimated for fitting the probability distribution.

If H_0 is true, then the statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i} = \sum_{i=1}^{k} \left(\frac{oi^2}{e_i}\right) - N$$

has χ^2 distribution with (k – p – 1) degrees of freedom. Degree of freedom is a parameter of the χ^2 distribution. In this case, the critical region at level of significance α is

$$\chi^2_{k-p-1} \geq \chi^2_{k-p-1;\,\alpha}$$

where, $\chi^2_{k-p-1;\,\alpha}$ is the table value corresponding to degrees of freedom k – p – 1 and level of significance (*l*.o.s) α. We say $\chi^2_{k-p-1;\,\alpha}$ as a critical value. It is shown by the shaded region in Fig. 6.26. Normally, the values of α are taken as 0.05 or 0.01.

Thus, we reject H_0 at *l*.o.s. α if,

$$\chi^2_{k-p-1} \geq \chi^2_{k-p-1;\,\alpha}$$

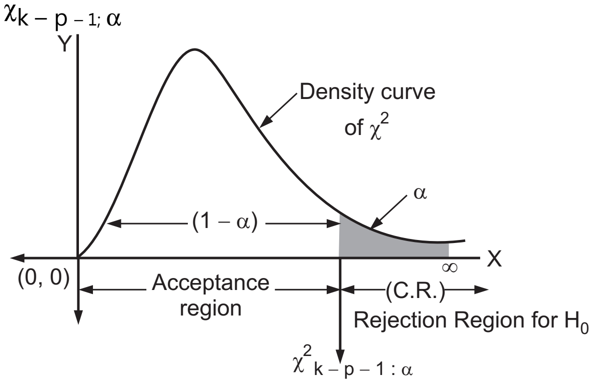


Fig. 6.26

Note :

1. We can apply this test if expected frequencies are greater than or equal to 5 (i.e. $e_i \geq 5$) and total of cell frequencies is sufficiently large (greater than 50).

2. When expected frequency of a class is less than 5, the class is merged into neighbouring class alongwith its observed and expected frequencies until total of expected frequencies becomes ≥ 5. This procedure is called '*pooling the classes*'. In this case, k is the number of class frequencies after pooling.
3. It is obvious that if any parameters are not estimated while fitting a probability distribution or obtaining expected frequencies, the value of p is zero.
4. This test is not applicable for testing goodness of fitting of straight line or curves such as second degree curve, exponential curve etc.

Remark :

Yate's Correction : If in a 2 × 2 contingency table, any cell frequency is less than 5 then the test statistic χ^2 is corrected in a specific way. This correction is due to Yate's and hence is known as Yate's correction. It is beyond the scope of this book.

ILLUSTRATIONS

Ex. 1 : *A nationalized bank utilizes four teller windows to render fast service to the customers. On a particular day, 800 customers were observed. They were given service at the different windows as follows :*

Window Number	*Expected Number of customers*
1	*150*
2	*250*
3	*170*
4	*230*

Test whether the customers are uniformly distributed over the windows.

Sol. : Here we want to test H_0 : Customers are uniformly distributed over the windows. i.e. H_0 : customers on all windows are equal against H_1 : They are not equal on all windows.

Under H_0, the expected frequencies are :

Window Number	Expected Number of Customers (e_i)
1	200
2	200
3	200
4	200

The test statistic is

$$\chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i} = \frac{(-50)^2}{200} + \frac{(50)^2}{200} + \frac{(-30)^2}{200} + \frac{(30)^2}{200}$$

Here number of parameters estimated = p = 0, k = 4.

$$\therefore \quad \chi^2_3 = 34 \qquad \text{(Calculated value)}$$

$$\chi^2_3 = 34 > \chi^2_{3;\,0.05} = 7.815 \qquad \text{(Critical or table value)}$$

We reject H_0 at 5 % *l*.o.s.

Conclusion : The customers in the nationalized bank may not be uniformly distributed over different windows.

Ex. 2 : *One hundred samples were drawn from a production process each after 5 hours. The number of defectives in these samples were noted. A Poisson distribution by estimating the parameter m was fitted to these data. The results obtained are as follows :*

Number of Defectives	*Number of Samples (observed)*	*Expected Number of Samples*
0	*63*	*60.65*
1	*28*	*30.33*
2	*6*	*7.58*
3	*2*	*1.26*
4	*1*	*0.16*
5 and above	*0*	*0.02*

Test the goodness of fit of Poisson distribution in above situation. [Use 5 % level of significance]

Sol. : We want to test

H_0 : Fitting of Poisson distribution is good (proper) against

H_1 : Fitting of Poisson distribution is not proper.

Here we pool expected frequencies until their sum becomes ≥ 5 and also pool corresponding observed frequencies. Thus the frequencies can be written as :

Observed Frequencies (o_i)	Expected Frequencies (e_i)
63	60.65
28	30.33
9	9.02

We use the test statistic

$$\chi^2_{k-p-1} = \sum_{i=1}^{k} \left(\frac{o_i^2}{e_i}\right) - N$$

Here number of parameters estimated = p = 1, N = 100, k = 3.

$\therefore \quad \chi^2_1 = 100.27009 - 100$

$\therefore \quad \chi^2_1 = 0.27009$ [Calculated value]

$\chi^2_1 = 0.27009 < \chi^2_{1;\ 0.05} = 3.841$ [Critical or Table value]

Hence we accept H_0 at 5 % *l*.o.s.

Conclusion : Fitting of Poisson distribution may be good to the given data.

Ex. 3 : *Among 64 offsprings of a certain cross between guinea pigs 34 were red, 10 were black and 20 were white. According to a genetic model, these numbers should be in the ratio 9 : 3 : 4.*

Are the data consistent with the model at 5 % level ?

Sol. : Here H_0 : The offsprings red, black and white in the colour are in the ratio 9 : 3 : 4.

In this problem, N = 64. Hence observed and expected frequencies are as follows :

Observed Frequencies (o_i)	34	10	20
Expected Frequencies (e_i)	$\frac{9}{16} \times 64 = 36$	$\frac{3}{16} \times 64 = 12$	$\frac{4}{16} \times 54 = 16$

To test H_0, the test statistic is

$$\chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i} \qquad \text{Here } p = 0 \text{ and } k = 3.$$

$\therefore \quad \chi^2_2 = 1.444444$ (Calculated value)

$\chi^2_{2\,:\,0.05} = 5.991$ (Critical or Table value)

$\chi^2_2 = 1.444444 < \chi^2_{2;\ 0.05} = 5.99$

We accept H_0 at 5 % *l*.o.s.

Conclusion : The data are consistent with the genetic model that the offsprings red, black and white in colour are in the ratio 9 : 3 : 4.

Ex. 4 : *The table below gives number of books issued from a certain library on the various days of a week.*

Days	***No. of Books Issued***	$(o_i - e_i)^2$
Mon.	*120*	*0*
Wed.	*130*	*100*
Thr.	*110*	*100*
Fri.	*115*	*25*
Sat.	*135*	*225*
Sun.	*110*	*100*

Test at 5 % l.o.s. whether issuing the book is day dependent.

Sol. : The issuing of the book is not dependent on the day of the week.

$$\chi^2_{k-p-1} = \chi^2_{6-0-1} = \chi^2_5$$

$$\therefore \quad \chi^2_5 = \frac{\sum (o_i - e_i)^2}{e_i}$$

$$\therefore \quad \chi^2_5 = \frac{550}{120} = 4.5833$$

$$\chi^2_{5,\,0.05} = 11.07$$

$$\chi^2_5 < \chi^2_{5,\,0.05}$$

Accept H_0.

Conclusion : i.e. issuing of book is independent of day.

Ex. 5 : *In experiment on pea breeding, the following frequencies of seeds were obtained :*

Round and green	***Wrinkled and Green***	***Round and Yellow***	***Wrinkled and Yellow***	***Total***
222	*120*	*32*	*150*	*524*

Theory predicts that the frequencies should be in proportion 8 : 2 : 2 : 1. Examine the correspondence between theory and experiment.

Sol. : From the given data the corresponding frequencies are, i.e. expected frequencies are

Expected Frequencies (e_i)	$\frac{8}{13} \times 524 = 323$	$\frac{2}{13} \times 524 = 81$	$\frac{2}{13} \times 524 = 81$	$\frac{1}{13} \times 524 = 40$

$$\chi^2_{k-p-1} = \chi^2_3 = \frac{\sum (o_i - e_i)^2}{e_i}$$

$$\chi^2_3 = \frac{(222-323)^2}{323} + \frac{(120-81)^2}{81} + \frac{(32-81)^2}{81} + \frac{(150-40)^2}{40}$$

$$\chi^2_3 = 31.5820 + 18.7778 + 29.64198 + 302.5$$

$$\chi^2_3 = 382.502$$

$$\chi^2_{3,\,0.05} = 7.815$$

The calculated value of χ^2 is much more than $\chi^2_{3,\,0.05}$, there is a very low degree of agreement between the theory and experiment.

Ex. 6 : *A set of five similar coins is tossed 210 times and the result is*

No. of Heads	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*2*	*5*	*20*	*60*	*100*	*23*

Test the hypothesis that the data follow a binomial distribution.

Sol. : Here k – p – 1 = 5.

$$p \;:\; \text{Probability of getting a head} = \frac{1}{2}$$

$$q \;:\; \text{Probability of getting a tail} = \frac{1}{2}.$$

Hence the theoretical frequencies of getting 0, 1, 2, 3, 4, 5 heads are the successive terms of the binomial expansion 210 $(p + q)^5$

$$= 210\,[p^5 + 5\,p^4 q + 10\,p^3 q^2 + 10\,p^2 q^3 + 5\,pq^4 + q^5]$$

$$= 210\left[\frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32}\right]$$

$$= 7 + 33 + 66 + 66 + 33 + 7$$

$\therefore$ The theoretical frequencies are 7, 33, 66, 66, 33, 7.

Hence,
$$\chi_5^2 = \frac{(2-7)^2}{7} + \frac{(5-33)^2}{33} + \frac{(20-66)^2}{66} + \frac{(60-66)^2}{66} + \frac{(100-33)^2}{33} + \frac{(23-7)^2}{7}$$

$$\chi_5^2 = 3.57143 + 23.7576 + 32.06061 + 0.5455 + 136.0303 + 36.5714$$

$$\chi_5^2 = 232.53684$$

$\therefore$
$$\chi_{5,\,0.05}^2 = 11.070$$

Since the calculated value of χ^2 is much greater than $\chi_{5,\,0.05}^2$, the hypothesis that the data follow the binomial distribution is rejected.

Ex. 7 : *The figures given below are (a) the theoretical frequencies of a distribution and (b) the frequencies of a normal distribution having the same mean, standard deviation and the total frequency as in (a).*

(a)	*1*	*5*	*20*	*28*	*42*	*22*	*15*	*5*	*2*
(b)	*1*	*6*	*18*	*25*	*40*	*25*	*18*	*6*	*1*

Apply the χ^2 test of goodness of fit.

Sol. : Since the observed and expected frequencies are less and 10 in the beginning and end of the series, we pull the classes and then apply the χ^2 test.

o_i	e_i	$(o_i - e_i)^2$	$(o_i - e_i)^2/E_i$
$\left.\begin{matrix}1\\5\end{matrix}\right\}$ 6	$\left.\begin{matrix}1\\6\end{matrix}\right\}$ 7	1	0.1429
20	18	4	0.2222
28	25	9	0.36
42	40	4	0.1
22	25	9	0.36
15	18	9	0.5
$\left.\begin{matrix}5\\2\end{matrix}\right\}$ 7	$\left.\begin{matrix}6\\1\end{matrix}\right\}$ 7	0	0

Now,
$$\chi_{6,\,0.05}^2 = \sum_i \frac{(o_i - e_i)^2}{e_i}$$

We take table value of χ_6^2. Because, we have total 9 classes, 2 classes are pulled, so after pulling number of classes is 7.

$\therefore$ The degree of freedom for this experiment is 6.

$\therefore$
$$\chi_{6,\,0.05}^2 = 12.592$$

Now, we have,

$$\chi_6^2 < \chi_{6,\,0.05}^2$$

$\therefore$ Accept H_0.

Conclusion : The fit is good.

Ex. 8 : *The demand for a particular spare part in a factory was found to vary from day-to-day. In a sample study the following information was obtained.*

Days	*Mon.*	*Tues.*	*Wed.*	*Thurs.*	*Fri.*	*Sat.*
No. of Parts Demanded	*1124*	*1125*	*1110*	*1120*	*1126*	*1115*

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

Sol. : To Test :

H_0 : The number of parts demanded does not depend on the day of the week.

Vs H_1 : The number of parts demanded depend on the day of the week.

The number of parts demanded during six days = 6720.

$\therefore$ Expected number of parts to be demanded each day of the week $= \dfrac{6720}{6} = 1120.$

Applying χ^2 Test :

Days	o_i	e_i	$(o_i - e_i)^2$	$(o_i - e_i)^2/e_i$
Monday	1124	1120	16	0.01429
Tuesday	1125	1120	25	0.0223
Wednesday	1110	1120	100	0.0893
Thursday	1120	1120	0	0
Friday	1126	1120	36	0.0321
Saturday	1115	1120	25	0.0223

Now, $\chi_5^2 = \sum \dfrac{(o_i - e_i)^2}{e_i}$

The critical value at 5% l.o.s. is $\chi^2_{5,\,0.05} = 11.07$

$\therefore \quad \chi_5^2 < \chi^2_{5,\,0.05}$

$\therefore$ Accept H_0.

Conclusion : The number of parts demanded does not depend on the day of the week.

6.25 STUDENTS T-DISTRIBUTION

In number of situations such as biological experiments, clinical trials etc. in which sample sizes are small ($n < 30$) exact tests can be used for testing hypothesis about a population parameter.

In large sample tests, the distribution of test statistic z is taken as standard normal which seems to be reasonable.

In small sample tests the statistic follows exact sampling distribution viz. χ^2, t or F for sample of any size. Hence small sample tests are regarded as exact tests. The procedure of testing a null hypothesis H_0 in small sample tests is almost identical to that of large sample test. We discuss below the test called t-distribution or student t-distribution.

Test for Population Mean :

Let $x_1, x_2, \ldots x_i, \ldots, x_n$ be a random sample of size n from a normal population with mean μ and variance σ^2. We desire to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$

where, μ_0 represents a specified value of population mean μ.

If the population variance σ^2 is known then the test statistic. Under $H_0 : \mu = \mu_0$ is,

$$\frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \rightarrow N(0, 1) \qquad \ldots \text{(i)}$$

The difficulty in testing H_0 arises when σ^2 is unknown. We shall discuss following two cases when σ^2 is *unknown*.

(i) $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. (ii) $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ or $\mu < \mu_0$

Case (i) : $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$

In order to overcome the difficulty that σ^2 is unknown, we propose the statistics U and V as follows :

$$U = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \quad \text{and} \quad V = \frac{\Sigma (x_i - \bar{x})^2}{\sigma^2}$$

Then $U \to N(0, 1)$ and $V \to \chi^2$ distribution with (n – 1) degrees of freedom. Moreover U and V are independent variates. Hence, we can define statistic as $t = \dfrac{U}{\sqrt{\dfrac{V}{n-1}}}$ which has t-distribution with (n – 1) degrees of freedom.

$$\therefore \quad t = \frac{U}{\sqrt{\frac{V}{n-1}}} = \frac{\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{\Sigma (x_i - \bar{x})^2}{\sigma^2}}{n-1}}}$$

$$= \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \quad \text{where,} \quad s^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1} \qquad \text{... (ii)}$$

$$\therefore \quad \text{Under } H_0 : \mu = \mu_0, \quad t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} \to \text{t-distribution with (n – 1) degrees of freedom.}$$

Thus we obtain test statistic which is free from unknown parameter σ.

Since, the alternative hypothesis $H_1 : \mu \neq \mu_0$ is two sided, the rejection region is $|t_{n-1}| \geq t_{n-1;\frac{\alpha}{2}}$ where value $t_{n-1;\frac{\alpha}{2}}$ is such that $P\left\{|t_{n-1}| \geq t_{n-1;\frac{\alpha}{2}}\right\} = \alpha$ This value is also shown in the Fig. 6.27.

So, the decision rule is, reject H_0 at level of significance α, if $|t_{n-1}| \geq t_{n-1;\frac{\alpha}{2}}$ and accept H_0 otherwise. Then the conclusion about the population mean can be drawn accordingly.

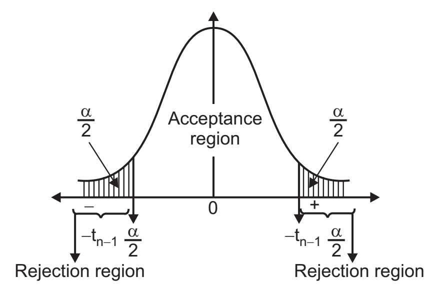


Fig. 6.27

Remarks : (I) If σ is known, we use test statistic (i) Which follows N(0, 1) distribution, where as if σ is unknown, we use test statistic (ii) Apparently we may think that the unknown σ is replaced by s but it is not so which is clear from the above discussion.

(II) For using any test based on t-distribution, one has to ensure that the observations in the parent population (population from which sample is drawn) follow normal distribution.

Case (ii) : $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$ or $\mu > \mu_0$. In this case we use the same test statistic as in case (i). If $H_1 : \mu < \mu_0$, then the critical region at level of significance α is $t_{n-1} \leq -t_{n-1;\alpha}$. It is shaded region as shown in figure.

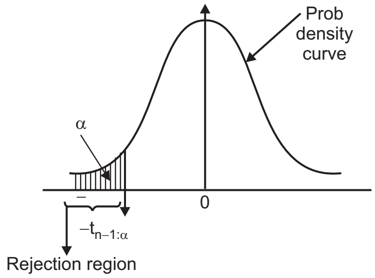


Fig. 6.28

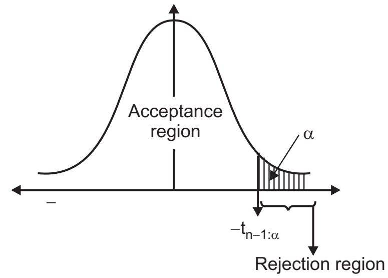


Fig. 6.29

On the other hand if $H_1 : \mu > \mu_0$ then critical region at level of significance α is $t_{n-1} \geq t_{n-1;\alpha}$. It is shaded region as shown in Fig. 6.29.

Illustrations

Ex. 1 : *In order to start new S.T. bus to a certain remote village it is required to get the average fare of ₹ 400 daily. Reports on number of passengers for 21 days revealed that the average daily collection of fare from the passengers was ₹ 390 with standard deviation of ₹ 40. Do these data support the demand of people for starting new bus to the village ? [Use 5% l.o.s.]*

Sol. : Here, we want to test $H_0 : \mu = \mu_0 = 400$ against $H_1 : \mu < 400$.

$\therefore$ The test statistic is given by

$$t_{n-1} = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}} \quad \text{where, } \bar{x} = \text{sample mean} = 390$$

and
$$s^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1} = \frac{n \text{ (sample variance)}}{n-1} = \frac{(21)\ (1600)}{20} = 1680$$

$$\therefore \quad s = 40.987803$$

$$\therefore \quad t_{20} = \frac{390 - 400}{\frac{40.987803}{\sqrt{21}}} = \frac{(-10)\sqrt{21}}{40.987803}$$

$$= -1.1180342 \quad \text{[calculated value]}$$

$$t_{20;\,0.05} = 1.725 \quad \text{[table value]}$$

[Since H_1 is one sided see the value in statistical table corresponding to n = 20 and probability 0.1].

As $H_1 : \mu < \mu_0$ the decision rule is reject H_0 if t_{20} (calculated value) $< -t_{20;\,0.05} = -1.725$ and accept H_0 otherwise. Here $t_{20} = -1.1180342 > -1.725$. Hence we accept H_0 at 5 % *l*.o.s.

Conclusion : These data support the demand of people for starting new bus to the village.

Ex. 2 : *Suppose that sweets are sold in packages of fixed weight of the contents. The producer of the packages is interested in testing that average weight of contents in packages in 1 kg. Hence a random sample of 12 packages is drawn and their contents found (in kg) as follows : 1.05, 1.01, 1.04, 0.98, 0.96, 1.01, 0.97, 0.99, 0.98, 0.95, 0.97, 0.95.*

Using the above data what should he conclude about the average weight of contents in the packets ? [Use 5% l.o.s.]

Sol. : Let μ denote the average weight of contents in packages. Producer wants to test $H_0 : \mu = \mu_0 = 1$ against $H_1 : \mu \neq 1$.

Test statistic is given by

$$t_{n-1} = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

where,
$$\bar{x} = \text{sample mean} = \frac{\Sigma x_i}{n} \quad \text{and} \quad s^2 = \frac{\Sigma x_i^2 - n\bar{x}^2}{n-1}$$

Here,
$$\bar{x} = 0.9883,$$

$$s^2 = \frac{11.7376 - 11.7208}{11} = 0.001164$$

$$\therefore \quad s = 0.034112$$

$$\therefore \quad t_{11} = \frac{(0.9883 - 1)\sqrt{12}}{0.034112} = -\frac{0.04053}{0.034112} = -1.188145$$

$\therefore |t_{11}| = 1.188145$ [calculated value] $< t_{11;\,0.05} = 2.201$ [Table value]

$\therefore$ We accept H_0 at 5 % *l*.o.s.

Conclusion : The producer should conclude that the average weight of contents of package may be taken as 1 kg.

Ex. 3 : *A random sample of 17 drinks from a soft drink machine has one average constant of 7.40 ounces with a standard deviation of 0.48 ounces. Test the hypothesis that μ = 7.5 ounces against the alternative hypothesis μ < 7.5 ounces. Use 5% level of significance. Symbols have their own meanings. State your assumptions if any.*

Sol. : Let x denote the variable under study.

Assumptions : (i) $x \sim N(\mu, \sigma^2)$, σ^2 unknown.

(ii) A r.s. of size n is drawn from distribution of x (n small).

To test : $H_0 = \mu = \mu_0 = 7.5$

$H_1 = \mu < 7.5$

Test statistics : $t_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t\,(n-1)$ under H_0

where, $n = 17$

$\bar{x} = 7.40$

Standard deviation $= s = 0.48$

$$s^2_{unknown} = \frac{ns^2}{n-1} = \frac{17\,(0.48)^2}{16} = \frac{17\,(0.2304)}{16} = 0.2448$$

$s_{unknown} = 0.4948$

Critical region : $C = \{t_0 \mid t_0 < t_{2\sigma,\, n-1}\}$

i.e. H_0 is rejected if $t_0 < -t_{2\sigma,\, n-1}$ and is accepted otherwise

$\therefore$ $\alpha = 0.05$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{7.40 - 7.5}{0.4948/4.1231} = \frac{-0.1}{0.1200} = -0.8333$$

$-t_{2\alpha,\, n-1} = -t_{0.1,\,16} = -1.746$

$t_0 > -t_{0.1,\,16}$

$\therefore$ Accept H_0.

Conclusion : $\mu = \mu_0 = 7.5$ ounces

Ex. 4 : *A roil order photo finishing company charges ₹ 50 per roll as handling cost. The manager suspects that handling cost has gone up. To verify this suspicion, he takes a random sample of 16 orders and finds that the average cost is ₹55 with a standard deviation of ₹ 8/-. Do these data confirm manager's suspension at 1% l.o.s. ?*

Sol. : Let x denote the heading cost per roil.

Assumptions :

(i) $x \sim N(\mu, \sigma^2)$, σ^2 unknown.

(ii) A random sample of size n is drawn from distribution of x (n small).

To test : $H_0 = \mu = \mu_0 = 50$ Vs $H_1 : \mu > 50$

Test statistics : $t_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$ under H_0.

where, $\bar{x}$ = Sample mean x $= \dfrac{\sum x_i}{n}$

$$s^2_{unknown} = \frac{ns^2}{n-1}$$

Critical region : $C = \{t_0 \mid t_0 > t_{2\alpha, n-1}\}$

i.e. H_0 is rejected if $t_0 > t_{2\alpha, n-1}$ and accepted otherwise.

Here,

$$n = 16$$

$$\bar{x} = 55$$

$$\text{Standard deviation} = s = 8$$

$$s^2_{\text{unknown}} = \frac{ns^2}{n-1} = \frac{16\,(64)}{15} = 68.2667$$

$$s_{\text{unknown}} = 8.2624$$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{55 - 50}{8.2624/4} = \frac{5}{2.0656} = 2.4206$$

$$t_{2\alpha, n-1} = t_{2\,(0.01),\,15} = 2.602$$

$$t_0 < t_{0.02,\,15} \quad \therefore \text{ Accept } H_0.$$

Conclusion : The data do not confirm the manager's suspension.

EXERCISE 6.2

1. 10 coins are thrown simultaneously. Find the probability that
(i) Exactly 3 Heads will appear. (ii) Three or less Heads will appear. **Ans.** (i) 0.1172; (ii) 0.1718.

2. Probability of man now aged 60 years will live upto 70 years of age is 0.65. Find the probability of out of 10 men sixty years old 6 or more will live upto the age of 70 years. **Ans.** 0.2377

3. Six dice are thrown 729 times. How many times do you expect at least 3 dice to show a five or six ?
$\left[\textbf{Hint :}\ p = \frac{2}{6} = \frac{1}{3},\ q = \frac{2}{3},\ 729\ P(r \geq 3) = 729\ [1 - P(r < 3)\right]$ **Ans.** 233

4. Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys, (ii) 5 girls, (iii) either 2 or 3 boys. Assume equal probabilities for boys and girls. **Ans.** (i) 250, (ii) 25, (iii) 500.

5. In sampling the large numbers of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2 out of 1000 such samples. How many would be expected to contain at least 3 defective parts ? **Ans.** 323.

6. According to past record of one day internationals between India and Pakistan, India has won 15 matches and lost 10. If they decide to play a series of 6 matches now, what is the probability of India winning the series ? (Draw is ruled out). **Ans.** 0.5443

7. Two dice are thrown 100 times and the number of nines recorded. What is the probability that 'r' nines occur ? Find the probability that at least 3 nines occur. $\left[\textbf{Hint :}\ z = 2,\ P(r) = \frac{e^{-2}\,(2)^r}{r!}\right]$ **Ans.** 0.00045

8. Between 2 p.m. and 3 p.m. the average number of phone calls per minute coming into the company are 2. Find the probability that during one particular minute, there will be:

(i) No phone calls at all. (ii) 2 or less calls. **Ans.** (i) 0.1353; 6/65

9. A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimetre equal to 2. Five 1 c.c. test tubes are filled with the liquid, assuming that Poisson distribution is applicable, calculate the probability that all test tubes show growth. **Ans.** 0.036

10. In a Telephone exchange, the probability that any one call is wrongly connected is 0.02. What is the minimum number of calls required to ensure a probability 0.1 that at least one call is wrongly connected ? **Ans.** 6 calls approximately

11. Fit a Binomial distribution to the following data.

x	0	1	2	3	4	5
f	2	22	63	76	96	56

Ans. $[315\,(0.34 + 0.66)^5]$

12. A manufacturer of electronic goods has 4 % of his product defective. He sells the articles in packets of 300 and guarantees 90 % good quality. Determine the probability that a particular packet will violate the guarantee.

[**Hint :** z = np = 300 × 0.04 = 12)] **Ans.** $1 - \sum_{r=1}^{30} \frac{e^{-12} (12)^r}{r!}$

13. Prove that the following data represents Poisson distribution.

x	0	1	2	3	4
y	109	65	22	3	1

Ans. Mean = Variance = 0.61

14. Fit a Poisson distribution to the following frequency distribution and compare the theoretical frequencies with observed frequencies.

x	0	1	2	3	4	5
f	158	160	60	25	10	2

15. X is normally distributed and the mean of X is 15 and standard deviation 3. Determine the probability of (i) $0 < X < 10$; (ii) $X \geq 18$ **Ans.** (i) 0.10483; (ii) 0.1587

16. In a certain examination, the percentage of passes and distinction were 48 and 10 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. **Ans.** 38.5772

17. In a normal distribution, 31 % of the items are under 45 and 8 % are over 64. Find the mean and standard deviation of distribution. **Ans.** σ = 10.0529; μ = 49.9259

18. 5000 candidates appeared in a certain paper carrying a maximum of 100 marks. It was found that marks were normally distributed with mean 39.5 and standard deviation 12.5. Determine approximately the number of candidates who secured a first class for which a minimum of 60 marks is necessary. **Ans.** 253

19. A random sample of 200 screws is drawn from a population which represents the size of screws. If a sample is distributed normally with a mean 3.15 cm and standard deviation 0.025 cm, find expected number of screws whose size falls between 3.12 cm and 3.2 cm.Area corresponding to 1.2 → 0.3849; Area corresponding to 2.0 → 0.4772 **(Nov. 2014)**

[**Note :** $p(-1.2 < z < 2) = p(0 < z < 1.2) + p(0 < z < 2)$]. **Ans.** 172 Approximately.

20. For a normal distribution, N = 300, μ = 75 and σ = 15. How many values lie between x = 60 and x = 70 ?

The area under the normal curve for various values of Z is given as,

Z	**Area**
0.33	0.12930
0.34	0.13307
1.0	0.34134

Ans. 63 approximately.

21. In a certain city, 2000 electric lamps are installed. If the lamps have average life of 1000 burning hours with standard deviation of 200 hours,

(i) What number of lamps might be expected to fail in first 700 burning hours ?

(ii) After what period of burning hours, 10% of lamps would still be burning ?

Given that if F (1.5) =0.933 and F (1.28) = 0.900 **Ans.** (i) 866, (ii) 1256

22. Obtain the equation of normal curve that may be fitted to the following distribution.

x	50	60	70	80	90	100
f	5	20	120	250	240	5

Also obtain expected normal frequencies.

23. Explain the test procedure for testing the independence of two attributes in an $r \times s$ contingency table.

24. Describe χ^2 test for goodness of fit. State the assumptions we make while applying the test.

25. The table below gives the number of accidents that occurred in the certain factory on the various days of a particular week.

Days of Week	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
No. of Accidents	6	4	9	7	8	10	12

Test at 5% level whether accidents are uniformly distributed over the different days

$\chi_6^2 = 5.25$; $\chi_{6\,;0.05}^2 = 15.592$ Accept 40. **Ans.** $\chi_6^2 = 5.25$, $\chi_{6;\,0.05}^6 = 15.592$ Accept H_0

26. The following is a 2×2 contingency table :

Eye Colour in Father	**Eye Colour in Son**	
	Not Light	**Light**
Not light	23	15
Light	15	47

Test whether the eye colour in son is associated with the eye colour in father. **Ans.** $\chi_1^2 = 13.2$ $\chi_{1;\,0.05}^2 = 3.841$ Reject H_0

27. A die when tossed 300 times gave the following results :

Score	1	2	3	4	5	6
Frequency	43	49	56	45	66	41

Are the data consistent at 5 % level of significance with the hypothesis that the die is true ?

Ans. $\chi_5^2 = 8.56$, $\chi_{5;\,0.05}^2 = 11.07$ Accept H_0

28. The table below gives the number of books issued from a certain library on the various days of a week.

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
No. of Books Issued	120	130	110	115	135	110

Test at 5% *l*.o.s. whether the issuing of books is independent of a day. **Ans.** $\chi_5^2 = 4.583333$, $\chi_{5;\,0.05}^2 = 11.07$ Accept H_0

29. In a locality, 100 persons were randomly selected and asked for their educational achievements. The results are given as under.

Sex	**Education**		
	Primary School	**High School**	**College**
Male	10	15	25
Female	25	10	15

Test whether education depends on sex at 1% level of significance. **Ans.** $\chi_2^2 = 9.929$ $\chi_{2;\,0.05}^2 = 5.991$ Reject H_0

30. In an experiment on pea breeding, a scientist obtained the following frequencies of seeds : 316 round and yellow, 102 wrinkled and yellow, 109 round and green and 33 wrinkled and green. Theory predicts that the frequencies of seeds should be in the proportion 9 : 3 : 3 : 1 respectively. Set a proper hypothesis and test it at 5 % *l*.o.s.

Ans. $\chi_3^2 = 0.3555554$, $\chi_{3;\,0.5}^2 = 7.815$ Accept H_0

31. A newspaper publisher is interested in testing whether newspaper readership in the society is associated with readers' educational achievement. A related survey showed the followed results :

	Level of Education			
Type of Readership	**Post Graduate**	**Graduate**	**Passed S.S.C**	**Not passed S.S.C.**
Never	09	12	30	60
Sometimes	25	20	15	20
Daily	68	48	40	10

Test whether type of newspaper readership depends on level of education.

[Take $\alpha = 0.05$] **Ans.** $\chi_6^2 = 97.651$, $\chi_{6;\,0.05}^2 = 12.592$ Reject H_0

32. From the information given below, test whether the type of occupation and attitude towards the social laws are independent. [Use 1 % *l*.o.s.]

Attitude towards Social Laws			
Occupation	**Favourable**	**Neutral**	**Opposite**
Blue-collar	29	26	37
White-collar	25	32	56
Professional	34	21	42

Ans. $\chi_4^2 = 5.415$, $\chi_{4;\,0.01}^2 = 9.488$ Accept H_0

Degree of Freedom	Distribution of χ^2	
	5 %	**1 %**
1	3.841	6.635
2	5.991	9.210
3	7.815	11.345
4	9.488	13.277
5	11.070	15.086
11	19.675	24.725
12	21.026	26.217
13	22.362	27.668
14	23.685	29.141
15	24.996	30.578
21	32.671	38.932
22	33.924	40.289
23	35.172	41.638
24	36.415	42.980
25	37.652	44.314
40	55.759	63.691
60	79.082	88.379
∞	–	–

Degree of Freedom	Distribution of χ^2	
	5 %	**1 %**
6	15.592	16.812
7	15.067	18.475
8	15.507	20.090
9	16.919	21.666
10	18.307	23.209
16	26.296	32.000
17	27.587	33.409
18	28.869	34.191
19	30.144	36.191
20	31.410	37.566
26	38.885	45.642
27	40.113	46.963
28	41.337	48.278
29	42.557	49.588
30	43.773	50.892

33. Ten individuals are chosen at random from a population whose heights are found to be (in cms) 159, 162, 164, 170, 169, 171, 172, 168, 171, 175 cms. Can this sample be regarded as drawn from a population with mean height more than 170 cms at 5% level of significance ? **Ans.** $|t| = 1.833$, Accept H_0.

34. The height of 8 persons in a office is found to be 68, 64, 67, 70, 62, 64, 66, 70 inches. Is it reasonable to believe that average height is 64 inches ? Test at 5% level of significance. **Ans.** $|t| = 3$, Reject H_0

UNIT V : NUMERICAL METHODS

CHAPTER-7
NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

7.1 INTRODUCTION

Solution of algebraic or transcendental equation is a problem of common occurrence in mathematics. Solution of linear algebraic equation ax + b = 0 is given by $x = -\frac{b}{a}$. Roots of quadratic equation $ax^2 + bx^2 + c = 0$ are given by,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Roots are real or complex, depending upon whether $b^2 \geq 4ac$ or $b^2 < 4ac$.

Formulae also exist to find the roots of cubic and biquadratic equations. If f(x) represents a polynomial of degree greater than four, ready formulae are not available to find roots of algebraic equation f(x) = 0. Again, if f(x) is transcendental (i.e. contains functions such as trigonometric, logarithmic, exponential etc.) then finding roots of f(x) = 0 is not an easy task. Exact solutions of the equations f(x) = 0 are not generally possible. Using Numerical techniques, approximate solutions of f(x) = 0 can almost be obtained. Numerical methods usually start with some crude approximation of the root i.e. some guessed value of the root. Definite procedure (Numerical Algorithm) brings about constant improvement in the initial approximation. Procedure terminates when required degree of accuracy is achieved.

Algorithm set to solve the equation f(x) = 0 must give close approximation to the root in finite number of steps. Computational effort should also be minimum. Numerical methods which start with ***initial guess*** value and bring about closer approximations in subsequent steps, are called ***iterative methods***. In work that follows we shall discuss different iterative methods to solve the equation f(x) = 0.

Based on the number of initial approximations used, iterative methods are divided into two categories namely Bracketing Method and Open-end Methods. Bracketing methods begin with two initial approximations which bracket the root. Then the width of this bracket is systematically reduced until the root is approached to desired accuracy. Open-end methods are formulae based which require a single initial value or two initial values which do not necessarily bracket the root.

Geometrically, roots of f(x) = 0 are the points where graph of y = f(x) cuts x axis. This fact helps to start with initial approximation. Many times it is difficult to plot the graph of y = f(x), in such cases if f(x) = 0 is expressible as $f_l(x) = f_2(x)$ then points of intersection of graphs of $y = f_l(x)$ and $y = f_2(x)$ determine approximate roots of the equation f(x) = 0.

To find graphical solution of the equation $f(x) = x^3 + 2x - 5 = 0$, we write the equation as $x^3 = 5 - 2x$ and then plot the graphs of $y = x^3$ and $y = 5 - 2x$. The points of intersection of these graphs help to determine the roots of the equation f(x) = 0. Apart from using graphical method, most commonly used method to obtain the initial approximation is based upon ***intermediate value theorem*** which states that,

Theorem : If f(x) is continuous function on some interval [a, b] and f(a) f(b) < 0 [i.e. f(a), f(b) are of opposite signs] then the equation f(x) = 0 has at least one real root say ξ in the interval (a, b) (refer Fig. 7.1).

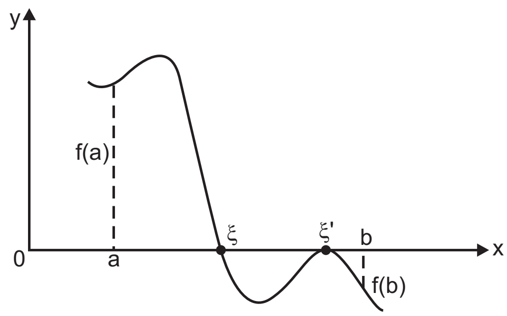


Fig. 7.1

Further, the root ξ will definitely be unique if the derivative f '(x) exists and preserves sign within the interval (a, b) that is, if f'(x) > 0 or (f '(x) < 0) for a < x < b. refer (Fig. 7.2).

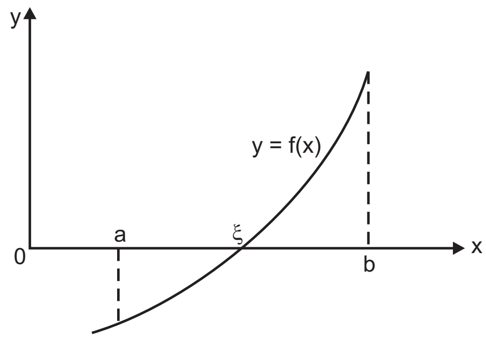


Fig. 7.2

We note here that if f(ξ) = 0 then ξ is called a root of the equation f(x) = 0 or zero of the function f(x).

Whenever essential, graphical method is used to find initial approximation, but in most cases, use of intermediate value theorem serves the purpose.

Success of any iterative method depends upon successive approximations reaching nearer to the exact root say ξ of the equation f(x) = 0 in finite number of steps.

Definition : A sequence of iterates x_i, i = 1, 2, 3 ... is said to converge to the root ξ if

$$\lim_{i \to \infty} |x_i - \xi| = 0$$

In practice, it is generally not possible to find exact value ξ of the root, we terminate the iterative procedure at i^{th} stage if $|f(x_i)| < \varepsilon$, where, ε is prescribed error tolerance, or we can stop when $|x_{i+1} - x_i| < \varepsilon$.

If $|x_{i+1}|$ is not less than $|x_i|$, then the iterative procedure is not convergent and is unsuitable for finding the root of the equation. We shall now consider following iterative methods which are commonly used for finding the roots of algebraic or transcedental equations.

1. Bisection method
2. Secant method
3. Regula falsi method or Method of false position
4. Newton-Raphson method
5. Method of successive approximation (simple iteration method)

7.2 BISECTION METHOD

This is a simple method based upon successive applications of intermediate value theorem. If function f(x) is continuous in [a, b] and f(a) f(b) < 0, then to find the root of f(x) = 0, lying in the interval (a, b), we find middle point $c = \frac{a + b}{2}$. If f(c) = 0, then

$\xi = c$ is the root of the equation. If $f(c) \neq 0$, then we find whether $f(a)\, f(c) < 0$ or $f(b)\, f(c) < 0$. If $f(a)\, f(c) < 0$, then the root lies in the interval $[a, c]$ and if $f(b)\, f(c) < 0$, then the root lies in the interval $[b, c]$. Thus, the interval $[a, b]$ is reduced to half interval $[a_1, b_1]$.

Employing the same line of investigation, the interval $[a_1, b_1]$ is halved to $[a_2, b_2]$ and the above process is repeated. Finally, at some stage either the exact root or an infinite sequence of nested intervals, $[a_1, b_1], [a_2, b_2] \ldots [a_n, b_n]$ is obtained. Above process ensures that $f(a_n)\, f(b_n) < 0$ $(n = 1, 2 \ldots)$ and $b_n - a_n = \frac{1}{2^n}(b - a)$.

Since $a_l, a_2 \ldots a_n$ form a monotonic non-decreasing bounded sequence and $b_1, b_2, \ldots b_n$ form monotonic non-increasing bounded sequence there exists a common limit,

$$\xi = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

Bisection method is thus computationally simple and the sequence of approximations always converges to the root for any f(x) which is continuous in the interval that contains the root. If the permissible error is ε then the approximate number of iterations required can be determined from the relation

$$\frac{b-a}{2^n} \leq \varepsilon$$

or $$\log(b-a) - n \log 2 \leq \log \varepsilon$$

or $$n \geq \frac{\log(b-a) - \log \varepsilon}{\log 2} \qquad \ldots (1)$$

In particular, the minimum number of iterations required for converging to a root in the interval (0, 1) for a given ε are as under

ε :	10^{-2}	10^{-3}	10^{-4}
n :	7	10	14

If the roots of $f(x) = 0$ are not isolated in the interval $[a, b]$, then Bisection method can be used to find one of the roots of the equation. Computational efforts increase substantially with increase in accuracy. Method requires one functional evaluation for each iteration and the number of iterations required are quite large to achieve reasonable degree of accuracy, nevertheless the method is quite suitable for computer programming.

ILLUSTRATIONS

Ex. 1 : *Use the method of Bisection to find a root of the equation*

$$f(x) = x^4 + 2x^3 - x - 1 = 0$$

lying in the interval [0, 1] at the end of sixth iteration. How many iterations are required if the permissible error is $\varepsilon = 0.0005$

Sol. : Here $f(0) = -1$

$$f(1) = 1 \left[\text{Root lies between 0 and 1, } \xi_1 = \frac{0+1}{2} = 0.5\right]$$

$$f(0.5) = -1.1875 \left[\text{Root lies between 0.5 and 1, } \xi_2 = \frac{0.5+1}{2} = 0.75\right]$$

$$f(0.75) = -0.5898 \left[\text{Root lies between 0.75 and 1, } \xi_3 = \frac{0.75+1}{2} = 0.875\right]$$

$$f(0.875) = 0.0510 \left[\text{Root lies between 0.875 and 0.75, } \xi_4 = \frac{0.75+0.875}{2} = 0.8125\right]$$

$$f(0.8125) = -0.3039 \left[\text{Root lies between 0.875 and 0.8125, } \xi_5 = \frac{0.875 + 0.8125}{2} = 0.84375\right]$$

$$f(0.84375) = -0.642 \left[\text{Root lies between 0.84375 and 0.875, } \xi_6 = \frac{0.84375 + 0.875}{2} = 0.859375\right]$$

Thus, approximate root at the end of sixth iteration is 0.859375

$$|\xi_6 - \xi_5| = |0.859375 - 0.844375| = 0.015622 < 0.02$$

Thus, if permissible error is $\varepsilon = 0.02$ then root at the end of sixth iteration gives required accuracy.

If the permissible error is 0.0005 then by equation (1)

$$n \geq \frac{\log (1 - 0) - \log 0.0005}{\log 2}$$

$$\geq 10.965$$

i.e. n = 11 iterations are required to achieve the degree of accuracy.

Ex. 2 : *Using method of Bisection, find the cube root of 100, using six iterations. Determine whether number of iterations are enough for three significant digit accuracy.*

Sol. : Cube root of 100, is obtained by solving the equation $x^3 - 100 = 0$.

$$f(x) = x^3 - 100$$

$$f(4) = 64 - 100 = -36, \quad f(5) = 125 - 100 = 25$$

$\therefore$ Root lies between 4 and 5, $\xi_1 = \dfrac{4 + 5}{2} = 4.5$

$$f(4.5) = -8.875, \text{ Root lies between 4.5 and 5}$$

$$\xi_2 = \frac{4.5 + 5}{2} = 4.75$$

$$f(4.75) = 7.17, \text{ Root lies between 4.5 and 4.75}$$

$$\xi_3 = \frac{4.5 + 4.75}{2} = 4.625$$

$$f(4.625) = -1.068, \text{ Root lies between 4.625 and 4.75}$$

$$\xi_4 = \frac{4.625 + 4.75}{2} = 4.6875$$

$$f(4.6875) = 2.99, \text{ Root lies between 4.625 and 4.6875}$$

$$\xi_5 = \frac{4.625 + 4.6875}{2} = 4.65625$$

$$f(4.65625) = 0.95, \text{ Root lies between 4.625 and 4.65625}$$

$$\therefore \quad \xi_6 = \frac{4.625 + 4.65625}{2} = 4.640625$$

Thus, $\xi_6 = 4.640625$ is the approximate value of cube root at the end of sixth iteration.

$$|\xi_6 - \xi_5| = |4.640625 - 4.65625| = 0.015625$$

ξ_5, ξ_6 will agree to three significant digit if

$$|\xi_6 - \xi_5| < \frac{1}{2} \times 10^{0-3+1}$$

i.e.
$$< \frac{1}{2} \times 10^{-2} < 0.005$$

but 0.015625 is not less than 0.005 therefore two approximations do not agree to three significant digits. Hence number of iterations are not sufficient for three significant digit accuracy.

If $\varepsilon = 0.05$ then by (1), number of iteration required,

$$n \geq \frac{\log (5-4) - \log 0.05}{\log 2}$$

or
$$n \geq 7.64$$

i.e. 8 iterations will be required for three significant digit accuracy.

Ex. 3 : *Using the Bisection method, find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places.*

Sol. : Let
$$f(x) = x^3 - 4x - 9$$

Since
$$f(0) = -9,\ f(1) = -12,\ f(2) = -9,\ f(3) = 6$$

$\therefore$ Root lies between 2 and 3. The first approximation to the root by bisection method.

$$\xi_1 = \frac{2+3}{2} = 2.5$$

$f(\xi_1) = -3.375$, Root lies between 2.5 and 3

$$\xi_2 = \frac{2.5+3}{2} = 2.75$$

$f(\xi_2) = 0.7969$, Root lies between 2.5 and 2.75

$$\xi_3 = \frac{2.5+2.75}{2} = 2.625$$

$f(\xi_3) = -1.4124$, Root lies between 2.625 and 2.75

$$\xi_4 = \frac{2.625+2.75}{2} = 2.6875$$

$f(\xi_4) = -0.3391$, Root lies between 2.6875 and 2.75

$$\xi_5 = \frac{2.6875+2.75}{2} = 2.71875$$

$f(\xi_5) = 0.2209$, Root lies between 2.6875 and 2.71875

$$\xi_6 = \frac{2.6875+2.71875}{2} = 2.70313$$

$f(\xi_6) = -0.061$, Root lies between 2.70313 and 2.71875

$$\xi_7 = \frac{2.70313+2.71875}{2} = 2.71094$$

$f(\xi_7) = 0.795$, Root lies between 2.70313 and 2.71094

$$\xi_8 = \frac{2.70313 + 2.71094}{2} = 2.70703$$

$$f(\xi_8) = 0.0903, \text{ Root lies between } 2.70313 \text{ and } 2.70703$$

$$\xi_9 = \frac{2.70313 + 2.70703}{2} = 2.70508$$

$$f(\xi_9) = -0.0260, \text{ Root lies between } 2.70508 \text{ and } 2.70703$$

$$\xi_{10} = \frac{2.70508 + 2.70703}{2} = 2.70605$$

$$f(\xi_{10}) = -0.0086, \text{ Root lies between } 2.70605 \text{ and } 2.70703$$

$$\xi_{11} = \frac{2.70605 + 2.70703}{2} = 2.70654$$

$$f(\xi_{11}) = 0.0002165, \text{ Root lies between } 2.70605 \text{ and } 2.70605$$

$$\xi_{12} = \frac{2.70605 + 2.70654}{2} = 2.706295$$

Hence, the root is 2.7063 correct to three decimal places.

Ex. 4 : *Find a root of the equation cos x = x* e^x *(measured in radian) using the bisection method at the end of sixth iteration.*

Sol. : Let $f(x) = \cos x = x\, e^x$

Since $f(0) = 1$ and $f(1) = -2.18$

$\therefore$ Root lies between 0 and 1. The first approximation to the root is

$$\xi_1 = \frac{0 + 1}{2} = 0.5$$

$$f(\xi_1) = 0.05, \text{ Root lies between } 0.5 \text{ and } 1$$

$$\xi_2 = \frac{0.5 + 1}{2} = 0.75$$

$$f(\xi_2) = -0.86, \text{ Root lies between } 0.5 \text{ and } 0.75$$

$$\xi_3 = \frac{0.5 + 0.75}{2} = 0.625$$

$$f(\xi_3) = -0.36, \text{ Root lies between } 0.5 \text{ and } 0.625$$

$$\xi_4 = \frac{0.5 + 0.625}{2} = 0.5625$$

$$f(\xi_4) = -0.14, \text{ Root lies between } 0.5 \text{ and } 0.5625$$

$$\xi_5 = \frac{0.5 + 0.5625}{2} = 0.5312$$

$$f(\xi_5) = -0.041, \text{ Root lies between } 0.5 \text{ and } 0.5312$$

$$\xi_6 = \frac{0.5 + 0.5312}{2} = 0.5156$$

Hence, the desired approximation to the root is 0.5156.

Ex. 5 : *Find a positive real root of $x \log_{10} x = 1.2$ using bisection method at the end of fifth iteration.*

Sol. : Let $f(x) = x \log_{10} x - 1.2$

Since $f(2) = -0.598$ and $f(3) = 0.231$

$\therefore$ Root lies between 2 and 3. The first approximation to the root is

$$\xi_1 = \frac{2+3}{2} = 2.5$$

$$f(\xi_1) = -0.205, \text{ Root lies between 2.5 and 3}$$

$$\xi_2 = \frac{2.5+3}{2} = 2.75$$

$$f(\xi_2) = 0.008, \text{ Root lies between 2.5 and 2.75}$$

$$\xi_3 = \frac{2.5+2.75}{2} = 2.625$$

$$f(\xi_3) = -0.099, \text{ Root lies between 2.625 and 2.75}$$

$$\xi_4 = \frac{2.625+2.75}{2} = 2.687$$

$$f(\xi_4) = -0.047, \text{ Root lies between 2.6875 and 2.75}$$

$$\xi_5 = \frac{2.6875+2.75}{2} = 2.7185$$

Hence, desired approximation to the root is 2.7185.

Ex. 6 : *Using the Bisection method, find an approximate root of the equation $\sin x = \frac{1}{x}$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computation upto 7^{th} iteration.*

Sol. : Let $f(x) = x \sin x - 1$

Since $f(1) = -0.1585$ and $f(1.5) = 0.4962$

$\therefore$ Root lies between 1 and 1.5. The first approximation to the root is

$$\xi_1 = \frac{1+1.5}{2} = 1.25$$

$$f(\xi_1) = 0.18623, \text{ Root lies between 1 and 1.25}$$

$$\xi_2 = \frac{1+1.25}{2} = 1.125$$

$$f(\xi_2) = 0.0151, \text{ Root lies between 1 and 1.125}$$

$$\xi_3 = \frac{1+1.125}{2} = 1.0625$$

$$f(\xi_3) = -0.0718, \text{ Root lies between 1.0625 and 1.125}$$

$$\xi_4 = \frac{1.0625+1.125}{2} = 1.09375$$

$$f(\xi_4) = -0.02836, \text{ Root lies between 1.09375 and 1.125}$$

$$\xi_5 = \frac{1.09375+1.125}{2} = 1.10937$$

$$f(\xi_5) = -0.00665, \text{ Root lies between 1.10937 and 1.125}$$

$$\xi_6 = \frac{1.10937 + 1.125}{2} = 1.11719$$

$$f(\xi_6) = 0.00421, \text{ Root lies between 1.10937 and 1.11719}$$

$$\xi_7 = \frac{1.10937 + 1.11719}{2} = 1.11328$$

Hence, approximation root at the seventh approximation is 1.11328.

7.3 ITERATION METHOD BASED ON FIRST DEGREE EQUATION

Consider first degree equation

$$f(x) = m_1 x + m_2 = 0$$

Solution of this equation is given by

$$x = -\frac{m_2}{m_1} \quad \text{if } m_1 \neq 0$$

Based on this analysis, iterative methods are developed to find approximation to the root of $f(x) = 0$. Prescribing two appropriate conditions on $f(x)$ or its derivative, the values of m_1 and m_2 are determined, from which root of $f(x)$ is calculated. Let us consider methods based on this.

7.4 SECANT METHOD

Let $x = a$, $x = b$ be two successive approximations to the root of $f(x) = 0$.

Consider $$f(x) = m_1 x + m_2 \quad \ldots (1)$$

$$f(x) = 0 \text{ has solution } x = -\frac{m_2}{m_1} \quad \ldots (2)$$

Putting $x = a$ and b in (1),

$$f(a) = m_1 a + m_2 \quad \ldots (3)$$

$$f(b) = m_1 b + m_2 \quad \ldots (4)$$

Subtraction gives $$m_1 (b - a) = f(b) - f(a)$$

or $$m_1 = \frac{f(b) - f(a)}{b - a}$$

Substituting for m_1 in (3), $$f(a) = a\,\frac{f(b) - f(a)}{b - a} + m_2$$

$\therefore$ $$m_2 = f(a) - \frac{a\,\{f(b) - f(a)\}}{b - a}$$

$$= \frac{b\,f(a) - a\,f(a) - a\,f(b) + a\,f(a)}{b - a}$$

$$= \frac{b\,f(a) - a\,f(b)}{b - a}$$

Equation (2) gives approximation as

$$x = -\frac{\left[\dfrac{b\,f(a) - a\,f(b)}{b - a}\right]}{\left[\dfrac{f(b) - f(a)}{b - a}\right]}$$

or $$x = -\frac{b\,f(a) - a\,f(b)}{f(b) - f(a)}$$

or $$\boxed{x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}}$$

which can also be written as

$$\boxed{x = a - \frac{(b-a)}{f(b)-f(a)}\, f(a)} \qquad \ldots (5)$$

Thus, the improved approximation to the root is given by (5).

Remarks : Alternatively, the equation of the chord (Secant) joining the two points [a, f(a)] and [b, f(b)] is

$$y - (a) = \frac{f(b)-f(a)}{b-a}\,(x-a)$$

Taking point of intersection of chord with x-axis (y = 0) as an approximation to the root is given by

$$x - a = -\frac{(b-a)}{f(b)-f(a)}\, f(a) \quad \text{or } x = a - \frac{(b-a)}{f(b)-f(a)}\, f(a)$$

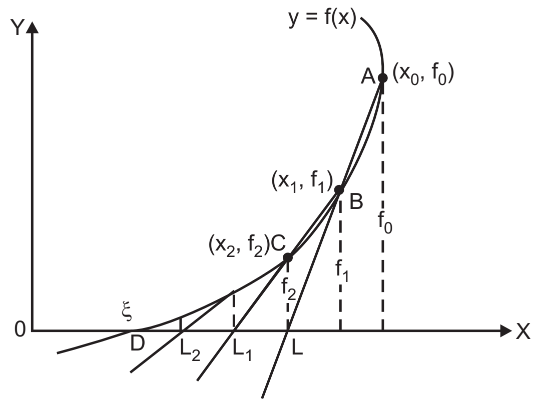


Fig. 7.3

If x_0, x_1 are two initial approximations to the root of f(x) = 0 then next approximation x_2 is given by

$$\boxed{x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_1)} \qquad \ldots (6)$$

If $f_1 = f(x_1)$, $f_0 = f(x_0)$, equation (6) can be written as

$$\boxed{x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0}\, f_1} \qquad \ldots (7)$$

Generalised form of (7) can be written as

$$\boxed{x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f_i - f_{i-1}}\, f_i} \qquad \ldots (8)$$

For geometrical interpretation of (8), consider the curve y = f(x) as shown in the Fig 7.3, A (x_0, f_0), B(x_1, f_1) are two points on the curve which cuts x-axis at the point D. Value of $x = \xi$ corresponding to the point D is the actual root of the equation f(x) = 0. Chord or Secant AB cuts x axis at L. Value of x corresponding to the point L is taken as approximation of the root of f(x) = 0.

Perpendicular to the x-axis at L, cuts the curve at C. Secant through BC cuts x-axis at L_1, which gives next approximation to the root. Geometrically, method of secant replaces the function y = f(x) by a secant or chord joining the points (x_{i-1}, f_{i-1}), (x_i, f_i) and approximation x_{i+1} to the root is given by formula (8).

Procedure is terminated when $|f(x_{i+1})| < \varepsilon$, where ε is permissible error.

Since (x_{i-1}, f_{i-1}), (x_i, f_i) are known before the start of the iteration, the secant method requires evaluation of function once at each step.

We note here that group of the function y = f(x) is approximated by a secant (chord) line and at each iteraction, two most recent approximations to the root are used to find next approximate. Also, it is not necessary that the interval must contains the root.

ILLUSTRATIONS

Ex. 1 : *Find a positive root of the equation : $x^3 - 2x^2 + 3x - 4 = 0$ at the end of fifth iteration by using secant method.*

Sol. : To obtain initial guess, we use intermediate value theorem.

Let $f(x) = x^3 - 2x^2 + 3x - 4$

Since $f(0) = -4,\ f(1) = -2,\ f(2) = 2$

$\therefore$ Root lies between 1 and 2. Taking the initial approximations as $x_0 = 1.00$, $x_1 = 2.0$ we proceed to obtain successive approximations.

In (8), putting i = 1,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \cdot f_1$$

Here $f_0 = -2,\ \ f_1 = 2$

$$x_2 = 2 - \frac{(2-1)}{(2+2)} \times 2$$

$$= 2 - \frac{1}{4} \times 2 = 1.5$$

$$f_2 = f(x_2) = f(1.5) = (1.5)^3 - 2\,(1.5)^2 + 3\,(1.5) - 4 = -0.625$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 1.5 + \frac{(1.5 - 2)}{(-0.625 - 2)} \times (-0.625)$$

$$= 1.5 + \frac{0.5}{2.625} \times 0.625 = 1.619$$

$$f_3 = f(x_3) = f(1.619) = (1.619)^3 - 2\,(1.619)^2 + 3\,(1.619) - 4 = -0.1417$$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 1.619 - \frac{(1.619 - 1.5)}{(-0.1417 + 0.625)} \times (-0.1417)$$

$$= 1.619 + \frac{0.119}{0.4833} \times 0.1417 = 1.6539$$

or we may take $x_4 = 1.654$

$$f_4 = f(x_4) = f(1.654)^3 - 2\,(1.654)^2 + 3\,(1.654) - 4 = 0.015$$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 1.654 - \frac{(1.654 - 1.619)}{(0.015 + 0.1417)} \times 0.015$$

$$= 1.654 - \frac{0.035}{0.1567} \times 0.015$$

$$= 1.6506$$

$$f_5 = f(x_5) = f(1.6506) = (1.6506)^3 - 2(1.6506)^2 + 3(1.6506) - 4 = -0.00013$$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{(f_5 - f_4)} \times f_5$$

$$= 1.6506 - \frac{(1.6506 - 1.654)}{(-0.00013 - 0.0.015)} \times (-0.00013)$$

$$= 1.6506 + \frac{0.0034}{0.01513} \times 0.00013 = 1.65063$$

which is the root required at the end of fifth iteration. It is correct to four decimal places.

$$f_6 = (1.65063)^3 - 2(1.65063)^2 + 3(1.65063) - 4 = 0.000003696$$

If we fix $\varepsilon = 0.00001$ then $|f_6| < \varepsilon$. The accuracy required can thus be achieved.

Ex. 2 : *Use secant method to find a root of the equation $f(x) = x^3 - 5x - 7 = 0$ correct to three places of decimal.*

Sol. : To obtain initial guess, we use intermediate value theorem.

Let $\quad f(x) = x^3 - 5x - 7$

Since $\quad f(2) = -9,\ f(2.5) = -3.875,\ f(3) = 5$

$\therefore$ Root lies between 2.5 and 3. Taking the initial approximations as $x_0 = 2.5$, $x_1 = 3$, we proceed to obtain successive approximations by secant method,

$$\therefore \quad x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \cdot f_1$$

Here $\quad f_0 = f(x_0) = -3.875,\ \ f_1 = (f(x_1) = 5$

$$\therefore \quad x_2 = 3 - \frac{(3 - 2.5)}{[5 - (-3.875)]} \times 5$$

$$= 3 - \frac{0.5}{8.875} \times 5 = 2.7183$$

$$f_2 = f(x_2) = f(2.7183) = -0.5053 \qquad \text{(Root lies between 2.7183 and 3)}$$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 2.7183 - \frac{(2.7183 - 3)}{(-0.5053 - 5)} \times (-0.5053)$$

$$= 2.7183 + 0.02596 = 2.7442$$

$$f_3 = f(x_3) = -0.0554 \qquad \text{(Root lies between 2.7442 and 3)}$$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 2.7442 - \frac{(2.7442 - 2.7183)}{[-0.0554 - (-0.5033)]} \times (-0.0554) = 2.7442 + 0.0032 = 2.7474$$

$$f_4 = f(x_4) = 0.00094 \qquad \text{(Root lies between 2.7442 and 2.7474)}$$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 2.7474 - \frac{(2.7474 - 2.7442)}{[0.00094 - (-0.0554)]} \times 0.00094$$

$$= 2.7474 - 0.000034 = 2.7474$$

Hence the required root is 2.7474.

Ex. 3 : *Find a real root of the equation $x^3 - 2x - 5 = 0$ using secant method correct to three decimal places.*

Sol. : To obtain initial approximation

Let $$f(x) = x^3 - 2x - 5 = 0$$

Since $$f(1) = -7,\ f(2) = -1,\ f(3) = 16.$$

Taking the initial approximation as $x_0 = 2$, $x_1 = 3$, by secant method, we have

$$\therefore \quad x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \cdot f_1$$

$$= 3 - \frac{(3-2)}{[16-(-1)]} \times 16 = 3 - \frac{1}{17} \times 16 = 2.058823$$

$$f(x_2) = -0.390799$$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 2.058823 - \frac{(2.058823 - 3)}{(-0.390799 - 16)} \times (-0.390799) = 2.081263$$

$$f(x_3) = -0.147204$$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 2.081263 - \frac{(2.081263 - 2.058823)}{[-0.147204 - (-0.390799)]} \times (-0.147204) = 2.094824$$

$$f(x_4) = 0.003042$$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 2.094824 - \frac{(2.094824 - 2.081263)}{[0.003042 - (-0.147204)]} \times (0.003042) = 2.094549$$

Hence the root is 2.094 correct to three decimal places.

Ex. 4 : *Find the root of the equation $xe^x = \cos x$ using secant method correct to four decimal places.*

Sol. : To obtain initial guess,

Let $$f(x) = \cos x - x\,e^x = 0.$$

$$f(0) = 1,\ f(1) = -2.17798$$

Taking the initial approximations as $x_0 = 0$, $x_1 = 1$

By secant method,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \cdot f_1$$

$$x_2 = 1 - \frac{(1-0)}{[-2.17798 - 1]} \times (-2.17798) = 1 - 0.68533 = 0.31467$$

$$f(x_2) = 0.51986 \qquad \text{(Root lies between 0.31467 and 1)}$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 0.31467 - \frac{(0.31467 - 1)}{[0.51986 - (-2.17798)]} \times (0.51986)$$

$$= 0.31467 + 0.13206 = 0.44673$$

$$f(x_3) = 0.20354$$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 0.44673 - \frac{(0.44673 - 0.31467)}{(0.20354 - 0.51986)} \times (0.20354) = 0.44673 + 0.08498 = 0.53171$$

$$f(x_4) = -0.04294$$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 0.53171 - \frac{(0.53171 - 0.44673)}{(-0.04294 - 0.20354)} \times (-0.04294)$$

$$= 0.53171 - 0.01481 = 0.51690$$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{(f_5 - f_4)} \times f_5$$

$$= 0.51690 - \frac{(0.51690 - 0.53171)}{[0.00260 - (-0.04294)]} \times (0.00260)$$

$$= 0.51690 + 0.00085 = 0.517746$$

$$f(x_6) = 0.000034$$

$$x_7 = x_6 - \frac{(x_6 - x_5)}{(f_6 - f_5)} \times f_6$$

$$= 0.517746 - \frac{(0.517746 - 0.51690)}{(0.000034 - 0.00260)} \times (0.000034)$$

$$= 0.517746 + 0.000012 = 0.517757$$

Hence, the root is 0.5177 correct upto four decimal places.

7.5 REGULA-FALSI OR METHOD OF FALSE POSITION

This method is based on the same principle as that of method of secant described in previous section. Only change is while choosing the initial approximations x_0, x_1 it is necessary that $f(x_0)\, f(x_1) < 0$. Similarly for obtaining further approximations, it must be ensured that $f_i \cdot f_{i-1} < 0$ for i = 1, 2, 3, … (see Fig. 7.4). Formula (8) of 7.4 used for secant method is also used for Regula-Falsi method.

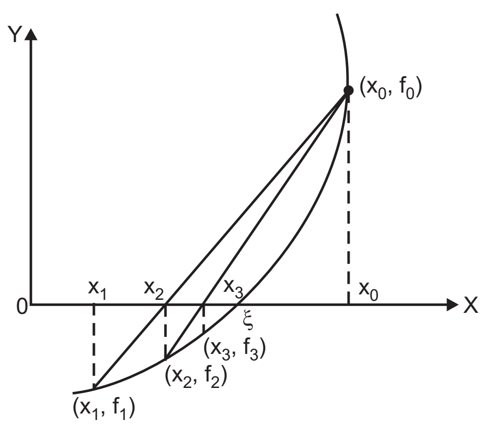


Fig. 7.4

ILLUSTRATIONS

Ex. 1 : Use Regula–Falsi method to find a real root of the equation $e^x - 4x = 0$, correct to three decimal places.

Sol. : Here $f(x) = e^x - 4x$,

Since $f(0) = 1$ $f(1) = e^1 - 4 \times 1 = -1.28$

$\therefore$ Root lies between x = 0 and x = 1.

Let $x_0 = 0$, $x_1 = 1$, $f_0 = 1$ and $f_1 = -1.28$

From (8) of art. 7.4,
$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$
$$= 1 - \frac{(1-0)}{(-1.28-1)} \times (-1.28)$$
$$= 1 - \frac{1}{2.28} \times (1.28) = 0.439$$
$$f_2 = e^{0.439} - 4 \times 0.439 = -0.2048$$

Now we take the interval (0, 0.439) as $f_0 = 1$ and $f_2 = -0.2048$, $f_0 \times f_2$ is – ve.

Let $x_1 = 0$, so that $f_1 = 1$

and $x_2 = 0.439$, $f_2 = -0.2048$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$
$$= 0.439 - \frac{(0.439-0)}{(-0.2048-1)} \times (-0.2048)$$
$$= 0.439 - \frac{0.439}{1.2048} \times (0.2048)$$
$$= 0.3643$$
$$f(0.3643) = e^{0.3643} - 4 \times 0.3643 = -0.0177$$

Now the root lies in the interval (0, 0.3643).

Let $x_2 = 0$, so that $f_2 = 1$

and $x_3 = 0.3643$, $f_3 = -0.0177$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times (-0.0177)$$
$$= 0.3643 - \frac{(0.3643-0)}{(-0.0177-1)} \times (-0.0177)$$
$$= 0.3643 - \frac{0.3643}{1.0177} \times (0.0177) = 0.35796$$
$$f_4 = e^{0.35796} - 4 \times 0.35796 = -0.00143$$

Root lies in the interval (0, 0.35796).

Let $x_3 = 0$, so that $f_3 = 1$

and $x_4 = 0.35796$, $f_4 = -0.00143$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$
$$= 0.35796 - \frac{(0.35796-0)}{(-0.00143-1)} \times (-0.00143)$$
$$= 0.35796 - \frac{0.35796}{1.00143} \times (0.00143) = 0.35745$$
$$f_5 = e^{0.35745} - 4 \times 0.35745 = -0.000121$$

Root lies in the interval (0, 0.35745)

Let $x_4 = 0,$ so that $f_4 = 1$

and $x_5 = 0.35745,$ $f_5 = -0.000121$

$$\therefore \quad x_6 = x_5 - \frac{(x_5 - x_4)}{(f_5 - f_4)} \times f_5$$

$$= 0.35745 - \frac{(0.35745 - 0)}{(-0.000121 - 1)} \times (-0.000121)$$

$$= 0.35745 - \frac{0.35745}{1.000121} \times 0.000121$$

$$= 0.35741$$

Here $f_6 = e^{0.35741} - 4 \times 0.35741 = -0.0000181$

Thus, $|f_6| < 0.00002$ which exhibits degree of accuracy.

Root at end of fifth iteration is 0.35741 which is correct to four decimal places.

Unlike secant method there is guaranteed convergence in method of false-position.

Ex. 2 : Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position at the end of sixth teration.

Sol. : Let $f(x) = x^3 - 2x - 5 = 0$

Since $f(0) = -5, f(a) = -6, f(2) = -1, f(3) = 16$

$\therefore$ A root lies between 2 and 3,

Taking $x_0 = 2$, $f_0 = -1$, and $x_1 = 3$, $f_1 = 16$ in method of false position, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$= 3 - \frac{(3 - 2)}{[16 - (-1)]} \times 16$$

$$= 3 - \frac{1}{17} \times 16 = 2.0588$$

$$f(x_2) = -0.3908$$

Now we take the interval (2.0588, 3) as $f_2 \times f_1 < 0$

Here $x_2 = 2.0588,$ $f_2 = -0.3908$

and let $x_1 = 3,$ $f_1 = 16$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 2.0588 - \frac{(2.0588 - 3)}{(-0.3908 - 16)} \times (-0.3908)$$

$$= 2.0588 + 0.0224 = 2.0812$$

$$f(x_3) = -0.1479$$

Now, we take the interval (2.0812, 3) as $f_3 \times f_1 < 0$

Here $x_3 = 2.0812,$ $f_3 = -0.1479$

and let $x_2 = 3,$ so that $f_2 = 16$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 2.0812 - \frac{(2.0812 - 3)}{(-0.1479 - 16)} \times (-0.1479)$$

$$= 2.0812 + 0.00842 = 2.0896$$

$$f(x_4) = -0.0511$$

Next, we take the interval (2.0896, 3) as $f_4 \times f_1 < 0$

Here $x_4 = 2.0896, \quad f_4 = -0.0577$

and let $x_3 = 3, \quad f_3 = 16$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 2.0896 - \frac{(2.0896 - 3)}{(-0.0511 - 16)} \times (-0.0511)$$

$$= 2.0896 + 0.0029 = 2.0925$$

$$f(x_5) = -0.0229$$

Next, we take the interval (2.0925, 3) as $f_5 \times f_1 < 0$

Here $x_5 = 2.0925, \quad f_5 = -0.0229$

and let $x_4 = 3, \quad f_4 = 16$

$$\therefore \quad x_6 = x_5 - \frac{(x_5 - x_4)}{(f_5 - f_4)} \times f_5$$

$$= 2.0925 - \frac{(2.0925 - 3)}{(-0.0229 - 16)} \times (-0.0229)$$

$$= 2.0925 + 0.0013 = 2.0938$$

Hence the root at the end of sixth iteration is 2.0938.

Ex. 3 : Solve the equation $f(x) = x - e^{-x} = 0$ by Regula-Falsi method with the initial approximations 0.5 and 1 correct upto three places of decimal.

Sol. : Let $f(x) = x - e^{-x}$

Since $f(0) = -1, f(0.5) = -0.1065, f(1) = 0.6321,$

$\therefore$ A root lies between 0.5 and 1.

Taking $x_0 = 0.5$, $f_0 = -0.1065$, and $x_1 = 1$, $f_1 = 0.6321$ in Regula-falsi method, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$= 1 - \frac{(1 - 0.5)}{[0.6321 - (-0.1065)]} \times (0.6321) = 1 - 0.4279 = 0.5721$$

$$f(x_2) = 0.007761$$

Now, we take the interval (0.5, 0.5721) as $f_0 \times f_2 < 0$

Here $x_2 = 0.5721, \quad f_2 = 0.00761$

and let $x_1 = 0.5, \quad f_1 = -0.1065$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 0.5721 - \frac{(0.5721 - 0.5)}{[0.00761 - (-0.1065)]} \times (0.00761)$$

$$= 0.5721 - 0.00481 = 0.5672$$

$$f(x_3) = 0.000088$$

Now, the root lies in the interval (0.5, 0.5672) as $f_0 \times f_3 < 0$

Here $x_3 = 0.5672, \quad f_3 = 0.000088$

and let $x_2 = 0.5$, so that $f_2 = -0.1065$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 0.5672 - \frac{(0.5672 - 0.5)}{[0.000088 - (-0.1065)]} \times (-0.000088)$$

$$= 0.5672 - 0.0000055 = 0.5671$$

Hence, the root of the equation correct upto three places of decimal is 0.567.

Ex. 4 : Use Regula-falsi method to determine the root of the equation $\cos x - x\, e^x = 0$ taking initial approximately as $x_0 = 0$ and $x_1 = 1$, at the end of sixth iterations.

Sol. : Let $f(x) = \cos x - x\, e^x$

Taking $x_0 = 0,\ x_1 = 1$ and $f(0) = 1,\ f(1) = \cos 1 - e = -2.17798$,

$\therefore$ a root lies between 0 and 1. By Regula-falsi method (from (6) of 7.3), we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$= 1 - \frac{(1 - 0)}{(-2.17798 - 1)} \times (-2.17798)$$

$$= 1 - 0.68533 = 0.31467$$

$$f(x_2) = 0.51986$$

Now, we take the interval (0.31467, 1) as $f_1 \times f_2 < 0$

Here $x_2 = 0.31467,\quad f_2 = 0.51986$

and let $x_1 = 1,\quad f_1 = -2.17798$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 0.31467 - \frac{(0.31467 - 1)}{[0.51986 - (-2.17798)]} \times (0.51986)$$

$$= 0.3467 + 0.13206 = 0.44673$$

$$f(x_3) = 0.20354$$

Now, the root lies in the interval (0.44673, 1) as $f_1 \times f_3 < 0$

Here $x_3 = 0.44673,\quad f_3 = 0.20354$

and let $x_2 = 1,\quad f_2 = -2.17798$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 0.44673 - \frac{(0.44673 - 1)}{(0.20354 - (-2.17798))} \times (0.20354)$$

$$= 0.44673 + 0.047293 = 0.49402$$

$$f(x_4) = 0.07078$$

Next, the root lies in the interval (0.49402, 1) as $f_1 \times f_4 < 0$

Here $x_4 = 0.49402,\quad f_4 = 0.07078$

and let $x_3 = 1,\quad f_3 = -2.17798$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 0.49402 - \frac{(0.49402 - 1)}{[0.07078 - (-2.17798)]} \times (0.07078)$$

$$= 0.49402 + 0.05193 = 0.50995$$

$$f(x_5) = 0.02360$$

Next, the root lies in the interval (0.50995, 1) as $f_1 \times f_5 < 0$

Here $x_5 = 0.50995, \quad f_5 = 0.02360$

and let $x_4 = 1, \quad f_4 = -2.17798$

$$\therefore \quad x_6 = x_5 - \frac{(x_5 - x_4)}{(f_5 - f_4)} \times f_5$$

$$= 0.50995 - \frac{(0.50995 - 1)}{[0.02360 - (-2.17798)]} \times (0.02360)$$

$$= 0.50995 + 0.00525 = 0.51520$$

Hence, the root at the end of sixth iteration is 0.51520. We note here that on repeating above processes, the successive approximations are $x_7 = 0.51692$, $x_8 = 0.51748$, $x_9 = 0.51767$, $x_{10} = 0.51775$.

Ex. 5 : Find a real root of the equation $x\log_{10} x = 1.2$ by Regula-falsi method correct to four decimal places.

Sol. : Let $f(x) = x \log_{10} x - 1.2$

Since $f(1) = -1.2, f(2) = -0.59794, f(3) = 0.23136,$

$\therefore$ A root lies between 2 and 3. Taking $x_0 = 2$, $f_0 = -0.59794$ and $x_1 = 3$, $f_1 = 0.23136$ in Regula-falsi method, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$= 3 - \frac{(3 - 2)}{[0.23136 - (-0.59794)]} \times (0.23136)$$

$$= 3 - 0.27898 = 2.72102$$

$$f(x_2) = -0.01709$$

Now, we take interval (2.72102, 3) as $f_2 \times f_1 < 0$

Here $x_2 = 2.72102, \quad f_2 = -0.01709$

and let $x_1 = 3, \quad f_1 = 0.23136$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 2.72102 - \frac{(2.72102 - 3)}{(-0.01709 - 0.23136)} \times (-0.01709)$$

$$= 2.72102 + 0.019190 = 2.74021$$

$$f(x_3) = -0.00038$$

Now, we take interval (2.74021, 3) as $f_3 \times f_1 < 0$

Here $x_3 = 2.74021, \quad f_3 = -0.00038$

and let $x_2 = 3, \quad f_2 = 0.23136$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 2.74021 - \frac{(2.74021 - 3)}{(-0.00038 - 0.23136)} \times (-0.00038)$$

$$= 2.74021 + 0.00043 = 2.74064$$

$$f(x_4) = -0.000005$$

Next, we take interval (2.74064, 3) as $f_4 \times f_1 < 0$

Here $x_4 = 2.74064, \quad f_4 = -0.000005$

and let $x_3 = 3, \quad f_3 = 0.23136$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 2.74064 - \frac{(2.74064 - 3)}{(-0.000005 - 0.23136)} \times (-0.000005)$$

$$= 2.74064 + 0.000005 = 2.74065$$

Hence, the required root is 2.7406 correct to four decimal places.

Ex. 6 : Use method of false position, to find the fourth root of 32 correct to three decimal places.

Sol. : Let $x = (32)^{1/4}$ or $f(x) = x^4 - 32 = 0$

Since $f(2) = -16,\ f(3) = 9,$

$\therefore$ A root lies between 2 and 3. Taking $x_0 = 2$, $f_0 = -16$ and $x_1 = 3$, $f_1 = 49$ in the method of false position,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$= 3 - \frac{(3 - 2)}{[49 - (-16)]} \times (49) = 3 - \frac{1}{65} \times 49 = 2.2462$$

$$f(x_2) = -6.5438$$

Now, we take the interval (2.2462, 3) as $f_2 \times f_1 < 0$

Here $x_2 = 2.2462, \quad f_2 = -6.5438$

and let $x_1 = 3, \quad f_1 = 49$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 2.2462 - \frac{(2.2462 - 3)}{(-6.5438 - 49)} \times (-6.5438)$$

$$= 2.2462 + 0.0888 = 2.335$$

$$f(x_3) = -2.2732$$

Now, we take the interval (2.335, 3) as $f_3 \times f_1 < 0$

Here $x_3 = 2.335, \quad f_3 = -2.2732$

and let $x_2 = 3, \quad f_2 = 49$

$$\therefore \quad x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$= 2.335 - \frac{(2.335 - 3)}{(-2.2732 - 49)} \times (-2.2732)$$

$$= 2.335 + 0.0295 = 2.3645$$

$$f(x_4) = -0.7423$$

Next, we take the interval (2.3645, 3) as $f_4 \times f_1 < 0$

Here $x_4 = 2.3645, \quad f_4 = -0.7423$

and let $x_3 = 3, \quad f_3 = 49$

$$\therefore \quad x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 2.3645 - \frac{(2.3645 - 3)}{(-0.7423 - 49)} \times (-0.7423)$$

$$= 2.3645 + 0.0095 = 2.374$$

$$f(x_5) = -0.2369$$

Next, we take the interval (2.374, 3) as $f_5 \times f_1 < 0$

Here $x_5 = 2.374, \quad f_5 = -0.2569$

and let $x_4 = 3, \quad f_4 = 49$

$$\therefore \quad x_6 = x_5 - \frac{(x_5 - x_4)}{(f_5 - f_4)} \times f_5$$

$$= 2.374 - \frac{(2.374 - 3)}{(-0.2369 - 49)} \times (-0.2369)$$

$$= 2.374 + 0.0030 = 2.3770$$

$$f(x_6) = -0.0760$$

Next, we take the interval (2.3770, 3) as $f_6 \times f_1 < 0$

Here $x_6 = 2.3770, \quad f_6 = -0.0760$

and let $x_5 = 3, \quad f_5 = 49$

$$\therefore \quad x_7 = x_6 - \frac{(x_6 - x_5)}{(f_6 - f_5)} \times f_6$$

$$= 2.3770 - \frac{(2.3770 - 3)}{(-0.0760 - 49)} \times (-0.0760)$$

$$= 2.3770 + 0.0009 = 2.3779$$

Since $x_6 = x_7$ current upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

7.6 NEWTON-RAPHSON METHOD

This is also a method based on first degree equaiton.

To solve the equation f(x) = 0,

let $$f(x) = m_1 x + m_2 \qquad \ldots(1)$$

Differentiating (1) w.r.t. x

$$f'(x) = m_1 \qquad \ldots(2)$$

If $x = x_0$ is initial approximation to the root from (1) and (2)

$$f(x_0) = m_1 x_0 + m_2 \qquad \ldots(3)$$

$$f'(x_0) = m_1 \qquad \ldots(4)$$

putting for m_1 in (3),

$$f(x_0) = x_0 f'(x_0) + m_2$$

or $$m_2 = f(x_0) - x_0 f'(x_0) \qquad \ldots(5)$$

For the equation, f(x) = 0, (1) gives

$$x = -\frac{m_2}{m_1} \qquad \ldots(6)$$

Substituting for m_1 and m_2 in (6), we get

$$x = -\frac{f(x_0) - x_0 f'(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

Representing approximate value of x by x_1,

$$\boxed{x_1 = x_0 - \frac{(x_0)}{f'(x_0)}} \qquad \ldots(7)$$

Thus, from initial approximation x_0, we get first approximation after one iteration by (7). Successive approximations are given by

$$x_2 = x_1 - \frac{(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{(x_2)}{f'(x_2)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \qquad \ldots (8)$$

which is a generalised form of Newton-Raphson formula for getting successive approximations, starting from initial approximation. Thus, Newton-Raphson method is generally used to improve the results obtained by other methods.

To give geometrical interpretation, consider the curve $y = f(x)$ as shown in the Fig. 7.5.

A (x_i, f_i), B(x_{i-1}, f_{i-1}) are two points on the curve. Exact root of the equation is $x = \xi$ which corresponds to the point of intersection of the curve with x-axis. Equation of chord AB is given by

$$y - f_{i-1} = \frac{f_i - f_{i-1}}{x_i - x_{i-1}} (x - x_{i-1}) \qquad \ldots (9)$$

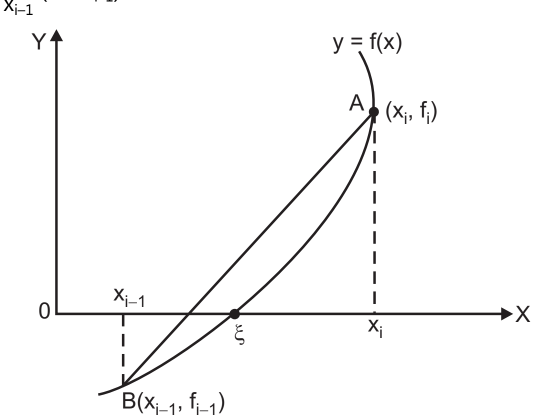


Fig. 7.5

In the limit when B is indefinitely closer to A, chord AB becomes tangent to the curve at A. Slope of the tangent at x_i is given by $f'(x_i)$ where,

$$\left.\frac{dy}{dx}\right|_{x = x_i} = f'(x_i)$$

Replacing equation of chord AB by tangent at A, we have

$$y - f_i = f'(x_i)(x - x_i) \qquad \ldots (10)$$

Taking intersection of this tangent with x-axis

i.e. putting $y = 0$ in (10), we get

$$0 - f_i = f'(x_i)(x - x_i)$$

or
$$x - x_i = \frac{-f_i}{f'(x_i)}$$

or
$$x = x_i - \frac{f(x_i)}{f'(x_i)}$$

Denoting point of intersection x by x_{i+1}, we can write

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \qquad (11)$$

which is same as (8). Thus geometrically, Newton's method is equivalent to replacing the curve y = f(x) by a tangent line drawn to a point on the curve.

For application of Newton's formula (8) or (11), choice of initial approximation is also important.

As we have discussed above, Newton's method replaces the curve y = f(x) by a tangent at point on the curve. A [a, f(a)),, B [b, f(b)] are two points on the curve y = f(x) as shown in Fig. 7.6.

Choosing $b = x_0$ as first approximation, tangent to the curve y = f(x) at $B(x_0, f_0)$ meets x-axis at $x = x_1$ which is the first approximation of the root. Through the point $B_1[x_1\ f(x_1)]$ we draw a tangent which meets x-axis at $x = x_2$ which is the second approximation. Thus we get closer and closer to the exact value $x = \xi$.

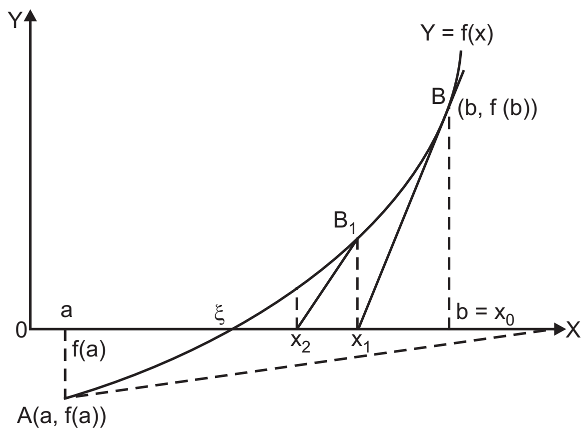


Fig. 7.6

On the contrary if we choose $a = x_0$ as first a pproximation, tangent to the curve at $A(x_0, f_0)$ meets the x-axis at a point x', which is outside the interval (a, b). In other words, Newton's method is impractical for such a choice of initial approximation. If we assume f "(x) > 0 for $a \le x \le b$ and f(b) > 0 (Fig. 7.6) then $f(x_0)\ f''(x_0) > 0$ for $x_0 = b$. In this case, tangent to the curve at B(b, f(b)), meets x-axis at a point which is within the interval (a, b). If we choose initial approximation $x_0 = a$ [f(a) < 0] then $f(x_0)\ f''(x_0) < 0$ and in this case tangent to the curve at A(a, f(a)), meets x-axis at a point outside the interval (a, b).

This observation provides general rule for choice of initial approximation. When application of intermediate value theorem shows that root is between (a, b), rule says that for the initial point x_0, choose the end of the interval (a, b) associated with an ordinate of the same sign as the sign of f "(x).

It should also be noted that, Newton's formula (11) is applicable only if $f'(x_i) \ne 0$. This should be checked not only at initially guessed value of x_0, but right through all the iterative cycles its f '(x) might become too small at any of the intermediate points. In developing the computer algorithm a safety exit out of the iterative loop should be provided after completing a pre-assigned number of iterations to prevent the program from getting caught up into an endless cycle.

ILLUSTRATIONS

Ex. 1 : *Find the real root of the equation $x^3 + 2x - 5 = 0$ by applying Newton-Raphson method at the end of fifth iteration.*

Sol. : Let $f(x) = x^3 + 2x - 5$, $f'(x) = 3x^2 + 2$, $f''(x) = 6x$

Since $f(0) = -5$, $f(1) = -2$, $f(2) = 7$

$\therefore$ A root lies between 1 and 2. Since f"(x) = 6x which is positive for interval (1, 2), hence we choose $x_0 = 2$ as $f(x_0)\ f''(x_0) > 0$ (usually, $x_0 = \frac{1+2}{2} = 1.5$ is safe choice). By Newton-Raphson formula (11),

Putting i = 0 in (11),

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{7}{14} = 1.5 \text{ and } f(x_1) = 1.375,\ f'(x_1) = 8.75$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{1.375}{8.75} = 1.343 \text{ and } f(x_2) = 0.1083,\ f'(x_2) = 7.411$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.343 - \frac{0.1083}{7.411} = 1.329 \text{ and } f(x_3) = 0.1083,\ f'(x_3) = 7.2987$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.329 - \frac{0.0053}{7.2987} = 1.3283 \text{ and } f(x_4) = 0.00227,\ f'(x_4) = 7.29314$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.3283 - \frac{0.000227}{7.29314} = 1.3283$$

Hence, the required root is 1.3283.

Note : In calculation of x_4 and x_5 rounding is done at fourth decimal place.

Ex. 2 : *Use Newton's method to find the smallest positive root of the equation tan x = x correct to three decimal places.*

Sol. : Plotting the graphs of the curve y = tan x and y = x (Fig. 7.7) it is clear that the desired root ξ lies in the interval $\pi < \xi < \frac{3\pi}{2}$.

Writing the equation tan x = x as sin x – x cos x = 0, we have

$$f(x) = \sin x - x\cos x, \quad f'(x) = x\sin x, \quad f''(x) = x\cos x + \sin x$$

Since $$f''(\pi) = -\pi, \quad f''\left(\frac{3\pi}{2}\right) = -1$$

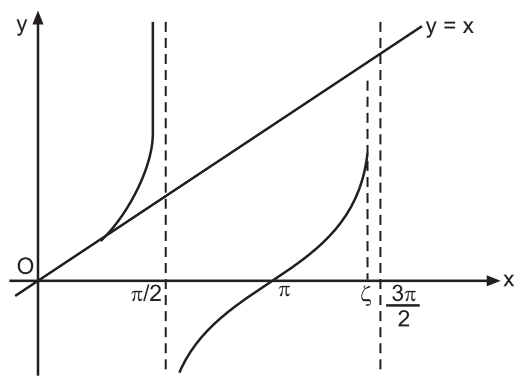


Fig. 7.7

Thus, f "(x) is –ve throughout the interval $\left(\pi, \frac{3\pi}{2}\right)$. Also, $f\left(\frac{3\pi}{2}\right) = -1$

Thus, we can take initial approximation as $x_0 = \frac{3\pi}{2}$ and $f'\left(\frac{3\pi}{2}\right) = -\frac{3\pi}{2} = -4.7124$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 4.7124 - \frac{(-1.000052)}{(-4.7125)} = 4.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 4.5 - \frac{(-0.023)}{(-4.399)} = 4.4948$$

$$x_3 = 4.4948 - \frac{(-0.0061)}{(-4.3888)} = 4.49341$$

$$x_4 = 4.49341 - \frac{(-0.0000024)}{(-4.3861)} = 4.493409$$

Rounding to fifth decimal place x_4 = 4.49341 (Radians)

which is the root correct to fourth decimal place.

or x_4 = 257.45342 (degree).

Ex. 3 : *Find the root of the equation $x - e^{-x} = 0$ that lies between 0.5 and 1 by Newton-Raphson theorem correct upto four places of decimal.*

Sol. : Here $f(x) = x - e^{-x}$, $f'(x) = 1 + e^{-x}$

Since $f(0.5) = -0.10653$, $f(1) = 0.63212$

$\therefore$ A root lies between 0.5 and 1 and it is closer to 0.5.

Since $f''(x) = -e^{-x}$ which is negative for interval (0.5, 1), hence we choose $x_0 = 0.5$ as $f(x_0)\, f''(x_0) > 0$.

By Newton-Raphson formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{(0.5 - e^{-0.5})}{(1 + e^{-0.5})} = 0.56631$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.56631 - \frac{(0.56631 - e^{-0.56631})}{(1 + e^{-0.56631})} = 0.56714$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.56714 - \frac{(0.56714 - e^{-0.56714})}{(1 + e^{-0.56714})} = 0.56714$$

Here $x_2 = x_3$. Hence desired root is 0.5671 correct to four decimal places.

Ex. 4 : *Find by Newton-Raphson method $3x - \cos x - 1 = 0$ correct to four decimal places.*

Sol. : Here $f(x) = 3x - \cos x - 1$, $f'(x) = 3 + \sin x$

Since $f(0) = -2$, and $f(1) = 1.4597$,

$\therefore$ a root lies between 0 and 1.

Since $f''(x) = \cos x$ which is positive for the interval (0, 1), hence we choose $x_0 = 0.6$ (nearer to 1) as $f(x_0)\, f''(x_0) > 0$.

By Newton-Raphson formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.6 - \frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6071$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)} = 0.6071$$

Here $x_2 = x_3$. Hence desired root is 0.6071 correct to four decimal places.

Ex. 5 : Obtain the root of the equation $x^3 - 4x - 9 = 0$ correct to four decimal places by using Newton-Raphson method.

Solution : Let $f(x) = x^3 - 4x - 9$

Since $f(0) = -9$, $f(1) = -12$, $f(2) = -9$, $f(3) = 6$,

$\therefore$ a root lies between 2 and 3.

To choose initial value, consider,

$$f(x) = 3x^2 - 4, \quad f'(x) = 6x, \quad f''(3) = 18$$

Since $f(3) = 6$ and $f''(3) = 18$, are the same sign, we choose initial approximation as $x_0 = 3$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f''(3)} = 3 - \frac{6}{23} = 2.739$$

$$f(x_1) = f(2.739) = (2.739)^3 - 4(2.739) - 9 = 0.59231$$

$$f'(x_1) = f'(2.739) = 3(2.739)^2 - 4 = 18.5064$$

$$x_2 = x_1 - \frac{(x_1)}{f'(x_1)} = 2.739 - \frac{0.59231}{18.5064} = 2.707$$

$$f(x_2) = f(2.707) = (2.707)^3 - 4(2.707) - 9 = 0.008487$$

$$f'(x_2) = f'(2.707) = 3(2.707)^2 - 4 = 17.983547$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.707 - \frac{0.008487}{17.983547} = 2.706528$$

$$f(x_3) = (2.7006528)^3 - 4(2.706528) - 9 = 8.179 \times 10^{-7}$$

Next iteration is not going to affect the fourth decimal place. No further iterations are required.

Root correct to fourth decimal place = 2.7065.

Ex. 6 : Using Newton-Raphson method, find a root of the following equation $x^3 - 3x^2 - 5.5x + 9.5 = 0$ correct up to fifth decimal place.

Sol. : The initial guess may be assumed as zero.

Let $$f(x) = x^3 - 3x^2 - 5.5x + 9.5$$

$\therefore$ $$f'(x) = 3x^2 - 6x - 5.5$$

Taking $x_0 = 0$,

$$x_1 = x_0 + \frac{f(x)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = \frac{9.5}{5.5} = 1.72727$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.72727 - 0.54925 = 1.17802$$

On repeating above process, the successive approximations are $x_3 = 1.2366$, $x_4 = 1.23686$, $x_5 = 1.2368574$.

Thus, root correct to fifth decimal place is 1.23686.

Ex. 7 : Show that the initial approximation x_0 for finding $\frac{1}{n}$, where n is a positive integer must satisfy $0 < x_0 < \frac{2}{n}$, for convergence, by the Newton's method.

Sol. : Taking $$f(x) = \frac{1}{x} - n$$

$$f'(x) = -\frac{1}{x^2}$$

Using Newton's formula, $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)},$$ we get

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - n\right)}{\left(-\frac{1}{x_i^2}\right)}$$

$$x_{i+1} = x_i + x_i^2\left(\frac{1}{x_i} - n\right)$$

$$= x_i + x_i - nx_i^2$$

$$x_{i+1} = 2x_i - nx_i^2$$

Let us now draw the graphs of curves

$$y = x \quad \text{and} \quad y = 2x - nx^2$$

For plotting the graph of $y = 2x - nx^2$, we write the equation as

$$\frac{y}{n} = \frac{2}{n}x - x^2$$

or $$x^2 - \frac{2}{n}x + \frac{1}{n^2} = \frac{1}{n^2} - \frac{y}{n}$$

i.e. $$\left(x - \frac{1}{n}\right)^2 = -\frac{1}{n}\left(y - \frac{1}{n}\right)$$

which is a parabola with vertex at $\left(\frac{1}{n}, \frac{1}{n}\right)$. Points of intersection of $y = x$ and $y = 2x - nx^2$ are (0, 0) and $\left(\frac{1}{n}, \frac{1}{n}\right)$.

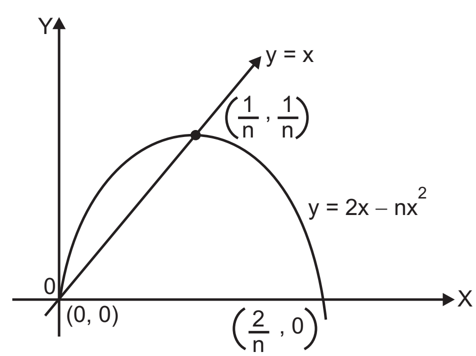


Fig. 7.8

Graphs are shown in Fig. 7.8.

The point of intersection is the required value $\frac{1}{n}$. From the graph, we find that any initial approximation outside the range $\left(0, \frac{2}{n}\right)$ cannot converge. If initial approximation x_0 is taken as '0' then it gives trivial root x = 0.

7.7 MODIFIED NEWTON-RAPHSON METHOD

If the derivative f '(x) varies but slightly on the interval [a, b] then in the formula (11) of article 7.6 we can put $f'(x_i) \approx f'(x_0)$.

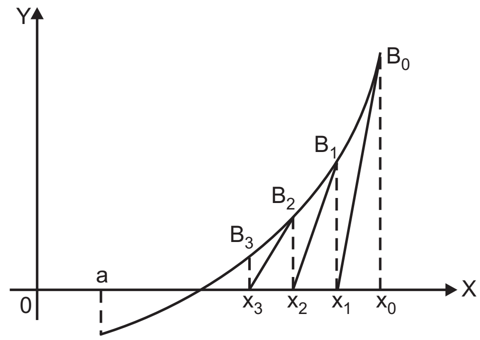


Fig. 7.9

From this for the root ξ of the equation f(x) = 0, we get the formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \approx x_i - \frac{f(x_i)}{f'(x_0)} \qquad \text{... (1)}$$

Geometrically, this method signifies that we replace the tangents at the points B_i $[x_i, f(x_i)]$ by straight lines parallel to the tangent to the curve y = f(x) at its fixed point B_0 $[x_0, f(x_0)]$ [Fig. 7.9].

In Newton's formula $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ we are required to compute $f'(x_i)$ for i = 0, 1, 2, … etc. In formula (1) $f'(x_i)$ is required to be computed only once for i = 0 [i.e. $f'(x_0)$].

Thus formula (1) saves lot of labour and is very useful when $f'(x_i)$ is complicated function.

7.8 METHOD OF SUCCESSIVE APPROXIMATION (ITERATION METHOD)

This method is also called method of simple iteration. To solve the equation f(x) = 0, it is arranged in the form x = φ(x). For example, the equation $x^2 - 3x + 1 = 0$ can be rearranged as

$$x = \frac{1}{3}(x^2 + 1) \text{ or } x = 3 - \frac{1}{x}$$

The iterative technique is to guess an initial approximation x_0 and to compute next approximation from the formula $x_1 = \phi(x_0)$. Similarly, $x_2 = \phi(x_1)$, $x_3 = \phi(x_2)$ … etc. give successive approximations. Computational scheme can be expressed as

$$x_{i+1} = \phi(x_i)$$

Procedure is terminated when $|x_{i+1} - x_i| < \varepsilon$, where ε is prescribed tolerance error.

ILLUSTRATIONS

Ex. 1 : Solve the equation $x^2 - 4x + 2 = 0$ using the method of simple iteration.

Sol. : Analytically, the roots of the equation are given by

$$x = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

Smaller root is $2 - \sqrt{2} = 0.586$

Larger root is $2 + \sqrt{2} = 3.414$

Rearranging the equation as

$$x = \frac{1}{4}(x^2 + 2)$$

Computational scheme is

$$x_{i+1} = \frac{1}{4}(x_i^2 + 2)$$

putting i = 0 and taking $x_0 = 1$, we get

$$x_1 = \frac{1}{4}(x_0^2 + 2) = \frac{1}{4}(1 + 2) = 0.75$$

$$x_2 = \frac{1}{4}(x_1^2 + 2) = \frac{1}{4}\{(0.75)^2 + 2)\} = 0.641$$

$$x_3 = \frac{1}{4}(x_2^2 + 2) = \frac{1}{4}\{(0.641)^2 + 2\} = 0.6027$$

$$x_4 = \frac{1}{4}(x_3^2 + 2) = \frac{1}{4}\{(0.6027)^2 + 2\} = 0.5908$$

$$x_5 = \frac{1}{4}\{(0.5908)^2 + 2\} = 0.5873$$

After fifth iteration we are sufficiently closer to the smaller root.

If we take $x_0 = 2$

$$x_1 = \frac{1}{4}(4 + 2) = 1.5, \qquad x_2 = 1.0625$$

$$x_3 = 0.7822, \qquad x_4 = 0.6529$$

Thus, initial approximation $x_0 = 2$ also takes us to smaller root. It does not lead to larger root which is now required.

To obtain larger root, we take the alternate arrangement computational scheme is

$$x = 4 - \frac{2}{x}$$

$$x_{i+1} = 4 - \frac{2}{x_i}$$

putting i = 0 and taking $x_0 = 1$, we get

$$x_1 = 4 - \frac{2}{1} = 2.0, \qquad x_2 = 4 - \frac{2}{2} = 3$$

$$x_3 = 4 - \frac{2}{3} = 3.33, \qquad x_4 = 4 - \frac{2}{3.33} = 3.34$$

Thus we get closer to the larger root. Convergence shows down after certain number of iterations.

It may be noted that all the arrangements of the equation in the form $x = \phi(x)$ may not give convergent solution. Similarly, all the initial guesses may not lead to the desired root. Since the equation f(x) = 0 can be arranged in the form $x = \phi(x)$ in various ways, it is preferable to test whether a particular computational scheme with initial guess will lead to convergent solution.

7.9 TEST FOR CONVERGENCE

We now look for the conditions of convergence of the computational scheme $x_{i+1} = \phi(x_i)$. The function $\phi(x)$ is called sequence generating function as it generate the sequence x_1, x_2 … etc. If this function varies rapidly with x there will be large difference in the values of $x_1, x_2, \dots x_n$ which is not desirable.

For convergence it is convenient to identify an interval that contain the root and for which $\phi'(x)$ has small magnitude. If this is not possible the sequence generating function is unsuitable and an alternative function should be found. By using an intermediate value theorem, we locate the interval [a, b] which contains the root ξ of f(x) = 0. To investigate convergence we consider the behaviour of the quantity $|x_i - \xi|$ as i increases and this will depend on the behaviour of the sequence generating function. Therefore, we use the equations $x_i = \phi(x_{i-1})$ and $\xi = \phi(\xi)$ to obtain

$$|x_i - \xi| = |\phi(x_{i-1}) - \phi(\xi)| \qquad \dots(1)$$

If $\phi(x)$ has continuous derivative on the interval I_{i-1} defined by the end points x_{i-1} and ξ then according to the mean value theorem, there exists a value of x in I_{i-1} which we denote by c_{i-1} such that

$$\phi'(c_{i-1}) = \frac{\phi(x_{i-1}) - \phi(\xi)}{x_{i-1} - \xi}$$

Substituting in (1), we get

$$|xi - \xi| = |\phi'(c_{i-1})| \, |x_{i-1} - \xi| \qquad \dots (2)$$

This relates the closeness of two successive approximations to the root ξ. If x_i is to be closer to the root than x_{i-1} then we must have

$$|\phi'(c_{i-1})| < 1$$

Similarly, we can show that

$$|x_{i-1} - \xi| = |\phi'(c_{i-2})| \, |x_{i-2} - \xi|$$

Substituting in (2), we have

$$|x_{i-1} - \xi| = |\phi'(c_{i-1})| \, \phi'(c_{i-2}) \, | \, |x_{i-2} - \xi|$$

proceeding in this way, we obtain

$$|x_i - \xi| = |\phi'(c_{i-1})| \, |\phi'(c_{i-2})| \dots |\phi'(\varepsilon_0)| \, |x_0 - \xi| \qquad \dots (3)$$

It is clear that little about the location of each c_j except that it lies in I_j (which has end points ξ and x_j), the condition (3) will only be use if

$$|\phi'(x)| < 1 \qquad \dots (4)$$

Over an interval sufficiently large to obtain each interval I_j. Consequently, not only should we identify an interval [a, b] containing ξ, but also the sequence generating function must satisfy the conditions.

(i) $|\phi'(x)| < 1$ for all x in [a, b].

(ii) $a \le \phi(x) \le b$ for all x in [a, b] … (5)

as well as having a continuous derivative on [a, b]. If the second condition is not satisfied, recalling that $x_j = \phi(x_{j-1})$, some of the x_js may lie outside [a, b]. This allows the possibility of c_j bounded by ξ and x_j, also lying outside [a, b] and the first condition may be violated when $x = c_j$.

We have seen conditions that are sufficient for the scheme $x_{i+1} = \phi(x_i)$ to converge to a root ξ of $f(x) = 0$. Let us now assume that condition (i) and (ii) hold and confirm that convergence is guaranteed. Since $|\phi'(x)| < 1$ on [a, b] we can find an upper bound k for $|\phi'(x)|$ on [a, b] which is also less than 1. For example,

$$k = \max |\phi'(x)| \quad \text{or} \quad k = (1 + \max |\phi'(x)|)/2$$

where $a \le x \le b$. Then since

$$|\phi'(x)| < x < 1$$

equation (3) leads to

$$|x_i - \xi| \le k^i \, |x_0 - \xi|$$

As $k < 1$, $k^i \to 0$ as $i \to \infty$ so that $x_1 \to \xi$ as $i \to \infty$, i.e. the scheme converges. In short, given an iterative scheme $x_{i+1} = \phi(x_i)$, an interval [a, b] must be found containing the root of $f(x) = 0$. The scheme will then converge to the root ξ, provided that

(i) The initial guess x_0 is selected from [a, b].

(ii) $\phi(x)$ has a continuous derivative on [a, b].

(iii) $|\phi'(x)| < 1$ for all x in [a, b].

(iv) $a \le \phi(x) \le b$ for all x in [a, b].

When using the convergence test to decide on an iterative scheme an interval [a, b] must be located which contains the root. This can be done by using intermediate value theorem or graphical representation of function. It must be then ensured that conditions (ii), (iii) and (iv) are satisfied. These conditions must be satisfied over the whole interval [a, b].

Geometrically, if x_0 is in the neighbourhood of the root ξ, and $0 < \phi'(x) < 1$ then convergent iteration method provides a staircase solution as shown in Fig. 7.10 (a) and it provides a spiral solution as shown Fig. 7. 10 (b) if $-1 < \phi'(x) < 0$.

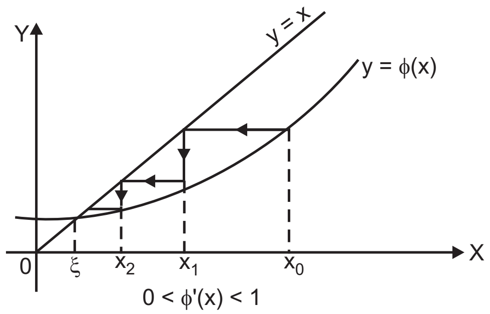


Fig. 7. 10 (a)

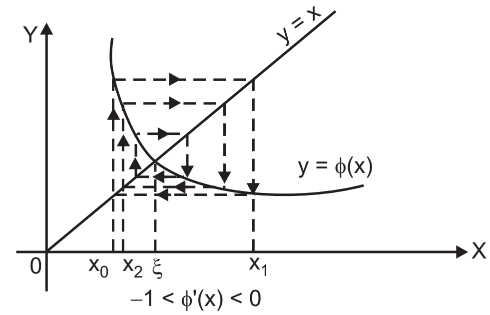


Fig. 7.10 (b)

ILLUSTRATIONS

Ex. 1 : Use a simple iterative method to solve $f(x) = x^3 + 2x + 1 = 0$

Sol. : I : Since $f(-1) = -2$, $f(0) = 1$, $\therefore$ a root lies between – 1 and 0.

II : We can write the equation as $x^3 = -2x - 1$

Roots will correspond to the points of intersection of curves $y = x^3$ and $y = -2x - 1$

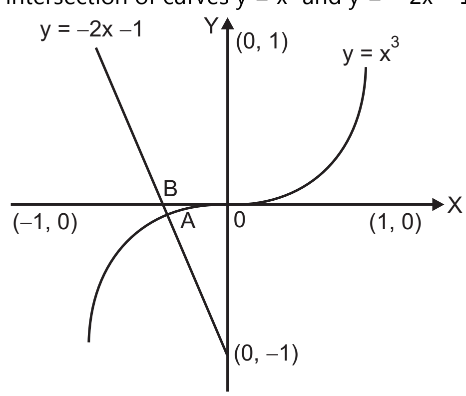


Fig. 7.11

From the Fig. 7.11 it is clear that real root lies in the interval [– 0.75, – 0.25].

One rearrangement of the equation is

$$x = -\frac{(1 + 2x)}{x^2} = \phi(x)$$

Thus $$\phi(x) = -\frac{(1 + 2x)}{x^2}, \quad \phi'(x) = \frac{2\left(1 + \frac{1}{x}\right)}{x^2}$$

Since $|\phi'(-0.4)| > 1$, convergence is not guaranteed.

We try alternative arrangement

$$x = -\frac{(1 + x^3)}{2}$$

Here $$\phi(x) = -\frac{(1 + x^3)}{2}, \quad \phi'(x) = -\frac{3x^2}{2}$$

and to have $|\phi'(x)| < 1$, we require

$$\frac{3x^2}{2} < 1 \qquad \text{or} \qquad x^2 - \frac{2}{3} < 0 \text{ i.e. } \left(x - \sqrt{\frac{2}{3}}\right)\left(x + \sqrt{\frac{2}{3}}\right) < 0$$

Thus, the two factors on left hand side should be of opposite signs.

This implies $x < \sqrt{\frac{2}{3}}$ and $x > -\sqrt{\frac{2}{3}}$

Hence $|\phi'(x)| \le 1$ when x lies in the interval $\left[-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right]$

We can restrict to the interval $\left[-\sqrt{\frac{2}{3}}, 0\right]$. Since it contains the root. $\phi(x)$ is monotonic and its maximum and minimum values occur at the ends of the interval $\left[-\sqrt{\frac{2}{3}}, 0\right]$. These extreme values are – 0.3519 and – 0.5 and it is evident that $-\sqrt{\frac{2}{3}} < \phi(x) < 0$. Thus convergence of the scheme $x_{i+1} = -\frac{(1 + x_i^3)}{2}$ is guaranteed provided that initial guess is taken from the interval $\left[-\sqrt{\frac{2}{3}}, 0\right]$. Taking $x_0 = -0.4$, we find after seven iterations, root correct to four decimal places as – 0.4534.

Ex. 2 : Obtain one root of the equation $8x^3 - 6x - 1 = 0$, correct to four decimal places considering the initial value as 0.95, using

(a) Newton-Raphson method.

(b) Successive approximation method.

Justify the iteration function used in the successive approximation method.

Solution : (a) Here initial value is given as 0.95.

Let $x_0 = 0.95$

$$f(x) = 8x^3 - 6x - 1, \qquad f'(x) = 24x^2 - 6$$

$$f(x_0) = f(0.95) = 0.159, \quad f'(x_0) = f'(0.95) = 15.66$$

Putting i = 0, 1, 2, … in (ii) of art. (7.6)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.95 - \frac{0.159}{15.66} = 0.93985$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.93985 - \frac{0.0023915}{15.199633} = 0.93969$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.93969$$

Thus, the root correct to four decimal places is x = 0.9397 (rounding to fourth decimal place).

(b) Equation f(x) = 0 can be arranged in the form

$$x = \phi(x)$$

or
$$x = \left(\frac{6x + 1}{8}\right)^{1/3}$$

Here
$$\phi(x) = \left(\frac{6x + 1}{8}\right)^{1/3} \text{ and } \phi'(x) = \frac{1}{3}\left(\frac{6x + 1}{8}\right)^{-2/3} \times \frac{6}{8} = \frac{1}{4}\left(\frac{8}{6x + 1}\right)^{2/3}$$

Root of the given equation clearly lies in the interval (0.9, 1) and for all the values of x in this interval $|\phi'(x)| < 1$. Therefore, Iteration function $\phi(x)$ should converge which justifies the use of iteration function $\phi(x)$.

Taking $x_0 = 0.95$

$$x_1 = \phi(x_0) = \left(\frac{6 \times 0.95 + 1}{8}\right)^{1/3} = 0.9426018$$

$$x_2 = \phi(x_1) = 0.9405155$$

$$x_3 = \phi(x_2) = 0.9399255$$

$$x_4 = \phi(x_3) = 0.9397585$$

$$x_5 = 0.9397112$$

$$x_6 = 0.9396979$$

$$x_7 = 0.9396941$$

which shows that root correct to four decimal places is = 0.9397

Ex. 3 : Find a root of second degree equation $x^2 - x - 2 = 0$ using successive approximation technique. Justify the iteration function selected.

Sol. : Let $f(x) = x^2 - x - 2$

Since $f(0) = -2$, $f(1) = -2$, $f(1.5) = -1.25$, $f(2.5) = 1.75$

$\therefore$ A root lies between x = 1.5 and 2.5. (In fact it is 2 as f(2) = 0)

We rearrange f(x) as

$$x = \frac{x+2}{x} = \phi(x)$$

Here $\phi(x) = \frac{x+2}{x} = 1 + \frac{2}{x}$, $\phi'(x) = -\frac{2}{x^2}$

Now $|\phi'(x)| < 1$ for $1.5 \le x \le 2.5$. Hence choice of $\phi(x) = \frac{x+2}{x}$ is justified as $\phi(x)$ has continuous derivatives over the interval (1.5, 2).

We can take any value of x in the interval (1.5, 2) as the initial approximation.

Taking $x_1 = 1.5$, we get $x_2 = \frac{1.5+2}{1.5} = 2.33$ similarly, $x_3 = 1.858$ and successive approximations are obtained as

2.076, 1.9634, 2.0188, 1.9907, 2.0047, 1.9976,

2.0012, 1.99940, 2.0003, 1.99985, 2.000075

which shows that the root converges to the value x = 2.

Another iteration scheme $x = (x+2)^{1/2}$ can also give the required result.

Ex. 4 : Solve $3x - 1 - \cos x = 0$ by using the method of successive approximations correct to three decimal places.

Sol. : Let $f(x) = 3x - 1 - \cos x = 0$

Since $f(0) = 3 \times 0 - 1 - \cos 0 = -2$, $f(1) = 3 \times 1 - 1 - \cos 1 = 1.459696$

$\therefore$ A root lies between 0 and 1 (Radians).

Expressing (rewriting) the equation in the form $x = \phi(x)$,

Consider, $$x = \frac{1 + \cos x}{3} = \phi(x)$$

we have $\phi'(x) = -\frac{1}{3}\sin x$ and $|f'(x)| = \left|-\frac{\sin x}{3}\right| < 1$ in the interval (0, 1).

Method of successive approximation will give convergent result. Starting with $x_0 = 0$.

$$x_1 = \frac{1 + \cos 0}{3} = \frac{2}{3} = 0.6667$$

$$x_2 = \frac{1 + \cos (0.6667)}{3} = 0.5953$$

$$x_3 = \frac{1 + \cos (0.5953)}{3} = 0.\ 6093$$

$$x_4 = \frac{1 + \cos (0.6093)}{3} = 0.6067$$

$$x_5 = \frac{1 + \cos (0.6067)}{3} = 0.6072$$

$$x_6 = \frac{1 + \cos (0.607182)}{3} = 0.6071$$

Since x_5 and x_6 are same upto three decimal places, we stop.

Ex. 5 : Find a real root of $2x - \log_{10} x = 7$ correct to four decimal places using iteration method.

Sol. : Let $f(x) = 2x - \log_{10} x - 7$

Since $f(3) = -1.4471$, $f(4) = 0.398$, $\therefore$ a root lies between 3 and 4.

Rewriting the given equation as

$$x = \frac{1}{2} (\log_{10} x + 7) = \phi(x) \quad \text{and} \quad \phi'(x) = \frac{1}{2}\left(\frac{1}{x} \log_{10} e\right)$$

and $|\phi'(x)| < 1$ in the interval (3, 4)

Since $|f(4)| < |f(3)|$, the root is near to 4. Hence the iteration method can be applied.

Taking $x_0 = 3.6$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{2} (\log_{10} 3.6 + 7) = 3.77815$$

$$x_2 = \phi(x_1) = \frac{1}{2} (\log_{10} 3.77815 + 7) = 3.78863$$

$$x_3 = \phi(x_2) = \frac{1}{2} (\log_{10} 3.78863 + 7) = 3.78924$$

$$x_4 = \phi(x_3) = \frac{1}{2} (\log_{10} 3.78924 + 7) = 3.78927$$

Hence, x_3 and x_4 are almost equal, the desired root is 3.7892 correct upto four decimal places.

7.10 CONVERGENCE OF NEWTON-RAPHSON SCHEME

Newton-Raphson formula is of the form

$$x_{i+1} = x_i - \frac{(x_i)}{f'(x_i)}$$

This can be put in the form $x_{i+1} = g(x_i)$ with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Differentiating (1), we get

$$g'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

and for convergence we require that

$$\left|\frac{f(x)\, f''(x)}{[f'(x)]^2}\right| < 1$$

for all x in some interval I containing the root. Since $f(\xi) = 0$ the above condition is definitely satisfied at $x = \xi$ provided that $f'(\xi) \neq 0$, then provided that g(x) is continuous, an interval I must exists in the neighbourhood of the root and over which (3) is satisfied. However, the difficulty is encountered when the interval I is small because the initial guess must be taken from this interval. This usually arises when f(x) and f '(x) have roots close together. This feature is already demonstrated in Fig. 7.9. In such case, tangent line is almost parallel to x-axis. Consequently x_{i+1} can differ considerably from x_i unless an extremely accurate initial guess is obtained. We have already considered this case in illustrative Example 4.

7.11 RATE OF CONVERGENCE

We have already considered various methods to find the root of an equation f(x) = 0. We have also seen conditions for convergence. Now we shall consider how fast the convergent scheme converges.

Definition :

An iterative method is said to be of order k or has the rate of convergence k, if k is the largest positive real number for which there exists a finite non-zero constant C such that

$$|\varepsilon_{i+1}| \leq C\, |\varepsilon_i|^k \qquad \dots (a)$$

where $\varepsilon_i = x_i - \xi$ [ξ is the exact value of the root] is the error at the i^{th} iteration stage. The constant C usually depends on derivatives of f(x) at $x = \xi$ and is called asymptotic error constant.

We shall now find rate of convergence for various iterative methods.

(i) Newton-Raphson Method :

In Newton's formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad \dots (1)$$

Put $\quad x_i = \varepsilon_i + \xi, \quad x_{i+1} = \varepsilon_{i+1} + \xi$ in (1) $\qquad \dots (2)$

$$\varepsilon_{i+1} + \xi = \varepsilon_i + \xi - \frac{f(\varepsilon_i + \xi)}{f'(\varepsilon_i + \xi)}$$

By Taylor's series expansion about ξ.

$$f(\varepsilon_i + \xi) = f(\xi) + \varepsilon_i\, f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi)$$

as ξ is the root $f(\xi) = 0$

$$f(\varepsilon_i + \xi) = \varepsilon_i\, f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \qquad \dots (3)$$

$$f'(\varepsilon_i + \xi) = f'(\xi) + \varepsilon_i\, f''(\xi) + \frac{\varepsilon_i^2}{2} f'''(\xi) \qquad \dots (4)$$

Substituting from (3) and (4) in (2),

$$\varepsilon_{i+1} = \varepsilon_i - \frac{\left\{\varepsilon_i\, f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \dots\right\}}{\left\{f'(\xi) + \varepsilon_i\, f''(\xi) + \frac{\varepsilon_i^2}{2} f'''(\xi) \dots\right\}}$$

$$= \varepsilon_i - \left\{\varepsilon_i\, f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \dots\right\} \{f'(\xi) + \varepsilon_i\, f''(\xi) \dots\}^{-1}$$

$$= \varepsilon_i - \left\{\varepsilon_i f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \ldots\right\} [f'(\xi)]^{-1} \left\{1 + \frac{\varepsilon_i f''(\xi)}{f'(\xi)} \ldots\right\}^{-1}$$

$$= \varepsilon_i - \frac{1}{f'(\xi)} \left\{\varepsilon_i f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \ldots\right\} \left\{1 + \frac{\varepsilon_i f''(\xi)}{f'(\xi)} \ldots\right\}$$

$$= \varepsilon_i - \frac{1}{f'(\xi)} \left\{\varepsilon_i f'(\xi) - \varepsilon_i^2 f''(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \ldots\right\}$$

$$= \varepsilon_i - \varepsilon_i + \frac{1}{2} \frac{f''(\xi)}{f'(\xi)} \varepsilon_i^2$$

+ terms containing third and higher power of ε_i

Neglecting third and higher powers of ε_i, we get

$$\varepsilon_{i+1} = \frac{1}{2} \frac{f''(\xi)}{f'(\xi)} \varepsilon_i^2$$

or $\varepsilon_{i+1} = C\,\varepsilon_i^2$, where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ if $f''(\xi)$ and $f'(\xi)$ are non-zero, comparison with formula (a) shows that $k = 2$.

Thus Newton-Raphson method has second order convergence or it is second order process.

(ii) Method of Simple Iteration : Computational scheme for the method of simple iteration is

$$x_{i+1} = \phi(x_i) \qquad \ldots (1)$$

As before we put $x_i = \varepsilon_i + \xi$, $x_{i+1} = \varepsilon_{i+1} + \xi$ in (1), this gives

$$\varepsilon_{i+1} + \xi = \phi(\varepsilon_i + \xi)$$

Expanding by Taylor's series

$$\varepsilon_{i+1} + \xi = \phi(\xi) + \varepsilon_i \phi'(\xi) + \frac{\varepsilon_i^2}{2} \phi''(\xi) \ldots$$

but $\xi = \phi(\xi)$ in simple iteration.

$$\therefore \qquad \varepsilon_{i+1} = \varepsilon_i \phi'(\xi) + \frac{\varepsilon_i^2}{2} \phi''(\xi)$$

Neglecting second and higher powers of ε and if $\phi'(\xi) \neq 0$, we get

$$\varepsilon_{i+1} = c\,\varepsilon_i, \quad \text{where} \quad c = \phi'(\xi).$$

Thus, simple iteration method has first order convergence.

(iii) Secant Method : Computational scheme for secant method is

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})\, f(x_i)}{f(x_i) - f(x_{i-1})} \qquad \ldots (1)$$

[equation (6) of 7.4]

Putting $x_i = \varepsilon_i + \xi$, $x_{i+1} = \varepsilon_{i+1} + \xi$, $x_{i-1} = \varepsilon_{i-1} + \xi$ in (1), we get

$$\varepsilon_{i+1} + \xi = \varepsilon_i + \xi - \frac{(\varepsilon_i + \xi - \varepsilon_{i-1} - \xi)}{\{f(\varepsilon_i + \xi) - f(\varepsilon_{i-1} + \xi)\}} f(\varepsilon_i + \xi)$$

$$\therefore \qquad \varepsilon_{i+1} = \varepsilon_i - \frac{(\varepsilon_i - \varepsilon_{i-1})}{\{f(\varepsilon_i + \xi) - f(\varepsilon_{i-1} + \xi)\}} f(\varepsilon_i + \xi) \qquad \ldots (2)$$

By Taylor's series expansion

$$f(\varepsilon_i + \xi) = f(\xi) + \varepsilon_i f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \ldots$$

$$f(\varepsilon_{i-1} + \xi) = f(\xi) + e_{i-1} f'(\xi) + \frac{\varepsilon_{i-1}^2}{2} f''(\xi) \ldots$$

$$f(\varepsilon_i + \xi) - f(\varepsilon_{i-1} + \xi) = (\varepsilon_i - \varepsilon_{i-1}) f'(\xi) + \frac{1}{2} (\varepsilon_i^2 - \varepsilon_{i-1}^2) f''(\xi) \ldots$$

$$= (\varepsilon_i - \varepsilon_{i-1}) f'(\xi) + \frac{1}{2} (\varepsilon_i - \varepsilon_{i-1}) (\varepsilon_i + \varepsilon_{i-1}) f''(\xi)$$

$$= (\varepsilon_i - \varepsilon_{i-1}) f'(\xi) \left\{1 + \frac{1}{2} (\varepsilon_i + \varepsilon_{i-1}) \frac{f''(\xi)}{f'(\xi)} \ldots\right\}$$

Substituting in (2), we have

$$\varepsilon_{i+1} = \varepsilon_i - \frac{\left\{f(\xi) + \varepsilon_i f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \ldots\right\}}{f'(\xi)} \left\{1 + \frac{1}{2} (\varepsilon_i + \varepsilon_{i-1}) \frac{f''(\xi)}{f'(\xi)} \ldots\right\}^{-1}$$

but $f(\xi) = 0$

$$\therefore \quad \varepsilon_{i+1} = \varepsilon_i - \frac{1}{f'(\xi)} \left\{\varepsilon_i f'(\xi) + \frac{\varepsilon_i^2}{2} f''(\xi) \ldots\right\} \left\{1 - \frac{1}{2} (\varepsilon_i + \varepsilon_{i-1}) \frac{f''(\xi)}{f'(\xi)} \ldots\right\}$$

$$= -\frac{\varepsilon_i^2}{2} f''(\xi) + \frac{1}{2} (\varepsilon_i + \varepsilon_{i-1}) \varepsilon_i \frac{f''(\xi)}{f'(\xi)} \ldots$$

$$= \frac{1}{2} \frac{f''(\xi)}{f'(\xi)} \varepsilon_i \varepsilon_{i-1} + O(\varepsilon_i^2)$$

[$O(\varepsilon_i^2)$ – terms containing second and higher power of ε_i]

Neglecting second and higher powers of ε_i

$$\varepsilon_{i+1} = c\, \varepsilon_i\, \varepsilon_{i-1} \qquad \ldots (3)$$

where, $$c = \frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$$

To obtain rate of convergence, we should have a relation of the form

$$\varepsilon_{i+1} = A\, \varepsilon_i^k \qquad \ldots (4)$$

or $$\varepsilon_i = A^k \varepsilon_{i-1} \quad \text{i.e.} \quad \varepsilon_{i-1}^k = \frac{\varepsilon_i}{A}$$

$$\therefore \quad \varepsilon_{i-1} = \left(\frac{\varepsilon_i}{A}\right)^{1/k}$$

$$\varepsilon_{i-1} = A^{-1/k} \varepsilon_i^{1/k} \qquad \ldots (5)$$

Putting from (5) and (4), we get

$$\varepsilon_{i+1} = c\, \varepsilon_i A^{-1/k} \varepsilon_i^{1/k}$$

or $$A\, \varepsilon_i^k = c\, A^{-1/k} \varepsilon_i^{1+1/k}$$

$$\therefore \quad \varepsilon_i^k = c\, A^{-(1/k+1)} \varepsilon_i^{1+1/k}$$

Comparing powers of ε_i on both sideds, we have

$$k = 1 + \frac{1}{k} \quad \text{or} \quad k^2 - k - 1 = 0$$

$$\therefore \qquad k = \frac{1 \pm \sqrt{1+4}}{2}$$

Neglecting negative value of k.

We get, $\quad k = \dfrac{1+\sqrt{5}}{2} = 1.618$ or $k = 1.62$ approximately which gives rate of convergence.

Comparison of (i), (ii), (iii) shows that Newton's method has most rapid convergence.

MISCELLANEOUS EXAMPLES

Ex. 1 : In a circuit containing inductance L, resistance R and a battery of 20 volts, current I = 0.2 A is displayed at the end of 0.01 sec for a particular value of R given by the relation.

$$I = \frac{E}{R}\left(1 - e^{-\frac{R}{0.5}t}\right)$$

Determine R, using method of bisection at the end of fifth iteration.

Solution : $\quad 0.2 = \dfrac{20}{R}\left(1 - e^{-\frac{R}{0.5} \times 0.01}\right)$

which reduces to $\quad R = 100\left(1 - e^{-\frac{R}{50}}\right)$

Let $\quad f(R) = R - 100\left(1 - e^{-\frac{R}{50}}\right)$

Since $\quad f(0) = -100,\ f(50) = -13.21,\ \ f(100) = 13.53$ $\therefore$ a root lies between 50 and 100.

First approximation $c_1 = \dfrac{100 + 50}{2} = 75, \qquad \therefore f(75) = -2.6$

Second approximation $c_2 = \dfrac{100+75}{2} = 87.5, \qquad \therefore f(87.5) = 4.88$

Third approximation $c_3 = \dfrac{75 + 87.5}{2} = 81.25, \qquad \therefore f(81.25) = 0.941$

Fourth approximation $= \dfrac{75 + 81.25}{2} = 78.125, \qquad \therefore f(78.125) = -0.913$

Fifth approximation $= \dfrac{78.125 + 81.25}{2} = 79.6875,$

Hence, R = 79.6875 ohms.

Ex. 2 : A car covers 310 km with some speed. If its speed is increased by 10 km/hr, it will cover the same distance in one hour less. Find out the algebraic equation governing the speed of the car and find the speed of the car by Bisection method.

Solution : Distance travelled s = v.t

For initial condition, $\quad s_1 = v_1 \cdot t_1$

i.e. $\quad 310 = v_1\, t_1 \qquad \ldots (1)$

for second condition, $\quad s_1 = (v_1 + 10)(t_1 - 1)$

$$s_1 = v_1\, t_1 + 10t_1 - v_1 - 10$$

$$s_1 = s_1 + 10\,(t_1 - 1) - v_1$$

$$\therefore \qquad v_1 = 10t_1 - 10$$

$$\therefore \qquad v_1 = 10\,(t_1 - 1)$$

$$\therefore \qquad t_1 = \frac{V_1 + 10}{10} \qquad \ldots (2)$$

Now, substituting t_1 from equation (1) to equation (2)

$$310 = v_1 t_1 \Rightarrow t_1 = \frac{310}{v_1}$$

$$\therefore \qquad \frac{310}{v_1} = \frac{v_1 + 10}{10}$$

$$3100 = v_1^2 + 10v_1$$

$$\therefore \qquad v_1^2 + 10v_1 - 3100 = 0$$

which is required governing equation for speed of car.

Let, $\qquad y = v_1^2 + 10v - 3100$

selecting interval (50, 51)

when $v_0 = 50$ and when $v_1 = 51$, we find

$$\Rightarrow \qquad y = 2601 + 510 - 3100 = 3111 - 3100 = 11$$

$$\Rightarrow \qquad y = 2500 + 500 - 3100 = -100$$

These are of opposite signs therefore interval selected is correct.

$$\therefore \text{ Let} \qquad v_2 = \frac{v_0 + v_1}{2} = \frac{50 + 51}{2} = 50.5$$

For $v_2 = 50.5$ km/hr $\Rightarrow \qquad y = -44.75$

$$\therefore \qquad y_3 = \frac{v_2 + v_1}{2} = \frac{50.5 + 51}{2} = 50.75$$

For $v_3 = 50.75$ km/hr $\Rightarrow \qquad y = -16.9375$

$$\therefore \qquad v_4 = \frac{v_3 + v_1}{2} = \frac{50.75 + 51}{2} = 50.875$$

For $v_4 = 50.875$ km/hr $\Rightarrow \qquad y = -2.98437$

$$\therefore \qquad v_5 = \frac{v_4 + v_1}{2} = \frac{50.875 + 51}{2} = 50.9375$$

For $v_5 = 50.9375$ km/hr $\Rightarrow \qquad y = 4.0039$

$$v_6 = \frac{v_5 + v_4}{2} = \frac{50.9375 + 50.875}{2} = 50.90625$$

$v_6 = 50.90625$ km/hr $\Rightarrow \qquad y = 0.508$

$$v_7 = \frac{v_6 + v_4}{2} = \frac{50.90625 + 50.875}{2} = 50.890625$$

For $v_7 = 50.890625$ km/hr $\Rightarrow$

$$y = -1.238$$

$$v_8 = \frac{v_7 + v_6}{2} = \frac{50.890625 + 50.90625}{2}$$

$$v_8 = 50.898428 \text{ km/hr}$$

For $v_8 \Rightarrow y = -0.36468$

$$v_9 = \frac{v_8 + v_6}{2} = \frac{50.898438 + 50.90625}{2}$$

$$v_9 = 50.902344 \text{ km/hr}$$

For $v_9 \Rightarrow$ $y = 0.0721$

$\therefore$ Required speed of car = 50.902344 km/hr.

Ex. 3 : Obtain root of the equation tan x + tan hx = 0 by Regula Falsi method upto 5 iterations.

Solution : $f(x) = \tan x + \tan hx$

f(1) = 2.319; f(1.5) = 15.00; f(2) = – 1.22

Root lies between x = 1.5 and x = 2.

$$\text{First approximation } x_1 = \frac{2f(1.5) - 1.5f(2)}{f(1.5) - f(2)} = \frac{30 + 1.5(1.22)}{15 + 1.22}$$

$$= 1.962$$

$$f(x_1) = f(1.962) = -1.463$$

Root lies between 1.5 and 1.962.

$$\text{Second approximation } x_2 = \frac{1.962 \times 15 + 1.5 \times 1.463}{15 + 1.463} = 1.921 \qquad f(1.921) = -1.78$$

$$\text{Third approximation } x_3 = \frac{1.921 \times 15 + 1.5 \times 1.78}{15 + 1.78} = 1.875 \qquad f(1.876) = -2.22$$

$$\text{Fourth approximation } x_4 = \frac{1.876 \times 15 + 1.5 \times 2.22}{15 + 2.22} = 1.827 \qquad f(1.827) = -2.868$$

$$\text{Fifth approximation } x_5 = \frac{1.827 \times 15 + 1.5 \times 2.868}{15 + 2.68} = 1.7745$$

Hence, the required root is 1.7745.

Ex. 4 : Solve using Newton Raphson method the equation sin x – x cos x = 0. Assume initial guess value for x = $3\frac{\pi}{2}$, correct upto five decimal places.

Sol. : Given : $f(x) = \sin x - x \cos x$

$\therefore$ $f'(x) = \cos x - [x(-\sin x) + \cos x] = \cos x + x \sin x - \cos x = x \sin x$

Given the initial value, $x_0 = \frac{3\pi}{2}$. Using Newton Raphson method,

$$x_1 = x_0 - \frac{(x_0)}{f'(x_0)}$$

$$x_1 = \frac{3\pi}{2} - \frac{\sin(3\pi/2) - 3\pi/2 \times \cos(3\pi/2)}{3\pi/2 \times \sin(3\pi/2)} = 4.5001824$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 4.5001824 - \frac{\sin(4.5001824) - 4.5001824 \times \cos(4.5001824)}{4.5001824 \times \sin(4.5001824)} = 4.4934195$$

As $x_1 - x_2 = 4.5001824 - 4.4934195 = 6.7629 \times 10^{-3} > 0.00001$

Find next, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 4.4934195 - \frac{\sin(4.4934195) - 4.4934195 \cos(4.4934195)}{(4.4934195) \times \sin(4.493195)} = 4.4934195$$

As $x_2 - x_3 = 4.4934195 - 4.493405$

$$= 0.0000145 > 0.00001$$

Find next, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$x_4 = 4.4934095 - \frac{\sin(4.4934095) - 4.493405 \cos(4.4934095)}{4.4934095 \sin(4.4934095)} = 4.4934095$$

$$x_3 - x_4 = 4.4934095 - 4.4934095$$

$$= 0.00000 < 0.00001$$

Since Required accuracy is achieved. ∴ Required root is x = 4.4934095.

Ex. 5 : Solve $xe^x - 4 \cos x = 0$ using false position method taking initial values of x as 0 and 1.5 at the end of fourth iteration..

Solution : Let $f(x) = xe^x - 4 \cos x = 0$

$$x_1 = 0, \quad x_2 = 1.5 \text{ and } f(x_1) = -4, f(x_2) = 6.4395$$

$$x_3 = \frac{x_1 \times f(x_2) - x_2 \times f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0 \times 6.4395 - 1.5 * (-4)}{6.4395 - (-4)} = 0.5747$$

$$f(x_3) = -2.335, \text{ Root lies between } x_2 \text{ and } x_3$$

$$x_4 = \frac{x_3 \times f(x_2) - x_2 \times f(x_3)}{f(x_2) - f(x_3)}$$

$$= \frac{0.5747 \times (6.4395) - 1.5 \times (-2.335)}{6.4395 - (-2.335)} = 0.8208$$

$$f(x_4) = -0.8614, \text{ Root lies between } x_2 \text{ and } x_4$$

$$x_5 = \frac{x_4 * f(x_2) - x_2 * f(x_4)}{f(x_2) - f(x_3)}$$

$$= \frac{0.8208 \times (6.4395) - 1.5 \times (-0.8614)}{6.4395 - (-0.8614)} = 0.9$$

$$f(x_5) = -0.2727, \text{ Root lies between } x_2 \text{ and } x_5$$

$$x_6 = \frac{x_5 \times f(x_2) - x_2 \times f(x_5)}{f(x_2) - f(x_5)}$$

$$= \frac{0.9 \times (6.4395) - 1.5 \times (-0.2727)}{6.4395 - (-0.2727)} = 0.92$$

After fourth iteration required root is x = 0.92

Ex. 6 : Find a root of the equation $x^4 - 1.8x^3 + 6x^2 - 18x - 13.9 = 0$ by Regula-Falsi method. How many iterations are required so that the answer repeats upto two decimal places.

Solution : Let $f(x) = x^4 - 1.8x^3 + 6x^2 - 18x - 13.9 = 0$

Since $f(0) = -13.9, \; f(-1) = 12.9$,

∴ A root lies between – 1 and 0.

Here, $x_0 = -1$ and $x_1 = 0, f_0 = 12.9$ and $f_1 = -13.9$

By Regula Falsi method,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$\therefore \quad x_2 = 0 - \frac{[0 - (-1)]}{(-13.9 - 12.9)} \times (-13.9) = -0.5186$$

$$\therefore \quad f(x_2) = -2.6266$$

∴ Root lies between – 1 and – 0.5186

Here, $x_2 = -0.5186,\ f_2 = -2.6266$

Let $x_1 = -1,\ f_1 = 12.9$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$\therefore \quad x_3 = -0.5186 - \frac{[-0.5186 - (-1)]}{(-2.6266 - 12.9)} \times (-2.6226) = -0.6$$

$$\therefore \quad f(x_3) = -0.4216$$

$\therefore$ Root lies between – 1 and – 0.6

Here, $x_3 = -0.6,\ f_3 = -0.4216$

Let $x_2 = -1,\ f_2 = 12.9$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3$$

$$\therefore \quad x_3 = -0.6 - \frac{[-0.6 - (-1)]}{(-0.4216 - 12.9)} \times (-0.4216) = -0.6127$$

$$\therefore \quad f(x_4) = -0.06405$$

$\therefore$ Root lies between – 1 and – 0.6127

Here, $x_4 = -0.6127,\ f_4 = -0.06405$

Let $x_3 = -1,\ f_3 = 12.9$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$\therefore \quad x_3 = -0.6127 - \frac{[-0.6127 - (-1)]}{(-0.06405 - 12.9)} \times (-0.06405) = -0.6146$$

$$\therefore \quad f(x_5) = -0.01024$$

$\therefore$ Five iterations are required to repeat the root upto two decimal places.

$\therefore$ Hence the required root is – 0.6146.

Ex. 7 : Solve $e^x \cos(x) - 1.2 = 0$ using Newton-Raphson method. Taking initial value of x = 1.2 at the end of four iterations.

Sol. :

Let $f(x) = e^x \cos(x) - 1.2 = 0, \quad f'(x) = -e^x \sin x + \cos x\, e^x$

Give, $x_0 = 1.2$, by Newton-Raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{f(1.2)}{f'(1.2)}$$

$$x_1 = 1.2 - \frac{e^{1.2} \cos 1.2}{-e^{1.2} \sin 1.2 + e^{1.2} \cos 1.2} = 1.836$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 1.836 - \frac{e^{1.836} \cos 1.836}{-e^{1.836} \sin 1.836 + e^{1.836} \cos 1.836} = 1.6224$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.6224 - \frac{e^{1.6224} \cos 1.6224}{(-e^{-1.6224} \sin 1.6224 + e^{1.6224} \cos 1.6224)} = 1.57328$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 1.57328 - \frac{e^{1.57328} \cos 157328}{(e^{-1.57328} \sin 1.57328 + e^{1.57328} \cos 1.57328)} = 1.5708$$

Hence, the required root is 1.5708.

Ex. 8 : Find root of the equation $3x + \sin x - e^x = 0$ by Newton Raphson method starting with $x_0 = 0$ as initial value with an accuracy of 0.001.

Sol. : Initial guess value $x_0 = 0$,

Let $f(x) = 3x + \sin x - e^x$

Since $f(x_0) = 3x + \sin x - e^x = 0 + 0 - 1 = (-1)$

$$f'(x_0) = 3 + \cos x - e^x = 3 + 1 - 1 = (3)$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{3} = 0.3333$$

$$x_1 = 0.3333$$

$$f(x_1) = 3(0.3333) + \sin(0.3333) - e^{0.3333} = -0.06842$$

$$f'(x_1) = 3 + \cos(0.3333) - e^{0.3333} = 2.5493$$

$$\therefore \quad x_2 = x_1 - \frac{(x_1)}{f'(x_1)} = 0.3333 - \frac{(-0.06842)}{2.5493}$$

$$x_2 = 0.3602$$

$$f(x_2) = 3(0.3602) + \sin(0.3602) - e^{0.3602} = -0.000634$$

$$f'(x_2) = 3 + \cos(0.3602) - e^{0.3602} = 2.5023$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.3602 - \frac{(-0.000634)}{2.5023}$$

$$x_3 = 0.36045$$

$$f(x_3) = 0.0000707$$

$$f'(x_3) = 2.501764$$

$$x_4 = x_3 - \frac{f(x_3)}{Df(x_3)} = 0.36045 - \frac{0.0000707}{2.501764}$$

$$x_4 = 0.36042$$

Hence, the required root is 0.36042

Ex. 9 : Solve the equation $e^x - 5x = 0$ using method of successive approximation that is by arranging the equation in the form $x = \phi(x)$. Assume initial guess for x = 0.15, by performing five iterations.

Solution : $e^x - 5x = 0$

Consider $x = \frac{e^x}{5} = \phi(x)$

Successive Iterations are as below :

Iteration No.	**x**	**ϕ(x)**
1	0.15	0.22324
2	0.2324	0.2523
3	0.2523	0.2574

4	0.2574	0.2587
5	0.2587	0.259

Hence the root is 0.2587.

Ex. 10 : Apply iteration method to find the negative root of the equation $x^3 - 2x + 5 = 0$ correct to four decimal places.

Sol. : If α, β, γ are the roots of $x^3 - 2x + 5 = 0$, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 2(-x) + 5 = 0$ or $x^3 - 2x - 5 = 0$. Thus, negative root of the given equation is the positive root of $x^3 - 2x - 5 = 0$.

Let $\quad f(x) = x^3 - 2x - 5$

Since $\quad f(2) = -1, f(3) = 16, \therefore$ a root lies between 2 and 3.

Rewriting (1) as $\quad x = (2x + 5)^{1/3} = \phi(x)$

Also, $\phi'(x) = \frac{1}{3}(2x + 5)^{-2/3}$ and $|\phi'(x)| < 1$ in the interval (2, 3). $\therefore$ The interaction method can be applied.

Starting with $x_0 = 2$. The successive approximations are

$$x_1 = \phi(x_0) = (2x_0 + 5)^{1/3} = 2.08008$$

$$x_2 = \phi(x_1) = (2x_1 + 5)^{1/3} = 2.09235$$

$$x_3 = \phi(x_2) = (2x_2 + 5)^{1/3} = 2.09422$$

$$x_4 = \phi(x_3) = (2x_3 + 5)^{1/3} = 2.09450$$

$$x_5 = \phi(x_4) = (2x_4 + 5)^{1/3} = 2.09454$$

Since x_4 and x_5 being almost the same, the root of (1) is 2.0945 correct to 4 decimal places.

Hence the negative root of the given equation $x^3 - 2x + 5 = 0$ is – 2.09454.

Ex. 11 : Find the smallest root of the equation

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots\ldots\ldots\ldots = 0$$

Sol. : Writing the given equation as

$$x = 1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots\ldots\ldots\ldots = \phi(x)$$

Neglecting x^2 and higher powers of x, we get x = 1 approximately.

Taking $x_0 = 1$, we obtain by simple iteration method.

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \ldots\ldots\ldots\ldots = 1.2239$$

$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \ldots\ldots\ldots\ldots = 1.3263$$

Similarly, $x_3 = 1.38$, $x_4 = 1.409$, $x_5 = 1.426$, $x_6 = 1.434$, $x_7 = 1.439$, $x_8 = 1.442$

The values of x_7 and x_8 indicate that the root is 1.44 correct to two decimal places.

EXERCISE 7.1

[A] Bisection Method :

1. Find a root of the following equations, using the bisection method correct to three decimal places :

(i) $x^3 - x - 1 = 0$ (ii) $x^3 - x^2 - 1 = 0$ (iii) $2x^3 + x^2 - 20x + 12 = 0$ (iv) $x^4 - x - 10 = 0$

Ans. (i) 1.321, (ii) 1.466, (iii) 2.875, (iv) 1.855

2. Evaluate a real root of the following equations by bisection method.

(i) $x - \cos x = 0$ (ii) $e^{-x} - x = 0$ (iii) $e^x = 4 \sin x$ **Ans.** (i) 0.0625, (ii) 0.567, (iii) 0.367

3. Use method of Bisection to find a root of the equation $x^3 - 1.8x^2 - 10x + 17 = 0$ that lies in the internal [1, 2] at the end of fifth equation. **Ans.** 1.662

4. Perform five iterations of the bisection method to obtain the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$.

[**Hint** : Since $f(0) > 0$ and $f(1) < 0$; a root lies in the interval (0, 1)] **Ans.** 0.203125

5. Find a real root of the equation $f(x) = x^3 - x - 1 = 0$ by bisection method. [Hint : Since $f(1) < 0$ and $f(2) > 0$, a root lies between 1 and 2.] **Ans.** 1.328125

[B] Second Method and Regula-Fasi Method :

6. Find a root of the following equations correct to the decimal places by secant method:

(i) $x^3 + x^2 + x + 7 = 0$ (ii) $x - e^{-x} = 0$ (iii) $x \log_{10} x = 1.9$

Ans. (i) – 2.0625, (ii) 0.567, (iii) 3.496

7. Use secant method to estimate the roots of the equation $x^3 - 6.37x^2 + 6.48x + 7.11 = 0$ which lies in the interval (4, 5) performing four iteration. **Ans.** 4.6456

8. Find a real root of the following equations correct to three decimal places by method of false position.

(i) $x^3 - 5x + 1 = 0$ (ii) $x^3 - 4x - 9 = 0$ (iii) $x^6 - x^4 - x^3 - 1 = 0$

Ans. (i) 2.128, (ii) 2.7065, (iii) 1.4036

9. Using Regula-Falsi method, compute the real root of the following equations correct to three decimal places.

(i) $xe^x = 2$ (ii) $\cos x = 3x - 1$ (iii) $x e^x = \sin x$

(iv) $x \tan x = 1$ (v) $2x - \log_{10} x = 7$ (vi) $3x + \sin x = e^x$

Ans. (i) 0.853, (ii) 0.6071, (iii) – 0.134, (iv) 2.798, (v) 3.789, (vi) 0.3604

10. Find negative root of the equation $x^4 + 2x^3 + 6x^2 + 20x - 15 = 0$ using method of false position at the end of fifth iteration.

[**Hint :** $f(0) = -15$, $f(-1) = -30$, $f(-2) = -31$, $f(-3) = 6$, root lies in the interval (–2, –3)] **Ans.** – 2.909

[C] Newton-Raphson Method :

11. Using Newton-Raphson method, find a root of the following equations correct to three decimal places

(i) $x^3 - 3x + 1 = 0$ (ii) $x^3 - 2x - 5 = 0$ (iii) $x^3 - 5x + 3 = 0$

(iv) $3x^3 - 9x^2 + 8 = 0$ (v) $\sqrt{10}$ (vi) $\sqrt{21}$ (vii) $x = \frac{1}{7}$

Ans. (i) 1.532, (ii) 2.095, (iii) 1.834, (iv) 1.226, (v) 3.162, (vi) 2.5713, (vii) 0.1429

12. Find by Newton-Raphson method, find a root of the following equations correct to three decimal places.

(i) $x^2 + 4 \sin x = 0$ (ii) $x \sin x + \cos x = 0$ or $x \tan x + 1 = 0$

(iii) $e^x = x^3 + \cos 25x$ which is neat to 4.5 (iv) $x \log_{10} x = 12.34$ start with $x_0 = 10$.

(v) $\cos x = x e^x$ (vi) $10^x + x - 4 = 0$ (vii) $e^x \sin x = 1$ (viii) $\sin x = 1 - x$

Ans. (i) – 1.9338, (ii) 2.798, (iii) 4.545, (iv) 0.052, (v) 0.518, (vi) 0.695, (vii) 0.5886, (vii) 0.5110

13. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places by Newton's method.

Ans. – 16.56

14. Use Newton's method to find the root of $2x - 3 \cos x = 0$ which is near to 0.9. **Ans.** 0.9149

15. Use Newton-Raphson method to determine the root of $(2x + 1)^2 = 4 \cos \pi x$ lying in the interval $\left[\frac{1}{4}, \frac{1}{3}\right]$ at the end of fourth iteration. **Ans.** 0.2872

16. Show that the equation $x e^x = 1$ has a root between 0.5 and 0.6. Find the root correct to four decimal places using Newton-Raphson method. **Ans.** 0.5671

[D] Method of Successive Approximation (Iteration method) :

17. Use Newton-Rapshon method to find square root of 30 correct to four decimal places. **Ans.** 5.4772

18. Find a real root of the following equations correct to four decimal places using iteration method.

(i) $x^3 + x^2 - 100 = 0$ (ii) $x^3 - 9x + 1 = 0$ (iii) $x^3 - 3x + 1 = 0$ (lying between 1 and 2)

(iv) $x = \frac{1}{2} + \sin x$ (v) $e^x = 5x$ (vi) $2^x - x - 3 = 0$ (which lies between –3 and –2)

Ans. (i) 0.6071, (ii) 2.9428, (iii) 1.532, (iv) 1.4973, (v) 0.2591, (vi) 2.8625

19. Use the iterative method to find a root of the following equations correct to four significant figures.

(i) $\cos x = 3x - 1$ (ii) $e^{-x} = 10x$ (iii) $\sin^2 x = x^2 - 1$

(iv) $2x = \cos x + 3$ (v) $x = 1 + 0.3 \cos x$ (vi) $\tan x = x$ (smallest positive root)

Ans. (i) 0.6071, (ii) 0.0913, (iii) 1.404, (iv) 1.524, (v) 1.1284, (vi) 4.4346

20. Solve $x = 0.24 \sin (x + 0.5)$ correct tap to 4 decimals by using method of simple iteration. **Ans.** 0.1441

21. Find initial guesses to the three roots of $x^4 - x - 0.1 = 0$. Express the equation in the form $x_{i+1} = \phi(x)$ and find the roots correct upto four decimal places. **Ans.** $x_0 = 1.2$ and root = 1.04668

22. Show that the iterative scheme $x_{i+1} = \frac{1}{3}(4x_i - ax_i^4)$ is a second order process for calculation of the cube root of a. Use this scheme to find $4^{-1/3}$ to four decimal places. **Ans.** 0.6299

23. Find the real root of the equation $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots\ldots\ldots\ldots = 0.443$ correct to three decimal places using iteration method. **Ans.** 0.477

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UNIT V : NUMERICAL METHODS

CHAPTER - 8
SYSTEM OF LINEAR EQUATIONS

8.1 INTRODUCTION

Solving linear system of equation is of common occurrence of Engineering problems. Methods of solution of the system of linear algebraic equations could be classified into two categories :

1. **Direct Methods :** These methods produce the exact solution after a finite number of steps. The important direct methods are
   (i) Cramer's Rule.
   (ii) Matrix Inversion Method.
   (iii) Gauss Elimination Method.
   (iv) Gauss Jordan Method.
   (v) Triangular Factorization (LU Decomposition Method)
   (vi) Cholesky Method (Square Root Method).

2. **Indirect Methods (Iterative Methods) :** These methods start with trial (initial) solutions and give rise to sequence of approximate solutions which converges after large number of steps. The important iterative methods are
   (i) Jacobi Iteration Method
   (ii) Gauss-Seidel Iteration Method

We shall now discuss these methods.

8.2 DIRECT METHODS

8.2.1 Cramer's Rule

Consider a system of three equations

$$a_1x_1 + b_1 x_2 + c_1 x_3 = d_1$$
$$a_2 x_1 + b_2 x_2 + c_2 x_3 = d_2$$
$$a_3 x_1 + b_3 x_2 + c_3 x_3 = d_3$$

We first write down coefficient determinant Δ as

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We next replace first, second and third columns successively by the column $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and form the determinants

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solution is then obtained as

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}$$

Obviously such solution will exist when $\Delta \neq 0$.

Remark : Although Cramer's rule is simple and easy to apply, its use requires great deal of labour when the number of equations exceeds four or five, because of the labour in evaluating the determinants involved.

ILLUSTRATION

Ex. 1 : *Solve the system*

$$2x + 3y + 4z = 5$$
$$x + 2y + z = 4$$
$$3x - y + z = 6$$

Sol. : Here

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} = -16$$

$$\Delta_1 = \begin{vmatrix} 5 & 3 & 4 \\ 4 & 2 & 1 \\ 6 & -1 & 1 \end{vmatrix} = -43, \quad \Delta_2 = \begin{vmatrix} 2 & 5 & 4 \\ 1 & 4 & 1 \\ 3 & 6 & 1 \end{vmatrix} = -18, \quad \Delta_3 = \begin{vmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 3 & -1 & 6 \end{vmatrix} = 15$$

$$\therefore \quad x = \frac{-43}{-16} = \frac{43}{16}, \quad y = \frac{-18}{-16} = \frac{18}{16}, \quad z = \frac{15}{-16} = -\frac{15}{16}.$$

8.2.2 Matrix Inversion Method

We consider the system of three equations :

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned} \qquad \text{...(1)}$$

The above system can be written in matrix form

$$AX = B \qquad \text{... (2)}$$

$$\text{where,} \quad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let A be non-singular so that A^{-1} exists. Then premultiplying both sides of (2) by A^{-1}, we obtain

$$A^{-1} AX = A^{-1}B$$

Or $\quad IX = A^{-1}B$

i.e. $\quad X = A^{-1}B \qquad \text{... (3)}$

If A^{-1} is known, then the solution of the system (1) can be found out from the above relation (3).

ILLUSTRATION

Ex. 1 : Solve the system $\quad x + 2y + 2z = 7$

$$2x - 4y + z = -5$$
$$x + y + 2z = 5$$

Sol. : The given system can be written in matrix form

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -4 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix}$$

Also

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & -4 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -3$$

We now form the matrix C containing co-factors of elements of matrix A

$$C = \begin{bmatrix} -9 & -3 & 6 \\ -2 & 0 & 1 \\ 10 & 3 & -8 \end{bmatrix}$$

and $$\text{Adj } A = C' = \begin{bmatrix} -9 & -2 & 10 \\ -3 & 0 & 3 \\ 6 & 1 & -8 \end{bmatrix}$$

Hence $$A^{-1} = \frac{1}{|A|} \text{ Adj } A = -\frac{1}{3}\begin{bmatrix} -9 & -2 & 10 \\ -3 & 0 & 3 \\ 6 & 1 & -8 \end{bmatrix} = \begin{bmatrix} 3 & 2/3 & -10/3 \\ 1 & 0 & -1 \\ -2 & -1/3 & 8/3 \end{bmatrix}$$

It follows therefore that

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2/3 & -10/3 \\ 1 & 0 & -1 \\ -2 & -1/3 & 8/3 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 - \frac{10}{3} - \frac{50}{3} \\ 7 - 0 - 5 \\ -14 + \frac{5}{3} + \frac{40}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

which gives $x = 1$, $y = 2$, $z = 1$, the required solution.

Remark : Methods described above are precise and give exact results. However, these methods are not much suitable when the size of the system is large.

We shall now discuss methods which are very useful when dealing with system of equations with larger size.

8.2.3 Gauss Elimination Method

This is the elementary elimination method and it reduces the given system of linear equations to an equivalent upper triangular system of linear equations which is then solved by backward substitution. To describe the method, we consider the system of three equations in three unknowns for the sake of clarity and simplicity.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned} \qquad \dots (1)$$

The **Gauss Elimination** method consists of the following steps.

Step 1 : Elimination of x_1 : Assuming $a_{11} \neq 0$ we divide coefficients of the first equation of system (1) by a_{11}, we get

$$x_1 + a'_{12} x_2 + a'_{13} x_3 = b'_1 \qquad \dots (2)$$

where, $a'_{12} = \dfrac{a_{12}}{a_{11}}$, $a'_{13} = \dfrac{a_{13}}{a_{11}}$ and $b'_1 = \dfrac{b_1}{a_{11}}$

Using equation (2), we now eliminate x_1 from the remaining equations [i.e. from second and third equations of (1)] by subtracting a_{21} times equation (2) from second equation of (1), and a_{31} times equation (2) from third equation of (1). We thus get system consisting of two equations in two unknowns, namely x_2 and x_3 as

$$\begin{aligned} a'_{22} x_2 + a'_{23} x_3 &= b'_2 \\ a'_{32} x_2 + a'_{33} x_3 &= b'_3 \end{aligned} \qquad \dots (3)$$

where, $a'_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} a_{12}$, $a'_{23} = a_{23} - \dfrac{a_{21}}{a_{11}} a_{13}$, $b'_2 = b_2 - \dfrac{a_{21}}{a_{11}} b_1$,

$a'_{32} = a_{32} - \dfrac{a_{31}}{a_{11}} a_{12}$, $a'_{33} = a_{33} - \dfrac{a_{31}}{a_{11}} a_{13}$, $b'_3 = b_3 - \dfrac{a_{31}}{a_{11}} b_1$.

Step 2 : Elimination of x_2 : Assuming $a'_{22} \neq 0$, we divide the coefficients of the first equation of system (3), we get

$$x_2 + a''_{23}x_3 = b''_2 \qquad \dots (4)$$

where,
$$a''_{23} = \frac{a'_{23}}{a'_{22}}, \quad b''_2 = \frac{b'_2}{a'_{22}}$$

Using equation (4), we next eliminate x_2 from second equation of (3) by subtracting a'_{32} times equation (4) from second equation of (3). We thus get

$$a''_{33}\, x_3 = b''_3 \qquad \dots (5)$$

where,
$$a''_{33} = a'_{33} - \frac{a'_{32}}{a'_{22}}\, a'_{23}, \qquad b''_3 = b'_3 - \frac{a'_{32}}{a'_{22}}\, b'_2$$

Step 3 : Determination of Unknowns x_3, x_2, x_1 : Collecting the first equation from each stage i.e. from equations (2), (4) and (5), we obtain

$$\begin{aligned} x_1 + a'_{12}\, x_2 + a'_{13}\, x_3 &= b_1 \\ x_2 + a''_{23}\, x_3 &= b''_3 \\ x_3 &= \frac{b''_3}{a''_{33}} \end{aligned} \qquad \dots (6)$$

The system (6) is an upper triangular system and can be solved by a process called back substitution. The last equation determines x_3, which is then substituted in the next last equation to determine x_2 and finally on substituting x_3 and x_2 in the first equation, we obtain x_1.

Remark 1 : The elements a_{11}, a'_{22} and a''_{33} which have been assumed to be non-zero are called **Pivot Elements** and the equations containing these pivot elements are called **Pivotal Equations.**

Remark 2 : The process of dividing by a_{11} to first equation of system (1) for making coefficient of x_1 unity is called **Normalization**.

Remark 3 : A necessary and sufficient condition for using Gauss elimination method is that all leading elements (pivot elements) be non-zero.

In the elimination process, if any one of the pivot elements a_{11}, a'_{22}, ... vanishes or becomes very small compared to other elements in that equation then we attempt to rearrange the remaining equation so as to obtain a non-vanishing pivot or to avoid the division by small pivot element (equivalently multiplication by large number) to every coefficient in the pivotal equation. This procedure is called **Partial Pivoting**. If the matrix A is diagonally dominant or real, symmetric and positive definite, then no pivoting is necessary.

We note that, if the numerators of any fractions contain rounding errors, the effects of such errors are diminished when the denominator is large pivot.

Partial Pivoting :

In the first step of elimination, the numerically largest coefficient of x_1 is chosen from all the equations and brought as first pivot by interchanging the first equation with equation having the largest coefficient of x_1. In the second step of elimination, the largest coefficient of x_2 in magnitude is chosen from the remaining equations (leaving the first equation) and brought as second pivot by interchanging the second equation with the equation having the largest coefficient of x_2. This process is continued till we arrive at the equation with the single variable. In other words, partial pivoting involves searching for largest coefficient of an unknown quantity amongst a system of equations at each step of the elimination.

Complete Pivoting :

A slightly better result may be obtain by disregarding the order of elimination of x_1, x_2, x_3 and solving at each step of elimination process, the equation in which largest coefficient in the entire set (or in the entire matrix of coefficient) occurs. This requires not only an interchange of equations but also interchange of the position of the variables. The equation is then solved for that unknown to which the largest coefficient is attached. This process is called as **Complete Pivoting** and is rarely employed as it changes the order of unknowns and consequently adds complexity.

Remark 4 : The Gauss elimination method can be interpreted in matrix form as reducing the augmented matrix to upper triangular form whose leading diagonal elements are unity by using row operations only. Thus,

$$[A \mid B] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U \mid B']$$

where [A | B] is the augmented matrix.

8.2.4 Ill-Conditioned Linear Systems

In practical applications, one encounters systems in which small changes in the coefficients of the system or right-hand side terms result in very large changes in the solution. Such systems are said to be **Ill-Conditioned**. If the corresponding changes in the solution are also small, then the system is **Well-Conditioned**.

Ill-conditioning can usually be expected when the determinant of the coefficient matrix in the system AX = B is very small in magnitude. When such a system is encountered then method of improving the accuracy is employed by means of working all the calculations to more number of significant digits. Iterative method of improving accuracy of the approximate solution can also be used.

For example, consider the system

$$\begin{aligned} x + 3y &= 4 \\ 2x + 5.9y &= 6 \end{aligned} \quad \text{...(1)}$$

Here the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 3 \\ 2 & 5.9 \end{vmatrix} = -0.1$$

The solution of the system is x = – 56, y = 20. If we alter the first coefficient slightly in the first equation of (1) and consider the system

$$\begin{aligned} 1.1x + 3y &= 4 \\ 2x + 5.9y &= 6 \end{aligned} \quad \text{...(2)}$$

then it is seen that the solution of this new system is x = 11.4286, y = – 2.857 which is far away from the earlier solution x = – 56, y = 20 of the system (1).

Another example of ill-conditioned system is the system

$$\begin{aligned} 2x + y &= 2 \\ 2x + 1.01y &= 2.01 \end{aligned} \quad \text{... (3)}$$

which has the solution $x = \frac{1}{2}$, y = 1, whereas the system,

$$\begin{aligned} 2x + y &= 2 \\ 2.01x + y &= 2.05 \end{aligned} \quad \text{... (4)}$$

has the solution x = 5, y = – 8.

Geometrically, equations of the systems (1) or (2) and (3) or (4) represent straight lines and solution of these systems, say (1) is the point of intersection of two straight lines. When determinant of the coefficient matrix is very small, the straight lines are almost parallel and for this reason approximate solution can be very deceptive. In system of larger size with more unknowns, the small value of the coefficient determinant means the equations tend to be linearly dependent and hence the existence of large number of approximate solutions. Solution of such systems are mostly physically meaningless.

ILLUSTRATIONS

Ex. 1 : *Solve the following system of equations by Gauss elimination method*

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 4 \\ x_1 + 4x_2 - 2x_3 &= 4 \\ 3x_1 + 2x_2 - 4x_3 &= 6 \end{aligned}$$

Sol. : The given system is

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 4 \\ x_1 + 4x_2 - 2x_3 &= 4 \\ 3x_1 + 2x_2 - 4x_3 &= 6 \end{aligned} \quad \text{... (1)}$$

Step 1 : Dividing the first equation of (1) by 4, we get

$$x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 = 1 \quad \text{... (2)}$$

Using equation (2), we now eliminate x_1 from second and third equations by subtracting equation (2) from second equation and 3 times from third equation of (1), we obtain

$$\begin{aligned} \frac{15}{4}x_2 - \frac{9}{4}x_3 &= 3 \\ \frac{5}{4}x_2 - \frac{19}{4}x_3 &= 3 \end{aligned} \quad \text{... (3)}$$

Step 2 : Dividing the first equation of (3) by $\frac{15}{4}$, we get

$$x_2 - \frac{3}{5}\ x_3 = \frac{4}{5} \qquad \text{... (4)}$$

Using equation (4), we next eliminate x_2 from second equation of (3) by subtracting 5/4 times (4) from second equation of (3), we get

$$-4x_3 = 2$$

Or
$$x_3 = -\frac{1}{2} \qquad \text{... (5)}$$

Collecting equations (2), (4) and (5), we obtain

$$\begin{aligned} x_1 + \frac{1}{4}\ x_2 + \frac{1}{4}\ x_3 &= 1 \\ x_2 - \frac{3}{5}\ x_3 &= \frac{4}{5} \\ x_3 &= -\frac{1}{2} \end{aligned} \qquad \text{... (6)}$$

The system (6) is upper triangular system. Using backward substitution, the solution is $x_1 = 1$, $x_2 = \frac{1}{2}$ and $x_3 = -\frac{1}{2}$

Ex. 2 : *Use Gauss elimination method to solve the following system of equations*

$$\begin{aligned} x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 6x_3 &= -12 \\ 3x_1 - x_2 - x_3 &= 4 \end{aligned}$$

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Sol. : Given system is

$$\begin{aligned} x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 6x_3 &= -12 \\ 3x_1 - x_2 - x_3 &= 4 \end{aligned} \qquad \text{... (1)}$$

Step 1 : Since coefficient of x_1 is unity in the first equation, we eliminate x_1 from second and third equations of (1) by subtracting first equation from second equation and 3 times first equation from third equation of (1), we obtain

$$\begin{aligned} -3x_2 - 5x_3 &= -7 \\ -13x_2 + 2x_3 &= 19 \end{aligned}$$

Step 2 : Dividing the first equation (2) by – 3, we get

$$x_2 + \frac{5}{3}\ x_3 = \frac{7}{3} \qquad \text{... (3)}$$

Using equation (3), we next eliminate x_2 from second equation of (2) by adding 13 times equation (3) in second equation of (2), we get

$$\frac{71}{3}\ x_3 = \frac{148}{3}$$

Or
$$x_3 = \frac{148}{71} \qquad \text{...(4)}$$

Step 3 : Collecting first equation of (1) and equations (3) and (4), we have

$$\begin{aligned} x_1 + 4x_2 - x_3 &= -5 \\ x_2 + \frac{5}{3}x_3 &= \frac{7}{3} \\ x_3 &= \frac{148}{71} \end{aligned} \qquad \text{... (5)}$$

The system (5) is equivalent upper triangular system. Using backward substitution, the solution is

$$x_1 = \frac{117}{71},\quad x_2 = \frac{-81}{71} \quad \text{and} \quad x_3 = \frac{148}{71}$$

Ex. 3 : *Apply Gauss elimination method to solve the equations*

$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16$$

Sol. : Given system is

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 10 \\ 3x_1 + 2x_2 + 3x_3 &= 18 \\ x_1 + 4x_2 + 9x_3 &= 16 \end{aligned} \qquad \text{... (1)}$$

Step 1 : Dividing the first equation of (1) by 2, we get

$$x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 5 \qquad \text{...(2)}$$

Using (2), we now eliminate x_1 from second equation of (1) by subtracting 3 times equation (2) from second equation of (1) and since coefficient of x_1 in third equation of (1) is one, subtracting equation (2) from third equation of (1), we obtain

$$\begin{aligned} \frac{1}{2}x_2 + \frac{3}{2}x_3 &= 3 \\ \frac{7}{2}x_2 + \frac{17}{2}x_3 &= 11 \end{aligned} \qquad \text{... (3)}$$

Step 2 : Dividing the first equation of (3) by 1/2 (i.e. multiplying by 2), we get

$$x_2 + 3x_3 = 6 \qquad \text{... (4)}$$

Using (4), we next eliminate x_2 from second equation of (3) by subtracting $\frac{7}{2}$ times (4) from second equation of (3). We thus get

$$-2x_3 = -10$$

Or $$x_3 = 5 \qquad \text{... (5)}$$

Step 3 : Collecting equations (2), (4) and (5), we get

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 &= 5 \\ x_2 + 3x_3 &= 6 \\ x_3 &= 5 \end{aligned} \qquad \text{... (6)}$$

The system (6) is equivalent upper triangular system. Using backward substitution, the solution is $x_1 = 7$, $x_2 = -9$, $x_3 = 5$

Ex. 4 : *Solve the system*

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1$$
$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = 0$$
$$\frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 = 0$$

Sol. : Given system is

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &= 1 \\ \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 &= 0 \\ \frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 &= 0 \end{aligned} \qquad \text{... (1)}$$

Step 1 : Since coefficient of x_1 in the first equation of the given system is unity, we eliminate x_1 from second and third equations by multiplying first equation of (1) by $\frac{1}{2}$ and $\frac{1}{3}$ and then subtracting from second and third equations respectively, we get

$$\begin{aligned} \frac{1}{12}x_2 + \frac{1}{12}x_3 &= -\frac{1}{2} \\ \frac{1}{12}x_2 + \frac{4}{45}x_3 &= -\frac{1}{3} \end{aligned} \qquad \text{... (2)}$$

Step 2 : Divide the first equation by $\frac{1}{12}$ i.e. multiply by 12, we get

$$x_2 + x_3 = -6 \qquad \text{... (3)}$$

Using equation (3), we next eliminate x_2 from second equation of (2), we get

$$\frac{1}{180} x_3 = \frac{1}{6}$$

Or
$$x_3 = 30 \qquad \text{... (4)}$$

Now collecting first equation of system (1), equation (3) and equation (4), we obtain

$$\begin{aligned} x_1 + \frac{1}{2} x_2 + \frac{1}{3} x_3 &= 1 \\ x_2 + x_3 &= -6 \\ x_3 &= 30 \end{aligned} \qquad \text{... (5)}$$

which is equivalent upper triangular system. Using backward substitution, the solution is
$x_1 = 9, \; x_2 = -36, \; x_3 = 30.$

Ex. 5 : *Solve the following system of equations using Gauss elimination method*

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 3x_2 + 4x_3 &= 20 \\ 2x_1 + x_2 + 3x_3 &= 13 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 3x_2 + 4x_3 &= 20 \\ 2x_1 + x_2 + 3x_3 &= 13 \end{aligned} \qquad \text{...(1)}$$

Step 1 : Since coefficient of x_1 in first equation is unity, we eliminate x_1 from second and third equations of (1) by subtracting 3 times first equation from second equation and 2 times first equation from third equation

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 &= 2 \\ -x_2 + x_3 &= 1 \end{aligned} \qquad \text{... (2)}$$

Step 2 : The pivot in the second equation of (2) is zero and so we cannot proceed as usual. Hence we interchange the second equation and third equation of (2) before second step. We thus obtain

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ -x_2 + x_3 &= 1 \\ x_3 &= 2 \end{aligned} \qquad \text{...(3)}$$

Step 3 : The system (3) is an upper triangular system which has the solution
$x_1 = 3, \; x_2 = 1$ and $x_3 = 2.$

Ex. 6 : *Use Gauss elimination method to solve the following system of equations*

$$\begin{aligned} 10x_1 - 7x_2 + 3x_3 + 5x_4 &= 6 \\ -6x_1 + 8x_2 - x_3 - 4x_4 &= 5 \\ 3x_1 + x_2 + 4x_3 + 11x_4 &= 2 \\ 5x_1 - 9x_2 - 2x_3 + 4x_4 &= 7 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} 10x_1 - 7x_2 + 3x_3 + 5x_4 &= 6 \\ -6x_1 + 8x_2 - x_3 - 4x_4 &= 5 \\ 3x_1 + x_2 + 4x_3 + 11x_4 &= 2 \\ 5x_1 - 9x_2 - 2x_3 + 4x_4 &= 7 \end{aligned} \qquad \text{...(1)}$$

Step 1 : Dividing the first equation of (1) by 10, we get

$$x_1 - 0.7 x_2 + 0.3 x_3 + 0.5 x_4 = 0.6 \qquad \text{... (2)}$$

Using (2), we now eliminate x_1 from the second, third and fourth equations by subtracting – 6 times equation (2) from second equation (equivalently adding 6 times (2) in second equation), 3 times equation (2) from third equation and 5 times equation (2) from fourth equation of system (1), we obtain

$$\begin{aligned} 3.8 x_2 + 0.8 x_3 - x_4 &= 8.6 \\ 3.1 x_2 + 3.1 x_3 + 9.5 x_4 &= 0.2 \\ -5.5 x_2 - 3.5 x_3 + 1.5 x_4 &= 4 \end{aligned} \qquad \text{... (3)}$$

Step 2 : Dividing the first equation of (3) by 3.8, we get

$$x_2 + 0.21053\, x_3 - 0.26316\, x_4 = 2.26316 \quad \text{...(4)}$$

Using (4), we next eliminate x_2 from second and third equations of (3).

Subtracting 3.1 times (4) from second equation and – 5.5 times (4) from third equation (equivalently adding 5.5 times (4) in third equation) of system (3), we have

$$\begin{aligned} 2.44736\, x_3 + 10.31580\, x_4 &= -6.81580 \\ -2.34209\, x_3 + 0.05262\, x_4 &= 16.44738 \end{aligned} \quad \text{...(5)}$$

Step 3 : Dividing the first equation of (5) by 2.44736, we get

$$x_3 + 4.21507\, x_4 = -2.78496 \quad \text{... (6)}$$

Using (6), we finally eliminate x_3 by adding 2.34209 times (6) in the second equation of (5),

$$\begin{aligned} 9.92469\, x_4 &= 9.92475 \\ x_4 &= 1 \text{ (correct to five decimal places)} \end{aligned} \quad \text{... (7)}$$

Step 4 : Collecting equations (2), (4), (6) and (7), we get

$$\begin{aligned} x_1 - 0.7\, x_2 + 0.3\, x_3 + 0.5\, x_4 &= 0.6 \\ x_2 + 0.21053\, x_3 - 0.26316\, x_4 &= 2.26316 \\ x_3 + 4.21507\, x_4 &= -2.78496 \\ x_4 &= 1 \end{aligned} \quad \text{... (8)}$$

The system (8) is equivalent upper triangular system. Using backward substitution, the solution is $x_1 = 5$, $x_2 = 4$, $x_3 = -7$, $x_4 = 1$.

Ex. 7 : *Solve the following equations*

$$\begin{aligned} 1.2\, x_1 + 2.1\, x_2 - 1.1\, x_3 + 4\, x_4 &= 6 \\ -1.1\, x_1 + 2.0\, x_1 + 3.1\, x_3 + 3.9\, x_4 &= 3.9 \\ -2.1\, x_1 - 2.2\, x_2 + 3.7\, x_3 + 16.0\, x_4 &= 12.2 \\ -1.0\, x_1 - 2.3\, x_2 + 4.7\, x_3 + 12.0\, x_4 &= 4.0 \end{aligned}$$

by Gauss elimination method.

Sol. : Given system is :

$$\begin{aligned} 1.2\, x_1 + 2.1\, x_2 - 1.1\, x_3 + 4\, x_4 &= 6 \\ -1.1\, x_1 + 2.0\, x_1 + 3.1\, x_3 + 3.9\, x_4 &= 3.9 \\ -2.1\, x_1 - 2.2\, x_2 + 3.7\, x_3 + 16.0\, x_4 &= 12.2 \\ -1.0\, x_1 - 2.3\, x_2 + 4.7\, x_3 + 12.0\, x_4 &= 4.0 \end{aligned} \quad \text{... (1)}$$

Step 1 : Divide the first equation of (1) by 1.2, thus

$$x_1 + 1.75\, x_2 - 0.9167 x_3 + 3.3333\, x_4 = 5. \quad \text{... (2)}$$

Using (2), we now eliminate x_1 from second, third and fourth equations by adding 1.1 times (2) in second equation, 2.1 times (2) in third equation and 1.0 times (2) in fourth equation of system (1), we obtain

$$\begin{aligned} 3.925\, x_2 + 2.0916\, x_3 + 7.5666\, x_4 &= 9.4 \\ 1.475\, x_2 + 1.7749\, x_3 + 22.9999\, x_4 &= 22.7 \\ -0.55\, x_2 + 3.7833\, x_3 + 15.3333\, x_4 &= 9.0 \end{aligned} \quad \text{... (3)}$$

Step 2 : Dividing the first equation of (3) by 3.925, we get

$$x_2 + 0.5329\, x_3 + 1.9278\, x_4 = 2.3949 \quad \text{... (4)}$$

Using (4), we next eliminate x_2 from second and third equations of (4) by subtracting 1.475 times (4) from second equation of (3) and adding 0.55 times (4) in third equation of (3), we get

$$\begin{aligned} 0.9889\, x_3 + 20.1564\, x_4 &= 19.1675 \\ 4.0764\, x_3 + 16.3936\, x_4 &= 10.3172 \end{aligned} \quad \text{... (5)}$$

Step 3 : Dividing the first equation of (5) by 0.9889, we get

$$x_3 + 20.3827\, x_4 = 19.3827 \quad \text{... (6)}$$

Using (6), finally we eliminate x_3 from second equation of (5) by subtracting 4.0764 times (6) from second equation of (5). Thus

$$-66.6944\, x_4 = -68.6944$$

Or

$$x_4 = 1.0300 \quad \text{... (7)}$$

Step 4 : Collecting equations (2), (4), (6) and (7),

$$\begin{aligned} x_1 + 1.75\, x_2 - 0.9167\, x_3 + 3.3333\, x_4 &= 5 \\ x_2 + 0.5329\, x_3 + 1.9278\, x_4 &= 2.3949 \\ x_3 + 20.3827\, x_4 &= 19.3827 \\ x_4 &= 1.0300 \end{aligned} \qquad \text{... (8)}$$

The system (8) is upper triangular system. Using backward substitution, the solution is $x_1 = -2.1298$, $x_2 = 1.2681$, $x_3 = -1.6115$ and $x_4 = 1.0300$.

Ex. 8 : *Solve the following system by Gauss elimination method*

$$\begin{aligned} 7.9\, x_1 + 5.6\, x_2 + 5.7\, x_3 - 7.2\, x_4 &= 6.68 \\ 8.5\, x_1 - 4.8\, x_2 + 0.8\, x_3 + 3.5\, x_4 &= 9.95 \\ 4.3\, x_1 + 4.2\, x_2 - 3.2\, x_3 + 9.3\, x_4 &= 8.6 \\ 3.2\, x_1 - 1.4\, x_2 - 8.9\, x_3 + 3.3\, x_4 &= 1 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} 7.9\, x_1 + 5.6\, x_2 + 5.7\, x_3 - 7.2\, x_4 &= 6.68 \\ 8.5\, x_1 - 4.8\, x_2 + 0.8\, x_3 + 3.5\, x_4 &= 9.95 \\ 4.3\, x_1 + 4.2\, x_2 - 3.2\, x_3 + 9.3\, x_4 &= 8.6 \\ 3.2\, x_1 - 1.4\, x_2 - 8.9\, x_3 + 3.3\, x_4 &= 1 \end{aligned} \qquad \text{...(1)}$$

Step 1 : Divide the first equation of (1) by 7.9, thus

$$x_1 + 0.7089\, x_2 + 0.7215\, x_3 - 0.9114\, x_4 = 0.8456 \qquad \text{... (2)}$$

Using (2), we now eliminate x_1 from second, third and fourth equations of (1) by subtracting 8.5 times (2) from second equation, 5.3 times (2) from third equation and 3.2 times (2) from third equation of system (1). We obtain

$$\begin{aligned} -10.8257\, x_2 - 5.3328\, x_3 + 11.2469\, x_4 &= 2.7624 \\ 1.1517\, x_2 - 6.3025\, x_3 + 13.2302\, x_4 &= 4.9639 \\ -3.6685\, x_2 - 11.2088\, x_3 + 6.2165\, x_4 &= -1.7060 \end{aligned} \qquad \text{... (3)}$$

Step 2 : Dividing the first equation of (3), we get

$$x_2 + 0.4926\, x_3 - 1.0389\, x_4 = -0.2552 \qquad \text{... (4)}$$

Using (4), we next eliminate x_2 from second and third equations of (3) by subtracting 1.1517 times (4) from second equation and adding 3.6685 times (4) in the third equation of (3), we obtain

$$\begin{aligned} -6.8698\, x_3 + 14.4267\, x_4 &= 5.2578 \\ -9.4017\, x_3 + 2.4053\, x_4 &= -2.6422 \end{aligned} \qquad \text{... (5)}$$

Step 3 : Dividing the first equation of (5) by – 6.8698, we get

$$x_3 - 2.1000\, x_4 = -0.7653 \qquad \text{... (6)}$$

Using (6), we finally eliminate x_3 by adding 9.4017 times (6) in the second equation of (5). We get

$$\begin{aligned} -17.3383\, x_4 &= -9.8373 \\ x_4 &= 0.5673 \end{aligned} \qquad \text{... (7)}$$

Step 4 : Collecting equations (2), (4), (6) and (7),

$$\begin{aligned} x_1 + 0.7089\, x_2 + 0.7215\, x_3 - 0.9114\, x_4 &= 0.8456 \\ x_2 + 0.4926\, x_3 - 1.0389\, x_4 &= -0.2552 \\ x_3 - 2.1\, x_4 &= -0.7653 \\ x_4 &= 0.5673 \end{aligned} \qquad \text{... (8)}$$

Using backward substitution, the solution is $x_1 = 0.9672$, $x_2 = 0.1243$, $x_3 = 0.4260$, $x_4 = 0.5673$.

Note : All calculations are made with four significant digits accuracy.

Ex. 9 : *Use Gauss elimination with partial pivoting to solve the following system of linear equations :*

$$\begin{aligned} 8x_2 + 2x_3 &= -7 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 6x_1 + 2x_2 + 8x_3 &= 26 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} 8x_2 + 2x_3 &= -7 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 6x_1 + 2x_2 + 8x_3 &= 26 \end{aligned} \qquad \ldots (1)$$

Since first equation has no x_1 term and the third equation has numerically largest coefficient of x_1, hence we interchange first equation and third equation as the first stage of partial pivoting.

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 8x_2 + 2x_3 &= -7 \end{aligned} \qquad \ldots (2)$$

Here first equation is the pivotal equation and element 6 is the first pivot element.

Step 1 : Dividing first equation of (2) by 6 (pivot), we get

$$x_1 + \frac{1}{3} x_2 + \frac{4}{3} x_3 = \frac{13}{3} \qquad \ldots (3)$$

Using (3), we eliminate x_1 from second equation by substituting 3 times (3) from second equation of (2),

$$\begin{aligned} 4x_2 - 2x_3 &= -5 \\ 8x_2 + 2x_3 &= -7 \end{aligned} \qquad \ldots (4)$$

Step 2 : The largest coefficient of x_2 is 8 which appears in second equation of (4). Hence we take the new second equation of (4) as the pivotal equation. Interchanging first equation and second equation, we have

$$\begin{aligned} 8x_2 + 2x_3 &= -7 \\ 4x_2 - 2x_3 &= -5 \end{aligned} \qquad \ldots (5)$$

Dividing first equation of (5) by 8 (pivot), we get

$$x_2 + \frac{1}{4} x_3 = -\frac{7}{8} \qquad \ldots (6)$$

Using (6), we eliminate x_2 from second equation by subtracting 4 times (6) from second equation of (6),

$$-3x_3 = \frac{-3}{2}$$

Or $$x_3 = \frac{1}{2} \qquad \ldots (7)$$

Step 3 : Collecting equations (3), (6) and (7), which is the upper triangular system,

$$\begin{aligned} x_1 + \frac{1}{3} x_2 + \frac{4}{3} x_3 &= \frac{13}{3} \\ x_2 + \frac{1}{4} x_3 &= -\frac{7}{8} \\ x_3 &= \frac{1}{2} \end{aligned} \qquad \ldots (8)$$

Using backward substitution, solution is $x_1 = 4$, $x_2 = -1$, $x_3 = \frac{1}{2}$.

Ex. 10 : *Solve the following linear system of equations by Gauss elimination with partial pivoting :*

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 9x_4 &= 5 \\ 3x_1 + 10x_2 + 4x_3 + 2x_4 &= 7 \\ 11x_1 + 5x_2 + 9x_3 + 2x_4 &= 13 \\ 2x_1 + 3x_2 + 7x_3 + 6x_4 &= 11 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 9x_4 &= 5 \\ 3x_1 + 10x_2 + 4x_3 + 2x_4 &= 7 \\ 11x_1 + 5x_2 + 9x_3 + 2x_4 &= 13 \\ 2x_1 + 3x_2 + 7x_3 + 6x_4 &= 11 \end{aligned} \quad \text{... (1)}$$

The third equation has numerically largest coefficient. Hence interchanging first equation and third equation, we get the new system of equations as follows :

$$\begin{aligned} 11x_1 + 5x_2 + 9x_3 + 2x_4 &= 13 \\ 3x_1 + 10x_2 + 4x_3 + 2x_4 &= 7 \\ x_1 - 2x_2 + 3x_3 + 9x_4 &= 5 \\ 2x_1 + 3x_2 + 7x_3 + 6x_4 &= 11 \end{aligned} \quad \text{... (2)}$$

Here first equation is the pivotal equation and the element 11 is the first pivot element.

Step 1 : Dividing first equation of (2) by 11 (pivot), we get

$$x_1 + 0.4545\, x_2 + 0.8182\, x_3 + 0.1818\, x_4 = 1.1818 \quad \text{... (3)}$$

Using (3), we now eliminate x_1 from second, third and fourth equations by subtracting 3 times (3) from second equation, equation (3) from third equation and 2 times (3) from fourth equations. We obtain

$$\begin{aligned} 8.6365\, x_2 + 1.5454\, x_3 + 1.4546\, x_4 &= 3.4545 \\ -2.4545\, x_2 + 2.1818\, x_3 + 8.8182\, x_4 &= 3.8181 \\ 2.0910\, x_2 + 5.3636\, x_3 + 5.6364\, x_4 &= 8.6363 \end{aligned} \quad \text{... (4)}$$

Step 2 : Since coefficient of x_1 in the first equation of (4) is the largest, therefore, no pivoting is required.

Dividing the first equation of (4) by 8.6363, we get

$$x_2 + 0.1789\, x_3 + 0.1684\, x_4 = 0.4 \quad \text{... (5)}$$

Using (5), we next eliminate x_2 from second and third equations of (4),

$$\begin{aligned} 2.6209\, x_3 + 9.2315\, x_4 &= 4.7999 \\ 4.9895\, x_3 + 5.2842\, x_4 &= 7.7999 \end{aligned} \quad \text{...(6)}$$

Step 3 : Since largest coefficient of x_3 appeared in second equation of (6), we interchange first and second equations.

$$\begin{aligned} 4.9895\, x_3 + 5.2842\, x_4 &= 7.7999 \\ 2.6209\, x_3 + 9.2315\, x_4 &= 4.7999 \end{aligned} \quad \text{...(7)}$$

Dividing the first equation of (7) by 4.9895, we get

$$x_3 + 1.0591\, x_4 = 1.5633 \quad \text{... (8)}$$

Step 4 : Using (8), eliminate x_3 from second equation of (7),

$$6.4557\, x_4 = 0.7026$$

Or

$$x_4 = 0.1088 \quad \text{... (9)}$$

Step 5 : Collecting equations (3), (5), (8) and (9), we obtain

$$\begin{aligned} x_1 + 0.4545\, x_2 + 0.8182\, x_3 + 0.1818\, x_4 &= 1.1818 \\ x_2 + 0.1789\, x_3 + 0.1684\, x_4 &= 0.4000 \\ x_3 + 1.0591\, x_4 &= 1.5633 \\ x_4 &= 0.1088 \end{aligned} \quad \text{... (10)}$$

System (10) is upper triangular system. Using backward substitution, the required solution is

$x_1 = 0.0785,\ x_2 = 0.1226,\ x_3 = 1.4481,\ x_4 = 0.1088.$

Note : All calculations are made with four significant digits accuracy.

8.2.5 Gauss Jordan Method

This is a modification of the Gauss elimination method. In this method, after first elimination at all steps, the elimination of unknowns is performed not only equations below but also in the equations above. In this way, we finally obtain identity (or diagonal) matrix form of the system, producing the solution directly without using backward substitution method. This modification of Gauss elimination is called the **Gauss Jordan Method**.

Remark 1 : Gauss Jordan method in matrix form can be interpreted as reducing the augmented matrix [A | B] to identity matrix form [I | B'] by applying successive row transformations only. Thus,

$$[A \mid B] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I \mid B']$$

where, [A | B] is augmented matrix.

Remark 2 : Gauss Jordan method is more expensive from the computational point of view than the Gauss elimination method and it is generally not used for solving system of simultaneous linear equations. However, it is simple technique for finding the inverse of a given matrix A. Inverse of a matrix can be obtain by converting augmented matrix [A | I] to the form $[I \mid A^{-1}]$ by applying successive row transformations.

Thus,

$$[A \mid I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I \mid A^{-1}]$$

where, I is the identity matrix.

ILLUSTRATIONS

Ex. 1 : *Apply Gauss Jordan method to solve*

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8 \\ 2x_1 + 3x_2 + 4x_3 &= 20 \\ 4x_1 + 3x_2 + 2x_3 &= 16 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8 \\ 2x_1 + 3x_2 + 4x_3 &= 20 \\ 4x_1 + 3x_2 + 2x_3 &= 16 \end{aligned} \qquad \ldots (1)$$

Step 1 : Using first equation, we eliminate x_1 from second and third equations of (1) by subtracting 2 times first equation from second and 4 times from third equation, we get

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8 \\ -x_2 + 2x_3 &= 4 \\ -5x_2 - 2x_3 &= -16 \end{aligned} \qquad \ldots (2)$$

Step 2 : Using second equation, we now eliminate x_2 from first and third equations of (2) by adding second equation 2 times to first equation and subtracting 5 times to third equation, we get

$$\begin{aligned} x_1 + 0 + 5x_3 &= 16 \\ -x_2 + 2x_3 &= 4 \\ 0 - 12x_3 &= -36 \end{aligned} \qquad \ldots (3)$$

Step 3 : Using third equation, we next eliminate x_3 from first and second equations by adding $\frac{1}{6}$ times third equation in second equation and $\frac{5}{12}$ times third equation in the first equation, we obtain

$$\begin{aligned} x_1 &= 1 \\ -x_2 &= -2 \\ -12x_3 &= -36 \end{aligned}$$

This gives the solution $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

Ex. 2 : *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

using Gauss Jordan method and hence solve the system

$$x_1 + x_2 + x_3 = 4$$
$$4x_1 + 3x_2 - x_3 = 12$$
$$3x_1 + 5x_2 + 3x_3 = 15$$

Sol. : Consider the augmented matrix [A | I] as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_2 - 4R_1,\ R_3 - 3R_1}{}} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_1 + R_2,\ R_3 + 2R_2}{}} \begin{bmatrix} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{bmatrix} \underset{\sim}{\overset{R_2 - 1/2\, R_3,\ R_1 - 2/5\, R_3}{}} \begin{bmatrix} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & -1 & 0 & 3/2 & 0 & -1/2 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_3(-1/10),\ R_2(-1)}{}} \begin{bmatrix} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{bmatrix} = [I \mid A^{-1}]$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix}$$

Hence the solution of the system is obtain as

$$X = A^{-1} Y$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 15 \end{bmatrix}$$

Thus $x_1 = 2,\ x_2 = \frac{3}{2}$ and $x_3 = \frac{1}{2}$.

Ex. 3 : *Find the inverse of the coefficient matrix of the system*

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

by the Gauss Jordan method with partial pivoting and hence solve the system.

Sol. : Consider the augmented matrix [A | I] as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_{12}}{}} \begin{bmatrix} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix}$$

[The largest coefficient in column 1 is 4. Hence we interchange first row and second row.]

$$\underset{\sim}{\overset{R_1(1/4)}{}} \begin{bmatrix} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_2 - R_1,\ R_3 - 3R_1}{}} \begin{bmatrix} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 5/4 & 0 & -1/4 & 0 \\ 0 & 11/4 & 15/4 & 1 & -3/4 & 1 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_{23}}{}} \begin{bmatrix} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 11/4 & 15/4 & 0 & -3/4 & 1 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \end{bmatrix} \underset{\sim}{\overset{R_2(4/11)}{}} \begin{bmatrix} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \end{bmatrix}$$

[The largest coefficient in column 2 is 11/4. Hence we interchange second row and third row.]

$$\underset{\sim}{\overset{R_1 - 3/4\,R_2,\; R_3 - 1/4R_2}{}} \begin{bmatrix} 1 & 0 & -14/11 & 0 & 5/11 & -3/11 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 0 & 10/11 & 1 & -2/11 & -1/11 \end{bmatrix} \underset{\sim}{\overset{R_3\,(11/10)}{}} \begin{bmatrix} 1 & 0 & -14/11 & 0 & 5/11 & -3/11 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_1 + 14/11\,R_3,\; R_2 - 15/11\,R_3}{}} \begin{bmatrix} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{bmatrix} = [I|A^{-1}]$$

Therefore the solution of the system is $X = A^{-1}B$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Thus, $x_1 = 1$, $x_2 = 1/2$, $x_3 = -1/2$.

Ex. 4 : *Solve the following system of equations using Gauss Jordan method :* ***(Dec. 2009)***

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10$$

Sol. : Consider the augmented matrix [A | B] as

$$\begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix} \underset{\sim}{\overset{R_1\,(1/2)}{}} \begin{bmatrix} 1 & 1/2 & 1 & 1/2 & 3 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_2 - 6R_1,\; R_3 - 4R_1,\; R_4 - 2R_1}{}} \begin{bmatrix} 1 & 1/2 & 1 & 1/2 & 3 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{bmatrix} \underset{\sim}{\overset{R_2\,(-1/9)}{}} \begin{bmatrix} 1 & 1/2 & 1 & 1/2 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_1 - (1/2)\,R_2,\; R - R_2,\; RR_2}{}} \begin{bmatrix} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & -3 & 1 & 6 \end{bmatrix} \underset{\sim}{\overset{R_3\,(-1)}{}} \begin{bmatrix} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & -3 & 1 & 6 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_4 + 3R_3,\; R_1 - R_3}{}} \begin{bmatrix} 1 & 0 & 0 & -3 & -7 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & 13 & 39 \end{bmatrix} \underset{\sim}{\overset{R_4\,(1/13)}{}} \begin{bmatrix} 1 & 0 & 0 & -3 & -7 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_1 + 3R_4,\; R_2 + R_4,\; R_3 - 4R_4}{}} \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} = [I \mid B']$$

Here we get solution of the system as $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = B' = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$ i.e. $x_1 = 2$, $x_2 = 1$, $x_3 = -1$, $x_4 = 3$.

8.2.6 Triangular Factorization (LU Decomposition Method)

We consider for definiteness, the linear system of three equations in three variables.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned} \qquad \dots (1)$$

The above system can be written in matrix form as

$$AX = B \qquad \dots (2)$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

In this method, the coefficient matrix A of the system of equation (2) is decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U.

We write as $A = LU$ ... (3)

where, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Hence (2) becomes

$$LUX = B \qquad \dots (4)$$

If we set

$$UX = Z \qquad \dots (5)$$

then (4) can be written as

$$LZ = B \qquad \dots (6)$$

which is equivalent to the system

$$\begin{aligned} z_1 &= b_1 \\ l_{21} z_1 + z_2 &= b_2 \\ l_{31} z_1 + l_{32} z_2 + z_3 &= b_3 \end{aligned}$$

and can therefore be solved for z_1, z_2, z_3 by the forward substitution. When Z is known, the system (5) becomes

$$\begin{aligned} u_{11}x_1 + u_{12} x_2 + u_{13} x_3 &= z_1 \\ u_{22} x_2 + u_{23} x_3 &= z_2 \\ u_{33} x_3 &= z_3 \end{aligned}$$

from which x_1, x_2, x_3 are obtained by backward substitution.

We shall now explain the procedure for finding the elements of the matrices L and U.

We write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and comparing the elements of resulting matrix with those of A, we obtain

(i) $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$,

(ii) $l_{21} u_{11} = a_{21} \Rightarrow l_{21} = \dfrac{a_{21}}{a_{11}}$, $l_{31} u_{11} = a_{31} \Rightarrow l_{31} = \dfrac{a_{31}}{a_{11}}$,

(iii) $l_{21} u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} a_{12}$,

$l_{21} u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}} a_{13}$,

(iv) $l_{31} u_{12} + l_{32} u_{22} = a_{32} \quad \Rightarrow \quad l_{32} = \frac{1}{u_{22}} \left[a_{32} - \frac{a_{33}}{a_{11}} a_{12} \right]$,

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = a_{33}$, from which u_{33} can be computed.

Thus we compute the elements of L and U in the following set of order : First, we determine the first row of U and the first column of L; then we determine the second row of U and the second column of L, and finally we compute the third row of U. This procedure can easily be generalised.

Remark 1 : The inverse of A can also be determined from the formula

$$A^{-1} = (LU)^{-1} = U^{-1} L^{-1} \qquad \text{... (7)}$$

Remark 2 : In general, matrices L and U which together involve $n^2 + n$ unknowns while A = LU provides only n^2 equations (n = 3 in our case). To overcome this problem and for unique solution, it is convenient to choose diagonal elements of L (or of U) as 1.

ILLUSTRATIONS

Ex. 1 : *Solve the system of equations by triangular factorization method :*

$$2x_1 + 2x_2 + 3x_3 = 4$$
$$4x_1 - 2x_2 + x_3 = 9$$
$$x_1 + 5x_2 + 4x_3 = 3$$

Sol. : Given system can be written in the matrix form as

$$AX = B \qquad \text{... (1)}$$

where, $A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & -2 & 1 \\ 1 & 5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$

Now, we express $A = LU$ ... (2)

$$\begin{bmatrix} 2 & 2 & 3 \\ 4 & -2 & 1 \\ 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Multiplying the right hand side and equating the elements in corresponding position, we get

(i) $u_{11} = 2, \quad u_{12} = 2, \quad u_{13} = 3$

(ii) $l_{21} u_{11} = 4 \Rightarrow l_{21} = \frac{4}{2} = 2, \; l_{31} u_{11} = 1 \Rightarrow l_{31} = \frac{1}{2}$

(iii) $l_{21} u_{12} + u_{22} = -2$, and $l_{21} u_{13} + u_{23} = 1$

$\therefore \; u_{22} = -2 - (2)(2) = -6$ and $u_{23} = 1 - (2)(3) = -5$

(iv) $l_{31} u_{12} + l_{32} u_{22} = 5$ and $l_{31} u_{13} + l_{32} u_{23} + u_{33} = 4$

$\therefore \; l_{32} = -\frac{1}{6}\left(5 - \frac{1}{2}(2)\right) = -\frac{2}{3}$ and $u_{33} = 4 - \frac{1}{2}(3) - \left(-\frac{2}{3}\right)(-5) = -\frac{5}{6}$.

Thus we have,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & -2/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5/6 \end{bmatrix} \qquad \text{... (3)}$$

Given system (1) can be written as

$$LUX = B \qquad \text{...(4)}$$

Let $UX = Z \quad \therefore \; LZ = B$, where $Z = [z_1, z_2, z_3]^T$

$LZ = B$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix} \qquad \text{... (5)}$$

which is equivalent to

$$z_1 = 4, \quad 2z_1 + z_2 = 9, \quad \frac{1}{2} z_1 - \frac{2}{3} z_2 + z_3 = 3$$

Using forward substitution, we obtain

$$z_1 = 4, \quad z_2 = 1, \text{ and } z_3 = 5/3$$

Hence the solution of original system is given by, UX = Z as

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5/3 \end{bmatrix} \quad \text{... (6)}$$

which is equivalent to

$2x_1 + 2y_1 + 3x_3 = 4, \quad -6x_2 - 5x_3 = 1, \quad -5/6\, x_3 = 5/3.$

Solving using backward substitution, we have

$x_1 = \frac{-7}{2}, \; x_2 = \frac{3}{2}, \quad x_3 = -2.$

Ex. 2 : *Solve the equations by LU decomposition method :*

$$2x_1 + 3x_2 + x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$

Sol. : Given system of equations can be written in matrix form as

$$AX = B \quad \text{... (1)}$$

where, $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Let $A = LU$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Multiplying the right hand side and equating the elements in corresponding position, we get

(i) $u_{11} = 2, \quad u_{12} = 3, \quad u_{13} = 1$

(ii) $l_{21} u_{11} = 1 \Rightarrow l_{21} = 1/2, \quad l_{31} u_{11} = 3 \Rightarrow l_{31} = \frac{3}{2}$

(iii) $l_{21} u_{12} + u_{22} = 2$ and $l_{21} u_{13} + u_{23} = 3$

$\therefore \; u_{22} = 2 - \left(\frac{1}{2}\right)(3) = \frac{1}{2}$ and $u_{23} = 3 - \left(\frac{1}{2}\right)(1) = \frac{5}{2}$.

(iv) $l_{31} u_{12} + l_{32} u_{22} = 1$ and $l_{31} u_{13} + l_{32} u_{23} + u_{33} = 2,$

$\therefore \; l_{32} = 2\left(1 - \left(\frac{3}{2}\right)(3)\right) = -7$ and $u_{33} = 2 - \left(\frac{3}{2}\right)(1) - (-7)\left(\frac{5}{2}\right) = 18$

It follows that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \quad \text{... (2)}$$

Given system (1) can be written as

$$LUX = B \quad \text{...(3)}$$

Let $UX = Z \quad \therefore \quad LZ = B$, where $Z = [z_1\, z_2\, z_3]^T$

$LZ = B$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad \text{... (4)}$$

Using forward substitution, we have

$$z_1 = 9, \quad z_2 = \frac{3}{2} \quad \text{and} \quad z_3 = 5$$

Hence the solution of original system is given by $UX = Z$ as

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix} \qquad \ldots (5)$$

Using backward substitution, we get $x_1 = \frac{35}{8}, \; x_2 = \frac{29}{18}, \; x_3 = \frac{5}{18}$

8.2.7 Cholesky Method

This method is also known as the square-root method. For understanding this method, consider the following definitions.

- **Hermitian Matrix :** Matrix A is called Hermitian if $A = (\bar{A})^T$, where, $\bar{A}$ is complex conjugate of A. From the definition it is clear that, Real symmetric matrix is Hermitian as the complex conjugate of a real number x is equal to x.
- **Positive Definite Matrix :** A matrix A is said to be a positive definite matrix if $X^* AX > 0$ for any vector X and $X^* = (\bar{X})^T$.

We shall now discuss Cholesky's method to solve the system of equation

$$AX = B \qquad \ldots (1)$$

If the coefficient matrix A is symmetric and positive definite then the matrix A can be decomposed as

$$A = LL^T \qquad \text{where, L is lower triangular matrix}$$

Or $$A = UU^T \qquad \text{where, U is upper triangular matrix}$$

Taking $A = LL^T$, (1) becomes

$$LL^T X = B \qquad \ldots (2)$$

If we set $$L^T X = Z \qquad \ldots (3)$$

then (3) can be written as

$$LZ = B \qquad \ldots (4)$$

The intermediate solution Z is obtained from (4) by forward substitution and solution X is determined from (3) by backward substitution.

Remark 1 : The inverse matrix A can also be determined from the formula

$$A^{-1} = (LL^T)^{-1} = (L^{-1})^T L^{-1}$$

Remark 2 : The advantage of Cholesky's method over LU decomposition method is substantial saving of computational work.

Remark 3 : If A is symmetric but not positive definite, this method could still be applied but then leads to a complex matrix L, so that it becomes impractical.

ILLUSTRATIONS

Ex. 1 : *Solve the following system of equations by Cholesky's method :*

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 3x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

Sol. : Given system can be expressed in matrix form as

$$AX = B \qquad \ldots (1)$$

where $$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, we express

$$A = LL^T \qquad \dots (2)$$

i.e.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Multiplying the right hand side and equating the elements in corresponding position, we get

$$l_{11}^2 = 2 \qquad \therefore \qquad l_{11} = \sqrt{2}$$

$$l_{11} l_{21} = -1 \qquad \therefore \qquad l_{21} = \frac{-1}{\sqrt{2}}$$

$$l_{11} l_{31} = 0 \qquad \therefore \qquad l_{31} = 0$$

$$l_{21}^2 + l_{22}^2 = 3 \qquad \therefore \qquad l_{22} = \sqrt{\frac{5}{2}}$$

$$l_{21} l_{31} + l_{22} l_{32} = 1 \qquad \therefore \qquad l_{32} = \sqrt{\frac{2}{5}}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 2 \qquad \therefore \qquad l_{33} = \sqrt{\frac{8}{5}}$$

Thus, we have

$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix}$$

Given system can be written as

$$LL^T X = B \qquad \dots (3)$$

Let $L^T X = Z \quad \therefore \quad LZ = B$ where, $Z = [z_1 \, z_2 \, z_3]^T$

Now consider $LZ = B$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \dots (4)$$

which is equivalent to

$$\sqrt{2}\, z_1 = 1, \qquad \frac{-1}{\sqrt{2}} z_1 + \sqrt{\frac{5}{2}}\, z_2 = 0, \qquad \sqrt{\frac{2}{5}}\, z_2 + \sqrt{\frac{8}{5}}\, z_3 = 0$$

Using forward substitution, we obtain

$$z_1 = \frac{1}{\sqrt{2}}, \quad z_2 = \frac{1}{\sqrt{10}} \quad \text{and} \quad z_3 = \frac{-1}{\sqrt{40}}.$$

Hence the solution of the original system is given by $L^T X = Z$ as

$$\begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{5}{2}} & \sqrt{\frac{2}{5}} \\ 0 & 0 & \sqrt{\frac{8}{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{40}} \end{bmatrix} \qquad \dots (5)$$

which is equivalent to

$$\sqrt{2}\ x_1 - \frac{1}{\sqrt{2}}\ x_2 = \frac{1}{\sqrt{2}},\quad \sqrt{\frac{5}{2}}\ x_2 + \sqrt{\frac{2}{5}}\ x_3 = \frac{1}{\sqrt{10}},\quad \sqrt{\frac{8}{5}}\ x_3 = \frac{-1}{\sqrt{40}}$$

By backward substitution, we get $x_1 = \frac{5}{8},\ x_2 = \frac{1}{4},\ x_3 = -\frac{1}{8}$

Ex. 2 : *Solve the following system by Cholesky's method :*

$$\begin{aligned} 4x_1 + 2x_2 + 14x_3 &= 14 \\ 2x_1 + 17x_2 - 5x_3 &= -101 \\ 14x_1 - 5x_2 + 83x_3 &= 155. \end{aligned}$$

Sol. : Given system can be expressed in matrix form as

$$AX = B \qquad \dots (1)$$

where, $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$

Now, we express

$$A = LL^T \qquad \dots (2)$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Multiplying right hand side matrices and using matrix equality,

$$\begin{aligned} l_{11}^2 &= 4 & \therefore\quad l_{11} &= 2 \\ l_{11} l_{21} &= 2 & \therefore\quad l_{21} &= 1 \\ l_{11} l_{31} &= 14 & \therefore\quad l_{31} &= 7 \\ l_{21}^2 + l_{22}^2 &= 17 & \therefore\quad l_{22} &= 4 \\ l_{21} l_{31} + l_{22} l_{32} &= -5 & \therefore\quad l_{32} &= -3 \\ l_{31}^2 + l_{32}^2 + l_{33}^2 &= 83 & \therefore\quad l_{33} &= 5 \end{aligned}$$

Thus, we have

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$$

Given system can be written as

$$LL^T X = B \qquad \dots (3)$$

Let $L^T X = Z \quad \therefore\ LZ = B$ where $Z = [z_1\ z_2\ z_3]^T$

We now have to solve

$$LZ = B$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

Using forward substitution, we get the solution as $z_1 = 7,\ z_2 = -27,$ and $z_3 = 5$

Finally, we have to solve

$$L^T X = Z$$

i.e.
$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

Using backward substitution, the required solution is $x_1 = 3,\ x_2 = -6$ and $x_3 = 1$.

Ex. 3 : *Solve the following system by Cholesky's method :*

$$9x_1 + 6x_2 + 12x_3 = 17.4$$

$$6x_1 + 13x_2 + 11x_3 = 23.6$$

$$12x_1 + 11x_2 + 26x_3 = 30.8$$

Sol. : Given system can be expressed in matrix form as

$$AX = B \qquad \ldots (1)$$

where, $A = \begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 17.4 \\ 23.6 \\ 30.8 \end{bmatrix}$

Now, we express $A = LL^T \qquad \ldots (2)$

$$\begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Multiplying right hand side matrices and using matrix equality,

$$l_{11}^{22} = 9 \qquad \therefore \qquad l_{11} = 2$$

$$l_{11}\, l_{21} = 6 \qquad \therefore \qquad l_{21} = 2$$

$$l_{11}\, l_{31} = 12 \qquad \therefore \qquad l_{31} = 4$$

$$l_{21}^2 + l_{22}^2 = 13 \qquad \therefore \qquad l_{22} = 3$$

$$l_{21}\, l_{31} + l_{22}\, l_{32} = 1 \qquad \therefore \qquad l_{32} = 1$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 26 \qquad \therefore \qquad l_{33} = 3$$

Thus, we have

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 3 \end{bmatrix}$$

Given system can be written as

$$LL^TX = B \qquad \ldots (3)$$

Let $L^TX = Z \quad \therefore \quad LZ = B$ where, $Z = [z_1, z_2, z_3]^T$

Now we first solve, $LZ = B$

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 17.4 \\ 23.6 \\ 30.8 \end{bmatrix}$$

Using forward substitution, we get solution as $z_1 = 5.8$, $z_2 = 4$, $z_3 = 1.2$.

Finally, we now solve

$$L^TX = Z$$

i.e. $$\begin{bmatrix} 3 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 4 \\ 1.2 \end{bmatrix}$$

Using backward substitution, the required solution is $x_1 = 0.6$, $x_2 = 1.2$, $x_3 = 0.4$

Ex. 4 : *Solve the following system of equations*

$$\begin{bmatrix} 9 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

using the Cholesky's method.

Sol. : Given system is $\quad AX = B \quad$ ... (1)

where, $A = \begin{bmatrix} 9 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Now, we express, $\quad A = LL^T \quad$ ... (2)

i.e.
$$\begin{bmatrix} 9 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix}$$

Multiplying right hand side matrices and using matrix equality,

$$\begin{aligned}
l_{11}^2 &= 9 & \therefore\ l_{11} &= 3 \\
l_{11}\, l_{21} &= 1 & \therefore\ l_{21} &= \frac{1}{3} \\
l_{11}\, l_{31} &= 0 & \therefore\ l_{31} &= 0 \\
l_{11}\, l_{41} &= 0 & \therefore\ l_{41} &= 0 \\
l_{21}^2 + l_{22}^2 &= 4 & \therefore\ l_{22} &= \sqrt{\frac{35}{9}} \\
l_{21}\, l_{31} + l_{22}\, l_{32} &= 1 & \therefore\ l_{32} &= \sqrt{\frac{9}{35}} \\
l_{21}\, l_{41} + l_{22}\, l_{42} &= 0 & \therefore\ l_{42} &= 0 \\
l_{31}^2 + l_{32}^2 + l_{33}^2 &= 4 & \therefore\ l_{33} &= \sqrt{\frac{131}{35}} \\
l_{31}\, l_{41} + l_{32}\, l_{42} + l_{33}\, l_{43} &= -1 & \therefore\ l_{43} &= -\sqrt{\frac{35}{131}} \\
l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 &= 9 & \therefore\ l_{44} &= \sqrt{\frac{1144}{131}}
\end{aligned}$$

Thus, we obtain matrix

$$L = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1/3 & \sqrt{35/9} & 0 & 0 \\ 0 & \sqrt{9/35} & \sqrt{131/35} & 0 \\ 0 & 0 & -\sqrt{35/131} & \sqrt{1144/131} \end{bmatrix}$$

Using $\quad LZ = B \quad$ ... (3)

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1/3 & \sqrt{35/9} & 0 & 0 \\ 0 & \sqrt{9/35} & \sqrt{131/35} & 0 \\ 0 & 0 & -\sqrt{35/131} & \sqrt{1144/131} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Solving by forward substitution, $z_1 = 1/3$, $z_2 = \dfrac{17}{3\sqrt{35}}$, $z_3 = -\dfrac{17}{\sqrt{4585}}$ and $z_4 = -\dfrac{17}{\sqrt{149864}}$

Now employing $L^T X = Z$ ... (4)

$$\begin{bmatrix} 3 & 1/3 & 0 & 0 \\ 0 & \sqrt{35/9} & \sqrt{9/35} & 0 \\ 0 & 0 & \sqrt{131/35} & -\sqrt{35/131} \\ 0 & 0 & 0 & \sqrt{1144/131} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 17/3\sqrt{35} \\ -17/\sqrt{4585} \\ -17/\sqrt{149864} \end{bmatrix}$$

Using backward substitution, we get the solution as $x_1 = \dfrac{61}{1144}$, $x_2 = \dfrac{595}{1144}$, $x_3 = \dfrac{-153}{1144}$ and $x_4 = -\dfrac{17}{1144}$

EXERCISE 8.1

1. Solve the following Systems of Linear Equations using Cramer's Rule :

(i)
$$\begin{aligned} x + y + z &= 6 \\ 3x + 3y + 4z &= 20 \\ 2x + y + 3z &= 13 \end{aligned}$$
Ans. $x = 3$, $y = 1$, $z = 2$

(ii)
$$\begin{aligned} x + 2y - z &= 2 \\ 3x + 6y + z &= 1 \\ 3x + 3y + 2z &= 3 \end{aligned}$$
Ans. $x = 35/12$, $y = -13/12$, $z = -15/12$

(iii)
$$\begin{aligned} 3x - 0.1y - 0.2z &= 7.85 \\ 0.1x + 7y - 0.3z &= -19.3 \\ 0.3x - 0.2y + 10z &= 71.4 \end{aligned}$$
Ans. $x = \dfrac{631.059}{210.353} = 3$, $y = -\dfrac{525.8825}{210.353} = -2.5$, $z = \dfrac{1472.471}{210.353} = 7$

(iv)
$$\begin{aligned} x + 2y + 2z &= 4 \\ 2x - y + 3z &= 5 \\ x + y + 2z &= 2 \end{aligned}$$
Ans. $x = 14$, $y = 2$, $z = -7$

(v)
$$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$
Ans. $x = \dfrac{35}{8}$, $y = \dfrac{29}{8}$, $z = \dfrac{5}{18}$

2. Using Matrix Inversion, Solve the following Linear Systems of Equations :

(i)
$$\begin{aligned} 2x - y + z &= 4 \\ x + 2y + 3z &= 6 \\ x + 4y + 6z &= 5 \end{aligned}$$
Ans. $x = 7$, $y = 29/5$, $z = -21/5$

(ii)
$$\begin{aligned} x + 2y - z &= 2 \\ 3x + 6y + z &= 1 \\ 3x + 3y + 2z &= 3 \end{aligned}$$
Ans. $x = \dfrac{35}{12}$, $y = \dfrac{-13}{12}$, $z = \dfrac{-15}{12}$

(iii)
$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -3 \\ x + 2y + z &= 4 \end{aligned}$$
Ans. $x = 1$, $y = 2$, $z = -1$

(iv)
$$\begin{aligned} 3x + 2y + 4z &= 7 \\ 2x + y + z &= 7 \\ x + 3y + 5z &= 2 \end{aligned}$$
Ans. $x = 9/4$, $y = -9/8$, $z = 5/8$

(v)
$$\begin{aligned} 2x + 4y + z &= 3 \\ 3x + 2y - 2z &= -2 \\ x - y + z &= 6 \end{aligned}$$
Ans. $x = 2$, $y = -1$, $z = 3$

3. Solve by using Gauss Elimination Method, the following System of Equations :

(i)
$$x_1 + 3x_2 + 5x_3 = -4.4$$
$$5x_1 + 10x_2 + 19x_3 = 6.7$$
$$30x_1 + 52x_2 + 100x_3 = -1$$
Ans. $x_1 = -14.9, \quad x_2 = -29.5, \quad x_3 = 19.8$

(ii)
$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 3x_3 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$
Ans. $x_1 = 1, \; x_2 = 0, \; x_3 = -1$

(iii)
$$2x + 2y + z = 12$$
$$3x + 2y + 2z = 8$$
$$5x + 10y - 8z = 10$$
Ans. $x_1 = -51/4, \quad y = 115/8, \; z = 35/4$

(iv)
$$2x_1 + 4x_2 + x_3 = 3$$
$$3x_1 + 2x_2 - 2x_3 = -2$$
$$x_1 - x_2 + x_3 = 6$$
Ans. $x_1 = 2, \; x_2 = -1, \; x_3 = 3$

(v)
$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22$$
Ans. $x = 1, \; y = -2, \; z = 3$

(vi)
$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
Ans. $x_1 = 3, \quad x_2 = 2, \; x_3 = 1$

(vii)
$$5x_1 - x_2 - 2x_3 = 142$$
$$x_1 - 3x_2 - x_3 = -30$$
$$2x_1 - x_2 - 3x_3 = 5$$
Ans. $x_1 = 39.2, \; x_2 = 16.7, \quad x_3 = 19$

(viii)
$$-12x_1 + x_2 - 8x_3 = 80$$
$$x_1 - 6x_2 + 4x_3 = 13$$
$$-2x_1 - x_2 + 10x_3 = 90$$
Ans. $x_1 = 0.651715, \; x_2 = 4.316620, \; x_3 = 9.562005$

(ix)
$$2x_1 + x_2 - 0.1\,x_3 + x_4 = 2.7$$
$$0.4\,x_1 + 0.5\,x_2 + 4x_3 - 8.5\,x_4 = 21.9$$
$$0.3\,x_1 - x_2 + x_3 + 5.2\,x_4 = -3.9$$
$$x_1 + 0.2\,x_2 + 2.5\,x_3 - x_4 = 9.9$$
Ans. $x_1 = 1, \; x_2 = 2, \; x_3 = 3, \; x_4 = -1$

(x)
$$2x_1 + x_2 + 4x_3 + 7x_4 = 1$$
$$-4x_1 + x_2 - 6x_3 - 13x_4 = -1$$
$$4x_1 + 5x_2 + 7x_3 + 7x_4 = 4$$
$$-2x_1 + 5x_2 - 4x_3 - 8x_4 = -5$$
Ans. $x_1 = -14.6529, \; x_2 = 2.3324, \; x_3 = -6.9944, \; x_4 = 7.9930$

(xi)
$$5x_1 - 2x_2 + x_3 = 4$$
$$7x_1 + x_2 - 5x_3 = 8$$
$$3x + 7x_2 + 4x_3 = 10$$
Ans. $x_1 = 122/109, \; x_2 = 284/327, \; x_3 = 46/327$

(xii)
$$2x_1 + 2x_2 + 4x_3 - 2x_4 = 10$$
$$x_1 + 3x_2 - 2x_3 + x_4 = 17$$
$$3x_1 + x_2 + 3x_3 + x_4 = 18$$
$$x_1 + 3x_2 + 4x_3 + 2x_4 = 27$$
Ans. $x_1 = 1, \; x_2 = 2, \; x_3 = 3, \; x_4 = 4$

(xiii)
$$2x_1 + 2x_2 + x_3 + 2x_4 = 7$$
$$-x_1 + 2x_2 + x_4 = -2$$
$$3x_1 + x_2 + 2x_3 + x_4 = -3$$
$$-x_1 + 2x_4 = 0$$
Ans. $x_1 = 2.1622,\ x_2 = 0.4555,\ x_3 = 1.4324,\ x_4 = 1.0811$

(xiv)
$$2.462\,x_1 + 1.349\,x_2 - 2.390\,x_3 - 3.4\,x_4 = 0.903$$
$$-x_1 + 2x_2 + 3x_3 + 2x_4 = 1.31$$
$$0.983\,x_1 + 0.22\,x_2 - 1.6\,x_3 - 0.93\,x_4 = 3$$
$$1.31\,x_1 - 3x_2 + 2.1\,x_3 + 1.2\,x_4 = 2.56$$
Ans. $x_1 = 3.721,\ x_2 = 1.141,\ x_3 = -1.871,\ x_4 = 4.197$

(xv)
$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$
Ans. $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$

(xvi)
$$5x_1 + 3x_2 + 7x_3 = 4$$
$$x_1 + 5x_2 + 3x_3 = 2$$
$$7x_1 + 2x_2 + 10x_3 = 5$$
Ans. $x_1 = 0.6374,\ x_2 = 0.2727,\ x_3 = 0$

(xvii)
$$-3x_1 + 6x_2 - 9x_3 = -46.725$$
$$x_1 - 4x_2 + 3x_3 = 19.571$$
$$2x_1 + 5x_2 - 7x_3 = -20.073$$
Ans. $x_1 = 3.908,\ x_2 = -1.998,\ x_3 = 2.557$

(xviii)
$$3.2\,x_1 + 1.6\,x_2 = -0.8$$
$$1.6\,x_1 - 0.8\,x_2 + 2.4\,x_3 = 16.0$$
$$2.4\,x_2 - 4.8\,x_3 + 3.6\,x_4 = -39.0$$
$$3.6\,x_3 + 2.4\,x_4 = 10.2$$
Ans. $x_1 = 1.5,\ x_2 = -3.5,\ x_3 = 4.5,\ x_4 = -2.5$

(xix)
$$4.405\,x_1 + 2.972\,x_2 - 3.954\,x_3 = 8.671$$
$$3.502\,x_1 + 9.617\,x_2 - 2.111\,x_3 = 12.345$$
$$7.199\,x_1 + 3.248\,x_2 + 1.924\,x_3 = 6.227$$
Ans. $x_1 = 0.6846,\ x_2 = 0.8628,\ x_3 = -0.7817$

4. Use Gauss Elimination with Partial Pivoting to Solve the Following System of Linear Equations :

(i)
$$10\,x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$
Ans. $x_1 = 0.375,\ x_2 = 0.289,\ x_3 = 0.269$

(ii)
$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$
Ans. $x_1 = 3,\ x_2 = 1,\ x_3 = 2$

(iii)
$$2.63\,x_1 + 5.21\,x_2 - 1.694\,x_3 + 0.938\,x_4 = 4.23$$
$$3.16\,x_1 - 2.95\,x_2 + 0.813\,x_3 - 4.21\,x_4 = -0.716$$
$$5.36\,x_1 + 1.88\,x_2 - 2.15\,x_3 - 4.95\,x_4 = 1.28$$
$$1.34\,x_1 + 2.98\,x_2 - 0.432\,x_3 - 1.768\,x_4 = 0.419$$
Ans. $x_1 = 1.03824,\ x_2 = 0.208901,\ x_3 = 0.22637,\ x_4 = 0.846773$

(iv)
$$2x_1 + 3x_2 + 5x_3 = 23$$
$$3x_1 + 4x_2 + x_3 = 14$$
$$6x_1 + 7x_2 + 2x_3 = 26$$
Ans. $x_1 = 0.846,\ x_2 = 2,\ x_3 = 3.4615$

5. Solve the following equations by Gauss Jordan method :

(i)
$$\begin{aligned} 2x - 3y + z &= -1 \\ x + 4y + 5z &= 25 \\ 3x - 4y + z &= 2 \end{aligned}$$
Ans. $x = 8.7,\ y = 5.7,\ z = -1.3$

(ii)
$$\begin{aligned} 10x + y + z &= 12 \\ x + 10y + z &= 12 \\ x + y + 10z &= 12 \end{aligned}$$
Ans. $x = 1,\ y = 1,\ z = 1$

(iii)
$$\begin{aligned} 2x - 3y + 4z &= 7 \\ 5x - 2y + 2z &= 7 \\ 6x - 3y + 10z &= 23 \end{aligned}$$
Ans. $x = 1,\ y = 1,\ z = 2$

(iv)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8 \\ 2x_1 + 3x_2 + 4x_3 &= 20 \\ 4x_1 + 3x_2 + 2x_3 &= 16 \end{aligned}$$
Ans. $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

(v)
$$\begin{aligned} x_1 + x_2 + x_3 &= 9 \\ 2x_1 - 3x_2 + 4x_3 &= 13 \\ 3x_1 + 4x_2 + 5x_3 &= 40 \end{aligned}$$
Ans. $x_1 = 1,\ x_2 = 3,\ x_3 = 5$

(vi)
$$\begin{aligned} 10x_1 - 7x_2 + 3x_3 + 5x_4 &= 6 \\ -6x_1 + 8x_2 - x_3 - 4x_4 &= 5 \\ 3x_1 + x_2 + 4x_3 + 11x_4 &= 2 \\ 5x_1 - 9x_2 - 2x_3 + 4x_4 &= 7 \end{aligned}$$
Ans. $x_1 = 5,\ x_2 = 4,\ x_3 = -7,\ x_4 = 1$

(vii)
$$\begin{aligned} 2x_1 + x_2 + 5x_3 + x_4 &= 5 \\ x_1 + x_2 - 3x_3 + 4x_4 &= -1 \\ 3x_1 + 6x_2 - 2x_3 + x_4 &= 8 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 &= 2 \end{aligned}$$
Ans. $x_1 = 2,\ x_2 = 1/5,\ x_3 = 0,\ x_4 = 4/5$

(viii)
$$\begin{aligned} 7x + 2y - 5z &= -18 \\ x + 5y - 3z &= -40 \\ 2x - y - 9z &= -26 \end{aligned}$$
Ans. $x = 2,\ y = -6,\ z = 4$

(ix)
$$\begin{aligned} 8x_1 + 4x_2 + 2x_3 &= 24 \\ 4x_1 + 10x_2 + 5x_3 + 4x_3 &= 32 \\ 2x_1 + 5x_2 + 6.5x_3 + 4x_4 &= 26 \\ 4x_2 + 4x_3 + 4x_4 &= 21 \end{aligned}$$
Ans. $x_1 = 2,\ x_2 = 1,\ x_3 = 2,\ x_4 = 1$

(x)
$$\begin{aligned} 2x_1 - 5x_2 + x_3 &= 12 \\ -x_1 + 3x_2 - x_3 &= -8 \\ 3x_1 - 4x_2 + 2x_3 &= 16 \end{aligned}$$
Ans. $x_1 = 2,\ x_2 = -1,\ x_3 = 3$

(xi)
$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ 3x_1 + 5x_2 + 3x_3 &= 4 \end{aligned}$$
Ans. $x_1 = 1,\ x_2 = 1/2,\ x_3 = -1/2$

(xii)
$$\begin{aligned} 9x_1 + 7x_2 + 3x_3 &= 6 \\ 5x_1 - x_2 + 4x_3 &= 1 \\ 3x_1 + 5x_2 + x_3 &= 2 \end{aligned}$$
Ans. $x_1 = 1,\ x_2 = 0,\ x_3 = -1$

(xiii)
$$\begin{aligned} x + y + z &= 6 \\ x - y + 2z &= 5 \\ 3x + y + z &= 8 \end{aligned}$$
Ans. $x = 1,\ y = 2, z = 3$

(xiv)
$$\begin{aligned} 3x - 2y + 5z &= 2 \\ 4x + y + 2z &= 4 \\ 2x - y + 4z &= 7 \end{aligned}$$
Ans. $x = -13/4,\ y = 41/6,\ z = 61/12$

6. Solve the following Systems by Triangular Factorization (LU Decomposition Method) :

(i)
$$3x_1 + x_2 + x_3 = 4$$
$$x_1 + 2x_2 + 2x_3 = 3$$
$$2x_1 + x_2 + 3x_3 = 4$$
Ans. $x_1 = 1,\ x_2 = 1/2,\ x_3 = 1/2$

(ii)
$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$
Ans. $x = 1,\ y = 1,\ z = 1$

(iii)
$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$3x_1 + 4x_2 + x_3 = 14$$
Ans. $x = 1, y = 2,\ z = 3$

(iv)
$$2x_1 + 3x_2 + x_3 = 9$$
$$x_2 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$
Ans. $x_1 = 35/18, x_2 = 29/18,\ x_3 = 5/18$

(v)
$$2x_1 - x_2 + x_3 = -1$$
$$2x_2 - x_3 + x_4 = 1$$
$$x_1 + 2x_3 - x_4 = -1$$
$$x_1 + x_2 + 2x_4 = 5$$
Ans. $x_1 = -1,\ x_2 = 0,\ x_3 = 1, x_4 = 2$

(vi)
$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$
Ans. $x = 7/8,\ y = 9/8,\ z = -1/8$

(vii)
$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$
Ans. $x_1 = 1,\ x_2 = 0.5,\ x_3 = -0.5$

7. Solve the following System of Equations by Cholesky's Method (Square Root Method) :

(i)
$$4x_1 - 2x_2 = 0$$
$$-2x_1 + 4x_2 - x_3 = 1$$
$$-x_2 + 4x_3 = 0$$
Ans. $x_1 = 2/11, x_2 = 4/11, x_3 = 1/11$

(ii)
$$4x_1 - x_2 = 1$$
$$-x_1 + 4x_2 - x_3 = 0$$
$$-x_2 + 4x_3 - x_4 = 0$$
$$-x_3 + 4x_4 = 0$$
Hint : $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1/2 & \sqrt{15/14} & 0 & 0 \\ 0 & -\sqrt{4/15} & \sqrt{56/15} & 0 \\ 0 & 0 & -\sqrt{15/56} & \sqrt{209/56} \end{bmatrix}$

Ans. $x_1 = 56/209,\ x_2 = 15/209,\ x_3 = 4/209,\ x_4 = 1/209$

(iii)
$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$
$$-6x_1 + 8x_2 - x_3 - 4x_4 = 5$$
$$3x_1 + x_2 + 4x_3 + 11x_4 = 2$$
$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7$$
Ans. $x_1 = 5,\ x_2 = 4,\ x_3 = -7,\ x_4 = 1$

(iv)
$$4x_1 + 6x_2 + 8x_3 = 0$$
$$6x_1 + 34x_2 + 52x_3 = -160$$
$$8x_1 + 52x_2 + 129x_3 = -452$$
Hint : $L = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 8 & 7 \end{bmatrix}$

Ans. $x_1 = 8,\ x_2 = 0,\ x_3 = -4$

(v)
$$4x_1 + 2x_2 + 4x_3 = 10$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = 18$$
$$4x_1 + 3x_2 + 6x_3 + 3x_4 = 30$$
$$2x_2 + 3x_3 + 9x_4 = 61$$
Hint : $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$

[**Ans.** $x_1 = 3,\ x_2 = -1, x_3 = 0, x_4 = 7$]

8. Find the Cholesky's Factorization of the following Matrix :

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Ans. $A = LL^T$, where, $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

8.3 ITERATIVE METHODS

We have so far discussed direct methods of solving system of linear equations and we have seen that these methods give solutions after an amount of computation that can be specified in advance. When a given system of linear equations is of very large size i.e. there are large number of unknowns, direct methods become unweildy. Under such conditions, it is more convenient to use approximate numerical methods for finding the solution of the system. One of these methods is the method of iteration. In an **Indirect** or **Iterative method**, we start from an approximation to the true solution and, if successful (convergent), obtain better and better approximations from a computational cycle repeated till the desired accuracy is obtained, so that the amount of computation depends on the accuracy required and varies from case to case. We prefer iterative methods if the convergence is rapid (if matrices have large main diagonal entries), so that we save operations compared to a direct method.

8.3.1 Jacobi Iteration Method

Consider the system of n linear equations in n unknowns.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots + a_{2n} x_n &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \ldots\ldots\ldots + a_{3n} x_n &= b_3 \\ &\ldots\ldots \\ &\ldots\ldots \\ a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \ldots\ldots + a_{nn} x_n &= b_n \end{aligned} \qquad \ldots (1)$$

which can be written in matrix from as

$$AX = B \qquad \ldots (2)$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ : \\ b_n \end{bmatrix}$

Assuming that the diagonal coefficients (pivot elements)

$$a_{ii} \neq 0 \qquad (i = 1, 2, \ldots n)$$

are large in magnitude as compared to other coefficients, otherwise the equation should be rearranged so that this condition is satisfied. We then solve the first equation for x_1, the second for x_2, third for x_3 etc.

Thus, we have

$$\begin{aligned} x_1 &= \frac{1}{a_{11}} (b_1 - a_{12} x_2 - a_{13} x_3 \ldots\ldots\ldots\ldots\ldots\ldots - a_{1n} x_n) \\ x_2 &= \frac{1}{a_{22}} (b_2 - a_{21} x_1 - a_{23} x_3 \ldots\ldots\ldots\ldots\ldots\ldots - a_{2n} x_n) \\ x_3 &= \frac{1}{a_{33}} (b_3 - a_{31} x_1 - a_{32} x_2 - a_{34} x_4 \ldots\ldots - a_{3n} x_n) \\ &\ldots\ldots \\ &\ldots\ldots \\ x_n &= \frac{1}{a_{nn}} (b_n - a_{n1} x_1 - a_{n2}\, x \ldots\ldots\ldots - a_{nn-1} x_{n-1}) \end{aligned} \qquad \ldots (3)$$

We solve the system (3) by the method of successive approximations.

Suppose $x_1^{(0)}$, $x_2^{(0)}$, $x_3^{(0)}$ .... $x_n^{(0)}$ are any first (initial) approximations (also called zeroth approximation) to unknowns $x_1, x_2, x_3, \ldots x_n$. Substituting in the right side of (3), we find a system of first approximations.

$$\begin{aligned}
x_1^{(1)} &= \frac{1}{a_{11}} (b_1 - a_{12} x_2^{(0)} - x_{13} x_3^{(0)} \;\ldots\ldots\ldots\ldots\ldots\; - a_{1n} x_n^{(0)}) \\
x_2^{(1)} &= \frac{1}{a_{22}} (b_2 - b_{21} x_1^{(0)} - a_{23} x_3^{(0)} \;\ldots\ldots\ldots\ldots\ldots\; - a_{2n} x_n^{(0)}) \\
x_3^{(1)} &= \frac{1}{a_{33}} (b_3 - a_{31} x_1^{(0)} - a_{32} x_2^{(0)} - a_{34} x_4^{(0)} \;\ldots\ldots\; - a_{3n} x_n^{(0)}) \qquad \ldots (4) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{(1)} &= \frac{1}{a_n} (b_n - a_{n1} x_1^{(0)} - a_{n2} x_2^{(0)} \;\ldots\ldots\ldots\; - a_{nn-1} x_{n-1}^{(0)})
\end{aligned}$$

Similarly, if $x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots x_n^{(k)}$ are a system of k^{th} approximations, then the next approximation is given by

$$\begin{aligned}
x_1^{(k+1)} &= \frac{1}{a_{11}} (b_1 - a_{12} x_2^{(k)} - a_{13} x_3^{(k)} \;\ldots\ldots\ldots\ldots\; a_{1n} x_n^{(k)}) \\
x_2^{(k+1)} &= \frac{1}{a_{22}} (b_2 - a_{21} x_1^{(k)} - a_{23} x_3^{(k)} \;\ldots\ldots\ldots\ldots\; a_{2n} x_n^{(k)}) \\
x_3^{(k+1)} &= \frac{1}{a_{33}} (b_3 - a_{31} x_1^{(k)} - a_{32} x_2^{(k)} - a_{34} x_4^{(k)} \;\ldots\ldots\; a_{3n} x_n^{(k)}) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots (5) \\
x_n^{(k+1)} &= \frac{1}{a_{nn}} (b_n - a_{n1} x_1^{(k)} - a_{n2} x_2^{(k)} \;\ldots\ldots\ldots\ldots\; a_{nn-1} x_n^{(k)})
\end{aligned}$$

Above system of approximations (5) can be considered as general iterative formula.

Remark 1 : When using this method, for the initial (zeroth) approximation, we may take $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = \ldots = x_n^{(1)} = 0$, but it is not necessary that the same initial approximation be taken. The convergence of the iteration process depends on the relation between $x_i^{(k)}$ and $x_i^{(k+1)}$. When certain conditions are met and if the iteration process converges for certain choice of initial approximation, it converges to the same limiting solution for any other choice of the initial approximation as well. For this reason, initial approximation can be chosen arbitrarily. A converging process is self-correcting, that is an individual computational error will not affect the final result, since any erroneous approximation may be regarded as a new initial approximation. In this method, at each stage, the result obtained is used as next trial solution for substitution, the method is also called method of **Simultaneous Correction**.

Remark 2 : Condition for the Convergence of the Iteration Process : For the system (1), the method of iteration converges if the inequalities

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad (i = 1, 2, \ldots n)$$

hold true, that is if the moduli of the diagonal coefficients are greater for each equation of the system than the sum of the moduli of all the remaining coefficients (disregarding the constant terms) in that equation.

Remark 3 : From system (3), we can write the equations as

$$\begin{aligned}
x_1 &= \beta_1 + \alpha_{12} x_2 + \alpha_{13} x_3 + \ldots\ldots\ldots\ldots + \alpha_{1n} x_n \\
x_2 &= \beta_2 + \alpha_{21} x_1 + \alpha_{23} x_3 + \ldots\ldots\ldots\ldots + \alpha_{2n} x_n \\
x_3 &= \beta_3 + \alpha_{31} x_1 + \alpha_{32} x_2 + \alpha_{34} x_4 + \ldots\ldots + \alpha_{3n} x_n \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots (6) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n &= \beta_n + \alpha_{n1} x_1 + \alpha_{n2} x_2 + \ldots\ldots\ldots + \alpha_{nn-1} x_{n-1}
\end{aligned}$$

where, $\beta_i = \dfrac{b_i}{a_{ii}}$, $\alpha_{ij} = \dfrac{-a_{ij}}{a_{ii}}$ and $i \neq j$ and $\alpha_{ij} = 0$ for $i = j$ $(i, j = 1, 2, \dots n)$

Writing
$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots\dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots\dots & \alpha_{2n} \\ \dots. & \dots & \dots\dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots\dots & \alpha_{nn} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ : \\ \beta_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_n \end{bmatrix}$$

we can write the above system in matrix form as

$$X = \beta + \alpha X \qquad \dots(7)$$

We solve the system (7) by successive approximations.

For the zeroth approximation, we may take $X^{(0)} = \beta$, then we successively construct the column matrices as

$$X^{(1)} = \beta + \alpha X^{(0)} \qquad \text{(first approximation)}$$
$$X^{(2)} = \beta + \alpha X^{(1)} \qquad \text{(second approximation)}$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$X^{(k+1)} = \beta + \alpha X^{(k)} \qquad (k^{th} \text{ approximation})$$

Thus $X^{(k+1)} = \beta + \alpha X^{(k)}$ $(k = 0, 1, 2, \dots)$ can be considered as general iterative formula.

If the sequence of approximations $X^{(0)}, X^{(1)}, \dots X^{(k)}$ has a limit, the limiting vector X is the solution of system (7) or system (1). The iteration formula can also be written as

$$x_i^{(k+1)} = \beta_i + \sum_{j=1}^{n} \alpha_{ij}\, x_j^{(k)} \qquad (\alpha_{ii} = 0;\ i = 1, 2, \dots n;\ k = 0, 1, 2, \dots)$$

If for the reduced system (6) at least one of the following two conditions is valid :

1. $$\sum_{j=1}^{n} |\alpha_{ij}| < 1 \qquad (i = 1, 2, \dots n)$$

2. $$\sum_{i=1}^{n} |\alpha_{ij}| < 1 \qquad (j = 1, 2, \dots n) \qquad \dots (8)$$

then the process of iteration (4) converges to a unique solution of the system irrespective of the choice of the initial approximation.

ILLUSTRATIONS

Ex. 1 : *Solve the following system of equations by Jacobi's iteration method :*

$$2x_1 + 12x_2 + x_3 - 4x_4 = 13$$
$$13x_1 + 5x_2 - 3x_3 + x_4 = 18$$
$$2x_1 + x_2 - 3x_3 + 9x_4 = 31$$
$$3x_1 - 4x_2 + 10x_3 + x_4 = 29$$

Sol. : Since large coefficients are not along the leading diagonal (i.e. given system of equations is not diagonally dominant), we therefore first rearrange the system as

$$\begin{aligned} 13x_1 + 5x_2 - 3x_3 + x_4 &= 18 \\ 2x_1 + 12x_2 + x_3 - 4x_4 &= 13 \\ 3x_1 - 4x_2 + 10x_3 + x_4 &= 29 \qquad \dots (1) \\ 2x_1 + x_2 - 3x_3 + 9x_4 &= 31 \end{aligned}$$

It can be verified that above system of equations (1) satisfies convergence conditions (i.e. the system (1) is diagonally dominant). Hence it can be solved by Jacobi's iteration method.

We now express the system (1) as

$$\begin{aligned} x_1 &= \frac{1}{13}\,[18 - 5x_2 + 3x_3 - x_4] \\ x_2 &= \frac{1}{12}\,[13 - 2x_1 - x_3 + 4x_4] \\ x_3 &= \frac{1}{10}\,[29 - 3x_1 + 4x_2 - x_4] \\ x_4 &= \frac{1}{9}\,[31 - 2x_1 - x_2 + 3x_3] \end{aligned} \qquad \text{... (2)}$$

First Iteration : Let us start iteration with trial (initial) approximation to the solution $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$ and substitute in the right hand side of this system of equations (working to three decimals), which gives

$$\begin{aligned} x_1^{(1)} &= \frac{1}{13}\,[18 - 5\,(0) + 3\,(0) - (0)] = 1.385 \\ x_2^{(1)} &= \frac{1}{12}\,[13 - 2\,(0) - (0) + 4\,(0)] = 1.083 \\ x_3^{(1)} &= \frac{1}{10}\,[29 - 3\,(0) + 4\,(0) - (0)] = 2.900 \\ x_4^{(1)} &= \frac{1}{9}\,[31 - 2\,(0) - (0) + 3\,(0)] = 3.444 \end{aligned}$$

Second Iteration : The set of values $x_1^{(1)} = 1.385$, $x_2^{(1)} = 1.083$, $x_3^{(1)} = 2.900$ and $x_4^{(1)} = 3.444$ then becomes our second trial solution. Substituting on the right hand side of our system of equation, we obtain

$$\begin{aligned} x_1^{(2)} &= \frac{1}{13}\,[18 - 5\,(1.083) + 3\,(2.900) - (3.444)] = 1.372 \\ x_2^{(2)} &= \frac{1}{12}\,[13 - 2\,(1.385) - (2.900) + 4\,(3.444)] = 1.759 \\ x_3^{(2)} &= \frac{1}{10}\,[29 - 3\,(1.385) + 4\,(1.083) - (3.444)] = 2.573 \\ x_4^{(2)} &= \frac{1}{9}\,[31 - 2\,(1.385) - (1.083) + 3\,(2.900)] = 3.983 \end{aligned}$$

By continuing in this manner, successive iterations can be tabulated as

n	x_1	x_2	x_3	x_4
1	1.385	1.083	2.900	3.444
2	1.372	1.759	2.573	3.983
3	0.995	1.968	2.794	3.802
4	0.980	1.952	3.009	3.936
5	1.025	1.981	2.993	4.013
6	1.005	2.001	2.984	3.914
7	0.996	1.999	3.000	3.993
8	1.001	1.998	3.002	4.001
9	1.001	2.000	2.999	4.001

It can be easily checked that the correct answer is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$.

Ex. 2 : *Solve by Jacobi's iteration method the following system of equations*

$$\begin{aligned} 20x_1 + x_2 - 2x_3 &= 17 \\ 3x_1 + 20x_2 - x_3 &= -18 \\ 2x_1 - 3x_2 + 20x_3 &= 25 \end{aligned}$$

Sol. : Since these equations meet the requirement for iteration (i.e. system of equation is diagonally dominant), we express for each unknown in terms of the others and thus get the system

$$x_1 = \frac{1}{20}[17 - x_2 + 2x_3]$$

$$x_2 = \frac{1}{20}[-18 - 3x_1 + x_3] \qquad \dots (1)$$

$$x_3 = \frac{1}{20}[25 - 2x_1 + 3x_2]$$

First Iteration : We start iteration with initial approximation $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$. Substituting these on the right hand side of equations (1), we obtain

$$x_1^{(1)} = \frac{1}{20}[17 - (0) + 2(0)] = \frac{17}{20} = 0.85$$

$$x_2^{(1)} = \frac{1}{20}[-18 - 3(0) + (0)] = -\frac{18}{20} = -0.9$$

$$x_3^{(1)} = \frac{1}{20}[25 - 2(0) + 3(0)] = \frac{25}{20} = 1.25$$

Second Iteration : Putting these values on the right hand side of equations (1), we have

$$x_1^{(2)} = \frac{1}{20}[17 - (-0.9) + 2(1.25)] = 1.02$$

$$x_2^{(2)} = \frac{1}{20}[-18 - 3(0.85) + (1.25)] = -0.965$$

$$x_3^{(2)} = \frac{1}{20}[25 - 2(0.85) + 3(-0.9)] = 1.1515$$

By continuing in this manner, successive iterations can be tabulated as

n	x_1	x_2	x_3
1	0.85	– 0.9	1.25
2	1.02	– 0.965	1.1515
3	1.0134	– 0.9954	1.0032
4	1.0009	– 1.0018	0.9993
5	1.0000	– 1.0002	0.9996
6	1.0000	– 1.0000	1.0000

The values in the 5th and 6th iterations being the same, the solution is $x_1 = 1$, $x_2 = -1$ and $x_3 = 1$.

Ex. 3 : *Apply Jacobi's iteration method to solve the following system of equations*

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Sol. : The given system of equations is diagonally dominant. Hence these equations meet the requirement for iteration. We now express given system as

$$x_1 = \frac{1}{10}[3 + 2x_2 + x_3 + x_4]$$

$$x_2 = \frac{1}{10}[15 + 2x_1 + x_3 + x_4]$$

$$x_3 = \frac{1}{10}[27 + x_1 + x_2 + 2x_4] \qquad \dots (1)$$

$$x_4 = \frac{1}{10}[-9 + x_1 + x_2 + 2x_3]$$

First Iteration : We start iteration with initial approximations $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$. Substituting these on the right hand side of equations (1), we obtain

$$x_1^{(1)} = \frac{1}{10} [3 + 2 (0) + (0) + (0)] = 0.3$$

$$x_2^{(1)} = \frac{1}{10} [15 + 2 (0) + (0) + (0)] = 1.5$$

$$x_3^{(1)} = \frac{1}{10} [27 + (0) + (0) + 2 (0)] = 2.7$$

$$x_4^{(1)} = \frac{1}{10} [-9 + (0) + (0) + 2 (0)] = -0.9$$

Second Iteration : Putting these values on the right hand side of equations (1), we have

$$x_1^{(2)} = \frac{1}{10} [3 + 2 (1.5) + (2.7) + (-0.9)] = 0.78$$

$$x_2^{(2)} = \frac{1}{10} [15 + 2 (0.3) + (2.7) + (-0.9)] = 1.74$$

$$x_3^{(2)} = \frac{1}{10} [27 + (0.3) + (1.5) + 2 (-0.9)] = 2.7$$

$$x_4^{(2)} = \frac{1}{10} [-9 + (0.3) + (1.5) + 2 (2.7)] = -0.18$$

By continuing in this manner, successive iterations can be tabulated as

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	– 0.9
2	0.78	1.74	2.7	– 0.18
3	0.9	1.908	2.916	– 0.108
4	0.9624	1.9608	2.9592	– 0.036
5	0.9845	1.9848	2.9851	– 0.0158
6	0.9939	1.9938	2.9938	– 0.006
7	0.9975	1.9975	2.9976	– 0.0025
8	0.9990	1.9990	2.9990	– 0.0010
9	0.9996	1.9996	2.9996	– 0.0004
10	0.9998	1.9998	2.9998	– 0.0002
11	0.9999	1.9999	2.9999	– 0.0001
12	1.0	2.0	3.0	0.0

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 0$.

8.3.2 The Gauss-Seidel Iteration Method

The Gauss-Seidel method is a modification of the Jacobi's iteration method. As in Jacobi's iteration method (refer 8.3.1), consider a system of equations in which each equation is first solved for unknown having large coefficient, thereby expressing it explicitly in terms of other unknowns as

$$x_1 = \frac{1}{a_{11}} (b_1 - a_{12} x_2 - a_{13} x_3 \; .... - a_{1n} x_n)$$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{11} x_1 - a_{13} x_3 \; .... - a_{2n} x_n)$$

$$x_3 = \frac{1}{a_{33}} (b_3 - a_{31} x_1 - a_{32} x_2 - a_{34} x_4 \; .... - a_{3n} x_n) \quad ... (1)$$

....

....

$$x_4 = \frac{1}{a_{nn}} (b_n - a_{n1} x_1 - a_{n2} x_2 \; .... - a_{nn-1} x_{n-1})$$

We again start with initial approximations ($x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)} = 0$). However, this time we substitute these only in the first equation on the right hand side of system (1) which gives improved result $x_1^{(1)}$. Then in the second equation of (1), we substitute ($x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)}$) and denote the improved result $x_2^{(1)}$. In the third equation of (1), we substitute ($x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, \ldots, x_n^{(0)}$) and call the result $x_3^{(1)}$. Proceeding like this we find first iteration values as $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots x_n^{(1)}$. This completes the first stage of iteration and the entire process is repeated till the values of $x_1, x_2, \ldots x_n$ are obtained to desired accuracy.

Thus, if the values of the variables in k^{th} iteration are $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ then the values in the next $(k+1)^{th}$ iteration are given by

$$\begin{aligned}
x_1^{(k+1)} &= \frac{1}{a_{11}} \left(b_1 - a_{12} x_2^{(k)} - a_{13} x_3^{(k)} \ldots\ldots\ldots - a_{1n} x_n^{(k)}\right) \\
x_2^{(k+1)} &= \frac{1}{a_{22}} \left(b_2 - a_{21} x_1^{(k+1)} - a_{23} x_3^{(k)} \ldots\ldots\ldots - a_{2n} x_n^{(k)}\right) \qquad \ldots (2) \\
x_3^{(k+1)} &= \frac{1}{a_{33}} \left(b_3 - a_{31} x_1^{(k+1)} - a_{32} x_2^{(k+1)} - a_{34} x_4^{(k)} \ldots\ldots - a_{3n} x_n^{(k)}\right) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{(k+1)} &= \frac{1}{a_{nn}} \left(b_n - a_{n1} x_1^{(k+1)} - a_{n2} x_2^{(k+1)} \ldots\ldots\ldots - a_{nn-1} x_{n-1}^{(k+1)}\right)
\end{aligned}$$

Above system of approximation (2) can be considered as a general formula for Gauss-Seidel iterative method.

Remark 1 : For any choice of the first (initial) approximation, Gauss-Seidel iterative method converges. Condition for convergence of iteration process is same as discussed in Jacobi's iteration method. That is "if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients in that equation". In this method of iteration, the result of any stage within a step is used in succeeding stages of the same step, the method is also called method of **Successive Correction**.

Remark 2 : Suppose that we have reduced linear system

$$x_i = \beta_i + \sum_{j=1}^{n} \alpha_{ij} x_j \quad (i = 1, 2, \ldots n)$$

Arbitrarily, choose the initial (zeroth) approximations as $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$.

Now assuming that the k^{th} approximations $x_i^{(k)}$ of the roots are known, then $(k+1)^{th}$ approximations of the roots are given by the following formulae :

$$\begin{aligned}
x_1^{(k+1)} &= \beta_1 + \sum_{j=1}^{n} \alpha_{1j} x_j^{(k)} \\
x_2^{(k+1)} &= \beta_2 + \alpha_{21} x_1^{(k+1)} + \sum_{j=2}^{n} \alpha_{2j} x_j^{(k)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
x_i^{(k+1)} &= \beta_i + \sum_{j=1}^{i-1} \alpha_{ij} x_j^{(k+1)} + \sum_{j=i}^{n} \alpha_{ij} x_j^{(k)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{(k+1)} &= \beta_n + \sum_{j=1}^{n-1} \alpha_{nj} x_j^{(k+1)} + \alpha_{nn} x_n^{(k)}, \quad (k = 0, 1, 2 \ldots)
\end{aligned}$$

ILLUSTRATIONS

Ex. 1 : *Solve the following system of equations by Gauss-Seidel method :*

$$\begin{aligned} 13x_1 + 5x_2 - 3x_3 + x_4 &= 18 \\ 2x_1 + 12x_2 + x_3 - 4x_4 &= 13 \\ 3x_1 - 4x_2 + 10x_3 + x_4 &= 29 \\ 2x_1 + x_2 - 3x_3 + 9x_4 &= 31 \end{aligned}$$

Sol. : Given system is

$$\begin{aligned} 13x_1 + 5x_2 - 3x_3 + x_4 &= 18 \\ 2x_1 + 12x_2 + x_3 - 4x_4 &= 13 \\ 3x_1 - 4x_2 + 10x_3 + x_4 &= 29 \\ 2x_1 + x_2 - 3x_3 + 9x_4 &= 31 \end{aligned} \qquad \dots (1)$$

Since these equations meet the requirement for iteration (i.e. given system of equations is diagonally dominant), hence it can be solved by Gauss-Seidel iteration method. We now use successive equations to express for each unknown in terms of the others. Thus we obtain,

$$\begin{aligned} x_1 &= \frac{1}{13}[18 - 5x_2 + 3x_3 - x_4] \\ x_2 &= \frac{1}{12}[13 - 2x_1 - x_3 + 4x_4] \\ x_3 &= \frac{1}{10}[29 - 3x_1 + 4x_2 - x_4] \\ x_4 &= \frac{1}{9}[31 - 2x_1 - x_2 + 3x_3] \end{aligned} \qquad \dots (2)$$

First Iteration : Let us start iteration with trial (initial) approximation to the solution $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$.

In the Gauss-Seidel method, we substitute these only in the first equation of (2) (working to three decimals), which gives

$$x_1^{(1)} = \frac{1}{13}[18 - 5(0) + 3(0) - (0)] = 1.385$$

We then substitute $x_1^{(1)} = 1.385$, and $x_3^{(0)} = 0$, $x_4^{(0)} = 0$ in the second equation of (2), we get

$$x_2^{(1)} = \frac{1}{12}[13 - 2(1.385) - (0) + 4(0)] = 0.853$$

Putting $x_1^{(1)} = 1.385$, $x_2^{(1)} = 0.853$ and $x_4^{(0)} = 0$ in the third equation, we get

$$x_3^{(1)} = \frac{1}{10}[29 - 3(1.385) + 4(0.853) - (0)] = 2.826$$

Finally, putting $x_1^{(1)} = 1.385$, $x_2^{(1)} = 0.853$ and $x_3^{(1)} = 2.826$ in the fourth equation, we get

$$x_4^{(1)} = \frac{1}{9}[31 - 2(1.385) - (0.853) + 3(2.826)] = 3.984$$

Second Iteration : The set of values $x_1^{(1)} = 1.385$, $x_2^{(1)} = 0.853$, $x_3^{(1)} = 2.826$ and $x_4^{(1)} = 3.984$ now becomes our second of trial solution. Using these values, we have

$$\begin{aligned} x_1^{(2)} &= \frac{1}{13}[18 - 5(0.853) + 3(2.826) - (3.984)] = 1.402 \\ x_2^{(2)} &= \frac{1}{12}[13 - 2(1.402) - (2.826) + 4(3.984)] = 1.942 \\ x_3^{(2)} &= \frac{1}{10}[29 - 3(1.402) + 4(1.942) - (3.984)] = 2.858 \\ x_4^{(2)} &= \frac{1}{9}[31 - 2(1.402) - (1.942) + 3(2.858)] = 3.870 \end{aligned}$$

By continuing in this manner, successive iterations can be tabulated as

n	x_1	x_2	x_3	x_4
1	1.385	0.853	2.826	3.984
2	1.402	1.942	2.858	3.870
3	1.000	1.969	3.001	4.004
4	1.012	1.999	2.996	3.996
5	1.000	1.999	3.000	4.000

It can be easily checked that the correct answer is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$.

Ex. 2 : *Solve by Gauss-Seidel method, the following system of equations*

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 + 20x_3 = 25$$

(Dec. 2011)

Sol. : Since the system of equations is diagonally dominant, we express for each unknown in terms of others and thus get the system

$$\begin{aligned} x_1 &= \frac{1}{20}[17 - x_2 + 2x_3] \\ x_2 &= \frac{1}{20}[-18 - 3x_1 + x_3] \qquad \dots (1) \\ x_3 &= \frac{1}{20}[25 - 2x_1 + 3x_2] \end{aligned}$$

First Iteration : We start iteration with initial approximations $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$. We now substitute $x_2^{(0)} = 0$, $x_3^{(0)} = 0$ in only the first equation of (1), we have

$$x_1^{(1)} = \frac{1}{20}[17 - (0) + 2(0)] = 0.8500$$

We next substitute $x_1^{(1)} = 0.8500$ and $x_3^{(0)} = 0$ in the second equation of (1), we obtain

$$x_2^{(1)} = \frac{1}{20}[-18 - 3(0.8500) + (0)] = -1.0275$$

Finally, putting $x_1^{(1)} = 0.8500$, $x_2^{(1)} = -1.0275$ in the third equation of (1), we get

$$x_3^{(1)} = \frac{1}{20}[25 - 2(0.8500) + 3(-1.0275)] = 1.0109$$

Second Iteration : The set of values $x_1^{(1)} = 0.8500$, $x_2^{(1)} = -1.0275$, $x_3^{(1)} = 1.0109$ now becomes second set of trial solution.

Using these values, we obtain

$$\begin{aligned} x_1^{(2)} &= \frac{1}{20}[17 - (-1.0275) + 2(1.0109)] = 1.0025 \\ x_2^{(2)} &= \frac{1}{20}[-18 - 3(1.0025) - (1.0109)] = -0.9998 \\ x_3^{(2)} &= \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998 \end{aligned}$$

Third Iteration : The set of values $x_1^{(2)} = 1.0075$, $x_2^{(2)} = -0.9998$ and $x_3^{(2)} = 0.9998$ are now third set of trial solution. Using these, we have

$$\begin{aligned} x_1^{(3)} &= \frac{1}{20}[17 - (-0.9998) + 2(0.9998)] = 1.0000 \\ x_2^{(3)} &= \frac{1}{20}[-18 - 3(1.0000) + (0.9998)] = -1.0000 \\ x_3^{(3)} &= \frac{1}{20}[25 - 2(1.0000) + 3(-1.0000)] = 1.0000 \end{aligned}$$

These iteration's results are tabulated as

n	x_1	x_2	x_3
1	0.8500	– 1.0275	1.0109
2	1.0025	– 0.9998	0.9998
3	1.0000	– 1.0000	1.0000

The values in the second and third iterations being the same, the solution is $x_1 = 1$, $x_2 = -1$ and $x_3 = 1$.

Ex. 3 : *Consider the system of equations*

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

Use Gauss-Seidel method to solve the system.

Sol. : The given system of equations is diagonally dominant. Hence these equations meet the requirement for iteration. We now express the given system as

$$\begin{aligned} x_1 &= \frac{1}{10}[3 + 2x_2 + x_3 + x_4] \\ x_2 &= \frac{1}{10}[15 + 2x_1 + x_3 + x_4] \\ x_3 &= \frac{1}{10}[27 + x_1 + x_2 + 2x_4] \\ x_4 &= \frac{1}{10}[-9 + x_1 + x_2 + 2x_3] \end{aligned} \qquad \text{...(1)}$$

First Iteration : We start iteration with initial approximations $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$. We now substitute these only in the first equation of (1), we get

$$x_1^{(1)} = \frac{1}{10}[3 + 2(0) + (0) + (0)] = 0.3$$

We next substitute $x_1^{(1)} = 0.3$, $x_3^{(0)} = x_4^{(0)} = 0$ in the second equation of (1), we have

$$x_2^{(1)} = \frac{1}{10}[15 + 2(0.3) + (0) + (0)] = 1.56$$

Putting $x_1^{(1)} = 0.3$, $x_2^{(1)} = 1.56$ and $x_4^{(0)} = 0$ in the third equation of (1), which gives

$$x_3^{(1)} = \frac{1}{10}[27 + (0.3) + (1.56) + 2(0)] = 2.886$$

Finally, putting $x_1^{(1)} = 0.3$, $x_2^{(1)} = 1.56$, $x_3^{(1)} = 2.886$ in the fourth equation, we obtain

$$x_4^{(1)} = \frac{1}{10}[-9 + (0.3) + (1.56) + 2(2.886)] = -0.1368$$

Second Iteration : This set of values $x_1^{(1)} = 0.3$, $x_2^{(1)} = 1.56$, $x_3^{(1)} = 2.886$ and $x_4^{(1)} = -0.1368$ now becomes second trial solution. Using these values, we obtain

$$\begin{aligned} x_1^{(2)} &= \frac{1}{10}[3 + 2(1.56) + (2.886) + (-0.1368)] = 0.8869 \\ x_2^{(2)} &= \frac{1}{10}[15 + 2(0.8869) + (2.886) + (-0.1368)] = 1.9523 \\ x_3^{(2)} &= \frac{1}{10}[27 + (0.8869) + (1.9523) + 2(-0.1368)] = 2.9566 \\ x_4^{(2)} &= \frac{1}{10}[-9 + (0.8869) + (1.9523) + 2(2.9566)] = -0.0248 \end{aligned}$$

By continuing in this manner, the successive iterations can be tabulated as

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	– 0.1368
2	0.8869	1.9523	2.9566	– 0.0248
3	0.9836	1.9899	2.9924	– 0.0042
4	0.9968	1.9982	2.9987	– 0.0008
5	0.9994	1.9997	2.9998	– 0.001
6	0.9999	1.9999	3.0	0.0
7	1.0000	2.0	3.0	0.0

The values in the sixth and seventh iterations being same, the solution of the system is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 0$.

Ex. 4 : *Solve the following system of equations by the Gauss-Seidel method :*

$$\begin{aligned} 10x_1 + x_2 + x_3 &= 12 \\ 2x_1 + 10x_2 + x_3 &= 13 \\ 2x_1 + 2x_2 + 10x_3 &= 14 \end{aligned}$$

Sol. : Since these equations meet the requirement for iteration, we now use successive equations to express for each unknown in terms of others and thus get the system

$$\begin{aligned} x_1 &= \frac{1}{10}[12 - x_2 - x_3] \\ x_2 &= \frac{1}{10}[13 - 2x_1 - x_3] \qquad \ldots (1) \\ x_3 &= \frac{1}{10}[14 - 2x_1 - 2x_2] \end{aligned}$$

First Iteration : Let us start iteration with initial (zeroth) approximation to the solution as

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0.$$

We now substitute these only in the first equation of (1), we have

$$x_1^{(1)} = \frac{1}{10}[12 - (0) - (0)] = 1.2$$

We then substitute $x_1^{(1)} = 1.2$, $x_3^{(0)} = 0$ in the second equation of (1), we get

$$x_2^{(1)} = \frac{1}{10}[13 - 2(1.2) - (0)] = 1.06$$

We next put $x_1^{(1)} = 1.2$, $x_2^{(1)} = 1.06$ in third equation of (1), we get

$$x_3^{(1)} = \frac{1}{10}[14 - 2(1.2) - 2(1.06)] = 0.948$$

Second Iteration : Using $x_1^{(1)} = 1.2$, $x_2^{(1)} = 1.06$ and $x_3^{(1)} = 0.948$ as second set of trial solution, we obtain

$$x_1^{(2)} = \frac{1}{10}[12 - 1.06 - 0.948] = 0.9992$$

$$x_2^{(2)} = \frac{1}{10}[13 - 2(0.9992) - (0.948)] = 1.00536$$

$$x_3^{(2)} = \frac{1}{10}[14 - 2(0.9992) - 2(1.00536)] = 0.999098$$

By continuing in this manner, the successive iterations computed correct to four decimal places are tabulated as

n	x_1	x_2	x_3
1	1.2	1.06	0.948
2	0.9992	1.0054	0.9991
3	0.9996	1.0001	1.0001
4	1.000	1.000	1.000

The solution of the given system is therefore $x_1 = 1$, $x_2 = 1$ and $x_3 = 1$.

Ex. 5 : *Solve the following system of equations by Gauss-Seidel method :*

$$\begin{aligned} 27x_1 + 6x_2 - x_3 &= 85 \\ 6x_1 + 15x_2 + 2x_3 &= 72 \\ x_1 + x_2 + 54x_3 &= 110 \end{aligned}$$

Sol. : Given system of equations is diagonally dominant. We express each equation for the unknown having the large coefficient and thus get the system

$$\begin{aligned} x_1 &= \frac{1}{27}[85 - 6x_2 + x_3] \\ x_2 &= \frac{1}{15}[72 - 6x_1 - 2x_3] \\ x_3 &= \frac{1}{54}[110 - x_1 - x_2] \end{aligned} \qquad \ldots (1)$$

First Iteration : We start iteration with initial (zeroth) approximations to the solution as

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0.$$

We now substituting these in the first equation of (1), thus getting

$$x_1^{(1)} = \frac{1}{27}[85 - 6(0) + (0)] = 3.15$$

Now substituting $x_1^{(1)} = 3.15$, $x_3^{(0)} = 0$ in the second equation of (1), we get

$$x_2^{(1)} = \frac{1}{15}[72 - 6(3.15) - 2(0)] = 3.54$$

Then putting $x_1^{(1)} = 3.15$, $x_2^{(1)} = 3.54$ in the third equation, we get

$$x_3^{(1)} = \frac{1}{54}[110 - (3.15) - (3.54)] = 1.91$$

Second Iteration : Using $x_1^{(1)} = 3.15$, $x_2^{(1)} = 3.54$, $x_3^{(1)} = 1.91$, for the second iteration, we have

$$x_1^{(2)} = \frac{1}{27}[85 - 6(3.54) + (1.91)] = 2.43$$

$$x_2^{(2)} = \frac{1}{15}[72 - 6(2.43) - 2(1.91)] = 3.57$$

$$x_3^{(2)} = \frac{1}{54}[110 - (2.43) - (3.57)] = 1.926$$

By continuing in this manner and denoting the successive iterations by n = 1, 2, 3, ..., we get the following table :

n	x_1	x_2	x_3
1	3.15	3.54	1.91
2	2.43	3.57	1.926
3	2.423	3.574	1.926
4	2.425	3.573	1.926
5	2.425	3.573	1.926

The solution of the given system is therefore $x_1 = 2.425$, $x_2 = 3.573$ and $x_3 = 1.926$.

Ex. 6 : *Solve the following system of equations using Gauss-Seidel iteration method :*

$$3.122\, x_1 + 0.5756\, x_2 - 0.1565\, x_3 - 0.0067\, x_4 = 1.571$$
$$0.5756\, x_1 + 2.938\, x_2 + 0.1103\, x_3 - 0.0015\, x_4 = -0.9275$$
$$-0.1565\, x_1 + 0.1103\, x_2 + 4.127\, x_3 + 0.2051\, x_4 = -0.0652$$
$$-0.0067\, x_1 - 0.0015\, x_2 + 0.2051\, x_3 + 4.133\, x_4 = -0.0178$$

Sol. : The given system of equations is diagonally dominant. Expressing each for the unknown having largest coefficient, we have

$$x_1 = \frac{1}{3.122}\,[1.571 - 0.5756\, x_2 + 0.1565\, x_3 + 0.0067\, x_4]$$
$$x_2 = \frac{1}{2.938}\,[-0.9275 - 0.5736\, x_1 - 0.1103\, x_3 + 0.0015\, x_4]$$
$$x_3 = \frac{1}{4.127}\,[-0.0652 + 0.1565\, x_1 - 0.1103\, x_2 - 0.2051\, x_4] \qquad \ldots (1)$$
$$x_4 = \frac{1}{4.138}\,[-0.0178 + 0.0067\, x_1 + 0.0015\, x_2 - 0.2051\, x_3]$$

First Iteration : Let initial approximation be $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0.$

Putting $x_2^{(0)} = x_3^{(0)} = x_4^{(0)}$ in the first equation of (1), we get

$$x_1^{(1)} = \frac{1}{3.122}\,[1.571 - 0.5756\,(0) + 0.1565\,(0) + 0.0067\,(0)] = 0.503$$

Then putting $x_1 = 0.503$, $x_3^{(0)} = x_4^{(0)} = 0$ in the second equation of (1), we get

$$x_2^{(1)} = \frac{1}{2.938}\,[-0.9275 - 0.5756\,(0.503) - 0.1103\,(0) + 0.0015\,(0)] = -0.414$$

To find $x_3^{(1)}$, we put $x_1^{(1)} = 0.503$, $x_2^{(1)} = -0.414$, $x_4^{(0)} = 0$ in the third equation,

$$x_3^{(1)} = \frac{1}{4.127}\,[-0.0652 + 0.1565\,(0.503) - 0.1103\,(-0.414) - 0.2051\,(0)]$$
$$= 0.0143$$

Then for $x_4^{(1)}$, we have

$$x_4^{(1)} = \frac{1}{4.133}\,[-0.0178 + 0.0067\,(0.503) - 0.0015\,(0.414) - 0.2051\,(0.0143)]$$
$$= -0.00435$$

Second Iteration : To start the second iteration, we use the above values of $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}$ and get

$$x_1^{(2)} = \frac{1}{3.122}\,[1.571 - 0.5756\,(-0.414) + 0.1565\,(0.0143) + (0.0067)\,(-0.00435)]$$
$$= 0.580$$

Then
$$x_2^{(2)} = \frac{1}{2.938}\,[-0.9275 - 0.5756\,(0.580) - (0.1103)\,(0.0143) - (0.0015)\,(0.00435)]$$
$$= -0.430$$

Then
$$x_3^{(2)} = \frac{1}{4.127}\,[-0.0652 + 0.1565\,(0.580) - 0.1103\,(-0.430) - (0.2051)\,(-0.00435)]$$
$$= 0.0179$$

and
$$x_4^{(2)} = \frac{1}{4.133}\,[-0.0178 + 0.0067\,(0.580) + 0.0015\,(-0.430) - 0.2051\,(0.0179)]$$
$$= -0.00441$$

By continuing the iteration as outlined above, we obtain the following table :

n	x_1	x_2	x_3	x_4
1	0.503	– 0.414	0.0143	– 0.00435
2	0.580	– 0.430	0.079	– 0.00441
3	0.5834	– 0.4307	0.01805	– 0.004413
4	0.5835	– 0.4307	0.01806	– 0.004413
5	0.5835	– 0.4307	0.01806	– 0.004413

The solution of the given system is thus $x_1 = 0.5835$, $x_2 = -0.4307$, $x_3 = 0.01806$ and $x_4 = -0.004413$.

EXERCISE 8.2

1. Solve the following system of equations by Jacobi's iteration method :

(i)
$$28x_1 + 4x_2 - x_3 = 32$$
$$x_1 + 3x_2 + 10x_3 = 24$$
$$2x_1 + 17x_2 + 4x_3 = 35$$

Ans. $x_1 = 0.9936$, $x_2 = 1.5069$, $x_3 = 1.8484$

(ii)
$$x_1 + 17x_2 - 2x_3 = 48$$
$$2x_1 + 2x_2 + 18x_3 = 30$$
$$30x_1 - 2x_2 + 3x_3 = 48$$

Ans. $x_1 = 2.580$, $x_2 = 2.798$, $x_3 = 1.069$

(iii)
$$2x_1 + x_2 + x_3 = 4$$
$$x_1 + x_2 + 2x_3 = 4$$
$$x_1 + 2x_2 + x_3 = 4$$

Ans. $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

(iv)
$$4x_1 + 2x_2 + x_3 = 14$$
$$x_1 + 5x_2 - x_3 = 10$$
$$x_1 + x_2 + 8x_3 = 20$$

Ans. $x_1 = 2$, $x_2 = 2$, $x_3 = 2$

2. Solve the system of equations by using Jocobi's iteration method :
$$10x_1 + 2x_2 + 3x_3 = 17$$
$$x_1 + 8x_2 - x_4 = -3$$
$$2x_1 + 7x_3 + 11x_4 = 30$$
$$2x_1 + 10x_3 + 2x_4 = 27$$

finding four successive approximations. **Ans.** $x_1 = 1.07192$, $x_2 = -0.38315$, $x_3 = 2.23108$, $x_4 = 1.0019$

3. Apply Jacobi's iteration technique to solve the system
$$17x_1 + 65x_2 - 13x_3 + 50x_4 = 84$$
$$12x_1 + 16x_2 + 37x_3 + 18x_4 = 25$$
$$56x_1 + 23x_2 + 11x_3 - 19x_4 = 36$$
$$3x_1 - 5x_2 + 47x_3 + 10x_4 = 18$$

finding four successive iterations. **Ans.** $x_1 = -0.473$, $x_2 = 2.86$, $x_3 = 0.89$, $x_4 = -0.315$

4. Solve by Jacobi's iteration method, the equations
$$5x_1 - x_2 + x_3 = 10$$
$$2x_1 + 4x_2 = 12$$
$$x_1 + x_2 + 5x_3 = -1$$

Start with the solution (2, 3, 0). **Ans.** $x_1 = 2.556$, $x_2 = 1.722$, $x_3 = -1.055$

5. Solve the following system of equations using Gauss-Seidel iteration method :

(i)
$$\begin{aligned} 28x_1 + 4x_2 - x_3 &= 32 \\ x_1 + 3x_2 + 10x_3 &= 24 \\ 2x_1 + 17x_2 + 4x_3 &= 35 \end{aligned}$$

Ans. $x_1 = 0.9936$, $x_2 = 1.5070$, $x_3 = 1.8485$

(ii)
$$\begin{aligned} 10x_1 - 2x_2 + x_3 &= 12 \\ x_1 + 9x_2 - x_3 &= 10 \\ 2x_1 - x_2 + 11x_3 &= 20 \end{aligned}$$

Ans. $x_1 = 1.262$, $x_2 = 1.159$, $x_3 = 1.694$

(iii)
$$\begin{aligned} 10x_1 + 2x_2 + x_3 &= 9 \\ 2x_1 + 20x_2 - 2x_3 &= -44 \\ -2x_1 + 3x_2 + 10x_3 &= 22 \end{aligned}$$

Ans. $x_1 = 1.0$, $x_2 = -2.0$, $x_3 = 3.0$

(iv)
$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 12 \\ x_1 + 4x_2 + 2x_3 &= 15 \\ x_1 + 2x_2 + 5x_3 &= 20 \end{aligned}$$

Ans. $x_1 = 0.996$, $x_2 = 2$, $x_3 = 3$

(v)
$$\begin{aligned} 2x_1 + x_2 + 6x_3 &= 9 \\ 8x_1 + 3x_2 + 2x_3 &= 13 \\ x_1 + 5x_2 + x_3 &= 7 \end{aligned}$$

Ans. $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

(vi)
$$\begin{aligned} 83x_1 + 11x_2 - 4x_3 &= 95 \\ 7x_1 + 52x_2 + 13x_3 &= 104 \\ 3x_1 + 8x_2 + 29x_3 &= 71 \end{aligned}$$

Ans. $x_1 = 1.052$, $x_2 = 1.369$, $x_3 = 1.962$

(vii)
$$\begin{aligned} 4x_1 - 2x_2 - x_3 &= 40 \\ x_1 - 6x_2 + 2x_3 &= -28 \\ x_1 - 2x_2 + 12x_3 &= -86 \end{aligned}$$

Ans. $x_1 = 10.109375$, $x_2 = 3.8984375$, $x_3 = -7.359375$

(viii)
$$\begin{aligned} 6x_1 + x_2 + x_3 &= 105 \\ 4x_1 + 8x_2 + 3x_3 &= 155 \\ 5x_1 + 4x_2 - 10x_3 &= 65 \end{aligned}$$

Ans. $x_1 = 15$, $x_2 = 10$, $x_3 = 5$

6. Solve by Gauss-Seidel Iteration Method

$$\begin{aligned} 83x_1 + 11x_2 - 4x_3 &= 95 \\ 7x_1 + 52x_2 + 13x_3 &= 104 \\ 3x_1 + 8x_2 + 29x_3 &= 71 \end{aligned}$$

finding solution at the end of fifth iteration. **Ans.** $x_1 = 1.0579$, $x_2 = 1.3672$, $x_3 = 1.9617$

7. Solve the system of following equations by using Gauss-Seidel iteration method. The answer should be correct to three significant digits.

$$\begin{aligned} 9x_1 + 2x_2 + 4x_3 &= 20 \\ x_1 + 10x_2 + 4x_3 &= 6 \\ 2x_1 - 4x_2 + 10x_3 &= -15 \end{aligned}$$

Ans. $x_1 = 2.737$, $x_2 = 0.987$, $x_3 = -1.652$

8. Solve the following equations using Gauss-Seidel method, obtaining the solution at the end of fifth iteration :

$$\begin{aligned} 7a + 2b - 5c &= -18 \\ a + 5b - 3c &= -40 \\ 2a - b - 9c &= -26 \end{aligned}$$

Ans. $a = 1.9996$, $b = -5.9968$, $c = 3.9996$

9. Solve the following system :

$$17x_1 + 65x_2 - 13x_3 + 50x_4 = 84$$
$$12x_1 + 16x_2 + 37x_3 + 18x_4 = 25$$
$$56x_1 + 23x_2 + 11x_3 - 19x_4 = 36$$
$$3x_1 - 5x_2 + 47x_3 + 10x_4 = 18$$

Ans. $x_1 = -0.473$, $x_2 = 2.86$, $x_3 = 0.89$, $x_4 = -0.315$

by Gauss-Seidel method. Carry out computations to two decimal places.

10. Solve the equations :

$$10.22\,x_1 + 1.25\,x_2 + 3.12\,x_3 = 21.047$$
$$1.25\,x_1 + 10.45\,x_2 + 4.15\,x_3 = 62.440$$
$$3.12\,x_1 + 4.15\,x_2 + 10.62\,x_3 = 109.726$$

by Gauss-Seidel iteration method, correct upto three decimal places. **Ans.** $x_1 = -1.21$, $x_2 = 2.220$, $x_3 = 9.819$

UNIT VI : NUMERICAL METHODS

CHAPTER-9

INTERPOLATION, NUMERICAL DIFFERENTIATION AND INTEGRATION

9.1 POLYNOMIAL APPROXIMATION

Many times we come across a situation where the function $y = f(x)$ is not known. All that is known to us is that the function takes certain numerical values f_i for $x = x_i$ ($i = 0, 1, 2, \ldots, n$) or $i = 0\ (1)\ n$. In other words, function is not explicitly defined but a set of values (x_i, y_i), $i = 0, 1, \ldots, n$ are available which satisfy the relation $y = f(x)$. In such case, we may consider a polynomial

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \qquad \ldots (1)$$

such that $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfy (1). All the points (x_i, y_i), $i = 0\ (1)\ n$ lie on the curve (1). Nevertheless we cannot say that equation (1) precisely defines the function $y = f(x)$, because values other than (x_i, y_i) may satisfy $y = f(x)$, but may not satisfy polynomial equation defined by (1). We call the polynomial (1) as polynomial approximation of the function $y = f(x)$.

Given two points $(x_0, y_0), (x_1, y_1)$, we can find a polynomial of degree one namely $y = a_0 + a_1x$ passing through $(x_0, y_0), (x_1, y_1)$. By solving two equations in two unknowns we can determine a_0, a_1 i.e. $y = a_0 + a_1x$ which is a straight line. We can find second degree polynomial (parabola) $y = a_0 + a_1x + a_2 x^2$ passing through three points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. This will require determination of three unknowns by solving three simultaneous equations. Thus, theoretically it should be possible to determine a polynomial

$$y = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$$

passing through $(n + 1)$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ by solving n simultaneous equations.

It is not an easy task to solve a system of n simultaneous equations. In sections to follow we shall discuss methods to find an approximating polynomial. It may also be understood that approximating polynomial passing through set of $(n + 1)$ points can be of first degree i.e. a straight line, or of second degree (parabola) or of any degree. Before we consider the methods to find polynomial approximations, let us examine Taylor's series expansion which is most useful for approximating a given function f (x).

Taylor's series expansion of function f(x) around the point $x = x_0$ in interval [a, b] is given by

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \frac{(x - x_0)^3}{3!} f'''(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^n(x_0) + \frac{(x - x_0)^{n+1}}{(n + 1)!} f^{n+1}(\xi) \quad \ldots (2)$$

Here $f'(x_0), f''(x_0), \ldots, f^n(x_0)$, etc. are the derivatives of first, second and n^{th} order etc. at $x = x_0$.

The term $\dfrac{(x - x_0)^{n-1}}{(n + 1)!} f^{n+1}(\xi)$ (ξ in [a, b]) is called the remainder term. It gives the truncation error when first n terms of the Taylor's series expansion are used to approximate the function f(x).

$$T \text{ (truncation error)} = \left| \frac{(x - x_0)^{n+1} f^{n+1}(\xi)}{(n + 1)!} \right|$$

If ξ is so chosen that $f^{n+1}(x)$ is maximum for $x = \xi$ in the interval (a, b), then

$$T \leq \left| \frac{(x - x_0)^{n+1} f^{n+1}(\xi)}{(n + 1)!} \right|$$

ILLUSTRATIONS

Ex. 1 : *Obtain polynomial approximation to $f(x) = e^x$ (around $x = 0$) using Taylor's series expansion. Find the number of terms in the approximation so that truncation error is less than 10^{-6} for $0 \leq x \leq 1$.*

Sol. : $f(x) = e^x$, $\quad f'(x) = e^x$, $\quad f''(x) = e^x$, …… etc.

$\therefore$ $f(0) = 1$, $\quad f'(0) = 1$, $\quad f''(0) = 1$ ……

Substituting in (2), Taylor's series expansion is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots$$

$$T \le \left| \frac{x^{n+1}}{(n+1)!} f^{n+1}(\xi) \right|$$

for $0 \le x \le 1$, $\xi = 1$ as f (x) is maximum for x = 1.

$$f^{n+1}(1) = e^1 = e$$

∴ For requisite accuracy,

$$\left| \frac{x^{n+1} e}{(n+1)!} \right| \le 10^{-6}$$

Taking maximum for x i.e. x = 1, we must have,

$$\frac{e}{(n+1)!} \le 10^{-6}$$

$$\therefore \quad \log e - \log (n+1)! \le -6 \log 10$$

$$\text{or} \quad \log (n+1)! \ge 1 + 6 \log 10$$

$$\ge 14.82$$

$$\therefore \quad n = 9$$

Thus nine terms are required for error to be less than 10^{-6}.

9.2 INTERPOLATING POLYNOMIAL

In previous section we have discussed polynomial approximation of the function f (x). When a function is defined by means of set of values (x_i, y_i), i = 0 (1) n polynomial approximation ideally should satisfy the given set of values. We may now use this approximating polynomial to find the values of y corresponding to the values of x which may lie between (x_i, x_{i+1}) for i = 0 (1) n. That means if x varies over the interval [a, b] and if table of values of (x, y) is available for some distinct values of x lying in the interval [a, b] we can find approximating polynomial function P(x) and use this to find values of y for any value of x in [a, b]. This is called interpolation. Approximating polynomial is called interpolating polynomial. We can also use this polynomial to find value of y for x lying outside the interval [a, b]. This is termed as extrapolation. Thus, interpolating polynomial may be used for both interpolation and extrapolation.

9.3 LAGRANGE'S INTERPOLATION

Lagrange's interpolation polynomial of n^{th} degree is defined as

Consider, $$L_i(x_k) = 0 \text{ when } k \ne i$$

$$= 1 \text{ when } k = i \qquad \ldots (1)$$

This ensures that $x_0, x_1, x_2, \ldots, x_n$ are the roots of $L_i(x)$ except $x = x_i$. Thus $L_i(x)$ can be written as

$$L_i(x) = A_i (x - x_0)(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n) \qquad \ldots (2)$$

where, A_i can be determined from the condition $L_i(x_i) = 1$.

Substituting $$x = x_i \text{ in (2)},$$

$$L_i(x_i) = 1 \text{ by virtue of (1)}$$

$$\therefore \quad 1 = A_i (x_i - x_0)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)$$

$$\therefore \quad A_i = \frac{1}{(x_i - x_0)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)}$$

Putting for A_i in (2), we get,

$$L_i(x) = \frac{(x - x_0)(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_0)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)} \qquad \ldots (3)$$

Using definition of $L_i(x)$ and from (3), putting i = 0, 1, 2, …, etc.

$$L_0(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_0)(x_2-x_3)\dots(x_2-x_n)}$$

$$L_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

In general, we can write,

$$L_i(x) = \frac{\prod\limits_{\substack{r=0\\(r\neq i)}}^{n}(x-x_r)}{\prod\limits_{\substack{r=0\\(r\neq i)}}^{n}(x_i-x_r)} \qquad \dots (4)$$

where infinite product notation means that we take continued product of the terms for r = 0, 1, 2, …, n excepting the factor corresponding to r = i.

With $L_i(x)$ well defined, we are in a position to find Lagrange's interpolating polynomial through (x_i, y_i) i = 0 (1) n. It is given by;

$$P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$$

or $$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + \dots + y_n L_n(x) \qquad \dots (5)$$

Putting $x = x_0$ in (5),

$$P_n(x_0) = y_0 L_0(x_0) + y_1 L_1(x_0) + \dots + y_n L_n(x_0)$$

But $L_1(x_0), L_2(x_0), \dots, L_n(x_0)$ are each equal to zero.

$\therefore \quad P_n(x_0) = y_0$

Thus, point (x_0, y_0) satisfies the equation (5).

Similarly, it can be easily seen that $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ satisfy (5).

Hence (5) is the interpolating polynomial of degree n and is called Lagrange's interpolating polynomial passing through the set of points (x_i, y_i) i = 0 (1) n, which can be used for interpolation or extrapolation.

ILLUSTRATIONS

Ex. 1 : *Find Lagrange's interpolating polynomial passing through set of points*

x	0	1	2
y	4	3	6

Use it to find y at x = 1.5, $\frac{dy}{dx}$ *at x = 0.5 and find* $\int_0^3 y\,dx$.

Sol. : Here from equation (5),

$$y = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) \qquad \dots (a)$$

$$L_0(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x^2-3x+2)$$

$$L_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -(x^2-2x)$$

$$L_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}(x^2-x)$$

[Given : $x_0 = 0,\ y_0 = 4,\ x_1 = 1,\ y_1 = 3,\ x_2 = 2,\ y_2 = 6$]

Putting in (a),
$$\begin{aligned} y &= 4\cdot\frac{1}{2}(x^2-3x+2) + 3[-(x^2-2x)] + 6\cdot\frac{1}{2}(x^2-x) \\ &= 2x^2-6x+4-3x^2+6x+3x^2-3x \\ &= 2x^2-3x+4 \end{aligned}$$

$\therefore$ y = $2x^2 - 3x + 4$ is the Lagrange's interpolating polynomial through given data.

$$y\Big]_{x = 1.5} = 2\,(1.5)^2 - 3\,(1.5) + 4 = 4$$

$$\frac{dy}{dx} = 4x - 3\Big]_{x = 0.5} = 4 \times (0.5) - 3 = -1$$

$$\int_0^3 y\,dx = \int_0^3 (2x^2 - 3x + 4)\,dx = \left[2 \cdot \frac{x^3}{3} - 3\,\frac{x^2}{2} + 4x\right]_0^3$$

$$= \frac{2}{3}(27) - \frac{3}{2}(9) + 4 \times 3 = 18 - \frac{27}{2} + 12 = 16.5$$

It may be noted that for interpolation or extrapolation, at some value of x, it is not necessary to find interpolating polynomial with explicit coefficients of various powers of x. We can substitute the value of x directly in equation (a). But for finding derivative or integral, above procedure should be followed.

Ex. 2 : *Given $(1.0)^3 = 1.000$, $(1.2)^3 = 1.728$, $(1.3)^3 = 2.197$ and $(1.5)^3 = 3.375$. Using Lagrange's interpolation formula, evaluate $(1.07)^3$.*

(Dec. 2010)

Sol. : Given data can be tabulated as

x	1.0	1.2	1.3	1.5
y	1.0	1.728	2.197	3.375

Using Lagrange's interpolation formula in equation (5),

$$y = \frac{(x - 1.2)\,(x - 1.3)\,(x - 1.5)}{(1.0 - 1.2)\,(1.0 - 1.3)\,(1.0 - 1.5)} \times (1) + \frac{(x - 1.0)\,(x - 1.3)\,(x - 1.5)}{(1.2 - 1.0)\,(1.2 - 1.3)\,(1.2 - 1.5)} \times (1.728)$$
$$+ \frac{(x - 1.0)\,(x - 1.2)\,(x - 1.5)}{(1.3 - 1.0)\,(1.3 - 1.2)\,(1.3 - 1.5)} \times (2.197) + \frac{(x - 1.0)\,(x - 1.2)\,(x - 1.3)}{(1.5 - 1.0)\,(1.5 - 1.2)\,(1.5 - 1.3)} \times (3.375)$$

Putting x = 1.07,

$$y\Big]_{(1.07)} = \frac{(1.07 - 1.2)\,(1.07 - 1.3)\,(1.07 - 1.5)}{(-0.2)\,(-0.3)\,(-0.5)} \times (1) + \frac{(1.07 - 1.0)\,(1.07 - 1.3)\,(1.07 - 1.5)}{(0.2)\,(-0.1)\,(-0.3)} \times (1.728)$$
$$+ \frac{(1.07 - 1.0)\,(1.07 - 1.2)\,(1.07 - 1.5)}{(0.3)\,(0.1)\,(-0.2)} \times (2.197) + \frac{(1.07 - 1.0)\,(1.07 - 1.2)\,(1.07 - 1.3)}{(0.5)\,(0.3)\,(0.2)} \times (3.375)$$

which after simplification gives

$$y = 0.4285666 + 1.126944 - 1.432810 + 0.23546$$
$$= 0.3581606$$

9.4 REMAINDER TERM IN LAGRANGE'S FORMULA

For the polynomial

$$P_n(x) = \sum_{i=0}^{n} Y_i\,L_i(x)$$

which approximates unknown function at (n + 1) points (x_i, y_i), i = 0 (1) n, the remainder term must have zero at $x_0, x_1, x_2, \ldots, x_n$.

$$\therefore \quad \text{Remainder term} = (x - x_0)\,(x - x_1) \ldots (x - x_n)\,\phi(x)$$

$$\therefore \quad f(x) = P_n(x) + \phi(x) \prod_{i=0}^{n} (x - x_i)$$

Now, consider a function

$$F(s) = f(s) - P_n(s) - \phi(x) \prod_{i=0}^{n} (s - x_i)$$

where, x is fixed.

Since F (s) has (n + 2) roots $x_0, x_1, x_2, \ldots, x_n$ and x, we have by Rolle's theorem,

$$F^{(n+1)}(\xi) = 0 \text{ for } x_0 < \xi < x_n$$

But $$F^{(n+1)}(s) = f^{n+1}(s) - (n + 1)!\,\phi(x)$$

as $$F^{(n+1)}(\xi) = 0$$

we get, $$\phi(x) = \frac{f^{n+1}(\xi)}{(n + 1)!}$$

Thus Lagrange's interpolation formula with remainder term is

$$f(x) = \sum_{i=0}^{n} Y_i L_i(x) + \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \qquad \dots (1)$$

Here remainder term is $R_n = \dfrac{f^{n+1}(\xi)}{(n+1)!} \displaystyle\prod_{i=0}^{n} (x - x_i)$

which determines the accuracy of the interpolation.

ILLUSTRATIONS

Ex. 1 : *Given*

x	*1*	*1.2*	*1.3*	*1.4*
$\sqrt{x}$	*1*	*1.095*	*1.140*	*1.183*

Find $\sqrt{1.1}$ *using Lagrange's interpolation. Determine the accuracy of the interpolation.*

Sol. : Lagrange's interpolating polynomial through given data is given by

$$y = \frac{(x-1.2)(x-1.3)(x-1.4)}{(1-1.2)(1-1.3)(1-1.4)} \times (1) + \frac{(x-1)(x-1.3)(x-1.4)}{(1.2-1)(1.2-1.3)(1.2-1.4)} \times (1.095)$$
$$+ \frac{(x-1)(x-1.2)(x-1.4)}{(1.3-1)(1.3-1.2)(1.3-1.4)} \times (1.140) + \frac{(x-1)(x-1.2)(x-1.3)}{(1.4-1)(1.4-1.2)(1.4-1.3)} \times (1.183)$$

$$y\Big]_{x=1.1} = \frac{(1.1-1.2)(1.1-1.3)(1.1-1.4)}{-(0.2)(0.3)(0.4)} \times (1) + \frac{(1.1-1)(1.1-1.3)(1.1-1.4)}{(0.2)(0.1)(0.2)} \times (1.095)$$
$$+ \frac{(1.1-1)(1.1-1.2)(1.1-1.4)}{-(0.3)(0.1)(0.1)} \times (1.140) + \frac{(1.1-1)(1.1-1.2)(1.1-1.3)}{(0.4)(0.2)(0.1)} \times (1.183)$$
$$= \frac{(0.1)(0.2)(0.3)}{0.024} \times (1) + \frac{(0.1)(0.2)(0.3)}{0.004} \times (1.095) - \frac{(0.1)(0.1)(0.3)}{0.003} \times (1.140) + \frac{(0.1)(0.1)(0.2)}{0.008} \times (1.183)$$
$$= 0.25 + 1.6425 - 1.140 + 0.29575$$
$$= 1.04825$$

Here $n = 3$. The remainder term is

$$R_n = \frac{f^4(\xi)}{4!} \prod_{i=0}^{3} (1.1 - x_i)$$
$$= \frac{(1.1-1)(1.1-1.2)(1.1-1.3)(1.1-1.4)}{24} f^4(\xi)$$
$$= -\frac{(0.1)(0.1)(0.2)(0.3)}{24} f^4(\xi) \quad \text{where, } 1 < \xi < 1.4$$

We note that, $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} x^{-1/2}$, $f''(x) = -\dfrac{1}{4} x^{-3/2}$, $f'''(x) = \dfrac{3}{8} x^{-5/2}$, $f^4(x) = -\dfrac{15}{16} \dfrac{1}{x^{7/2}}$

$$\text{Max. } f^4(x) = -\frac{15}{16} \text{ (at } x = 1)$$
$$f^4(\xi) = -\frac{15}{16}$$
$$\therefore \quad R_n = \frac{0.0006}{24} \times \frac{15}{16} = 2.34375 \times 10^{-5}$$

Thus the interpolation is correct upto 4th decimal place.

In our Ex., interpolation is correct upto 3rd decimal place because all the values of $\sqrt{x}$ are rounded at 3rd decimal place.

9.5 FINITE DIFFERENCES AND DIFFERENCE OPERATORS

Consider set of points (x_i, y_i) $i = 0\ (1)\ n$ which satisfy the function $y = f(x)$. Let us further assume that the set of values of x are evenly spaced and $x_i - x_{i-1} = h$ for $i = 0\ (1)\ n$.

The forward difference operator Δ is defined such that when it operates on f(x), it gives the difference

$$\Delta f(x) = f(x+h) - f(x) \qquad \dots (1)$$

This difference $\Delta f(x)$ being the difference between forward value of the function (i.e. the value of the function at next point x + h) and the present value (i.e. the value of the function at current point x) is called the first forward difference or forward difference of the first order.

If Δ operates on (1) again, then

$$\Delta(\Delta f(x)) = \Delta[f(x+h) - f(x)]$$

or $$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

Using (1) again, $$\Delta^2 f(x) = f(x+2h) - f(x+h) - f(x+h) + f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x) \quad \dots (2)$$

This gives forward difference of second order or second forward difference of f (x).

Similarly, we can obtain third and higher order differences $\Delta^3 f(x)$, $\Delta^4 f(x) \dots \Delta^n f(x)$.

Since (x_i, y_i), i = 0 (1) n satisfy y = f (x) and h is the difference between successive values of x, we have

$$y_0 = f(x_0), \quad y_1 = f(x_1) = f(x_0+h), \quad y_2 = f(x_2)$$

or $$y_2 = f(x_1+h) = f(x_0+2h) \text{ ...... etc.}$$

Now, $$\Delta f(x_0) = f(x_0+h) - f(x_0)$$

or $$\Delta y_0 = y_1 - y_0$$

Similarly, $$\Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_2$$

or in general, $$\Delta y_r = y_{r+1} - y_r \quad \dots (3)$$

Again, $$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$\therefore$ $$\Delta^2 y_0 = y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$

Similarly, we can find $\Delta^3 y_0$, $\Delta^4 y_0$ ...... etc.

Results such as $$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$\Delta^n y_0 = \Delta^{n-1} y_1 - \Delta^{n-1} y_0$$

help us in framing forward difference table for set of values (x_i, y_i).

ILLUSTRATIONS

Ex. 1 : *Prepare the forward difference table for*

$$y = x^3 - 2x + 5, \quad x = 0\ (1)\ 6$$

Sol. : Here x varies from 0 to 6 with h = 1, which is called differencing interval.

First we tabulate values of x and y. y is calculated from the relation $y = x^3 - 2x + 5$ for values of x from x = 0 to x = 6.

x	0	1	2	3	4	5	6
y	5	4	9	26	61	120	209

Forward difference table is now presented as

Table 9.1

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$0 = x_0$	$5 = y_0$						
		$-1 = \Delta y_0$					
$1 = x_1$	$4 = y_1$		$6 = \Delta^2 y_0$				
		$5 = \Delta y_1$		$6 = \Delta^3 y_0$			
$2 = x_2$	$9 = y_2$		$12 = \Delta^2 y_1$		$0 = \Delta^4 y_0$		
		$17 = \Delta y_2$		$6 = \Delta^3 y_1$		$0 = \Delta^5 y_0$	
$3 = x_3$	$26 = y_3$		$18 = \Delta^2 y_2$		$0 = \Delta^4 y_1$		$0 = \Delta^6 y_0$
		$35 = \Delta y_3$		$6 = \Delta^3 y_2$		$0 = \Delta^5 y_1$	
$4 = x_4$	$61 = y_4$		$24 = \Delta^2 y_3$		$0 = \Delta^4 y_2$		
		$59 = \Delta y_4$		$6 = \Delta^3 y_3$			
$5 = x_5$	$120 = y_5$		$30 = \Delta^2 y_4$				
		$89 = \Delta y_5$					
$6 = x_6$	$209 = y_6$						

Certain observations can be made from the forward difference table.

1. Third order differences are constants i.e. all the values in column headed by $\Delta^3 y$ are same. In this case, each value is 6.
2. Fourth and higher order differences are equal to zeroes. All the values in columns headed by $\Delta^4 y$, $\Delta^5 y$, $\Delta^6 y$ are zeroes.
3. For a table of values containing seven entries, we have obtained a forward difference table giving differences upto 6^{th} order.
4. There are maximum number of differences available for the first value. Here corresponding to $y_0 = 5$, we have

   $\Delta y_0 = -1, \quad \Delta^2 y_0 = 6, \quad \Delta^3 y_0 = 6, \quad \Delta^4 y_0 = 0, \quad \Delta^5 y_0 = 0, \quad \Delta^6 y_0 = 0$

Thus all the differences upto 6^{th} order are available for $y_0 = 5$.

For $y_1 = 4$, differences upto 5^{th} order are available and so on. For the last value y_6, no difference is available.

The observation that third order differences are constant is a general property. For polynomial of n^{th} degree, the n^{th} order differences are constants and higher order differences are zeroes. This property can be used to extend the forward difference table upto any desired value of x.

Ex. 2 : *Extend the table of previous problem upto x = 8.*

Sol. :

Table 9.2

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	5						
		–1					
1	4		6				
		5		6			
2	9		12		0		
		17		6		0	
3	26		18		0		0
		35		6		0	
4	61		24		0		0
		59		6		0	
5	120		30		0		0
		89		6		0	
6	209		36		0		
		125		6			
7	334		42				
		167					
8	501						

We add sufficient number of zeroes in columns headed by $\Delta^6 y$, $\Delta^5 y$, $\Delta^4 y$ and then add 6 in column headed by $\Delta^3 y$. Working backwards we arrive at the values of y for x = 7 and x = 8.

Ex. 3 : *Determine $\Delta f(x)$ for $f(x) = x^3 - 2x + 5$ with h = 1.*

Sol. :

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) = (x+h)^3 - 2(x+h) + 5 - x^3 + 2x - 5 \\ &= x^3 + 3hx^2 + 3xh^3 + h^2 - 2x - 2h + 5 - x^3 + 2x - 5 \\ &= 3hx^2 + 3xh^2 + h^3 - 2h \end{aligned}$$

Putting h = 1, we get

$$\Delta f(x) = 3x^2 + 3x - 1$$

which is a polynomial of second degree.

Thus it is observed that operation by Δ on a polynomial of degree n, reduces its degree by one.

We now introduce the operator ∇ called Backward difference operator and it is defined such that

$$\nabla f(x) = f(x) - f(x-h) \qquad \ldots (4)$$

which gives backward difference of the first order or first backward difference. It is the difference between value of f (x) at current point x and the preceding point x – h.

$$\begin{aligned} \nabla^2 f(x) &= \nabla [f(x) - f(x-h)] = \nabla f(x) - \nabla f(x-h) \\ &= f(x) - f(x-h) - f(x-h) + f(x-2h) = f(x) - 2 f(x-h) + f(x-2h) \qquad \ldots (5) \end{aligned}$$

which gives second order Backward difference.

Similarly we can obtain third and higher order differences $\nabla^3 f(x), \nabla^4 f(x), \ldots, \nabla^n f(x)$.

As seen earlier $y_0 = f(x_0), \quad y_1 = f(x_1) = f(x_0 + h)$

We can denote $f(x_0 - h)$ by y_{-1}, $f(x_0 - 2h)$ by y_{-2}, etc.

With these notations,

$$\nabla f(x_0) = f(x_0) - f(x_0 - h)$$

or $$\nabla y_0 = y_0 - y_{-1}$$

or in general, $$\nabla y_r = y_r - y_{r-1}$$

Again, $$\nabla^2 y_0 = \nabla(\nabla y_0) = \nabla(y_0 - y_{-1}) = \nabla y_0 - \nabla y_{-1}$$

$\therefore$ $$\nabla^2 y_0 = y_0 - y_{-1} - y_{-1} + y_{-2} = y_0 - 2y_{-1} + y_{-2}$$

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1$$

$\therefore$ $$\nabla^2 y_2 = y_2 - y_1 - y_1 + y_0 = y_2 - 2y_1 + y_0 = \Delta^2 y_0 \qquad \ldots (6)$$

In general, $$\nabla^n y_r = \nabla^{n-1} \nabla y_r = \nabla^{n-1}(y_r - y_{r-1})$$

or $$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}$$

Such type of relations enable us to prepare the backward difference table.

Ex. 4 : *Prepare backward difference table for $y = x^3 + 1$, $x = 0\ (1)\ 5$. Proceeding as in Ex. 1, we get the backward difference table as*

Table 9.3

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
$0 = x_0$	$1 = y_0$					
		$1 = \nabla y_1$				
$1 = x_1$	$2 = y_1$		$6 = \nabla^2 y_2$			
		$7 = \nabla y_2$		$6 = \nabla^3 y_3$		
$2 = x_2$	$9 = y_2$		$12 = \nabla^2 y_3$		$0 = \nabla^4 y_4$	
		$19 = \nabla y_3$		$6 = \nabla^3 y_4$		$0 = \nabla^5 y_5$
$3 = x_3$	$28 = y_3$		$18 = \nabla^2 y_4$		$0 = \nabla^4 y_5$	
		$37 = \nabla y_4$		$6 = \nabla^3 y_5$		
$4 = x_4$	$65 = y_4$		$24 = \nabla^2 y_5$			
		$61 = \nabla y_5$				
$5 = x_5$	$126 = y_5$					

Sol. : Tables 9.2 and 9.3 exhibit identical properties. Using the property of the table, namely the third order differences are constants, the table can be extended on upper side i.e. say for values of $x = -1$ and $x = -2$, etc.

Here the maximum number of differences are available for $y_5 = 126$ and none for $y_0 = 1$.

Comparison of Tables 9.2 and 9.3 show that there is correspondence between entries.

$$\Delta y_0 \longleftrightarrow \nabla y_1, \quad \Delta y_1 \longleftrightarrow \nabla y_2, \text{ etc.}$$
$$\Delta^2 y_0 \longleftrightarrow \nabla^2 y_2, \quad \Delta^2 y_1 \longleftrightarrow \nabla^2 y_3, \text{ etc.} \qquad \ldots (7)$$
$$\Delta^3 y_0 \longleftrightarrow \nabla^3 y_3, \quad \Delta^3 y_1 \longleftrightarrow \nabla^3 y_4, \text{ etc.}$$

Similarly for other entries.

Central Differences : The central difference operator denoted by δ is defined such that

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \qquad \ldots (8)$$

and it gives what we call as central difference of first order or first central difference.

Reoperating by δ on (1), we get

$$\delta[\delta f(x)] = \delta f\left(x + \frac{h}{2}\right) - \delta f\left(x - \frac{h}{2}\right)$$

or $$\delta^2 f(x) = f(x + h) - f(x) - f(x) + f(x - h)$$
$$= f(x + h) - 2 f(x) + f(x - h) \qquad \ldots (9)$$

which gives central difference of second order.

Similarly, we can find central differences of higher order.

If
$$y_r = f(x_r)$$
$$\delta y_r = f\left(x_r + \frac{h}{2}\right) - f\left(x_r - \frac{h}{2}\right)$$
$$= y_{r+1/2} - y_{r-1/2} \qquad \dots (10)$$

Putting
$$r = \frac{1}{2}$$
$$\delta y_{1/2} = y_1 - y_0 = \Delta y_0 = \nabla y_1 \qquad \dots (11)$$
$$\delta^2 y_{1/2} = \delta y_1 - \delta y_0$$
$$= y_{3/2} - y_{1/2} - y_{1/2} + y_{-1/2}$$
$$= y_{3/2} - 2y_{1/2} + y_{-1/2}$$

Ex. 5 : *Prepare central difference table for $y = x^3 + 1$, $x = 0$ (1) 5. Proceeding as in previous Ex.,*

Table 9.4

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
$0 = x_0$	$1 = y_0$					
		$1 = y_1 - y_0 = \delta y_{1/2}$				
$1 = x_1$	$2 = y_1$		$6 = \delta^2 y_1$			
		$7 = y_2 - y_1 = \delta y_{3/2}$		$6 = \delta^3 y_{3/2}$		
$2 = x_2$	$9 = y_2$		$12 = \delta^2 y_2$		$0 = \delta^4 y_2$	
		$19 = y_3 - y_2 = \delta y_{5/2}$		$6 = \delta^3 y_{5/2}$		$0 = \delta^5 y_{5/2}$
$3 = x_3$	$28 = y_3$		$18 = \delta^2 y_3$		$0 = \delta^4 y_3$	
		$37 = y_4 - y_3 = \delta y_{7/2}$		$6 = \delta^3 y_{7/2}$		
$4 = x_4$	$65 = y_4$		$24 = \delta^2 y_4$			
		$61 = y_5 - y_4 = \delta y_{9/2}$				
$5 = x_5$	$126 = y_5$					

Sol. : Comparison of Tables 9.2, 9.3, 9.4 show that there is correspondence between
$$\Delta y_0 \longleftrightarrow \nabla y_1 \longleftrightarrow \delta y_{1/2} = y_1 - y_0$$
$$\Delta^2 y_0 \longleftrightarrow \nabla^2 y_2 \longleftrightarrow \delta^2 y_1 = y_2 - 2y_1 + y_0 \qquad \dots (12)$$

Similarly for other differences.

Maximum number of central differences are available for a value located at the centre of the table.

In general, for a polynomial of n^{th} degree, forward, backward or central differences of n^{th} order are constant and higher order differences are zeroes.

Ex. 6 : *Show that*
$$\Delta\nabla = \nabla\Delta = \delta^2$$

Sol. : Consider the operation of $\Delta\nabla$ on f (x) with h as the differentiating interval.
$$\Delta\nabla f(x) = \Delta [f(x) - f(x-h)]$$
$$= \Delta f(x) - \Delta f(x-h)$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h) \qquad \dots (a)$$
$$\nabla\Delta f(x) = \nabla [f(x+h) - f(x)]$$
$$= \nabla f(x+h) - \nabla f(x)$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h) \qquad \dots (b)$$

$$\delta^2 f(x) = \delta[\delta f(x)] = \delta\left\{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right\}$$
$$= \delta f\left(x+\frac{h}{2}\right) - \delta f\left(x-\frac{h}{2}\right)$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h) \qquad \ldots (c)$$

(a), (b), (c) are same, which establishes the operational equivalence between three operators.

Average Operator : Averaging operator denoted by μ is defined such that

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right] \qquad \ldots (13)$$

Ex. 7 : *Show that* $\mu^2 = 1 + \dfrac{\delta^2}{4}$.

Sol. : Consider

$$\mu^2 f(x) = \mu[\mu f(x)] = \mu\left[\frac{1}{2}\left\{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right\}\right]$$
$$= \frac{1}{2}\left[\mu f\left(x+\frac{h}{2}\right) + \mu f\left(x-\frac{h}{2}\right)\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}\{f(x+h) + f(x)\} + \frac{1}{2}\{f(x) + f(x-h)\}\right]$$
$$= \frac{1}{4}[f(x+h) + 2f(x) + f(x-h)] \qquad \ldots (a)$$

In previous Ex., we have seen that

$$\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h)$$
$$\left(1+\frac{\delta^2}{4}\right) f(x) = f(x) + \frac{1}{4}[f(x+h) - 2f(x) + f(x-h)] = \frac{1}{4}[f(x+h) + 2f(x) + f(x-h)] \qquad \ldots (b)$$

(a) and (b) show that $\mu^2 = 1 + \dfrac{\delta^2}{4}$

Shift Operator : Shift operator denoted by E is defined such that

$$E f(x) = f(x+h) \qquad \ldots (14)$$

In general, $E^r f(x) = f(x+rh)$ … (15)

or $E^{-r} f(x) = f(x-rh)$ … (16)

We can easily establish relationship between Δ, ∇, E or E^{-1} and δ.

By definition, $\Delta f(x) = f(x+h) - f(x)$

$$= E f(x) - f(x)$$
$$= (E-1) f(x)$$
$$\therefore \quad \Delta = E - 1 \qquad \ldots (17)$$
$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$= (1 - E^{-1}) f(x)$$
$$\therefore \quad \nabla = 1 - E^{-1} \qquad \ldots (18)$$
$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$
$$= E^{1/2} f(x) - E^{1/2} f(x-h)$$
$$= E^{1/2}[f(x) - f(x-h)]$$
$$= E^{1/2} \nabla f(x)$$
$$\therefore \quad \delta = E^{1/2} \nabla \qquad \ldots (19)$$

Again,
$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$
$$= E^{-1/2} f(x + h) - E^{-1/2} f(x)$$
$$= E^{-1/2} [f(x + h) - f(x)]$$
$$= E^{-1/2} \Delta f(x)$$
$$\therefore \quad \delta = E^{-1/2} \Delta \qquad \ldots (20)$$

9.6 NEWTON'S FORWARD DIFFERENCE INTERPOLATION

For evenly spaced data, formula can be obtained as follows.

We have already seen that
$$E f(x_0) = f(x_0 + h)$$
or in general,
$$E^u f(x_0) = f(x_0 + uh)$$
$$\therefore \quad f(x_0 + uh) = E^u f(x_0) = E^u y_0$$

But, $E = 1 + \Delta$
$$\therefore \quad f(x_0 + uh) = (1 + \Delta)^u y_0$$

Expanding by Binomial theorem,
$$f(x_0 + uh) = \left\{1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{(u-1)(u-2)}{3!}\Delta^3 + \ldots\right\} y_0$$
$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \qquad \ldots (1)$$

Since $x = x_0 + uh$ or $u = \dfrac{x - x_0}{h}$ putting for u in (1), we get Newton-Gregory interpolating polynomial passing through a set of points (x_i, y_i), i = 0 (1) n.

9.7 NEWTON'S BACKWARD DIFFERENCE INTERPOLATION

In sec. 9.5, we have seen that $E^{-1} = 1 - \nabla$

If x is expressed as $x = x_n + uh$.

then
$$f(x) = f(x_n + uh) = E^u f(x_n)$$
$$\therefore \quad y = (E^{-1})^{-u} y_n$$
or
$$y = (1 - \nabla)^{-u} y_n$$

Expanding by Binomial theorem, we get
$$y = \left\{1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \frac{u(u+1)(u+2)}{3!}\nabla^3 + \ldots\right\} y_n$$
$$= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots (1)$$

This gives Newton's Backward difference interpolation formula.

We note here that in the above formula x is expressed in the form $x = x_n + uh$, if we express x as $x_0 - uh$, then
$$E^{-1} = 1 - \nabla$$
$$f(x_0 - uh) = E^{-u} f(x_0) = E^{-u} y_0$$
$$\therefore \quad f(x_0 - uh) = (E^{-1})^u y_0$$
$$= (1 - \nabla)^u y_0$$

Expanding by Binomial theorem,
$$f(x_0 - uh) = \left\{1 - u\nabla + \frac{u(u-1)}{2!}\nabla^2 - \frac{u(u-1)(u-2)}{3!}\nabla^3 + \ldots\ldots\right\} y_0$$
$$= y_0 - u\nabla y_0 + \frac{u(u-1)}{2!}\nabla^2 y_0 - \frac{u(u-1)(u-2)}{3!}\nabla^3 y_0 + \ldots \qquad \ldots (2)$$

Which is another form of Newton's Backward difference formula for interpolation.

ILLUSTRATIONS

Ex. 1 : *Find Newton's interpolating polynomial satisfying the data*

x	0	1	2	3	4
y	–4	–4	0	14	44

Sol. : We first construct forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–4				
		0			
1	–4		4		
		4		6	
2	0		10		0
		14		6	
3	14		16		
		30			
4	44				

Now, $x = x_0 + uh$, where $x_0 = 0$, $h = 1$

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

From the table, $y_0 = -4$, $\Delta y_0 = 0$, $\Delta^2 y_0 = 4$, $\Delta^3 y_0 = 6$, $\Delta^4 y_0 = 0$

Newton's interpolating polynomial is given by

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\,\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\,\Delta^3 y_0 \qquad \ldots \text{[From 1 of 9.6]}$$

Putting for u, y_0, Δy_0, …, etc.

$$y = -4 + x \times 0 + \frac{x(x-1)}{2} \times 4 + \frac{x(x-1)(x-2)}{6} \times 6$$

$$= -4 + 2x^2 - 2x + x^3 - 3x^2 + 2x$$

$$= x^3 - x^2 - 4$$

which is the required interpolating polynomial.

Ex. 2 : *Construct the difference table from the following data :*

x	50	51	52	53	54
f (x)	39.1961	39.7981	40.3942	40.9843	41.5687

Obtain f(50.5) using Newton's forward difference formula and f(53.4) using Newton's backward difference formula correct to 4 decimal places. Write the interpolation equation for the curve passing through the points given in the table.

We first construct forward difference table. ***(May 92)***

x	f (x)	Δ f (x)	Δ^2 f (x)	Δ^3 f (x)	Δ^4 f (x)
50	39.1961				
		0.602			
51	39.7981		– 0.0059		
		0.5961		– 0.0001	
52	40.3942		– 0.0060		0.0004
		0.5901		+ 0.0003	
53	40.9843		– 0.0057		
		0.5844			
54	41.5687				

Sol. : For finding f (50.5) we use Newton's forward difference formula.

Now, $x = x_0 + uh \quad \therefore \quad u = \dfrac{x - x_0}{h}$

Taking $x_0 = 50, \quad h = 1, \quad x = 50.5$

$\therefore \quad u = \dfrac{50.5 - 50}{1} = 0.5$

By Newton's forward difference formula for $y = f(x), \quad y_0 = f(x_0)$

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \qquad \ldots(1)$$

Putting in (1), $y_0 = 39.1961, \quad \Delta y_0 = 0.602, \quad \Delta^2 y_0 = -0.0059, \quad \Delta^3 y_0 = -0.0001, \quad \Delta^4 y_0 = 0.0004, \quad u = 0.5$

we get,
$$y = f(50.5) = 39.1961 + (0.5)(0.602) + \frac{1}{2}(0.5)(0.5-1)(-0.0059) + \frac{1}{6}(0.5)(0.5-1)(0.5-2)(-0.0001)$$
$$+ \frac{1}{24}(0.5)(0.5-1)(0.5-2)(0.5-3)(0.0004) + \ldots$$
$$= 39.497847$$

To find f (53.4), we use Newton's backward differences formula

$$x = x_n - uh \quad \therefore \quad u = \frac{x - x_n}{h}$$

Taking $x_n = 54, \quad x = 53.4, \quad h = 1$

$$u = \frac{53.4 - 54}{1} = -0.6$$

By Newton's backward difference formula for $y = f(x)$, $y_n = f(x_n)$

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \ldots \qquad \ldots(2)$$

Putting in (2) $y_n = 41.5687, \quad \nabla y_n = 0.5844, \quad \nabla^2 y_n = -0.0057, \quad \nabla^3 y_n = -0.0003, \quad \nabla^4 y_n = 0.0004$

$$f(53.4) = 41.5687 + (-0.6)(0.5844) + \frac{1}{2}(-0.6)(-0.6+1)(-0.0057) + \frac{1}{6}(-0.6)(-0.6+1)(-0.6+2)(-0.0003)$$
$$+ \frac{1}{24}(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(0.0004) + \ldots$$

we get $y = f(53.4) = 41.218713$

To find the interpolation equation for the curve passing through the points given in the table, we use Newton's forward difference formula (we can as well use Newton's backward difference formula).

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

Now, $x = x_0 + uh$, where $x_0 = 50, \quad h = 1$

$$\therefore \quad u = \frac{x - x_0}{h} = x - 50$$

$$y = y_0 + (x-50)\Delta y_0 + \frac{(x-50)(x-51)}{2!}\Delta^2 y_0 + \frac{(x-50)(x-51)(x-52)}{3!}\Delta^3 y_0$$
$$+ \frac{(x-50)(x-51)(x-52)(x-53)}{4!}\Delta^4 y_0$$

Let $X = x - 50$ or $x = X + 50$

$$\therefore \quad y = y_0 + X\Delta y_0 + \frac{X(X-1)}{2}\Delta^2 y_0 + \frac{X(X-1)(X-2)}{6}\Delta^3 y_0 + \frac{X(X-1)(X-2)(X-3)}{24}\Delta^2 y_0$$

Substituting for $y_0 = 39.1961, \quad \Delta y_0 = 0.602, \quad \Delta^2 y_0 = -0.0059, \quad \Delta^3 y_0 = 0.0001, \quad \Delta^4 y_0 = 0.0004$

we get, $y = 0.0000167\,X^4 - 0.0001167\,X^3 - 0.002717\,X^2 + 0.6048\,X + 39.1961$

Putting back $X = x - 50$

we get the required equation of the curve as

$$y = 0.0000167\,x^4 - 0.00034567\,x^3 + 0.265288\,x^2 - 8.34875\,x + 121.1261$$

9.8 STIRLING'S INTERPOLATION

Stirling's interpolation formula is given by

$$y = y_0 + u\,\mu\,\delta y_0 + \frac{1}{2!}\,u^2\,\delta^2\,y_0 + \frac{1}{3!}\,u\,(u^2-1)\,\mu\,\delta^3\,y_0 + \ldots\ldots$$
$$+ \frac{u\,(u^2-1)\,(u^2-4)\ldots[u^2-(k-1)^2]\,\mu\,\delta^{2k-1}\,y_0}{(2k-1)!} + \frac{u^2\,(u^2-1)\,(u^2-4)\ldots[u^2-(k-1)^2]}{(2k)!}\,\delta^{2k}\,y_0 \qquad \ldots (1)$$

Here, $u = \dfrac{x-x_0}{h}$

μ is the averaging operator and δ is the central difference operator as defined in Section 9.5.

Formula (1) can also be written in the form

$$y = y_0 + \frac{u}{2}\,[\delta y_{1/2} + \delta y_{-1/2}] + \frac{u^2}{2!}\,\delta^2 y_0 + \frac{u\,(u^2-1)}{3!}\cdot\frac{1}{2}\,[\delta^3 y_{1/2} + \delta^3\,y_{-1/2}] + \ldots$$
$$+ \frac{u\,(u^2-1^2)\,(u^2-2^2)\ldots[u^2-(k-1)^2]}{(2k-1)!}\cdot\frac{1}{2}\,[\delta^{2k-1}\,y_{1/2} + \delta^{2k-1}\,y_{-1/2}]$$
$$+ \frac{u^2\,(u^2-1^2)\,(u^2-2^2)\ldots[u^2-(k-1)^2]}{(2k)!}\,\delta^{2k}\,y_0 \qquad \ldots (2)$$

where, $y_0 = f_0$

9.9 ERROR PROPAGATION IN DIFFERENCE TABLE

In sec. 9.5, we have already seen, how to construct the difference table. Error may occur in the difference table due to wrong entry made initially in the column which gives functional values of y = f (x). It can also occur due to rounding or truncation.

Consider the forward difference table for y = f (x).

Table 9.5

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0 + \varepsilon$		
		Δy_2		$\Delta^3 y_1 + \varepsilon$		$\Delta^5 y_0 - 5\varepsilon$	
x_3	y_3		$\Delta^2 y_2 + \varepsilon$		$\Delta^4 y_1 - 4\varepsilon$		$\Delta^6 y_0 + 15\varepsilon$
		$\Delta y_3 + \varepsilon$		$\Delta^3 y_2 - 3\varepsilon$		$\Delta^5 y_1 + 10\varepsilon$	
x_4	$y_4 + \varepsilon$		$\Delta^2 y_3 - 2\varepsilon$		$\Delta^4 y_2 + 6\varepsilon$		$\Delta^6 y_1 - 20\varepsilon$
		$\Delta y_4 - \varepsilon$		$\Delta^3 y_3 + 3\varepsilon$		$\Delta^5 y_2 - 10\varepsilon$	
x_5	y_5		$\Delta^2 y_4 + \varepsilon$		$\Delta^4 y_3 - 4\varepsilon$		$\Delta^6 y_2 + 15\varepsilon$
		Δy_5		$\Delta^3 y_4 - \varepsilon$		$\Delta^5 y_3 + 5\varepsilon$	
x_6	y_6		$\Delta^2 y_5$		$\Delta^4 y_4 + \varepsilon$		
		Δy_6		$\Delta^3 y_5$			
x_7	y_7		$\Delta^2 y_6$				
		Δy_7					
x_8	y_8						

Above table shows forward differences upto 6^{th} order. ε is the error introduced in the column headed by y in the calculation of y_4. All the other entries y_1, y_2, y_3, etc. upto y_8 are correct. Table 9.4 shows how the error spreads and increases in magnitude as the differences of the higher order are calculated. It can also be observed that errors in each column have binomial coefficients.

$$(1-\varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2$$
$$(1-\varepsilon)^3 = 1 - 3\varepsilon + 3\varepsilon^2 - \varepsilon^3$$
$$(1-\varepsilon)^4 = 1 - 4\varepsilon + 6\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4$$

(i) 1, −2, 1 are the coefficients of errors in the column for $\Delta^2 y$.

(ii) 1, −3, 3, −1 are the coefficients of errors in the column for $\Delta^3 y$.

Same observation cannot be made in columns for higher order differences than four as all the erroneous entries cannot be considered. For Ex., in column of $\Delta^5 y$ there are only four entries which are erroneous, actually there would have been more entries with error if initially there had been entries for y_9 and y_{10}. If a difference column has all possible erroneous entries then these errors will exhibit the binomial coefficients.

If y = f (x) is a polynomial of degree n, then differences of tenth order are constant. Such property is also exhibited when f (x) has continuous derivatives of order n and h the differencing interval is sufficiently small. However, if there is an error at initial stage then this property is not observed. This gives evidence of the presence of error in the difference table. The error can be corrected with the help of differences. To illustrate this point, suppose initially there is an error in y_4 and the incorrect entry is $y_4 + \varepsilon$, we want to find ε. Let us assume that the third order differences are almost constant. In such case, second order differences are in arithmetic progression. $\Delta^2 y_3$ will be the arithmetic mean of three successive differences.

$$\Delta^2 y_3 = \frac{1}{3}\left[(\Delta^2 y_2 + \varepsilon) + (\Delta^2 y_3 - 2\varepsilon) + (\Delta^2 y_4 + \varepsilon)\right] \qquad \text{(Refer Table 9.4)}$$

Using the value of $\Delta^2 y_3$ thus found, ε can be calculated from the relation

$$\varepsilon = \frac{1}{2}\left[\Delta^2 y_3 - (\Delta^2 y_3 - 2\varepsilon)\right]$$

Once ε is known, y_4 can be corrected and difference table can be reconstructed.

9.9.1 Choice of Interpolation

When the data is evenly spaced with respect to x, usually Stirling's central difference formula is used which gives more accurate results. Central difference formula should be used when interpolation is required for the values of x near to the centre of the table. When the point of interpolation is very close to the top of the difference table, Newton-Gregory forward difference formula is used for the simple reason that maximum number of forward differences are available for the value y_0 which is at the top of the table. It may be advisable to consider the value at the top or at the most next from top value. Gregory-Newton Backward difference interpolation formula must be used for interpolation close to the bottom of a difference table. The difference patterns associated with other interpolation formulae quickly run out of the table. Lagrange's interpolation formula can be used for unevenly spaced data.

9.9.2 Remainder Term in Newton-Gregory Formula

We have already seen that, Newton-Gregory forward difference interpolation formula is given by

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

Introducing the notation

$$u^{(r)} = u(u-1)(u-2)\ldots(u-r+1)$$

[called u factorial r]

Newton's formula can be written as

$$y = \sum_{r=0}^{n} \frac{u^{(r)}}{r!}\Delta^r y_0$$

Remainder term is given by $\dfrac{h^{n+1}}{(n+1)!}\, u^{(n+1)} f^{(n+1)}(\xi)$, where $x_0 < \xi < x_n$

If Newton's polynomial $y = P_n(x)$ passing through (n + 1) points is approximating the function f (x), then,

$$f(x) = \sum_{r=0}^{n} \frac{u^{(r)}}{r!}\Delta^r y_0 + \frac{h^{n+1}}{(n+1)!}\, u^{(n+1)} f^{(n+1)}(\xi) \qquad \ldots (1)$$

Thus the truncation error due to replacing the tabulated function f(x), which possesses continuous $(n+1)^{th}$ derivative by a polynomial $P_n(x)$ of n^{th} degree is given by the remainder term in equation (1).

9.10 NUMERICAL DIFFERENTIATION

When the data is unevenly spaced, Lagrange's interpolating polynomial can be used for obtaining derivative at intermediate values of x. We have already considered one Ex. in section 9.5. Newton's forward or backward difference formulae or Stirling's formulae can be used for obtaining derivatives at the top of the table, at the bottom of the table or at the centre of the table.

If Newton's interpolating polynomial approximates a function y = f (x), then from equation (1) of previous section,

$$y = f(x) = \sum_{r=0}^{n} \frac{u^{(r)}}{r!} \Delta^r y_0 + \frac{h^{n+1}}{(n+1)!} u^{(n+1)} f^{(n+1)}(\xi), \text{ where } x_0 < \xi < x_n \quad \ldots (1)$$

where, $u = \dfrac{x - x_0}{h}$

$$y = f(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots + \frac{h^{n+1}}{(n+1)!} u^{(n+1)} f^{(n+1)}(\xi) \quad \ldots (2)$$

or we can write it as

$$y = f(x) = y_0 + u\,\Delta y_0 + \frac{(u^2 - u)}{2!}\Delta^2 y_0 + \frac{(u^3 - 3u^2 + 2u)}{3!}\Delta^3 y_0 + \frac{(u^4 - 6u^3 + 11u^2 - 6u)}{4!}\Delta^4 y_0 + \ldots + \frac{h^{n+1}}{(n+1)!} u^{(n+1)} f^{(n+1)}(\xi) \quad \ldots (3)$$

Differentiating (3) successively w.r.t. x,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

but $\dfrac{du}{dx} = \dfrac{1}{h}\left[\because u = \dfrac{x - x_0}{h}\right]$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\cdot\frac{dy}{du}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2 - 6u + 2)}{3!}\Delta^3 y_0 + \frac{(4u^3 - 18u^2 + 22u - 6)}{4!}\Delta^4 y_0 + \ldots\ldots\right] + \frac{h^n}{(n+1)!} f^{(n+1)}(\xi)\frac{d}{du}[u^{(n+1)}] \quad \ldots (4)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2 - 36u + 22)}{4!}\Delta^4 y_0 + \ldots\ldots\right] + \frac{h^{n-1}}{(n+1)!} f^{(n+1)}(\xi)\frac{d^2}{du^2}[u^{(n+1)}] \quad \ldots (5)$$

Higher order derivatives can be similarly obtained.

We note here that $\left.\dfrac{dy}{dx}\right]_{x = x_0} = \dfrac{1}{h}\left[\Delta y_0 - \dfrac{1}{2}\Delta^2 y_0 + \dfrac{1}{3}\Delta^3 y_0 - \dfrac{1}{4}\Delta^4 y_0 + \ldots\right] \quad \ldots (6)$

Similarly, Newton's backward difference formula is given by

$$y = f(x) = y_n + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \ldots \quad (7)$$

Here $u = \dfrac{x - x_n}{h}$

or we can write it as

$$y = f(x) = y_n + u\,\nabla y_n + \frac{(u^2 + u)}{2!}\nabla^2 y_n + \frac{(u^3 + 3u^2 + 2u)}{3!}\nabla^3 y_n + \frac{(u^4 + 6u^3 + 11u^2 + 6u)}{4!}\nabla^4 y_n + \ldots \quad \ldots (8)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2u+1)}{2!}\nabla^2 y_n + \frac{(3u^2 + 6u + 2)}{3!}\nabla^3 y_n + \frac{(4u^3 + 18u^2 + 22u + 6)}{4!}\nabla^4 y_n + \ldots\right] \quad \ldots (9)$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{(6u+6)}{3!}\nabla^3 y_n + \frac{(12u^2 + 36u + 22)}{4!} + \ldots\right] \quad \ldots (10)$$

Higher order derivatives can be similarly obtained.

We note here that

$$\left.\frac{dy}{dx}\right]_{x = x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right] \qquad \ldots (11)$$

From Stirling's formula,

$$y = y_0 + u\,\mu\,\delta y_0 + \frac{1}{2!}\,u^2\,\delta^2\,y_0 + \frac{1}{3!}\,u\,(u^2 - 1)\,\mu\,\delta^3\,y_0 + \frac{u^2\,(u^2 - 1)}{4!}\,\delta^4\,y_0 + \ldots$$

where, $x = x_0 + uh$ or $u = \frac{x - x_0}{h}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\cdot\frac{dy}{du}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\mu\,\delta y_0 + u\,\delta^2\,y_0 + \frac{1}{3!}(3u^2 - 1)\,\mu\,\delta^3\,y_0 + \frac{1}{4!}(4u^3 - 2u)\,\delta^4\,y_0 + \ldots\right] \qquad \ldots (12)$$

Thus to evaluate $\frac{dy}{dx}$ at $x = x_0$, $u = 0$.

$$\frac{dy}{dx} = \frac{1}{h}\left[\mu\,\delta y_0 - \frac{1}{3!}\,\mu\,\delta^3\,y_0\,\ldots\right] \qquad \ldots (13)$$

Higher order derivatives can be similarly obtained.

ILLUSTRATIONS

Ex. 1 : *For the tabulated data*

x	*0*	*1*	*2*	*3*	*4*	*5*
y	*1.12*	*3.45*	*6.67*	*10.8*	*16.12*	*24.52*

Find y at $x = 0.5$, $\frac{dy}{dx}$ at $x = 5.5$.

Sol. : First we construct the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1.12					
		2.33				
1	3.45		0.89			
		3.22		0.02		
2	6.67		0.91		0.26	
		4.13		0.28		1.35
3	10.8		1.19		1.61	
		5.32		1.89		
4	16.12		3.08			
		8.40				
5	24.52					

$$u = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

Since the interpolation is at the beginning of the table, we use Newton's forward difference formula

$$y = y_0 + u\,\Delta y_0 + \frac{u\,(u - 1)}{2!}\,\Delta^2 y_0 + \frac{u\,(u - 1)\,(u - 2)}{3!}\,\Delta^3 y_0 + \frac{u\,(u - 1)\,(u - 2)\,(u - 3)}{4!}\,\Delta^4 y_0$$

$$+ \frac{u\,(u - 1)\,(u - 2)\,(u - 3)\,(u - 4)}{5!}\,\Delta^5 y_0 + \ldots$$

Putting $u = 0.5, \quad y_0 = 1.12, \quad \Delta y_0 = 2.33, \quad \Delta^2 y_0 = 0.89, \quad \Delta^3 y_0 = 0.02, \quad \Delta^4 y_0 = 0.26, \quad \Delta^5 y_0 = 1.35$

$$\begin{aligned} y]_{x=0.5} &= 1.12 + 0.5\,(2.33) + \frac{(0.5)\,(0.5-1)}{2} \times 0.89 + \frac{(0.5)\,(0.5-1)\,(0.5-2)}{6} \times 0.02 \\ &\quad + \frac{(0.5)\,(0.5-1)\,(0.5-2)\,(0.5-3)}{24} \times 0.26 + \frac{(0.5)\,(0.5-1)\,(0.5-2)\,(0.5-3)\,(0.5-4)}{120} \times 1.35 \\ &= 1.12 + 1.165 - 0.11125 + 0.00125 - 0.0101562 + 0.036914 \\ &= 2.2017579 \end{aligned}$$

To find $\frac{dy}{dx}$ at x = 5.5, we use the formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2u+1)}{2}\nabla^2 y_n + \frac{(3u^2+6u+2)}{6}\nabla^3 y_n + \frac{(4u^3+18u^2+22u+6)}{24}\nabla^4 y_n + \frac{(5u^4+40u^3+105u^2+100u+24)}{120}\nabla^5 y_n + \ldots\right]$$

Put $h = 1, \quad u = \frac{x - x_n}{h} = \frac{5.5-5}{1} = 0.5,$

and $\nabla y_n = 8.40, \quad \nabla^2 y_n = 3.08, \quad \nabla^3 y_n = 1.89, \quad \nabla^4 y_n = 1.61, \quad \nabla^5 y_n = 1.35$

$$\begin{aligned} \therefore \quad \left.\frac{dy}{dx}\right|_{x=5.5} &= 8.40 + \frac{[2\,(0.5)+1]}{2}(3.08) + \frac{[3\,(0.5)^2 + 6\,(0.5) + 2]}{6}(1.89) + \frac{[4\,(0.5)^3 + 18\,(0.5)^2 + 22\,(0.5) + 6]}{24}(1.61) \\ &\quad + \frac{[5(0.5)^4 + 40\,(0.5)^3 + 105\,(0.5)^2 + 100\,(0.5) + 24]}{120}(1.35) \\ &= 8.40 + 3.08 + 1.81125 + 1.106875 + 1.187578 \\ &= 15.58 \end{aligned}$$

Ex. 2 : *State the order of polynomial which might be suitable for following function. Calculate f (3.5) using forward difference formula.*

x	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
y	*19*	*48*	*99*	*178*	*291*	*444*	*643*	*894*

Sol. :

x	y	Δy	Δy²	Δy³	Δy⁴	Δy⁵	Δy⁶	Δy⁷
2	19							
		29						
3	48		22					
		51		6				
4	99		8		0			
		79		6		0		
5	178		34		0		0	
		113		6		0		0
6	291		40		0		0	
		153		6		0		
7	444		46		0			
		199		6				
8	643		52					
		251						
9	894							

Now, $x = x_0 + uh, \quad x_0 = 2, \quad h = 1, \quad x = 3.5$

Here, $u = \frac{x - x_0}{h} = \frac{3.5-2}{1} = 1.5$

From table, $y_0 = 19, \quad \Delta y_0 = 29, \quad \Delta^2 y_0 = 22, \quad \Delta^3 y_0 = 6, \quad \Delta^4 y_0 = 0$

Newton's interpolating polynomial is given by

$$y = y_0 + uy_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 + \ldots\ldots$$

$$\therefore \quad y = 19 + 1.5 \times 29 + \frac{1.6(1.5-1)}{2} \times 22 + \frac{1.5(1.5-1)(1.5-2)}{6} \times 6 + 0$$

$$= 70.375$$

$\therefore$ At $x = 3.5$, $y = 70.375$

Ex. 3 : *Find value of y for x = 0.5 for the following table of x, y values using Newton's forward difference formula.*

x	0	1	2	3	4
y	1	5	25	100	250

Sol. : Preparing forward difference table :

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
0	1				
		4			
1	5		16		
		20		39	
2	25		55		– 19
		75		20	
3	100		75		
		150			
4	250				

Here, $h = 1$, $x_{given} = 0.5$, $x_0 = 0$, $y_0 = 1$

$$u = \frac{x_{given} - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

Newton's forward difference formula is

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

At $x = 0.5$, $$= 1 + 0.5 \times 4 + \frac{0.5(0.5-1)}{2} \times 16 + \frac{0.5(0.5-1)(0.5-2)}{6} \times 39 + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} \times (-19) + \ldots$$

$\therefore$ At $x = 0.5$, $y = 4.1796$

Ex. 4 : *From the tabulated values of x and y given below, prepare forward difference table. Find the polynomial passing through the points and estimate the value of y when x = 1.5. Also find the slope of the curve at x = 1.5.*

x	0	2	4	6	8
y	5	29	125	341	725

Sol. : Preparing forward difference table :

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
0	5				
		24			
2	29		72		
		96		48	
4	125		120		0
		216		48	
6	341		168		
		384			
8	725				

Now, $u = \dfrac{x - x_0}{h} = \dfrac{x}{2}$, here $x_0 = 0$ and $h = 2$

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

Put value of $u = \dfrac{x}{2}$, $(u - 1) = \left(\dfrac{x}{2} - 1\right)$, $(u - 2) = \left(\dfrac{x}{2} - 2\right)$ in above equation.

Newton's interpolating polynomial is given by,

$$y = 5 + \frac{x}{2} \times 24 + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)}{2!} \times 72 + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{3!} \times 48$$

$\therefore$ $y = x^3 + 3x^2 + 2x + 5$, which is the equation of the polynomial.

At x = 1.5, $y = 18.125$

At x = 1.5, $\dfrac{dy}{dx} = 3x^2 + 6x + 2 = 17.75$

Ex. 5 : *The distance travelled by a point P in x-y plane in a mechanism is as shown in the table below. Estimate the distance travelled, velocity and acceleration of point P when x = 4.5.*

x in mm	*1*	*2*	*3*	*4*	*5*
y in mm	*14*	*30*	*62*	*116*	*198*

Sol. : Preparing backward difference table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	14				
		16			
2	30		16		
		32		6	
3	62		22		0
		54		6	
4	116		28		
		82			
5	198				

Since the interpolation is near end of table, we use Newton's backward difference formula,

$$y = y_5 + u\nabla y_5 + \frac{u(u+1)}{2!}\nabla^2 y_5 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_5 + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_5 + \ldots$$

Here, $n = 5$, $y_5 = 198$, $x = x_n + uh$ where, $x = 4.5$, $h = 1$, $x_n = x_5 = 5$

$$\therefore \quad u = \frac{x - x_n}{h} = (x - 5)$$

$$\therefore \quad y = 198 + (x-5) \times (82) + \frac{(x-5)(x-4)}{2} \times 28 + \frac{(x-5)(x-4)(x-3)}{6} \times 6 + 0$$

$$= 198 + 82x - 410 + (x^2 - 9x + 20) \times 14 + (x-5)(x^2 - 7x + 12)$$

$$\therefore \quad y = x^3 + 2x^2 + 3x + 8 \qquad \ldots(1)$$

Distance travelled by P when x = 4.5 is

$$y_{\text{at } x = 4.5} = (4.5)^3 + 2(4.5)^2 + 3(4.5) + 8 = 153.125$$

From (1), velocity is given by

$$v = \frac{dy}{dx} = \frac{d}{dx}(x^3 + 2x^2 + 3x + 8) = 3x^2 + 4x + 3$$

$$\therefore \quad v_{\text{at } x = 4.5} = 3(4.5)^2 + 4(4.5) + 3 = .75$$

Acceleration is given by $a_{\text{at } x = 4.5} = \dfrac{dv}{dx} = \dfrac{d}{dx}(3x^2 + 4x + 3) = 6x + 4 = 31$

Ex. 6 : *Find a polynomial passing through points (0, 1), (1, 1), (2, 7), (3, 25), (4, 61), (5, 121) using Newton's interpolation formula and hence find y and* $\frac{dy}{dx}$ *at x = 0.5.* ***(Dec. 2016)***

Sol. : We first construct forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0			
1	1		6		
		6		6	
2	7		12		0
		18		6	
3	25		18		0
		36		6	
4	61		24		
		60			
5	121				

Here, $u = \frac{x - x_0}{h}$ where $x = 0.5, \quad x_0 = 0, \quad h = 1$

$$\therefore \quad u = \frac{x-0}{1} = x$$

Newton's forward interpolating polynomial is,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 1 + x(0) + \frac{x(x-1)}{2} \times 6 + \frac{x(x-1)(x-2)}{6} \times 6 + \frac{x(x-1)(x-2)(x-3)}{24} \times 0$$

$$= 1 + 3(x^2 - x) + x(x^2 - 3x + 2) = 1 + 3x^2 - 3x + x^3 - 3x^2 + 2x$$

$$\therefore \quad y = 1 - x + x^3$$

and $$\frac{dy}{dx} = -1 + 3x^2$$

$$\therefore \quad y_{\text{at } x = 0.5} = 0.625$$

and $$\left(\frac{dy}{dx}\right)_{\text{at } x = 0.5} = -0.25$$

Ex. 7 : *Given the table of square roots. Calculate the values of* $\sqrt{151}$ *and* $\sqrt{155}$ *by Newton's interpolation formula.*

x	*150*	*152*	*154*	*156*
$y = \sqrt{x}$	*12.247*	*12.329*	*12.410*	*12.490*

Sol. : We first construct forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
150	12.247			
		0.0820		
152	12.329		– 0.001	
		0.0810		0
154	12.410		– 0.001	
		0.0800		
156	12.490			

Here $h = 2, \quad x_0 = 150, \quad x = 151$ and $u = 0.5 \quad \therefore \quad x = x_0 + uh$

$$u = \frac{x - x_0}{h} = \frac{151 - 150}{2} = 0.5$$

Using Newton's forward difference formula,

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\,\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\,\Delta^3 y_0 + \ldots$$

$$y(151) = 12.247 + 0.5\,(0.0820) + \frac{0.5\,(0.5-1)}{2!}(-0.001) + 0$$

$$y(151) = 12.288$$

Again, $h = 2$, $x_n = 156$, $x = 155$ and $x = x_n + uh$

Using Newton's backward difference formula,

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \ldots \quad \therefore \quad u = \frac{x - x_n}{h} = \frac{155-156}{2} = -0.5$$

Again, $h = 2$, $x_n = 156$, $x = 165$ and $x = x_n + uh$

$$\therefore \quad y(155) = 12.490 + (-0.5 \times 0.800) + \left[\frac{-0.5\,(-0.5+1)}{2!} \times (-0.001)\right] + \ldots$$

$$\therefore \quad y(155) = 12.4496$$

Ex. 8 : *The velocity distribution of a fluid near a flat surface is given below :*

x	*0.1*	*0.3*	*0.6*	*0.8*
v	*0.72*	*1.81*	*2.73*	*3.47*

x is the distance from the surface (mm) and v is the velocity (mm/sec). Use Lagrange's interpolation polynomial to obtain the velocity at x = 0.4.

Sol. : Using Lagrange's interpolation formula,

$$v = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,v_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,v_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\,v_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,v_3$$

$$= \frac{0.72\,(0.4-0.3)(0.4-0.6)(0.4-0.8)}{(0.1-0.3)(0.1-0.6)(0.1-0.8)} + \frac{1.81\,(0.4-0.1)(0.4-0.6)(0.4-0.8)}{(0.3-0.1)(0.3-0.6)(0.3-0.8)} + \frac{2.73\,(0.4-0.1)(0.4-0.3)(0.4-0.8)}{(0.6-0.1)(0.6-0.3)(0.6-0.8)}$$

$$+ \frac{3.47\,(0.4-0.1)(0.4-0.3)(0.4-0.6)}{(0.8-0.1)(0.8-0.3)(0.8-0.6)}$$

$$= 2.16028$$

Ex. 9 : *Following table shows enthalpy at different pressures.*

Pressure (bar)	*1.9*	*2.2*	*2.4*	*2.6*
Enthalpy (kJ/kg·K)	*497.9*	*517.6*	*529.6*	*540.9*

Find out the enthalpy at pressure 2.1 bar by Lagrange's interpolation method.

Sol. : By Lagrange's method, we have

$$y = y_1\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + y_2\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + y_3\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + y_4\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

Here pressure is denoted by variable x while enthalpy is denoted by variable y. We find at $x = 2.1$, $y = ?$

Substituting values in above equation from given data of x and y values, we have for y at $x = 2.1$.

$$y = 497.9\frac{(2.1-2.2)(2.1-2.4)(2.1-2.6)}{(1.9-2.2)(1.9-2.4)(1.9-2.6)} + 517.6\frac{(2.1-1.9)(2.1-2.4)(2.1-2.6)}{(2.2-1.9)(2.2-2.4)(2.2-2.6)}$$

$$+ 529.6\frac{(2.1-1.9)(2.1-2.2)(2.2-2.6)}{(2.4-1.9)(2.4-2.2)(2.4-2.6)} + 540.9\frac{(2.1-1.9)(2.1-2.2)(2.1-2.4)}{(2.6-1.9)(2.6-2.2)(2.6-2.4)}$$

$$= (71.12857) + (647) - (264.8) + (57.954)$$

$$y = 511.282$$

$\therefore$ Enthalpy (y) at pressure (x) = 2.1 bar = 511.282 kJ/kg·K

Ex. 10 : *For the following data :*

x	*2*	*6*	*10*	*14*	*18*
f (x)	*21.857*	*21.025*	*20.132*	*19.145*	*18.057*

Find f(11) using Stirling's formula.

Sol. : We first construct the central difference table.

x	y	δ	δ^2	δ^3	δ^4
$x_{-2} = 2$	21.857				
		– 0.832			
$x_{-1} = 6$	21.025		– 0.061		
		– 0.893		– 0.033	
$x_0 = 10$	20.132		– 0.094		0.026
		– 0.987		– 0.007	
$x_1 = 14$	19.145		– 0.101		
		– 1.088			
$x_2 = 18$	18.057				

Here $x_0 = 10$, $x = 11$, $h = 4$ and $x = x_0 + uh$

$$\therefore \qquad u = \frac{11 - 10}{4} = 0.25$$

Stirling's formula [sec 9.8, equation (2)] is

$$y = y_0 + \frac{u}{2}[\delta y_{1/2} + \delta y_{-1/2}] + \frac{u^2}{2!}\delta^2 y_0 + \frac{u(u^2-1)}{3!}\cdot\frac{1}{2}\cdot[\delta^3 y_{1/2} + \delta^3 y_{-1/2}] + \frac{u^2(u^2-1)}{4!}\delta^4 y_0 + \ldots \qquad \ldots (1)$$

From above table,

$$y_0 = 20.132, \quad \delta y_{1/2} = -0.987, \quad \delta y_{-1/2} = -0.893$$

$$\delta^2 y_0 = -0.094, \quad \delta^3 y_{1/2} = -0.007, \quad \delta^3 y_{-1/2} = -0.033, \quad \delta^4 y_0 = 0.026$$

Substituting these values in (1),

$$\begin{aligned} y]_{x=11} = f(11) &= 20.132 + 0.125(-0.987 - 0.893)] + 0.3125(-0.094) - 0.01953(-0.007 - 0.033) + 0.03845(0.026) \\ &= 20.132 - 0.235 - 0.00294 + 0.00078 + 0.001 \\ &= 19.896 \end{aligned}$$

Ex. 11 : *From the following data obtain the first derivatives of $y = \log_e x$ (i) at $x = 500$, (ii) at $x = 550$.*

x	*500*	*510*	*520*	*530*	*540*	*550*
$\boldsymbol{y = \log_e x}$	*6.2146*	*6.2344*	*6.2538*	*6.2729*	*6.2916*	*6.3099*

Also, calculate the actual values of the derivatives at these points.

Sol. : Here $h = 10$. (i) $x_0 = 500$ at $x = 500$, $u = \frac{x - x_0}{h} = 0$.

We construct forward difference table.

x	$y = \log_e x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
500	6.2146					
		0.0198				
510	6.2344		– 0.0004			
		0.0194		0.0001		
520	6.2538		– 0.0003		– 0.0002	
		0.0191		– 0.0001		0.0003
530	6.2729		– 0.0004		0.0001	
		0.0187		0		
540	6.2916		– 0.0004			
		0.0183				
550	6.3099					

We apply Newton's forward difference formula for derivatives.

$$\left(\frac{dy}{dx}\right)_{x\,=\,500} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$$= \frac{1}{10}\left[0.0198 - \frac{1}{2}(-0.0004) + \frac{1}{3}(0.0001) - \frac{1}{4}(-0.0002) + \frac{1}{5}(0.0003)\right]$$

$$= \frac{1}{10}[0.0198 + 0.0002 + 0.00003 + 0.00005 + 0.00006] = 0.002014$$

Actual value $\left(\frac{dy}{dx}\right)_{x\,=\,500} = 0.0002$

(ii) Here $x_n = 550$. At $x = 550$; $u = \frac{x - x_n}{10} = 0$.

$$\left(\frac{dy}{dx}\right)_{x\,=\,550} = \frac{1}{h}\left[\nabla y_5 + \frac{1}{2}\nabla^2 y_5 + \frac{1}{3}\nabla^3 y_5 + \frac{1}{4}\nabla^4 y_5 + \frac{1}{5}\nabla^5 y_5\right]$$

$$= \frac{1}{10}\left[0.0183 + \frac{1}{2}(-0.0004) + \frac{1}{4}(0.0001) + \frac{1}{5}(0.0003)\right]$$

$$= \frac{1}{10}[0.0183 - 0.0002 + 0.000025 + 0.00006] = \frac{1}{10}[0.01819] = 0.001819$$

Actual value $\left(\frac{dy}{dx}\right)_{x\,=\,550} = 0.001818$

EXERCISE 9.1

1. Establish the following :

(i) $\sum_{r=1}^{n} y_r = {}^nC_1\, y_1 + {}^nC_2\, \Delta y_1 + {}^nC_3\, \Delta^2 y_1 \ldots\ldots \Delta^{n-1} y_1$

(ii) $y_6 = y_5 + \Delta y_4 + \Delta^2 y_3 + \Delta^3 y_2 + \Delta^4 y_1 + \Delta^5 y_0 + \Delta^6 y_0$

(iii) $\Delta^{2m+1} = E^m\left(\mu\delta^{2m+1} + \frac{1}{2}\delta^{2m+2}\right)$

(iv) $hD = \log E = 2\sinh^{-1}\left(\frac{\delta}{2}\right)$

(v) $(\Delta^2 - 2\mu\delta + \nabla^2)\, x^2 = 4(1 - x)$ taking $h = 1$

(vi) $1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$

2. Establish the following :

(i) $E = \left(\frac{\delta}{2} + \sqrt{1 + \frac{\delta^2}{4}}\right)^2$

(ii) $\Delta = \mu\delta + \frac{1}{2}\delta^2$

(iii) $\frac{1}{\nabla} y_n - \frac{1}{\Delta} y_0 = y_0 + y_1 + y_2 + \ldots + y_n$

(iv) $E = 1 + \frac{\delta^2}{2} + \delta\left[1 + \frac{\delta^2}{4}\right]^{1/2}$

3. Show that

$$\nabla^3 f_k = f_k - 3f_{k-1} + 3f_{k-2} - f_{k-3}$$

$$\delta^3 f_k = f_{k+3/2} - 3f_{k+1/2} + 3f_{k-1/2} - f_{k-3/2}$$

4. Given : log 2 = 0.3010, log 3 = 0.4771, log 5 = 0.6990, log 7 = 0.8451. Find log 47 using Lagrange's interpolation formula. Determine the accuracy of the result. **Ans.** 1.6721

5. Given : $y_0 = 1.23$, $y_2 = 3.78$, $y_3 = 6.9$, $y_4 = 10.9$. Find y_1. **Ans.** 1.8025

6. Prepare the difference table for sin x for $x = 10^\circ\ (5^\circ)\ 35^\circ$ and estimate sin x for $x = 12^\circ 30'$ and for $x = 21^\circ$.

7. $f(x) = e^x \sin x$ is to be approximated over the interval [0, 1] by Newton-Gregory interpolating polynomial passing through (n + 1) equally spaced points. Determine n so that truncation error will be less than 0.0001 over this interval.

Ans. n > 5

8. For the following tabulated data :

x	1	2	3	4	5
y	3.47	6.92	11.25	16.75	22.94

Find y at x = 4.5 and $\frac{dy}{dx}$ at x = 1.5 **Ans.** 19.81875; 3.40584

9. From the tabulated data :

x	0	0.1	0.2	0.3	0.4	0.5
f (x)	1.0	1.046	1.09423	1.44	1.297	1.252

Estimate the values of f (0.05), f (0.25), f (0.47) using appropriate interpolation formulae.

10. Compute cosh (4.25) from the table :

x	4.0	4.1	4.2	4.3	4.4	4.5
cosh x	27.3081	30.1784	33.3507	36.8567	40.7316	45.0141

Ans. 35.059

11. Find y'(0) and y'(5) for the following data :

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Ans. – 27.9, – 31.23.

9.11 NUMERICAL INTEGRATION

Evaluation of integral,

$$I = \int_a^b f(x)\,dx$$

is of common occurrence in Mathematics or Engineering Sciences. In many cases, function y = f (x) is such that the integral I cannot be evaluated analytically. Numerical integration is required to be used in such instances.

If function y = f (x) is defined by set of points (x_i, y_i), i = 1, 2, … n, then also numerical technique must be employed. To develop numerical methods of integration, we use Geometrical interpretation of I.

In elementary integral calculus it is already seen that 'I' represents the area under the curve y = f (x), bounded by the ordinates x = a, x = b, and the X-axis i.e. the area ABCD (See Fig. 9.1)

When function y = f (x) is not known but the set of points (x_i, y_i), i = 0 (1) n lying on the curve y = f (x) are given, area ABCD is determined by breaking it into several strips, determination of area gives the value of the integral I. Various methods to determine area ABCD are described in following sections.

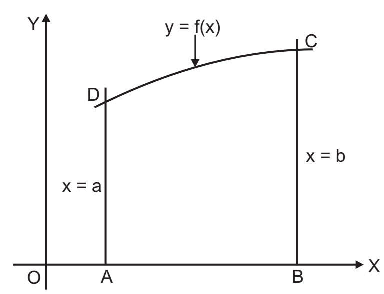

Fig. 9.1

9.12 TRAPEZOIDAL RULE

Consider the area A_1, bounded by the ordinates $x = x_0$, $x = x_1$, X-axis and the curve joining the points A_3 and A_4 (See Fig. 9.2) Approximating curve joining the two points (x_0, y_0), (x_1, y_1) is given by Newton's interpolating polynomial $y = y_0 + u\,\Delta y_0$.

Where, $$u = \frac{x - x_0}{h}, \quad \Delta y_0 = y_1 - y_0, \quad x_1 - x_0 = h$$

$$A_1 = \int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} (y_0 + u\,\Delta y_0)\,dx$$

When $x = x_0$, $u = 0$; $x = x_1$, $u = \frac{x_1 - x_0}{h} = \frac{h}{h} = 1$, $dx = h\,du$

$$\therefore \quad A_1 = \int_0^1 \{y_0 + u(y_1 - y_0)\}\,h\,du$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0)\right]_0^1 = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}[y_0 + y_1] \qquad \ldots (1)$$

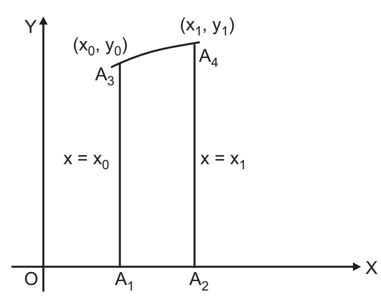

Fig. 9.2

Error involved in this process is given by,

$$\frac{h^3}{2!}\int_0^1 u\,(u-1)\,f''(\xi)\,du = \frac{h^3}{2}\,f''(\xi_1)\int_0^1 (u^2-u)\,du$$

by mean value theorem, ξ_1 is any point in the range (x_0, x_1).

Error term or remainder term is

$$\frac{h^3}{2}\,f''(\xi_1)\left[\frac{u^3}{3}-\frac{u^2}{2}\right]_0^1 = -\frac{h^3}{12}\,f''(\xi_1) \qquad \ldots (2)$$

Just as we have obtained.

$A_1 = \frac{h}{2}[y_0 + y_1]$ with error term given by (2) we can find area under the curve joining the points (x_1, y_1) (x_2, y_2) from the relation

$$A_2 = \frac{h}{2}[y_1 + y_2]$$

$$\text{with an error term} = -\frac{h^3}{12}\,f''(\xi_2)$$

Proceeding in this manner, we shall get

$$A_3 = \frac{h}{2}[y_2 + y_3]$$

$$\ldots\ldots$$

$$A_r = \frac{h}{2}[y_{r-1} + y_r] \text{ with an error term } -\frac{h^3}{12}\,f''(\xi_r)$$

$\therefore$ $\int_{x_0}^{x_n} y\,dx$ = Total area under the curve passing through the points $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n)$

$$= \frac{h}{2}[(y_0 + y_1) + (y_1 + y_2) + \ldots (y_{r-1} + y_r) \ldots (y_{n-1} + y_n)]$$

$$= \frac{h}{2}[(y_0 + y_n) + 2\,(y_1 + y_2\ \ldots y_{n-1})] \qquad \ldots(3)$$

with an error,

$$= -\frac{h^3}{12}[f''(\xi_1) + f''(\xi_2) \ldots f''(\xi_n)]$$

Replacing the quantity in the bracket by using the relation

$$\text{the average } f''(\bar{\xi}) = \frac{f''(\xi_1) + f''(\xi_2) \ldots f''(\xi_n)}{n}$$

$$\therefore \quad \text{Error term} = -\frac{h^3}{12}\,nf''(\bar{\xi})$$

$$= -\frac{h^3}{12}\frac{(x_n - x_0)}{h}\,f''(\bar{\xi}) \quad \left[h = \frac{x_n - x_0}{n}\right]$$

$$= -\frac{h^2}{12}(x_n - x_0)\,f''(\bar{\xi})$$

Thus,

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[(y_0 + y_n) + 2\,(y_1 + y_2 + \ldots + y_{n-1})]} \qquad \ldots (4)$$

Above formula can also be written as

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[\begin{pmatrix}\text{sum of the first and}\\ \text{last ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of the remaining}\\ \text{ordinates}\end{pmatrix}\right]} \quad \dots (5)$$

$$\text{with an error term} = -\frac{h^2}{12}(x_n - x_0)\ f''(\bar{\xi}) \quad \dots (6)$$

Formulae (4) or (5) are known as Trapezoidal rule for numerical evaluation of the integral.

9.13 SIMPSON'S $\frac{1}{3}^{rd}$ RULE

In this method, we find the value of the integral $\int_{x_0}^{x_2} y\,dx$ first by finding the area of the double strip (See Fig. 9.3) under the curve passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2).

Equation of curve is given by Newton's interpolating polynomial through (x_0, y_0), (x_1, y_1), (x_2, y_2).

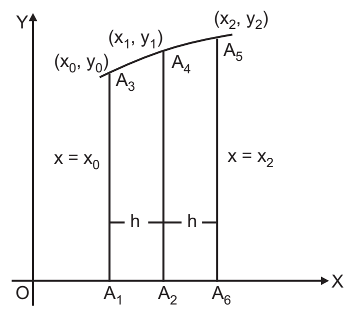

Fig. 9.3

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0,$$

where, $u = \dfrac{x - x_0}{h}$ with an error term $\dfrac{h^4}{3!}\,u(u-1)(u-2)\,f'''(\xi_1)$

Area A_1 of the double strip is

$$A_1 = \int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2} y\frac{dx}{du}\,du$$

when $x = x_0$, $u = 0$, $x = x_2$, $u = 2$.

$$A_1 = \int_0^2 \left\{y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0\right\} h\,du$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0) + \left(\frac{u^3}{6} - \frac{u^2}{4}\right)(y_2 - 2y_1 + y_0)\right]_0^2$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{8}{6} - 1\right)(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right] = \frac{h}{3}[6y_1 + y_2 - 2y_1 + y_0]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2] \quad \dots (1)$$

Error involved in the process is given by

$$\frac{h^4}{3!}\int_0^2 u(u-1)(u-2)\,f'''(\xi)\,du = \frac{h^4}{3!}\,f'''(\xi_1)\int_0^2 u(u-1)(u-2)\,du$$

$$= \frac{h^4}{3!}\,f'''(\xi_1)\left[\frac{u^4}{4} - 3\frac{u^3}{3} + 2\frac{u^2}{2}\right]_0^2 = 0$$

Since the error term (remainder term) is zero, we have to consider the next term of the remainder.

$$\begin{aligned} R_n &= \text{Remainder term (error term)} \\ &= \frac{h^5}{4!} \int_0^2 u\,(u-1)\,(u-2)\,(u-3)\, f^{iv}(\xi_1)\, du \\ &= \frac{h^5}{4!} f^{iv}(\xi_1) \int_0^2 u\,(u-1)\,(u-2)\,(u-3)\, du \end{aligned}$$

(when ξ_1 is the point in the range of integration)

$$\begin{aligned} &= \frac{h^5}{4!} f^{iv}(\xi_1) \left(-\frac{4}{15}\right) \\ &= -\frac{1}{90} h^5 f^{iv}(\xi_1) \qquad \text{...(2)} \end{aligned}$$

Similarly, the area A_2, of the next double strip bounded by the curve passing through the points (x_2, y_2), (x_3, y_3), (x_4, y_4) is given by,

$$A_2 = \frac{h}{3} [y_2 + 4y_3 + y_4] \qquad \text{...(3)}$$

with an error term $-\frac{1}{90} h^5 f^{iv}(\xi_2)$

Proceeding in this manner we can find the areas of consecutive strips A_3, A_4, ...

Thus, A_{last} = Area of last strip $= \frac{h}{3} [y_{n-2} + 4y_{n-1} + y_n]$ ...(4)

with an error $-\frac{1}{90} h^5 f^{iv}(\xi_n)$ ...(5)

Total area under the curve is given by,

$$\begin{aligned} A &= A_1 + A_2 + A_3 + \dots \\ &= \frac{h}{3} [(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \dots + (y_{n-2} + 4y_{n-1} + y_n)] \end{aligned}$$

$$\boxed{\int_{x_0}^{x_n} f(x)\, dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]} \qquad \text{...(6)}$$

Here y_0, y_n are the first and last ordinates; y_1, y_3 ... are second, fourth ... or even ordinates; y_2, y_4 ... are third, fifth ... or odd ordinates

$$\therefore \quad \boxed{A = \frac{h}{3} \Big[(\text{sum of first + last ordinate}) + 4\,(\text{sum of even ordinates}) + 2\,(\text{sum of odd ordinates})\Big]} \qquad \text{...(7)}$$

Net error involved in this process is $-\frac{h^5}{90} [f^{iv}(\xi_1) + f^{iv}(\xi_2) + \dots f^{iv}(\xi_n)]$

$$f^{iv}(\bar{\xi}) = \text{average of } f^{iv}(\xi) = \frac{f^{iv}(\xi_1) + f^{iv}(\xi_2) + \dots + f^{iv}(\xi_n)}{n}$$

$$\therefore \quad \text{Net error} = -\frac{h^5}{90} \times n\, f^{iv}(\bar{\xi}) \qquad \text{...(8)}$$

For application of above rule the total number of ordinates required are odd, therefore the range $(x_n - x_0)$ should be divided into even number of ordinates 2n.

$$\therefore \quad h = \frac{(x_n - x_0)}{2n} \quad \text{or} \quad n = \frac{x_n - x_0}{2h}$$

$$\therefore \quad \text{Net error} = -\frac{h^5}{90} \times \frac{x_n - x_0}{2h} f^{iv}(\bar{\xi})$$

$$= -\frac{h^4}{180}(x_n - x_0)\ f^{iv}(\bar{\xi}) \qquad \text{...(9)}$$

where, $x_0 < \bar{\xi} < x_n$

Formulae (6) or (7) give Simpson's $\frac{1}{3}^{rd}$ rule for evaluating integral numerically and (9) giving the bound of error involved.

9.14 SIMPSON'S $\frac{3}{8}^{th}$ RULE

For developing this rule, we first determine $\int_{x_0}^{x_3} y\,dx$ by finding the area under the curve passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) bounded by the ordinates $x = x_0$, $x = x_3$ and X-axis (See Fig. 9.4).

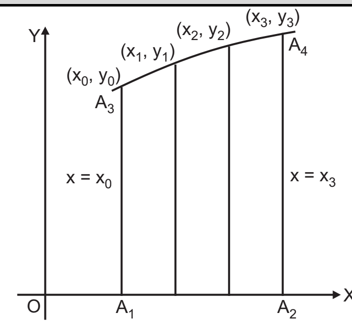

Fig. 9.4

Newton's polynomial through these points is given by :

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where, $u = \frac{x - x_0}{h}$

As in previous sections,

$$\int_{x_0}^{x_3} y\,dx = h\int_0^3 \left\{y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0\right\} du$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0) + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)(y_2 - 2y_1 + y_0) + \frac{1}{6}\left(\frac{u^4}{4} - 3\,\frac{u^3}{3} + 2\,\frac{u^2}{2}\right)(y_3 - 3y_2 + 3y_1 - y_0)\right]_0^3$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{h}{8}\left[24\,y_0 + 36y_1 - 36y_0 + 18y_2 - 36y_1 + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0\right]$$

$$= \frac{h}{8}\left[3y_0 + 9y_1 + 9y_2 + 3y_3\right] = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Thus, area A_1 under the curve bounded by $x = x_0$ and $x = x_3$ is

$$A_1 = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Error involved in the process is given by the remainder term R_n.

$$R_n = \frac{h^5}{4!}\int_0^3 u(u-1)(u-2)(u-3)\,f^{iv}(\xi)\,du$$

$$= \frac{h^5}{4!}f^{iv}(\xi_1)\int_0^3 u(u-1)(u-2)(u-3)\,du = \frac{h^5}{4!}f^{iv}(\xi_1)\left(-\frac{9}{10}\right)$$

$$= -\frac{3}{80}h^5 f^{iv}(\xi_1)$$

Similarly, the next area bounded by the curve passing through the points (x_3, y_3), (x_4, y_4), (x_5, y_5) and (x_6, y_6) is given by;

$$A_2 = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

with an error term $-\frac{3}{80} h^5 f^{iv}(\xi_2)$

Finding successive areas

$$A_{last} = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

with error involved as $-\frac{3h^5}{80} f^{iv}(\xi_n)$

Application of this procedure requires the interval $(x_n - x_0)$ to be divided into 3n parts.

$$\therefore \qquad h = \frac{x_n - x_0}{3n}$$

$$\begin{aligned} \text{Total amount of error} &= -\frac{3h^5}{80}[f^{iv}(\xi_1) + f^{iv}(\xi_2) \ldots\ldots f^{iv}(\xi_n)] \\ &= -\frac{3h^5}{80}[n f^{iv}(\bar{\xi})] \\ &= -\frac{3h^5}{80}\left[\frac{(x_n - x_0)}{3h} f^{iv}(\bar{\xi})\right] \\ &= -\frac{h^4}{80}(x_n - x_0)\ f^{iv}(\bar{\xi}) \qquad \ldots (1) \end{aligned}$$

where $x_0 < \bar{\xi} < x_n$

Total area under the curve is given by,

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \ldots + \ldots (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \ldots) + 3(y_1 + y_2 + y_4 + y_5 + \ldots)\right]} \qquad \ldots (2)$$

Formula (2) gives Simpson's $\frac{3^{th}}{8}$ rule for computing the integral $\int_{x_0}^{x_n} y\,dx$ numerically. Amount of error involved in the process is given by (1).

ILLUSTRATIONS

Ex. 1 : *Use Trapezoidal rule to numerically evaluate*

$$I = \int_0^1 x \cdot e^{x^2}\,dx \text{ by taking } h = 0.1$$

Sol. : Compare the approximate value with exact value. Tabulating values (x, y).

x	0	0.1	0.2	0.3	0.4	0.5
y	$y_0 = 0$	$y_1 = 0.101005$	$0.2081621\ y_2$	$0.3282522 y_3$	$0.4694043 y_4$	$0.6420127 y_5$
x	0.6	0.7	0.8	0.9	1.00	
y	$0.8599976 y_6$	$1.1426214 y_7$	$1.5171847 y_8$	$2.0231172 y_9$	$2.7182818 y_{10}$	

$$I = \frac{0.1}{2}\ [(0 + 2.7182818) + 2(0.101005 + 0.2081621 + 0.3282522 + 0.4694043$$
$$+ 0.6420127 + 0.8599976 + 1.1426214 + 1.5171847 + 2.0231172)]$$
$$= 0.8650898$$

Exact integration gives,

$$\int_0^1 x\, e^{x^2}\, dx = \frac{1}{2} \int_0^1 2x\, e^{x^2}\, dx = \frac{1}{2} \int_0^1 e^{x^2}\, d(x^2)$$
$$= \frac{1}{2} [e^{x^2}]_0^1 = \frac{1}{2}\ [e - 1] = 0.8591409$$

Thus, the two values compare well.

Ex. 2 : *Use Simpson's $\frac{1}{3}^{rd}$ rule to obtain $\int_0^{\pi/2} \left(\frac{\sin x}{x}\right) dx$, by dividing the interval into four parts.*

Sol. : Here $h = \frac{\pi}{8}$

Using $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ we tabulate the values of x and $y = \frac{\sin x}{x}$

x	0	π/8	π/4	3π/8	π/2
y	1.0 y_0	0.9744953 y_1	0.9003163 y_2	0.7842133 y_3	0.6366197 y_4

Using 9.13 (6),

$$I = \frac{h}{3}\ [(y_0 + y_4) + 4\ (y_1 + y_3) + 2\ (y_2)]$$
$$= \frac{\pi}{24}\ [(1.0 + 0.6366197) + 4\ (0.9744953 + 0.7842133) + 2\ (0.9003163)]$$
$$= \frac{\pi}{24}\ [10.472087]$$
$$= 1.3707929$$

Ex. 3 : *Evaluate $I = \int_1^2 \frac{dx}{x^2}$ by dividing the integral into equally spaced intervals of width (i) 0.5, (ii) 0.25 and fitting quadratics through the set of three points. Compare the result with the exact value.* ***(May 2010, Dec. 2018)***

Sol. : Since, the integral is to be evaluated by fitting quadratics through the set of three points, we employ Simpson's $\frac{1}{3}^{rd}$ rule.

(i) h = 0.5 :

x	1	1.5	2.0
y = 1/x²	1	0.44444	0.25

$$I = \frac{0.5}{3}\{ (1 + 0.25) + 4\ (0.44444)\}$$
$$= 0.5046266$$

(ii) h = 0.25 :

x	1.0	1.25	1.5	1.75	2.0
y = 1/x²	1.0	0.64	0.44444	0.32653	0.25

$$I = \frac{0.25}{3}\ [(1.0 + 0.25) + 4\ (0.64 + 0.32653) + 2\ (0.44444)]$$
$$= 0.500416$$

Exact value is given by direct integration as,

$$\int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2} = 0.5$$

Comparison of (1), (2) with (3) shows that error is much less when h = 0.25.

Ex. 4 : *A curve is drawn to pass through the points given by the following table*

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, the X-axis and the ordinates x = 1, x = 4.

Sol. : Area A is given by $A = \int_1^4 y\, dx$

Employing Simpson's $\frac{1}{3}^{rd}$ rule, h = 0.5

$$A = \frac{0.5}{3}[(2 + 2.1) + 4(2.4 + 2.8 + 2.6) + 2(2.7 + 3)] = \frac{0.5}{3}[4.1 + 4(7.8) + 2(5.7)]$$

$$= 7.8$$

Ex. 5 : *A function f(t) is described by the following experimental data at equally spaced intervals.*

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f(t)	93	87	68	55	42	37	35	39	48	53	51

Evaluate the integral, $I = \int_0^1 f(t)\, dt.$

Sol. : Here h = 0.1, number of ordinates = 11

Using Simpson's $\frac{1}{3}^{rd}$ rule,

$$I = \frac{0.1}{3}[(93 + 51) + 4(87 + 55 + 37 + 39 + 53) + 2(68 + 42 + 35 + 48)]$$

$$= \frac{0.1}{3}[144 + 1084 + 386] = 53.8$$

Ex. 6 : *The speed of a train which starts from rest is given by the following table, the time being recorded in minutes from the start and the speed in kilometers per hour.*

t	2	4	6	8	10	12	14	16	18	20
v = speed	10	18	25	29	32	20	11	5	2	0

Find approximately the total distance run in 20 minutes.

Sol. : $\frac{ds}{dt} = v \quad \therefore s = \int_0^t v\, dt$, where, t = 20 min = $\frac{1}{3}$ hr.

h = 2 minutes $= \frac{1}{30}$ hours

We take $h = \frac{1}{30}$

Here, $v_0 = 0$ [the starting speed]

$v_1 = 10, \quad v_2 = 18, \quad v_3 = 25, \quad v_4 = 29, \quad v_5 = 32,$

$v_6 = 20, \quad v_7 = 11, \quad v_8 = 5, \quad v_9 = 2, \quad v_{10} = 0$

Total ordinates are 11, we use Simpson's $\frac{1}{3}^{rd}$ rule.

$$s = \frac{h}{3} [(0 + 0) + 4 (10 + 25 + 32 + 11 + 2) + 2 (18 + 29 + 20 + 5)]$$

$$= \frac{1}{90} [0 + 4(80) + 2 (72)] = \frac{1}{90} (464) = 5.16$$

Distance covered = 5.16 kilometers.

Ex. 7 : *A function f(x) is described by following data :*

x	1	1.1	1.2	1.4	1.6	1.9	2.2
f (x)	3.123	4.247	5.635	9.299	14.307	24.759	39.319

Find numerical integration of the function in the limits 1.0 to 2.2 using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}^{rd}$ rule.

Sol. : (i) Trapezoidal Method :

For Trapezoidal rule, initially the area of strip (da) will be

da = $[f(x_i) + f(x_{i+1})]$ in the present case h = 1.

Area of the first strip is 0.05 [3.123 + 4.247] = 0.3685, in the similar manner areas of all strips are found out and added to give the total sum which is the value of the integral as shown below :

x	1	1.1	1.2	1.4	1.6	1.9	2.0
f (x)	3.123	4.247	5.635	9.299	14.307	24.759	39.319
da		0.3685	0.4941	1.4934	2.3600	5.8599	9.6117

Total area = 20.187

(ii) Simpson's $\frac{1}{3}^{rd}$ Rule :

The area of strip in this case will be

$$da = \frac{h}{3} [f(x_i) + 4f(x_{i+1}) + f (x_{i+2})]$$

The areas of all strips are calculated and added to give value of the integral.

x	1	1.1	1.2	1.4	1.6	1.9	2.0
f (x)	3.123	4.247	5.635	9.299	14.307	24.759	39.319
da			0.8582		3.8092		15.2662

Total Area = 19.9336

Ex. 8 : *Determine h so that the value of the integral $\int_0^1 x\, e^x\, dx$ obtained by Simpson's $\frac{1}{3}^{rd}$ rule is correct upto four decimal places.*

Sol. : The error in Simpson's $\frac{1}{3}^{rd}$ rule from Sec. 9.13 (9) is $\frac{-h^4}{180} (x_n - x_0)\ f^{iv} (\bar{\xi})$

$f(x) = xe^x$, $f'(x) = e^x + xe^x$, $f''(x) = 2e^x + xe^x$, $f'''(x) = 3e^x + xe^x$, $f^{iv}(x) = 4e^x + xe^x$

Maximum of $f^{iv}(x)$ is at x = 1 on the interval (0, 1)

$$\therefore \quad f^{iv} (\xi) = 4e + e = 5e = 13.59$$

$$E_{max} = \frac{h^4}{180} (1 - 0) \times 13.59$$

Since, the error should be in fifth decimal place

$$\frac{h^4}{180} \times 13.59 < 0.00001$$

or

$$h^4 < \frac{180 \times 0.00001}{13.59}$$

$$< 0.00013245$$

or $\quad < 0.000133$

$\therefore \quad h < 0.10734$

For Simpson's $\frac{1}{3}^{rd}$ rule, $\quad h = \dfrac{x_n - x_0}{2n}$

or $\quad \dfrac{1-0}{2n} < 0.10734$

$\therefore \quad n > \dfrac{1}{2 \times (0.10734)}$

or $\quad n > 4.658$

Thus, number of subdivisions could be taken as

$$2n = 2 \times 5 = 10 \text{ or } h = 0.1$$

Thus, by taking h = 0.1 the error in the calculation of the integral will not occur upto fourth decimal place.

Ex. 9 : *Use Simpson's $\frac{1}{3}^{rd}$ rule with two, four and ten intervals to evaluate $\int_1^2 \frac{1}{x}\, dx = \log 2$ and comment upon the results as the number of intervals is increased.*

Sol. : Using Two Intervals :

$$h = \frac{2-1}{2} = 0.5 \quad \text{and} \quad y = f(x) = \frac{1}{x}$$

$$y_0 = \frac{1}{1} = 1,\ y_1 = \frac{1}{1.5} = 0.666,\ y_2 = \frac{1}{2} = 0.5$$

$$I_1 = \int_1^2 f(x)\, dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$= \frac{0.5}{3}[1 + 4 \times 0.666 + 0.5]$$

$$= 0.694$$

Using Four Intervals :

$$h = \frac{2-1}{4} = 0.25$$

$$y_0 = 1,\quad y_1 = \frac{1}{1.25} = 0.8,\quad y_2 = \frac{1}{1.5} = 0.667,\quad y_3 = \frac{1}{1.75} = 0.57143,\quad y_4 = 0.5$$

$$I_1 = \int_1^2 \frac{1}{x}\, dx$$

$$= \frac{0.25}{3}[(1 + 0.5) + 4(0.8 + 0.57143) + 2(0.667)]$$

$$= \frac{0.25}{3}(8.31972) = 0.69331$$

Using Ten Intervals :

$$h = \frac{2-1}{10} = 0.1$$

$y_0 = 1,\quad y_1 = \frac{1}{1.1} = 0.909,\quad y_2 = \frac{1}{1.2} = 0.833,\quad y_3 = \frac{1}{1.3} = 0.769$

$y_4 = \frac{1}{1.4} = 0.714,\quad y_5 = \frac{1}{1.5} = 0.667,\quad y_6 = \frac{1}{1.6} = 0.625,\quad y_7 = \frac{1}{1.7} = 0.588,$

$y_8 = \frac{1}{1.8} = 0.556,\quad y_9 = \frac{1}{1.9} = 0.526,\quad y_{10} = \frac{1}{2} = 0.5$

$$I_3 = \frac{0.1}{3}\,[(1 + 0.5) + 4\,(0.909 + 0.769 + 0.667 + 0.588 + 0.526) + 2(0.833 + 0.714 + 0.625 + 0.556)]$$

$$= \frac{0.1}{3}\,[1.5 + 13.836 + 5.456]$$

$$= 0.693067$$

Actual value of log 2 = 0.69314718.

As the number of intervals are increased, the accuracy of the result is also increased.

Ex. 10 : *Evaluate* $\int_0^{\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\,d\theta$. *By Simpson's* $\frac{3^{th}}{8}$ *rule, taking* $h = \frac{\pi}{6}$.

Sol. : Tabulating the values for $y = \frac{\sin^2\theta}{5 + 4\cos\theta}$

θ	0	π/6	π/3	π/2	2π/3	5π/6	π
y	$0 y_0$	$0.0295365\ y_1$	$0.1071428\ y_2$	$0.2\ y_3$	$0.25\ y_4$	$0.1627711 y_5$	$0.0\ y_6$

$$I = \frac{3h}{8}\,[(y_0 + y_6) + 3\,(y_1 + y_4) + 3\,(y_2 + y_5) + 2y_3]$$

$$= \frac{3}{8} \times \frac{\pi}{6}\,[(0 + 0) + 3\,(0.0295365 + 0.25) + 3\,(0.1071428 + 0.162711 + 2\,(0.2)]$$

$$= \frac{\pi}{16}\,[2.0483512] = 0.4021928$$

Ex. 11 : *Evaluate* $\int_0^{3} \frac{dx}{1 + x}$ *with 7 ordinates by using Simpson's* $\frac{3^{th}}{8}$ *rule and hence calculate log 2. Estimate the bound of error involved in the process.*

Sol. : Here $h = \frac{3 - 0}{6} = \frac{1}{2} = 0.5;\ y = \frac{1}{1 + x}$

Tabulating the values of x and y,

x	0	0.5	1.0	1.5	2.0	2.5	3
y	1	0.6667	0.5	0.4	0.3333	0.2857	0.25

By Simpson's $\frac{3^{th}}{8}$ rule [5.14 (2)]

$$\int_0^3 \frac{1}{1 + x}\,dx = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_4) + 3\,(y_2 + y_5) + 2y_3]$$

$$= \frac{3 \times 0.5}{8}\,[(1 + 0.25) + 3\,(0.6667 + 0.3333) + 3\,(0.5 + 0.2857) + 2 \times 0.4]$$

$$= 0.1875\,[4.9001 + 2.507]$$

$$= 0.1875 \times 7.4071 = 1.3888$$

By direct integration,

$$\left[\int_0^3 \frac{1}{1 + x}\,dx = \log\,(1 + x)\right]_0^3 = \log 4 = 2 \log 2$$

$$\therefore \quad 2 \log 2 = 1.3888$$

$$\log 2 = 0.6944$$

Bound of truncation error involved is given by [5.14 (1)]

$$\frac{h^4}{80}(x_n - x_0)\; f^{iv}(\bar{\xi}) = \frac{(0.5)^4}{80}(3-0)\; f^{iv}(\bar{\xi})$$

$$f(x) = \frac{1}{1+x},\; f'(x) = -\frac{1}{(1+x)^2},\; f''(x) = \frac{2}{(1+x)^3},\; f'''(x) = \frac{-6}{(1+x)^4},\; f^{iv}(x) = \frac{24}{(1+x)^5}$$

for the interval (0, 3) $f^{iv}(x)$ is maximum,

for $x = 0$ i.e. $f^{iv}(\bar{\xi}) = \frac{24}{(1+0)^5} = 24$

$\therefore$ Truncation error $\leq \frac{(0.5)^4}{80}(3) \times 24 \leq 0.05625$

Ex. 12 : *Compute the value of the definite integral $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)dx$ by Simpson's $\frac{3}{8}^{th}$ rule. Take h = 0.1.*

Sol. : $y = \sin x - \log_e x + e^x$

Tabulating values of x and y

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
y	3.0295	2.849	2.7975	2.82129	2.8975	3.0146	3.1660	3.3483	3.559	3.80	4.06	4.37	3.704
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}

Using Simpson's $\frac{3}{8}^{th}$ rule,

$$\int_{0.2}^{1.4} y\,dx = \frac{3h}{8}[(y_0 + y_{12}) + 3(y_1 + y_4 + y_7 + y_{10}) + 3(y_2 + y_5 + y_8 + y_{11}) + 2(y_3 + y_6 + y_9)]$$

$$= \frac{3 \times 0.1}{2}\;[(3.0295 + 4.704) + 3(2.849 + 2.8975 + 3.3483 + 4.06)]$$

$$+ 3[(2.7975 + 3.0146 + 3.559 + 4.37) + 2(2.82129 + 3.1660 + 3.80)]$$

$$= \frac{0.3}{8}[7.7335 + 3(13.1548) + 3(13.7411) + 2(9.78729)]$$

$$= \frac{0.3}{8}[107.99578]$$

$$= 4.04984 \text{ [Exact value = 4.0506]}$$

Ex. 13 : *The table below shows the temperature f(t) as function of time.*

Time t	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Temperature f (t)	*81*	*75*	*80*	*83*	*78*	*70*	*60*

Use (i) Simpson's $\frac{1}{3}^{rd}$ method and (ii) Simpson's $\frac{3}{8}^{rd}$ method to estimate $\int_1^7 f(t)\,dt$.

Sol. :

Time t	1	2	3	4	5	6	7
Temperature f (t)	$t_0 = 81$	$t_1 = 75$	$t_2 = 80$	$t_3 = 83$	$t_4 = 78$	$t_5 = 70$	$t_6 = 60$

Here, h = 1.

(i) Using Simpson's $\frac{1}{3}^{rd}$ method,

$$A = \int_{x_n}^{x_0} y\,dx = \frac{h}{3}\{(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + y_6 + \ldots)\}$$

$$A = \frac{1}{3}\{(t_0 + t_6) + 4(t_1 + t_3 + t_5) + 2(t_2 + t_4)\}$$

$$= \frac{1}{3}\{(81 + 60) + 4(75 + 83 + 70) + 2(80 + 78)\}$$

$$= 456.3333$$

(ii) Using Simpson's $\frac{3^{th}}{8}$ method,

$$A = \frac{3h}{8}\{(t_0 + t_6) + 3(t_1 + t_2 + t_4 + t_5) + 2(t_3)\}$$

$$= \frac{3}{8}\{(81 + 60) + 3(75 + 80 + 78 + 70) + 2(83)\}$$

$$= 456.0000$$

Ex. 14 : *A gas is expanded according to law $PV^{1.3}$ = constant, from pressure of 10 N/m^2. Assuming initial volume of gas 1 m^3 and final volume 7 m^3. Calculate work done using Simpson's $\frac{1}{3}^{rd}$ rule. Divide volume in 6 equal intervals.*

Sol. : Given : $P_1 = 10$ bar, $V_1 = 1$ m^3 and $V_{final} = 7$ m^3

$$PV^{1.3} = \text{Constant}$$

We have, Work done $= \int p\, dv$

$$\therefore \quad P_1 V_1^{1.3} = P_i V_i^{1.3}$$

where, $i = 1$ to 7

$$\therefore \quad P_i = P_1 \left(\frac{V_1}{V_i}\right)^{1.3}$$

$$n = 6, \; h = 7 - 1/6 = 1$$

V	1	2	3	4	5	6	7
P	10	4.06126	2.39741	1.64938	1.23406	0.977365	0.79684

$$P_2 = P_1\left(\frac{V_1}{V_2}\right)^{1.2} = 10\left(\frac{1}{2}\right)^{1.3}$$

$$= 4.06126$$

$$P_3 = P_1\left(\frac{V_1}{V_3}\right)^{1.3} = 10\left(\frac{1}{3}\right)^{1.3}$$

$$= 2.39741$$

$$P_4 = P_1\left(\frac{V_1}{V_4}\right)^{1.3} = 10\left(\frac{1}{4}\right) = 1.64938$$

Similarly, we get, $P_5 = 1.23406$, $P_6 = 0.977365$, $P_7 = 0.79684$.

We have, Simpson's $\frac{1}{3}^{rd}$ rule,

$$\text{Work done} = \frac{h}{3}\left(P_1 + P_7 + 4\underbrace{\sum_{i=2}^{n-1} P_i}_{\text{step } 2} + 2\underbrace{\sum_{i=3}^{n-2} P_i}_{\text{step } 2}\right)$$

$$= \frac{1}{3}[10 + 0.79684 + 4(4.06126 + 1.64938 + 0.97365) + 2(2.39741 + 1.23406)]$$

$$= 14.932314$$

$\therefore$ Hence, required work done = 14.932314

Ex. 15 : *Time required for cooling an object is given by the relation :*

$$\text{Time} = \int_{400}^{700} \frac{9.085\ dt}{7.895\,(T - 293) + 3.4 \times 10^{-8}\,(T^4 - 293^4)}$$

Compute the time using Simpson's $\frac{1}{3}^{rd}$ rule and taking four intervals.

Sol. : The total interval is (700 – 400) = 300. Since, it is to be divided into four intervals, each interval is 300/4 = 75 units. The values of the function is therefore, calculated and tabulated as under :

T	400	475	550	625	700
F (T)	0.006203	0.003114	0.001858	0.001201	0.006060
da		0.512964		0.187059	

$$\text{Area} = 0.512964 + 0.187059 = 0.700023$$

$$\text{Sample calculation – da} = \frac{h}{3}(f_1 + 4f_2 + f_3)$$

$$= \frac{75}{3}(0.006203 + 4 \times 0.003114 + 0.001858)$$

$$= 0.512964$$

Ex. 16 : *Evaluate $\int_0^{0.8} [\log_e (x + 1) + \sin (2x)]dx$ where x is in radian. Using Simpson's $\frac{1}{3}^{rd}$ rule, divide the entire interval into 8 strips.*

Sol. : $y = f(x) = \log_e (x + 1) + \sin (2x)\ dx$

$$h = \text{strip width} = \frac{x_n - x_0}{n} = \frac{0.8 - 0}{0.8} = 0.1$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y = f(x)	0	0.2939	0.5717	0.827	1.0538	1.2469	1.402	1.516	1.5873
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Simpson's $\frac{1}{3}^{rd}$ rule.

$$I = \int_{x_0}^{x_2 = 0.2} f(x)\,dx + \int_{x_2 = 0.2}^{x = 0.4} f(x)\,dx + \int_{x = 0.4}^{x = 0.6} f(x)\,dx + \int_{x = 0.6}^{x = 0.6} f(x)\,dx$$

where, $f(x) = \log_e (x + 1) + \sin 2x$

$$= \frac{h}{3}\,[y_0 + 4y_1 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + y_6 + 4y_7 + y_8]$$

$$= \frac{h}{3}\,[y_0 + y_8 + 4\,(y_1 + y_3 + y_5 + y_7) + 2\,(y_2 + y_4 + y_6)]$$

$$= \frac{0.1}{3}\,[0 + 1.5873 + 4\,(0.2939 + 0.827 + 1.2469 + 1.516) + 2\,(0.5717 + 1.0538 + 1.402)]$$

$$I = 0.7725833$$

Ex. 17 : *A function f (x) is described by the following data :*

x	1	1.1	1.2	1.4	1.6	1.9	2.2
f (x)	3.123	4.247	5.635	9.299	14.307	24.759	39.319

Find numerical integration of the function in the limits from 1 to 2.2 using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}^{rd}$ *rule.*

Sol. : (i) Trapezoidal Rule :

For first two strips, h = 0.1

For next two strips, h = 0.2

and for next strips, h = 0.3

Area by Trapezoidal Rule is given by

$$A_1 = \frac{h_1}{2}[y_0 + y_1] + \frac{h_1}{2}[y_1 + y_2] \quad h_1 = 0.1$$

$$A_2 = \frac{h_2}{2}[y_2 + y_3] + \frac{h_2}{2}[y_3 + y_4] \quad h_2 = 0.2$$

$$A_3 = \frac{h_3}{2}[y_4 + y_5] + \frac{h_3}{2}[y_5 + y_6] \quad h_3 = 0.3$$

x	1	1.1	1.2	1.4	1.6	1.9	2.2
f(x)	3.123	4.247	5.635	9.299	14.307	24.759	39.319
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\therefore \quad A_1 = \frac{h_1}{2}[y_0 + 2y_1 + y_2] = \frac{0.1}{2}[3.123 + 2(4.247) + 5.635]$$

$$= 0.8626$$

$$A_2 = \frac{h_2}{2}[y_2 + 2y_3 + y_4] = \frac{0.2}{2}[5.635 + 2(9.299) + 14.307]$$

$$= 3.854$$

$$A_3 = \frac{h_3}{2}[y_4 + 2y_5 + y_6]$$

$$= \frac{0.3}{2}[14.307 + 2(24.759) + 39.319]$$

$$= 15.4716$$

$$A = A_1 + A_2 + A_3 = 0.8626 + 3.854 + 15.4716$$

$$A = 20.1882$$

(ii) Simpson's $\frac{1}{3}^{rd}$ Rule :

$$A_1 = \frac{h_1}{3}(y_0 + 4y_1 + y_2) = \frac{0.1}{3}[3.123 + 4(4.247) + 5.635]$$

$$A_1 = 0.8582$$

$$A_2 = \frac{h_2}{3}[y_2 + 4y_3 + y_4] = \frac{0.2}{3}[5.635 + 4(9.299) + 114.307]$$

$$A_2 = 3.8092$$

$$A_3 = \frac{h_3}{3}[y_4 + 4y_5 + y_6] = \frac{0.3}{3}[14.307 + 4(24.759) + 39.319]$$

$$A_3 = 15.2662$$

$\therefore$ Total area is

$$A = A_1 + A_2 + A_3$$

$$= 0.8582 + 3.8092 + 15.2662$$

$$A = 19.9336$$

Ex. 18 : *A cylinder of radius 150 mm and length 900 mm with a movable piston is filled with gas at a pressure of 10 bar. Find out work required to compress the gas to one third of its initial volume by Simpson's* $\frac{3^{th}}{8}$ *rule. (**Hint :** Work done W = JPdV). Take number of strips = 2.*

Sol. : Given : Cylinder radius = 150 mm = 0.15 m;

Cylinder length = 900 mm = 0.9 m, Gas pressure = 10 bar = 10^6 N/m^2

Work done for $V_3 = \frac{V_1}{3} = 0$

$$\text{Number of strips} = 2$$

$$W = JP\,dV$$

Now,
$$V_1 = \pi r^2 l$$
$$= \pi (0.15)^2\, 0.9$$
$$V_1 = 0.063617 \text{ m}^3$$
$$V_2 = \frac{V_1}{2} = 0.0318086 \text{ m}^3$$
$$V_3 = \frac{V_1}{3} = 0.0212057 \text{m}^3$$
$$V_4 = \frac{V_1}{4} = 0.01590 \text{ m}^3$$
$$W_1 = JPdV_1 = JP(V_1 - V_2) = 10^6\ (0.0318084)$$
$$= 31808.4 \text{ J}$$
$$W_2 = JP\,(V_1 - V_3) = 10^6\,(0.0424113)$$
$$JPdV_2 = 42411.3 \text{ J}$$
$$W_3 = JP\,(V_1 - V_4) = 10^6\ (0.047717)$$
$$JPdV_3 = 47717 \text{ J}$$

V	dV_1	dV_2	dV_3
W	31808.4	42411.3	47717
	W_0	W_1	W_2

By Simpson's $\frac{3^{th}}{8}$ rule, take h = 2.

$$W = \frac{3h}{8}\,[(W_0 + W_2) + 3\,(W_2) + 2\,(0)] = \frac{3 \times 2}{8}\,[(31808.4) + 42411.3 + 3(47717)]$$
$$W = 163028.03 \text{ J}$$

Ex. 19 : *The data listed in the table gives measurements of heat flux q at the surface of a solar collector. Estimate the total heat absorbed by a 2 × 10^5 cm^2 collector panel during 14 hours period. The panel has an absorption efficiency ε = 42%. The total heat absorbed is given by $H = \varepsilon \int_0^t qA\, dt$, where A is area, q is heat flux and t is time.*

t (hr)	*0*	*1*	*2*	*3*	*4*	*6*	*8*	*11*	*14*
q cal/cm².hr	*0.05*	*1.72*	*5.23*	*6.38*	*7.86*	*8.05*	*8.03*	*5.82*	*0.24*

Use Simpson's $\frac{1^{rd}}{3}$ *rule.*

Sol. : Total heat absorbed is,

$$H = \varepsilon \int_0^t qA \, dt$$

$A = 2 \times 10^5 \text{ cm}^2$, $t = 14$ hr.

$\varepsilon = 42\%$

t (hr)	0	1	2	3	4	6	8	11	14
q cal/cm^2.hr	0.05	1.72	5.23	6.38	7.86	8.05	8.03	5.82	0.24

For first 4 strips, $h = 1$

For next two strips, $h = 2$

For next strips, $h = 3$

By Simpson's $\frac{1}{3}^{rd}$ rule,

$$A = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + y_{n-2}) + y_n]$$

$$A_1 = \frac{1}{3}\,[(0.05 + 4(1.72) + 6.38) + 2(5.25) + 7.86]$$

$$= 16.9233$$

$$A_2 = \frac{2}{3}\,[7.86 + 4(8.05) + 8.03]$$

$$= 32.06$$

$$A_4 = \frac{3}{3}\,[8.03 + 4(5.82) + 0.24]$$

$$= 31.55$$

Total heat absorbed,

$$H = 0.42\,(2 \times 10^5)\left\{\int_0^{14} q \, dt\right\}$$

$$= 0.42 \times 2 \times 10^5 \times [16.9233 + 32.06 + 31.55]$$

$$= 6764797.2 \text{ cal.}$$

$$H = 6764.797 \text{ kcal.}$$

Ex. 20 : *A solid of revolution is formed by rotating about x-axis, the area between x – axis, the lines x = 0 and x = 1 and a curve through the points*

x	*0.00*	*0.25*	*0.50*	*0.75*	*1.00*
y	*1.000*	*0.9886*	*0.9589*	*0.8489*	*0.9415*

Estimate the volume of the solid formed.

Sol. : When one strip of length y and width dx is rotated about x-axis, the volume generated by that strip = $\pi y^2 dx$

$$\therefore \text{ Total volume generated} = \sum_{x=0}^{1} \pi y^2 \, dx = \int_0^1 \pi y^2 \, dx$$

$$= \pi \cdot \frac{h}{3}\left[\left(y_0^2 + 4y_1^2 + y_2^2\right) + \left(y_2^2 + 4y_3^2 + y_4^2\right)\right], \quad h = 0.25$$

$$= \pi \cdot \frac{0.25}{3}\,[(1 + 4 \times 0.9773 + 0.9195) + (0.9195 + 4 \times 0.7206 + 0.8864)]$$

$$= 2.7553 \text{ cubic units.}$$

EXERCISE 9.2

1. Calculate $\int_{4}^{5.2} \log_e x \, dx$ [taking h = 0.2]. Using Simpson's rule, compare the value thus calculated with exact value.

Ans. Approximate value = 1.82785, Exact value = 1.82804

2. Evaluate numerically $\int_{0}^{1} \frac{1}{1+x^2} dx$ [h = 0.25] to calculate value of π. **Ans.** 3.1416.

3. Determine h so that the value of integral $\int_{0} e^x \, dx$ obtained by Trapezoidal rule is correct to four decimal places.

Ans. : h < 0.0066 or n > 152 (subdivisions)

4. Distance covered and corresponding speeds of motor vehicle are as under :

s	0	10	20	30	40	50	60
v	21	39	62	64	72	56	45

Find the time taken to travel the whole distance. **Ans. :** 1.2213.

5. Use Trapezium rule with four step lengths to estimate the value of the integral $\int_{0}^{2} \frac{x}{\sqrt{2+x^2}} dx$. Compare the estimate with exact value. **Ans.** Approximate value = 1.023275, Exact value = 1.035276.

6. Use Simpson's $\frac{1}{3}^{rd}$ rule to evaluate $\int_{0}^{1} \frac{\sin x}{2 + 3 \sin x} dx$ by dividing the interval into six parts. **Ans.** 0.125067

7. Evaluate $\int_{0}^{6} \frac{1}{1+x} dx$. Using Simpson's $\frac{3}{8}^{th}$ rule dividing the interval into 6 parts, compare the numerical value with exact value. **Ans.** 1.966071, Exact value = 1.9459

8. Use Simpson's $\frac{3}{8}^{th}$ rule to evaluate $\int_{0}^{\pi/2} \sqrt{\sin x + \cos x} \, dx$ taking $h = \frac{\pi}{6}$. **Ans.** 1.7702.

9. Evaluate $\int_{0}^{3\pi/20} (1 + 2 \sin x) \, dx$. Using Simpson's $\frac{1}{3}^{rd}$ rule, take 4 segments. Compare this value with analytical value and calculate the percentage relative error.

Ans. : Numerical value = 0.6892260, Analytical value = 0.6892258. Relative percentage error = 2.902×10^{-5}]

10. Determine the maximum error in evaluating the integral $\int_{0}^{\pi/2} \cos x \, dx$ using both Trapezoidal rule and Simpson's $\frac{1}{3}^{rd}$ rule, taking four sub-intervals. **Ans.** 0.0128842

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CHAPTER-10
NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

10.1 INTRODUCTION

Differential Equations occur in Engineering Sciences as mathematical models of physical phenomena. When differential equations are formulated under simplifying assumptions they are easy to solve and yield analytical solution. However, differential equations representing detailed analysis are complicated in nature and quite often it is not possible to obtain an exact solution. Numerical methods are required to be used to solve such class of differential equations. In this chapter, we shall consider the solution of ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y) \qquad \dots (1)$$

with specified initial condition $y(x_0) = y_0$.

When x varies in the interval (x_0, x_{n+1}), the specification of the condition at the starting value x_0 of the interval is called initial condition. Problem of solution of a differential equation with given initial condition is called an ***Initial Value Problem***. From an initial condition of the form $y(x_0) = y_0$, we develop methods for finding $y(x_1) = y(x_0 + h)$, $y(x_2) = y(x_1 + h)$, $y(x_3) = y(x_2 + h)$, ... , etc. where, h is a suitably chosen step length. These methods can be extended to solve system of first order differential equations together with a set of initial conditions. Such system of equations is usually expressed in the form

$$\begin{aligned} \frac{dy_1}{dx} &= f_1(x, y_1, y_2, \dots y_n) \\ \frac{dy_2}{dx} &= f_2(x, y_1, y_2, \dots y_n) \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ \frac{dy_n}{dx} &= f_n(x, y_1, y_2, \dots y_n) \end{aligned} \qquad \dots (2)$$

with initial conditions $\quad y_1(x_0) = a_1, \quad y_2(x_0) = a_2, \ \dots\dots, y_n(x_0) = a_n \qquad \dots (3)$

As we have seen earlier, our attempt is to find $y_1 = y(x_1)$, $y_2 = y(x_2)$, ... $y_{n+1} = y(x_{n+1})$ (where $x_{n+1} = x_n + h$) from the initial condition $y_0 = y(x_0)$, where h is the step length.

If a method which evaluates y at $x = x_{n+1}$ requires knowledge of only y at $x = x_n$ and no where else, then such a method is called *single step method*. In this method, from the given initial condition $y(x_0) = y_0$, we obtain y_1 at $x = x_1 = x_0 + h$ then knowing the value of y at $x = x_1$, value of y at $x = x_2 = x_1 + h$, ... and so on are found out. The solution is developed step-by-step.

If the estimate of $y(x_{n+1})$ does not depend solely on estimate of $y(x_n)$ but depends on some other intermediate values between the interval $x_0 < x < x_n$ then such a method of solving the differential equation is called *multi-step method*. In Engineering problems we are also encountered with boundary value problems where solution of differential equation has to satisfy the conditions at end points x_0 and x_{n+1} of the interval (x_0, x_{n+1}).

In what follows we shall now discuss following methods of numerical solution of ordinary differential equations :

1. Taylor's Series Method.
2. Euler's Method.
3. Modified Euler's Method.
4. Runge-Kutta Method.
5. Predictor-Corrector Methods.

10.2 TAYLOR'S SERIES METHOD

We consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \dots (1)$$

which is to be solved, subject to the initial condition $x = x_0$, $y = y_0$.

If $y = f(x)$ is the solution of (1), then the Taylor's series expansion about $x = x_0$, gives

$$y = f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \dots \qquad \dots (2)$$

or we can write,

$$y = y_0 + (x - x_0) y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \dots \dots \qquad (3)$$

Derivatives y'_0, y''_0, ... can be found from the equation (1) by successive differentiation while y_0 is known from the initial condition.

Remark : Taylor's series method is powerful single step method if we can find the successive derivatives easily.

ILLUSTRATIONS

Ex. 1 : *Solve the equation* $5x \frac{dy}{dx} + y^2 - 2 = 0$, *subject to the conditions at* $x = 4$, $y = 1$ *finding the solution for* $4 < x < 4.4$.

Sol. : Given differential equation is

$$5x\, y' + y^2 - 2 = 0, \quad x_0 = 4, \quad y_0 = 1 \qquad \dots (1)$$

Differentiating,

$$\begin{aligned} 5x\, y'' + 5y' + 2y\, y' &= 0 \\ 5x\, y''' + 10y'' + 2y\, y'' + 2y'^2 &= 0 \\ 5x\, y^{iv} + 15y''' + 2y\, y''' + 6y'\, y'' &= 0 \\ 5x\, y^{v} + 20y^{iv} + 2y\, y^{iv} + 8y'\, y''' + 6y''^2 &= 0 \end{aligned} \qquad \dots (2)$$

Putting $x_0 = 4$, $y_0 = 1$ in (1), we get

$$y'_0 = \frac{2 - y_0^2}{5x_0} = 0.05$$

Similarly by putting $x = 4$, $y = 1$ in above equations and using the values of y'_0, y''_0, y'''_0, ... calculated from (2), we get,

$$y''_0 = -0.0175, \quad y'''_0 = 0.01015, \quad y^{iv}_0 = -0.0845, \quad y^{v}_0 = 0.008998125.$$

By Taylor's series, we have

$$y = y_0 + (x - x_0) y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \dots \qquad \dots (3)$$

Substituting the values of y_0, y'_0, y''_0, y'''_0, y^{iv}_0, y^{v}_0 in (3), we get the series

$$y = f(x) = 1 + 0.05 (x - 4) - 0.00875 (x - 4)^2 + 0.0017083 (x - 4)^3 - 0.0003521 (x - 4)^4 + 0.00007498 (x - 4)^5 \qquad \dots (4)$$

Substituting $x = 4.1, 4.2, 4.3, 4.4$ in (4), we tabulate

x	4	4.1	4.2	4.3	4.4
y	1.000000	1.004914	1.009663	1.014256	1.018701

Remark : In obtaining above result we have used step lengths as h = 0.1, 0.2, 0.3 and 0.4.

In fact more accurate procedure would be to find y_1 at $x = x_0 + h = 4 + 0.1 = 4.1$ as 1.004914 and then again obtain Taylor's series with x_0, y_0 as 4.1 and 1.004914 respectively for obtaining solution $x_2 = 4.2$, $y_2 = 1.009663$ and so on.

To find y at x = 4.5, we use $x_0 = 4.4$, $y_0 = 1.018701$ and obtain Taylor's expansion by calculating y'_0, y''_0, … etc. as

$$y'_0 = 0.043739, \quad y''_0 = -0.01399, \quad y'''_0 = 0.00748, \ldots$$

$$\therefore \quad y]_{x=4.5} = 1.018701 + 0.043739\,(x-4.4) - 0.00700\,(x-4.4)^2 + 0.00125\,(x-4.4)^3 + \ldots$$

For x = 4.4, we get

$$y]_{x=4.4} = 1.018701 + 0.004374 - 0.000070 - 0.000001 = 1.023006$$

Here all the computations are performed correct upto six decimal places.

Ex. 2 : *Solve the equation* $2\dfrac{d^2y}{dx^2} = 3x\dfrac{dy}{dx} - 9y + 9$ *with* $y(0) = 1$, $y'(0) = -2$ *to estimate y for* $x = 0.1, 0.2$.

Sol. : Given differential equation is

$$2y'' = 3xy' - 9y + 9, \quad x_0 = 0, \quad y_0 = 1, \quad y'_0 = -2. \qquad \ldots (1)$$

Differentiating,

$$\begin{aligned} 2y''' &= 3x\,y'' - 6y' \\ 2y^{iv} &= 3x\,y''' - 3y'' \\ 2y^{v} &= 3x\,y^{iv} \\ 2y^{vi} &= 3x\,y^{v} + 3y^{iv} \\ 2y^{vii} &= 3x\,y^{vi} + 6y^{v} \end{aligned} \qquad \ldots (2)$$

Putting $x_0 = 0$, $y_0 = 1$, and y'_0 in (1), we get

$$y''_0 = \frac{1}{2}\,[3x_0\,y'_0 - 9y_0 + 9] = 0$$

Similarly, by putting $x_0 = 0$, $y_0 = 1$, $y'_0 = -2$ and $y''_0 = 0$ in above equations (2), we obtain

$$y'''_0 = -3y'_0 = 6, \quad y^{iv}_0 = 0, \quad y^{v}_0 = 0, \quad y^{vi}_0 = 0, \quad y^{vii}_0 = 0.$$

Hence Taylor's expansion is

$$y = y_0 + (x - x_0)\,y'_0 + \frac{(x-x_0)^2}{2!}\,y''_0 + \frac{(x-x_0)^3}{3!}\,y'''_0 + \ldots$$

$$y = 1 + xy'_0 + \frac{x^2}{2}\,y''_0 + \frac{x^3}{6}\,y'''_0$$

$$y = 1 - 2x + x^3$$

$$y]_{x=0.1} = 1 - 2\,(0.1) + (0.1)^3 = 0.801$$

$$y]_{x=0.2} = 1 - 2\,(0.2) + (0.2)^3 = 0.68$$

10.3 EULER'S METHOD

Taylor's series method discussed in the previous section yield the solution of a differential equation in the form of power series. We will now describe methods which give the solution in the form of tabulated values (i.e. discrete set of y_n values for argument values x_n).

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots (1)$$

which is to be solved, subject to initial condition $x = x_0$, $y = y_0$.

Suppose we want to obtain the value of y at $x = x_1 = x_0 + h$, $x = x_2 = x_1 + h$, $x = x_3 = x_2 + h$, … $x = x_{n+1} = x_n + h$. Writing (1) as

$$dy = f(x, y)\,dx$$

Integrating both the sides, with limits of y as y_0 to y_1 which correspond to limits of x as x_0 to x_1, we get

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

We assume that for the interval (x_0, x_1), f(x, y) remains stationary as $f(x_0, y_0)$.

$$\therefore \quad y_1 - y_0 = f(x_0, y_0)\, [x]_{x_0}^{x_1}$$

$$= (x_1 - x_0)\, f(x_0, y_0)$$

$$= h\, f(x_0, y_0)$$

$$\therefore \quad y_1 = y_0 + h\, f(x_0, y_0) \qquad \text{... (2)}$$

Thus value of y at $x = x_1 = x_0 + h$ is calculated from the formula (2).

Proceeding in similar fashion, we shall get,

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_3 = y_2 + h\, f(x_2, y_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_{n+1} = y_n + h\, f(x_n, y_n) \qquad \text{... (3)}$$

which gives Euler's formula to find y at $x = x_1, x_2, \ldots\ldots, x_n$.

ILLUSTRATIONS

***Ex. 1 :** Use Euler's method to solve the equation*

$$\frac{dy}{dx} = 1 + xy$$

Subject to the conditions at x = 0, y = 1 and tabulate y for x = 0 (0.1) 0.5.

Sol. : Here $f(x, y) = 1 + xy$, $h = 0.1$, $x_0 = 0$, $y_0 = 1$.

Successive application of formula (3) with h = 0.1 gives

Step 1 : $f(x_0, y_0) = 1 + x_0 y_0 = 1 + (0)(1) = 1$ $\quad [x_0 = 0,\ y_0 = 1]$

$\therefore$ $y_1 = y_0 + h\, f(x_0, y_0) = 1 + (0.1)\ (1) = 1.1$

Thus $y_1 = y]_{x = 0.1} = 1.1$

Step 2 : $f(x_1, y_1) = 1 + x_1 y_1 = 1 + (0.1)(1.1) = 1.11$ $\quad [x_1 = 0.1,\ y_1 = 1.1]$

$\therefore$ $y_2 = y_1 + h\, f(x_1, y_1) = 1.1 + (0.1)(1.11) = 1.211$

Thus $y_2 = y]_{x = 0.2} = 1.211$

Step 3 : $f(x_2, y_2) = 1 + x_2 y_2 = 1 + (0.2)(1.211) = 1.2422$ $\quad [x_2 = 0.2,\ y_2 = 1.211]$

$\therefore$ $y_3 = y_2 + h\, f(x_2, y_2) = 1.211 + (0.1)(1.2422) = 1.33522$

Thus $y_3 = y]_{x = 0.3} = 1.3352$ (rounding to fourth decimal place)

Step 4 : $f(x_3, y_3) = 1 + x_3 y_3 = 1 + (0.3)(1.3352) = 1.40056$ $\quad [x_3 = 0.3,\ y_3 = 1.3352]$

$\therefore$ $y_4 = y_3 + h\, f(x_3, y_3) = 1.3352 + (0.1)(1.40056) = 1.475246$

Thus, $y_4 = y]_{x = 0.4} = 1.4753$

Step 5 : $f(x_4, y_4) = 1 + x_4 y_4 = 1 + (0.4)(1.4753) = 1.59012$ $\quad [x_4 = 0.4,\ y_4 = 1.4753]$

$\therefore$ $y_5 = y_4 + h\, f(x_4, y_4) = 1.4753 + (0.1)(1.59012) = 1.634312$

Thus $y_5 = y]_{x = 0.5} = 1.6343$

The tabulated solution is

x	0	0.1	0.2	0.3	0.4	0.5
y	1	1.1	1.211	1.3352	1.4753	1.6343

Ex. 2 : *(a) Use Euler's method to solve the ordinary differential equation*

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from x = 0 to x = 4 with step size of 0.5. The initial condition is y(0) = 1.

(b) Giving proper example illustrate the effect of step size on stability of Euler's method.

Sol. : (a) Here $f(x, y) = -2x^3 + 12x^2 - 20x + 8.5, \quad h = 0.5, \; x_0 = 0, \; y_0 = 1$

Successive application of formula (3) with h = 0.5 gives

Step 1 : $y_1 = y_0 + h\, f(x_0, y_0)$ $[x_0 = 0, \; y_0 = 1]$

$= 1 + (0.5)\,[-2\,(0) + 12\,(0) - 20\,(0) + 8.5]$

$= 5.25$

Step 2 : $y_2 = y_1 + h\, f(x_1, y_1)$ $[x_1 = 0.5, y_1 = 5.25]$

$= 5.25 + (0.5)\,[-2\,(0.5)^3 + 12\,(0.5)^2 - 20\,(0.5) + 8.5]$

$= 5.875$

Step 3 : $y_3 = y_2 + h\, f(x_2, y_2)$ $[x_2 = 1, \; y_2 = 5.875]$

$= 5.875 + (0.5)\,[-2\,(1)^3 + 12\,(1)^2 - 20\,(1) + 8.5]$

$= 5.125$

Step 4 : $y_4 = y_3 + h\, f(x_3, y_3)$ $[x_3 = 1.5, \; y_3 = 5.125]$

$= 5.125 + (0.5)\,[-2\,(1.5)^3 + 12\,(1.5)^2 - 20\,(1.5) + 8.5]$

$= 4.5$

Step 5 : $y_5 = y_4 + h\, f(x_4, y_4)$ $[x_4 = 2, \; y_4 = 4.5]$

$= 4.5 + (0.5)\,[-2\,(2)^3 + 12\,(2)^2 - 20\,(2) + 8.5]$

$= 4.75$

Step 6 : $y_6 = y_5 + h\, f(x_5, y_5)$ $[x_5 = 2.5, \; y_5 = 4.75]$

$= 4.75 + (0.5)\,[-2\,(2.5)^3 + 12\,(2.5)^2 - 20\,(2.5) + 8.5]$

$= 5.875$

Step 7 : $y_7 = y_6 + h\, f(x_6, y_6)$ $[x_6 = 3, \; y_6 = 5.875]$

$= 5.875 + (0.5)\,[-2\,(3)^3 + 12\,(3)^2 - 20\,(3) + 8.5]$

$= 7.125$

Step 8 : $y_8 = y_7 + h\, f(x_7, y_7)$ $[x_7 = 3.5, \; y_7 = 7.125]$

$= 7.125 + (0.5)\,[-2\,(3.5)^3 + 12\,(3.5)^2 - 20\,(3.5) + 8.5]$

$= 7$

The tabulated solution is solution by Euler's method.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	1	5.25	5.875	5.125	4.5	4.75	5.875	7.125	7

It may be interesting to note that the exact solution obtained by direct integration satisfying the initial condition is

$$y = -\frac{x^4}{2} + 4x^3 - 10x^2 + 8.5x + 1$$

Exact Solution : Solution when tabulated is

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	1	3.219	3	2.219	2	2.719	4	4.719	3

Comparison of the two solutions shows that approximate solution by Euler's method deviates too much from exact solution. This is because step size h = 0.5 is too large.

(b) Let us take step size h = 0.1 in the above problem. We shall calculate y at x = 0.5.

$$\begin{aligned}
y_1 &= y_0 + h\, f(x_0, y_0) && [x_0 = 0,\ y_0 = 1] \\
&= 1 + (0.1)\,[-2\,(0)^3 + 12\,(0)^2 - 20\,(0) + 8.5] \\
&= 1.85 \\
y_2 &= y_1 + h\, f(x_1, y_1) && [x_1 = 0.1,\ y_1 = 1.85] \\
&= 1.85 + (0.1)\,[-2\,(0.1)^3 + 12\,(0.1)^2 - 20\,(0.1) + 8.5] \\
&= 2.5118 \\
y_3 &= y_2 + h\, f(x_2, y_2) && [x_2 = 0.2,\ y_2 = 2.5118] \\
&= 2.5118 + (0.1)\,[-2\,(0.2)^3 + 12\,(0.2)^2 - 20\,(0.1) + 8.5] \\
&= 3.2082 \\
y_4 &= y_3 + h\, f(x_3, y_3) && [x_3 = 0.3,\ y_3 = 3.2082] \\
&= 3.2082 + (0.1)\,[-2\,(0.3)^3 + 12\,(0.3)^2 - 20\,(0.3) + 8.5] \\
&= 3.5608 \\
y_5 &= y_4 + h\, f(x_4, y_4) && [x_4 = 0.4,\ y_4 = 3.5608] \\
&= 3.5608 + (0.1)\,[-2\,(0.4)^3 + 12\,(0.4)^2 - 20\,(0.4) + 8.5] \\
&= 3.79
\end{aligned}$$

Thus, we have

$$\begin{aligned}
y_{app} \text{ at } x = 0.5 &= 3.79 \text{ when } h = 0.1 \\
y_{app} \text{ at } x = 0.5 &= 5.25 \text{ when } h = 0.5 \\
y_{exact} \text{ at } x = 0.5 &= 3.219
\end{aligned}$$

It is observed that there is significant fall in the magnitude of the error when h is taken equal to 0.1 instead of 0.5.

This illustrates the effect of step size on the stability of Euler's method.

Remark : By taking h small enough, we tabulated the solution as a set of corresponding values of x and y. It is observed that the Euler's method is either too slow (in case of h small) or too inaccurate (in case h is not small) for practical use and a modification of it, known as the modified Euler's method, gives more accurate results.

10.3.1 Euler's Method to Solve Differential Equation of Higher Order

Consider the differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \qquad \dots (1)$$

with condition at $x = x_0$, $y = y_0$, $y'(x_0) = y_0'$ specified.

To solve such an equation, we may take

$$\frac{dy}{dx} = z \quad \text{so that}$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = f(x, y, z)$$

Given equation of higher order is equivalent to system of simultaneous equations of first such as

$$\frac{dz}{dx} = f(x, y, z) \qquad \dots (2)$$

$$\frac{dy}{dx} = z = \phi\,(x, y, z) \ \text{(say)} \qquad \dots (3)$$

with conditions at $x = x_0$, $y = y_0$, $z = z_0$ ($z_0 = y_0'$).

If h is the increment in x then values of z_1, y_1 at $x = x_0 + h$ are given by Euler's method discussed earlier as;

$$\begin{aligned}
z_1 &= z_0 + h\, f\,(x_0, y_0, z_0) \\
y_1 &= y_0 + h\, \phi\,(x_0, y_0, z_0)
\end{aligned}$$

In general, we have

$$z_{n+1} = z_n + h\, f\,(x_n, y_n, z_n) \qquad \dots (4)$$

$$y_{n+1} = y_n + h\, \phi\,(x_n, y_n, z_n) \qquad \dots (5)$$

ILLUSTRATION

Ex. 1 : *Solve the equation*

$$2\frac{d^2y}{dx^2} = 3x\frac{dy}{dx} - 9y + 9$$

subject to the conditions y(0) = 1, y'(0) = – 2 using Euler's method and compute y for x = 0.1 (0.1) (0.4).

Sol. : Given equation can be written as

$$\frac{d^2y}{dx^2} = \frac{3}{2}\ x\ \frac{dy}{dx} - \frac{9}{2}\ y + \frac{9}{2} \qquad \text{... (1)}$$

Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ and therefore equation (1) is equivalent to the system

$$\frac{dz}{dx} = 1.5x\,z - 4.5y + 4.5 \qquad \text{... (2)}$$

$$\frac{dy}{dx} = z \qquad \text{...(3)}$$

With $x_0 = 0$, $y_0 = 1$, $z_0 = y'(0) = -2$.

Taking $h = 0.1$, we have

Step 1 :

$$\begin{aligned} z_1 &= z_0 + h\,f(x_0, y_0, z_0) \qquad [x_0 = 0,\ y_0 = 1,\ z_0 = -2] \\ &= -2 + (0.1)\,[1.5\,(0)\,(-2) - 4.5\,(1) + 4.5] \\ &= -2 \\ y_1 &= y_0 + h\,z_0 \\ &= 1 + (0.1)\,(-2) \\ &= 0.8 \end{aligned}$$

Step 2 :

$$\begin{aligned} z_2 &= z_1 + h\,f(x_1, y_1, z_1) \qquad [x_1 = 0.1,\ y_1 = 0.8,\ z_1 = -2] \\ &= -2 + (0.1)\,[1.5\,(0.1)\,(-2) - 4.5\,(0.8) + 4.5] \\ &= -1.94 \\ y_2 &= y_1 + h\,z_1 \\ &= 0.8 + (0.1)\,(-2) \\ &= 0.6 \end{aligned}$$

Step 3 :

$$\begin{aligned} z_3 &= z_2 + h\,f(x_2, y_2, z_2) \qquad [x_2 = 0.2,\ y_2 = 0.6,\ z_2 = -1.94] \\ &= -1.94 + (0.1)\,[1.5\,(0.2)\,(-1.94) - 4.5\,(0.6) + 4.5] \\ &= -1.8182 \\ y_3 &= y_2 + h\,z_2 \\ &= 0.6 + (0.1)\,(-1.94) = 0.406 \end{aligned}$$

Step 4 :

$$\begin{aligned} z_4 &= z_3 + h\,f(x_3, y_3, z_3) \qquad [x_3 = 0.3,\ y_3 = 0.406,\ z_3 = -1.8182] \\ &= -1.8182 + (0.1)\,[1.5\,(0.3)\,(-1.8182) - 4.5\,(0.406) + 4.5] \\ &= -1.6327 \\ y_4 &= y_3 + h\,z_3 \\ &= 0.41818 + (0.1)\,(-1.8182) \\ &= 0.23636 \end{aligned}$$

Values when tabulated are

x	0	0.1	0.2	0.3	0.4
y	1	0.8	0.6	0.406	0.23636
dy/dx = z	– 2	– 2	– 1.94	– 1.8182	– 1.6327

10.4 MODIFIED EULER'S METHOD

In modification of Euler's method while integrating

$$dy = f(x, y)\, dx$$

$f(x, y)$ is replaced by $\dfrac{f(x_0, y_0) + f(x_1, y_1)}{2}$ instead of $f(x_0, y_0)$.

$f(x_0, y_0)$ can be computed from known initial conditions. Initially y_1 is computed by using Euler's formula $y_1 = y_0 + h\, f(x_0, y_0)$ then $f(x_1, y_1)$ is computed.

The first modification of y_1 is obtained from the formula

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1)]$$

Second modification is given by

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

The n^{th} modification is given by

$$y_1^{(n)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(n-1)})]$$

The procedure is terminated when successive modification do not show any change at prefixed decimal place, depending upon the desired accuracy of the result.

After finding y_1 with desired accuracy, same procedure is repeated to find y_2. Here again first computation of y_2 is made from Euler's formula $y_2 = y_1 + h\, f(x_1, y_1)$, where y_1 is already determined by modified Euler's formula.

Successive modification of y_2 are given by

$$y_2^{(n)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{n-1})]$$

In this way, we can tabulate (x, y) for the desired interval $x_0 < x < x_n$.

ILLUSTRATIONS

As an illustration we shall solve the same problem (Ex. 1) which we have already solved by Euler's method.

Ex. 1 : *Solve the equation*

$$\frac{dy}{dx} = 1 + xy, \qquad x_0 = 0, \quad y_0 = 1$$

to find y at x = 0.1 and x = 0.2 using modified Euler's method taking h = 0.1.

Here $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

Sol. : As per our calculations of the previous problem,

$$f(x_0, y_0) = 1 + x_0 y_0 = 1$$
$$y_1 = y_0 + h\, f(x_0, y_0) = 1.1$$
$$f(x_1, y_1) = 1 + x_1 y_1 = 1 + (0.1)(1.1) = 1.11$$

(a) Now using modified Euler's method, improved values of y_1 are given by

Step 1 :

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1)] \qquad [x_0 = 0, y_0 = 1, x_1 = 0.1, y_1 = 1.1]$$
$$= 1.0 + \frac{0.1}{2} [1 + 1.11]$$
$$= 1.1055$$

Step 2 :

$$\because \quad f(x_1, y_1^{(1)}) = 1 + (0.1)(1.1055) = 1.11055$$

$$\therefore \quad y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$
$$= 1 + \frac{(0.1)}{2} [1 + 1.11055] = 1.1055275$$

If the accuracy is required upto fourth decimal place, we can accept $y_1 = 1.1055$. For better accuracy, we find further modification.

Step 3 : $\because$ $f(x_1, y_1^{(2)}) = 1 + (0.1)(1.1055275) = 1.11055275$

$$\therefore \quad y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}[1 + 1.11055275]$$

$$= 1.105527638$$

Step 4 : $\because$ $f(x_1, y_1^{(3)}) = 1 + (0.1)(1.105527638) = 1.1105527638$

$$\therefore \quad y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.1}{2}[1 + 1.1105527638]$$

$$= 1.105527638$$

$y_1^{(3)}$ and $y_1^{(4)}$ do not show any change and the procedure is determined. Rounding at seventh decimal place, we can take

$$y_1 = 1.1055276$$

(b) To obtain y_2, i.e., the value of y when $x = 0.2$, we first use Euler's formula.

$$\because \quad f(x_1, y_1) = 1 + x_1 y_1 = 1 + (0.1)(1.1055276) = 1.11055276$$

$$\therefore \quad y_2 = y_1 + h\, f(x_1, y_1)$$

$$= 1.1055276 + (0.1)(1.11055276)$$

$$= 1.216582876$$

$$= 1.21658 \quad \text{(rounding at fifth decimal place)}$$

$$f(x_2, y_2) = 1 + x_2 y_2 = 1 + (0.2)(1.21658) = 1.243316$$

Now using modified Euler's method, improved values of y_2 are given by;

Step 1 : $y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$

$[x_1 = 0.1, y_1 = 1.1055276, x_2 = 0.2, y_2 = 1.21658]$

$$= 1.1055276 + \frac{0.1}{2}[1.11055276 + 1.243316]$$

$$= 1.223221038$$

$$= 1.22322 \quad \text{(rounding at fifth decimal place)}$$

Step 2 : $\because$ $f(x_2, y_2^{(1)}) = 1 + (0.2)(1.22322) = 1.244644$

$$\therefore \quad y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.1055276 + \frac{0.1}{2}[1.11055276 + 1.244644]$$

$$= 1.223287438$$

$$= 1.22329 \quad \text{(rounding at fifth decimal place)}$$

Step 3 : $\because$ $f(x_2, y_2^{(2)}) = 1 + (0.2)(1.22329) = 1.244658$

$$\therefore \quad y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 1.1055276 + \frac{0.1}{2}[1.11055276 + 1.244658]$$

$$= 1.223288138$$

$$= 1.22329 \quad \text{(rounding at fifth decimal place)}$$

Procedure is thus terminated as $y_2^{(2)}$ and $y_2^{(3)}$ agree upto fifth decimal place.

Ex. 2 : *Determine using modified Euler's method the value of y when x = 0.1, given that*

$$\frac{dy}{dx} = x^2 + y, \quad y(0) = 1$$

Sol. : Here, $f(x, y) = x^2 + y, \quad x_0 = 0, \quad y_0 = 1$ and we take $h = 0.05$

$$f(x_0, y_0) = 1$$

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + (0.05)(1) = 1.05$$

$$f(x_1, y_1) = x_1^2 + y_1 = (0.05)^2 + 1.05 = 1.0525$$

(a) Using modified Euler's method, improved values of y_1 are given by

Step 1 : $$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

$[x_0 = 0, y_0 = 1, x_1 = 0.05, y_1 = 1.05]$

$$= 1 + \frac{0.05}{2}\,[1 + 1.0525] = 1.0513$$

Step 2 : $\because$ $$f(x_1, y_1^{(1)}) = (0.05)^2 + 1.0513 = 1.0538$$

$\therefore$ $$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.05}{2}\,[1 + 1.0538] = 1.0513$$

Hence we take $y_1 = 1.0513$, which is correct to four decimal places.

(b) To obtain y_2, i.e. the value of y when $x = 0.1$, we first use Euler's formula.

$\because$ $$f(x_1, y_1) = x_1^2 + y_1 = (0.05)^2 + 1.0513 = 1.0538$$

$\therefore$ $$y_2 = y_1 + h\, f(x_1, y_1)$$

$$= 1.0513 + (0.05)(1.0538) = 1.1040$$

$$f(x_2, y_2) = x_2^2 + y_2 = (0.1)^2 + 1.1040 = 1.114$$

Improved values of y_2, using modified Euler's method are

Step 1 : $$y_2^{(1)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2)]$$ $[x_1 = 0.05, y_1 = 1.0513, x_2 = 0.1, y_2 = 1.1040]$

$$= 1.0513 + \frac{0.05}{2}\,[1.0538 + 1.114] = 1.1055$$

Step 2 : $\because$ $$f(x_2, y_2^{(1)}) = x_2^2 + y_2^{(1)} = (0.1)^2 + 1.1055 = 1.1155$$

$\therefore$ $$y_2^{(2)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.0513 + \frac{0.05}{2}\,[1.0538 + 1.1155] = 1.1055$$

Procedure is thus terminated as $y_2^{(1)}$ and $y_2^{(2)}$ agree upto fifth decimal place and we conclude that the value of y when x = 0.1 is 1.1055.

10.5 RUNGE-KUTTA METHODS

The Runge-Kutta methods are designed to give more greater accuracy with the advantage of requiring function values only at some selected points on the subinterval. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \qquad \text{... (1)}$$

(A) Second Order Runge-Kutta Method :

If we substitute $y_1 = y_0 + h\, f(x_0, y_0)$ in the right hand side of modified Euler's formula, we obtain

$$y_1 = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

$$y_1 = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h,\ y_0 + h\, f(x_0, y_0))]$$

$$= y_0 + \frac{1}{2}\,[h\, f(x_0, y_0) + h\, f(x_0 + h, y_0 + h\, f(x_0, y_0))]$$

If we now set, $k_1 = h\, f(x_0, y_0)$

$$k_2 = h\, f(x_0 + h, y_0 + k_1)$$

and $k = \frac{1}{2}\,[k_1 + k_2]$

then the above equation becomes

$$y]_{x = x_0 + h} = y_0 + k \qquad \text{... (2)}$$

which is the second order Runge-Kutta formula. The error in this formula can be shown to be of order h^3.

ILLUSTRATION

Ex. 1 : *Solve the equation*

$$\frac{dy}{dx} = y - x$$

by second order Runge-Kutta method, subject to the condition y(0) = 2 and calculate y at x = 0.2 taking h = 0.1.

Sol. : Here we take $f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$ and $h = 0.1$ and carry out the calculations in two steps.

Step 1 : $k_1 = h\, f(x_0, y_0) = h\,[y_0 - x_0] = (0.1)\,[2 - 0] = 0.2$

$$k_2 = h\, f(x_0 + h, y_0 + k_1) = 0.1\,[(2 + 0.2) - (0 + 0.1)] = 0.21$$

and $k = \frac{1}{2}\,[k_1 + k_2] = \frac{1}{2}\,[0.2 + 0.21] = 0.205$

$\therefore$ $y]_{x = 0.1} = y_1 = y_0 + k = 2 + 0.205 = 2.205$

Step 2 : For calculating y at x = 0.2, now $x_0 = 0.1$, $y_0 = 2.205$, $h = 0.1$

$$k_1 = h\, f(x_0, y_0) = 0.1\,[2.205 - 0.1] = 0.2105$$

$$k_2 = h\, f(x_0 + h,\ y_0 + k_1) = 0.1\,[(2.205 + 0.2105) - (0.1 + 0.1)]$$

$$= 0.22155$$

and $k = \frac{1}{2}\,[k_1 + k_2] = \frac{1}{2}\,[0.2105 + 0.22155] = 0.216025$

$\therefore$ $y]_{x = 0.2} = y_2 = y_0 + k$

$$= 2.205 + 0.216025 = 2.421025$$

(B) Fourth Order Runge-Kutta Method :

A still more accurate and commonly used method of great practical importance is the classical Runge-Kutta method of fourth order which we call briefly Runge-Kutta method.

For computing the increment k of y corresponding to an increment h of x, fourth order Runge-Kutta formulae are

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h,\ y_0 + k_3)$$

and $k = \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$

Then the required approximate value is given by

$$y]_{x = x_0 + h} = y_0 + k \qquad \text{... (3)}$$

The error in this formula can be shown to be of order h^5.

Remark : The advantage of these methods is that the operation is identical whether the differential equation is linear or non-linear.

ILLUSTRATIONS

Ex. 1 : *Using fourth order Runge-Kutta method, solve the equation* $\frac{dy}{dx} = \sqrt{x+y}$ *subject to the conditions x = 0, y = 1 and find y at x = 0.2 taking h = 0.2.*

Sol. : To determine y at x = 0.2, we have $x_0 = 0$, $y_0 = 1$ and h = 0.2.

We then obtain

$$k_1 = h\,f(x_0, y_0) = h\sqrt{x_0 + y_0} = (0.2)\sqrt{0+1} = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\sqrt{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)}$$

$$= 0.2\sqrt{(0 + 0.1) + (1 + 0.1)} = 0.2191$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\sqrt{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)}$$

$$= 0.2\sqrt{(0 + 0.1) + (1 + 0.10955)} = 0.2120$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = h\sqrt{(x_0 + h) + (y_0 + k_3)}$$

$$= 0.2\sqrt{(0 + 0.2) + (1 + 0.2120)} = 0.2377$$

and

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.2191) + 2(0.2120) + 0.2377]$$

$$= 0.2167$$

Hence

$$y]_{x = 0.2} = y_0 + k = 1 + 0.2167 = 1.2167$$

Ex. 2 : *Using fourth order Runge-Kutta method, solve the equation* $\frac{dy}{dx} = \sqrt{x+y}$ *subject to the conditions x = 0, y = 1 to find y at x = 0.2 taking h = 0.1.* ***(Dec. 2010)***

Sol. : We carry out the calculation in two steps.

Step 1 : Taking h = 0.1, and $x_0 = 0$, $y_0 = 1$, we shall first determine y at x = 0.1

$$k_1 = h\,f(x_0, y_0) = h\sqrt{x_0 + y_0} = (0.1)\sqrt{0+1} = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\sqrt{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)}$$

$$= (0.1)\sqrt{(0 + 0.05) + (1 + 0.05)} = 0.1049$$

$$k_3 = h\,f(x_0 + h, y_0 + k_3) = h\sqrt{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)}$$

$$= (0.1)\sqrt{(0 + 0.05) + (1 + 0.05245)} = 0.1050$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h\sqrt{(x_0 + h) + (y_0 + k_3)}$$

$$= (0.1)\sqrt{(0 + 0.1) + (1 + 0.1050)} = 0.1098$$

and

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[(0.1) + 2(0.1049) + 2(0.1050) + 0.1098]$$

$$= 0.1049$$

giving

$$y]_{x = 0.1} = y_0 + k = 1 + 0.1049 = 1.1049$$

Step 2 : To calculate y at x = 0.2, we take $x_0 = 0.1$, $y_0 = 1.1049$ and $h = 0.1$

$$k_1 = h\, f(x_0, y_0) = h\sqrt{x_0 + y_0} = (0.1)\sqrt{0.1 + 1.1049} = 0.1098$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\sqrt{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)}$$

$$= (0.1)\sqrt{(0.1 + 0.05) + (1.1049 + 0.0549)} = 0.1144$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\sqrt{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)}$$

$$= (0.1)\sqrt{(0.1 + 0.05) + (1.1049 + 0.0572)} = 0.1145$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = h\sqrt{(x_0 + h) + (y_0 + k_3)}$$

$$= (0.1)\sqrt{(0.1 + 0.1) + (1.1049 + 0.1145)} = 0.1191$$

and
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1098 + 2(0.1144) + 2(0.1145) + 0.1191]$$

$$= 0.1145$$

Hence,
$$y]_{x = 0.2} = y_0 + k = 1.1049 + 0.1145 = 1.2194$$

10.5.1 Runge-Kutta Method for Simultaneous First Order Differential Equations

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z)$$

$$\frac{dz}{dx} = \phi(x, y, z) \qquad \text{... (4)}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ and can be solved by Runge-Kutta method.

Starting with initial conditions $x = x_0$, $y = y_0$, $z = z_0$ and taking step sizes for x, y, z to be h, k, l respectively, we have the following formulae for the Runge-Kutta method of fourth order.

$$k_1 = h\, f(x_0, y_0, z_0)$$

$$l_1 = h\, \phi(x_0, y_0, z_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_2 = h\, \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_3 = h\, \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = h\, \phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

and
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Then the required approximate values are given by

$$y]_{x = x_0 + h} = y_0 + k \qquad \text{... (5)}$$

$$z]_{x = x_0 + h} = z_0 + l$$

Remark : Second order differential equations can be broken up into system of simultaneous first order differential equations of type (4) and can be solved by using formula (5).

ILLUSTRATIONS

Ex. 1 : *Using Runge-Kutta method, solve the system of equations*

$$\frac{dy}{dx} = x + yz$$

$$\frac{dz}{dx} = x^2 - y^2$$

subject to $x_0 = 0,\ y_0 = 1,\ z_0 = \frac{1}{2}$ *to find y and z at x = 0.2 taking* $h = 0.2$.

Sol. : Here $f(x, y, z) = x + yz$ and $\phi(x, y, z) = x^2 - y^2$.

Initial conditions are $x_0 = 0,\ y_0 = 1,\ z_0 = \frac{1}{2}$ and $h = 0.2$.

$$\begin{aligned} k_1 &= h\, f(x_0, y_0, z_0) \\ &= (0.2)\,[0 + (1)\,(1/2)] \\ &= 0.1 \end{aligned} \qquad \begin{aligned} l_1 &= h\,\phi\,(x, y, z) \\ &= (0.2)\,[(0)^2 - (1)^2] \\ &= -0.2 \end{aligned}$$

$$\begin{aligned} k_2 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \\ &= (0.2)\,[(0 + 0.1) + (1 + 0.05)\,(0.5 - 0.1)] \\ &= 0.1040 \end{aligned} \qquad \begin{aligned} l_2 &= h\,\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \\ &= (0.2)\,[(0 + 0.1)^2 - (1 + 0.05)^2] \\ &= -0.16 \end{aligned}$$

$$\begin{aligned} k_3 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= (0.2)\,[(0 + 0.1) + (1 + 0.052)\,(0.5 - 0.08)] \\ &= 0.1084 \end{aligned} \qquad \begin{aligned} l_3 &= h\,\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= (0.2)\,[(0 + 0.1)^2 - (1 + 0.052)^2] \\ &= -0.2193 \end{aligned}$$

$$\begin{aligned} k_4 &= h\, f(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= (0.2)\,[(0 + 0.2) + (1 + 0.1084)\,(0.5 - 0.2193)] \\ &= 0.1022 \end{aligned} \qquad \begin{aligned} l_4 &= h\,\phi\,(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= (0.2)\,[(0 + 0.2)^2 - (1 + 0.1084)^2] \\ &= -0.2377 \end{aligned}$$

and

$$\begin{aligned} k &= \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4) \\ &= \frac{1}{6}\,[0.1 + 2\,(0.1040) + 2\,(0.1084) + 0.1022] = 0.1045 \\ l &= \frac{1}{6}\,(l_1 + 2l_2 + 2l_3 + l_4) \\ &= \frac{1}{6}\,[-0.2 - 2\,(0.16) - 2\,(0.2193) - 0.2377] = -0.1994 \end{aligned}$$

Hence,

$$\begin{aligned} y]_{x = 0.2} &= y_0 + k = 1 + 0.1045 = 1.1045 \\ z]_{x = 0.2} &= z_0 + l = 0.5 - 0.1994 = 0.3006 \end{aligned}$$

Ex. 2 : *Solve the equation* $\frac{d^2y}{dx^2} - y^2\frac{dy}{dx} = x$, *subject to the conditions at x = 1, y = 2,* $\frac{dy}{dx} = 1$ *and calculate y at x = 1.2 taking* $h = 0.2$.

Sol. : To solve the above equation, we put $z = \frac{dy}{dx}$. Given equation then becomes

$$\frac{dz}{dx} - y^2 z = x$$

Second order equation is thus converted into simultaneous system

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = x + y^2 z$$

Here, $f(x, y, z) = z$ and $\phi\,(x, y, z) = x + y^2 z$

Initial conditions are $x_0 = 1,\ y_0 = 2,\ z_0 = \frac{dy}{dx} = 1$ and $h = 0.2$.

The system can be solved by the method used to solve the previous problem.

$$\begin{aligned} k_1 &= h\, f(x_0, y_0, z_0) & l_1 &= h\, \phi(x_0, y_0, z_0) \\ &= 0.2\,(1) = 0.2 & &= (0.2)\,[1 + (2)^2\,(1)] = 1.0 \\ k_2 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) & l_2 &= h\, \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \\ &= 0.2\,(1 + 0.5) & &= 0.2\,[(1 + 0.1) + (2 + 0.1)^2\,(1 + 0.5)] \\ &= 0.3 & &= 1.543 \\ k_3 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) & l_3 &= h\, \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= 0.2\,(1 + 0.7715) & &= 0.2\,[(1 + 0.1) + (2 + 0.15)^2\,(1 + 0.7715)] \\ &= 0.3543 & &= 1.8578 \\ k_4 &= h\, f(x_0 + h, y_0 + k_3, z_0 + l_3) & l_4 &= h\, \phi(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= 0.2\,(1 + 1.8578) & &= 0.2\,[(1 + 0.2) + (2 + 0.3543)^2\,(1 + 1.8578)] \\ &= 0.5716 & &= 3.4080 \end{aligned}$$

$$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + l_4] = \frac{1}{6}\,[0.2 + 2\,(0.3) + 2\,(0.3543) + 0.5716] = 0.3467$$

$$l = \frac{1}{6}\,[l_1 + 2l_2 + 2l_3 + l_4] = \frac{1}{6}\,[1.0 + 2\,(1.543) + 2\,(1.8578) + 3.4080] = 1.8683$$

Hence, $$y]_{x = 1.2} = y_0 + k = 2 + 0.3467 = 2.3467$$

$$z]_{x = 1.2} = \left.\frac{dy}{dx}\right]_{x = 1.2} = z_0 + l = 1 + 1.8683 = 2.8683$$

10.6 PREDICTOR-CORRECTOR METHODS

We have so far discussed the methods to solve a differential equation over an interval, say from $x = x_n$ to $x = x_{n+1}$, which require function value only at the beginning of the interval, i.e. at $x = x_n$. Predictor-Corrector methods are *multistep methods* for the solution of initial value problems which require function values at $x_n, x_{n-1}, x_{n-2}, \ldots$ for the computation of the function value at x_{n+1}. We use the explicit predictor formula for predicting a value y_{n+1} and then use implicit corrector formula to improve the value of y_{n+1}, iteratively until the convergence is obtained.

10.6.1 Milne's Method

Consider an initial value problem

$$\frac{dy}{dx} = f(x, y), \quad x = x_0, \; y = y \qquad \ldots(1)$$

To find an approximate value of y_{n+1} (approximation of the solution of (1) at x_{n+1}) using Newton-Gregory forward difference interpolation formula, **Milne's Predictor Formula** can be obtained as

$$y_4^P = y_0 + \frac{4\,h}{3}\,[2f_1 - f_2 + 2f_3] \qquad \ldots(2)$$

and **Corrector Formula** is given by

$$y_4^C = y_2 + \frac{h}{3}\,[f_2 + 4f_3 + f_4] \qquad \ldots(3)$$

here, $f_1 = f(x_1, y_1)$, $y_2 = f(x_2, y_2)$ etc.

General Forms of Milne's formulae are

Predictor Formula : $$y_{n+1}^P = y_{n-3} + \frac{4h}{3}\,[2\,f_{n-2} - f_{n-1} + 2f_n] \qquad \ldots(4)$$

Corrector Formula : $$y_{n+1}^C = y_{n-1} + \frac{h}{3}\,[2\,f_{n-1} - 4f_n + f_{n+1}] \qquad \ldots(5)$$

The superscripts P and C are used to indicate predicted and corrected values.

For solving differential equation (1) using Milne's predictor-corrector method, it is necessary to obtain y_1, y_2, y_3 by using Taylor's series or other methods discussed earlier.

Knowing (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3), we obtain f_1, f_2, f_3 and use formula (2) to predict y_4. f_4 is then calculated as we know x_4 and y_4.

Formula (3) now corrects the value of y_4 which was predicted by formula (2). f_4 is now recalculated from connected y_4 and corrector formula is again applied to obtain better approximation of y_4. Successive approximations usually converge into two or three steps. Once the value of y_4 is fixed, formulae (2) and (3) are again employed to calculate $y_5, y_6, \ldots, y_{n+1}$.

Remark : We note that single step methods are "self starting" (need no starting data beyond the given initial condition), whereas multi-step methods require four starting values y_0, y_1, y_2, y_3 by means of Taylor's series, or Euler's method or Runge-Kutta method.

ILLUSTRATION

Ex. 1 : *Solution of the equation* $5x \dfrac{dy}{dx} + y^2 - 2 = 0$ *is tabulated as*

x	4	4.1	4.2	4.3
y	1.0	1.0049	1.0097	1.0143

Use Milne's predictor-corrector method to find y at x = 4.4 and 4.5.

Sol. : Given differential equation is

$$\frac{dy}{dx} = f(x, y) = \frac{2 - y^2}{5x}$$

We first determine starting values of Milne's method from the table as

$$x_0 = 4, \quad y_0 = 1.0 \quad \therefore f_0 = f(x_0, y_0) = 0.05$$
$$x_1 = 4.1, \quad y_1 = 1.0049 \quad \therefore f_1 = f(x_1, y_1) = 0.0483$$
$$x_2 = 4.2, \quad y_2 = 1.0097 \quad \therefore f_2 = f(x_2, y_2) = 0.0467$$
$$x_3 = 4.3, \quad y_3 = 1.0143 \quad \therefore f_3 = f(x_3, y_3) = 0.0452$$

Using predictor formula,

$$y_4^P = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$
$$= 1 + \frac{4\,(0.1)}{3}[2\,(0.0483) - 0.0467 + 2\,(0.0452)]$$
$$= 1.01871$$

This predicted value is used to calculate f_4 as

$x_4 = 4.4$, $y_4 = 1.01871$ $\quad\therefore\ f_4 = f(x_4, y_4) = 0.04374$

Using corrector formula,

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$
$$= 1.0097 + \frac{0.1}{3}[0.0467 + 4\,(0.0452) + 0.04374]$$
$$= 1.01871$$

Since the predicted value and corrected value is same, we terminate the procedure and accept

$$y_4 = 1.01871$$

To find y_5, we use predictor formula

$$y_5^P = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4]$$
$$= 1.0049 + \frac{4\,(0.1)}{3}[2\,(0.0467) - 0.0452 + 2\,(0.04374)$$
$$= 1.023$$

This predicted value is used to calculate f_5 as

$x_5 = 4.5$, $y_5 = 1.023$ $\therefore f_5 = f(x_5, y_5) = 0.04238$

By using corrector formula

$$y_5^C = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$$

$$= 1.0143 + \frac{0.1}{3}[0.0452 + 4(0.04374) + 0.04238]$$

$$= 1.02305$$

If we want to modify this value

$$f_5 = f(x_5, y_5^C) = \frac{2-(1.02305)^2}{5(4.5)} = 0.04238$$

Hence $y_5 = 1.02305$ as f_5 is same as before, thus the complete table is

x	4	4.1	4.2	4.3	4.4	4.5
y	1.0	1.0049	1.0097	1.0143	1.01871	1.02305

10.6.2 Adams-Moulton Method

We consider an initial value problem

$$\frac{dy}{dx} = f(x, y), \quad x = x_0, \; y = y_0 \qquad \text{... (1)}$$

To find an approximate value of y_{n+1} (approximation to the solution of (1) at x_{n+1}) using Newton-Gregory backward difference interpolation formula, **Predictor Formula** can be obtained as

$$y_4^P = y_3 + \frac{h}{24}[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0] \qquad \text{... (2)}$$

Here $f_0 = f(x_0, y_0)$, $f_1 = f(x_1, y_1)$, $f_2 = f(x_2, y_2)$ and so on.

Formula (2) is called **Adams-Bashforth Predictor Formula**. And the corrector formula is given by

$$y_4^C = y_3 + \frac{h}{24}[9 f_4^P + 19 f_3 - 5 f_2 + f_1] \qquad \text{... (3)}$$

where, $f_4^P = f(x_4, y_4^P)$

Formula (3) is called **Adams-Moulton Corrector Formula**.

Predictor Formula : $$y_{n+1}^P = y_n + \frac{h}{24}[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}] \qquad \text{... (4)}$$

Corrector Formula : $$y_{n+1}^C = y_n + \frac{h}{24}[9 f_{n+1}^P + 19 f_n - 5 f_{n-1} + f_{n-2}] \qquad \text{... (5)}$$

where, $f_{n+1}^P = f\left(x_{n+1}, y_{n+1}^P\right)$

This predictor-corrector formulae (4), (5) is often called the **Adams-Moulton Method.**

In predictor formula (2) for computing y_4, we need f_0, f_1, f_2, f_3, thus, we must first compute y_1, y_2, y_3 by some other method, for instance, by Taylor's series, or a Euler's method, or a Runge-Kutta method. f_4 and f_4^P are then calculated as we know x_4 and y_4.

Formula (3) now corrects the value of y_4. Corrector formula is again applied to obtain better approximation of y_4. Formulae (2) and (3) are again employed to calculate y_5, y_6, ... y_{n+1}.

ILLUSTRATION

Ex. 1 : *Solution of the equation* $\frac{dy}{dx} = 1 + y^2$ *is tabulated as*

x	*0*	*0.2*	*0.4*	*0.6*
y	*0*	*0.2027*	*0.4228*	*0.6841*

Use Adams-Moulton method to find y at x = 0.8 taking h = 0.2.

Sol. : Given differential equation is

$$\frac{dy}{dx} = 1 + y^2$$

Starting values are given in the table as

$x_0 = 0,$	$y_0 = 0$	$\therefore\ f_0 = f(x_0, y_0) = 1$
$x_1 = 0.2,$	$y_1 = 0.2027$	$\therefore\ f_1 = f(x_1, y_1) = 1.0411$
$x_2 = 0.4,$	$y_2 = 0.4228$	$\therefore\ f_2 = f(x_2, y_2) = 1.1787$
$x_3 = 0.6$	$y_3 = 0.6841$	$\therefore\ f_3 = f(x_3, y_3) = 1.4861$

Using predictor formula,

$$\begin{aligned} y_4^P &= y_3 + \frac{h}{24}\,[55\,f_3 - 59\,f_2 + 37\,f_1 - 9\,f_0] \\ &= 0.6841 + \frac{0.2}{24}\,[55\,(1.4681) - 59\,(1.1787) + 37\,(1.0411) - 9\,(1)] \\ &= 1.0233 \end{aligned}$$

This predicted value is used to calculate f_4^P as

$x_4 = 0.8, \quad y_4^P = 1.0233 \quad \therefore\ f_4^P = f(x_4, y_4^P) = 2.047465$

To improve y_4, we use corrector formula

$$\begin{aligned} y_4^C &= y_3 + \frac{h}{24}\,[9\,f_4^P + 19\,f_3 - 5\,f_2 + f_1] \\ &= 0.6841 + \frac{0.2}{24}\,[9\,(2.047465 + 19\,(1.4681) - 5\,(1.1787) + 1.0411] \\ &= 1.0296 \end{aligned}$$

The corrected value is correct upto two decimal places.

Hence at $x_4 = 0.8$, $y = 1.0296$.

MISCELLANEOUS ILLUSTRATION

Ex. 1 : *Solve* $\dfrac{dy}{dx} = x - y^2$ *by Taylor's series method to calculate y at x = 0.4 in two steps. Initial values are x = 0, y = 1.*

Sol. : Given differential equation is

$$y' = x - y^2, \quad x_0 = 0, \quad y_1 = 1$$

(a) To find y at x = 0.2, we use $x_0 = 0$, $y_0 = 1$ and $h = 0.2$.

The Taylor's series for y is given by

$$y = y_0 + h\,y_0' + \frac{h^2}{2!}\,y_0'' + \frac{h^3}{3!}\,y_0''' + \frac{h^4}{4!}\,y_0^{iv} + \ldots\ldots$$

The derivatives y_0', y_0'', ... etc. are obtained thus

$y' = x - y^2,$	$y_0' = 0 - 1 = -1$
$y'' = 1 - 2yy'$	$y_0'' = 1 - 2\,(1)\,(-1) = 3$
$y''' = -2yy'' - 2\,(y')^2$	$y_0''' = -2\,(1)\,(3) - 2\,(-1)^2 = -8$
$y^{iv} = -2yy''' - 6y'y''$	$y_0^{iv} = -2\,(1)\,(-8) - 6\,(-1)\,(3) = 34$

Using these values, the Taylor's series becomes

$$\begin{aligned} y]_{x\,=\,0.2} &= 1 + (0.2)\,(-1) + \frac{(0.2)^2}{2}\,(3) + \frac{(0.2)^3}{6}\,(-8) + \frac{(0.2)^4}{24}\,(34) + \ldots \\ &= 1 - 0.2 + 0.06 - 0.01067 + 0.002267 \\ &= 0.8516 \end{aligned}$$

(b) To find y at x = 0.4, we use $x_0 = 0.2$, $y_0 = 0.8516$ and $h = 0.2$.

$$y_0' = (0.2) - (0.8516)^2 = -0.525$$

$$y_0'' = 1 - 2\,(0.8516)\,(-0.525) = 1.8942$$

$$y_0''' = -2(-0.525)(1.8942) - 2(-0.525)^2 = -3.78$$

$$y_0^{iv} = -2(0.8516)(-3.78) - 6(-0.525)(1.8942) = 12.405$$

Using these values, the Taylor's series becomes

$$\begin{aligned} y]_{x=0.4} &= 0.8516 + (0.2)(-0.525) + \frac{(0.2)^2}{2}(1.8942) + \frac{(0.2)^3}{6}(-3.78) + \frac{(0.2)^4}{24}(12.405) + \ldots \\ &= 0.8516 - 0.105 + 0.0379 - 0.00504 + 0.000827 \\ &= 0.7803 \end{aligned}$$

Ex. 2 : *Obtain by Taylor's series starting values for the numerical solution of*

$$\frac{dy}{dx} = 2y + 3e^x$$

at $x = 0.3$ with $h = 0.1$ and $x_0 = 0$, $y_0 = 1$. Compare the result with analytical solutions.

Sol. : Given differential equation is

$$y' = 2y + 3e^x$$

(a) To find y at $x = 0.1$, we use $x_0 = 0$, $y_0 = 1$ and $h = 0.1$. The derivatives y_0, y_0', y_0'', … etc. are thus obtained as

$$\begin{aligned} y' &= 2y + 3e^x & y_0' &= 2(1) + 3(1) = 5 \\ y'' &= 2y' + 3e^x & y_0'' &= 2(5) + 3 = 13 \\ y''' &= 2y'' + 3e^x & y_0''' &= 2(13) + 3 = 29 \\ y^{iv} &= 2y''' + 3e^x & y_0^{iv} &= 2(29) + 3 = 61 \end{aligned}$$

Thus the Taylor's series for the differential equation is

$$\begin{aligned} y &= y_0 + h\,y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots \\ &= 1 + (0.1)(5) + \frac{(0.1)^2}{2}(13) + \frac{(0.1)^3}{6}(29) + \frac{(0.1)^4}{24}(61) + \ldots \end{aligned}$$

Thus, $$y]_{x=0.1} = 1.5701$$

(b) To find y at $x = 0.2$, we use $x_0 = 0.1$, $y_0 = 1.5701$, $h = 0.1$ and obtain

$$\begin{aligned} y_0' &= 2(1.5701) + 3.3155 = 6.4557 \\ y_0'' &= 2(6.4557) + 3.3155 = 16.2270 \\ y_0''' &= 2(16.2270) + 3.3155 = 35.7695 \\ y_0^{iv} &= 2(35.7695) + 3.3155 = 74.8545 \end{aligned}$$

Thus by Taylor's series, we have

$$\begin{aligned} y]_{x=0.2} &= 1.5701 + (0.1)(6.4557) + \frac{(0.1)^2}{2!}(16.2270) + \frac{(0.1)^3}{6}(35.7695) + \frac{(0.1)^4}{24}(74.8545) \\ &= 2.3025 \end{aligned}$$

(c) To find y at $x = 0.3$, we use $x_0 = 0.2$, $y_0 = 2.3025$, $h = 0.01$ and obtain

$$\begin{aligned} y_0' &= 2(2.3025) + 3.6642 = 8.2692 \\ y_0'' &= 2(8.2692) + 3.6642 = 20.2026 \\ y_0''' &= 2(20.2026) + 3.6642 = 44.0694 \\ y_0^{iv} &= 2(44.0694) + 3.6642 = 91.8030 \end{aligned}$$

By Taylor's series, we have

$$y]_{x=0.3} = 2.3025 + (0.1)(8.2692) + \frac{(0.1)^2}{2}(20.2026) + \frac{(0.1)^3}{6}(44.0694) + \frac{(0.1)^4}{24}(91.8030)$$

$$= 3.2382$$

Analytical Solution : The solution of the equation for x = 0, y = 1 is

$$y = 4e^{2x} - 3e^{x}$$

at x = 0.3

$$y]_{x=0.3} = 3.2389$$

Comparing numerical solution with analytical solution, we conclude that numerical solution $y]_{x=0.3} = 3.238$ is corrected upto three decimal places.

Ex. 3 : *Use modified Euler's method to solve* $\frac{dy}{dx} = x - y^2$*, y (0) = 1 to calculate y (0.4) taking h = 0.2.* ***(Dec. 2011)***

Sol. : Here $x_0 = 0$, $y_0 = 1$ and h = 0.2

$f(x, y) = x - y^2$, $f(x_0, y_0) = 0 - (1)^2 = -1$

(a) To find y_1 (i.e. the value of y at x = 0.2), we first use Euler's formula

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0.2\,[0 - (1)^2] = 0.8$$

Applying modified Euler's formula, improved values of y_1 are given by;

Step 1 :

$$f(x_1, y_1) = 0.2 - (0.8)^2 = -0.44$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \qquad [x_0 = 0, y_0 = 1, x_1 = 0.2, y_1 = 0.8]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.44] = 0.8560$$

Step 2 :

$$f(x_1, y_1^{(1)}) = 0.2 - (0.8560)^2 = -0.5327$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5327] = 0.8467$$

Step 3 :

$$f(x_1, y_1^{(2)}) = 0.2 - (0.8467)^2 = -0.5169$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5169] = 0.8483$$

Step 4 :

$$f(x_1, y_1^{(3)}) = 0.2 - (0.8483)^2 = -0.5196$$

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5196] = 0.8480$$

Step 5 :

$$f(x_1, y_1^{(4)}) = 0.2 - (0.8480)^2 = -0.5191$$

$$y_1^{(5)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(4)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5191] = 0.8481$$

Step 6 : $f(x_1, y_1^{(5)}) = 0.2 - (0.8481)^2 = -0.5193$

$$y_1^{(6)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(5)})]$$

$$= 1 + \frac{0.2}{2}\,[-1 - 0.5193] = 0.8481$$

$\because$ $y_1^{(5)} = y_1^{(6)} = 0.8481$

$\therefore$ $y]_{x=0.2} = y_1 = 0.8481$

(b) Here $x_1 = 0.2$, $y_1 = 0.8481$, $h = 0.2$ and $f(x_1, y_1) = 0.2 - (0.8481)^2 = -0.5193$

To find y_2 (i.e. the value of y at x = 0.4), we first use Euler's formula.

$$y_2 = y_1 + h\,f(x_1, y_1) = 0.8481 + 0.2\,[0.2 - (0.8481)^2]$$

$$= 0.7962$$

Applying modified Euler's formula, improved values of y_2 are

Step 1 : $f(x_2, y_2) = 0.4 - (0.7962)^2 = -0.2340$

$$y_2^{(1)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 0.8481 + \frac{0.2}{2}\,[-0.5193 - 0.2340] = 0.7728$$

Step 2 : $f(x_2, y_2^{(1)}) = 0.4 - (0.7728)^2 = -0.1972$

$$y_2^{(2)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 0.8481 + \frac{0.2}{2}\,[-0.5193 - 0.1972] = 0.7765$$

Step 3 : $f(x_2, y_2^{(2)}) = 0.4 - (0.7765)^2 = -0.2030$

$$y_2^{(3)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 0.8481 + \frac{0.2}{2}\,[-0.5193 - 0.2030] = 0.7759$$

Step 4 : $f(x_2, y_2^{(3)}) = 0.4 - (0.7759)^2 = -0.2020$

$$y_2^{(4)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(3)})]$$

$$= 0.8481 + \frac{0.2}{2}\,[-0.5193 - 0.2020] = 0.7760$$

Step 5 : $f(x_2, y_2^{(4)}) = 0.4 - (0.7760)^2 = -0.2022$

$$y_2^{(5)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(4)})]$$

$$= 0.8481 + \frac{0.2}{2}\,[-0.5193 - 0.2022] = 0.7760$$

$\because$ $y_2^{(5)} = y_2^{(4)} = 0.7760$

$\therefore$ $y]_{x=0.4} = y_2 = 0.7760$

Ex. 4 : *Given* $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, *y (1) = 1. Evaluate y (1.3) by modified Euler's method.*

Sol. : Given equation is

$$\frac{dy}{dx} = \frac{1 - xy}{x^2}$$

Here $$f(x, y) = \frac{1 - xy}{x^2}, \; f(x_0, y_0) = 0,$$

$$x_0 = 1, \; y_0 = 1 \text{ and we take } h = 0.1$$

(a) To find y_1 (i.e. the value of y at x = 1.1), we first use Euler's formula

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0.1\,(0) = 1$$

Applying modified Euler's formula, improved values of y_1 are given by;

Step 1 : $$f(x_1, y_1) = \frac{1 - (1.1)\,(1)}{(1.1)^2} = -\,0.0826 \qquad [x_0 = 1, y_0 = 1, x_1 = 1.1, \; y_1 = 1]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0.1}{2}\,[0 - 0.0826] = 0.9959$$

Step 2 : $$f(x_1, y_1^{(1)}) = \frac{1 - (1.1)\,(0.9959)}{(1.1)^2} = -\,0.0789$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.1}{2}\,[0 - 0.0789] = 0.9961$$

Step 3 : $$f(x_1, y_1^{(2)}) = \frac{1 - (1.1)\,(0.9961)}{(1.1)^2} = -\,0.0791$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}\,[0 - 0.0791] = 0.9960$$

Step 4 : $$f(x_1, y_1^{(3)}) = \frac{1 - (1.1)\,(0.9960)}{(1.1)^2} = -\,0.0790$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.1}{2}\,[0 - 0.0790] = 0.9960$$

Final value of $y_1 = 0.9960$.

(b) Here $x_1 = 1.1$, $y_1 = 0.9960$, $f(x_1, y_1) = \frac{1 - (1.1)\,(0.9960)}{(1.1)^2} = -\,0.0790$

To find y_2 (i.e. the value of y at x = 1.2), we first use Euler's formula

$$y_2 = y_1 + h\, f(x_1, y_1) = 0.9960 + (0.1)\,(-\,0.0790) = 0.9881$$

Applying modified Euler's formula, improved values of y_2 are given by;

Step 1 : $$f(x_2, y_2) = \frac{1 - (1.2)\,(0.9881)}{(1.2)^2} = -\,0.1290$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 0.9960 + \frac{0.1}{2}\,[-\,0.0790 - 0.1290] = 0.9856$$

Step 2 : $f(x_2, y_2^{(1)}) = \dfrac{1 - (1.2)\,(0.9856)}{(1.2)^2} = -\,0.1269$

$$y_2^{(2)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 0.9960 + \frac{0.1}{2}\,[-\,0.0790 - 0.1269] = 0.9857$$

Step 3 : $f(x_2, y_2^{(2)}) = \dfrac{1 - (1.2)\,(0.9856)}{(1.2)^2} = -\,0.1269$

$\therefore$ $y_2^{(3)} = 0.9857$

Final value is $y_2 = 0.9857$.

(c) Here $x_2 = 1.2$, $y_2 = 0.9857$, $f(x_2, y_2) = \dfrac{1 - (1.2)\,(0.9857)}{(1.2)^2} = -\,0.1269$

Finally to find y_3 (i.e. the value of y at x = 1.3), we first use Euler's formula

$$y_3 = y_2 + h\,f(x_2, y_2) = 0.9857 + (0.1)\,(-\,0.1269) = 0.9730$$

Applying modified Euler's formula, improved values of y_3 are given by;

Step 1 : $f(x_3, y_3) = \dfrac{1 - (1.3)\,(0.9730)}{(1.3)^2} = -\,0.1567$

$$y_3^{(1)} = y_2 + \frac{h}{2}\,[f(x_2, y_2) + f(x_3, y_3)]$$

$$= 0.9857 + \frac{0.1}{2}\,[-\,0.1269 - 0.1567] = 0.9715$$

Step 2 : $f(x_3, y_3^{(1)}) = \dfrac{1 - (1.3)\,(0.9715)}{(1.3)^2} = -\,0.1556$

$$y_3^{(2)} = y_2 + \frac{h}{2}\,[f(x_2, y_2) + f(x_3, y_3^{(1)})]$$

$$= 0.9857 + \frac{0.1}{2}\,[-\,0.1269 - 0.1556] = 0.9716$$

Step 3 : $f(x_3, y_3^{(2)}) = \dfrac{1 - (1.3)\,(0.9716)}{(1.3)^3} = -\,0.1557$

$$y_3^{(3)} = y_2 + \frac{h}{2}\,[f(x_2, y_2) + f(x_3, y_3^{(2)})]$$

$$= 0.9857 + \frac{0.1}{2}\,[-\,0.1269 - 0.1557] = 0.9716$$

Final value of $y_3 = 0.9716$

Thus, $y]_{x\,=\,1.3} = 0.9716$

Ex. 5 : *Use Euler's modified method to find the value of y satisfying the equation*

$$\frac{dy}{dx} = \log\,(x + y), \qquad y\,(1) = 2$$

for x = 1.2 and x = 1.4 correct to three decimal places by taking h = 0.2.

Sol. : (a) For y at x = 1.2, we take $x_0 = 1$, $y_0 = 2$, $h = 0.2$ and $f(x_0, y_0) = \log_e\,(1 + 2) = 1.0986$.

By Euler's formula, $y_1 = y_0 + h\,f(x_0, y_0) = 2 + (0.2)\,\log_e\,(1 + 2) = 2.2197$

Applying modified Euler's formula, improved values of y are given by

Step 1 : $f(x_1, y_1) = \log\,(x_1 + y_1) = \log_e (1.2 + 2.2197) = 1.2296$

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 2 + \frac{0.2}{2}\,[1.0986 + 1.2296] = 2.2328$$

Step 2 : $f(x_1, y_1^{(1)}) = \log(x_1 + y_1^{(1)}) = \log_e(1.2 + 2.2328) = 1.2334$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 2 + \frac{0.2}{2}[1.0986 + 1.2334] = 2.2332$$

Step 3 : $f(x_1, y_1^{(2)}) = \log(x_1 + y_1^{(2)}) = \log_e(1.2 + 2.2332) = 1.2335$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 2 + \frac{0.2}{2}[1.0986 + 1.2335] = 2.2332$$

Hence there is no change in the value of y_1 in second and third iterations, the correct value of y at x = 1.2 is

$$y_1 = 2.2332$$

(b) To find y at x = 1.4, we take for h = 0.2,

$$x_1 = 1.2, \quad y_1 = 2.2332 \quad \text{and} \quad f(x_1, y_1) = \log(1.2 + 2.2332) = 1.2335$$

By using Euler's formula,

$$y_2 = y_1 + h\,f(x_1, y_1) = 2.2332 + (0.2)(1.2335) = 2.4799$$

Now to get improved value of y_2, we use modified Euler's formula,

Step 1 : $f(x_2, y_2) = \log_e(1.4 + 2.4799) = 1.3558$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + 1.3558] = 2.4921$$

Step 2 : $f(x_2, y_1^{(1)}) = \log_e(1.4 + 2.4921) = 1.3589$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + 1.3589] = 2.492$$

Thus, as there is no change in third decimal in results $y_2^{(1)}$ and $y_2^{(2)}$, the solution correct upto three decimal place is

$$y]_{x=1.4} = y_2 = 2.492$$

Ex. 6 : *Use Runge-Kutta method of second order to solve*

$$\frac{dy}{dx} = \frac{1}{x+y}, \qquad x_0 = 0, \quad y_0 = 1$$

to find y at x = 0.4 taking h = 0.2.

Sol. : Here $f(x, y) = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$, h = 0.2 and carry out the calculations in two steps.

Step 1 : $k_1 = h\,f(x_0, y_0) = 0.2\left(\frac{1}{0+1}\right) = 0.2$

$$k_2 = h\,f(x_0 + h, y_0 + k_1) = 0.2\left(\frac{1}{0.2 + 1.2}\right) = 0.14286$$

and $k = \frac{1}{2}[k_1 + k_2] = \frac{1}{2}[0.2 + 0.14286] = 0.17143$

$\therefore$ $y]_{x=0.2} = y_0 + k = 1 + 0.17143 = 1.17143$

Step 2 : For calculating y at x = 0.4, we take

$$x_0 = 0.2, \quad y_0 = 1.17143 \text{ and } h = 0.2$$

$$k_1 = h\, f(x_0, y_0) = 0.2 \left(\frac{1}{0.2 + 1.17143}\right) = 0.14583$$

$$k_2 = h\, f(x_0 + h, y_0 + k_1) = 0.2 \left(\frac{1}{0.4 + 1.31726}\right) = 0.11646$$

and
$$k = \frac{1}{2}[k_1 + k_2] = \frac{1}{2}[0.14583 + 0.11646] = 0.131145$$

$$\therefore \quad y]_{x=0.4} = y_0 + k = 1.17143 + 0.131145 = 1.302575$$

Ex. 7 : *Use Runge-Kutta method of fourth order to solve*

$$\frac{dy}{dx} = \frac{1}{x+y} \qquad x_0 = 0, \; y_0 = 1$$

to find y at x = 0.4 taking h = 0.2.

Sol. : Here $f(x, y) = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$ and carry out the calculations in two steps.

Step 1 :
$$k_1 = h\, f(x_0, y_0) = 0.2\left(\frac{1}{0 + 1}\right) = 0.2$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left(\frac{1}{0.1 + 1.1}\right) = 0.167$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\left(\frac{1}{0.1 + 1.0835}\right) = 0.169$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.2\left(\frac{1}{0.2 + 1.169}\right) = 0.1461$$

and
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 2\,(0.167) + 2\,(0.169) + 0.1461)$$

$$= 0.1697$$

$$\therefore \quad y]_{x=0.2} = y_0 + k = 1 + 0.1697 = 1.1697$$

Step 2 : For calculating y at x = 0.4, we take $x_0 = 0.2$

$$y_0 = 1.1697 \quad \text{and} \quad h = 0.2$$

$$k_1 = h\, f(x_0, y_0) = 0.2\left(\frac{1}{0.2 + 1.1697}\right) = 0.146$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left(\frac{1}{0.3 + 1.2497}\right) = 0.1296$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\left(\frac{1}{0.3 + 1.2345}\right) = 0.1303$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.2\left(\frac{1}{0.4 + 1.3}\right) = 0.1176$$

and
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.146 + 2\,(0.1296) + 2\,(0.1303) + 0.1176]$$

$$= 0.13057$$

$$\therefore \quad y]_{x=0.4} = y_0 + k = 1.1697 + 0.13057 = 1.30027$$

Ex. 8 : *Use Runge-Kutta method of fourth order to obtain the numerical solutions of*

$$\frac{dy}{dx} = x^2 + y^2, \; y(1) = 1.5$$

in the interval (1, 1.2) with h = 0.1.

Sol. : (a) For y at x = 1.1, $x_0 = 1$, $y_0 = 1.5$, $h = 0.1$ and $f(x, y) = x^2 + y^2$

$$f(x_0, y_0) = x_0^2 + y_0^2 = (1)^2 + (1.5)^2 = 3.25$$

$$k_1 = h\,f(x_0, y_0) = 0.1\,(3.25) = 0.325$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,[(1 + 0.05)^2 + (1.5 + 0.1625))]$$

$$= 0.38664$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\,[(1 + 0.05)^2 + (1.5 + 0.1933)^2]$$

$$= 0.39698$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.1\,[(1 + 0.1)^2 + (1.5 + 0.39698)^2]$$

$$= 0.48086$$

and
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.325 + 2\,(0.38664) + 2\,(0.39698) + 0.48086]$$

$$= 0.3955$$

$$y]_{x = 1.1} = y_0 + k = 1.5 + 0.3955 = 1.8955$$

(b) For y at x = 1.2, $x_0 = 1.1$, $y_0 = 1.8955$, $h = 0.1$, and $f(x, y) = x^2 + y^2$

$$f(x_0, y_0) = (1.1)^2 + (1.8955)^2 = 4.8029$$

$$k_1 = h\,f(x_0, y_0) = 0.1\,(4.8029) = 0.48029$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\,[(1.1 + 0.05)^2 + (1.8955 + 0.24015)^2]$$

$$= 0.58835$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\,[(1.1 + 0.05)^2 + (1.8955 + 0.29418)^2]$$

$$= 0.61172$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

$$= 0.1\,[(1.1 + 0.1)^2 + (1.8955 + 0.61172)^2] = 0.77262$$

and
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.48029 + 2\,(0.58835) + 2\,(0.61172) + 0.77262]$$

$$= 0.60887$$

$$y]_{x = 1.2} = y_0 + k = 1.8955 + 0.60887 = 2.50437$$

Ex. 9 : *Compute y (0.1) and y (0.2) by Runge-Kutta method of 4^{th} order for the differential equation*

$$\frac{dy}{dx} = xy + y^2, \; y(0) = 1.$$

Sol. : (a) For y at x = 0.1, $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $f(x, y) = xy + y^2$

$$f(x_0, y_0) = x_0 y_0 + y_0^2 = (0)(1) + (1)^2 = 1$$

$$k_1 = h\,f(x_0, y_0) = 0.1\,(1) = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,[(0.05)\,(1.05) + (1.05)^2] = 0.1155$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\,[(0.05)\,(1.05775) + (1.05775)^2]$$

$$= 0.1172$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.1\,[(0.1)\,(1.1172) + (1.1172)^2] = 0.1360$$

and $$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.1 + 2\,(0.1155) + 2\,(0.1172) + (0.1360)]$$

$$= 0.1169$$

$\therefore$ $$y]_{x=0.1} = y_0 + k = 1 + 0.1169 = 1.1169$$

(b) For y at $x = 0.2$, $x_0 = 0.1$, $y_0 = 1.1169$, $h = 0.1$, $f(x, h) = xy + y^2$

$$f(x_0, y_0) = x_0\,y_0 + y_0^2 = (0.1)\,(1.1169) + (1.1169)^2 = 1.359$$

$$k_1 = h\,f(x_0, y_0) = 0.1\,(1.359) = 0.1359$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,[(0.15)\,(1.1849) + (1.1849)^2]$$

$$= 0.1582$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\,[(0.15)\,(1.196) + (1.196)^2] = 0.1610$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.1\,[(0.2)\,(1.2779) + (1.2779)^2] = 0.1889$$

and $$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.1359 + 2\,(0.1582) + 2\,(0.1610) + (0.1889)]$$

$$= 0.1605$$

$\therefore$ $$y]_{x=0.2} = y_0 + k = 1.1169 + 0.1605 = 1.2774$$

Ex. 10 : *Using fourth order Runge-Kutta method, evaluate the value of y when x = 1.1 given that*

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \quad y\,(1) = 1$$

Sol. : For y at $x = 1.1$, $x_0 = 1$, $y_0 = 1$, $h = 0.1$, and $f(x, y) = \dfrac{1 - xy}{x^2}$.

$$f(x_0, y_0) = \frac{1 - (1)\,(1)}{(1)^2} = 0$$

$$k_1 = h\,f(x_0, y_0) = (0.1)\,(0) = 0$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\left[\frac{1 - (1.05)\,(1)}{(1.05)^2}\right] = -0.00454$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\left[\frac{1 - (1.05)\,(0.9977)}{(1.05)^2}\right]$$

$$= -0.00432$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = (0.1)\left[\frac{1 - (1.1)\,(0.99568)}{(1.1)^2}\right]$$

$$= -0.00788$$

and $$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0 + 2\,(-0.00454) + 2\,(-0.00432) + (-0.00788)]$$

$$= -0.0042667$$

$\therefore$ $$y]_{x=1.1} = 1 - 0.0042667 = 0.9957$$

Ex. 11 : *A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. If initially the current in the circuit is zero, find the current in the circuit at t = 0.001, 0.01 using*

(i) Runge-Kutta fourth order method, (ii) Euler's method, (iii) Analytical method.

Compare the approximate value with exact values.

Sol. : Voltage drops across inductance L and resistance R are respectively $L\frac{dI}{dt}$ and RI. Sum of these voltage drops = Applied e.m.f. This gives the differential equation of the circuit as

$$L\frac{dI}{dt} + RI = E, \qquad I = 0 \text{ at } t = 0$$

(i) Analytical Method :

$$L\frac{dI}{dt} + RI = E$$

This is a linear differential equation with integrating factor I.F. = $e^{(R/L)\,t}$. Multiplying the equation by I.F., it can be written as

$$\frac{d}{dt}\left(I\, e^{(R/L)\,t}\right) = \frac{E}{L}\, e^{(R/L)\,t}$$

Integrating, $$I\, e^{(R/L)\,t} = \frac{E}{L} \times \frac{L}{R}\, e^{(R/L)\,t} + C$$

Or $$I = \frac{E}{R} + C\, e^{-(R/L)\,t}$$

at t = 0, I = 0 gives $C = -\frac{E}{R}$

$$\therefore \qquad I = \frac{E}{R}\left(1 - e^{-(R/L)\,t}\right)$$

Putting E = 20, R = 100, L = 0.5

$$I = \frac{1}{5}\left(1 - e^{-200\,t}\right)$$

at t = 0.001, I = 0.036254

at t = 0.01, I = 0.172933

(ii) Euler's Method :

$$\frac{dI}{dt} = \frac{E}{L} - \frac{R}{L} I, \qquad I_0 = 0, \quad t_0 = 0$$

Taking h = 0.001 (increment in t) and $f(I \cdot t) = \frac{E}{L} - \frac{R}{L} I = (40 - 200\, I)$

$$I_1 = I_0 + h\, f(I_0, t_0) = 0 + (0.001)\,[40 - 200\,(0)] = 0.04$$

Thus I = 0.04 at t = 0.001

Taking h = 0.01 (increment in t)

$$I_2 = I_0 + h\, f(I_0, t_0) = 0 + 0.01\,[40 - 200\,(0)] = 0.4$$

Comparison with analytical solution shows that at t = 0.01, Euler's solution = 0.4, while analytical solution = 0.172933. Thus error is substantial. At t = 0.001, Euler's solution = 0.04, while analytical solution = 0.036254. Thus error is much reduced with reduction in step size.

(iii) Runge-Kutta Method of Fourth Order :

$$\frac{dI}{dt} = \frac{E}{L} - \frac{R}{L} I = (40 - 200\, I), \quad I_0 = 0, \quad t_0 = 0$$

Step I : Taking h = 0.001 (increment in t) and f(I, t) = (40 – 200 I)

$$k_1 = h\, f(I_0, t_0) = (0.001)\,[40 - 200\,(0)] = 0.04$$

$$k_2 = h\, f\left(I_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}\right) = (0.001)\,[40 - 200\,(0 + 0.02)] = 0.036$$

$$k_3 = h\,f\left(I_0 + \frac{k_2}{2}, t_0 + \frac{h}{2}\right)$$

$$= 0.001\,[40 - 200\,(0 + 0.018)] = 0.0364$$

$$k_4 = h\,f\,(I_0 + k_3, t_0 + h) = 0.001\,[40 - 200\,(0 + 0.0364)] = 0.03272$$

and $$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.04 + 2\,(0.036) + 2\,(0.0364) + 0.03272]$$

$$= 0.036254$$

$\therefore$ $$I]_{t = 0.001} = I_0 + k = 0 + 0.036254 = 0.036254$$

which is the same as analytical solution. Thus Runge-Kutta method of fourth order with step-size h = 0.001 produces accurate result upto fifth decimal place.

Step II : Taking $h = 0.01$ (increment in t), $I_0 = 0$, $t_0 = 0$

$$k_1 = h\,f\,(I_0, t_0) = 0.01\,[40 - 200\,(0)] = 0.4$$

$$k_2 = h\,f\left(I_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}\right) = 0.01\,[40 - 200\,(0 + 0.2)] = 0$$

$$k_3 = h\,f\left(I_0 + \frac{k_2}{2}, t_0 + \frac{h}{2}\right) = 0.01\,[40 - 200\,(0 + 0)] = 0.4$$

$$k_4 = h\,f\,(I_0 + k_3, t_0 + h) = 0.01\,[40 - 200\,(0 + 0.4)] = -0.4$$

and $$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.4 + 2\,(0) + 2\,(0.4) - 0.4]$$

$$= 0.134$$

$\therefore$ $$I]_{t = 0.01} = I_0 + k = 0 + 0.134 = 0.134$$

Analytical solution at t = 0.01 is 0.172933. Thus with increment in step size there is significant rise in the error.

Ex. 12 : *If $\frac{dx}{dt} = t - x$ and $x = 3$ for $t = 0$, tabulate values of t and x for t = 0 to 3 with increment in t as 1 using Runge-Kutta fourth order method.*

Sol. : Given, $$\frac{dx}{dt} = t - x \qquad \therefore\ f(t, x) = -t - x$$

Also, $t_0 = 0$, $x_0 = 3$

and $h = 1$

$$k_1 = hf(t_0, x_0) = 1(0 - 3) = -3$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right) = 1\left[\left(0 + \frac{1}{2}\right) - \left(3 - \frac{3}{2}\right)\right] = -1$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right) = 1\left[\frac{1}{2} - \left(3 - \frac{1}{2}\right)\right] = -2$$

$$k_4 = hf(t_0 + h, x_0 + k_3) = 1\,[1 - (3 - 2)] = 0$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{-3 - 2 - 4 + 0}{6} = -1.5$$

At t = 1, $$x = 3 - 1.5 = 1.5$$

Now, $t_0 = 1$, $x_0 = 1.5$

$$k_1 = 1\,(1 - 1.5) = -0.5$$

$$k_2 = 1\left[1.5 - \left(1.5 - \frac{0.2}{2}\right)\right] = 0.25$$

$$k_3 = 1\left[1.5 - \left(1.5 + \frac{0.25}{2}\right)\right] = -0.125$$

$$k_4 = 1\ [2 - (1.5 - 0.125)] = 0.625$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{-0.5 + 2 \times 0.25 - 2 \times 0.125 + 0.625}{6} = 0.0625$$

At $t = 2$, $\quad x = 1.5 + 0.0625 = 1.5625$

Now, $\quad t_0 = 2,\ x_0 = 1.5625$

$$k_1 = 1\ (2 - 1.5625) = 0.4375$$

$$k_2 = 1\ [2.5 - (1.5625 + 0.4375/2)] = 0.71875$$

$$k_3 = 1\ [2.5 - (1.5625 + 0.71875/2)] = 0.57125$$

$$k_4 = 1\ [3 - (1.5625 + 0.578125)] = 0.859375$$

$$k = \frac{0.4375 + 2\ (0.71875) + 2\ (0.578125) + 0.859375}{6} = 0.6484375$$

At $t = 3$, $\quad x = 1.5625 + 0.6484375 = 2.2109375$

Ex. 13 : *Solve the following differential equation to get y (0.1).*

$$\frac{dy}{dx} = x + y + xy, \qquad y(0) = 1$$

Use (i) Modified Euler Method with h = 0.05, (ii) Runge-Kutta fourth order with h = 0.1 ***(May 2017)***

Sol. : (i) Modified Euler Method : $h = 0.05,\ x_0 = 0,\ y_0 = 1.$

First Iteration : $\quad x_1 = x_0 + h = 0.05$

$$y_1^{(0)} = y_0 + h\ f(x_0, y_0) = 1.05$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1)\ (0)\} = 1.05381$$

To achieve more accuracy,

$$y_1^{(2)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

i.e. $\quad y\ (0.05) = 1.05391$

Second Iteration : $\quad x_2 = x_1 + h = 0.1$

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1) = 1.05783$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\{f(x_1, y_1) + f(x_0, y_2^{(0)})\}$$

$$= 1.11441$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\{f(x_1, y_1) + f(x_2, y_2^{(1)})\}$$

$$y(0.1) = 1.11597$$

(ii) Runge-Kutta Fourth Order : $h = 0.1,\ x_0 = 0,\ y_0 = 1.$

$$k_1 = f(x_0, y_0) = 1$$

$$k_2 = f\left\{x_0 + \frac{h}{2}, y_0 + \frac{k_1 h}{2}\right\} = f(0.05, 1.05) = 1.1525$$

$$k_3 = f\left\{x_0 + \frac{h}{2}, y_0 + \frac{k_2 h}{2}\right\} = f(0.05, 1.0576) = 1.16048$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = f(0.1, 1.11605) = 1.32765$$

$$\therefore \quad y(0.1) = y_0 + \left\{\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right\} h = 1.11589$$

Ex. 14 : *Using fourth order Runge-Kutta method solve the differential equation* $\frac{dy}{dx} = x^2 + y^2$. *Assuming y(0) = 0 estimate y (0.2) and y (0.4).*

Sol. : Given, $f(x, y) = x^2 + y^2$ and $h = 0.2$

Now, First Iteration,

$$k_1 = f(x_0, y_0) = 0$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1 h}{2}\right) = f(0.1, 0) = 0.01$$

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2 h}{2}\right) = f(0.1, 0.0001) = 0.01$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = f(0.2, 0.002) = 0.04$$

$$\therefore \quad y(0.2) = 0 + \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right) h = 0.002667$$

Second Iteration : $x_1 = 0.2, \; y_1 = 0.002667$

$$k_1 = f(0.2, 0.002667) = 0.04$$

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1 h}{2}\right) = 0.090044$$

$$k_3 = f\left(0.3, 0.002667 + \frac{0.090044 \times 0.2}{2}\right) = 0.090136$$

and $k_4 = 0.160428$

$$\therefore \quad y(0.4) = y_1 + \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right) h = 0.021360224$$

Ex. 15 : *Solve* $\frac{dy}{dx} = \sqrt{x^2 + y}$ *using Runge-Kutta fourth order method to find y at x = 0.4, given y(0.0) = 1.0, take h = 0.2.*

Sol. : Given differential equation,

$$\frac{dy}{dx} = \sqrt{x^2 + y} = f(x, y) \quad x_0 = 0, \quad y_0 = 1.0, \quad h = 0.2$$

$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y_1 = y_0 + k$$

$$\therefore \quad k_1 = hf(k_0, y_0) = 0.2\sqrt{x_0^2 + y_0} = 0.2\sqrt{0^2 + 1} = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\sqrt{(0.1)^2 + (0.6)} = 0.156204$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\sqrt{(0.1)^2 + (1.078102)} = 0.208624$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\sqrt{(0.2)^2 + (1.208624)} = 0.223483$$

$$k = \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4] = \frac{1}{6}[1.153139] = 0.1921898$$

$$\therefore \quad y_1 = y_0 + k = 1 + 0.2112639$$

$$\therefore \quad y_1 = 1.1921898$$

Next, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4, \quad x_1 = 0.2$ and $y_2 = y_1 + k, \quad y_1 = 1.1921898$

$$\therefore \quad k_1 = h\,f(x_1, y_1) = 0.2\sqrt{x_1^2 + y_1} = 0.2\sqrt{0.2^2 + 1.1921898} = -\,0.222008089$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2\sqrt{(0.3)^2 + 1.3231238} = 0.278780408$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2\sqrt{(0.3)^2 + 1.3301388} = 0.284316$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.2\sqrt{(0.4)^2 + 1.4496031} = 0.32730116$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

$$= \frac{1}{6}[0.222008089 + 2(0.278780408 + 0.284316) + 0.32730116] = 0.279250344$$

$\therefore \quad y_2 = y_1 + k = 1.1921898 + 0.279250344$

$y_2 = 1.471440144$

x	0	0.2	0.4
y	1	1.1921898	1.471440144

Ex. 16 : *Numerical solution of the differential equation* $\dfrac{dy}{dx} = x - y^2$ *is tabulated as*

x	*0.0*	*0.2*	*0.4*	*0.6*
y	*0.0*	*0.020*	*0.0795*	*0.1762*

Find y at x = 0.8 and 1.0 by Milne's predictor-corrector method taking h = 0.2.

Sol. : Given differential equation is

$$\frac{dy}{dx} = x - y^2$$

We first determine starting values of Milne's method from table as

$x_0 = 0, \quad y_0 = 0.0, \quad \therefore \quad f_0 = f(x_0, y_0) = 0$

$x_1 = 0.2, \quad y_1 = 0.02, \quad \therefore \quad f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4, \quad y_2 = 0.0795, \quad \therefore \quad f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6, \quad y_3 = 0.1762, \quad \therefore \quad f_3 = f(x_3, y_3) = 0.5689$

Using predictor formula,

$$y_4^P = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 0.0 + \frac{4\,(0.2)}{3}[2\,(0.1996) - 0.3937 + 2(0.5689)]$$

$$= 0.3049$$

This predicted value is used to calculate f_4 as $x_4 = 0.8$, $y_4 = 0.3049$

$$\therefore \quad f_4 = f\,(x_4, y_4) = 0.7070$$

Using corrector formula,

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.0795 + \frac{0.2}{3}[0.3937 + 4\,(0.5689) + (0.7070)]$$

$$= 0.3046$$

Since the predicted and corrected values do not match at fourth decimal place, we recalculate f_4 and modified corrected values by using corrector formula.

$$f_4 = f\,(x_4, y_4^C) = 0.7072$$

Again using corrector formula,

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.0795 + \frac{0.2}{3}[0.3937 + 4\,(0.5689) + 0.7072]$$

$$= 0.3046$$

Since two successive values match, we accept

$$y_4 = 0.3046$$

Next, to find y_5 at $x_5 = 1.0$, we have

$$f_2 = 0.3937, \quad f_3 = 0.5689, \quad f_4 = 0.7072$$

Using predictor formula,

$$y_5^P = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

$$= 0.02 + \frac{4\,(0.2)}{3}[2\,(0.3937) - (0.5689) + 2\,(0.7072)]$$

$$= 0.4554$$

This predicted value is used to calculate f_5 as

$x_5 = 1.0$, $y_5 = 0.4554$, $f_5 = f\,(x_5, y_5) = 0.7926$

Using corrector formula,

$$y_5^C = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$$

$$= 0.1762 + \frac{0.2}{3}[0.5689 + 4\,(0.7072) + 0.7926]$$

$$= 0.4555$$

Since the predicted and corrected values do not match at fourth decimal place, we recalculate f_5 and modified corrected value again using corrector formula

$$f_5\left(x_5, y_5^C\right) = 0.7925$$

$$\therefore \quad y_5^C = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$$

$$= 0.1762 + \frac{0.2}{3}[0.5689 + 4\,(0.7072) + 0.7925]$$

$$= 0.4555$$

Value of $y_5^C = 0.4555$, a value which is same as before, we accept

$$y_5 = 0.4555$$

Ex. 17 : *Numerical solution of the differential equation* $\frac{dy}{dx} = 2 + \sqrt{xy}$ *is tabulated as*

x	*1.0*	*1.2*	*1.4*	*1.6*
y	*1.0*	*1.6*	*2.2771*	*3.0342*

Find y at x = 1.8 by Milne's predictor-corrector method taking h = 0.2.

Sol. : Given differential equation is

$$\frac{dy}{dx} = 2 + \sqrt{xy}$$

We first determine starting values of Milne's method from the table as

$x_0 = 1.0$ $\quad y_0 = 1.0$ $\quad \therefore f_0 = f(x_0, y_0) = 3.0$

$x_1 = 1.2$ $\quad y_1 = 1.6$ $\quad \therefore f_1 = f(x_1, y_1) = 3.3856$

$x_2 = 1.4$ $\quad y_2 = 2.2771$ $\quad \therefore f_2 = f(x_2, y_2) = 3.7855$

$x_3 = 1.6$ $\quad y_3 = 3.0342$ $\quad \therefore f_3 = f(x_3, y_3) = 4.2033$

Using predictor formula,

$$y_4^P = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4\,(0.2)}{3}[2\,(3.3856) - 3.7855 + 2\,(4.2033)]$$

$$= 4.0379$$

This predicted value is used to calculate f_4 as

$x_4 = 1.8,$ $\quad y_4 = 4.0379$ $\quad \therefore f_4 = f(x_1, y_4) = 4.696$

Using corrector formula,

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.2771 + \frac{0.2}{3}[3.7855 + 4\,(4.2033) + 4.696] = 3.9634$$

Since the predicted and corrected values do not match, we recalculate f_4 and modified corrected value by using corrector formula

$$f_4 = (x_4, y_4) = 2 + \sqrt{(1.8)\,(3.9634)} = 4.671$$

Again using corrector formula

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.2771 + \frac{0.2}{3}[3.7855 + 4\,(4.2033) + 4.671] = 3.9617$$

Repeating the procedure

$$f_4 = f(x_4, y_4) = 2 + \sqrt{1.8 \times 3.9617} = 4.67$$

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.2771 + \frac{0.2}{3}[3.7855 + 4\,(4.2033) + 4.67] = 3.9617$$

Since two successive values match, we accept $y_4 = 3.9617$.

Ex. 18 : *Solve numerically* $\frac{dy}{dx} = 2e^x - y$ *at* $x = 0.5$ *by Milne's predictor and corrector method given that* $y(0.1) = 2$.

Sol. : Given differential equation is

$$\frac{dy}{dx} = 2e^x - y, \quad x_0 = 0.1, \quad y_0 = 2$$

To find y(0.5), we first calculate starting values y_1, y_2, y_3 by Taylor's series about $x = x_0 = 0.1$ i.e.

$$y = y_0 + (x - x_0)\,y_0' + \frac{(x - x_0)^2}{2!}\,y_0'' + \frac{(x - x_0)^3}{3!}\,y_0''' + \frac{(x - x_0)^4}{4!}\,y_0^{iv} + \ldots$$

To determine y_0', y_0'', etc., we have

$$y' = 2e^x - y \qquad \therefore \qquad y_0' = 2e^{0.1} - 2 = 0.2103$$

$$y'' = 2e^x - y' \qquad \therefore \qquad y_0'' = 2e^{0.1} - 0.2103 = 2.0000$$

$$y''' = 2e^x - y'' \qquad \therefore \qquad y_0''' = 2e^{0.1} - 2 = 0.2103$$

$$y^{iv} = 2e^x - y''' \qquad \therefore \qquad y^{iv} = 2e^{0.1} - 0.2103 = 2.0000$$

$$y = 2 + (x - 0.1) + (0.2103) + \frac{(x - 0.1)^2}{2}(2.0) + \frac{(x - 0.1)^3}{6}(0.2103) + \frac{(x - 0.1)^4}{24}(2)$$

$$\therefore \quad x_0 = 0.1,\ y_0 = 2 \qquad \therefore \qquad f_0 = f(x_0, y_0) = 2e^{0.1} - 2 = 0.2103$$

$$x_1 = 0.2,\ y_1 = 2.0310 \qquad f_1 = f(x_1, y_1) = 2e^{0.2} - 2.0310 = 0.4118$$

$$x_2 = 0.3,\ y_2 = 2.0825 \qquad f_2 = f(x_2, y_2) = 2e^{0.3} - 2.0825 = 0.6172$$

$$x_3 = 0.4,\ y_3 = 2.1548 \qquad f_3 = f(x_3, y_3) = 2e^{0.4} - 2.1548 = 0.8289$$

Using Milne's predictor formula,

$$y_4^P = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 2 + \frac{4\,(0.1)}{3}[2\,(0.4118) - (0.6172) + (0.8289)] = 1.8642$$

Now we first determine f_4 using predicted value i.e. $f_4 = f(x_4, y_4) = 2e^{0.5} - 1.8642 = 1.4332$

Using Milne's corrector formula,

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.0825 + \frac{0.1}{3}[0.6172 + 4(0.8289) + 1.4332] = 2.2614$$

Since predicted and corrected values do not match, we recalculate f_4 and modified corrected value as

$$f_4 = f(x_4, y_4) = 2e^{0.5} - 2.2614 = 1.0360$$

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.0825 + \frac{0.1}{3}[0.6172 + 4(0.8289) + 1.0360] = 2.2481$$

Repeating the procedure

$$f_4 = f(x_4, y_4) = 2e^{0.5} - 2.2481 = 1.0493$$

$$y_4^C = y_2 + \frac{h}{2}[f_2 + 4f_3 + f_4]$$

$$= 2.0825 + \frac{0.1}{3}[0.6172 + 4(0.8289) + 1.0493]$$

$$= 2.2485$$

Since two successive values match, we accept the solution y (0.5) = 2.2485.

Ex. 19 : *Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2) y^2$ and y (0) = 1, y (0.1) = 1.06, y (0.2) = 1.12, y (0.3) = 1.21. Evaluate y (0.4) by Milne's predictor-corrector method.*

Sol. : Given differential equation is

$$\frac{dy}{dx} = \frac{1}{2}(1 + x^2) y^2$$

Starting values are given as

$$x_0 = 0, \quad y_0 = 1 \quad \therefore f_0 = f(x_0, y_0) = \frac{1}{2}(1 + 0)(1)^2 = 0.5$$

$$x_1 = 0.1, \quad y_1 = 1.06 \quad \therefore f_1 = f(x_1, y_1) = \frac{1}{2}[1 + (0.1)^2](1.06)^2 = 0.5674$$

$$x_2 = 0.2, \quad y_2 = 1.12 \quad \therefore f_2 = f(x_2, y_2) = \frac{1}{2}[1 + (0.2)^2](1.12)^2 = 0.6523$$

$$x_3 = 0.3, \quad y_3 = 1.21 \quad \therefore f_3 = f(x_3, y_3) = \frac{1}{2}[1 + (0.3)^2](1.21)^2 = 0.7979$$

Using Milne's predictor formula,

$$y_4^P = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4(0.1)}{3}[2(0.5674) - 0.6523 + 2(0.7979)] = 1.2771$$

This predicted value is used to calculate f_4 as

$$f_4 = f(x_4, y_4) = \frac{1}{2}[1 + (0.4)^2](1.2771)^2 = 0.9460$$

Using corrector formula,

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.6523 + \frac{0.1}{3}[0.6523 + 4(0.7979) + 0.9460] = 1.2797$$

Recalculate f_4 and modified corrected value by corrector formula,

$$f_4 = f(x_4, y_4) = \frac{1}{2}[1 + (0.4)^2](1.2797)^2 = 0.9498$$

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.6523 + \frac{0.1}{3}[0.6523 + 4(0.7979) + 0.9498] = 1.2798$$

Repeating the procedure

$$f_4 = f(x_4, y_4) = \frac{1}{2}[1 + (0.4)^2](1.2798)^2 = 0.9500$$

$$y_4^C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 0.6523 + \frac{0.1}{3}[0.6523 + 4(0.7979) + 0.9500]$$

$$= 1.2798$$

Since two successive values match, we accept the solution y (0.4) = 1.2798.

Ex. 20 : *Using Adam's Bashforth method, determine y(1.4), given that* $\frac{dy}{dx} = x^2(1 + y)$, $y(1) = 1$. *Obtain the starting values from Euler's method.*

Sol. : Given differential equation is

$$\frac{dy}{dx} = x^2(1 + y)$$

We calculate starting values using Euler's formula,

$x_0 = 1,\quad y_0 = 1 \quad \therefore \quad f_0 = f(x_0, y_0) = 2$

$x_1 = 1.1,\quad y_1 = y_0 + h\,f(x_0, y_0) = 1.2 \quad \therefore \quad f_1 = f(x_1, y_1) = 2.662$

$x_2 = 1.2,\quad y_2 = y_1 + h\,f(x_1, y_1) = 1.4662 \quad \therefore \quad f_2 = f(x_2, y_2) = 3.5513$

$x_3 = 1.3,\quad y_3 = y_2 + h\,f(x_2, y_2) = 1.8213 \quad \therefore \quad f_3 = f(x_3, y_3) = 4.7680$

Using Adam's predictor formula,

$$y_4^P = y_3 + \frac{h}{24}[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0]$$

$$= 1.8213 + \frac{0.1}{24}[55(4.768) - 59(3.5513) + 37(2.662) - 9(2)]$$

$$= 2.3763$$

Predicted value is used to calculate f_4^P (using $x_4 = 1.4$, $y_4^P = 2.3763$) as

$$f_4^P = f(x_4, y_4^P) = (1.4)^2[1 + 2.3763] = 6.6175$$

To improve y_4, we use corrector formula

$$y_4^C = y_3 + \frac{h}{24}[9 f_4^P + 19 f_3 - 5 f_2 + f_1]$$

$$= 1.8213 + \frac{0.1}{24}[9(6.6175) + 19(4.7680) - 5(3.5513) + 2.662]$$

$$= 2.3840$$

Recalculating f_4, modified corrected value of y_4 is obtained as

$$f_4 = f(x_4, y_4^C) = (1.4)^2[1 + 2.3840] = 6.63264$$

$$y_4^C = y_3 + \frac{h}{24}[9 f_4 + 19 f_3 - 5 f_2 + f_1]$$

$$= 1.8213 + \frac{0.1}{24}[9(6.63264) + 19(4.7680) - 5(3.5513) + 2.662]$$

$$= 2.3846$$

Repeating the process,

$$f_4 = f(x_4, y_4^C) = (1.4)^2[1 + 2.3846] = 6.6338$$

$$y_4^C = y_3 + \frac{h}{24}[9 f_4 + 19 f_3 - 5 f_2 + f_1]$$

$$= 1.8213 + \frac{0.1}{24}[9 (6.6338) + 19 (4.7680) - 5 (3.5513) + 2.662]$$

$$= 2.3846$$

Since two successive values match, we accept

$$y_4 = 2.3846$$

Ex. 21 : *Solve the initial value problem* $\frac{dy}{dx} = x - y^2$, *$y(0) = 1$ to find $y(0.4)$ by Adam's method. Use $y(0) = 1$, $y(0.1) = 0.9117$, $y(0.2) = 0.8494$, $y(0.3) = 0.8061$ to obtain starting solutions.*

Sol. : Given differential equation is

$$\frac{dy}{dx} = x - y^2$$

The starting values of Adam's method with $h = 0.1$ are

$$\begin{array}{llll} x_0 = 0.0, & y_0 = 1 & \therefore & f_0 = f(x_0, y_0) = -1.0000 \\ x_1 = 0.1, & y_1 = 0.9117 & \therefore & f_1 = f(x_1, y_1) = -0.7312 \\ x_2 = 0.2, & y_2 = 0.8494 & \therefore & f_2 = f(x_2, y_2) = -0.5215 \\ x_3 = 0.3, & y_3 = 0.8061 & \therefore & f_3 = f(x_3, y_3) = -0.3498 \end{array}$$

Using Adam's predictor formula,

$$y_4^P = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 0.8061 + \frac{0.1}{24}[55 (-0.3498) - 59 (-0.5215) + 37 (-0.7312) - 9 (-1.0000)]$$

$$= 0.7789$$

Predicted value is used to calculate y_4^P (using $x_4 = 0.4$, $y_4^P = 0.7789$).

$$f_P^4 = f(x_4, y_4^P) = -0.2067$$

To improve y_4, we use corrector formula,

$$y_4^C = y_3 + \frac{h}{24}(9f_4^P + 19f_3 - 5f_2 + f_1)$$

$$= 0.8061 + \frac{0.1}{24}[9 (-0.2067) + 19 (-0.3498) - 5 (-0.5215) + (-0.7312)]$$

$$= 0.7785$$

$$y(0.4) = 0.7785 \text{ is correct upto three decimals.}$$

Ex. 22 : *Using Adam's Bashforth method find $y(0.4)$ given that $\frac{dy}{dx} = 1 + xy$, $y(0) = 2$, $y(0.1) = 2.1103$, $y(0.2) = 2.243$, $y(0.3) = 2.4011$.*

Sol. : Given differential equation is

$$\frac{dy}{dx} = 1 + xy$$

We calculate starting values as

$$\begin{array}{llll} x_0 = 0, & y_0 = 2, & \therefore & f_0 = f(x_0, y_0) = 1 \\ x_1 = 0.1, & y_1 = 2.1103 & \therefore & f_1 = f(x_1, y_1) = 1.21103 \\ x_2 = 0.2, & y_2 = 2.243 & \therefore & f_2 = f(x_2, y_2) = 1.4486 \\ x_3 = 0.3, & y_3 = 2.4011 & \therefore & f_3 = f(x_3, y_3) = 1.72033 \end{array}$$

Using Adam's predictor formula,

$$y_4^P = y_3 + \frac{h}{24}[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0]$$

$$= 2.4011 + \frac{0.1}{24}[55 (1.72033) - 59 (1.4486) + 37 (1.21103) - 9]$$

$$= 2.5884$$

Predicted value is used to calculate f_4^P as

$$f_4^P = f(x_4, y_4^P) = 1 + (0.4)(2.5884) = 2.0354$$

To improve y_4, we use corrector formula

$$y_4^C = y_3 + \frac{h}{24}[9 f_4^P + 19 f_3 - 5 f_2 + f_1]$$

$$= 2.4011 + \frac{0.1}{24}[9(2.0354) + 19(1.72033) - 5(1.4486) + 1.21103]$$

$$= 2.5885$$

Since corrected value is same as predicted values to three decimal places, we accept

$$y(0.4) = 2.588$$

Ex. 23 : *Solve $\frac{dy}{dx} = x + y$. Taylor's series method to find y at x = 3.0. Given x = 2, y = 2 take h = 0.5; accuracy upto third term of the Taylor's series.*

Sol. : Given differential equation, $\frac{dy}{dx} = x + y$, $x = 2$, $y = 2$, $h = 0.5$

$$\therefore \quad y' = x + y$$

$$\therefore \quad y'_0 = y_0 + x_0 = 2 + 2 = 4$$

$$\therefore \quad y'_0 = 4$$

Consider, $y' - y - x = 0$

Differentiating, we get

$$y'' - y' - 1 = 0$$

$$y''_0 - y'_0 - 1 = 0$$

$$\therefore \quad y''_0 = y'_0 + 1$$

$$= 4 + 1$$

$$\therefore \quad y''_0 = 5$$

Differentiating again, we get

$$y''' - y'' - 0 = 0$$

$$\therefore \quad y''' = y''$$

$$\therefore \quad y'''_0 = y''_0 = 5$$

Taylor's series, $$y = y_0 = (x - x_0)\, y'_0 + \frac{(x - x_0)^2}{2!} y''_0 + \frac{(x - x_0)^3\, y'''_0}{3!}$$

$$\therefore \quad y = 2 + (x - 2) \times 4 + \frac{(x - 2)^2}{2} \times 5 + \frac{(x - 2)^3 \times 5}{6}$$

$$\therefore \quad y_{x=3} = 2 + (3 - 2) \times 4 + \frac{(3 - 2)^2 \times 5}{2} + \frac{(3 - 2)^3 \times 5}{6}$$

$$\therefore \quad y_{x=3} = 9.3333$$

Ex. 24 : *Solve the equation $\frac{dy}{dx} + xy = 0$, in the range x = 0 to x = 0.25 with step size 0.05 using Runge-Kutta second order method. Assume at x = 0, y = 1.*

Sol. : Given : $$\frac{dy}{dx} = -xy$$

$$x_0 = 0, \quad y_0 = 1$$

$$h = 0.05$$

$$k_1 = h\, f(x_0, y_0) = 0.05\,(0 \times 1) = 0$$

$$k_2 = h\, f(x_0 + h, y_0 + k_1) = 0.05\,(-(0.05) \times (1 + 0)) = -2.5 \times 10^{-3}$$

$$k = \frac{k_1 + k_2}{2} = -1.25 \times 10^{-3}$$

At x = 0.05, $y = 1 - 1.25 \times 10^{-3} = 0.99875$

Now, $x_0 = 0.05, \quad y_0 = 0.99875$

$k_1 = 0.05\,(-0.05 \times 0.99875) = -2.496875 \times 10^{-3}$

$k_2 = 0.05\,[-0.1 \times (0.99875 - 2.496875 \times 10^{-3})] = -4.9812656 \times 10^{-3}$

$$k = \frac{k_1 + k_2}{2} = \frac{-2.496875 \times 10^{-3} - 4.9812656 \times 10^{-3}}{2} = -3.7390703 \times 10^{-3}$$

At x = 0.1, $y = 0.99875 - 3.7390703 \times 10^{-3} = 0.9950109$

Now, $x_0 = 0.1\; y_0 = 0.9950109$

$k_1 = 0.05 \times (-0.9950109) = -4.975054 \times 10^{-3}$

$k_2 = 0.05 \times [-0.15 \times (0.9950109 - 4.975054 \times 10^{-3})] = -7.4252 \times 10^{-3}$

$$k = \frac{-4.975 \times 10^{-3} - 7.4252 \times 10^{-3}}{2} = -6.2 \times 10^{-3}$$

At x = 0.15, $y = 0.9950109 - 6.2 \times 10^{-3} = 0.9888$

Now, $x_0 = 0.15\; y_0 = 0.9888$

$k_1 = 0.05\,(-0.15 \times 0.988) = -7.416 \times 10^{-3}$

$k_2 = 0.05\,(-0.2 \times (0.988 - 7.416 \times 10^{-3}) = -9.8 \times 10^{-3}$

$$k = \frac{-7.416 \times 10^{-3} - 9.8 \times 10^{-3}}{2} = -8.61 \times 10^{-3}$$

At x = 0.2, $y = 0.9888 - 8.61 \times 10^{-3} = 0.9801$

Now, $x_0 = 0.2, \quad y_0 = 0.9801$

$k_1 = 0.05\,(-0.2 \times 0.9801) = -9.8 \times 10^{-3}$

$k_2 = 0.05\,[-0.25 \times (0.9801 - 9.8 \times 10^{-3})] = -0.0121287$

$$k = \frac{-9.8 \times 10^{-3} - 0.121287}{2} = -0.01096$$

At x = 0.25 $y = 0.9801 - 0.01096 = 0.96913$

***

Ex. 25 : *Using Runge-Kutta method of fourth order, find the approximate values of x and y at t = 0.2 for the following system*

$$\frac{dx}{dt} = 2x + y, \quad \frac{dy}{dt} = x - 3y$$

$t = 0, \quad x = 1, \quad y = 0.5, \quad$ taking $h = 0.1$.

Sol. : Given $\frac{dx}{dt} = f(t, x, y) = 2x + y, \quad \frac{dy}{dt} = \phi\,(t, x, y) = x - 3y.$

Initial conditions are $t_0 = 0,\; x_0 = 1,\; y_0 = 0.5$ and $h = 0.1$.

The computing x and y at t = 0.1, we have

$$\begin{aligned} k_1 &= h\,f\,(t_0, x_0, y_0) \\ &= 0.1\,[2\,(1) + 0.5] \\ &= 0.25 \\ k_2 &= h\,f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right) \\ &= (0.1)\,[2\,(1 + 0.125) + (0.5 - 0.025)] \\ &= 0.2725 \\ k_3 &= h\,f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right) \\ &= (0.1)\,[2\,(1 + 0.13625) + (0.5 - 0.015)] \\ &= 0.2758 \\ k_4 &= h\,f\,(t_0 + h, x_0 + k_3, y_0 + l_3) \\ &= (0.1)\,[2\,(1 + 0.2758) + (0.5 - 0.0319)] \\ &= 0.302 \end{aligned}$$

$$\begin{aligned} l_1 &= h\,\phi\,(t_0, x_0, y_0) \\ &= 0.1\,[1 - 3\,(0.5)] \\ &= -0.05 \\ l_2 &= h\,\phi\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right) \\ &= (0.1)\,[(1 + 0.125) - 3\,(0.5 - 0.025)] \\ &= -0.03 \\ l_3 &= h\,\phi\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right) \\ &= (0.1)\,[(1 + 0.13625) - 3\,(0.5 - 0.015)] \\ &= -0.0319 \\ l_4 &= h\,\phi\,(t_0 + h, x_0 + k_3, y_0 + l_3) \\ &= (0.1)\,[(1 + 0.2758) - 3\,(0.5 - 0.0319)] \\ &= -0.0129 \end{aligned}$$

and $k = \frac{1}{6}(k_1 + 2k_1 + 2k_3 + k_4) = \frac{1}{6}[0.25 + 2(0.2725) + 2(0.2758) + 0.302]$

$= 0.2748$

$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= \frac{1}{6}[-0.05 - 2(0.03) - 2(0.0319) - 0.0129] = -0.0311$

Hence, $x]_{t=0.1} = x_0 + k = 1 + 0.2748 = 1.2748$

$y]_{t=0.1} = y_0 + k = 0.5 - 0.0311 = 0.4689$

Replacing t_0, x_0, y_0 by t_1, x_1, y_1 in the above calculations,

$k_1 = 0.3019, \quad k_2 = 0.3314, \quad k_3 = 0.3352, \quad k_4 = 0.3692;$

$l_1 = -0.0132, \quad l_2 = 0.0039, \quad l_3 = 0.0028, \quad l_4 = 0.0195$

$k = 0.33405$ and $l = 0.0033$

$x]_{t=0.2} = x_1 + k = 1.2748 + 0.33405 = 1.6089$

$y]_{t=0.2} = y_1 + l = 0.4689 + 0.0033 = 0.4722$

***Ex. 26 :** Using Runge-Kutta fourth order method, solve* $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ *for x = 0.2 correct to 4 decimal places. Initial conditions are x = 0, y = 1,* $\frac{dy}{dx} = 0$.

Sol. : To solve the above equation, we put $z = \frac{dy}{dx}$. Given equation then becomes

$$\frac{dz}{dx} = xz^2 - y^2$$

Second order differential equation is thus converted into simultaneous system.

$$\frac{dy}{dx} = z = f(x, y, z), \qquad \frac{dz}{dx} = xz^2 - y^2 = \phi(x, y, z)$$

Initial conditions are $x_0 = 0$, $y_0 = 1$, $z_0 = \left(\frac{dy}{dx}\right)_{x=0} = 0$; $h = 0.2$.

$k_1 = h\,f(x_0, y_0, z_0)$
$= 0.2\,(0) = 0$

$l_1 = h\,\phi(x_0, y_0, z_0)$
$= 0.2\,(-1) = -0.2$

$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.2\,(-0.1) = -0.02$

$l_2 = h\,\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.2\,(-0.999) = -0.1998$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.2\,(-0.0999) = -0.02$

$l_3 = h\,\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.2\,(-0.9791) = -0.1958$

$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.2\,(-0.1958)$
$= -0.03916$

$l_4 = h\,\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.2\,(-0.9527) = -0.1905$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$= -0.0199$

$l = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$= -0.1970$

Hence, $y]_{x=0.2} = y_0 + k = 1 - 0.0199 = 0.9801$

and $z]_{x=0.2} = \left.\frac{dy}{dx}\right]_{x=0.2} = z_0 + l = 0 - 0.1970 = -0.1970$

***Ex. 27 :** Solve the differential equation $\frac{d^2y}{dx^2} - 1.5x\frac{dy}{dx} + 4.5y = 4.5$. Using Euler's method, assume at $x = 0$, $\frac{dy}{dx} = -2$ and $y = 1$. Tabulate the results for $x = 0.1, 0.2$ and 0.3.*

Sol. : Given differential equation, $\frac{d^2y}{dx^2} - 1.5x\frac{dy}{dx} + 4.5y = 4.5$

$$\therefore \quad \frac{d^2y}{dx} = 1.5x\frac{dy}{dx} - 4.5y + 4.5$$

$$\text{Let } \frac{dy}{dx} = z \quad \therefore \quad \frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\therefore \quad \frac{dz}{dx} = 1.5xz - 4.5y + 4.5 = f(x, y, z) \quad \ldots \text{(I)}$$

$$\frac{dy}{dx} = z = g(x, y, z) \quad \ldots \text{(II)}$$

$$y_0 = 1, \quad y'(0) = -2, \quad x_0 = 0$$

$$\therefore \quad z_0 = -2, \quad h = 0.1$$

$$z_1 = z_0 + hf(x_0, y_0, z_0) = -2 + 0.1 \times f(0, 1, -2) = -2 + 0.1\,[1.5 \times 0 \times (-2) - 4.5 \times 1 \times 4.5] = -2$$

$$y_1 = y_0 + hg(x_0, y_0, z_0) = 1 + 0.1\,g(0, 1, -2) = 1 + 0.1\,z_0 = 1 + 0.1\,(-2) = 0.8$$

Similarly,

$$z_2 = z_1 + hf(x_1, y_1, z_1) = -2 + 0.1\,f(0.1, 0.8, -2) = -2 + 0.1\,[1.5 \times 0.1 \times -2 - 4.5 \times 0.8 + 4.5] = -1.94$$

$$y_2 = y_1 + hg(x_1, y_1, z_1) = y_1 + hg(0.1, 0.8, -2) = 0.8 + 0.1 \times -2 = 0.6$$

and

$$z_3 = z_2 + hf(x_2, y_2, z_2) = z_2 + hf(0.2, 0.6, -1.94)$$

$$= -1.94 + 0.1 \times [1.5 \times 0.2 \times -1.94 - 4.5 \times 0.6 + 4.5] = -1.8182$$

$$y_3 = y_2 + hg(x_2, y_2, z_2) = 0.6 + 0.1 \times z_2 = 0.6 + 0.1 \times -1.94 = 0.406$$

x	0.1	0.2	0.3
y	0.8	0.6	0.406
$\frac{dy}{dx}$	–2	–1.94	–1.8182

***Ex. 28 :** Solve the equation : $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$.*

Subject to initial conditions $y(0) = 1$, $y'(0) = 0$ using Runge-Kutta fourth order method. Compute value of y at $x = 0.2$.

Sol. :

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$$

Let,

$$\frac{dy}{dx} = z$$

$$\therefore \quad \frac{dz}{dx} = xz^2 - y^2$$

and $y_0 = 1$, $z_0 = 0$ for $x_0 = 0$.

Here,

$$f(x, y, z) = z$$

$$g(x, y, z) = xz^2 - y^2 \text{ take } h = 0.2$$

$$x_1 = x_0 + h = 0.2$$

$$y_1 = y_0 + k \text{ and } z_1 = z_0 + l$$

To find k and l.

$$k_1 = h\,f(x_0, y_0, z_0) = (0.2)\,z_0 = 0$$

$$l_1 = h\,g(x_0, y_0, z_0) = (0.2)\,(x_0, z_0^2 - y_0^2) = -0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = h\,f(0.1, 1 - 0.1) = 0.2\,(-0.1) = -0.02$$

$$l_2 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right] = hg\,(0.1, 1, -0.1) = 0.2\,[(0.1)\,(-0.1)^2 - 1^2] = -0.1998$$

$$k_3 = h\,f\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right] = (0.2)\,f(0.1, 0.99, -0.0999) = (0.2)\,(-0.0999) = -0.01998$$

$$l_3 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$

$$= (0.2)\ g(0.1, 0.99, -0.0999) = (0.2)\ [(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$$

$$k_4 = h\ f\ [x_0 + h, y_0 + k_3, z_0 + l_3] = h\ f\ [0.2, 0.98002, -0.1958] = (0.2)\ (-0.1958) = -0.0392$$

$$l_4 = h\ g\ [x_0 + h, y_0 + k_3, z_0 + l_3]$$

$$= h\ g\ [0.2, 0.98002, -0.1958] = (0.2)\ [0.2\ (-0.1958)^2 - (0.98002)^2] = -0.1906$$

$$k = \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4] = \frac{1}{6}[0 + 2(-0.02 - 0.0198) - 0.0392] = -0.01986$$

$$l = \frac{1}{6}[l_1 + 2(l_2 + l_3) + l_4] = \frac{1}{6}[-0.2 + 2(-0.1998 - 0.1958) - 0.1906] = -0.1970$$

At x = 0.2, $y_1 = y_0 + k = 0.9801$

$$z_1 = z_0 + l = -0.1970$$

At $x = 0.2$, $y = 0.9801$.

Ex. 29 : *A body of mass 2 kg is attached to a spring with a spring constant of 10. The differential equation governing the displacement of the body y and time t is given by,*

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

Find the displacement y at time t = 1.5, given that y(0) = 2 and y'(0) = – 4.

Sol. : Given differential equation, $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$

and $y(0) = 2$

$y'(0) = -4 \quad y = ?$ at $t = 1.5$

Above differential equation is second order, converting it into two first order equations. Solving by RK method.

Let, $\frac{dy}{dt} = p = f_1(t, y, p)$

$$\frac{dp}{dt} = -2p - 5y = f_2(t, y, p)$$

$t_0 = 0$, $y_0 = 2$, $p_0 = y_0 = -4$

Taking h = k = 1.5

By RK method, $k_1 = h\ f_1(t_0, y_0, p_0) = 1.5\ (-4) = -6$

$$l_1 = h\ f_2(t_0, y_0, p_0) = 1.5\ [-2(-4) - 5(2)] = -3$$

$$k_2 = h \cdot f_1\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) = 1.5\ (-5.5) = -8.25$$

$$l_2 = h \cdot f_2\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) = 1.5\ [-2(-5.5) - 5(-1)] = 24$$

$$k_3 = h \cdot f_1\left(y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right)$$

$$l_3 = h \cdot f_2\left(y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right) = 1.5\ [-2(8) - 5(-2.125)] = 39.9375$$

$$k_4 = h \cdot f_1\left(p_0 + \frac{l_3}{2}\right) = 1.5\ (35.9375) = 53.90625$$

$$l_4 = h \cdot f_2(y_0 + k_3, p_0 + l_3) = 1.5\ [-2(35.9375) - 5(14)] = -109.5$$

$\therefore$
$$y(1.5) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 2 + \frac{1}{6}[-6 + 2(-8.25 + 12) + 53.90625]$$

$$y(1.5) = 11.23437$$

and
$$y'(1.5) = p_0 + \frac{1}{6}[l_1 + 2(l_2 + l_3)\ l_4] = -4 + \frac{1}{6}[-3 + 2(24 + 39.9375) - 109.5]$$

$$y'(1.5) = -1.4375$$

Ex. 30 : *The rate of disintegration of a radioactive substance is proportional to the amount of substance (m) remaining at any time (t) is governed by the relation* $\frac{dm}{dt} = -Km$*, where, K is a constant of disintegration and is equal to 0.01. If initial mass of the substance is 100 gm, evaluate the amount of substance remaining at the end of 10 sec. (take increment in time as 2.5 sec.). Use Euler's method.*

Sol. : $m_0 = 100$ gm, $K = 0.01$, $h = 2.5$ sec, $t_0 = 0$ sec.

$$t_1 = t_0 + h = 0 + 2.5 = 2.5 \text{ sec}$$

$$\begin{aligned} m_1 &= m_0 + h \cdot f(t_0, m_0) && \text{… Using Euler method} \\ &= 100 + 2.5(-Km_0) \\ &= 100 - 2.5 \times 0.01 \times 100 \\ m_1 &= 97.5 \text{ gm} \end{aligned}$$

Now, $$t_2 = t_1 + h = 2.5 + 2.5 = 5 \text{ sec}$$

$$\begin{aligned} m_2 &= m_1 + hf(t_1, m_1) \\ &= 97.5 + 2.5[-0.01 \times 97.5] \\ m_2 &= 95.0625 \text{ gm} \end{aligned}$$

Now, $$t_3 = t_2 + h = 5 + 2.5 = 7.5 \text{ sec}$$

$$\begin{aligned} m_3 &= m_2 + hf(t_2, m_2) \\ &= 95.0625 + 2.5[-0.01 \times 95.0625] \\ m_3 &= 92.6859 \text{ gm} \end{aligned}$$

Now, $$t_4 = t_3 + h = 7.5 + 2.5 = 10 \text{ sec}$$

$\therefore$ $$\begin{aligned} m_4 &= m_3 + hf(t_3, m_3) \\ &= 92.6859 + 2.5[-0.01 \times 92.6859] \\ m_4 &= 90.3688 \text{ gm} \end{aligned}$$

which is mass of substance remaining at the end of 10 sec.

10.6 ERROR PROPAGATION - STABILITY

With modern computational methods, it is usual to find the solution of differential equation by integrating over very large number of step lengths. Errors are introduced in the numerical solution at all stages of the calculation. Errors that are introduced at an early stage of the integration may die away or may be propagated with increasing magnitude through the stepwise integration. Whether the error propagates or dies away can depend upon the differential equation. Stability of the numerical technique means whether an initial value problem diverges away from the exact solution or converges to the exact solution. Numerical instability can be inherent in the initial value problem itself regardless of which numerical scheme is used to obtain the solution. To explain this point, consider the differential equation;

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 14y = 0$$

with the initial conditions,

$$y(0) = 1, \quad y'(0) = -2$$

Exact solution of this problem can be easily found out to be $y = e^{-2x}$.

To examine the stability of this solution we change one of the initial conditions by small amount ε.

Consider the initial conditions,

$$y(0) = 1 + \varepsilon, \quad y'(0) = -2$$

Solution with these initial conditions is

$$y(x) = \left(1 + \frac{7}{9}\varepsilon\right) e^{-2x} + \frac{2}{9}\varepsilon\, e^{7x}$$

for any $\varepsilon > 0$ this solution tends to infinity as $x \to \infty$. Thus we see that a small change in the initial condition produces a large change in the solution i.e. the solution $y = e^{-2x}$ is an unstable solution. Such problems are often termed as ill conditioned and are difficult to solve numerically. Since truncation error and round-off error have the same effect as changing the boundary conditions.

Whereas some multi-step methods exhibit numerical instability, this problem usually does not arise with single-step methods such as Runge-Kutta methods provided h is sufficiently small.

Let us consider stability aspects of some numerical methods.

Euler's Method : Euler's formula

$$y_{n+1} = y_n + h\, f(x_n\, y_n)$$

is basically a piecewise linear approximation to the solution of $\frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$. At any step, the error in the above formula is of the order $\frac{h^2}{2}\, y^2 (\xi)$, where, ξ is the point in the interval (x_0, x_{n+1}).

This point is clear from the Taylor's expansion

$$y(x + h) = y_0 + h\, f(x_0, y_0) + \frac{h^2}{2} \{f_x + f(x_0, y_0)\, f_y\} + \ldots$$

Euler's method takes into account only two terms and therefore truncation error is of the order h^2. An improvement of the Euler's method includes h^2 term in the Taylor's expansion and use the expansion

$$y = y_0 + h\, f(x_0, y_0) + \frac{h^2}{2} [f_x (x_0, y_0) + f(x_0, y_0)\, f_y (x_0, y_0)]$$

The truncation error is thus of the order h^3. To consider the stability aspect of Euler's method, let the n^{th} step error in y_n is e_n then error e_{n+1} is given by,

$$y_{n+1} = Y_{n+1} + e_{n+1} = (Y_n + e_n) + h\, f(x_n, Y_n + e_n)$$

where, Y_n and Y_{n+1} are the exact solutions of the differential equation.

Expanding $f(x_n, Y_n + e_n)$ in a Taylor's series

$$Y_{n+1} + e_{n+1} = Y_n + e_n + h \left\{ f(x_n, Y_n) + e_n \left. \frac{\partial f}{\partial y} \right|_{x_n, Y_n} \right\} \qquad \ldots (1)$$

$$= Y_n + h\, f(x_n, Y_n) + e_n \left(1 + h \frac{\partial f}{\partial y}\right)$$

As $Y_{n+1} = Y_n + h\, f(x_n, Y_n)$, equation (1) gives

$$e_{n+1} = e_n \left(1 + h \frac{\partial f}{\partial y}\right) \qquad \ldots (2)$$

If $\left|1 + h \frac{\partial f}{\partial y}\right| < 1$, then the error will die down with successive iterations, in which case Euler's method could be stable. But if the condition (2) is not satisfied, stability of the method is not guaranteed. Detailed analysis will show that Euler's method is relatively stable.

Runge-Kutta Method : Runge-Kutta formula given earlier match the Taylor's series expansion for y upto h^4 term and thus has a truncation error of the order h^5. Runge-Kutta method is stable if h is sufficiently small and relative stability of the method is generally guaranteed.

EXERCISE 10.1

Type I : Taylor's Series Methods :

1. Using Taylor's series, solve the equation $\frac{dy}{dx} = x + y^2$, $y(0) = 0$ to tabulate the solution for $x = 0\ (0.1)\ (0.5)$.

Ans. $y(0) = 0.0$, $y(0.1) = 0.005005$, $y(0.2) = 0.020016$, $y(0.3) = 0.0451215$, $y(0.4) = 0.080512$

2. Using Taylor's series method, solve $\frac{dy}{dx} = 1 + xy$ with $x_0 = 0$, $y_0 = 2$. Find (i) $y(0.1)$, (ii) $y(0.2)$, and (iii) $y(0.3)$.

Ans. (i) $y(0.1) = 2.1103$, (ii) $y(0.2) = 2.2430$, (iii) $y(0.3) = 2.4011$

3. Using Taylor's series method, find y(0.1) correct to 3 decimal places from

$\frac{dy}{dx} + 2xy = 1$, $x_0 = 0$, $y_0 = 0$. **Ans.** $y(0.1) = 0.0993$

4. Using Taylor's series method, find y at x = 1.1 and 1.2 by solving $\frac{dy}{dx} = x^2 + y^2$ given $x_0 = 1$, $y_0 = 2.3$.

Ans. $y(1.1) = 3.1209$, $y(1.2) = 4.8623$

5. Use Taylor's series method to find the value of y(1.1), y(1.2) and y(1.3) correct to three decimal places, given that $\frac{dy}{dx} = xy^{1/3}$, y (1) = 1, taking the first three terms of the Taylor's series expansion.

Ans. y (1.1) = 1.107, y(1.2) = 1.228, y (1.3) = 1.364

6. From the Taylor's series for y(x), find y(0.1) correct to four decimal places if y (x) satisfies $\frac{dy}{dx} = x - y^2$ and y (0) = 1.

Ans. y (0.1) = 0.9138

7. Use Taylor's series method to find y at x = ± 0.1, ± 0.2 for the initial value problem $\frac{dy}{dx} - y = 1 - 2x$, y (0) = 2.17.

Ans. y (0.1) = 2.28413, y (– 0.1) = 1.6858, y (0.2) = 2.5328, y (– 0.2) = 1.34617

8. Tabulate the solution of the equation $\frac{dy}{dx} - x = 0.1y^2$, y(0) = 0 for the range 0 < x < 0.5 at intervals of 0.1. Obtain the solution correct to four decimal places and compare it with the Taylor's series solution.

Ans. 0.0050, 0.0200, 0.0450, 0.0800, 0.1252

Type II : Euler's Method and Modified Euler's Method :

9. Use Euler's method to solve the equation $\frac{dy}{dx} = x^2 + y$, subject to the conditions x = 0, y = 1 and tabulate the solution for x = 0 (0.1) 0.5. **Ans.** y (0) = 1, y (0.1) = 1.1, y (0.2) = 1.211, y (0.3) = 1.3361, y (0.4) = 1.4748, y (0.5) = 1.64257

10. Use modified Euler's method to solve the above problem and find y at x = 0.1, 0.2 and 0.3.

Ans. y (0.1) = 1.1058, y (0.2) = 1.2245, y (0.3) = 1.3606

11. Use Euler's method to solve the equation $\frac{dy}{dx} = -y$, y (0) = 1 and tabulate the solution for x = 0 (0.01) 0.04.

Ans. y (0.01) = 0.99, y (0.02) = 0.9801, y (0.03) = 0.9703, y (0.04) = 0.9606 Exact solution is $y = e^{-x}$

12. Solve by Euler's method, the equation $\frac{dy}{dx} = x + y$, y(0) = 0, choose h = 0.2 and compute y(0.4) and y(0.6).

Ans. y (0.4) = 0.0938, y (0.6) = 0.2258

13. Solve $\frac{dy}{dx} = 1 - y$, y (0) = 0 using Euler's method. Find x = 0.1 and x = 0.2. Compare the result with exact solution.

Ans. Euler's method : y (0.1) = 0.1, y (0.2) = 0.19; Exact solutions : y (0.1) = 0.1052, y (0.2) = 0.214

14. Using Euler's method solve for y at x = 0.1 from $\frac{dy}{dx} = x + y + xy$, y(0) = 1, taking step size h = 0.025.

Ans. y (0.1) = 1.1448

15. Solve numerically the differential equation $\frac{dy}{dx} = y^2 - \frac{y}{x}$, y(1) = 1 for the interval 1 (0.1) 1.5 by (a) Euler's method, (b) Modified Euler's method.

Ans. (a) 1, 1.0091, 1.0269, 1.0534, 1.0891, 1.1357, (b) 1, 1.0142, 1.0376, 1.0704, 1.1141, 1.1747

16. Given $\frac{dy}{dx} = x^2 + y$, y (0) = 1. Determine y (0.02), y (0.04) and y (0.06) using modified Euler's method.

Ans. y (0.02) = 1.0202, y (0.04) = 1.0408, y (0.06) = 1.0619

17. Given $\frac{dy}{dx} - \sqrt{xy} = 2$, y(1) = 1. Find the value of y(2) in steps of 0.1 using modified Euler's method.

Ans. y (2) = 5.0524 when h = 0.1 and note that y (2) = 5.051 when h = 0.2

18. Using modified Euler's method, find an approximate value of y when x = 0.3, given that $\frac{dy}{dx} = x + y$, y = 1 when x = 0 and h = 0.1. Compare the result with the true value.

Ans. y (0.3) = 1.4004, exact solution $y = 2e^x - x - 1$; y (0.3) = 1.3997

19. Solve the following by Euler's modified method $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ at $x = 1.2$ and 1.4 with $h = 0.2$.

Ans. $y(1.2) = 2.5351$, $y(1.4) = 2.6531$

20. Solve the differential equation $\frac{dy}{dx} = -xy^2$, $y = 2$ at $x = 0$, by modified Euler's method and obtain y at $x = 0.2$ in two stages of 0.1 each. **Ans.** $y(0.2) = 1.9227$

Type III : Runge-Kutta Method of Fourth Order :

21. Solve the following initial value problem using Runge-Kutta method of fourth order.

$\frac{dy}{dx} = (1 + x)\,y$, $y(0) = 1$, $x = 0\,(0.2)\,0.6$. **Ans.** $y(0.2) = 1.2247$, $y(0.4) = 1.5240$, $y(0.6) = 1.9581$

22. Given the differential equation $\frac{dy}{dx} = \frac{x + y}{x}$ with $y(2) = 2$. Estimate $y(2.5)$ using the fourth order Runge-Kutta method with $h = 0.5$. **Ans.** $y(2.5) = 3.058$

23. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, given $y(0) = 1$. Find the value of y at $x = 0.2$ and 0.4.

Ans. $y(0.2) = 1.1966$, $y(0.4) = 1.3744$

24. Using Runge-Kutta method of order 4, find $y(0.2)$, given that $\frac{dy}{dx} = 3x + \frac{1}{2}y$, $y(0) = 1$ taking $h = 0.1$.

25. Apply Runge-Kutta method to find an approximate value of y for $x = 0.2$ in steps in 0.1, if $\frac{dy}{dx} = x + y^2$, given that $y = 1$, when $x = 0$. **Ans.** $y(0.1) = 1.1165$, $y(0.2) = 1.2736$

26. Given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$. Find $y(0.1)$ using Runge-Kutta method of fourth order. **Ans.** $y(0.1) = 0.9052$

27. Using Runge-Kutta method of fourth order, find $y(0.1)$, $y(0.2)$ and $y(0.3)$, given that $\frac{dy}{dx} = 1 + xy$; $y(0) = 2$.

Ans. $y(0.1) = 2.1086$, $y(0.2) = 2.2416$, $y(0.3) = 2.3997$

28. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = xy$, $y(1) = 2$ at $x = 1.2$ with $h = 0.2$. **Ans.** $y(1.2) = 2.4921$

29. Find $y(0.2)$ with $h = 0.1$ from $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ by Runge-Kutta method of fourth order. **Ans.** $y(1.2) = 1.1832$

30. Use the Runge-Kutta method of fourth order to solve the equation $\frac{dy}{dx} = \frac{y - x}{y + x}$, $x_0 = 0$, $y_0 = 1$ to find the value of y at $x = 0.2$, $x = 0.4$ and $x = 1$ taking $h = 0.2$. **Ans.** $y(0.2) = 1.16787$, $y(0.4) = 1.2925$, $y(1) = 1.4983$

31. Use the Runge-Kutta method of fourth order to solve $\frac{dy}{dx} = 1 + y^2$, subject to conditions $x = 0$, $y = 0$ to compute y at $x = 0.2$ taking $h = 0.1$.

32. Use the Runge-Kutta method to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ for the interval $0 < x \le 0.4$ with $h = 0.1$.

Ans. 1.0101, 1.0207, 1.0318, 1.0438

33. Apply the Runge-Kutta fourth order method, to find an approximate value of y when $x = 0.2$, given that $\frac{dy}{dx} = x + y$ and $y = 1$, when $x = 0$. **Ans.** $y(0.2) = 1.2428$

34. Use the Runge-Kutta fourth order method to solve $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ to find $y(0.4)$ taking $h = 0.1$.

Ans. $y(0.1) = 0.9117$, $y(0.2) = 0.8494$, $y(0.3) = 0.8061$

35. Use the Runge-Kutta method of second order $\frac{dy}{dx} = x + y$, $x_0 = 0$, $y_0 = 1$ to find y at 0.2 taking $h = 0.2$.

Ans. $y(0.2) = 1.24$

36. Given $\frac{dy}{dx} = y - x$ with y (0) = 2, find y (0.1) and y (0.2) correct to four decimal places by second and fourth order Runge-Kutta methods. **Ans.** Second order : y (0.1) = 2.2050, y (0.2) = 2.4210,

Fourth order : y (0.1) = 2.2052, y (0.2) = 2.4210

37. Solve the differential equations $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for x = 0.3 using fourth order Runge-Kutta method. Initial values are x = 0, y = 0, z = 1. **Ans.** y = 0.3448, z = 0.99

38. Numerical solution of the differential equation $\frac{dy}{dx} = 0.2x + 0.1y$ is tabulated as

x	0.0	0.05	0.10	0.15
y	2	2.0103	2.0211	2.0323

Find y at x = 0.2 by Milne's predictor-corrector method. **Ans.** y (0.2) = 2.0444

39. Solve the following initial value problem by Milne's predictor-corrector method with the values of y being given at the four points by the table

x	0	0.5	1	1.5
y	2	2.636	3.595	4.968

$\frac{dy}{dx} = \frac{1}{2}(x + y)$, find y at x = 2. **Ans.** y (2) = 6.8733

40. Find y (0.8) by Milne's method for the equation $y' = y - x^2$, y (0) = 1 obtaining the starting values by Taylor's series.

Ans. y(0.8) = 2.0583, starting values : y (0.2) = 1.2187, y (0.4) = 1.4684, y (0.6) = 1.7383

41. Apply Milne's method to find a solution of the differential equation $\frac{dy}{dx} = x - y^2$, y (0) = 0 in the range $0 \le x \le 1$.

Ans. y (0.8) = 0.3046, y (1) = 0.4555

42. Use the predictor-corrector formulae for tabulating a solution of $10\frac{dy}{dx} = x^2 + y^2$, y (0) = 1 for the range $0.5 \le x \le 1.0$.

Ans. 1.0569, 1.0713, 1.0871, 1.1048, 1.1244, 1.1464

43. Tabulate the solution of $\frac{dy}{dx} = x + y$, y (0) = 0 for $0.4 < x \le 1.0$ with h = 0.1 using the predictor-corrector formulae.

Ans. 0.0918, 0.1487, 0.2221, 0.3138, 0.4255, 0.5596, 0.7183

44. Using Milne's predictor-corrector method find y(0.4), for the differential equation $\frac{dy}{dx} = 1 + xy$, y (0) = 2.

Ans. y (0.4) = 2.5885, starting values : $y_1 = 2.1103$, $y_2 = 2.2340$, $y_3 = 2.4011$

45. Given $\frac{dy}{dx} = \frac{1}{x + y}$, y (0) = 2. If y(0.2) = 2.09, y(0.4) = 2.17 and y(0.6) = 2.24, find y (0.8) using Milne's method.

Ans. y (0.8) = 2.3112

46. Find y (0.8) using Milne's predictor-corrector method correct to four decimal places, given $\frac{dy}{dx} = (1 + x^2)y$, y (0) = 1, y (0.2) = 1.2246696, y (0.4) = 1.5239936, y (0.6) = 1.9581377. **Ans.** y (0.8) = 3.128

47. Find y(0.4) by Milne's method, given $\frac{dy}{dx} = y - \frac{2x}{y}$, y (0) = 1 with h = 0.1. **Ans.** y (0.4) = 1.3428

48. Solve the initial value problem $\frac{dy}{dx} = 1 + xy^2$, $y(0) = 1$ for $x = 0.4$ by using Milne's method, when it is given that

x	0.1	0.2	0.3
y	1.105	1.223	1.355

Ans. $y(0.4) = 1.5$

49. Evaluate y (1.4), given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$ by using Adam's Moulton predictor-corrector method. **Ans.** $y(1.4) = 0.949$

50. Given that $\frac{dy}{dx} = 2x^2 + 2y$ with $y(-0.6) = 0.1918$, $y(-0.4) = 0.4140$, $y(-0.2) = 0.6655$, $y(0) = 1$. Estimate y (0.2) by Adam's-Bashforth method. **Ans.** $y(0.2) = 1.4982$

51. Using the Adam's-Bashforth method, obtain the solution of $\frac{dy}{dx} = x - y^2$ at $x = 0.8$, given the values

x	0	0.2	0.4	0.6
y	0	0.0200	0.0795	0.1762

Ans. $y(0.8) = 0.3049$

52. Using Adam's-Bashforth formulae, determine y (0.4), given differential equation $\frac{dy}{dx} = \frac{1}{2}xy$ and the data

x	0	0.1	0.2	0.3
y	1	1.0025	1.0101	1.0228

Ans. $y(0.4) = 1.0408$

53. Given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate y (0.4) using Adam's–Bashforth method. **Ans.** $y(0.4) = 0.6897$

54. Using Runge-Kutta method of order four, solve $\frac{d^2y}{dx^2} = y + x\frac{dy}{dx}$, $y(0) = 1$, $y'(0) = 0$ to find y (0.2) and y' (0.2).

Ans. $y(0.2) = 0.9802$, $y'(0.2) = -0.196$

55. Using fourth order Runge-Kutta method, evaluate y (1.1) and z (1.1), given that

$\frac{dy}{dx} = xyz$, $\frac{dz}{dx} = y^2$, $y(1) = \frac{1}{3}$, $z(1) = 1$. **Ans.** $y(1.1) = 0.3707$, $z(1.1) = 1.03615$

56. Solve $\frac{dy}{dx} + xz = 0$, $\frac{dz}{dx} = y^2$ with $y(0) = 1$, $z(0) = 1$ for $x = 0.2\ (0.2)\ 0.4$ by fourth order Runge-Kutta method.

Ans. $y(0.2) = 0.978$, $z(0.2) = 1.2$, $y(0.4) = 0.9003$, $z(0.4) = 1.382$

57. Solve the system of differential equations by fourth order Runge-Kutta method

$\frac{dx}{dt} = y - t$, $\frac{dy}{dt} = x + t$ with $x = 1$, $y = 1$ when $t = 0$ taking $h = 0.1$ **Ans.** $x(0.1) = 1.1003$, $y(0.1) = 1.1102$

58. Solve the system of differential equations by fourth order Runge-Kutta method

$\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$ with $y = 0$ and $z = 1$ at $x = 0$ for $x = 0.3\ (0.3)\ 0.9$.

Ans. $y_1 = 0.3448$, $y_2 = 0.7738$, $y_3 = 1.255$, $z_1 = 0.99$, $z_2 = 0.9121$, $z_3 = 0.6806$

APPENDIX

MULTIPLE CHOICE QUESTIONS (MCQ'S)

Ch. 1 Linear Differential Equations with Constant Coefficient

Type I : Complementary Functions : **Marks**

1. If the roots $m_1, m_2, m_3, \ldots, m_n$ of auxiliary equation $\phi(D) = 0$ are real and distinct, then solution of $\phi(D)\, y = 0$ is .... (1)

(A) $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$
(B) $c_1 \cos m_1 x + c_2 \cos m_2 x + \ldots + c_n \cos m_n x$
(C) $m_1 e^{c_1 x} + m_2 e^{c_2 x} + \ldots + m_n e^{c_n x}$
(D) $c_1 \sin m_1 x + c_2 \sin m_2 x + \ldots + c_n \sin m_n x$

2. The roots $m_1, m_2, m_3 \ldots, m_n$ of auxiliary equation $\phi(D) = 0$ are real. If two of these roots are repeated say $m_1 = m_2$ and the remaining roots $m_3, m_4, \ldots, m_n$ are distinct then solution of $\phi(D)\, y = 0$ is .... (1)

(A) $c_1 e^{m_1 x} + c_2^{m_2 x} + \ldots + c_n^{m_n x}$
(B) $(c_1 x + c_2) \cos m_1 x + c_3 \cos m_3 x + x \ldots + c_n \cos m_n x$
(C) $(c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$
(D) $(c_1 x + c_2) \sin m_1 x + c_3 \sin m_3 x + \ldots + c_n \sin m_n x$

3. The roots $m_1, m_2, m_3 \ldots, m_n$ of auxiliary equation $\phi(D) = 0$ are real. If three of these roots are repeated, say, $m_1 = m_2 = m_3$ and the remaining roots $m_4, m_5, \ldots m_n$ are distinct then solution of $\phi(D)\, y = 0$ is .... (1)

(A) $c_1 e^{m_1 x} + c_2^{m_2 x} + \ldots + c_n\, e^{m_n x}$
(B) $(c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$
(C) $(c_1 x^2 + c_2 x + c_3) \cos m_1 x + c_4 \cos m_4 x + \ldots + c_n \cos m_n x$
(D) $(c_1 x^2 + c_2 x + c_3) \sin m_1 x + c_4 \sin m_4 x + \ldots + c_n \sin m_n x$

4. If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are two complex roots of auxiliary equation of second order DE $\phi(D)\, y = 0$ then it's solution is .... (1)

(A) $e^{\beta x} [c_1 \cos \alpha x + c_2 \sin \alpha x]$
(B) $e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$
(C) $c_1 e^{\alpha x} + c_2 e^{\beta x}$
(D) $e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

5. If the complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ of auxiliary equation of fourth order DE $\phi(D)\, y = 0$ are repeated twice then it's solution is .... (1)

(A) $e^{\beta x} [c_1 \cos \alpha x + c_2 \sin \alpha x]$
(B) $e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$
(C) $(c_1 x + c_2)\, e^{\alpha x} + (c_3 x + c_4)\, e^{\beta x}$
(D) $e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

6. The solution of differential equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ is .... (1)

(A) $c_1 e^{2x} + c_2 e^{-3x}$ (B) $c_1 e^{-2x} + c_2 e^{3x}$ (C) $c_1 e^{-2x} + c_2 e^{-3x}$ (D) $c_1 e^{2x} + c_2 e^{3x}$

7. The solution of differential equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 6y = 0$ is .... (1)

(A) $c_1 e^{-x} + c_2 e^{6x}$ (B) $c_1 e^{-2x} + c_2 e^{-3x}$ (C) $c_1 e^{3x} + c_2 e^{2x}$ (D) $c_1 e^{-3x} + c_2 e^{-2x}$

8. The solution of differential equation $2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 10y = 0$ is .... (1)

(A) $c_1 e^{2x} + c_2 e^{\frac{5}{2}x}$ (B) $c_1 e^{-2x} + c_2 e^{-\frac{5}{2}x}$ (C) $c_1 e^{-2x} + c_2 e^{\frac{5}{2}x}$ (D) $c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$

9. The solution of differential equation $\dfrac{d^2y}{dx^2} - 4y = 0$ is .... (1)

(A) $(c_1 x + c_2)\, e^{2x}$ (B) $c_1 e^{4x} + c_2 e^{-4x}$ (C) $c_1 \cos 2x + c_2 \sin 2x$ (D) $c_1 e^{2x} + c_2 e^{-2x}$

10. The solution of differential equation $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$ is .... (1)

(A) $c_1 e^{2x} + c_2 e^{x}$ (B) $c_1 e^{2x} + c_2 e^{-x}$ (C) $c_1 e^{-2x} + c_2 e^{x}$ (D) $c_1 e^{-2x} + c_2 e^{-x}$

11. The solution of differential equation $2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = 0$ is .... (1)

(A) $c_1 e^{x} + c_2 e^{\frac{3}{2}x}$ (B) $c_1 e^{2x} + c_2 e^{-3x}$ (C) $c_1 e^{-x} + c_2 e^{\frac{3}{2}x}$ (D) $c_1 e^{\frac{x}{2}} + c_2 e^{\frac{3}{2}x}$

12. The solution of differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ is .... (1)

(A) $c_1e^{2x} + c_2e^{x}$ (B) $c_1e^{x} + c_2e^{-x}$ (C) $(c_1x + c_2)\, e^{-x}$ (D) $(c_1x + c_2)\, e^{x}$

13. The solution of differential equation $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$ is .... (1)

(A) $c_1e^{\frac{x}{2}} + c_2e^{-\frac{x}{2}}$ (B) $(c_1 + c_2x)\, e^{-2x}$ (C) $c_1 \cos 2x + c_2 \sin 2x$ (D) $(c_1 + c_2x)\, e^{\frac{x}{2}}$

14. The solution of differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ is .... (1)

(A) $(c_1x + c_2)\, e^{2x}$ (B) $(c_1x + c_2)\, e^{-2x}$ (C) $c_1e^{4x} + c_2e^{-4x}$ (D) $c_1e^{2x} + c_2e^{-2x}$

15. The solution of differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ is (1)

(A) $c_1e^{-6x} + c_2e^{-9x}$ (B) $(c_1x + c_2)\, e^{-3x}$ (C) $(c_1x + c_2)\, e^{3x}$ (D) $c_1e^{3x} + c_2e^{2x}$

16. The solution of differential equation $\frac{d^2y}{dx^2} + y = 0$ is .... (1)

(A) $c_1e^{x} + c_2e^{-x}$ (B) $(c_1x + c_2)\, e^{-x}$

(C) $c_1 \cos x + c_2 \sin x$ (D) $e^{x}\,(c_1 \cos x + c_2 \sin x)$

17. The solution of differential equation $\frac{d^2y}{dx^2} + 9y = 0$ is .... (1)

(A) $c_1 \cos 2x + c_2 \sin 2x$ (B) $(c_1x + c_2)\, e^{-3x}$

(C) $c_1e^{3x} + c_2e^{-3x}$ (D) $c_1 \cos 3x + c_2 \sin 3x$

18. The solution of differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$ is .... (1)

(A) $e^{-3x}\,(c_1 \cos x + c_2 \sin x)$ (B) $e^{x}\,(c_1 \cos 3x + c_2 \sin 3x)$

(C) $c_1e^{5x} + c_2e^{2x}$ (D) $e^{x}\,(c_1 \cos x + c_2 \sin x)$

19. The solution of differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ is .... (1)

(A) $e^{x}\,(c_1 \cos x + c_2 \sin x)$ (B) $e^{x/2}\left[c_1 \cos\left(\frac{3}{2}\right)x + c_2 \sin\left(\frac{3}{2}\right)x\right]$

(C) $e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$ (D) $c_1e^{x} + c_2e^{-x}$

20. The solution of differential equation $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$ is .... (1)

(A) $e^{-x}\,(c_1 \cos 2x + c_2 \sin 2x)$ (B) $e^{-x/2}\,[c_1 \cos x + c_2 \sin x]$

(C) $e^{-2x}\,(c_1 \cos x + c_2 \sin x)$ (D) $c_1e^{-4x} + c_2e^{-5x}$

21. The solution of differential equation $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ is .... (2)

(A) $c_1e^{x} + c_2e^{2x} + c_3e^{3x}$ (B) $c_1e^{-x} + c_2e^{2x} + c_3e^{-3x}$

(C) $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$ (D) $c_1e^{x} + c_2e^{-2x} + c_3e^{3x}$

22. The solution of differential equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$ is .... (2)

(A) $c_1e^{x} + c_2e^{2x} + c_3e^{3x}$ (B) $c_1e^{-x} + c_2e^{-2x} + c_33^{6x}$

(C) $c_1e^{-x} + c_2e^{2x} + c_3e^{x}$ (D) $c_1e^{-x} + c_2e^{-2x} + c_3e^{3x}$

23. The solution of differential equation $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ is .... (2)

(A) $c_1 + e^{x}(c_2x + c_3)$ (B) $c_1 + e^{-x}(c_2x + c_3)$

(C) $e^{-x}(c_2x + c_3)$ (D) $c_1 + c_2e^{x} + c_3e^{-x}$

24. The solution of differential equation $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$ is .... (2)

(A) $c_1e^{x} + (c_2x + c_3)e^{2x}$ (B) $c_1e^{x} + c_2e^{2x} + c_3e^{3x}$

(C) $(c_2x + c_3)e^{2x}$ (D) $c_1e^{-x} + (c_2x + c_3)e^{-2x}$

25. The solution of differential equation $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0$ is .... (2)

(A) $c_1e^{2x} + c_2e^{-2x}$ (B) $c_1 + c_2\cos 2x + c_3\sin 2x$

(C) $c_1e^{x} + c_2e^{-2x} + c_3e^{-3x}$ (D) $c_1 + c_2e^{2x} + c_3e^{-2x}$

26. The solution of differential equation $\frac{d^3y}{dx^3} + y = 0$ is .... (2)

(A) $c_1e^{x} + e^{x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$ (B) $c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\frac{1}{2}x + c_3\sin\frac{1}{2}x\right)$

(C) $c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$ (D) $(c_1 + c_2x + c_3x^2)e^{-x}$

27. The solution of differential equation $\frac{d^3y}{dx^3} + 3\frac{dy}{dx} = 0$ is .... (2)

(A) $c_1 + c_2\cos x + c_3\sin x$ (B) $c_1 + c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x$

(C) $c_1 + c_2e^{\sqrt{3}x} + c_3e^{-\sqrt{3}x}$ (D) $c_1\cos x + c_2\sin x$

28. The solution of differential equation $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 12y = 0$ is .... (2)

(A) $c_1e^{-3x} + e^{x}\left(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right)$ (B) $c_1e^{-3x} + (c_2\cos 3x + c_3\sin 3x)$

(C) $c_1e^{3x} + e^{-x}\left(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right)$ (D) $c_1e^{-x} + c_2e^{-\sqrt{3}x} + c_3e^{\sqrt{3}x}$

29. The solution of differential equation $(D^3 - D^2 + 3D + 5)y = 0$ where $D = \frac{d}{dx}$ is .... (2)

(A) $c_1e^{-x} + e^{x}(c_2\cos 2x + c_3\sin 2x)$ (B) $c_1e^{-x} + (c_2\cos 3x + c_3\sin 3x)$

(C) $c_1e^{x} + e^{-x}(c_2\cos 2x + c_3\sin 2x)$ (D) $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

30. The solution of differential equation $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$ is (2)

(A) $(c_1 + c_2x)e^{-2x} + c_3e^{-x}$ (B) $c_1e^{x} + c_2\cos 4x + c_3\sin 4x$

(C) $c_1e^{x} + c_2\cos 2x + c_3\sin 2x$ (D) $c_1e^{x} + c_2e^{2x} + c_3e^{-2x}$

31. The solution of differential equation $\frac{d^4y}{dx^4} - y = 0$ is .... (2)

(A) $(c_1x + c_2)e^{-x} + c_3\cos x + c_4\sin x$ (B) $(c_1x + c_2)\cos x + (c_3x + c_4)\sin x$

(C) $(c_1 + c_2x + c_3x^2 + c_4x^3)e^{x}$ (D) $c_1e^{x} + c_2e^{-x} + c_3\cos x + c_4\sin x$

32. The solution of differential equation $(D^4 + 2D^2 + 1)y = 0$ where $D = \frac{d}{dx}$ is .... (2)

(A) $(c_1x + c_2)e^{x} + (c_3x + c_4)e^{-x}$ (B) $(c_1x + c_2)\cos x + (c_3x + c_4)\sin x$

(C) $c_1e^{x} + c_2e^{-x} + c_3\cos x + c_4\sin x$ (D) $(c_1x + c_2)\cos 2x + (c_3x + c_4)\sin 2x$

33. The solution of differential equation $(D^2 + 9)^2 y = 0$, where $D = \frac{d}{dx}$ is .... (2)

(A) $(c_1x + c_2)e^{3x} + (c_3x + c_4)e^{-3x}$ (B) $(c_1x + c_2)\cos 3x + (c_3x + c_4)\sin 3x$

(C) $(c_1x + c_2)\cos 9x + (c_3x + c_4)\sin 9x$ (D) $(c_1x + c_2)\cos x + (c_3x + c_4)\sin x$

34. The solution of differential equation $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$ is .... (2)

(A) $c_1e^{2x} + c_2e^{-x} + c_3e^{x} + c_4e^{-2x}$

(B) $(c_1x + c_2)\, e^{2x} + (c_3x + c_4)\, e^{-2x}$

(C) $(c_1x + c_2) \cos 4x + (c_3x + c_4) \sin 4x$

(D) $(c_1x + c_2) \cos 2x + (c_3x + c_4) \sin 2x$

35. The solution of differential equation $\frac{d^6y}{dx^6} + 6\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 0$ is .... (2)

(A) $c_1x + c_2 + (c_3x + c_4) \cos \sqrt{3}x + (c_3x + c_6) \sin \sqrt{3}x$

(B) $c_1x + c_2 + (c_3x + c_4) \cos 3x + (c_5x + c_6) \sin 3x$

(C) $(c_1x + c_2) \cos \sqrt{3}x + (c_3x + c_4) \sin \sqrt{3}x$

(D) $c_1x + c_2 + (c_3x + c_4)\, e^{\sqrt{3}x}$.

Answers

1. (A)	2. (C)	3. (B)	4. (D)	5. (B)	6. (D)	7. (A)	8. (C)
9. (D)	10. (B)	11. (A)	12. (C)	13. (D)	14. (A)	15. (B)	16. (C)
17. (D)	18. (A)	19. (C)	20. (B)	21. (C)	22. (D)	23. (B)	24. (A)
25. (D)	26. (C)	27. (B)	28. (A)	29. (A)	30. (C)	31. (D)	32. (B)
33. (B)	34. (D)	35. (A)					

Type II : Particular Integral : **Marks**

1. Particular Integral of linear differential equation with constant coefficient $\phi(D)\, y = f(x)$ is given by .... (1)

(A) $\frac{1}{\phi(D)} f(x)$

(B) $\frac{1}{\phi(D)\, f(x)}$

(C) $\phi(D) \frac{1}{f(x)}$

(D) $\frac{1}{\phi(D^2)} f(x)$

2. $\frac{1}{D - m} f(x)$, where $D \equiv \frac{d}{dx}$ and m is constant, is equal to .... (1)

(A) $e^{mx} \int e^{-mx}\, dx$

(B) $\int e^{-mx} f(x)\, dx$

(C) $e^{mx} \int e^{-mx} f(x)\, dx$

(D) $e^{-mx} \int e^{mx} f(x)\, dx$

3. $\frac{1}{D + m} f(x)$, where $D \equiv \frac{d}{dx}$ and m is constant, is equal to .... (1)

(A) $e^{-mx} \int e^{mx}\, dx$

(B) $\int e^{mx} f(x)\, dx$

(C) $e^{mx} \int e^{-mx} f(x)\, dx$

(D) $e^{-mx} \int e^{mx} f(x)\, dx$

4. Particular Integral $\frac{1}{\phi(D)} e^{ax}$, where $D \equiv \frac{d}{dx}$ and $\phi(a) \neq 0$ is .... (1)

(A) $\frac{1}{\phi(-a)} e^{ax}$

(B) $x \frac{1}{\phi(a)} e^{ax}$

(C) $\frac{1}{\phi(a^2)} e^{ax}$

(D) $\frac{1}{\phi(a)} e^{ax}$

5. Particular Integral $\frac{1}{(D - a)^r} e^{ax}$ where $D \equiv \frac{d}{dx}$ is .... (1)

(A) $\frac{1}{r!} e^{ax}$

(B) $\frac{x^r}{r} e^{ax}$

(C) $\frac{x^r}{r!} e^{ax}$

(D) $x^r e^{ax}$

6. Particular Integral $\frac{1}{\phi(D^2)} \sin (ax + b)$, where $D \equiv \frac{d}{dx}$ and $\phi(-a^2) \neq 0$ is .... (1)

(A) $\frac{1}{\phi(-a^2)} \cos (ax + b)$

(B) $\frac{1}{\phi(-a^2)} \sin (ax + b)$

(C) $x \frac{1}{\phi(-a^2)} \sin (ax + b)$

(D) $\frac{1}{\phi(a^2)} \sin (ax + b)$

7. Particular Integral $\frac{1}{\phi(D^2)} \sin (ax + b)$, where $D \equiv \frac{d}{dx}$ and $\phi(-a^2) = 0$, $\phi'(-a^2) \neq 0$ is .... (1)

(A) $x \frac{1}{\phi'(-a^2)} \cos (ax + b)$

(B) $x \frac{1}{\phi'(-a^2)} \sin (ax + b)$

(C) $\frac{1}{\phi(-a^2)} \sin(ax + b)$

(D) $\frac{1}{\phi'(-a^2)} \sin (ax + b)$

8. Particular Integral $\frac{1}{\phi(D^2)}\cos(ax+b)$, where $D \equiv \frac{d}{dx}$ and $\phi(-a^2) \neq 0$ is .... (1)

(A) $\frac{1}{\phi(-a^2)}\cos(ax+b)$ (B) $\frac{1}{\phi(-a^2)}\sin(ax+b)$

(C) $x\frac{1}{\phi'(-a^2)}\cos(ax+b)$ (D) $\frac{1}{\phi(a^2)}\cos(ax+b)$

9. Particular Integral $\frac{1}{\phi(D^2)}\cos(ax+b)$, where $D \equiv \frac{d}{dx}$ and $\phi(-a^2) = 0$, $\phi'(-a^2) \neq 0$ is .... (1)

(A) $\frac{1}{\phi'(-a^2)}\cos(ax+b)$ (B) $\frac{1}{\phi'(-a^2)}\cos(ax+b)$

(C) $x\frac{1}{\phi'(-a^2)}\sin(ax+b)$ (D) $x\frac{1}{\phi'(-a^2)}\cos(ax+b)$

10. Particular Integral $\frac{1}{\phi(D^2)}\sinh(ax+b)$, where $D \equiv \frac{d}{dx}$ and $\phi(a^2) \neq 0$ is .... (1)

(A) $\frac{1}{\phi(a^2)}\cosh(ax+b)$ (B) $x\frac{1}{\phi'(a^2)}\sinh(ax+b)$

(C) $\frac{1}{\phi(a^2)}\sinh(ax+b)$ (D) $\frac{1}{\phi(-a^2)}\sinh(ax+b)$

11. Particular Integral $\frac{1}{\phi(D^2)}\cosh(ax+b)$, where $D \equiv \frac{d}{dx}$ and $\phi(a^2) \neq 0$ is .... (1)

(A) $\frac{1}{\phi(a^2)}\cosh(ax+b)$ (B) $x\frac{1}{\phi'(a^2)}\cosh(ax+b)$

(C) $\frac{1}{\phi(a^2)}\sinh(ax+b)$ (D) $\frac{1}{\phi(-a^2)}\cosh(ax+b)$

12. Particular Integral $\frac{1}{\phi(D)}e^{ax}V$ where, V is any function of x and $D \equiv \frac{d}{dx}$ is .... (1)

(A) $e^{ax}\frac{1}{\phi(D-a)}V$ (B) $e^{ax}\frac{1}{\phi(a)}V$ (C) $e^{ax}\frac{1}{\phi(D+a)}V$ (D) $\frac{1}{\phi(D+a)}V$

13. Particular Integral $\frac{1}{\phi(D)}xV$ where, V is a function of x and $D \equiv \frac{d}{dx}$ is .... (1)

(A) $\left[x-\frac{1}{\phi(D)}\right]\frac{1}{\phi(D)}V$ (B) $\left[x-\frac{\phi'(D)}{\phi(D)}\right]\phi(D)V$

(C) $\left[x+\frac{\phi'(D)}{\phi(D)}\right]V$ (D) $\left[x-\frac{\phi'(D)}{\phi(D)}\right]\frac{1}{\phi(D)}V$

14. Particular integral $\frac{1}{D+1}e^{e^x}$, where $D \equiv \frac{d}{dx}$ is .... (2)

(A) $e^{-x}e^{e^x}$ (B) e^{e^x} (C) $e^x e^{e^x}$ (D) $e^{-2x}e^{e^x}$

15. Particular Integral $\frac{1}{D+2}e^{-x}e^{e^x}$ where $D \equiv \frac{d}{dx}$ is .... (2)

(A) $e^{2x}e^{e^x}$ (B) $e^{-2x}e^{e^x}$ (C) e^{e^x} (D) $e^{-x}e^{e^x}$

16. particular Integral $\frac{1}{D+1}\sin e^x$, where $D \equiv \frac{d}{dx}$ is .... (2)

(A) $-e^{-x}\sin e^x$ (B) $e^x\cos e^x$ (C) $-e^{-x}\cos e^x$ (D) $e^{-x}\cos e^x$

17. Particular Integral $\frac{1}{D+2}e^{-x}\cos e^x$, where $D \equiv \frac{d}{dx}$ is .... (2)

(A) $e^{-x}\cos e^x$ (B) $e^{-x}\sin e^x$ (C) $e^{-2x}\cos e^x$ (D) $e^{-2x}\sin e^x$

18. Particular Integral $\frac{1}{D+2} e^{-2x} \sec^2 x\,(1 + 2\tan x)$, (use $\tan x = t$ and $D \equiv \frac{d}{dx}$) is .... (2)

(A) $e^{-2x}(1 + 2\tan^2 x)$ (B) $e^{-2x}(\tan x + \tan^2 x)$

(C) $e^{2x}(\tan x + 2\tan^2 x)$ (D) $e^{-2x}(\tan x + \sec x)$

19. Particular Integral $\frac{1}{D+1}\left(\frac{1}{1+e^x}\right)$ where $D \equiv \frac{d}{dx}$ is .... (2)

(A) $e^x \log(1 - e^x)$ (B) $\log(1 + e^x)$

(C) $e^x \log(1 + e^x)$ (D) $e^{-x} \log(1 + e^x)$

20. Particular Integral of differential equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$ is .... (2)

(A) $-\frac{xe^{2x}}{3}$ (B) $-\frac{e^{2x}}{4}$ (C) $\frac{e^{2x}}{4}$ (D) $\frac{e^{2x}}{24}$

21. Particular Integral of differential equation $(D^2 - 5D + 6)\,y = 3e^{5x}$ is .... (2)

(A) $\frac{e^{5x}}{2}$ (B) $\frac{e^{5x}}{6}$ (C) $-\frac{e^{5x}}{14}$ (D) $-\frac{e^{2x}}{2}$

22. Particular Integral of differential equation $(D^2 - 9)\,y = e^{3x} + 1$ is .... (2)

(A) $\frac{3x}{2} e^{3x} - \frac{1}{9}$ (B) $x\frac{e^{3x}}{6} + \frac{3}{8}$

(C) $x\frac{e^{3x}}{6} - \frac{1}{9}$ (D) $xe^{3x} + \frac{1}{8}$

23. Particular Integral differential equation $(D^2 + 4D + 3)\,y = e^{-3x}$ is .... (2)

(A) xe^{-3x} (B) $-\frac{1}{2} e^{-3x}$ (C) $-\frac{x}{10} e^{-3x}$ (D) $-\frac{x}{2} e^{-3x}$

24. Particular Integral of differential equation $(D - 2)^3\,y = e^{2x} + 3^x$ is .... (2)

(A) $\frac{x^3}{3!} e^{2x} + \frac{1}{(\log 3 - 2)^3} 3^x$ (B) $\frac{x^3}{3!} e^{2x} + \frac{1}{(e^3 - 2)^3} 3^x$

(C) $\frac{x}{3!} e^{2x} + \frac{1}{(\log 3 - 2)^3} 3^x$ (D) $\frac{x^3}{3!} e^{2x} + \frac{1}{(\log 3 - 2)^3}$

25. Particular Integral of differential equation $(D^5 - D)\,y = 12e^x$ is .... (2)

(A) $3e^x$ (B) $\frac{12}{5} xe^x$ (C) $12xe^x$ (D) $3xe^x$

26. Particular Integral of differential equation $(D^2 + 1)(D - 1)\,y = e^x$ is .... (2)

(A) xe^x (B) $\frac{1}{2} x^2 e^x$ (C) $\frac{1}{2} xe^x$ (D) $x^2 e^x$

27. Particular Integral of differential equation $(D^2 - 4D + 4)\,y = \sin 2x$ is .... (2)

(A) $-\frac{\cos 2x}{8}$ (B) $\frac{\cos 2x}{8}$ (C) $\frac{\sin 2x}{8}$ (D) $x\frac{\cos 2x}{8}$

28. Particular Integral of differential equation $(D^3 + D)\,y = \cos x$ is .... (2)

(A) $-\frac{x}{2} \sin x$ (B) $\frac{x}{4} \cos x$ (C) $-\frac{1}{2} \cos x$ (D) $-\frac{x}{2} \cos x$

29. Particular Integral of differential equation $(D^2 + 1)\,y = \sin x$ is .... (2)

(A) $-\frac{x}{2} \cos x$ (B) $-\frac{x}{4} \cos x$ (C) $-\frac{x}{2} \sin x$ (D) $-\frac{1}{2} \cos x$

30. Particular Integral of differential equation $(D^3 + 9D)\,y = \sin 3x$ is .... (2)

(A) $-\frac{x}{18} \cos 3x$ (B) $-\frac{x}{18} \sin 3x$ (C) $-x \sin 3x$ (D) $-\frac{1}{18} \sin 3x$

31. Particular integral of differential equation $(D^4 + 10D^2 + 9)\, y = \sin 2x + \cos 4x$ is .... (2)

(A) $-\frac{1}{23}\sin 2x - \frac{1}{105}\cos 4x$
(B) $\frac{1}{15}\sin 2x + \cos 4x$
(C) $-\frac{1}{15}\sin 2x + \frac{1}{105}\cos 4x$
(D) $-\frac{1}{15}\sin 2x + \frac{1}{87}\cos 4x$

32. Particular Integral of differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10 \sin x$ is .... (2)

(A) $\frac{8}{3}\sin x$
(B) $\sin x - 2\cos x$
(C) $4\sin x + 2\cos x$
(D) $2\sin x + \cos x$

33. Particular Integral of differential equation $(D^4 - m^4)\, y = \cos mx$ is .... (2)

(A) $\frac{-x}{4m^3}\cos mx$
(B) $\frac{x}{m^3}\sin mx$
(C) $-x \sin mx$
(D) $\frac{-x}{4m^3}\sin mx$

34. Particular Integral of differential equation $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh 2x$ is .... (2)

(A) $\frac{1}{4}\cosh 2x$
(B) $\frac{x}{8}\cosh 2x$
(C) $\frac{x}{4}\cosh 2x$
(D) $\frac{x}{4}\sinh 2x$

35. Particular Integral of differential equation $(D^2 + 6D - 9)\, y = \sinh 3x$ is .... (2)

(A) $\frac{1}{18}\cosh 3x$
(B) $\frac{1}{2}\cosh 3x$
(C) $\frac{1}{18}\sinh 3x$
(D) $-\frac{1}{18}\cosh 3x$

36. Particular Integral of differential equation $\frac{d^3y}{dx^3} + 8y = x^4 + 2x + 1$ is .... (2)

(A) $\frac{1}{8}(x^4 + 5x + 1)$
(B) $\frac{1}{8}(x^3 - 3x^2 + 1)$
(C) $x^4 - x + 1$
(D) $\frac{1}{8}(x^4 - x + 1)$

37. Particular Integral of differential equation $(D^4 + D^2 + 1)\, y = 53x^2 + 17$ is .... (2)

(A) $53x^2 + 17$
(B) $53x^2 - 89$
(C) $53x^2 + 113$
(D) $3x^2 - 17$

38. Particular integral of differential equation $(D^2 - D + 1)\, y = 3x^2 - 1$ is .... (2)

(A) $3x^2 + 6x + 5$
(B) $x^2 - 6x + 1$
(C) $3x^2 + 6x - 1$
(D) $x^2 + 18x - 11$

39. Particular Integral of differential equation $(D^2 - 1)\, y = x^3$ is .... (2)

(A) $-x^3 + 6x$
(B) $x^2 + 6$
(C) $x^3 + 6x$
(D) $-x^3 - 6x$

40. Particular Integral of differential equation $(D^3 + 3D^2 - 4)\, y = x^2$ is.............. (2)

(A) $-\frac{1}{4}\left(x^2 + \frac{3}{2}\right)$
(B) $\frac{1}{4}\left(x^2 + \frac{3}{2}x\right)$
(C) $\left(x^2 + \frac{3}{2}\right)$
(D) $-\frac{1}{4}\left(x^2 - \frac{3}{2}\right)$

41. Particular Integral of differential equation $(D^4 + 25)\, y = x^4 + x^2 + 1$ is .... (2)

(A) $\left(x^4 + x^2 - \frac{1}{25}\right)$
(B) $\left(x^4 + x^2 + \frac{49}{25}\right)$
(C) $\frac{1}{25}(x^4 + x^2 + 24x + 1)$
(D) $\frac{1}{25}\left(x^4 + x^2 + \frac{1}{25}\right)$

42. Particular Integral of differential equation $(D^2 - 4D + 4)\, y = e^{2x} x^4$ is .... (2)

(A) $\frac{x^6}{120}e^{2x}$
(B) $\frac{x^6}{60}e^{2x}$
(C) $\frac{x^6}{30}e^{2x}$
(D) $\frac{x^5}{20}e^{2x}$

43. Particular Integral of differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} \cos x$ is .... (2)

(A) $e^x \cos x$ (B) $-e^{-x} \sin x$ (C) $-e^{-x} \cos x$ (D) $(c_1x + c_2) e^{-x}$

44. Particular integral of differential equation $(D^2 + 6D + 9) y = e^{-3x} x^{-3}$ is .... (2)

(A) $\frac{e^{-3x}}{2x}$ (B) $e^{-3x} x$ (C) $\frac{e^{-3x}}{12x}$ (D) $(c_1x + c_2) e^{-3x}$

45. Particular Integral of differential equation $(D^2 + 2D + 1) y = e^{-x} (1 + x^2)$ is .... (2)

(A) $e^{-x}\left(\frac{x^2}{2} - \frac{x^4}{12}\right)$ (B) $e^{-x}\left(x + \frac{x^3}{3}\right)$ (C) $e^{-x}\left(\frac{x^2}{2} + \frac{x^4}{12}\right)$ (D) $\left(\frac{x^2}{2} + \frac{x^4}{12}\right)$

46. Particular Integral of differential equation $(D - 1)^3 y = e^x \sqrt{x}$ is .... (2)

(A) $\frac{4}{15} e^x x^{5/2}$ (B) $\frac{8}{105} e^x x^{7/2}$ (C) $e^x x^{7/2}$ (D) $\frac{3}{8} e^x x^{-5/2}$

47. Particular integral of differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$ is .... (2)

(A) $-e^x (x \sin x + 2 \cos x)$ (B) $e^x (x \sin x - 2 \cos x)$

(C) $(x \sin x + 2 \cos x)$ (D) $-e^x (x \cos x + 2 \sin x)$

48. Solution of differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{2x}$ is .... (2)

(A) $e^x\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) - \frac{1}{7} e^{2x}$ (B) $e^{\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{5} e^{2x}$

(C) $e^{-\frac{1}{2}x}\left(c_1 \cos \frac{1}{2} x + c_2 \sin \frac{1}{2} x\right) + \frac{1}{7} e^{x}$ (D) $e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{7} e^{2x}$

49. Solution of differential equation $(D^2 + 1) y = x$ is .... (2)

(A) $c_1 \cos x + c_2 \sin x - x$ (B) $c_1 \cos x + c_2 \sin x + x$

(C) $c_1 \cos x + c_2 \sin x + 2x$ (D) $c_1 \cos x + c_2 \sin x - 2x$

Answers

1. (A)	2. (C)	3. (D)	4. (D)	5. (C)	6. (B)	7. (B)	8. (A)	9. (D)
10. (C)	11. (A)	12. (C)	13. (D)	14. (A)	15. (B)	16. (C)	17. (D)	18. (B)
19. (D)	20. (B)	21. (A)	22. (C)	23. (D)	24. (A)	25. (D)	26. (C)	27. (B)
28. (D)	29. (A)	30.(B)	31. (C)	32. (D)	33.(D)	34. (C)	35. (A)	36. (D)
37. (B)	38. (C)	39. (D)	40. (A)	41. (D)	42. (C)	43. (C)	44. (A)	45. (C)
46. (B)	47. (A)	48. (D)	49. (B)					

Type III : Cauchy's and Legendre's Linear Differential Equations : **Marks**

1. The general form of Cauchy's linear differential equation is .... (1)

(A) $a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = f(x)$, where, $a_0, a_1, a_2, \ldots, a_n$ are constants.

(B) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where, P, Q, R are functions of x, y, z.

(C) $a_0 x^n \frac{d^ny}{dx^n} + a_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}}{dx^{n-2}} + \ldots + a_ny = f(x)$, where $a_0, a_1, a_2 \ldots a_n$ are constants

(D) $a_0 (ax + b)^n \frac{d^ny}{dx^n} + a_1 (ax + b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 (ax + b)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x)$, where $a_0, a_1, a_2 \ldots, a_n$ are constant.

2. Cauchy's linear differential equation $a_0x^n \frac{d^ny}{dx^n} + a_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x)$ can be reduced to linear differential equation with constant coefficients by using substitution .... (1)

(A) $x = e^z$ (B) $y = e^z$

(C) $x = \log z$ (D) $x = e^{z^2}$

3. The general form of Legendre's linear differential equation is .... (1)

(A) $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = f(x)$, where, $a_0, a_1, a_2 \ldots, a_n$ are constant.

(B) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where, P, Q, R are functions of x, y, z.

(C) $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}}{dx^{n-2}} + \ldots + a_n y = f(x)$, where, $a_0, a_1, a_2 \ldots, a_n$ are constant

(D) $a_0 (ax + b)^n \frac{d^n y}{dx^n} + a_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 (ax + b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = f(x)$, where, $a_0, a_1, a_2, \ldots, a_n$ are constant.

4. Legendre's linear differential equation $a_0 (ax + b)^n \frac{d^n y}{dx^n} + a_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 (ax + b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = f(x)$ can be reduced to linear differential equation with constant coefficients by using substitution .... (1)

(A) $x = e^z$ (B) $ax + b = e^z$

(C) $ax + b = \log z$ (D) $ax + b = e^{z^2}$

5. To reduce the differential equation $x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^4$ to linear differential equation with constant coefficients, substitutions is .... (1)

(A) $x = z^2 + 1$ (B) $x = e^z$

(C) $x = \log z$ (D) $x^2 = \log z$

6. To reduce the differential equation $(x + 2)^2 \frac{d^2 y}{dx^2} - (x + 2) \frac{dy}{dx} + y = 4x + 7$ to linear differential equation with constant coefficients, substitution is .... (1)

(A) $x + 2 = e^{-z}$ (B) $x = z + 1$

(C) $x + 2 = e^z$ (D) $x + 2 = \log z$

7. To reduce the differential equation $(3x + 2)^2 \frac{d^2 y}{dx^2} + 3 (3x + 2) \frac{dy}{dx} - 36y = x^2 + 3x + 1$ to linear differential equation with constant coefficients, substitution is.............. (1)

(A) $3x + 2 = e^z$ (B) $3x + 2 = z$

(C) $x = e^z$ (D) $3x + 2 = \log z$

8. On putting $x = e^z$ and using $D \equiv \frac{d}{dz}$ the differential equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x$ is transformed into .... (1)

(A) $(D^2 - 1) y = e^z$ (B) $(D^2 + 1) y = e^z$

(C) $(D^2 + 1) y = x$ (D) $(D^2 + D + 1) y = e^z$

9. The differential equation $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 4y = \cos (\log x) + x \sin (\log x)$, on putting $x = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 - D + 4) y = \sin z + e^z \cos z$ (B) $(D^2 - 2D + 4) y = \cos (\log x) + x \sin (\log x)$

(C) $(D^2 + 2D + 4) y = \cos z + e^{-z} \sin z$ (D) $(D^2 - 2D + 4) y = \cos z + e^z \sin z$

10. On putting $x = e^z$ the transformed differential equation of $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin (\log x)$ using $D \equiv \frac{d}{dz}$ is .... (1)

(A) $(D^2 - 4D + 5) y = e^{2z} \sin z$ (B) $(D^2 - 4D + 5) y = x^2 \sin (\log x)$

(C) $(D^2 - 4D - 4) y = e^z \sin z$ (D) $(D^2 - 3D + 5) y = e^{z^2} \sin z$

11. The differential equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = \frac{x^3}{1 + x^2}$, on putting $x = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 - 1)\, y = \frac{x^3}{1 + x^2}$ (B) $(D^2 - 2D - 1)\, y = \frac{e^{3z}}{1 + e^{2z}}$

(C) $(D^2 - 1)\, y = \frac{e^{3z}}{1 + e^{2z}}$ (D) $(D^2 - 1)\, y = \frac{e^{z^3}}{1 + e^{z^2}}$

12. The differential equation $x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} + 5y = x^2 \log x$, on putting $x = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 - 5D + 5)\, y = z\, e^{z^2}$ (B) $(D^2 - 5D - 5)\, y = e^{2z}\, z$

(C) $(D^2 - 6D + 5)\, y = x^2 \log x$ (D) $(D^2 - 6D + 5)\, y = z\, e^{2z}$

13. The differential equation $(2x + 1)^2 \frac{d^2y}{dx^2} - 2\,(2x + 1) \frac{dy}{dx} - 12y = 6x$, on putting $2x + 1 = e^z$ and putting $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 - 2D - 3)\, y = \frac{3}{4}(e^z - 1)$ (B) $(D^2 + 2D + 3)\, y = 3\,(e^z - 1)$

(C) $(D^2 + 2D - 12)\, y = \frac{3}{4}(e^z - 1)$ (D) $(D^2 - 2D - 3)\, y = 6x$

14. The differential equation $(3x + 2)^2 \frac{d^2y}{dx^2} + 3(3x + 2) \frac{dy}{dx} - 36y = \frac{1}{3}[(3x + 2)^2 - 1]$. On putting $3x + 2 = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 + 3D - 36)\, y = \frac{1}{27}(e^{2z} - 1)$ (B) $(D^2 + 4)\, y = \frac{1}{9}(e^{2z} - 1)$

(C) $(D^2 - 4)\, y = \frac{1}{27}(e^{2z} - 1)$ (D) $(D^2 - 9)\, y = (e^{2z} - 1)$

15. The differential equation $(1 + x)^2 \frac{d^2y}{dx^2} + 3(1 + x) \frac{dy}{dx} - 36y = 4 \cos [\log (1 + x)]$ on putting $1 + x = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 + 2D - 36)\, y = 4 \cos [\log (1+ x)]$ (B) $(D^2 + 2D - 36)\, y = 4 \cos z$

(C) $(D^2 + 3D - 36)\, y = 4 \cos z$ (D) $(D^2 - 2D - 36)\, y = 4 \cos (\log z)$

16. The differential equation $(4x + 1)^2 \frac{d^2y}{dx^2} + 2\,(4x + 1) \frac{dy}{dx} + 2y = 2x + 1$ on putting $4x + 1 = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 + D + 2)\, y = \frac{1}{2}(e^z + 1)$ (B) $(16D^2 + 8D + 2)\, y = (e^z + 1)$

(C) $(16D^2 - 8D + 2)\, y = \frac{1}{2}(e^z + 1)$ (D) $(D^2 + 2D + 2)\, y = (e^z - 1)$

17. The differential equation $(x + 2)^2 \frac{d^2y}{dx^2} + 3\,(x + 2) \frac{dy}{dx} + y = 4 \sin [\log (x + 2)]$ on putting $x + 2 = e^z$ and using $D \equiv \frac{d}{dz}$ is transformed into .... (1)

(A) $(D^2 + 3D + 1)\, y = 4 \sin (\log z)$ (B) $(D^2 + 1)\, y = 4 \sin z$

(C) $(D^2 + 2D + 1)\, y = 4 \sin [\log (x + 2)]$ (D) $(D^2 + 2D + 1)\, y = 4 \sin z$

18. For the differential equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x^2 + x^{-2}$, complimentary function is given by .... (2)

(A) $c_1 x + c_2$ (B) $c_1 \log x + c_2$

(C) $c_1 \cos x + c_2 \sin x$ (D) $c_1 \cos (\log x) + c_2 \sin (\log x)$

19. For the differential equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = A + B \log x$, complimentary function is given by .... (2)

(A) $c_1 x + c_2$ (B) $c_1 x^2 + c_2$

(C) $c_1 \log x + c_2$ (D) $\frac{c_1}{x} + c_2$

20. For the differential equation $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$, complimentary function is given by .... (2)

(A) $c_1x^2 + c_2x^3$ (B) $c_1x^2 + c_2x$
(C) $c_1x^{-2} + c_2x^{-3}$ (D) $c_1x^5 + c_2x$

21. For the differential equation $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$, complimentary function is given by .... (2)

(A) $\left[c_1 \cos\sqrt{3}\,(\log x) + c_2 \sin\sqrt{3}\,(\log x)\right]$ (B) $x\left[c_1 \cos\sqrt{2}\,(\log x) + c_2 \sin\sqrt{2}\,(\log x)\right]$
(C) $x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right]$ (D) $x\left[c_1 \cos\sqrt{3}\,(\log x) + c_2 \sin\sqrt{3}\,(\log x)\right]$

22. For the differential equation $r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - u = -kr^3$, complimentary function is given by .... (2)

(A) $(c_1 \log r + c_2)\, r$ (B) $c_1r + \frac{c_2}{r}$
(C) $[c_1 \cos(\log r) + c_2 \sin(\log r)]$ (D) $c_1r^2 + \frac{c_2}{r^2}$

23. For the differential equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x$, particular integral is given by .... (2)

(A) x (B) $\frac{x}{2}$
(C) $\frac{x}{3}$ (D) $2x$

24. For the differential equation $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$, particular integral is given by .... (2)

(A) $\frac{x^5}{6}$ (B) $\frac{x^5}{56}$
(C) $\frac{x^4}{6}$ (D) $-\frac{x^5}{44}$

25. Solution of differential equation $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = x$ is .... (2)

(A) $(c_1x + c_2) - \frac{x^2}{4}$ (B) $(c_1x^2 + c_2) + \frac{x^2}{4}$
(C) $(c_1 \log x + c_2) - \frac{x^2}{4}$ (D) $(c_1 \log x + c_2) + \frac{x^2}{4}$

26. Solution of differential equation $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = \frac{1}{x^2}$ is .... (2)

(A) $(c_1x + c_2) - \frac{x^2}{4}$ (B) $(c_1x^2 + c_2) + \frac{x^2}{4}$
(C) $c_1 + c_2 \frac{1}{x} + \frac{1}{2x^2}$ (D) $(c_1 \log x + c_2) + \frac{x^2}{4}$

27. For the differential equation $(x + 1)^2 \frac{d^2y}{dx^2} + (x + 1) \frac{dy}{dx} + y = 2 \sin[\log(x + 1)]$, complimentary function is given by .... (2)

(A) $c_1 (x + 1) + c_2 (x + 1)^{-1}$ (B) $c_1 \cos[\log(x + 1)] + c_2 \sin[\log(x + 1)]$
(C) $[c_1 \log(x + 1) + c_2]\,(x + 1)$ (D) $c_1 \cos(\log x) + c_2 \sin(\log x)$

28. For the differential equation $(2x + 3)^2 \frac{d^2y}{dx^2} - 2(2x + 3) \frac{dy}{dx} - 12y = 6x$, complimentary function is given by .... (2)

(A) $c_1 (2x + 3)^3 + c_2 (2x + 3)^{-1}$ (B) $c_1 (2x + 3)^{-3} + c_2 (2x + 3)$
(C) $c_1 (2x + 3)^3 + c_2 (2x + 3)^2$ (D) $c_1 (2x - 3)^2 + c_2 (2x - 3)^{-1}$

29. For the differential equation $(3x + 2)^2 \frac{d^2y}{dx^2} + 3(3x + 2) \frac{dy}{dx} - 36y = (3x + 2)^2$, complimentary function is given by .... (2)

(A) $c_1 (3x + 2)^3 + c_2 (3x + 2)^{-3}$ (B) $[c_1 \log(3x + 2) + c_2]\,(3x + 2)^{-2}$
(C) $c_1 (3x + 2)^2 + c_2 (3x + 2)^{-2}$ (D) $c_1 (3x - 2)^2 + c_2 (3x - 2)^{-2}$

30. For the differential equation $(x + 2)^2 \frac{d^2y}{dx^2} - (x + 2) \frac{dy}{dx} + y = (3x + 6)$, complimentary function is given by .... (2)

(A) $c_1 (x + 2) + c_2 (x + 2)^{-1}$ (B) $c_1 \log (x + 2) + c_2$

(C) $c_1 (x - 2) + c_2 (x - 2)^{-1}$ (D) $[c_1 \log (x + 2) + c_2] (x + 2)$

Answers

1. (C)	2. (A)	3. (D)	4. (B)	5. (B)	6. (C)	7. (A)	8. (B)
9. (D)	10. (A)	11. (C)	12. (D)	13. (A)	14. (C)	15. (B)	16. (C)
17. (D)	18. (D)	19. (C)	20. (A)	21. (D)	22. (B)	23. (B)	24. (A)
25. (D)	26. (C)	27. (B)	28. (A)	29. (C)	30. (D)		

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Ch. 2 Simultaneous Linear Differential Equations Symmetrical Simultaneous D.E.

Type I : Simultaneous Linear Differential Equations : **Marks**

1. For the simultaneous linear differential equations

$\frac{dx}{dt} + 2x - 3y = t, \frac{dy}{dx} - 3x + 2y = e^{2t}$ solution of x using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(D^2 + 4D - 5)\, x = 1 + 2t + 3e^{2t}$ (B) $(D^2 - 4D - 5)\, x = 1 + 2t - 3e^{2t}$

(C) $(D^2 + 4D - 5)\, x = 3t + 3e^{2t}$ (D) $(D^2 + 4D - 5)\, y = 3t + 4e^{2t}$

2. For the system of linear differential equations $\frac{dx}{dt} + 2x - 3y = t, \frac{dy}{dt} - 3x + 2y = e^{2t}$ elimination of x results in $\left(\text{use } D \equiv \frac{d}{dt}\right)$ .... (2)

(A) $(D^2 + 4D - 5)\, x = 1 + 2t + 3e^{2t}$ (B) $(D^2 - 4D - 5)\, y = t - 4e^{2t}$

(C) $(D^2 - 4D + 5)\, y = 3t - 2e^{2t}$ (D) $(D^2 + 4D - 5)\, y = 3t + 4e^{2t}$

3. For the simultaneous Linear DE $\frac{du}{dx} + v = \sin x, \frac{dv}{dx} + u = \cos x$ solution of u using $D \equiv \frac{d}{dx}$ is obtain from .... (2)

(A) $(D^2 + 1)\, u = 2 \cos x$ (B) $(D^2 - 1)\, u = 0$

(C) $(D^2 - 1)\, u = \sin x - \cos x$ (D) $(D^2 - 1)\, v = -2 \sin x$

4. For the simultaneous Linear DE $\frac{du}{dx} + v = \sin x, \frac{dv}{dx} + u = \cos x$ eliminating u results in $\left(\text{use } D \equiv \frac{d}{dx}\right)$ .... (2)

(A) $(D^2 + 1)\, v = 0$ (B) $(D^2 - 1)\, u = 0$

(C) $(D^2 - 1)\, v = -2 \sin x$ (D) $(D^2 + 1)\, v = \sin x + \cos x$

5. For the simultaneous Linear DE $\frac{dx}{dt} - 3x - 6y = t^2, \frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t$ solution of x using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(D^2 + 9)\, x = 6e^t - 3t^2 + 2t$ (B) $(D^2 + 9)\, y = -2e^t - 2t$

(C) $(D^2 - 9)\, x = 6e^t - 3t^2$ (D) $(D^2 + 12D + 9)\, x = 6e^t + 3t^2 + 2t$

6. For the simultaneous Linear DE $L \frac{dx}{dt} + Rx + R(x - y) = E, L \frac{dy}{dt} + Ry - R(x - y) = 0$ where L, R and E are constants, solution of x using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(L^2D^2 + 4RLD + 5R^2)\, x = 2RE + 2R$ (B) $(L^2D^2 + 4RLD + 3R^2)\, y = RE$

(C) $(L^2D^2 + 4RLD + 3R^2)\, x = 2RE$ (D) $(L^2D^2 + 2RLD + 5R^2)\, x = 2RE$

7. For the simultaneous Liner DE $L\frac{dx}{dt} + Rx + R(x - y) = E$, $L\frac{dy}{dt} + Ry - R(x - y) = 0$ where L, R and E are constants, solution of y using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(L^2D^2 + 4RLD + 5R^2)\, y = RE + 2R$
(B) $(L^2D^2 + 4RLD + 3R^2)\, y = RE$
(C) $(L^2D^2 + 4RLD + 3R^2)\, x = 2RE$
(D) $(L^2D^2 + 2RLD + 5R^2)\, y = 2RE$

8. For the simultaneous Linear DE $\frac{dx}{dt} + y = e^t$, $\frac{dy}{dt} + x = e^{-t}$ solution of x using $D = \frac{d}{dt}$ is obtain from .... (2)

(A) $(D^2 - 1)\, x = 2e^t$
(B) $(D^2 - 1)\, y = - e^t - e^{-t}$
(C) $(D^2 + 1)\, x = e^{-t} + e^t$
(D) $(D^2 - 1)\, x = e^t - e^{-t}$

9. From the simultaneous Linear DE $\frac{dx}{dt} + y = e^t$, $\frac{dy}{dt} + x = e^{-t}$, solution of y using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(D^2 - 1)\, y = 2e^t$
(B) $(D^2 - 1)\, y = -e^t - e^{-t}$
(C) $(D^2 + 1)\, y = e^{-t} + e^t$
(D) $(D^2 - 1)\, x = e^t - e^{-t}$

10. For the simultaneous Linear DE $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, solution of x using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(D^2 + 6D + 9)\, x = 1 + t$
(B) $(D^2 - 6D + 9)\, x = 2t$
(C) $(D^2 + 6D + 1)\, x = t$
(D) $(D^2 + 6D + 9)\, y = 2t$

11. For the simultaneous Linear DE $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, solution of y using $D \equiv \frac{d}{dt}$ is obtain from .... (2)

(A) $(D^2 - 6D - 9)\, y = 2t$
(B) $(D^2 + 6D + 9)\, x = 1 + t$
(C) $(D^2 + 6D + 1)\, y = t$
(D) $(D^2 + 6D + 9)\, y = - 2t$

Answers

1. (A)	2. (D)	3. (B)	4. (C)	5. (A)	6. (C)	7. (B)	8. (D)
9. (B)	10. (A)	11. (D)					

Type II : Symmetrical Simultaneous Differential Equations : **Marks**

1. The general form of symmetric simultaneous DE is .... (1)

(A) $a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x)$, where, $a_0, a_1, a_2 \ldots, a_n$ are constant

(B) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where, P, Q, R are function of x, y, z

(C) $a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x)$, where, $a_0, a_1, a_2 \ldots, a_n$ are constant

(D) $a_0(ax + b)^n\frac{d^ny}{dx^n} + a_1(ax + b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(ax + b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = f(x)$, where, $a_0, a_1, a_2 \ldots, a_n$ are constant

2. Solution of symmetric simultaneous DE $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$ is .... (1)

(A) $x + y = 0, y + z = 0$
(B) $x - y = c_1, y + z = c_2$
(C) $x + y = c_1, y - z = c_2$
(D) $x - z = c_1, y - z = c_2$

3. Solution of symmetric simultaneous DE $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ is .... (1)

(A) $x = c_1y, y = c_2z$
(B) $xy = c_1z, yz = c_2x$
(C) $x + y = c_1, y + z = c_2$
(D) $x + y = c_1, y - z = c_2$

4. Considering the first two ratio of the symmetrical simultaneous DE $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$, one of the relation in the solution is DE is .... (1)

(A) $\frac{1}{x} - \frac{1}{y} = c$ (B) $x - y = c$ (C) $x^2 - y^2 = c$ (D) $x^3 - y^3 = c$

5. Considering the first two ratio of the symmetrical simultaneous DE $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}$, one of the relation in the solution of DE is .... (2)

(A) $x^2 + y^2 = c$ (B) $x^3 + y^3 = c$ (C) $-\frac{x^2}{2} = \frac{y^3}{3} + c$ (D) $x^2 - y^2 = c$

6. Considering the first two ratio of the symmetrical simultaneous DE $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$, one of the relation in the solution of DE is .... (2)

(A) $x^2 - y^2 = c$ (B) $x - y = c$ (C) $x^3 - y^3 = c$ (D) $x^3 + y^3 = c$

7. Considering the first and third ratio of the symmetrical simultaneous DE $\frac{xdx}{y^3z} = \frac{dy}{x^2z} = \frac{dz}{y^3}$, one of the relation in the solution of DE is .... (2)

(A) $x^2 - z^2 = c$ (B) $x^4 - y^4 = c$ (C) $x^3 - z^3 = c$ (D) $x - z = c$

8. Considering the second and third ratio of the symmetrical simultaneous DE $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$, one of the relation in the solution of DE is .... (2)

(A) $\frac{1}{y^2} - \frac{1}{z^2} = c$ (B) $y^2 - z^2 = c$ (C) $y = cz$ (D) $x - z = c$

9. Using a set of multiplier as 1, 1, 1 the solution of DE $\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}$ is .... (2)

(A) $x^2 + y^2 + z^2 = c$ (B) $x - y - z = c$
(C) $x + y + z = c$ (D) $-x + y - z = c$

10. Using a set of multiplier as x, y, z the solution of DE $\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$ is .... (2)

(A) $x^3 + y^3 + z^3 = c$ (B) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c$
(C) $x + y + z = c$ (D) $x^2 + y^2 + z^2 = c$

11. Using a set of multiplier as x^3, y^3, z^3 the solution of DE $\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$ is .... (2)

(A) $x^3 + y^3 + z^3 = c$ (B) $x^4 + y^4 + z^4 = c$
(C) $x + y + z = c$ (D) $xyz = c$

12. Using a set of multiplier as 3, 2, 1 the solution of DE $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x - 3y}$ is .... (2)

(A) $3x^2 + 2y^2 + z^2 = c$ (B) $\frac{3}{x} + \frac{2}{y} + \frac{1}{z} = c$
(C) $3x - 2y - z = c$ (D) $3x + 2y + z = c$

13. Using a set of multiplier as 1, y, z the solution of DE $\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{y + z} = \frac{dz}{y - z}$ is .... (2)

(A) $x^2 + y^2 + z^2 = c$ (B) $x + \frac{y^2}{2} + \frac{z^2}{2} = c$
(C) $x + y + z = c$ (D) $x + y^2 + z^2 = c$

Answers

1. (B)	2. (D)	3. (A)	4. (D)	5. (A)	6. (C)	7. (A)	8. (C)
9. (C)	10. (D)	11. (B)	12. (D)	13. (B)			

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Ch. 3 Fourier Transform

Type I : Fourier Integral Representation Fourier Transform and Inverse Fourier Transform **Marks**

1. The fourier integral representation of f(x) defined in the interval $-\infty < x < \infty$ is .... (1)

(A) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$

(B) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$

(C) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\lambda u}\, du\, dx$

(D) $\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\lambda(u-x)}\, du\, d\lambda$

2. The Fourier transform F(λ) of function f(x) defined in the interval $-\infty < x < \infty$ is .... (1)

(A) $\int_{-\infty}^{\infty} f(u)\, e^{iu}\, du$

(B) $\int_{-\infty}^{\infty} f(u)\, e^{-\lambda u}\, du$

(C) $\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$

(D) $\int_{0}^{\infty} f(u)\, e^{-i\lambda u}\, du$

3. The inverse Fourier transform f(x) defined in $-\propto < x < \infty$ of F(λ) is .... (1)

(A) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$

(B) $\frac{2}{\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}\, d\lambda$

(C) $\frac{1}{2\pi}\int_{-\infty}^{0} F(\lambda)\, e^{ix}\, d\lambda$

(D) $\frac{1}{2\pi}\int_{0}^{\infty} F(\lambda)\, e^{i\lambda x}\, dx$

4. In the Fourier integral representation of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-i\lambda}{1+\lambda^2}\right) e^{i\lambda x}\, d\lambda = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$, F(λ) is .... (1)

(A) $\frac{1+\lambda^2}{1-i\lambda}$ (B) $\frac{\sin\lambda}{1+\lambda^2}$ (C) $\frac{\cos\lambda}{1+\lambda^2}$ (D) $\frac{1-i\lambda}{1+\lambda^2}$

5. In the Fourier integral representation of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{-i\lambda\pi}+1}{1-\lambda^2}\right) e^{i\lambda x}\, d\lambda = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x < 0 \text{ and } x > \pi \end{cases}$, F(λ) is .... (1)

(A) $\frac{1+\lambda^2}{1-i\lambda}$ (B) $\frac{e^{-i\lambda}}{1-\lambda^2}$ (C) $\frac{e^{-i\lambda\pi}+1}{1-\lambda^2}$ (D) $\frac{\sin\lambda}{1-\lambda^2}$

6. In the Fourier integral representation $\frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\left(\frac{1-i\lambda}{1+\lambda^2}\right) e^{i\lambda x}\, d\lambda = \begin{cases} 0, & x < 0 \\ e^{-\pi}, & x > 0 \end{cases}$, F(λ) is .... (2)

(A) $\frac{1+\lambda^2}{1-i\lambda}$ (B) $\frac{\sin\lambda}{1+\lambda^2}$ (C) $\frac{\cos\lambda}{1+\lambda^2}$ (D) $\pi\frac{1-i\lambda}{1+\lambda^2}$

7. The Fourier transform F(λ) of $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) iλ (B) $\frac{1}{i\lambda}$ (C) $\frac{1}{\lambda}$ (D) λ

8. The Fourier transform F(λ) of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ is .... (2)

(A) $\dfrac{2\sin\lambda a}{\lambda}$ (B) $\dfrac{e^{-i\lambda a}}{\lambda}$ (C) $\dfrac{e^{i\lambda a}}{\lambda}$ (D) $\dfrac{2\cos\lambda a}{\lambda}$

9. The Fourier transform F(λ) of $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) $\dfrac{1-\lambda}{1+\lambda^2}$ (B) $\dfrac{1-i\lambda}{1+\lambda^2}$ (C) $\dfrac{1-i\lambda}{1-\lambda^2}$ (D) $\dfrac{1}{1+\lambda^2}$

10. The Fourier transform F(λ) of $f(x) = e^{-|x|}$ is given by .... (2)

(A) $\dfrac{3}{1+\lambda^2}$ (B) $\dfrac{1}{1-\lambda^2}$ (C) $\dfrac{2}{1-\lambda^2}$ (D) $\dfrac{2}{1+\lambda^2}$

11. If $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x < 0 \text{ and } x > \pi \end{cases}$ then Fourier transform F(λ) of f(x) is .... (2)

(A) $\dfrac{e^{i\lambda\pi}+1}{1+\lambda^2}$ (B) $\dfrac{e^{i\lambda\pi}+1}{1-\lambda^2}$ (C) $\dfrac{e^{-i\lambda\pi}+1}{1-\lambda^2}$ (D) $\dfrac{e^{-i\lambda\pi}+1}{1+\lambda^2}$

12. The Fourier transform F(λ) of $f(x) = \begin{cases} \cos x, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) $\dfrac{i\lambda}{1-\lambda^2}$ (B) $-\dfrac{i\lambda}{1-\lambda^2}$ (C) $-\dfrac{i\lambda}{1+\lambda^2}$ (D) $\dfrac{i\lambda}{1+\lambda^2}$

13. The Fourier transform F(λ) of $f(x) = \begin{cases} \sin x, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) $\dfrac{1}{1-\lambda^2}$ (B) $\dfrac{1}{1+\lambda^2}$ (C) $\dfrac{i\lambda}{1-\lambda^2}$ (D) $\dfrac{i\lambda}{1+\lambda^2}$

14. The Fourier transform F(λ) of $f(x) = \begin{cases} x, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) 0 (B) $\dfrac{1}{\lambda^2}$ (C) λ^2 (D) $-\dfrac{1}{\lambda^2}$

15. If $f(x) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ then Fourier transform F(λ) of f(x) is given by .... (2)

(A) $\dfrac{4\cos\lambda}{\lambda^2}$ (B) $\dfrac{4\sin\lambda}{\lambda}$ (C) $\dfrac{2\sin 2\lambda}{\lambda}$ (D) $\dfrac{\sin\lambda}{\lambda}$

16. The Fourier transform F(λ) of $f(x) = \begin{cases} x^2, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) $-\dfrac{2i}{\lambda^3}$ (B) $\dfrac{1}{i\lambda^3}$ (C) $\dfrac{2i}{\lambda^3}$ (D) $-\dfrac{1}{i\lambda^3}$

17. The Fourier transform F(λ) of $f(x) = \begin{cases} x - x^2, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) $\dfrac{2}{\lambda^2} + i\dfrac{1}{\lambda^3}$ (B) $\dfrac{1}{\lambda^2} - i\dfrac{2}{\lambda^3}$ (C) $\dfrac{1}{\lambda^2} + i\dfrac{2}{\lambda^3}$ (D) $-\dfrac{1}{\lambda^2} - i\dfrac{2}{\lambda^3}$

18. The Fourier transform F(λ) of $f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ is .... (2)

(A) $-\dfrac{4}{\lambda^3}(\sin\lambda - \lambda\cos\lambda)$ (B) $\dfrac{4}{\lambda^3}(\sin\lambda - \lambda\cos\lambda)$

(C) $\dfrac{4}{\lambda^2}(\sin\lambda - \lambda\cos\lambda)$ (D) $\dfrac{4}{\lambda^3}(\sin\lambda + \lambda\cos\lambda)$

19. The Fourier transform F(λ) of $f(x) = \begin{cases} 2 + x, & x > 0 \\ 0, & x < 0 \end{cases}$ is .... (2)

(A) $-\dfrac{1}{\lambda^2} - i\dfrac{2}{\lambda}$ (B) $\dfrac{1}{\lambda^2} - i\dfrac{2}{\lambda}$ (C) $\dfrac{1}{\lambda^2} + i\dfrac{2}{\lambda}$ (D) $-\dfrac{1}{\lambda^2} + i\dfrac{2}{\lambda}$

20. The inverse Fourier transform, f(x) defined in $-\infty < x < \infty$ of $F(\lambda) = \left[\frac{1 - i\lambda}{1 + \lambda^2}\right]$ is .... (2)

(A) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 + \lambda^2}\right] d\lambda$

(B) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x - \lambda \sin \lambda x}{1 + \lambda^2} + i \frac{-\lambda \cos \lambda x - \sin \lambda x}{1 + \lambda^2}\right] d\lambda$

(C) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1 + \lambda^2} + i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 + \lambda^2}\right] d\lambda$

(D) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1 - \lambda^2} + i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 - \lambda^2}\right] d\lambda$

21. The inverse Fourier transform f(x) defined in $-\infty < x < \infty$ of $F(\lambda) = \pi \left[\frac{1 - i\lambda}{1 + \lambda^2}\right]$ is .... (2)

(A) $\frac{1}{2} \int_{0}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1 + \lambda^2} + i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 + \lambda^2}\right] d\lambda$

(B) $\frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1 + \lambda^2} + i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 + \lambda^2}\right] d\lambda$

(C) $\frac{1}{2} \int_{-\infty}^{\infty} \left[i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 + \lambda^2}\right] d\lambda$

(D) $\frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1 - \lambda^2} + i \frac{-\lambda \cos \lambda x + \sin \lambda x}{1 - \lambda^2}\right] d\lambda$

22. The inverse Fourier transform f(x) defined in $-\infty < x < \infty$ of $F(\lambda) = \frac{e^{-i\lambda\pi} + 1}{1 - \lambda^2}$ is .... (2)

(A) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1 + \cos \lambda x}{1 - \lambda^2}\right] (\cos \lambda x + i \sin \lambda x)\, d\lambda$

(B) $\frac{1}{2\pi} \int_{0}^{\infty} \left[\frac{(1 + \cos \lambda x) - i \sin \lambda\pi}{1 - \lambda^2}\right] (\cos \lambda x + i \sin \lambda x)\, d\lambda$

(C) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{(1 + \cos \lambda\pi) - i \sin \lambda\pi}{1 - \lambda^2}\right] (\cos \lambda x + i \sin \lambda x)\, d\lambda$

(D) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\sin \lambda\pi}{1 - \lambda^2}\right] (\cos \lambda x + i \sin \lambda x)\, d\lambda$

23. If the Fourier integral representation of f(x) is $\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ then value of integral $\int_{0}^{\infty} \frac{\sin \lambda}{\lambda} d\lambda$ is .... (2)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) 0 (D) 1

24. If the Fourier integral representation of f(x) is

$\frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \lambda x + \cos [\lambda(\pi - x)]}{1 - \lambda^2} d\lambda = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x < 0 \text{ and } x > \pi \end{cases}$ then value of the integral $\int_{0}^{\infty} \frac{\cos \frac{\lambda\pi}{2}}{1 - \lambda^2} d\lambda$ is .... (2)

(A) $\frac{\pi}{4}$ (B) 1 (C) 0 (D) $\frac{\pi}{2}$

Answers

1. (A)	2. (C)	3. (A)	4. (D)	5. (C)	6. (D)	7. (B)	8. (A)
9. (B)	10. (D)	11. (C)	12. (A)	13. (A)	14. (D)	15. (B)	16. (C)
17. (D)	18. (B)	19. (A)	20. (C)	21. (B)	22. (C)	23. (B)	24. (D)

Type II : Fourier Sine and Cosine Integral Representations, Transform and Inverse Transform **Marks**

1. The Fourier consine integral representation of an even function f(x) defined in the interval $-\infty < x < \infty$ is .... (1)

(A) $\int_{0}^{\infty} \int_{0}^{\infty} f(u) \cos \lambda u \sin \lambda x \, du \, d\lambda$

(B) $\frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \cos \lambda u \cos \lambda x \, du \, d\lambda$

(C) $\frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \sin \lambda u \cos \lambda x \, du \, d\lambda$

(D) $\frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \sin \lambda u \sin \lambda x \, du \, d\lambda$

2. The Fourier sine integral representation of an odd function f(x) defined in the interval $-\infty < x < \infty$ is .... (1)

(A) $\int_0^\infty \int_0^\infty f(u) \sin \lambda u \cos \lambda x \, du \, d\lambda$

(B) $\int_0^\infty \int_0^\infty f(u) \cos \lambda u \sin \lambda x \, du \, d\lambda$

(C) $\frac{2}{\pi} \int_0^\infty \int_0^\infty f(u) \cos \lambda u \cos \lambda x \, du \, d\lambda$

(D) $\frac{2}{\pi} \int_0^\infty \int_0^\infty f(u) \sin \lambda u \sin \lambda x \, du \, d\lambda$

3. The Fourier cosine transform $F_c(\lambda)$ of an even function f(x) defined in the interval $-\infty < x < \infty$ is .... (1)

(A) $\int_0^\infty f(u) \sec \lambda u \, du$

(B) $\int_0^\infty f(u) \cos \lambda u \, d\lambda$

(C) $\int_0^\infty f(u) \cos \lambda u \, du$

(D) $\int_0^\infty f(u) \sin\lambda u \, du$

4. The Fourier sine transform $F_s(\lambda)$ of an odd function f(x) defined in the interval $-\infty < x < \infty$ is .... (1)

(A) $\int_0^\infty f(u) \sin \lambda u \, du$

(B) $\int_0^\infty f(u) \operatorname{cosec} \lambda u \, du$

(C) $\int_0^\infty f(u) \sin \lambda u \, d\lambda$

(D) $\int_0^\infty f(u) \cos \lambda u \, du$

5. The inverse Fourier cosine transform f(x) of $F_c(\lambda)$ is .... (1)

(A) $\int_0^\infty F_c(\lambda) \sin \lambda x \, d\lambda$

(B) $\frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, dx$

(C) $\int_0^\infty F_c(\lambda) \sec \lambda x \, d\lambda$

(D) $\frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda$

6. The inverse Fourier sine transform f(x) of $F_s(\lambda)$ is .... (1)

(A) $\frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$

(B) $\frac{2}{\pi} \int_0^\infty F_s(\lambda) \cos \lambda x \, d\lambda$

(C) $\frac{2}{\pi} \int_0^\infty F_s(\lambda) \operatorname{cosec} \lambda x \, d\lambda$

(D) $\int_0^\infty F_s(\lambda) \sin \lambda x \, dx$

7. For the Fourier sine integral representation $e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^3}{\lambda^4 + 4} \sin \lambda x \, d\lambda$, $F_s(\lambda)$ is .... (1)

(A) $\frac{\lambda}{\lambda^4 + 4}$ (B) $\frac{\lambda^3}{\lambda^4 + 4}$ (C) $\frac{\lambda^4 + 4}{\lambda^3}$ (D) $\frac{1}{\lambda^4 + 4}$

8. For the Fourier cosine integral representation $\frac{2}{\pi} \int_0^\infty \frac{\cos \frac{\pi\lambda}{2}}{1 - \lambda^2} \cos \lambda x \, d\lambda = \begin{cases} \cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$, then Fourier cosine transform $F_c(\lambda)$ is .... (1)

(A) $\frac{1 - \lambda^2}{\cos \frac{\pi\lambda}{2}}$ (B) $\frac{\sin \frac{\pi\lambda}{2}}{1 - \lambda^2}$ (C) $\frac{\cos \frac{\pi\lambda}{2}}{1 - \lambda^2}$ (D) $\frac{\cos \frac{\pi\lambda}{2}}{1 + \lambda^2}$

9. For the Fourier sine integral representation $\frac{2}{\pi}\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda = \begin{cases} 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$, $F_s(\lambda)$ is .... (1)

(A) $\frac{1-\cos \pi\lambda}{\lambda^2}$ (B) $\frac{\lambda}{1-\cos \pi\lambda}$ (C) $\frac{1-\sin \pi\lambda}{\lambda}$ (D) $\frac{1-\cos \pi\lambda}{\lambda}$

10. For the Fourier sine integral representation $\frac{2}{\pi}\int_0^\infty \frac{\sin \pi\lambda}{1-\lambda^2} \sin \lambda x \, d\lambda = \begin{cases} \sin x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$, $F_s(\lambda)$ is .... (1)

(A) $\frac{\sin \pi\lambda}{1-\lambda^2}$ (B) $\frac{1-\cos \pi\lambda}{1-\lambda^2}$ (C) $\frac{\sin \pi\lambda}{1+\lambda^2}$ (D) $\frac{1-\lambda^2}{\sin \lambda\pi}$

11. For the Fourier sine integral representation $\frac{6}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+1)(\lambda^2+4)} d\lambda = e^{-x} - e^{-2x}$, $x > 0$, $F_s(\lambda)$ is .... (1)

(A) $\frac{(\lambda^2+1)(\lambda^2+4)}{3\lambda}$ (B) $\frac{\lambda}{(\lambda^2+1)(\lambda^2+4)}$

(C) $\frac{3\lambda}{(\lambda^2+1)(\lambda^2+4)}$ (D) $\frac{\lambda \sin \lambda x}{(\lambda^2+1)(\lambda^2+4)}$

12. For the Fourier sine integral representation $\frac{2}{\pi}\int_0^\infty \frac{2\lambda \sin \lambda x}{\lambda^4+4} d\lambda = e^{-x} \sin x$, $x > 0$, $F_s(\lambda)$ is .... (1)

(A) $\frac{\lambda^4+4}{2\lambda \sin \lambda x}$ (B) $\frac{2\lambda}{\lambda^4+4}$ (C) $\frac{2\lambda \sin \lambda x}{\lambda^4+4}$ (D) $\frac{2\lambda \cos \lambda x}{\lambda^4+4}$

13. For the Fourier sine integral representation $\frac{12}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+4)(\lambda^2+16)} d\lambda = e^{-3x} \sinh x$, $x > 0$, $F_s(\lambda)$ is .... (1)

(A) $\frac{6\lambda}{(\lambda^2+4)(\lambda^2+16)}$ (B) $\frac{\lambda}{(\lambda^2+4)(\lambda^2+16)}$ (C) $\frac{6 \cos \lambda x}{(\lambda^2+4)(\lambda^2+16)}$ (D) $\frac{1}{(\lambda^2+4)(\lambda^2+16)}$

14. For the Fourier cosine integral representation $\frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \pi\lambda}{1-\lambda^2} \cos \lambda x \, d\lambda = \begin{cases} \cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$, $F_c(\lambda)$ is .... (1)

(A) $\frac{\sin \pi\lambda}{1-\lambda^2}$ (B) $\frac{\lambda \sin \pi\lambda}{1-\lambda^2}$ (C) $\frac{\lambda \cos \pi\lambda}{1-\lambda^2}$ (D) $\frac{1-\lambda^2}{\sin \lambda\pi}$

15. For the Fourier cosine integral representation $\frac{20}{\pi}\int_0^\infty \left(\frac{1}{\lambda^2+5} + \frac{1}{\lambda^2+4}\right) \cos \lambda x \, d\lambda = 2e^{-5x} + 5e^{-2x}$, $F_c(\lambda)$ is .... (1)

(A) $2e^{-5\lambda} + 5e^{-2\lambda}$ (B) $\left(\frac{1}{\lambda^2+5} + \frac{1}{\lambda^2+4}\right) \cos \lambda x$

(C) $\left(\frac{1}{\lambda^2+5} + \frac{1}{\lambda^2+4}\right)$ (D) $10\left(\frac{1}{\lambda^2+5} + \frac{1}{\lambda^2+4}\right)$

16. For the Fourier sine transform of $f(x) = e^{-mx}$, $m > 0$, $x > 0$ is $F_s(\lambda) = \frac{\lambda}{\lambda^2+m^2}$ then its inverse Fourier sine transform is .... (1)

(A) $\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin \lambda x \, dm$ (B) $\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin \lambda x \, dx$

(C) $\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \cos \lambda x \, d\lambda$ (D) $\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin \lambda x \, d\lambda$

17. If the Fourier cosine integral representation of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ is

$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ then the value of integral $\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$ is equal to .... (1)

(A) $\frac{\pi}{2}$ (B) $\frac{2}{\pi}$ (C) 1 (D) 0

18. The Fourier sine transform $F_s(\lambda)$ of $f(x) = \begin{cases}\pi/2, & 0 < x < \pi\\ 0, & x > \pi\end{cases}$ is .... (2)

(A) $\frac{\pi}{2}\left(\frac{1-\sin\lambda\pi}{\lambda}\right)$ (B) $\frac{\pi}{2}\left(\frac{\cos\lambda\pi-1}{\lambda}\right)$ (C) $\frac{\pi}{2}\left(\frac{1-\cos\lambda\pi}{\lambda}\right)$ (D) $\left(\frac{\cos\lambda\pi}{\lambda}\right)$

19. The Fourier sine transform $F_s(\lambda)$ of $f(x) = \begin{cases}1, & 0 \le x \le 1\\ 0, & x > 1\end{cases}$ is .... (2)

(A) $\left(\frac{\cos\lambda\pi-1}{\lambda}\right)$ (B) $\left(\frac{1-\cos\lambda}{\lambda}\right)$ (C) $\left(\frac{1-\sin\lambda}{\lambda}\right)$ (D) $\left(\frac{\cos\lambda\pi}{\lambda}\right)$

20. If $f(x) = \begin{cases}x, & 0 < x < 1\\ 0, & x > 1\end{cases}$ then Fourier cosine transform $F_c(\lambda)$ of f(x) is given by .... (2)

(A) $\frac{\lambda\sin\lambda+\cos\lambda-1}{\lambda^2}$ (B) $\frac{\cos\lambda-\lambda\sin\lambda-1}{\lambda^2}$

(C) $\frac{\cos\lambda-\lambda\sin\lambda+1}{\lambda^2}$ (D) $\frac{\lambda\sin\lambda+1}{\lambda^2}$

21. If $f(x) = \begin{cases}x, & 0 < x < 1\\ 0, & x > 1\end{cases}$ then Fourier sine transform $F_s(\lambda)$ of f(x) is given by .... (2)

(A) $\frac{\lambda\cos\lambda+\sin\lambda}{\lambda^2}$ (B) $\frac{-\lambda\cos\lambda-\sin\lambda}{\lambda^2}$

(C) $\frac{-\lambda\cos\lambda+\sin\lambda}{\lambda^2}$ (D) $\frac{\cos\lambda}{\lambda^2}$

22. If $f(x) = \begin{cases}x^2, & 0 < x < 1\\ 0, & x > 1\end{cases}$ then Fourier cosine transform $F_c(\lambda)$ of f(x) is given by .... (2)

(A) $\frac{-\lambda^2\sin\lambda+2\lambda\cos\lambda-2\sin\lambda}{\lambda^3}$ (B) $\frac{\lambda^2\sin\lambda-2\lambda\cos\lambda-2\sin\lambda}{\lambda^3}$

(C) $\frac{\lambda^2\sin\lambda-2\lambda\cos\lambda+2\sin\lambda}{\lambda^3}$ (D) $\frac{\lambda^2\sin\lambda+2\lambda\cos\lambda-2\sin\lambda}{\lambda^3}$

23. If $f(x) = \begin{cases}x^2, & 0 < x < 1\\ 0, & x > 1\end{cases}$ then Fourier sine transform $F_s(\lambda)$ of f(x) is given by .... (2)

(A) $\frac{-\lambda^2\cos\lambda+2\lambda\sin\lambda+2(\cos\lambda-1)}{\lambda^3}$ (B) $\frac{\lambda^2\cos\lambda+2\lambda\sin\lambda+2(\cos\lambda-1)}{\lambda^3}$

(C) $\frac{\lambda^2\cos\lambda-2\lambda\sin\lambda+2(\cos\lambda-1)}{\lambda^3}$ (D) $\frac{\lambda^2\cos\lambda-2\lambda\sin\lambda-2(\cos\lambda-1)}{\lambda^3}$

24. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = \begin{cases}1-x^2, & |x| \le 1\\ 0, & |x| > 1\end{cases}$ is .... (2)

(A) $-\frac{2}{\lambda^3}(\sin\lambda-\lambda\cos\lambda)$ (B) $\frac{2}{\lambda^3}(\sin\lambda-\lambda\cos\lambda)$

(C) $\frac{2}{\lambda^2}(\sin\lambda-\lambda\cos\lambda)$ (D) $\frac{2}{\lambda^3}(\sin\lambda+\lambda\cos\lambda)$

25. The Fourier cosine transform $f_c(\lambda)$ of $f(x) = \begin{cases}\pi/2, & 0 < x < \pi\\ 0, & x > \pi\end{cases}$ is .... (2)

(A) $\frac{\pi}{2}\left(\frac{1-\sin\lambda\pi}{\lambda}\right)$ (B) $\left(\frac{1-\sin\lambda\pi}{\lambda}\right)$ (C) $\left(\frac{\pi\sin\lambda\pi}{2\lambda}\right)$ (D) $\left(\frac{\sin\lambda\pi}{\lambda}\right)$

26. The Fourier sine transform $F_s(\lambda)$ of $f(x) = e^{-x}$, $x > 0$ is given by .... (2)

(A) $\frac{3\lambda}{1+\lambda^2}$ (B) $\frac{\lambda}{1-\lambda^2}$ (C) $\frac{\lambda}{1+\lambda^2}$ (D) $\frac{\lambda}{1-\lambda^2}$

27. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = e^{-x}$, $x > 0$ is given by .... (2)

(A) $\frac{2}{1-\lambda^2}$ (B) $\frac{1}{1-\lambda^2}$ (C) $\frac{2}{1+\lambda^2}$ (D) $\frac{1}{1+\lambda^2}$

28. If $f(x) = e^{-kx}$, $x > 0$, $k > 0$ then Fourier sine transform $F_s(\lambda)$ of f(x) is given by .... (2)

(A) $\frac{\lambda}{k^2+\lambda^2}$ (B) $\frac{k}{k^2+\lambda^2}$ (C) $\frac{1}{k^2+\lambda^2}$ (D) $-\frac{k}{k^2+\lambda^2}$

29. If $f(x) = e^{-kx}$, $x > 0$ then Fourier cosine transform $F_c(\lambda)$ of f(x) is given by .... (2)

(A) $-\frac{k}{k^2+\lambda^2}$ (B) $\frac{k}{k^2+\lambda^2}$ (C) $\frac{\lambda}{k^2+\lambda^2}$ (D) $\frac{1}{k^2+\lambda^2}$

30. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = e^{-|x|}$, $-\infty < x < \infty$ is .... (2)

(A) $\frac{\lambda}{1+\lambda^2}$ (B) $\frac{1}{1+\lambda^2}$ (C) $\frac{1}{1-\lambda^2}$ (D) $-\frac{1}{1+\lambda^2}$

31. The Fourier sine transform $F_s(\lambda)$ of $f(x) = e^{-|x|}$, $0 < x < \infty$ is (2)

(A) $\frac{\lambda}{1+\lambda^2}$ (B) $\frac{1}{1+\lambda^2}$ (C) $\frac{1}{1-\lambda^2}$ (D) $-\frac{1}{1+\lambda^2}$

32. If $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$ then Fourier cosine transform $F_c(\lambda)$ of f(x) is given by .... (2)

(A) $\frac{\cos\lambda}{\lambda}$ (B) $\frac{\cos 2\lambda}{\lambda}$ (C) $\frac{\sin\lambda}{\lambda}$ (D) $\frac{\sin 2\lambda}{\lambda}$

33. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ is .... (2)

(A) $\frac{1-\cos\lambda a}{\lambda}$ (B) $\frac{\cos\lambda a - 1}{\lambda}$ (C) $\frac{\sin\lambda a}{a}$ (D) $\frac{\sin\lambda a}{\lambda}$

34. The Fourier sine transform $F_s(\lambda)$ of $f(x) = \begin{cases} 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$ is .... (2)

(A) $\frac{1-\cos\lambda a}{\lambda}$ (B) $\frac{\sin\lambda a}{\lambda}$ (C) $\frac{\cos\lambda a - 1}{\lambda}$ (D) $\frac{\sin\lambda a}{a}$

35. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ is .... (2)

(A) $\frac{1}{2}\left[-\frac{\sin(1+\lambda)u}{1+\lambda} - \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$ (B) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$

(C) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\cos(1-\lambda)u}{1-\lambda}\right]_0^\pi$ (D) $\frac{1}{2}\left[-\frac{\sin(1+\lambda)u}{1+\lambda} - \frac{\cos(1-\lambda)u}{1-\lambda}\right]_0^\pi$

36. The Fourier sine transform $F_s(\lambda)$ of $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ is .... (2)

(A) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$ (B) $\frac{1}{2}\left[\frac{\sin(1-\lambda)u}{1-\lambda} - \frac{\sin(1+\lambda)u}{1+\lambda}\right]_0^\pi$

(C) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\cos(1-\lambda)u}{1-\lambda}\right]_0^\pi$ (D) $\frac{1}{2}\left[-\frac{\sin(1+\lambda)u}{1+\lambda} - \frac{\cos(1-\lambda)u}{1-\lambda}\right]_0^\pi$

37. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ is .... (2)

(A) $\frac{1}{2}\left[\frac{\sin(1-\lambda)u}{1-\lambda} - \frac{\cos(1+\lambda)u}{1+\lambda}\right]_0^\pi$ (B) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$

(C) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\cos(1-\lambda)u}{1-\lambda}\right]_0^\pi$ (D) $\frac{1}{2}\left[\frac{\sin(1+\lambda)u}{1+\lambda} + \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$

38. The Fourier sine transform $F_s(\lambda)$ of $f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ is .... (2)

(A) $\frac{1}{2}\left[\frac{\sin(1-\lambda)u}{1-\lambda} - \frac{\cos(1+\lambda)u}{1+\lambda}\right]_0^\pi$

(B) $\frac{1}{2}\left[-\frac{\cos(\lambda+1)u}{\lambda+1} - \frac{\cos(\lambda-1)u}{\lambda-1}\right]_0^\pi$

(C) $\frac{1}{2}\left[-\frac{\cos(1+\lambda)u}{1+\lambda} - \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$

(D) $\frac{1}{2}\left[\frac{\sin(1+\lambda)u}{1+\lambda} - \frac{\sin(1-\lambda)u}{1-\lambda}\right]_0^\pi$

39. The Fourier cosine transform $F_c(\lambda)$ of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$ is .... (2)

(A) $\frac{1}{2}\left[\frac{\sin(\lambda+1)a}{\lambda+1} - \frac{\sin(\lambda-1)a}{\lambda-1}\right]$

(B) $\frac{1}{2}\left[\frac{\sin(\lambda-1)a}{\lambda-1} - \frac{\sin(\lambda+1)a}{\lambda+1}\right]$

(C) $\frac{1}{2}\left[\frac{\sin(\lambda+1)a}{\lambda+1} + \frac{\sin(\lambda-1)a}{\lambda-1}\right]$

(D) $\frac{\sin(\lambda+1)a}{\lambda+1}$

40. The solution f(x) of integral equation $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}, \lambda > 0$ is .... (2)

(A) $\frac{2}{\pi}\left(\frac{e^{-x}}{1+x^2}\right)$ (B) $\frac{2}{\pi}\left(\frac{x}{1+x^2}\right)$ (C) $\frac{2}{\pi}\left(\frac{1}{1-x^2}\right)$ (D) $\frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$

41. The solution of integral equation $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ is

$f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda$ then the value of f(x) is equal to .... (2)

(A) $\frac{2}{\pi}\left(\frac{1}{x} - \frac{\sin x}{x^2}\right)$ (B) $\frac{2}{\pi}\left(\frac{1}{x} - \frac{\cos x}{x^2}\right)$ (C) $\frac{2}{\pi}\left(\frac{1}{x} + \frac{\sin x}{x^2}\right)$ (D) $\frac{2}{\pi}\left(-\frac{1}{x} + \frac{\sin x}{x^2}\right)$

42. The solution of integral equation $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ is

$f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda$ then the value of f(x) is equal to .... (2)

(A) $\frac{2}{\pi}\left(\frac{1+\cos x}{x^2}\right)$ (B) $\frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$ (C) $\frac{2}{\pi}\left(\frac{1+\sin x}{x^2}\right)$ (D) $\frac{2}{\pi}\left(\frac{1-\sin x}{x^2}\right)$

43. The solution f(x) of integral $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1, & 0 \le \lambda \le 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \ge 2 \end{cases}$ is .... (2)

(A) $\frac{2}{\pi}\left[\left(\frac{1-\sin x}{x}\right) + 2\left(\frac{\sin x - \sin 2x}{x}\right)\right]$

(B) $\frac{2}{\pi}\left[\left(\frac{-1+\cos x}{x}\right) + 2\left(\frac{-\cos x + \cos 2x}{x}\right)\right]$

(C) $\frac{2}{\pi}\left[\left(\frac{1-\cos x}{x}\right) + 2\left(\frac{\cos x - \cos 2x}{x}\right)\right]$

(D) $\frac{2}{\pi}\left[\left(\frac{1-\cos x}{x^2}\right) + 2\left(\frac{\cos x - \cos 2x}{x^2}\right)\right]$

44. The solution f(x) of integral equation $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ is .... (2)

(A) $\frac{2}{\pi}\left(\frac{1+\cos x}{x}\right)$ (B) $\frac{2}{\pi}\left(\frac{1+\sin x}{x}\right)$ (C) $\frac{2}{\pi}\left(\frac{1-\sin x}{x}\right)$ (D) $\frac{2}{\pi}\left(\frac{1-\cos x}{x}\right)$

45. The solution f(x) of integral equation $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ is .... (2)

(A) $\frac{2}{\pi}\left(\frac{\sin x}{x}\right)$ (B) $\frac{2}{\pi}\left(\frac{\cos x}{x}\right)$ (C) $\frac{2}{\pi}\left(\frac{1-\cos x}{x}\right)$ (D) $\frac{2}{\pi}\left(\frac{1+\sin x}{x}\right)$

46. The inverse Fourier cosine transform f(x) of $F_c(\lambda) = \dfrac{\sin a\lambda}{\lambda}$ is .... (2)

(A) $\dfrac{1}{\pi}\int_0^\infty \dfrac{\cos (a + x)\lambda + \sin (a - x)\lambda}{\lambda} d\lambda$ (B) $\dfrac{1}{\pi}\int_0^\infty \dfrac{\cos (a + x)\lambda + \cos (a - x)\lambda}{\lambda} d\lambda$

(C) $\dfrac{1}{\pi}\int_0^\infty \dfrac{\sin (a + x)\lambda + \sin (a - x)\lambda}{\lambda} d\lambda$ (D) $\dfrac{1}{\pi}\int_0^\infty \dfrac{\sin (a + x)\lambda + \cos (a - x)\lambda}{\lambda} d\lambda$

47. If the Fourier cosine integral representation of $f(x) = \begin{cases} 1 - x^2, & 0 < x < 1 \\ 0 & , x > 1 \end{cases}$ is

$f(x) = \dfrac{4}{\pi}\int_0^\infty \left(\dfrac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \cos \lambda x \, d\lambda$ then the value of integral $\int_0^\infty \left(\dfrac{\sin \lambda - \lambda \cos \lambda}{\lambda^3}\right) \cos \dfrac{\lambda}{2} d\lambda$ is equal to .... (2)

(A) $-\dfrac{3\pi}{16}$ (B) $\dfrac{3\pi}{16}$ (C) $\dfrac{3\pi}{8}$ (D) $\dfrac{3\pi}{4}$

48. Given that $\int_0^\infty \dfrac{\sin t}{t} dt = \dfrac{\pi}{2}$, then Fourier sine transform $F_s(\lambda)$ of $f(x) = \dfrac{1}{x}$, x > 0 is given by .... (2)

(A) π (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{2}$ (D) $-\pi$

49. For the Fourier cosine transform $\int_0^\infty \left(\dfrac{1 - \cos u}{u^2}\right) \cos \lambda u \, du = \begin{cases} \dfrac{\pi}{2}(1 - \lambda), & 0 < \lambda < 1 \\ 0 & , \lambda > 1 \end{cases}$ the value of integral $\int_0^\infty \dfrac{\sin^2 z}{z^2} dz$ is .... (2)

(A) 1 (B) $\dfrac{\pi}{2}$ (C) 0 (D) $\dfrac{\pi}{4}$

50. For the Fourier sine integral representation

$\dfrac{2}{\pi}\int_0^\infty \left(\dfrac{1 - \cos \lambda}{\lambda}\right) \sin \lambda x \, d\lambda = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$, the value of integral $\int_0^\infty \dfrac{\sin^3 t}{t} dt$ is .... (2)

(A) $\dfrac{\pi}{2}$ (B) 1 (C) 0 (D) $\dfrac{\pi}{4}$

51. Given that $F_c(\lambda) = \int_0^\infty u^{m-1} \cos \lambda u \, du = \dfrac{\overline{|m}}{\lambda^m} \cos \dfrac{m\pi}{2}$, then Fourier cosine transform $F_c(\lambda)$ of $f(x) = x^3$, x > 0 is given by .... (2)

(A) $\dfrac{6}{\lambda^4}$ (B) $\dfrac{3}{\lambda^3}$ (C) $\dfrac{4}{\lambda^2}$ (D) $\dfrac{1}{\lambda^2}$

52. Given that $F_s(\lambda) = \int_0^\infty u^{m-1} \sin \lambda u \, du = \dfrac{\overline{|m}}{\lambda^m} \sin \dfrac{m\pi}{2}$, then Fourier sine transform $F_s(\lambda)$ of $f(x) = x^2$, x > 0 is given by .... (2)

(A) $\dfrac{2}{\lambda^3}$ (B) $-\dfrac{2}{\lambda^3}$ (C) $\dfrac{3}{\lambda^2}$ (D) $-\dfrac{3}{\lambda^2}$

Answers

1. (B)	2. (D)	3. (C)	4. (A)	5. (D)	6. (A)	7. (B)	8. (C)
9. (D)	10. (A)	11. (C)	12. (B)	13. (A)	14. (B)	15. (D)	16. (D)
17. (A)	18. (C)	19. (B)	20. (A)	21. (C)	22. (D)	23. (A)	24. (B)
25. (C)	26. (C)	27. (D)	28. (A)	29. (B)	30. (B)	31. (A)	32. (C)
33. (D)	34. (A)	35. (C)	36. (B)	37. (D)	38. (B)	39. (C)	40. (D)
41. (A)	42. (B)	43. (C)	44. (D)	45. (A)	46. (C)	47. (B)	48. (C)
49. (B)	50. (D)	51. (A)	52. (B)				

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Ch. 4 The Z-Transform

Type I : Z-transform **Marks**

1. Z-transform of sequence {f(k)} is defined as .... (1)

(A) $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ (B) $\sum_{k=-\infty}^{\infty} f(k)\, z^{k}$ (C) $\sum_{k=-\infty}^{\infty} f(k)\, z^{-2k}$ (D) $\sum_{k=-\infty}^{\infty} f(k)\, z^{2k}$

2. Z-transform of causal sequence {f(k)}, k ≥ 0 is defined as .... (1)

(A) $\sum_{k=0}^{\infty} f(k)\, z^{k}$ (B) $\sum_{k=0}^{\infty} f(k)\, z^{-k}$ (C) $\sum_{k=0}^{\infty} f(-k)\, z^{-k}$ (D) $\sum_{k=0}^{\infty} f(-k)\, z^{k}$

3. If $U(k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$, then Z-transform of U(k) is given by .... (2)

(A) $-\frac{z}{z-1}$, $|z| > 1$ (B) $\frac{1}{z-1}$, $|z| > 1$ (C) $\frac{z}{z-1}$, $|z| > 1$ (D) $\frac{2}{z-1}$, $|z| > 1$

4. If $\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$, then Z-transform of δ(k) is given by .... (2)

(A) $\frac{1}{z}$ (B) $\frac{1}{z-1}$ (C) $\frac{2}{z-2}$ (D) 1

5. If $f(k) = a^k$, k ≥ 0, then Z-transform of $\{a^K\}$ is given by .... (1)

(A) $\frac{z}{z-a}$, $|z| < |a|$ (B) $\frac{z}{z-a}$, $|z| > |a|$

(C) $\frac{1}{z-a}$, $|z| > |a|$ (D) $-\frac{z}{z-a}$, $|z| > |a|$

6. If $f(k) = a^k$, k < 0, then Z-transform of $\{a^k\}$ is given by .... (1)

(A) $\frac{z}{a-z}$, $|z| < |a|$ (B) $\frac{z}{z-a}$, $|z| < |a|$

(C) $\frac{1}{a-z}$, $|z| > |a|$ (D) $\frac{z}{a-z}$, $|z| > |a|$

7. If $f(k) = 2^k$, k ≥ 0, then Z-transform of $\{2^k\}$ is given by .... (1)

(A) $\frac{z}{z-2}$, $|z| < |2|$ (B) $\frac{1}{z-2}$, $|z| > |2|$

(C) $\frac{z}{z-2}$, $|z| > |2|$ (D) $-\frac{z}{z-2}$, $|z| > |2|$

8. If $f(k) = 3^k$, k < 0, then Z-transform of $\{3^k\}$ is given by .... (1)

(A) $\frac{z}{3-z}$, $|z| > |3|$ (B) $\frac{z}{z-3}$, $|z| < |3|$

(B) $\frac{1}{3-z}$, $|z| > |3|$ (D) $\frac{z}{3-z}$, $|z| < |3|$

9. If f(k) = cos αk, k ≥ 0, then Z-transform of {cos αk} is given by .... (1)

(A) $\frac{z(z+\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$, $|z| > 1$ (B) $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$, $|z| < 1$

(C) $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$, $|z| > 1$ (D) $\frac{z\cos\alpha}{z^2 + 2z\cos\alpha + 1}$, $|z| > 1$

10. If f(k) = sin αk, k ≥ 0, then Z-transform of {sin αk} is given by .... (1)

(A) $\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$, $|z| > 1$ (B) $\frac{z\sin\alpha}{z^2 + 2z\cos\alpha + 1}$, $|z| > 1$

(C) $\frac{z(z-\sin\alpha)}{z^2 - 2z\cos\alpha + 1}$, $|z| > 1$ (D) $\frac{z\sin\alpha}{z^2 + 2z\cos\alpha + 1}$, $|z| < 1$

11. If f(k) = cosh αk, k ≥ 0, then Z-transform of {cosh αk} is given by .... (1)

(A) $\dfrac{z(z - \sinh\alpha)}{z^2 - 2z\cosh\alpha + 1}$, $|z| > \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$ (B) $\dfrac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$, $|z| > \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$

(C) $\dfrac{z(z + \cosh\alpha)}{z^2 + 2z\cosh\alpha + 1}$, $|z| > \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$ (D) $\dfrac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$, $|z| < \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$

12. If f(k) = sinh αk, k ≥ 0, then Z-transform of {sinh αk} is given by .... (1)

(A) $\dfrac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$, $|z| < \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$ (B) $\dfrac{z(z - \sinh\alpha)}{z^2 - 2z\cosh\alpha + 1}$, $|z| > \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$

(C) $\dfrac{z(z + \sinh\alpha)}{z^2 + 2z\cosh\alpha + 1}$, $|z| > \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$ (D) $\dfrac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$, $|z| > \max(|e^{\alpha}| \text{ or } |e^{-\alpha}|)$

13. If f(k) = cosh 2k, k ≥ 0, then Z-transform of {cosh 2k} is given by .... (1)

(A) $\dfrac{z\sinh 2}{z^2 - 2z\cosh 2 + 1}$, $|z| > \max(|e^{2}| \text{ or } |e^{-2}|)$ (B) $\dfrac{z(z - \cosh 2)}{z^2 - 2z\cosh 2 + 1}$, $|z| > \max(|e^{2}| \text{ or } |e^{-2}|)$

(C) $\dfrac{z(z + \cosh 2)}{z^2 + 2z\cosh 2 + 1}$, $|z| > \max(|e^{2}| \text{ or } |e^{-2}|)$ (D) $\dfrac{z(z - \cosh 2)}{z^2 - 2z\cosh 2 + 1}$, $|z| < \max(|e^{2}| \text{ or } |e^{-2}|)$

14. If f(k) = sinh 2k, k ≥ 0, then Z-transform of {sinh 2k} is given by .... (1)

(A) $\dfrac{z\sinh 2}{z^2 + 2z\cosh 2 - 1}$, $|z| > \max(|e^{2}| \text{ or } |e^{-2}|)$ (B) $\dfrac{z(z - \cosh 2)}{z^2 - 2z\cosh 2 + 1}$, $|z| > \max(|e^{2}| \text{ or } |e^{-2}|)$

(C) $\dfrac{z\sinh 2}{z^2 - 2z\cosh 2 + 1}$, $|z| > \max(|e^{2}| \text{ or } |e^{-2}|)$ (D) $\dfrac{z(z - \cosh 2)}{z^2 - 2z\cosh 2 + 1}$, $|z| < \max(|e^{2}| \text{ or } |e^{-2}|)$

15. If f(k) = cos 2k, k ≥ 0, then Z-transform of {cos 2k} is given by .... (1)

(A) $\dfrac{z(z + \cos 2)}{z^2 - 2z\cos 2 + 1}$, $|z| > 1$ (B) $\dfrac{z\cos 2}{z^2 + 2z\cos 2 + 1}$, $|z| > 1$

(C) $\dfrac{z(z - \cos 2)}{z^2 - 2z\cos 2 + 1}$, $|z| < 1$ (D) $\dfrac{z(z - \cos 2)}{z^2 - 2z\cos 2 + 1}$, $|z| > 1$

16. If f(k) = sin 2k, k ≥ 0, then Z-transform of {sin 2k} is given by .... (1)

(A) $\dfrac{z\sin 2}{z^2 - 2z\cos 2 + 1}$, $|z| > 1$ (B) $\dfrac{z\sin 2}{z^2 + 2z\cos 2 + 1}$, $|z| > 1$

(C) $\dfrac{z(z - \sin 2)}{z^2 - 2z\cos 2 + 1}$, $|z| > 1$ (D) $\dfrac{z\sin 2}{z^2 + 2z\cos 2 + 1}$, $|z| < 1$

17. If Z {f(k)} = F(z), then Z {a^k f(k)}, a constant, is equal to .... (1)

(A) $F\left(\dfrac{a}{z}\right)$ (B) $F\left(\dfrac{z}{a}\right)$ (C) F(az) (D) $\dfrac{F(z)}{a}$

18. If Z {f(k)} = F(z), then Z {e^{-ak} f(k)}, a constant, is equal to .... (1)

(A) $F\left(\dfrac{z}{e^a}\right)$ (B) $F(e^{-a} z)$ (C) $F(e^a z)$ (D) $\dfrac{F(z)}{e^a}$

19. If Z {f(k)} = F(z), then Z {k^n f(k)}, is equal to .... (1)

(A) $\left(-z\dfrac{d}{dz}\right)^n F(z)$ (B) $\left(z\dfrac{d}{dz}\right)^n F(z)$ (C) $(-z)^n \dfrac{d}{dz} F(z)$ (D) $\left(z\dfrac{d}{dz}\right)^{n-1} F(z)$

20. Z-transform of {f(k)} = $\dfrac{a^k}{k!}$, k ≥ 0 is given by .... (1)

(A) $e^{z/a}$ (B) e^{az} (C) ze^a (D) $e^{a/z}$

21. If Z {f(k)} = F(z), k ≥ 0 then Z {f(k + 1)} is given by .... (1)

(A) zF(z) + zf(0) (B) zF(z) – zf(0) (C) zF(z) – f(0) (D) $z^2F(z) - zf(0)$

22. If Z {f(k)} = F(z), $k \geq 0$ then Z {f(k + 2)} is given by .... (1)

(A) $z^2F(z) - zf(0) - f(1)$ (B) $z^2F(z) + z^2f(0) + zf(1)$

(C) $z^2F(z) + zf(0) + f(1)$ (D) $z^2F(z) - z^2f(0) - zf(1)$

23. If Z {f(k)} = F(z), $k \geq 0$ then Z {f(k – 1)} is given by .... (1)

(A) $z^{-1}F(z)$ (B) $z^{-1}(F) - f(0)$

(C) $zF(z)$ (D) $z^{-2}F(z) - z^{-1}(0)$

24. If Z {f(k)} = F(z), $k \geq 0$ then Z {f(k – 2)} is given by .... (1)

(A) $z^2F(z) - zf(0)$ (B) $z^{-1}F(z) - f(0)$

(C) $z^{-2}F(z)$ (D) $z^{-2}F(z) - z^{-1}f(0)$

25. Convolution of two sequences {f(k)} and {g(k)} is {h(k)} = {f(k)} $*$ {g(k)}. Then Z[{h(k)}] is given by .... (1)

(A) F(z) G(z) (B) F(z) + G(z) (C) F(z) – G(z) (D) $\dfrac{F(z)}{G(z)}$

26. For $\{f(k)\} = \{-2, \underset{\uparrow}{-1}, 2\}$, F(z) is given by .... (2)

(A) $2z + 1 + 2z^{-1}$ (B) $-2z - 1 + 2z^{-1}$ (C) $2z + 1 - 2z^{-1}$ (D) $2z - 1 + 2z^{-1}$

27. For $\{f(k)\} = \{2, 1, \underset{\uparrow}{3}, 2, -4\}$, F(z) is given by .... (2)

(A) $2z^2 - z - 3 + 2z^{-1} - 4z^{-2}$ (B) $2z^2 + z + 3 - 2z^{-1} + 4z^{-2}$

(C) $2z^2 + z + 3 + 2z^{-1} - 4z^{-2}$ (D) $2z^2 + z + 3 + 2z^{-1} + 4z^{-2}$

28. If $f(k) = a^{|k|}$, $\forall$ k, then Z-transofrm of $\{a^{|k|}\}$ is given by .... (2)

(A) $\left(\dfrac{az}{1 + az} + \dfrac{z}{z - a}\right)$, $|a| < |z| < \dfrac{1}{|a|}$ (B) $\left(\dfrac{az}{1 - az} - \dfrac{z}{z - a}\right)$, $|a| < |z| < \dfrac{1}{|a|}$

(C) $\left(\dfrac{az}{1 + az} + \dfrac{z}{z + a}\right)$, $|a| < |z| < \dfrac{1}{|a|}$ (D) $\left(\dfrac{az}{1 - az} + \dfrac{z}{z - a}\right)$, $|a| < |z| < \dfrac{1}{|a|}$

29. Z-transform of $\{f(k)\} = \dfrac{2^k}{k!}$, $k \geq 0$ is given by (1)

(A) $e^{z/2}$ (B) e^{2z} (C) e^z (D) $e^{2/z}$

30. If $f(k) = \cos \pi k$, $k \geq 0$, then Z-transform of $\{\cos \pi k\}$ is given by .... (2)

(A) $\dfrac{z(z - 1)}{(z + 1)^2}$, $|z| > 1$ (B) $\dfrac{z - 1}{z + 1}$, $|z| > 1$

(C) $\dfrac{z(z + 1)}{(z - 1)^2}$, $|z| > 1$ (D) $\dfrac{z}{z + 1}$, $|z| > 1$

31. If $f(k) = \cos \dfrac{\pi}{2} k$, $k \geq 0$, then Z-transform of $\left\{\cos \dfrac{\pi}{2} k\right\}$ is given by .... (2)

(A) $\dfrac{z^2}{z^2 + 1}$, $|z| > 1$ (B) $\dfrac{z^2}{z^2 - 1}$, $|z| > 1$

(C) $\dfrac{z}{z + 1}$, $|z| > 1$ (D) $\dfrac{z}{z - 1}$, $|z| < 1$

32. If $f(k) = \sin \dfrac{\pi}{2} k$, $k \geq 0$, then Z-transform of $\left(\sin \dfrac{\pi}{2} k\right)$ is given by .... (2)

(A) $\dfrac{z}{z^2 - 1}$, $|z| < 1$ (B) $\dfrac{z^2}{z^2 + 1}$, $|z| > 1$

(C) $\dfrac{z}{z^2 + 1}$, $|z| > 1$ (D) $\dfrac{z}{z^2 - 1}$, $|z| > 1$

33. If $f(k) = \left(\frac{\pi}{2}\right)^k \cos\frac{\pi}{2}k$, $k \geq 0$, then Z-transform of $\left\{\left(\frac{\pi}{2}\right)^k \cos\frac{\pi}{2}k\right\}$ is given by .... (2)

(A) $\frac{z^2}{z^2 + \frac{\pi^2}{4}}$, $|z| > \frac{\pi}{2}$ (B) $\frac{z^2}{z^2 - \frac{\pi^2}{4}}$, $|z| < \frac{\pi}{2}$

(C) $\frac{z}{z^2 + \frac{\pi^2}{4}}$, $|z| > \frac{\pi}{2}$ (D) $\frac{z}{z^2 - \frac{\pi^2}{4}}$, $|z| > \frac{\pi}{2}$

34. If $f(k) = 2^k \sin\frac{\pi}{2}k$, $k \geq 0$, then Z-transform of $\left\{2^k \sin\frac{\pi}{2}k\right\}$ is given by .... (2)

(A) $\frac{2z}{z^2 - 4}$, $|z| > 2$ (B) $\frac{2z}{z^2 - 4}$, $|z| < 2$ (C) $\frac{2z}{z^2 + 4}$, $|z| < 2$ (D) $\frac{2z}{z^2 + 4}$, $|z| > 2$

35. If $f(k) = 2^k \sin\frac{\pi}{3}k$, $k \geq 0$, then Z-transform of $\left\{2^k \sin\frac{\pi}{3}k\right\}$ is given by .... (2)

(A) $\frac{\sqrt{3}z}{z^2 - 2z + 4}$, $|z| > 2$ (B) $\frac{\sqrt{3}\,z}{z^2 - 2z + 4}$, $|z| < 2$

(C) $\frac{\sqrt{3}\,z}{z^2 + 2z + 4}$, $|z| > 2$ (D) $\frac{\sqrt{3}\,z}{z^2 + 2z + 4}$, $|z| < 2$

36. If $f(k) = 2^k \cosh 3k$, $k \geq 0$, then Z-transform of $\{2^k \cosh 3k\}$ is given by .... (2)

(A) $\frac{z\,(z - 2\cosh 3)}{z^2 - 4z\cosh 3 + 4}$, $|z| > \max(|e^2| \text{ or } |e^{-2}|)$ (B) $\frac{z\,(z - 2\cosh 3)}{z^2 - 4z\cosh 3 + 4}$, $|z| > \max(|e^3| \text{ or } |e^{-3}|)$

(C) $\frac{z\,(z + 2\cosh 3)}{z^2 + 4z\cosh 3 + 4}$, $|z| < \max(|e^3| \text{ or } |e^{-3}|)$ (D) $\frac{z\,(z - 2\sinh 3)}{z^2 - 4z\sinh 3 + 4}$, $|z| > \max(|e^3| \text{ or } |e^{-3}|)$

37. If $f(k) = 3^k \sinh 2k$, $k \geq 0$, then Z-transform of $\{3^k \sinh 2k\}$ is given by .... (2)

(A) $\frac{3z\sinh 2}{z^2 + 6z\cosh 2 - 9}$, $|z| > \max(|e^2| \text{ or } |e^{-2}|)$ (B) $\frac{3z\sinh 2}{z^2 - 6z\cosh 2 + 9}$, $|z| > \max(|e^3| \text{ or } |e^{-3}|)$

(C) $\frac{3z\sinh 2}{z^2 - 6z\cosh 2 + 9}$, $|z| > \max(|e^2| \text{ or } |e^{-2}|)$ (D) $\frac{3z\,(z - \sinh 2)}{z^2 - 6z\cosh 2 + 9}$, $|z| < \max(|e^2| \text{ or } |e^{-2}|)$

38. If $f(k) = k$, $k \geq 0$ then Z-transform of $\{k\}$ is given by .... (2)

(A) $\frac{z}{(z-1)^2}$, $|z| > 1$ (B) $\frac{(z-1)^2}{z^2}$, $|z| > 1$

(C) $\frac{(z+1)^2}{z^2}$, $|z| > 1$ (D) $\frac{z^2}{(z+1)^2}$, $|z| > 1$

39. If $f(k) = k5^k$, $k \geq 0$ then Z-transform of $\{k5^k\}$ is given by .... (2)

(A) $\frac{(z-5)^2}{5z}$, $|z| > 5$ (B) $\frac{(z-5)^2}{z}$, $|z| > 5$

(C) $\frac{5z}{(z-5)^2}$, $|z| > 5$ (D) $\frac{5z}{(z+5)^2}$, $|z| > 5$

40. If $f(k) = (k+1)\,2^k$, $k \geq 0$, then Z-transform of $\{(k+1)\,2^k\}$ is given by .... (2)

(A) $\frac{2}{(z+2)^2} + \frac{z}{z-2}$, $|z| > 2$ (B) $-\frac{2z}{(z-2)^2} - \frac{z}{z-2}$, $|z| > 2$

(C) $-\frac{2z}{(z-2)^2} + \frac{z}{z-2}$, $|z| > 2$ (D) $\frac{2z}{(z-2)^2} + \frac{z}{z-2}$, $|z| > 2$

41. $Z\{3^k e^{-2k}\}$, $k \geq 0$ is given by .... (2)

(A) $\frac{z}{(z-3e)^2}$ (B) $\frac{z}{z - 3e^{-2}}$ (C) $\frac{z}{z - 2e^3}$ (D) $\frac{z}{z + 3e^2}$

42. Z $\{ke^{-k}\}$, $k \geq 0$ is given by .... (2)

(A) $\dfrac{ez}{(ez+1)^2}$ (B) $\dfrac{e^{-1}z}{(z-e^{-1})}$ (C) $\dfrac{e^{-1}z}{(z-e^{-1})^2}$ (D) $\dfrac{e^{-1}z}{(z+e^{-1})^2}$

43. Z $\{\cos(2k+3)\}$, $k \geq 0$ is given by .... (2)

(A) $\cos 3 \dfrac{z(z-\cos 2)}{z^2-2z\cos 2+1} + \sin 3 \dfrac{z\sin 2}{z^2-2z\cos 2+1}$ (B) $\cos 3 \dfrac{z(z-\cos 2)}{z^2-2z\cos 2+1} - \sin 3 \dfrac{z\sin 2}{z^2-2z\cos 2+1}$

(C) $\sin 3 \dfrac{z(z-\cos 2)}{z^2-2z\cos 2+1} - \cos 3 \dfrac{z\sin 2}{z^2-2z\cos 2+1}$ (D) $\cos 3 \dfrac{z(z+\cos 2)}{z^2+2z\cos 2+1} + \sin 3 \dfrac{z\sin 2}{z^2+2z\cos 2+1}$

44. Z $\{\sinh(bk+c)\}$, $k \geq 0$ is given by .... (2)

(A) $\cosh c \dfrac{z\sinh b}{z^2-2z\cosh b+1} + \sinh c \dfrac{z(z-\cosh b)}{z^2-2z\cosh b+1}$ (B) $\cosh c \dfrac{z(z-\cosh b)}{z^2-2z\cosh b+1} + \sinh c \dfrac{z\sinh b}{z^2-2z\cosh b+1}$

(C) $\cosh c \dfrac{z(z-\cosh b)}{z^2-2z\cosh b+1} - \sinh c \dfrac{z\sinh b}{z^2-2z\cosh b+1}$ (D) $\cosh c \dfrac{z\sinh b}{z^2+2z\cosh b+1} + \sinh c \dfrac{z(z+\cosh b)}{z^2+2z\cosh b+1}$

45. Z $\{e^{-2k}\sin 3k\}$, $k \geq 0$ is given by .... (2)

(A) $\dfrac{(ze^3)\sin 2}{(ze^3)^2+2(ze^3)\cos 2-1}$ (B) $\dfrac{(ze^2)(ze^2-\cos 3)}{(ze^2)^2-2(ze^2)\cos 3+1}$

(C) $\dfrac{(ze^3)\sin 2}{(ze^3)^2-2(ze^3)\cos 2+1}$ (D) $\dfrac{(ze^2)\sin 3}{(ze^2)^2-2(ze^2)\cos 3+1}$

46. If $f(k) = {}^2C_k$, $0 \leq k \leq 2$ then Z $\{{}^2C_k\}$ is given by .... (2)

(A) $(1-z^{-1})^2$, $|z| > 0$ (B) $(1+z^{-1})^2$, $|z| > 0$

(C) $(1+z^{-1})$, $|z| > 0$ (D) $(1-z^{-1})$, $|z| > 0$

47. If $f(k) = a^k\,U(k)$ then Z $\{f(k)\}$ is given by .... (2)

(A) $\dfrac{z}{z-1}$, $|z| > |a|$ (B) $\dfrac{z-1}{z}$, $|z| > |a|$

(C) $\dfrac{z^2}{z-1}$, $|z| > |a|$ (D) $\dfrac{z}{z-a}$, $|z| > |a|$

48. If $\{x(k)\} = \left\{\dfrac{1}{1^k}\right\} * \left\{\dfrac{1}{2^k}\right\}$ then Z $\{x(k)\}$ is given by .... (2)

(A) $\left(\dfrac{z}{z-1}\right)\left(\dfrac{2z}{2z-1}\right)$, $|z| > 1$ (B) $\left(\dfrac{z}{z-1}\right)+\left(\dfrac{2z}{2z-1}\right)$, $|z| > 1$

(C) $\left(\dfrac{z}{z-1}\right)-\left(\dfrac{2z}{2z-1}\right)$, $|z| > 1$ (d) $\left(\dfrac{z}{z-1}\right)\div\left(\dfrac{2z}{2z-1}\right)$, $|z| > 1$

Answers

1. (A)	2. (B)	3. (C)	4. (D)	5. (B)	6. (A)	7. (C)	8. (D)
9. (C)	10. (A)	11. (B)	12. (D)	13. (B)	14. (C)	15. (D)	16. (A)
17. (B)	18. (C)	19. (A)	20. (D)	21. (B)	22. (D)	23. (A)	24. (C)
25. (A)	26. (B)	27. (C)	28. (D)	29. (D)	30. (D)	31. (A)	32. (C)
33. (A)	34. (D)	35. (A)	36. (B)	37. (C)	38. (A)	39. (C)	40. (D)
41. (B)	42. (C)	43. (B)	44. (A)	45. (D)	46. (B)	47. (D)	48. (A)

Type II : Inverse Z-transform and Difference Equation : **Marks**

1. If $|z| > |a|$, inverse Z-transform of $\frac{z}{z-a}$ is given by .... (1)

(A) $a^k, k \geq 0$ (B) $a^k, k < 0$ (C) $a^{k-1}, k \geq 0$ (D) $-a^k, k \geq 0$

2. If $|z| < |a|$, inverse Z-transform of $\frac{z}{z-a}$ is given by .... (1)

(A) $a^k, k \geq 0$ (B) $a^k, k < 0$ (C) $a^{k-1}, k \geq 0$ (D) $-a^k, k < 0$

3. If $|z| > |a|$, inverse Z-transform of $\frac{1}{z-a}$ is given by .... (1)

(A) $a^{k-1}, k \geq 0$ (B) $a^{k-1}, k < 0$ (C) $a^{k-1}, k \geq 1$ (D) $-a^k, k \geq 0$

4. If $|z| < |a|$, inverse Z-transform of $\frac{1}{z-a}$ is given by .... (1)

(A) $a^{k-1}, k \geq 0$ (B) $-a^{k-1}, k \leq 0$ (C) $a^{k-1}, k \geq 1$ (D) $-a^k, k \geq 0$

5. If $|z| > 2$, inverse Z-transform of $\frac{z}{z-2}$ is given by .... (1)

(A) $2^k, k \leq 0$ (B) $2^{k-1}, k > 0$ (C) $2^k, k \geq 0$ (D) $-2^k, k \geq 0$

6. If $|z| < 3$, inverse Z-transform of $\frac{z}{z-3}$ is given by .... (1)

(A) $-3^k, k < 0$ (B) $3^{k-1}, k < 0$ (C) $-3^{k-1}, k \geq 0$ (D) $3^k, k \geq 0$

7. If $|z| > 5$, inverse Z-transform of $\frac{1}{z-5}$ is given by .... (1)

(A) $5^{k-1}, k \leq 1$ (B) $5^{k-1}, k \geq 1$ (C) $5^k, k \geq 0$ (D) $-5^k, k \geq 1$

8. If $|z| < 5$, inverse Z-transform of $\frac{1}{z-5}$ is given by .... (1)

(A) $5^{k+1}, k \geq 0$ (B) $5^k, k \leq 0$ (C) $5^{k+1}, k \geq 1$ (D) $-5^{k-1}, k \leq 0$

9. If $|z| > |a|$, inverse Z-transform of $\frac{z}{(z-a)^2}$ is given by .... (1)

(A) $k\,a^{k-1}, k \geq 0$ (B) $a^{k-1}, k \geq 0$ (C) $k\,a^{k-1}, k < 0$ (D) $(k-1)\,a^k, k \leq 0$

10. If $|z| > 1$, $k \geq 0$, $Z^{-1}\left[\frac{z}{z-1}\right]$ is given by .... (1)

(A) $U(-k)$ (B) $U(k)$ (C) $U(k+1)$ (D) $\delta(k)$

11. $Z^{-1}[1]$ for all k is given by .... (1)

(A) $\delta(k+1)$ (B) $U(k)$ (C) $\delta(k)$ (D) $U(k-1)$

12. Inverse Z-transform of F(z) by inversion integral method is .... (1)

(A) $f(k) = \sum$ [Residues of z^k F(z) at the poles of F(z)]
(B) $f(k) = \sum$ [Residues of z^{k+2} F(z) at the poles of F(z)]
(C) $f(k) = \sum$ [Residues of z^{k+1} F(z) at the poles of F(z)]
(D) $f(k) = \sum$ [Residues of z^{k-1} F(z) at the poles of F(z)]

13. If $|z| > 10$, $k \geq 0$, inverse Z-transform of $\frac{z(z-\cosh 2)}{z^2 - 2z\cosh 2 + 1}$ is given by .... (1)

(A) cosh 2k (B) cosh 3k (C) sinh 2k (D) sinh 3k

14. If $|z| > 21$, $k \geq 0$, inverse Z-transform of $\frac{z \sinh 3}{z^2 - 2z\cosh 3 + 1}$ is given by .... (1)

(A) cosh 2k (B) cosh 3k (C) sinh 2k (D) sinh 3k

15. If $|z| < 2$, inverse Z-transform $Z^{-1}\left(\frac{3}{(z-2)^2}\right)$ is given by .... (2)

(A) $\left(\frac{-k}{2^{-k+1}}\right)$, $k \le 0$ (B) $\left(\frac{-k+1}{2^{-k+2}}\right)$, $k \le 0$

(C) $3\left(\frac{-k+1}{2^{-k+2}}\right)$, $k \le 0$ (D) $\left(\frac{-k+1}{2^{-k+2}}\right)$, $k \ge 0$

16. If $|z| > 3$, $k \ge 0$, inverse Z-transform $Z^{-1}\left[\frac{z^2}{(z-3)^2}\right]$ is given by .... (2)

(A) $-(k+1)\,3^k$ (B) $(k+1)\,3^k$ (C) $(k+1)\,3^{-k}$ (D) $(k-1)\,3^k$

17. If $|z| < 2$, $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ is given by .... (2)

(A) $2^{k-1} + 3^{k-1}$, $k \le 0$ (B) $-2^{k-1} - 3^{k-1}$, $k \le 0$

(C) $-2^{k-1} + 3^{k-1}$, $k \le 0$ (D) $2^{k-1} - 3^{k-1}$, $k \le 0$

18. If $2 < |z| < 3$, $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ is given by .... (2)

(A) $-3^{k-1} - 2^{k-1}$
$(k \le 0)$ $(k \ge 1)$

(B) $3^{k-1} + 2^{k-1}$
$(k \le 0)$ $(k \ge 2)$

(C) $3^{k+1} - 2^{k+1}$
$(k \le 0)$ $(k \le 0)$

(D) $\left(\frac{1}{3}\right)^{k+1} - \left(\frac{1}{2}\right)^{k+1}$
$(k \le 1)$ $(k \le 2)$

19. If $|z| > 2$, $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$ is given by .... (2)

(A) $1 - 2^k$, $k \ge 0$ (B) $2^k - 1$, $k \ge 0$ (C) $\frac{1^k}{2} - 1$, $k \ge 0$ (D) $k - 1$, $k \ge 0$

20. If $|z| < 1$, $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$ is given by .... (2)

(A) $2^k - 1$, $k \ge 0$ (B) $2^{k+1} - 1$, $k > 1$ (C) $1 - 2^k$, $k < 0$ (D) $2 - 3^k$, $k < 0$

21. If $1 < |z| < 2$, $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$ is given by .... (2)

(A) $1 + 2^k$, $k > 0$ (B) $3^k + 2^k$, $k < 0$

(C) $3^k - 1$, $k < 0$ (D) $-2^k \;-\; 1$ $(k \le 0)$ $(k > 0)$

22. If $|z| > 1$, $k \ge 0$, $Z^{-1}\left[\frac{z^2}{z^2+1}\right]$ is given by .... (2)

(A) $\cos \pi k$ (B) $\sin \frac{\pi}{2} k$ (C) $\cos \frac{\pi}{2} k$ (D) $\sin \pi k$

23. If $|z| > 1$, $k \ge 0$, $Z^{-1}\left[\frac{z}{z^2+1}\right]$ is given by .... (2)

(A) $\sin \frac{\pi}{2} k$ (B) $\sin \frac{\pi}{4} k$ (C) $\cos \frac{\pi}{2} k$ (D) $\cos \frac{\pi}{4} k$

24. For finding inverse Z-transform by inversion integral method of $F(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$ the residue of $z^{k-1}F(z)$ at the pole $z = \frac{1}{4}$ is .... (2)

(A) $-\frac{1}{20}\left(\frac{1}{4}\right)^k$ (B) $20\left(\frac{1}{4}\right)^k$ (C) $-20\left(\frac{1}{4}\right)^k$ (D) $\frac{1}{20}\left(\frac{1}{4}\right)^k$

25. For finding inverse Z-transform by inversion integral method of $F(z) = \dfrac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$ the residue of z^{k-1} F(z) at the pole $z = \frac{1}{2}$ is .... (2)

(A) $-\frac{1}{2}\left(\frac{1}{2}\right)^k$ (B) $\frac{1}{6}\left(\frac{1}{2}\right)^k$ (C) $-3\left(\frac{1}{2}\right)^k$ (D) $6\left(\frac{1}{2}\right)^k$

26. For finding inverse Z-transform by inversion integral method of $F(z) = \dfrac{10z}{(z-1)(z-2)}$ the residue of z^{k-1} F(z) at the pole z = 1 is .... (2)

(A) 10 (B) 10^{k-1} (C) – 10 (D) 10^k

27. For finding inverse Z-transform by inversion integral method of $F(z) = \dfrac{1}{(z-2)(z-3)}$ the residue of z^{k-1} F(z) at the pole z = 2 is .... (2)

(A) -2^{k-1} (B) 2^{k-1} (C) – 1 (D) -2^k

28. For the difference equation $f(k+1) + \frac{1}{2}f(k) = \left(\frac{1}{2}\right)^k$, $k \geq 0$, f(0) = 0, F(z) is given by .... (2)

(A) $\dfrac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$ (B) $\dfrac{z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$ (C) $\dfrac{z}{\left(z+\frac{1}{3}\right)\left(z+\frac{1}{2}\right)}$ (D) $\dfrac{z}{\left(z-\frac{1}{2}\right)^2}$

29. For the difference equation $12f(k+2) - 7f(k+1) + f(k) = 0$, f(0) = 0, f(1) = 3, F(z) is given by .... (2)

(A) $\dfrac{36z}{12z^2 - 7z - 1}$ (B) $\dfrac{36z}{12z^2 + 7z + 1}$ (C) $\dfrac{36z}{12z^2 - 7z + 1}$ (D) $\dfrac{36z}{12z^2 + 7z - 1}$

30. For the difference equation $y_k - 4y_{k-2} = 1$, $k \geq 0$, Y(z) is given by .... (2)

(A) $\dfrac{z}{(z-1)(z^2-4)}$ (B) $\dfrac{1}{(1-4z^2)}$ (C) $\dfrac{z}{(z-1)(1-4z^2)}$ (D) $\dfrac{z^3}{(z-1)(z^2-4)}$

Answers

1. (A)	2. (D)	3. (C)	4. (B)	5. (C)	6. (A)	7. (B)	8.(D)
9. (A)	10. (B)	11. (C)	12. (D)	13. (A)	14. (D)	15. (C)	16. (B)
17. (D)	18. (A)	19. (B)	20. (C)	21. (D)	22.(C)	23. (A)	24. (B)
25. (D)	26. (C)	27. (A)	28. (B)	29. (C)	30. (D)		

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Ch. 5 Statistics, Correlation and Regression

Type : Measures of Central Tendencies, Dispersion and Moments : **Marks**

1. If the data is presented in the forms of frequency distribution then arithmetic mean $\bar{x}$ is given by $(N = \sum f)$ .... (1)

(A) $\dfrac{\sum fx}{N}$ (b) $\frac{1}{N}\Sigma f|x - A|$ (C) $N\sum fx$ (D) $\dfrac{\sum fx^2}{N}$

2. For the data presented in the form of frequency distribution, mean deviation (M.D.) from the average A is given by $(N = \sum f)$ .... (1)

(A) $\dfrac{\sum fx}{N}$ (B) $\sum f|x - A|$ (C) $\frac{1}{N}\sum f|x - A|$ (D) $\frac{1}{N}\Sigma f|x - A|^2$

3. If the data is presented in the form of frequency distribution then standard deviation σ is given by ($\bar{x}$ is arithmetic mean and $N = \sum f$) .... (1)

(A) $\frac{1}{N}\sum f(x - \bar{x})^2$ (B) $\sqrt{\frac{1}{N}\sum f(x - \bar{x})^2}$

(C) $\frac{\sum fx}{N}$ (D) $\frac{1}{N}\sum f|x - \bar{x}|$

4. If the data is presented in the form of frequency distribution then variance V is given by ($\bar{x}$ is arithmetic mean and $N = \sum f$) .... (1)

(A) $\frac{1}{N}\sum f|x - \bar{x}|$ (B) $\sqrt{\frac{1}{N}\sum f(x - \bar{x})^2}$

(C) $\frac{\sum fx}{N}$ (D) $\frac{1}{N}\sum f(x - \bar{x})^2$

5. To compare the variability of two or more than two series, coefficient of variation (C.V.) is obtained using ($\bar{x}$ is arithmetic mean and σ is standard deviation) .... (1)

(A) $\frac{\bar{x}}{\sigma} \times 100$ (B) $\frac{\sigma}{\bar{x}} \times 100$ (C) $\sigma \times \bar{x} \times 100$ (D) $\frac{\bar{x}}{\sigma^2} \times 100$

6. If the data is presented in the form of frequency distribution then r^th^ moment μ, about the arithmetic mean $\bar{x}$ of distribution is given by ($N = \sum f$) .... (1)

(A) $\frac{1}{N}\sum f^r (x + \bar{x})^r$ (B) $N \times \sum f(x - \bar{x})^r$

(C) $\frac{1}{N}\sum f^r (x - \bar{x})$ (D) $\frac{1}{N}\sum f(x - \bar{x})^r$

7. If the data is presented in the form of frequency distribution then 1^st^ moment μ_1 about the arithmetic mean $\bar{x}$ of distribution is ($N = \sum f$) .... (1)

(A) 1 (B) σ^2 (C) 0 (D) $\frac{1}{N}\sum f(x - \bar{x})^3$

8. If μ_1' and μ_2' are the first two moments of the distribution about certain number then second moment μ_2 of the distribution about the arithmetic mean is given by .... (1)

(A) $\mu_2' - (\mu_1')^2$ (B) $2\mu_2' - \mu_1'$

(C) $\mu_2' + (\mu_1')^2$ (D) $\mu_2' + 2(\mu_1')^2$

9. If μ_1', μ_2', μ_3' are the first three moments of the distribution about certain number then third moment μ_3 of the distribution about the arithmetic mean is given by .... (1)

(A) $\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$ (B) $\mu_3' - 3\mu_1' + (\mu_2')^3$

(C) $\mu_3' + 2\mu_2'\mu_1' + (\mu_3')^3$ (D) $\mu_3' + 3\mu_2'\mu_1' + (\mu_1')^2$

10. If $\mu_1', \mu_2', \mu_3', \mu_4'$ are the first four moments of the distribution about certain number then fourth moment μ_4 of the distribution about the arithmetic mean is given by .... (1)

(1) $\mu_4' + 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^4 + 3(\mu_1')^4$ (B) $\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

(C) $\mu_4' + 4\mu_3'\mu_1' - 6\mu_2'(\mu_1')^4 - 3(\mu_1')^4$ (D) $\mu_4' + 2\mu_3'\mu_1' - 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

11. If μ_1' be the first moment of the distribution about any number A then arithmetic mean $\bar{x}$ is given by .... (1)

(A) $\mu_1' + A$ (B) μ_1' (C) $\mu_1' - A$ (D) $\mu_1' A$

12. Second moment μ_2 about mean is .... (1)

(A) Mean (B) Standard deviation

(C) Variance (D) Mean deviation

13. Coefficient of skewness β_1 is given by .... (1)

(A) $\frac{\mu_2^3}{\mu_3^2}$ (B) $\frac{\mu_1^2}{\mu_2^3}$ (C) $\frac{\mu_2^2}{\mu_3^2}$ (D) $\frac{\mu_3^2}{\mu_2^3}$

14. Coefficient of kurtosis β_2 is given by .... (1)

(A) $\frac{\mu_4}{\mu_3}$ (B) $\frac{\mu_4}{\mu_2^2}$ (C) $\frac{\mu_3}{\mu_2^2}$ (D) $\frac{\mu_4}{\mu_2^3}$

15. For a distribution coefficient of kurtosis $\beta_2 = 2.5$, this distribution is .... (1)

(A) Leptokurtic (B) Mesokurtic (C) Platykurtic (D) None of these

16. For a distribution coefficient of kurtosis $\beta_2 = 3.9$, this distribution is .... (1)

(A) Leptokurtic (B) Mesokurtic (C) Platykurtic (D) None of these

17. The first four moments of a distribution about the mean are 0, 16, −64 and 162. Standard deviation of a distribution is .... (1)

(A) 21 (B) 12 (C) 16 (D) 4

18. Standard deviation of three numbers 9, 10, 11 is .... (2)

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\sqrt{\frac{2}{3}}$ (D) $\sqrt{2}$

19. Standard deviation of four numbers 9, 11, 13, 15 is .... (2)

(A) 2 (B) 4 (C) $\sqrt{6}$ (D) $\sqrt{5}$

20. From the given information $\sum x = 235$, $\sum x^2 = 6750$, $n = 10$. Standard deviation of x is .... (2)

(A) 11.08 (B) 13.08 (C) 8.08 (D) 7.6

21. Coefficient of variation of the data 1, 3, 5, 7, 9 is .... (2)

(A) 54.23 (B) 56.57 (C) 55.41 (D) 60.19

22. The standard deviation and arithmetic mean of the distribution are 12 and 45.5 respectively. Coefficient of variation of the distribution is .... (2)

(A) 26.37 (B) 32.43 (C) 12.11 (D) 22.15

23. The Standard Deviation and Arithmetic Mean of three distribution x, y, z are as follow:

	Arithmetic mean	**Standard deviation**
x	18.0	5.4
y	22.5	4.5
z	24.0	6.0

The more stable distribution is .... (2)

(A) x (B) y (C) z (D) x and z

24. The standard deviation and arithmetic mean of scores of three batsman x, y, z in ten inning during a certain season are

	Arithmetic mean	**Standard deviation**
x	50	24.43
y	46	25.495
z	40	27

The more consistent batsman is .... (2)

(A) y and z (B) y (C) z (D) x

25. The standard deviation and arithmetic mean of aggregate marks obtained three group of students x, y, z are as follow :

	Arithmetic mean	Standard deviation
x	532	11
y	831	9
z	650	10

The more variable group is .... (2)

(A) y and z (B) z (C) y (D) x

26. Arithmetic mean of four numbers is 16, one item 20 is replaced by 24, what is the new arithmetic mean .... (2)

(A) 15 (B) 17 (C) 18 (D) 16

27. The first moment of the distribution about the value 5 is 2. Arithmetic mean of the distribution is .... (2)

(A) 5 (B) 2 (C) 4 (D) 7

28. The first and second moments of the distribution about the value 3 are 2 and 20. Second moment about the mean is .... (2)

(A) 12 (B) 14 (C) 16 (D) 20

29. The first three moments of a distribution about the value 5 are 2, 20 and 40. Third moment about the mean is .... (2)

(A) −64 (B) 64 (C) 32 (D) −32

30. The first four moments of a distribution about the value 5 are 2, 20, 40 and 50. Fourth moment about the mean is .... (2)

(A) 160 (B) 162 (C) 210 (D) 180

31. The first moments of a distribution about the value 2 are −2, 12, −20 and 100. Fourth moment about the mean is .... (2)

(A) 200 (B) 190 (C) 170 (D) 180

32. The first three moments of a distribution about the value 2 are −2, 12, −20. Third moment about the mean is .... (2)

(A) 36 (B) 30 (C) 22 (D) 8

33. The first and second moments of the distribution about the value 2 are 1 and 16. Variance of the distribution is .... (2)

(A) 12 (B) 3 (C) 15 (D) 17

34. The second and third moments of a distribution about the arithmetic mean are 16 and −64 respectively. Coefficient of skewness β_1 is given by .... (2)

(A) −0.25 (B) 1 (C) 4 (D) −1

35. The second and fourth moments of a distribution about the arithmetic mean are 16 and 162 respectively. Coefficient of kurtosis β_2 is given by .... (2)

(A) 1 (B) 1.51 (C) 0.63 (D) 1.69

Answers

1. (A)	2. (C)	3. (B)	4. (D)	5. (B)	6. (D)	7. (C)	8. (A)
9. (A)	10. (B)	11. (A)	12. (C)	13. (D)	14. (B)	15. (C)	16. (A)
17. (D)	18. (C)	19. (D)	20. (A)	21. (B)	22. (A)	23. (B)	24. (D)
25. (D)	26. (B)	27. (D)	28. (C)	29. (A)	30. (B)	31. (D)	32. (A)
33. (C)	34. (B)	35. (C)					

Type II : Curve Fitting : **Marks**

1. For least square fit of the straight line y = ax + b with n points, the normal equations are .... (1)

(A) $a \sum x + nb = \sum y$,
$a \sum x^2 + b \sum x = \sum xy$

(B) $a \sum x^2 + nb = \sum x$,
$a\sum x + nb = \sum y$

(C) $a \sum y^2 + nb = \sum y$,
$a \sum y + b\sum x = \sum x$

(D) $a \sum y + b \sum x = nb$,
$a \sum x^2 + nb = \sum y$

2. For least square fit of the straight line x = ay + b with n points, the normal equations are .... (1)

(A) $a\sum x + nb = \sum y$, $a\sum x^2 + b\sum x = \sum xy$

(B) $a\sum y + nb = \sum x$, $a\sum y^2 + b\sum y = \sum xy$

(C) $a\sum x^2 + nb = \sum xy$, $a\sum y^2 + n\sum x = \sum x^2$

(D) $a\sum x + b\sum y = \sum x$, $a\sum x^2 + b\sum y^2 = \sum y$

3. For least square fit of the straight line ax + by = c with n points, the normal equation are .... (1)

(A) $\frac{c}{b}\sum x - n\frac{c}{b}\sum x$, $-\frac{a}{b}\sum x^2 + n\frac{c}{b} = \sum xy$

(B) $\frac{c}{b}\sum x + n\frac{c}{b} = \sum y$, $\frac{a}{b}\sum y + \frac{c}{b}\sum y^2 = \sum x$

(C) $\frac{c}{b}\sum x^2 + \frac{a}{b}\sum y = \sum x^2$, $\frac{c}{b}\sum x^2 + \frac{a}{b}\sum x = \sum y^2$

(D) $-\frac{a}{b}\sum x + n\frac{c}{b} = \sum y$, $-\frac{a}{b}\sum x^2 + \frac{c}{b}\sum x = \sum xy$

4. Least square fit for the straight line y = ax + b to the data (2)

x	1	2	3
y	5	7	9

is ....

(A) y = 2x + 4 (B) y = 2x – 3 (C) y = 2x + 3 (D) y = 3x – 4

5. Least square fit for the straight line x = ay + b to the data (2)

y	1	2	3
x	–1	1	3

is ....

(A) x = y + 1 (B) x = y + 5 (C) x = y – 5 (D) x = 2y – 3

6. Least square fit for the straight line y = ax + b to the data (2)

x	2	3	4
y	1	4	7

is ....

(A) y = 2x – 5 (B) y = 3x – 5 (C) y = 2x + 3 (D) y = 2x – 3

7. Least square fit for the straight line x = ay + b to the data (2)

y	0	1	2
x	2	5	8

is ....

(A) x = 3y – 1 (B) x = 3y + 1 (C) x = 3y + 2 (D) x = 3y – 4

8. Least square fit for the straight line y = ax + b to the data (2)

x	0	1	2
y	–1	1	3

is ....

(A) y = 2x – 1 (B) y = 2x + 3 (C) y = 2x – 4 (D) y = x + 3

9. Least square fit for the straight line x = ay + b to the data (2)

y	1	2	3
x	–1	3	7

is ....

(A) x = 2y – 5 (B) x = 4y + 4 (C) x = 4y – 5 (D) x = y + 2

10. Least square fit for the straight line ax + by = c to the data (2)

x	0	1	2
y	$-\frac{4}{3}$	$-\frac{2}{3}$	0

is ....

(A) 2x + 3y = 4 (B) x – 3y = 4 (C) 2x + y = 4 (D) 2x – 3y = 4

11. For least square fit of the straight line y = ax + b to the data

x	0	1	2
y	– 1	1	3

the normal equations are .... (2)

(A) 3a + 3b = 3
5a + 3b = 7

(B) 3a + 3b = 3
3a + 5b = 7

(C) 3a + 3b = 3
5a + 7b = 3

(D) 3a + 3b = 7
5a + 3b = 3

12. For least square fit of the straight line y = ax + b to the data

x	2	3	4
y	1	4	7

the normal equations are .... (2)

(A) 9a + 3b = 42
29a + 9b = 12

(B) 9a + 3b = 12
9a + 29b = 42

(C) 9a + 3b = 12
29a + 9b = 42

(D) 9a + 3b = 12
29a + 42b = 9

13. For least square fit of the straight line x = ay + b to the data

y	1	4	7
x	2	3	4

the normal equations are .... (2)

(A) 12a + 3b = 9
12a + 66b = 42

(B) 12a + 3b = 9
66a + 12b = 42

(C) 12a + 3b = 9
66a + 42b = 12

(D) 12a + 3b = 42
66a + 12b = 9

14. For least square fit of the straight line x = ay + b to the data

y	1	3	5
x	5	9	13

the normal equations are .... (2)

(A) 9a + 3b = 27
9a + 35b = 97

(B) 9a + 3b = 97
35a + 9b = 27

(C) 9a + 3b = 27
35a + 97b = 9

(D) 9a + 3b = 27
35a + 9b = 97

15. Least square fit for the curve $y = ax^b$ to the data (2)

y	1	2	3
x	2	16	54

is ....

(A) $y = 2x^3$ (B) $y = 2x^2$ (C) $y = 3x^2$ (D) $y = 4x^3$

16. Least square fit for the curve $y = ax^b$ to the data (2)

x	1	2	3
y	3	12	27

is ....

(A) $y = 3x^3$ (B) $y = 2x^3$ (C) $y = 3x^2$ (D) $y = 2x^2$

17. Least square fit for the curve $y = ax^b$ to the data (2)

x	2	4	6
y	2	16	54

is ....

(A) $y = \frac{1}{4}x^3$ (B) $y = \frac{1}{4}x^2$ (C) $y = 2x^3$ (D) $y = \frac{1}{2}x^3$

18. Least square fit for the curve $y = ax^b$ to the data (2)

x	1	3	5
y	2	18	50

is ....

(A) $y = 2x^3$ (B) $y = 2x^2$ (C) $y = 3x^2$ (D) $y = 4x^2$

19. Least square fit for the curve $x = ay^b$ to the data (2)

y	2	4	6
x	8	32	72

is ....

(A) $x = 3y^2$ (B) $x = 2y^3$ (C) $x = y^3$ (D) $x = 2y^2$

20. Least square fit for the curve $x = ay^b$ to the data (2)

y	1	2	3
x	3	12	27

is

(A) $x = 2y^3$ (B) $x = 3y^3$ (C) $x = 3y^2$ (D) $x = 2y^2$

21. Least square fit for the curve $x = ay^b$ to the data (2)

y	1	3	5
y	4	36	100

is ....

(A) $x = 3y^2$ (B) $x = 2y^4$ (C) $x = 4y^2$ (D) $x = 4y^3$

22. Least square fit for the curve $x = ay^b$ to the data (2)

y	2	4	6
x	2	16	54

is ....

(A) $x = \frac{1}{4}y^3$ (B) $x = \frac{1}{4}y^4$ (C) $x = \frac{1}{2}y^3$ (D) $x = \frac{1}{4}y^2$

23. For the least square fit of the parabola $y = ax^2 + bx + c$ with n points, the normal equations are .... (1)

(A) $a\sum x^2 + b\sum x + nc = y$
$a\sum x^3 + b\sum x^2 + c\sum x = xy$
$a\sum x^4 + b\sum x^3 + c\sum x^2 = x^2y$

(B) $a\sum x^2 + b\sum x + nc = \sum y$
$a\sum y^3 + b\sum y^2 + c\sum y = \sum xy$
$a\sum y^4 + b\sum y^3 + c\sum y^2 = \sum y^2x$

(C) $a\sum x^2 + b\sum x + nc = \sum y$
$a\sum x^3 + b\sum x^2 + c\sum x = \sum xy$
$a\sum x^4 + b\sum x^3 + c\sum x^2 = \sum x^2y$

(D) $a\sum x^2 + b\sum x + nc = \sum y$
$a\sum x^3 + b\sum x^2 + ny = \sum x$
$a\sum x^4 + b\sum x^3 + c\sum x^2 = x^2y$

24. For the least square fit of the parabola $x = ay^2 + by + c$ with n points, the normal equations are .... (1)

(A) $a\Sigma x^2 + b\Sigma x + nc = \Sigma y$
$a\Sigma x^3 + b\Sigma x^2 + c\Sigma x = \Sigma xy$
$a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 = \Sigma x^2y$

(B) $a\Sigma y^2 + b\Sigma y + nc = \Sigma x$
$a\Sigma y^3 + b\Sigma y^2 + c\Sigma y = \Sigma xy$
$a\,\Sigma y^4 + b\Sigma y^3 + c\Sigma y^2 = \Sigma y^2x$

(C) $a\Sigma y^2 + b\Sigma y + nc = x$
$a\Sigma y^3 + b\Sigma y^2 + c\Sigma y = xy$
$a\Sigma y^4 + b\Sigma y^3 + c\Sigma y^2 = y^2x$

(D) $a\Sigma y^2 + b\Sigma y + nc = \Sigma x$
$a\Sigma y^3 + b\Sigma y^2 + c\Sigma y = \Sigma y$
$a\,\Sigma y^4 + b\Sigma y^3 + nc = \Sigma y^2$

25. For least square fit of the parabola $y = ax^2 + bx + c$ to the data

x	0	1	2
y	4	3	6

the normal equations are .... (2)

(A) $5a + 3b + 3c = 0$
$9a + 5b + 3c = 0$
$17a + 9b + 5c = 0$

(B) $5a + 3b + 3c = 15$
$9a + 5b + 3c = 27$
$17a + 9b + 5c = 13$

(C) $13a + 3b + 3c = 13$
$9a + 13b + 3c = 15$
$17a + 9b + 13c = 27$

(D) $5a + 3b + 3c = 13$
$9a + 5b + 3c = 15$
$17a + 9b + 5c = 27$

26. For least square fit of the parabola $y = ax^2 + bx + c$ to the data

x	0	1	2
y	2	2	4

the normal equations are .... (2)

(A) $5a + 3b + 3c = 8$
$9a + 5b + 3c = 10$
$17a + 9b + 5c = 18$

(B) $5a + 3b + 3c = 18$
$9a + 5b + 3c = 8$
$17a + 9b + 5c = 10$

(C) $17a + 3b + 3c = 8$
$9a + 17b + 3c = 10$
$17a + 9b + 17c = 18$

(D) $5a + 3b + 3c = 0$
$9a + 5b + 3c = 0$
$17a + 9b + 5c = 0$

27. For least square fit of the parabola $x = ay^2 + by + c$ to the data

y	1	2	3
x	3	7	13

the normal equations are .... (2)

(A) $3a + 6b + 3c = 23$
$36a + 3b + 6c = 56$
$98a + 36b + 3c = 148$

(B) $14a + 6b + 3c = 0$
$36a + 14b + 6c = 0$
$98a + 36b + 14c = 0$

(C) $14a + 6b + 3c = 23$
$36a + 14b + 6c = 56$
$98a + 36b + 14c = 148$

(D) $14a + 6b + 3c = 148$
$36a + 14b + 6c = 23$
$98a + 36b + 14c = 56$

28. For least square fit of the parabola $x = ay^2 + by + c$ to the data

y	0	1	3
x	3	6	24

is normal equations are .... (2)

(A) $10a + 4b + 3c = 0$
$28a + 10b + 4c = 0$
$82a + 28b + 10c = 0$

(B) $4a + 10b + 3c = 33$
$28a + 4b + 10c = 78$
$82a + 28b + 4c = 222$

(C) $10a + 4b + 3c = 78$
$28a + 10b + 4c = 33$
$82a + 28b + 10c = 222$

(D) $10a + 4b + 3c = 33$
$28a + 10b + 4c = 78$
$82a + 28b + 10c = 222$

29. For the least square parabolic fit of the parabola $y = ax^2 + bx + c$ with 3 points data given as $\sum x = 3$, $\sum x^2 = 5$, $\sum x^3 = 9$, $\sum x^4 = 17$, $\sum y = 13$, $\sum xy = 15$, $\sum x^2 y = 27$, the normal equations are .... (2)

(A) $5a + 3b + 3c = 13$
$9a + 5b + 3c = 15$
$9a + 17b + 5c = 27$

(B) $5a + 3b + 3c = 13$
$9a + 5b + 3c = 15$
$17a + 9b + 5c = 27$

(C) $5a + 3b + 3c = 13$
$9a + 5b + 3c = 15$
$17a + 9b + 27c = 5$

(D) $5a + 3b + 3c = 13$
$9a + 5b + 3c = 15$
$9a + 17b + 5c = 27$

30. For the least square parabolic fit of the parabola $y = ax^2 + bx + c$ with 3 points data given as $\sum x = 3$, $\sum x^2 = 5$, $\sum x^3 = 9$, $\sum x^4 = 17$, $\sum y = 8$, $\sum xy = 10$, $\sum xy^2 = 18$, the normal equations are .... (2)

(A) $5a + 3b + 3c = 8$
$9a + 5b + 3c = 10$
$17a + 9b + 5c = 18$

(B) $5a + 3b + 3c = 8$
$9a + 5b + 3c = 10$
$9a + 17b + 5c = 18$

(C) $5a + 3b + 3c = 8$
$9a + 5b + 3c = 10$
$17a + 9b + 18c = 5$

(D) $5a + 3b + 3c = 8$
$9a + 5b + 3c = 10$
$9a + 17b + 5c = 18$

31. For the least square parabolic fit of the parabola $x = ay^2 + by + c$ with 3 points data given as $\sum y = 9$, $\sum y^2 = 35$, $\sum y^3 = 153$, $\sum y^4 = 707$, $\sum x = 82$, $\sum xy = 350$, $\sum y^2 x = 1602$, the normal equations are .... (2)

(A) $35a + 9b + 3c = 82$
$153a + 35b + 9c = 350$
$707a + 153b + 35c = 1602$

(B) $35a + 9b + 3c = 82$
$153a + 35b + 9c = 350$
$707a + 153b + 1602c = 35$

(C) $35a + 9b + 3c = 350$
$153a + 35b + 9c = 1602$
$707a + 153b + 35c = 82$

(D) $35a + 9b + 3c = 82$
$153a + 35b + 9c = 350$
$153a + 707b + 35c = 1602$

32. For the least square parabolic fit of the parabola $x = ay^2 + by + c$ with 3 points data given as $\sum y = 6$, $\sum y^2 = 14$, $\sum y^3 = 36$, $\sum y^4 = 96$, $\sum x = 30$, $\sum xy = 70$, $\sum y^2 x = 180$, the normal equations are .... (2)

(A) $14a + 6b + 3c = 30$
$36a + 14b + 6c = 70$
$96a + 14b + 36c = 180$

(B) $14a + 6b + 3c = 70$
$36a + 14b + 6c = 180$
$96a + 36b + 14c = 30$

(C) $14a + 6b + 3c = 30$
$36a + 14b + 6c = 70$
$96a + 36b + 14c = 180$

(D) $14a + 6b + 3c = 30$
$36a + 14b + 6c = 70$
$36a + 96b + 14c = 180$

Answers

1. (A)	2. (B)	3. (D)	4. (C)	5. (D)	6. (B)	7. (C)	8. (A)
9. (C)	10. (D)	11. (A)	12. (C)	13. (B)	14. (D)	15. (A)	16. (C)
17. (A)	18. (B)	19. (D)	20. (C)	21. (C)	22. (A)	23. (C)	24. (B)
25. (D)	26. (A)	27. (C)	28. (D)	29. (B)	30. (A)	31. (A)	32. (C)

•••

Type III : Correlation and Regression : **Marks**

1. Covariance between two variables x and y is given by .... (1)

(A) $\frac{1}{n}\sum (x - \bar{x})(y - \bar{y})$ (B) $\frac{1}{n}\sum (x + \bar{x})(y + \bar{y})$

(C) $n\sum (x - \bar{x})(y - \bar{y})$ (D) $\frac{1}{n}\sum [(x - \bar{x}) + (y - \bar{y})]$

2. Correlation coefficient r between two variables x and y is given by .... (1)

(A) $\frac{\text{cov}(x, y)}{\sigma_x^2 \sigma_y^2}$ (B) $\frac{\sigma_y}{\sigma_x}$ (C) $\frac{\sigma_x}{\sigma_y}$ (D) $\frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$

3. Range of coefficient of correlation r is .... (1)

(A) $-\infty < \frac{1}{r} < \infty$ (B) $-\infty < r < \infty$ (C) $-1 \le r \le 1$ (D) $0 \le r \le 1$

4. Probable error of coefficient of correlation r is .... (1)

(A) $0.6745\left(\frac{1 + r^2}{\sqrt{N}}\right)$ (B) $0.6745\left(\frac{1 - r^2}{\sqrt{N}}\right)$

(C) $0.6745\left(\frac{1 - r^2}{N}\right)$ (D) $0.6547\left(\frac{1 - r^2}{N}\right)$

5. Line of regression y on x is .... (1)

(A) $y + \bar{y} = r\frac{\sigma_x}{\sigma_y}(x + \bar{x})$ (B) $x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$

(C) $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ (D) $y - \bar{y} = r\frac{\sigma_x}{\sigma_y}(x - \bar{x})$

6. Line of regression x on y is .... (1)

(A) $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ (B) $x + \bar{x} = r\frac{\sigma_x}{\sigma_y}(y + \bar{y})$

(C) $x - \bar{x} = r\frac{\sigma_y}{\sigma_x}(y - \bar{y})$ (D) $x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$

7. Slope of regression line of y on x is .... (1)

(A) r (x, y) (B) $r\frac{\sigma_y}{\sigma_x}$ (C) $r\frac{\sigma_x}{\sigma_y}$ (D) $\frac{\sigma_y}{\sigma_x}$

8. Slope of regression line of x on y is .... (1)

(A) $r\frac{\sigma_x}{\sigma_y}$ (B) r (x, y) (C) $\frac{\sigma_x}{\sigma_y}$ (D) $r\frac{\sigma_y}{\sigma_x}$

9. In regression line y on x, b_{yx} is given by .... (1)

(A) cov (x, y) (B) r (x, y) (C) $\frac{\text{cov}(x, y)}{\sigma_x^2}$ (D) $\frac{\text{cov}(x, y)}{\sigma_y^2}$

10. In regression line x on y, b_{xy} is given by .... (1)

(A) cov (x, y) (B) r (x, y) (C) $\frac{\text{cov}(x, y)}{\sigma_x^2}$ (D) $\frac{\text{cov}(x, y)}{\sigma_y^2}$

11. If b_{xy} and b_{yx} are the regression coefficient x on y and y on x respectively then the coefficient of correlation r(x, y) is given by .... (1)

(A) $\sqrt{b_{xy} + b_{yx}}$ (B) $b_{xy}\, b_{yx}$ (C) $\sqrt{\frac{b_{xy}}{b_{yx}}}$ (D) $\sqrt{b_{xy}\, b_{yx}}$

12. If θ is the acute angle between the regression line of y on x and the regression line of x on y, then tan θ is .... (1)

(A) $\frac{(1-r^2)}{|r|} \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}$ (B) $\frac{|r|}{(1-r^2)} \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}$

(C) $|r| \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}$ (D) $\frac{1}{|r|} \frac{\sigma_x^2 + \sigma_y^2}{\sigma_x\sigma_y}$

13. If $\sum xy = 2638$, $\bar{x} = 14$, $\bar{y} = 17$, n = 10 then cov (x, y) is ....

(A) 24.2 (B) 25.8 (C) 23.9 (D) 20.5

14. If $\sum xy = 1242$, $\bar{x} = -5.1$, $\bar{y} = -10$, n = 10, then cov (x, y) is .... (2)

(A) 67.4 (B) 83.9 (C) 58.5 (D) 73.2

15. If $\sum x^2 = 2291$, $\sum y^2 = 3056$, $\sum (x+y)^2 = 10623$, n = 10, $\bar{x} = 14.7$, $\bar{y} = 17$ then cov (x, y) is .... (2)

(A) 1.39 (B) 13.9 (C) 139 (D) – 13.9

16. If the two regression coefficient are 0.16 and 4 then the correlation coefficient is .... (2)

(A) 0.08 (B) – 0.8 (C) 0.8 (D) 0.64

17. If the two regression coefficient are $-\frac{8}{15}$ and $-\frac{5}{6}$ then the correlation coefficient is .... (2)

(A) – 0.667 (B) 0.5 (C) – 1.5 (D) 0.537

18. If covariance between x and y is 10 and the variance of x and y are 16 and 9 respectively then coefficient of correlation r(x, y) is .... (2)

(A) 0.833 (B) 0.633 (C) 0.527 (D) 0.745

19. If cov (x, y) = 25.8 $\sigma_x = 6$, $\sigma_y = 5$ then correlation coefficient r(x, y) is equal to .... (2)

(A) 0.5 (B) 0.75 (C) 0.91 (D) 0.86

20. If $\sum xy = 190$, $\bar{x} = 4$, $\bar{y} = 4$, n = 10, $\sigma_x = 1.732$, $\sigma_y = 2$ then correlation coefficientr(x, y) is equal to .... (2)

(A) 0.91287 (B) 0.8660 (C) 0.7548 (D) 0.5324

21. If $\sum xy = 2800$, $\bar{x} = 16$, $\bar{y} = 16$, n = 10, variance of x is 36 and variance of y is 25 then correlation coefficient r(x, y) is equal to .... (2)

(A) 0.95 (B) 0.73 (C) 0.8 (D) 0.65

22. The correlation coefficient for the following data n = 10, $\sum x = 140$, $\sum y = 150$, $\sum x^2 = 1980$, $\sum y^2 = 2465$, $\sum xy = 2160$ is .... (2)

(A) 0.753 (B) 0.4325 (C) 0.556 (D) 0.9013

23. You are given the following information related to a distribution comprising 10 observation $\bar{x} = 5.5$, $\bar{y} = 4$, $\sum x^2 = 385$, $\sum y^2 = 192$, $\sum (x+y)^2 = 947$. The correlation coefficient r(x, y) is .... (2)

(A) – 0.924 (B) – 0.681 (C) – 0.542 (D) – 0.813

24. Given the following data r = 0.022, $\sum xy = 33799$, $\sigma_x = 4.5$, $\sigma_y = 64.605$, $\bar{x} = 68$, $\bar{y} = 62.125$. The value of n (number of observation) is .... (2)

(A) 5 (B) 7 (C) 8 (D) 10

25. Given the following data r = 0.5, $\sum xy = 350$, $\sigma_x = 1$, $\sigma_y = 4$, $\bar{x} = 3$, $\bar{y} = 4$. The value of n (number of observation) is .... (2)

(A) 25 (B) 5 (C) 20 (D) 15

26. Coefficient of correlation between the variables x and y is 0.8 and their covariance is 20, the variance of x is 16. Standard deviation of y is .... (2)

(A) 6.75 (B) 6.25 (C) 7.5 (D) 8.25

27. Line of regression y on x is 8x − 10y + 66 = 0. Lie of regression x on y is 40x − 18y − 214 = 0. Mean values of x and y are .... (2)

(A) $\bar{x} = 12$, $\bar{y} = 15$ (B) $\bar{x} = 10$, $\bar{y} = 11$

(C) $\bar{x} = 13$, $\bar{y} = 17$ (D) $\bar{x} = 9$, $\bar{y} = 8$

28. If the two lines of regression of $9x + y - \lambda = 0$ and $4x + y = \mu$ and the mean of x and y are 2 and −3 respectively then the values of λ and μ are (2)

(A) $\lambda = 15$ and $\mu = 5$ (B) $\lambda = -15$ and $\mu = -5$

(C) $\lambda = 5$ and $\mu = 15$ (D) $\lambda = 15$ and $\mu = -5$

29. Line of regression y on x is 8x − 10y + 66 = 0. Line of regression x on y is 40x − 18y − 214 = 0. Correlation coefficient r(x, y) is given by .... (2)

(A) 0.6 (B) 0.5 (C) 0.75 (D) 0.45

30. The regression lines are 9x + y = 15 and 4x + y = 5. Correlation r(x, y) is given by .... (2)

(A) 0.444 (B) − 0.11 (C) 0.663 (D) 0.7

31. Line of regression y on x is 8x − 10y + 66 = 0. Line of regression x on y is 40x − 18y − 214 = 0. The value of variance of x is 9. The standard deviation of y is equal to (2)

(A) 2 (B) 5 (C) 6 (D) 4

32. Line of regression y on x is 8x − 10y + 66 = 0. Line of regression x on y is 40x − 18y − 214 = 0. The value of variance of y is 16. The standard deviation of x is equal to .... (2)

(A) 3 (B) 2 (C) 6 (D) 7

33. Line of regression y on x is 3x + 2y = 26, line of regression x on y is 6x + y = 31. The value of variance of x is 25. Then the standard deviation of y is .... (2)

(A) − 15 (B) 15

(C) 1.5 (D) − 1.5

34. The correlation coefficient between two variable x and y is 0.6. If $\sigma_x = 1.5$, $\sigma_y = 2.00$, $\bar{x} = 10$, $\bar{y} = 20$ then the lines of regression are .... (2)

(A) x = 0.45y + 12 and y = 0.8x + 1(B) x = 0.45y + 1 and y = 0.8x + 12

(C) x = 0.65y + 10 and y = 0.4x + 12 (D)x = 0.8y + 1 and y = 0.45x + 12

35. The correlation coefficient between two variable x and y is 0.711. If $\sigma_x = 4$, $\sigma_y = 1.8$, $\bar{x} = 5$, $\bar{y} = 4$ then the lines of regression are .... (2)

(A) x − 5 = 1.58 (y − 4) and y − 4 = 0.32 (x − 5) (B) x + 5 = 1.58 (y + 4) and y + 4 = 0.32 (x + 5)

(C) x − 5 = 0.32 (y − 4) and y − 4 = 1.58 (x − 5) (D) x − 4 = 1.58 (y − 5) and y − 5 = 0.32 (x − 4)

36. Your are given below the following information about advertisement expenditure and sales

	Adv. Expenditure (X) ₹ (Crore)	Sales (Y) ₹ (Crore)
Mean	10	90
Standard Deviation	3	12

Correlation coefficient = 0.8. The two lines of regression are .... (2)

(A) x = 58 + 3.2y and y = −8 + 0.2x (B) x = −8 + 2.2y and y = 8 + 1.2x

(C) x = −8 + 3.2y and y = 58 + 0.2x (D) x = −8 + 0.2y and y = 58 + 3.2x

37. You are given below the following information about rainfall and production of rice

	Rainfall (X) in inches	Production of Rice (Y) in Kg
Mean	30	500
Standard Deviation	5	100

Correlation coefficient = 0.8. The two lines of regression are .... (2)

(A) $x + 30 = 0.04 (y + 500)$ and $y + 500 = 6 (x + 30)$
(B) $x - 30 = 0.4 (y - 500)$ and $y - 500 = 1.6 (x - 30)$
(C) $x - 30 = 0.04 (y - 500)$ and $y - 500 = 16 (x - 30)$
(D) $x - 30 = 16 (y - 500)$ and $y - 500 = 0.04 (x - 30)$

38. Given $b_{xy} = 0.85$, $b_{yx} = 0.89$ and the standard deviation of x is 6 then the value of correlation coefficient r(x, y) and standard deviation of y is .... (2)

(A) $r = 0.87, \sigma_y = 6.l4$
(B) $r = -0.87, \sigma_y = 0.614$
(C) $r = 0.75, \sigma_y = 6.14$
(D) $r = 0.89, \sigma_y = 4.64$

39. Given $b_{xy} = 0.8411$, $b_{yx} = 0.4821$ and the standard deviation of y is 1.7916 then the value of correlation coefficient r(x, y) and standard deviation of x is .... (2)

(A) $r = -0.6368$ and $\sigma_x = -2.366$
(B) $r = 0.63678$ and $\sigma_x = 2.366$
(C) $r = 0.40549$ and $\sigma_x = 2.366$
(D) $r = 0.63678$ and $\sigma_x = 5.6$

40. For a given set of Bivariate data $\bar{x} = 53.2$, $\bar{y} = 27.9$ Regression coefficient of y on x = −1.5. By using line of regression y on x the most probable value of y when x is 60 is .... (2)

(A) 15.7
(B) 13.7
(C) 17.7
(D) 21.7

41. Given the following data $\bar{x} = 36$, $\bar{y} = 85$, $\sigma_x = 11$, $\sigma_y = 8$, $r = 0.66$. By using line of regression x on y, the most probable value of x when y = 75 is .... (2)

(A) 29.143
(B) 24.325
(C) 31.453
(D) 26.925

42. For a given set of Bivariate data $\bar{x} = 2$, $\bar{y} = -3$ Regression coefficient of x on y = −0.11. By using line of regression x on y the most probable value of x when y is 10 is .... (2)

(A) 0.77
(B) 0.57
(C) 1.77
(D) 0.87

Answers

1. (A)	2. (D)	3. (C)	4. (B)	5. (C)	6. (D)	7. (B)	8. (A)
9. (C)	10. (D)	11. (D)	12. (A)	13. (B)	14. (D)	15. (B)	16. (C)
17. (A)	18. (A)	19. (D)	20. (B)	21. (C)	22. (D)	23. (B)	24. (C)
25. (A)	26. (B)	27. (C)	28. (A)	29. (A)	30. (C)	31. (D)	32. (A)
33. (B)	34. (B)	35. (A)	36. (D)	37. (C)	38. (A)	39. (B)	40. (C)
41. (D)	42. (B)						

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Ch. 6 Probability and Porbability Distributions

Type I : Probability **Marks**

1. A throw is made with two dice. The probability of getting a score of 10 points is .... (1)

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{5}$ (D) $\frac{2}{3}$

2. A throw is made with two dice. The probability of getting a score of at least 10 points is .... (1)

(A) $\frac{1}{12}$ (B) $\frac{5}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{6}$

3. In a single throw of two dice, the probability of getting more than 7 points is .... (2)

(A) $\frac{7}{36}$ (B) $\frac{7}{12}$ (C) $\frac{5}{12}$ (D) $\frac{5}{36}$

4. In a single throw of two dice, the probability that the total score is a prime number is .... (2)

(A) $\frac{1}{6}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{5}{36}$

5. A throw is made with two dice. The probability of getting score a perfect square is .... (2)

(A) $\frac{11}{36}$ (B) $\frac{7}{36}$ (C) $\frac{10}{36}$ (D) $\frac{1}{4}$

6. A card is drawn from a well shuffled a pack of 52 cards, the probability of getting a club card is .... (2)

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

7. Two cards are drawn from a well shuffled a pack of 52 cards, the probability that both the cards are spade is .... (2)

(A) $\frac{1}{26}$ (B) $\frac{1}{4}$ (C) $\frac{1}{17}$ (D) $\frac{1}{13}$

8. Three cards are drawn from a well shuffled a pack of 52 cards, the probability of getting all of them red is .... (2)

(A) $\frac{3}{17}$ (B) $\frac{5}{17}$ (C) $\frac{4}{17}$ (D) $\frac{2}{17}$

9. A card is drawn from a well shuffled a pack of 52 cards. The probability of getting a queen of club or king of heart is .... (2)

(A) $\frac{1}{52}$ (B) $\frac{1}{26}$ (C) $\frac{1}{18}$ (D) $\frac{1}{12}$

10. Two cards are drawn from a well shuffled a pack of 52 cards. If the first card drawn is replaced, the probability that they are both kings is .... (2)

(A) $\frac{1}{15}$ (B) $\frac{1}{442}$ (C) $\frac{1}{169}$ (D) $\frac{2}{221}$

11. Two cards are drawn from a well shuffled a pack of 52 cards. If the first card drawn is not replaced, the probability that they are both kings is .... (2)

(A) $\frac{1}{221}$ (B) $\frac{1}{17}$ (C) $\frac{1}{15}$ (D) $\frac{2}{221}$

12. If A and B are two events such that P(A) = 0.4, P $(A \cup B)$ = 0.7 and P $(A \cap B)$ = 0.2 then P (B) = ... (2)

(A) 0.1 (B) 0.3 (C) 0.7 (D) 0.5

13. If A and B are any two mutually exclusive events such that P (A) = 0.4, P(B) = 0.2 then P $(A \cup B)$ = ... (2)

(A) 0.8 (B) 0.4 (C) 0.6 (D) 0.7

14. A ball is drawn from a box containing 6 red balls, 4 white balls and 5 black balls. The probability that it is not red is .... (2)

(A) $\frac{4}{15}$ (B) $\frac{1}{3}$ (C) $\frac{2}{5}$ (D) $\frac{3}{5}$

15. The probability of drawing a white ball from a bag containing 3 black and 4 white balls is .... (2)

(A) $\frac{3}{7}$ (B) $\frac{4}{7}$ (C) $\frac{1}{7}$ (D) $\frac{2}{7}$

16. The chances to fail in physics are 20% and the chances to fail in mathematics are 10%. The chances to fail in at least one subject is .... (2)

(A) 28% (B) 38% (C) 52% (D) 62%

17. Probability that a leap year selected at random will contain 53 Sunday is .... (2)

(A) $\frac{1}{7}$ (B) $\frac{6}{7}$ (C) $\frac{3}{7}$ (D) $\frac{2}{7}$

18. Probability that a non leap year (ordinary year) has 53 Sunday is .... (2)

(A) $\frac{6}{7}$ (B) $\frac{1}{7}$ (C) $\frac{3}{7}$ (D) $\frac{2}{7}$

19. In a simultaneous throw of three coins the probability of getting at least two tail is .... (2)

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$

20. Three coins are tossed simultaneously. The probability of getting at most two head is .... (2)

(A) $\frac{7}{8}$ (B) $\frac{3}{8}$ (C) $\frac{5}{8}$ (D) $\frac{1}{8}$

21. A coin is tossed and a dice is rolled. The probability that the coin shows the head and dice shows 6 is .... (2)

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{2}{3}$

22. An envelope contains six tickets with numbers 1, 2, 3, 5, 6, 7. Another envelope contains four tickets with numbers 1, 3, 5, 7. An envelope is chosen at random and ticket is drawn from it. Probability that the ticket bears the numbers 2 or 7 is .... (2)

(A) $\frac{1}{6}$ (B) $\frac{7}{24}$ (C) $\frac{1}{8}$ (D) $\frac{5}{24}$

23. There are six married couples in a room. If two persons are chosen at random, the probability that they are of different sex is .... (2)

(A) $\frac{3}{11}$ (B) $\frac{1}{11}$ (C) $\frac{5}{11}$ (D) $\frac{6}{11}$

24. A, B play a game of alternate tossing a coin, one who gets head first wins the game. The probability of B winning the game if A has start is .... (2)

(A) $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ (B) $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots$

(C) $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots$ (D) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$

25. A, B play a game of alternate tossing a coin, one who gets head first wins the game. The probability of A winning the game if A has start is .... (2)

(A) $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots$ (B) $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

(C) $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots$ (D) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$

26. If A and B are two independent events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{5}$ then $P(A \cap B) =$ .... (2)

(A) $\frac{1}{15}$ (B) $\frac{1}{5}$ (C) $\frac{2}{5}$ (D) $\frac{1}{10}$

27. If A and B are two independent events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ then $P(A \cup B) =$ .... (2)

(A) $\frac{3}{5}$ (B) $\frac{2}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

28. If A and B are two independent events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ then $P(\bar{A} \cap \bar{B})$ [i.e. P (neither A nor B)] = ... (2)

(A) $\frac{5}{6}$ (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

29. A can hit the target 2 out of 5 times, B can hit the target 1 out of 3 times, C can hit the target 3 out of 4 times. The probability that all of them hit the target is .... (2)

(A) $\frac{9}{10}$ (B) $\frac{4}{10}$ (C) $\frac{1}{10}$ (D) $\frac{7}{10}$

30. A can hit the target 3 out of 5 times, B can hit the target 1 out of 3 times. The probability that no one can hit the target is .... (2)

(A) $\frac{7}{15}$ (B) $\frac{3}{5}$ (C) $\frac{1}{15}$ (D) $\frac{4}{15}$

31. A problem in statistics is given to three students A, B, C whose chance of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. The probability that all of them can solved the problem is .... (2)

(A) $\frac{1}{8}$ (B) $\frac{1}{24}$ (C) $\frac{1}{12}$ (D) $\frac{1}{6}$

32. The probability that A can solve a problem is $\frac{2}{3}$ and B can solve it is problem is $\frac{3}{4}$. If both attempt the problem, then the probability that the problem get solved is .... (2)

(A) $\frac{11}{12}$ (B) $\frac{7}{12}$ (C) $\frac{5}{12}$ (D) $\frac{9}{12}$

33. If A and B are any two events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$ then $P(A|B)$ = ... (2)

(A) $\frac{1}{3}$ (B) $\frac{3}{4}$ (C) $\frac{1}{4}$ (D) $\frac{2}{3}$

34. If A and B are any two events with $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{1}{2}$ then $P(A|B)$ = ... (2)

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{2}{3}$ (D) $\frac{1}{4}$

35. If A and B are any two events with P(A) = 0.25, P(B) = 0.15 and $P(A \cup B) = 0.3$ then $P(B|A)$ = ... (2)

(A) 0.1 (B) 0.6 (C) 0.4 (D) 0.5

36. In a class 40% students read statistics, 25% read mathematics and 15% read both statistics and mathematics. One student is selected at random. The probability that he read statistics if it is known that he read mathematics is .... (2)

(A) 0.6 (B) 0.7 (C) 0.5 (D) 0.4

Answers

1. (A)	2. (D)	3. (C)	4. (B)	5. (B)	6. (A)	7. (C)	8. (D)
9. (B)	10. (C)	11. (A)	12. (D)	13. (C)	14. (D)	15. (B)	16. (A)
17. (D)	18. (B)	19. (C)	20. (A)	21. (A)	22. (B)	23. (D)	24. (C)
25. (A)	26. (D)	27. (B)	28. (C)	29. (C)	30. (D)	31. (B)	32. (A)
33. (B)	34. (D)	35. (C)	36. (A)				

Type II : Mathematical Expectation. **Marks**

1. Three coins are tossed together, x the random variable which denote the number of heads with distribution give (2)

x	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

the mathematical expectation E (x) is ....

(A) $\frac{13}{8}$ (B) $\frac{3}{2}$ (C) $\frac{9}{8}$ (D) $\frac{2}{3}$

2. The probability distribution of x is (2)

x	1	2	3	4
P(x)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

the mathematical expectation E (x) is ....

(A) $\frac{11}{8}$ (B) $\frac{13}{8}$ (C) $\frac{15}{8}$ (D) $\frac{9}{8}$

3. The probability distribution of x is (2)

x	1	2	3	4
P(x)	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

the mathematical expectation E (x) is ....

(A) 2 (B) 3 (C) 5 (D) 7

4. If x is random variable with distribution given below (2)

x	0	1	2	3
P(x)	k	3k	3k	k

the value of k is ....

(A) $\frac{1}{4}$ (B) $\frac{1}{6}$ (C) $\frac{1}{8}$ (D) $\frac{2}{3}$

5. If x is random variable with distribution given below (2)

x	2	3	4	5
P(x)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

the value of k is ....

(A) 16 (B) 8 (C) 48 (D) 32

6. Let f (x) be the continuous probability density function of random variable x then $P(a \le x \le b)$ is (1)

(A) $\int_a^b f(x)\,dx$ (B) f (b) – f (a) (C) f (b – a) (D) $\int_a^b x\,f(x)\,dx$

7. If probability density function f (x) of a continuous random variable x is defined by $f(x) = \begin{cases} \frac{1}{4}, & -2 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$, then $P(x \le 1)$ is .... (2)

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{3}{4}$

8. If probability density function f (x) of a continuous random variable x is defined by $f(x) = \begin{cases} \frac{3}{2}x^2, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$, then $P\left(-\frac{1}{3} \le x \le \frac{1}{3}\right)$ is .... (2)

(A) $\frac{2}{27}$ (B) $\frac{1}{27}$ (C) $\frac{1}{3}$ (D) $\frac{1}{9}$

9. If probability density function f (x) of a continuous random variable x is defined by $f(x) = \begin{cases} \frac{A}{x^3}, & 5 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$, then the value of A is .... (2)

(A) $\frac{50}{3}$ (B) $\frac{250}{3}$ (C) $\frac{100}{3}$ (D) $\frac{200}{3}$

Answers

1. (B)	2. (C)	3. (B)	4. (C)	5. (D)	6. (A)	7. (D)	8. (B)	9. (D)

Type III : Probability Distributions. **Marks**

1. In binomial probability distribution, probability of r successes in n trials is (where p probability of successes and q probability of failure in a single trial) .... (1)

(A) $p^r q^{n-r}$ (B) $^nC_r\, p^r\, q^{n+r}$ (C) $^nC_r p^r q^{n-r}$ (D) $^rC_n p^n q^{n-r}$

2. Mean of binomial probability distribution is .... (1)

(A) nq (B) n^2p (C) npq (D) np

3. Variance of binomial probability distribution is .... (1)

(A) npq (B) np (C) np^2q (D) npq^2

4. Standard deviation of binomial probability distribution is .... (1)

(A) $\sqrt{pq}$ (B) $\sqrt{npq}$ (C) $\sqrt{np}$ (D) np

5. An unbiased coin is thrown five times. Probability of getting three heads is .... (2)

(A) $\frac{1}{16}$ (B) $\frac{3}{16}$ (C) $\frac{5}{16}$ (D) $\frac{5}{8}$

6. 20% of bolts produced by machine are defective. The probability that out of three bolts chosen at random 1 is defective is .... (2)

(A) 0.384 (B) 0.9728 (C) 0.5069 (D) 0.6325

7. Probability of man now aged 60 years will live upto 70 years of age is 0.65. The probability that out of 10 men 60 years old 2 men will live upto 70 is .... (2)

(A) 0.5 (B) 0.002281 (C) 0.003281 (D) 0.004281

8. The probability that a person hit a target in shooting practice is 0.3. If the shoots 10 times, the probability that he hits the target is .... (2)

(A) 1 (B) $1 - (0.7)^{10}$ (C) $(0.7)^{10}$ (D) $(0.3)^{10}$

9. An unbiased coin is tossed five times. The probability of getting at least one head is .... (2)

(A) $\frac{1}{32}$ (B) $\frac{31}{32}$ (C) $\frac{16}{32}$ (D) $\frac{13}{32}$

10. A box contains 100 bulbs out of which 10 are defective. A sample of 5 bulbs is drawn. The probability that none is defective is .... (2)

(A) $\left(\frac{1}{10}\right)^5$ (B) $\left(\frac{1}{2}\right)^5$ (C) $\left(\frac{9}{10}\right)^5$ (D) $\frac{9}{10}$

11. On an average a packet containing 10 blades is likely to have two defective blades. In a box containing 100 packets, number of packets expected to contain less than two defective blades is .... (2)

(A) 38 (B) 52 (C) 26 (D) 47

12. Out of 2000 families with 4 children each, the number of families you would expect to have no girls is

p = probability of having a boy = $\frac{1}{2}$, q = probability of having a girl = $1 - \frac{1}{2} = \frac{1}{2}$ .... (2)

(A) 300 (B) 150 (C) 200 (D) 125

13. In 100 set of 10 tosses of a coin, the number of cases you expect 7 head and 3 tail is .... (2)

(A) 8 (B) 12 (C) 15 (D) 17

14. 20% of bolts produced by machine are defective. The mean and standard deviation of defective bolts in total of 900 bolts are respectively .... (2)

(A) 180 and 12 (B) 12 and 180 (C) 90 and 12 (D) 9 and 81

15. The mean and variance of binomial probability distribution are $\frac{5}{4}$ and $\frac{15}{16}$ respectively. Probability of success in a single trial p is equal to .... (2)

(A) $\frac{1}{2}$ (B) $\frac{15}{16}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

16. The mean and variance of binomial probability distribution are 6 and 4 respectively. Number of trials n is given by .... (2)

(A) 14 (B) 10 (C) 12 (D) 18

17. The mean and standard derivation of binomial probability distribution are 36 and 3 respectively. Number of trials n is given by .... (2)

(A) 42 (B) 36 (C) 48 (D) 24

18. The mean and variance of binomial probability distribution are 6 and 2 respectively. p $(r \geq 2)$ is .... (2)

(A) 0.66 (B) 0.88 (C) 0.77 (D) 0.99

19. If X follows the binomial distribution with parameter n = 6 and p and 9P (X = 4) = P(X = 2), then p is equal to .... (2)

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{3}{4}$ (D) $\frac{2}{3}$

20. If X follows the binomial distribution with parameter n and p = $\frac{1}{2}$ and P(X = 6) = P(X = 8), then n is equal to .... (2)

(A) 10 (B) 14 (C) 12 (D) 7

21. If X follows the binomial distribution with parameter n and p = $\frac{1}{2}$ and P (X = 4) = P(X = 5), then P(X = 2) is equal to .... (2)

(A) ${}^{7}C_2\left(\frac{1}{2}\right)^{7}$ (B) ${}^{11}C_2\left(\frac{1}{2}\right)^{11}$ (C) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$ (D) ${}^{9}C_2\left(\frac{1}{2}\right)^{9}$

22. The mean and variance of binomial probability distribution are 1 and $\frac{2}{3}$ respectively. Then p(r < 1) is .... (2)

(A) $\frac{4}{27}$ (B) $\frac{8}{27}$ (C) $\frac{5}{27}$ (D) $\frac{1}{27}$

23. In a binomial probability distribution, the probability of getting a success is $\frac{1}{4}$ and standard deviation is 3. Then its mean is .... (2)

(A) 6 (B) 8 (C) 12 (D) 10

24. A dice is thrown 10 times. If getting even number is considered as success, then the probability of getting four successes is .... (2)

(A) ${}^{10}C_4\left(\frac{1}{2}\right)^{10}$ (B) ${}^{10}C_4\left(\frac{1}{2}\right)^{4}$ (C) ${}^{10}C_4\left(\frac{1}{2}\right)^{8}$ (D) ${}^{10}C_4\left(\frac{1}{2}\right)^{6}$

25. A fair coin is tossed n number of times. In a binomial probability distribution, if the probability of getting 7 heads is equal to that of getting 9 then n is equal to .... (2)

(A) 7 (B) 2 (C) 9 (D) 16

26. If z = np where n the number of trials is very large and p the probability of success at each trial, then in Poisson's probability distribution p(r) the probability of r successes is given by .... (1)

(A) $\frac{e^{z} z}{r!}$ (B) $\frac{e^{-z} z^{r}}{r}$ (C) $\frac{e^{-z} z^{r}}{r!}$ (D) $\frac{e^{z} z^{r}}{r!}$

27. In a Poisson's probability distribution if n = 100, p = 0.01, p(r = 0) is given by .... (2)

(A) $\frac{1}{e}$ (B) $\frac{2}{e}$ (C) $\frac{3}{e}$ (D) $\frac{4}{e}$

28. In a Poisson's probability distribution if n = 100, p = 0.02, p(r = 1) is given by .... (2)

(A) $\frac{1}{e^2}$ (B) $\frac{2}{e^2}$ (C) $\frac{2}{e}$ (D) $\frac{1}{e}$

29. For a tabular data (2)

x	0	1	2	3
F	2	4	6	8

Poisson's fit p(r) is given by ....

(A) $\frac{e^{-1} 2^{r}}{r!}$ (B) $\frac{e^{-2} 2^{r}}{r!}$ (C) $\frac{e^{-2} 2^{3}}{r!}$ (D) $\frac{e^{-3} 3^{r}}{r!}$

30. For a tabulated data : (2)

x	0	1	2	3
f	1	4	15	24

Poisson's fit p(r) is given by ....

(A) $\frac{e^{-4.609}(4.609)^r}{r!}$ (B) $\frac{e^{-6.709}(6.709)^r}{r!}$ (C) $\frac{e^{-3.509}(3.509)^r}{r!}$ (D) $\frac{e^{-2.409}(2.409)^r}{r!}$

31. In a Poisson's probability distribution if p (r = 1) = 2p (r = 2) and p (r = 3) is given by .... (2)

(A) $\frac{1}{6e}$ (B) $\frac{2}{3e}$ (C) $\frac{1}{8e}$ (D) $\frac{1}{9e}$

32. In a Poisson's probability distribution if 3p (r = 4) = p (r = 5) and p (r = 6) is given by .... (2)

(A) $\frac{e^{-12}(12)^6}{6!}$ (B) $\frac{e^{-18}(18)^6}{6!}$ (C) $\frac{e^{-15}(15)^6}{6!}$ (D) $\frac{e^{-10}(10)^6}{6!}$

33. In a Poisson's probability distribution if p (r = 2) = 9p (r = 4) + 90p (r = 6) then mean of the distribution is .... (2)

(A) ± 1 (B) ± 2 (C) ± 3 (D) ± 4

34. Number of road accidents on a highway during a month follows a Poisson distribution with mean 2. Probability that in a certain month number of accidents on the highway will be equal to 2 is .... (2)

(A) 0354 (B) 0.2707 (C) 0.435 (D) 0.521

35. Between 2 P.M. and 3 P.M. the average number of phone calls per minute coming into company are 2. Using Poisson's probability distribution, the probability that during one particular minute there will be no phone call at all, is given by .... (2)

(A) 0.354 (B) 0.356 (C) 0.135 (D) 0.457

36. Average number of phone calls per minute coming into company are 3, during certain period. These calls follows Poisson's probability distribution. Probability that during one particular minute there will be less than two calls, is given by .... (2)

(A) 0.299 (B) 0.333 (C) 0.444 (D) 0.199

37. In a certain factory turning out razor blades, there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in a packets of 10. Using Poisson distribution, the probability that a packet contain one defective blade is .... (2)

(A) 0.0196 (B) 0.0396 (C) 0.0596 (D) 0.0496

38. The average number of misprints per page of a book is 1.5. Assuming the distribution of number of misprints to be Poisson. The probability that a particular book is free from misprints, is .... (2)

(A) 0.329 (B) 0.435 (C) 0.549 (D) 0.2231

39. The probability density function of normal variable x with mean μ and variance σ^2 is .... (2)

(A) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$ (B) $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$ (C) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ (D) $f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

40. Normal distribution curve is given by the equation $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Integral $\int_{\mu}^{\infty} y\,dx$ has the value .... (1)

(A) 0.025 (B) 1 (C) 0.5 (D) 0.75

41. Normal distribution curve is given by the equation $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Integral $\int_{-\infty}^{\infty} y\,dx$ has the value .... (1)

(A) 0.025 (B) 1 (C) 0.5 (D) 0.75

42. X is normally distributed. The mean of X is 15 and standard deviation 3. Given that for z = 1, A = 0.3413, p (X ≥ 18) is given by .... (2)

(A) 0.1587 (B) 0.4231 (C) 0.2231 (D) 0.3413

43. X is normally distributed. The mean of X is 15 and standard deviation 3. Given that for z = 1, A = 0.3413, p (X ≥ 12) is given by .... (2)

(A) 0.6587 (B) 0.8413 (C) 0.9413 (D) 0.7083

44. X is normally distributed. The mean of X is 15 and standard deviation 3. Given that for z = 1.666, A = 0.4515, p $(x \le 10)$ is given by …………… (2)

(A) 0.0585 (B) 0.0673 (C) 0.0485 (D) 0.1235

45. X is normally distributed. The mean of X is 30 and variance 25. The probability p $(26 \le x \le 40)$ is ……………

(Given : Area corresponding to z = 0.8 is 0.2881 and Area corresponding to z = 2 is 0.4772) (2)

(A) 0.8562 (B) 0.6574 (C) 0.3745 (D) 0.7653

46. In a sample of 1000 candidates, the mean of certain test is 14 and standard deviation is 2.5. Assuming Normal distribution, the probability of candidates getting less than eight marks i.e. p $(x \le 8)$ is ……………

(Given : Area corresponding to z = 2.4 is 0.4918) …………… (2)

(A) 0.0054 (B) 0.0075 (C) 0.0082 (D) 0.0035

47. In a normally distributed group of 450 students with mean 42 and standard deviation 8, the number of students scoring less than 48 marks is ……………

(Given : Area corresponding to z = 0.75 is 0.2734). (2)

(A) 348 (B) 102 (C) 127 (D) 250

48. In a certain examination test 10000 students appeared in a subject of mathematics. Average marks obtained were 50% with standard deviation 5%. Marks are normally distributed. Number of students expected to get more than 60% marks is equal to …………… (2)

(z = 2, A = 0.4772)

(A) 200 (B) 300 (C) 325 (D) 228

49. For normal variable x with probability density function $f(x) = \dfrac{1}{3\sqrt{2\pi}} e^{-\frac{1}{18}(x-6)^2}$ the mean μ and standard deviation σ are …………… (2)

(A) 3, 9 (B) 9, 6 (C) 6, 3 (D) 18, 6

Answers

1. (C)	2. (D)	3. (A)	4. (B)	5. (C)	6. (A)	7. (D)	8. (B)	9. (B)	10. (C)
11. (A)	12. (D)	13. (B)	14. (A)	15. (C)	16. (D)	17. (C)	18. (D)	19. (A)	20. (B)
21. (D)	22.(B)	23.(C)	24.(A)	25.(D)	26. (C)	27. (A)	28. (B)	29. (B)	30. (D)
31. (A)	32. (C)	33. (A)	34. (B)	35. (A)	36.(D)	37. (A)	38. (D)	39. (C)	40. (C)
41. (B)	42. (A)	43. (B)	44. (C)	45. (D)	46. (C)	47. (A)	48. (D)	49. (C)	

Type IV : Chi-square Distribution :

1. A bank utilizes three teller windows to render service to the customer. On a particular day 600 customer were served. If the customers are uniformly distributed over the counters. Expected numbers of customer served on each counter is …………… (2)

(A) 100 (B) 200 (C) 300 (D) 150

2. 200 digits are chosen at random from a set of tables. The frequencies of the digits are as follows :

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	18	19	23	21	16	25	22	20	21	15

The expected frequency and degree of freedom for uniform distribution is …………… (2)

(A) 20 and 10 (B) 21 and 9 (C) 20 and 9 (D) 15 and 8

3. In experiment on pea breeding, the observed frequencies are 222, 120, 32, 150 and expected frequencies are 323, 81, 81, 40, then χ_3^2 has the value …………… (2)

(A) 382.502 (B) 380.50 (C) 429.59 (D) 303.82

4. If observed frequencies O_1, O_2, O_3 are 5, 10, 15 and expected frequencies e_1, e_2, e_3 are each equal to 10, then χ_2^2 has the value …………… (2)

(A) 20 (B) 10 (C) 15 (D) 5

5. Number of books issued on six days of the week, excluding Sunday which is holiday are given as 120, 130, 110, 115, 135, 110 and expectation is 120 books on each day, then χ_5^2 is .... (2)

(A) 2.58 (B) 3.56 (C) 6.56 (D) 4.58

6. A coin is tossed 160 times and following are expected and observed frequencies for number of heads (2)

No. of heads	0	1	2	3	4
Observed frequency	17	52	54	31	6
Expected Frequency	10	40	60	40	10

Then χ_4^2 is ....

(A) 12.72 (B) 9.49 (C) 12.8 (D) 9.00

7. Among 64 offspring's of a certain cross between guinea pig 34 were red, 10 were black and 20 were white. Acceding to genetic model, these number should in the ratio 9 : 3 : 4. Expected frequencies in the order .... (2)

(A) 36, 12, 16 (B) 12, 36, 16 (C) 20, 12, 16 (D) 36, 12, 25

8. A sample analysis of examination results of 500 students was made. The observed frequencies are 220, 170, 90 and 20 and the numbers are in the ratio 4 : 3 : 2 : 1 for the various categories. Then the expected frequencies are ....

(A) 150, 150, 50, 25 (B) 200, 100, 50, 10 (C) 200, 150, 100, 50 (D) 400, 300, 200, 100

9. In experiment on pea breeding, the observed frequencies are 222, 120, 32, 150 and the theory predicts that the frequencies should be in proportion 8 : 2 : 2 : 1. Then the expected frequencies are .... (2)

(A) 323, 81, 40, 81 (B) 81, 323, 40, 81 (C) 323, 81, 81, 40 (D) 433, 81, 81, 35

Answers

1. (B)	2. (C)	3. (A)	4. (D)	5. (D)	6. (A)	7. (A)	8. (C)	9. (C)

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Ch. 7 Numerical Solution of Algebric and Transcendental Equations

1. Geometrically roots of equation f(x) = 0 are the points where .... (1)

(A) Graph of y = f(x) cuts x-axis (B) Graph of y = f(x) is parallel to x-axis

(C) Graph of y = f(x) is perpendicular to y-axis (D) Graph of y = f(x) is parallel to y-axis

2. By intermediate value theorem, for equation f(x) = 0 to have at least one root say ξ in the internal (a, b) are .... (1)

(A) f(x) is continuous on [a, b] and f(a) f(b) > 0 (B) f(x) is continuous on [a, b] and f(a) and (b) < 0

(C) f(x) is continuous on (a, b) and f(a) = f(b) (D) f(x) is continuous on (a, b) and f(a) f(b) = 0

3. If f(x) is continuous on [a, b] and f(a) f(b) < 0, then to find a root of f(x) = 0, initial approximation x_0 by bisection method is .... (1)

(A) $x_0 = \dfrac{a-b}{2}$ (B) $x_0 = \dfrac{f(a)+f(b)}{2}$

(C) $x_0 = \dfrac{a+b}{2}$ (D) $x_0 = \dfrac{a-b}{a+b}$

4. In bisection method, if permissible error is E for finding a root of f(x) = 0 then the approximate number of iterations required can be determined from relation .... (1)

(A) $\dfrac{b+a}{2} < \varepsilon$ (B) $\dfrac{a+b}{a-b} < \varepsilon$ (C) $\dfrac{b+a}{2^n} \le \varepsilon$ (D) $\dfrac{b-a}{2^n} \le \varepsilon$

5. If x_0, x_1 are two initial approximations to the root of f(x) = 0, by secant method next approximations x_2 is given by .... (1)

(A) $x_2 = x_1 - \dfrac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$ (B) $x_2 = \dfrac{x_0 + x_1}{2}$

(C) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ (D) $x_2 = x_1 + \dfrac{(x_1 + x_0)}{(f_1 + f_0)} \times f_1$

6. If x_0 is initial approximation to the root of the equation f(x) = 0, by Newton-Raphson method, first approximation x_1 is given by .... (1)

(A) $x_0 = \frac{x_0 + x_1}{2}$ (B) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

(C) $x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$ (D) $x_1 = x_0 + \frac{f'(x_0)}{f(x_0)}$

7. The condition of convergence of the computational scheme $x = \phi(x)$ in method of successive approximation (simple iteration method) for finding root of the equation f(x) = 0 is .... (1)

(A) $|\phi'(x)| >$ for all x in [a, b] (B) $|\phi'(x)| > = 1$ for all x

(C) $|\phi'(x)| <$ for all x in [a, b] (D) $|\phi'(x)| > = 0$ for all x

8. A root of the equation $x\log_{10} - 1.2 = 0$ by using bisection method lies between .... (1)

(A) 1 and 2 (B) 0 and 1

(C) 0.5 and 1 (D) 2 and 3

9. A root of the equation $x^3 - 4x - 9 = 0$ using bisection method lies between .... (2)

(A) 2 and 3 (B) 1 and 2 (C) 0 and 1 (D) 3 and 4

10. A cube root of 100 using secant method, lies between .... (2)

(A) 0 and 1 (B) 4 and 5 (C) 1 and 2 (D) 5 and 6

11. A root of the equation $\cos x - xe^x = 0$ (measure in radian) by using secant method lies between .... (2)

(A) 2 and 3 (B) 1 and 2

(C) 0 and 1 (D) 2.5 and 3

12. Using bisection method, the first approximation to root ξ of the equation $x \sin x - 1 = 0$ (measured in radian), that lies between x = 1 and x = 1.5 is .... (2)

(A) 1.5 (B) 0.25 (C) 1 (D) 1.25

13. Using secant method, the first approximation to a root x_2 of the equation $x^3 - 5x - 7 = 0$, if initial approximations are given as $x_0 = 2.5$ and $x_1 = 3$, is .... (2)

(A) 2.7183 (B) 3

(C) 2 (D) 0

14. Using Newton-Raphson method, the first approximation to a root x_1 of the equation $x^3 + 2x - 5 = 0$ in (1, 2) if initial approximation $x_0 = 2$, is .... (2)

(A) 0 (B) 1.5

(C) 3 (D) 4

15. Using successive approximation method (iterative method), the first approximation to a root x_1 of the equation $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$ in [3, 4], taking initial approximation $x_0 = 3.6$ is .... (2)

(A) 0 (B) 1

(C) 3.77815 (D) 2

Answers

1. (A)	2. (B)	3. (C)	4. (D)	5. (A)	6. (B)	7. (C)	8. (D)	9. (A)
10. (B)	11. (C)	12. (D)	13. (A)	14. (B)	15. (C)			

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Ch. 8 Numerical Solution of System of Linear Equations

Type I : Numerical Solution System of Linear Equations

(Direct Methods : Gauss Elimination, LU Decomposition, Cholesky Method) : **Marks**

1. For solving the system of equations 5x + y + 2z = 34, 4y − 3z = 12, 10x − 2y + z = − 4 by Gauss elimination method using partial pivoting, the pivots for elimination of x and y are .... (2)

(A) 10 and 4 (B) 5 and 4

(C) 10 and 2 (D) 5 and – 4

2. For solving the system of equations 8y + 2z = −7, 3x + 5y + 2z = 8, 6x + 2y + 8z = 26 by Gauss elimination method using partial pivoting, the pivots for elimination of x and y are .... (2)

(A) 6 and 3 (B) 6 and 8

(C) 8 and 5 (D) 6 and 4

3. For solving the system of equations $3x - y - z = 4$, $x + y - 6z = -12$, $x + 4y - z = -5$ by Gauss elimination method using partial pivoting, the pivots for elimination of x and y are .... (2)

(A) 3 and 1 (B) 3 and 4

(C) 1 and $\frac{4}{3}$ (D) 3 and $\frac{13}{3}$

4. The given system of equations is $x + 4y - z = -5$, $x + y - 6z = -12$, $3x - y - z = 4$. In Gauss elimination method on eliminating x from second and third equations, the system reduces to .... (2)

(A) $$\begin{aligned} x + 4y - z &= -5 \\ 5y - 7z &= -7 \\ 13y - 2z &= 9 \end{aligned}$$

(B) $$\begin{aligned} x + 4y - z &= -5 \\ -3y - 5z &= -17 \\ -13y + 2z &= 9 \end{aligned}$$

(C) $$\begin{aligned} x + 4y - z &= -5 \\ -3y - 5z &= -7 \\ -13y + 2z &= 19 \end{aligned}$$

(D) $$\begin{aligned} x + 4y - z &= -5 \\ y - 6z &= -12 \\ y - z &= 4 \end{aligned}$$

5. The given system of equations is $x + 2y + z = 4$, $2x + 3y + 4z = 5$, $3x - y + z = 6$. In Gauss elimination method, on eliminating x from second and third equations, the system reduces to .... (2)

(A) $$\begin{aligned} x + 2y + z &= 4 \\ -y + 2z &= -3 \\ -7y - 2z &= -6 \end{aligned}$$

(B) $$\begin{aligned} x + 2y + z &= 4 \\ -3y + 2z &= 13 \\ -7y - 2z &= 18 \end{aligned}$$

(C) $$\begin{aligned} x + 2y + z &= 4 \\ -3y + 2z &= 1 \\ -7y - 2z &= 2 \end{aligned}$$

(D) $$\begin{aligned} x + 2y + z &= 4 \\ -y + 2z &= -3 \\ -y + z &= 6 \end{aligned}$$

6. The given system of equations is $4x + y + z = 4$, $x + 4y - 2z = 4$, $3x + 2y - 4z = 6$. In Gauss elimination method, on eliminating x from second and third equations, the system reduces to .... (2)

(A) $$\begin{aligned} x + \frac{1}{4}y + \frac{1}{4}z &= 1 \\ \frac{15}{4}y + \frac{7}{4}z &= 5 \\ \frac{7}{4}y - \frac{17}{4}z &= 5 \end{aligned}$$

(B) $$\begin{aligned} x + \frac{1}{4}y + z &= 4 \\ 4y - 2z &= 4 \\ 3y + 2z &= 6 \end{aligned}$$

(C) $$\begin{aligned} x + \frac{1}{4}y + \frac{1}{4}z &= 1 \\ \frac{15}{4}y - \frac{9}{4}z &= 3 \\ \frac{5}{4}y - \frac{19}{4}z &= 3 \end{aligned}$$

(D) $$\begin{aligned} x + \frac{1}{4}y + \frac{1}{4}z &= 4 \\ 15y + 7z &= 5 \\ 7y - 17z &= 5 \end{aligned}$$

7. The given system of equation is $x + \frac{1}{2}y + \frac{1}{3}z = 1$, $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 0$, $\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 0$. In Gauss elimination method, on eliminating x from second and third equations, the system reduces to .... (2)

(A) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ \frac{1}{6}y - \frac{1}{12}z &= -1 \\ -\frac{1}{2}y - \frac{2}{15}z &= -1 \end{aligned}$$

(B) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ \frac{1}{12}y + \frac{1}{2}z &= -\frac{1}{2} \\ \frac{1}{12}y + \frac{4}{45}z &= -\frac{1}{3} \end{aligned}$$

(C) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ \frac{1}{3}y + \frac{1}{4}z &= 0 \\ \frac{1}{4}y + \frac{4}{5}z &= 0 \end{aligned}$$

(D) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ y + z &= 1 \\ 12 + 45z &= -3 \end{aligned}$$

8. The given system of equation is $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$. In Gauss elimination method, on eliminating x from second and third equations, the system reduces to .... (2)

(A) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ \frac{5}{2}y + \frac{7}{2}z &= 23 \\ \frac{9}{2}y + \frac{19}{2}z &= 21 \end{aligned}$$

(B) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ \frac{3}{2}y + \frac{3}{2}z &= 13 \\ \frac{7}{2}y + \frac{17}{2}z &= 11 \end{aligned}$$

(C) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ 2y + 3z &= 8 \\ 4y + 9z &= 6 \end{aligned}$$

(D) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{2}z &= 5 \\ \frac{1}{2}y + \frac{3}{2}z &= 3 \\ \frac{7}{2}y + \frac{17}{2}z &= 11 \end{aligned}$$

9. The given system of equation is $2x + 2y + z = 12$, $3x + 2y + 2z = 8$, $2x + 10y + z = 12$. In Gauss elimination method, on eliminating x from second and third equations, the system reduces to .... (2)

(A) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{2}z &= 5 \\ 2y + 2z &= 8 \\ 2y + z &= 12 \end{aligned}$$

(B) $$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 1 \\ 3y + \frac{5}{2}z &= 14 \\ 11y + \frac{15}{2}z &= 16 \end{aligned}$$

(C) $$\begin{aligned} x + y + \frac{1}{2}z &= 6 \\ -y + \frac{1}{2}z &= -10 \\ 8y &= 0 \end{aligned}$$

(D) $$\begin{aligned} x + y + \frac{1}{2}z &= 6 \\ y + \frac{3}{2}z &= 2 \\ 9y - \frac{17}{2}z &= 4 \end{aligned}$$

10. Using gauss elimination method, the solution of system of equations (2)

$$\begin{aligned} x + 2y + z &= 4 \\ -3y + 2z &= -3 \\ -7y - 2z &= -6 \end{aligned}$$ is ....

(A) $x = -\frac{43}{16}$, $y = -\frac{9}{8}$, $z = \frac{15}{16}$

(B) $x = \frac{47}{20}$, $y = \frac{9}{10}$, $z = -\frac{3}{20}$

(C) $x = \frac{4}{3}$, $y = \frac{3}{8}$, $z = -\frac{5}{6}$

(D) $x = \frac{16}{43}$, $y = \frac{8}{9}$, $z = -5$

11. Using Gauss elimination method, the solution of system of equations (2)

$$x + \frac{1}{4}y + \frac{1}{4}z = 1, \quad \frac{15}{4}y - \frac{9}{4}z = 3, \quad \frac{5}{4}y - \frac{19}{4}z = 3$$ is ....

(A) $x = 1$, $y = 2$, $z = 3$

(B) $x = \frac{1}{2}$, $y = 1$, $z = \frac{1}{2}$

(C) $x = 2$, $y = \frac{1}{2}$, $z = 2$

(D) $x = 1$, $y = \frac{1}{2}$, $z = -\frac{1}{2}$

12. Using Gauss elimination method, the solution of system of equations (2)

$$x + \frac{1}{2}y + \frac{1}{3}z = 1, \quad \frac{1}{12}y + \frac{1}{12}z = -\frac{1}{2}. \quad \frac{1}{12}y + \frac{4}{45}z = -\frac{1}{3}$$ is ....

(A) $x = 9$, $y = -36$, $z = 30$

(B) $x = 6$, $y = 0$, $z = 9$

(C) $x = -9$, $y = 36$, $z = -30$

(D) $x = 36$, $y = 30$, $z = 9$

13. Using Gauss elimination method, the solution of system of equations (2)

$$x + \frac{1}{2}y + \frac{1}{2}z = 5, \quad \frac{1}{2}y + \frac{3}{2}z = 3, \quad \frac{7}{2}y + \frac{17}{2}z = 11 \text{ is ....}$$

(A) $x = 9,\ y = 5,\ z = 7$ (B) $x = 1,\ y = \frac{1}{2},\ z = 3$

(C) $x = \frac{1}{7},\ y = -\frac{1}{9},\ z = \frac{1}{5}$ (D) $x = 7,\ y = -9,\ z = 5$

14. Using Gauss elimination method, the solution of system of equations (2)

$x + y + \frac{1}{2}z = 6 \quad -y + \frac{1}{2}z = -10 \quad 5y - \frac{21}{2}z = -20$ is ....

(A) $x = -\frac{4}{51},\ y = \frac{8}{115},\ z = \frac{4}{35}$ (B) $x = -\frac{51}{4},\ y = \frac{115}{8},\ z = \frac{35}{4}$

(C) $x = -51,\ y = 115,\ z = 35$ (D) $x = \frac{35}{4},\ y = -\frac{51}{4},\ z = \frac{115}{8}$

15. Using Gauss elimination method, the solution of system of equations (2)

$x + 4y - z = -5 \quad y + \frac{5}{3}z = \frac{7}{3} \quad -13y + 2z = 19$ is ....

(A) $x = \frac{117}{71},\ y = -\frac{81}{71},\ z = \frac{148}{71}$ (B) $x = \frac{71}{117},\ y = -\frac{71}{81},\ z = \frac{71}{148}$

(C) $x = -\frac{117}{71},\ y = \frac{81}{71},\ z = -\frac{148}{71}$ (D) $x = 1,\ y = 2,\ z = 0$

16. In solving system of equations by Cholesky's method, the system is expressed as

$AX = B$ where, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A = LL^T$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then l_{11} is given by .... (2)

(A) $\frac{1}{2}$ (B) 2 (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

17. In solving system of equations by Cholesky's method, the system is expressed as

$AX + B$ where, $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we express $A = LL^T$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then l_{11} is given by .... … (2)

(A) $l_{11} = \frac{1}{2}$ (B) $l_{11} = \frac{1}{\sqrt{2}}$ (C) $l_{11} = 2$ (D) $l_{11} = \sqrt{2}$

18. In solving system of equations by Cholesky's method, the system is expressed as

$AX = B$ where, $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$. If we express $A = LL^T$ where, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ then l_{11} is given by .... … (2)

(A) $l_{11} = 4$ (B) $l_{11} = -2$ (C) $l_{11} = \sqrt{2}$ (D) $l_{11} = 2$

19. In solving system of equations by Cholesky's method, the system is expressed as

$AX = B$ where $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$. If we express $A = LL^T$ where, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then values of l_{11} and l_{21} are given by …… (2)

(A) $l_{11} = 1,\ l_{21} = 2$ (B) $l_{11} = 2,\ l_{21} = 1$ (C) $l_{11} = -2,\ l_{21} = -\frac{1}{2}$ (D) $l_{11} = \frac{1}{2},\ l_{21} = -1$

20. In solving system of equations by LU decomposition method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & -2 & 1 \\ 1 & 5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$. If we express A = LU where, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ then .... (2)

(A) $u_{11} = 2,\ u_{12} = 2,\ u_{13} = 3$ (B) $u_{11} = -2,\ u_{12} = -2,\ u_{13} = -3$

(C) $u_{11} = 2,\ u_{12} = 4,\ u_{13} = 1$ (D) $u_{11} = 2,\ u_{12} = -2,\ u_{13} = 4$

21. In solving system of equations by LU decomposition method, the system is expressed as AX = B where,

$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$. If we express A = LU where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ then .... (2)

(A) $u_{11} = 2,\ u_{12} = 2,\ u_{13} = 2$ (B) $u_{11} = 2,\ u_{12} = 1,\ u_{13} = 3$

(C) $u_{11} = 2,\ u_{12} = 3,\ u_{13} = 1$ (D) $u_{11} = 1,\ u_{12} = 2,\ u_{13} = 3$

22. In solving system of equations by Cholesky's method, the system is expressed as

AX = B where $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & -2 & 1 \\ 3 & 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$. If we express $A = LL^T$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$, then .... (2)

(A) $l_{11} = \sqrt{2},\ l_{21} = \sqrt{2},\ l_{31} = \dfrac{3}{\sqrt{2}}$ (B) $l_{11} = 2,\ l_{21} = 2,\ l_{31} = 3$

(C) $l_{11} = \dfrac{1}{\sqrt{2}},\ l_{21} = \dfrac{1}{\sqrt{2}},\ l_{31} = \dfrac{3}{\sqrt{2}}$ (D) $l_{11} = 2,\ l_{21} = 4,\ l_{31} = 1$

23. In solving system of equations by LU decomposition method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 4 \\ 1 & 3 & 1 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ and A = LU then l_{21} is equal to .... (2)

(A) −1 (B) 1 (C) 2 (D) 4

24. In solving system of equations by LU decompositions method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ and A = LU, then l_{21} is given by .... (2)

(A) − 2 (B) 2 (C) $-\dfrac{1}{2}$ (D) $\dfrac{1}{2}$

25. In solving system of equations by LU decomposition method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ and A = LU then value of u_{22} is $\left(\text{Given that : } l_{21} = \dfrac{1}{3}\right)$ .... (2)

(A) $u_{22} = \dfrac{5}{3}$ (B) $u_{22} = 6$ (C) $u_{22} = \dfrac{2}{3}$ (D) $u_{22} = \dfrac{7}{3}$

26. In solving system of equations by LU decomposition method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ and A = LU, then values of l_{21} and l_{31} are given by .... (2)

(A) $l_{21} = 1, l_{31} = 2$ (B) $l_{21} = 3, l_{31} = \dfrac{3}{2}$ (C) $l_{21} = \dfrac{1}{3},\ l_{31} = \dfrac{2}{3}$ (D) $l_{21} = -2,\ l_{31} = -1$

27. In solving system of equations by Cholesky's method, the system is express as

AX = B where, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A = LL^T$ where, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then values of l_{11} and l_{21} are given by .... (2)

(A) $l_{11} = \frac{1}{2}$, $l_{21} = -2$

(B) $l_{11} = \frac{1}{\sqrt{2}}$, $l_{21} = -\sqrt{2}$

(C) $l_{11} = 2$, $l_{21} = -\frac{1}{2}$

(D) $l_{11} = \sqrt{2}$, $l_{21} = -\frac{1}{\sqrt{2}}$

28. In solving system of equations by Cholesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A = LL^T$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then values of l_{31} and l_{22} are given by $\left(\text{Given : } l_{11} = \sqrt{2}, l_{21} = -\frac{1}{\sqrt{2}}\right)$ .... (2)

(A) $l_{31} = \sqrt{2}$, $l_{22} = \sqrt{5}$

(B) $l_{31} = 0$, $l_{22} = \sqrt{\frac{5}{2}}$

(C) $l_{31} = 2$, $l_{22} = 5$

D) $l_{31} = \sqrt{5}$, $l_{22} = \sqrt{2}$

29. In solving of equations by Cholesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$. If we express $A = LL^T$ where, L $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then values of l_{31} and l_{22} are given by (Given : $l_{11} = 2$, $l_{21} = 1$). (2)

(A) $l_{31} = -7$, $l_{22} = -4$

(B) $l_{31} = \sqrt{7}$, $l_{22} = 2$

(C) $l_{31} = 7$, $l_{22} = 4$

(D) $l_{31} = 14$, $l_{22} = 17$

30. In solving system of equations by Cholesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we express $A = LL^T$ where, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then values of l_{11} and l_{21} are given by .... (2)

(A) $l_{11} = \frac{1}{2}$, $l_{21} = -\frac{1}{\sqrt{2}}$

(B) $l_{11} = 0$, $l_{21} = 2$

(C) $l_{11} = -2$, $l_{21} = -\frac{1}{2}$

(D) $l_{11} = 2$, $l_{21} = -1$

31. In solving system of equations by Cholesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we express $A = LL^T$ where, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & I_{33} \end{bmatrix}$ then values of l_{31} and l_{22} are given by .... (Given : $l_{11} = 2$, $l_{21} = -1$) (2)

(A) $l_{31} = 0$, $l_{22} = \sqrt{3}$

(B) $l_{31} = 2$, $l_{22} = -1$

(C) $l_{31} = 4$, $l_{22} = \sqrt{\frac{1}{2}}$

(D) $l_{31} = -1$, $l_{22} = 4$

32. In solving system of equations by LU decomposition method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix}$ and A = LU where, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$. If UX = Z then $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is given by then …… (2)

(A) $Z = \begin{bmatrix} 14 \\ 2 \\ 3 \end{bmatrix}$

(B) $Z = \begin{bmatrix} 14 \\ -8 \\ -12 \end{bmatrix}$

(C) $Z = \begin{bmatrix} 14 \\ -8 \\ 12 \end{bmatrix}$

(D) $Z = \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix}$

33. In solving system of equations by Cholesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A = LL^T$,

$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix}$. If $L^T X = Z$ then $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is given by .... (2)

(A) $Z = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \sqrt{10} \\ -\sqrt{40} \end{bmatrix}$ (B) $Z = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{40}} \end{bmatrix}$ (C) $Z = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{40}} \end{bmatrix}$ (D) $Z = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{10} \\ \frac{1}{40} \end{bmatrix}$

34. In solving system of equations by Cholesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $A = LL^T$, L^T is transpose of $L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \sqrt{\frac{11}{3}} \end{bmatrix}$.

If $L^T X = Z$ then $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is given by .... (2)

(A) $Z = \begin{bmatrix} 0 \\ 1 \\ 33 \end{bmatrix}$ (B) $Z = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{\sqrt{3}}{11} \end{bmatrix}$ (C) $Z = \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{33} \end{bmatrix}$ (D) $Z = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{33}} \end{bmatrix}$

35. In solving system of equations by Cyolesky's method, the system is expressed as

AX = B where, $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$ and $A = LL^T$, L^T is transpose of $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$. If $L^T X = Z$ then $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is given by …… (2)

(A) $Z = \begin{bmatrix} 7 \\ -27 \\ 7 \end{bmatrix}$ (B) $Z = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ (C) $Z = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$ (D) $Z = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$

36. The system of equations AX = B is solved by Cholesky's method where,

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A = LL^T$, $L \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix}$.

If $L^T X = Z$ and $Z = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{40}} \end{bmatrix}^T$ then solution of given system is .... (2)

(A) $x_1 = \frac{5}{8}$, $x_2 = \frac{1}{4}$, $x_3 = -\frac{1}{8}$
(B) $x_1 = 5$, $x_2 = 1$, $x_3 = -8$
(C) $x_1 = -\frac{1}{8}$, $x_2 = \frac{5}{4}$, $x_3 = -\frac{3}{8}$
(D) $x_1 = -\frac{1}{8}$, $x_2 = \frac{5}{8}$, $x_3 = \frac{1}{4}$

37. The system of equations is solved by Cholesky's method. The system is expressed as

$AX = B$ where, $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $A = LL^T$ where, $L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \sqrt{\frac{11}{3}} \end{bmatrix}$. If $L^T X = Z$ and

$Z = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{33}} \end{bmatrix}^T$ then solution of given system is .... (2)

(A) $x_1 = 2$, $x_2 = 4$, $x_3 = 1$
(B) $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{4}$, $x_3 = 1$
(C) $x_1 = \frac{2}{11}$, $x_2 = \frac{4}{11}$, $x_3 = \frac{1}{11}$
(D) $x_1 = \frac{11}{2}$, $x_2 = \frac{11}{4}$, $x_3 = 11$

38. The system of equations is solved by Cholesky's method. The system is expressed as

$AX = B$ where, $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$ and $A = LL^T$ where, $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$. If $L^T X = Z$ and $Z = [8 \ \ -27 \ \ 5]^T$ then solution of given system is .... (2)

(A) $x_1 = 6$, $x_2 = 1$, $x_3 = 3$
(B) $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{6}$, $x_3 = 1$
(C) $x_1 = \sqrt{3}$, $x_2 = \sqrt{6}$, $x_3 = 1$
(D) $x_1 = 3$, $x_2 = -6$, $x_3 = 1$

ANSWERS

1. (A)	2. (B)	3. (D)	4. (C)	5. (A)	6. (C)	7. (B)	8. (D)
9. (C)	10. (B)	11. (D)	12. (A)	13. (D)	14. (B)	15. (A)	16. (D)
17. (C)	18. (D)	19. (B)	20. (A)	21. (C)	22. (A)	23. (B)	24. (D)
25. (A)	26. (C)	27. (D)	28. (B)	29. (C)	30. (D)	31. (A)	32. (B)
33. (B)	34. (D)	35. (C)	36. (A)	37. (C)	38. (D)		

Type II : Numerical Solutions of System of Linear Equations (Iterative Method : Gauss-Seidel Method) **Marks**

1. The system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 20$ with initial approximation $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$. Using Gauss-Seidel method, first iterative solution $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ is given by .... (2)

(A) 0.8000, −1.0375, 1.0900
(B) 0.8500, −0.9, 1.1009
(C) 0.8000, −1.0275, 1.0109
(D) 0.8500, −1.0275, 0.7609

2. The system of equations $10x + y + z = 12$, $2x + 10y + z = 13$, $2x + 2y + 10z = 14$ with initial approximation $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$. Using Gauss-Seidel method, first iterative solution $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ is given by .... (2)

(A) 1.2, 1.06, 0.948
(B) 1.2, 1.30, 0.498
(C) 1.1, 1.46, 0.648
(D) 0.12, 1.8, 0.849

3. The system of equations $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$ with initial approximation $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$. Using Gauss-Seidel method, first iterative solution $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ is given by .... (2)

(A) 3, 3.65, 1.19
(B) 3.1481, 3.5408, 1.9132
(C) 3, 3.40, 1.29
(D) 3.1481, 3.5408, 1.1132

4. The system of equations $28x + 4y - z = 32$, $2x + 17y + 4z = 35$, $x + 3y + 10z = 24$ with initial approximation $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$. Using Gauss-Seidel method, first iterative solution $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ is given by .... (2)

(A) 0.875, 1.4429, 1.0719
(B) 0.875, 1.2944, 1.1907
(C) 1.1428, 1.9244, 1.7084
(D) 1.1428, 2.0588, 1.0784

5. The system of equations $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7$ with initial approximation $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$. Using Gauss-Seidel method, first iterative solution $x^{(1)}, y^{(1)}, z^{(1)}$ is given by .... (2)

(A) 1.1182, 1.057, 1.7007 (B) 1.625, 1.075, 0.7792

(C) 1.625, 1.4, 0.9277 (D) 1.1248, 1.570, 0.2279

6. The system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 20$ is solved by Gauss-Seidel method. If values of x, y, z in certain iteration are $x = 0.8500$, $y = -1.0275$, $z = 1.0109$ then values of x, y, z in next iteration are given by .... (2)

(A) 1.0025, −0.9998, 0.7498 (B) 1.2500, −0.9769, 1.0109

(C) 1.0025, −1.0780, 1.0232 (D) 1.2500, −0.9546, 0.9728

7. The system of equations $10x + y + z = 12$, $2x + 10y + z = 13$, $2x + 2y + 10z = 14$ is solved by Gauss-Seidel method. If values of x, y, z in certain iteration are $x = 1.2$, $y = 1.06$, $z = 0.948$ then values of x, y, z in next successive iteration are given by .... (2)

(A) 0.9299, 1.4054, 0.6661 (B) 0.2999, 1.5040, 0.9191

(C) 0.9992, 1.0054, 0.9991 (D) 0.9992, 0.9652, 0.1999

8. The system of equations $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$ is solved by Gauss-Seidel method. If values of x, y, z in certain iteration are $x = 2.43$, $y = 3.57$, $z = 1.926$ then values of x, y, z in next successive iteration are given by .... (2)

(A) 2.423, 3.5712, 1.296 (B) 0.2999, 3.754, 1.269

(C) 0.2999, 3.475, 1.629 (D) 2.4261, 3.5728, 1.926

9. The system of equations $28x + 4y - z = 32$, $2x + 17y + 4z = 35$, $x + 3y + 10z = 24$ is solved by Gauss-Seidel method. If values of x, y, z in certain iteration are $x = 0.9289$, $y = 1.5476$, $z = 1.8428$ then values of x, y, z in next successive iteration are given by .... (2)

(A) 0.8975, 1.9050, 1.4885 (B) 0.7589, 1.0509, 1.5885

(C) 0.9875, 1.5090, 1.8485 (D) 0.9875, 1.5159, 1.8845

10. The system of equations $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7$ is solved by Gauss-Seidel method. If values of x, y, z in certain iteration are $x = 1.0271$, $y = 1.0387$, $z = 0.9765$ then values of x, y, z in next successive iteration are given by .... (2)

(A) 0.7941, 1.6400, 1.1080 (B) 0.9914, 1.0064, 0.9993

(C) 0.5785, 1.4600, 1.8010 (D) 0.9914, 1.0064, 1.0018

ANSWERS

1. (D)	2. (A)	3. (B)	4. (C)	5. (B)	6. (A)	7. (C)	8. (D)
9. (C)	10. (D)						

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Ch. 9 Interpolation, Numerical Differentiation and Integration

Type I : Interpolation. **Marks**

1. Lagrange's interpolation polynomial formula passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2) is given by .... (1)

(A) $\frac{(x-x_1)(x-x_2)}{(x_0+x_1)(x_0+x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1+x_0)(x_1+x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2+x_0)(x_2+x_1)} y_2$

(B) $\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 - \frac{(x-x_0)(x_1-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 - \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$

(C) $\frac{(x+x_1)(x+x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x+x_0)(x+x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x+x_0)(x+x_1)}{(x_2-x_0)(x_2-x_1)} y_2$

(D) $\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$

2. If E is the shifting operator then E f(x) is equal to .... (1)

(A) $f(x - h)$ (B) $f(x + h) - f(x)$ (C) $f(x + h)$ (D $f(x) - f(x + h)$

3. If Δ is the forward difference operator then $\Delta f(x)$ is equal to .... (1)

(A) $f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$ (B) $f(x+h)-f(x)$ (C) $f(x+h)$ (D) $f(x)-f(x+h)$

4. If ∇ is the backward difference operator then $\nabla f(x)$ is equal to .... (1)

(A) $f(x)-f(x-h)$ (B) $f(x+h)-f(x)$ (C) $f(x+h)$ (D) $f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$

5. If δ is the central difference operator then $\delta f(x)$ is equal to .... (1)

(A) $f(x+h)$ (B) $f(x+h)-f(x)$ (C) $f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$ (D) $f(x)-f(x-h)$

6. If μ is the average difference operator then $\mu f(x)$ is equal to .... (1)

(A) $\frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$ (B) $f(x+h)-f(x)$ (C) $f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$ (D) $f(x)-f(x-h)$

7. Shifting operator E is equivalent to .... (1)

(A) $1-\delta$ (B) $1+\Delta$ (C) $1+\delta^2$ (D) $2-\delta$

8. Inverse shifting operator E^{-1} is equivalent to .... (1)

(A) $1-\delta$ (B) $1+\delta^2$ (C) $1+\delta$ (D) $1-\nabla$

9. If $f(x)=x^2+1$, $h=2$, $Ef(x)$ is given by .... (1)

(A) x^2+3 (B) x^2+4x (C) x^2-4x+5 (D) x^2+4x+5

10. If $f(x)=x^3$, $h=1$, $Ef(x)$ is given by .... (1)

(A) x^3-3x^2+3x-1 (B) x^3+3x^2+3x+1 (C) x^3-3x^2-3x-1 (D) x^3+3x^2-3x-1

11. For $f(x)=x^2$, $h=1$, first forward difference of $f(x)$ is given by .... (1)

(A) $2x-1$ (B) $3x+1$ (C) $2x+1$ (D) $3x-1$

12. If $f(x)=x^2-2$, $h=1$, first backward difference $\nabla f(x)$ is given by .... (1)

(A) $2x-1$ (B) $3x+2$ (C) $x-5$ (D) $2x-5$

13. For $f(x)=x^2$, $h=1$, $\delta f(x)$ is given by .... (1)

(A) $-2x$ (B) $2x^2$ (C) $2x$ (D) $3x$

14. For $f(x)=x^2$, $h=1$, $\mu f(x)$ is given by .... (1)

(A) $2x+\frac{1}{2}$ (B) $2x^2+\frac{1}{2}$ (C) $x^2-\frac{1}{4}$ (D) $x^2+\frac{1}{4}$

15. Newton's Gregory forward finite difference interpolation formula for a set of data points (x_i, y_i), $i = 0, 1, 2, 3, \ldots, n$ with $x_i = x_0 + ih$, $i = 0, 1, 2, 3, \ldots, n$ is $(x = x_0 + uh)$ .... (1)

(A) $y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$

(B) $y = y_0 + u\Delta y_0 + \frac{u(u+1)}{2!}\Delta^2 y_0 + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_0 + \ldots$

(C) $y = y_0 - u\Delta y_0 + \frac{u(u+1)}{2!}\Delta^2 y_0 - \frac{u(u+1)(u+2)}{3!}\Delta^3 y_0 + \ldots$

(D) $y - y_0 - u\Delta y_0 + u(u-1)\Delta^2 y_0 - u(u-1)(u-2)\Delta^3 y_0 - \ldots$

16. Newton's Gregory backward finite difference interpolation formula for a set of data points (x_i, y_i), $i = 0, 1, 2, 3, \ldots, n$ with $x_1 = x_0 + ih$, $i = 0, 1, 2, 3, \ldots, n$ and $x_0 \le x \le x_n$ is (where $x = x_n + uh$) .... (1)

(A) $y = y_n + u\nabla y_n + \frac{(u)(u-1)}{2!}\nabla^2 y_n + \frac{(u)(u-1)(u-2)}{3!}\nabla^3 y_n + \ldots$

(B) $y = y_n + u\nabla y_n + \frac{(u)(u+1)}{2!}\nabla^2 y_n + \frac{(u)(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots$

(C) $y = -y_n - u\nabla y_n - \frac{(u)(u-1)}{2!}\nabla^2 y_n - \frac{(u)(u-1)(u-2)}{3!}\nabla^3 y_n - \ldots$

(D) $y = y_n - u\nabla y_n + (u)(u+1)\nabla^2 y_n - (u)(u+1)(u+2)\nabla^3 y_n + \ldots$

17. Differentiation formula for $\frac{dy}{dx}$ at $x = x_0$ using forward differences for a set of data points (x_i, y_i), $i = 0, 1, 2, 3, \ldots, n$ with $x_i = x_0 + ih$, $i = 0, 1, 2, 3, \ldots, n$ is $(x = x_0 + uh)$ .... (1)

(A) $\frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 \ldots\right]$

(B) $h\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + \frac{1}{6}\Delta^6 y_0 \ldots\right]$

(C) $[\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 \ldots]$

(D) $\frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$

18. Differentiation formula for $\frac{dy}{dx}$ at $x = x_n$ using backward differences for a set of data points (x_i, y_i), $i = 0, 1, 2, 3, \ldots, n$ with $x_1 = x_0 + ih$, $i = 0, 1, 2, 3, \ldots, n$ is $(x = x_n + uh)$ .... (1)

(A) $\frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 \ldots\right]$

(B) $\frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$

(C) $\frac{1}{h}\left[\nabla y_n - \frac{1}{2!}\nabla^2 y_n + \frac{1}{3!}\nabla^3 y_n - \frac{1}{4!}\nabla^4 y_n + \ldots\right]$

(D) $[\nabla y_n + \nabla^2 y_n + \nabla^3 y_n + \nabla^4 y_n + \ldots]$

19. Lagrange's polynomial through the points (1)

x	0	1
y	4	3

is given by ....

(A) $y = 2x - 3$ (B) $y = x + 4$ (C) $y = -x + 3$ (D) $y = -x + 4$

20. Lagrange's polynomial through the points (2)

x	0	1	2
y	4	0	6

is given by ....

(A) $y = 5x^2 - 3x + 4$ (B) $y = 5x^2 + 3x + 4$ (C) $y = 5x^2 - 9x + 4$ (D) $y = x^2 - 9x + 4$

21. If Lagrange's interpolation polynomial passing through the points (2)

x	0	2	3
y	1	3	2

then the value of y at x = 1 is given by ....

(A) $\frac{7}{3}$ (B) $\frac{5}{3}$ (C) $\frac{8}{3}$ (D) $\frac{5}{4}$

22. If Lagrange's polynomial passes through (2)

x	0	2
y	2	3

then $\frac{dy}{dx}$ at x = 2 is given by ....

(A) $\frac{4}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) $\frac{1}{2}$

23. If Lagrange's polynomial passes through (2)

x	0	1
y	– 4	– 4

then $\frac{dy}{dx}$ at x = 1 is given by ....

(A) 0 (B) 2 (C) 1 (D) $\frac{1}{2}$

24. If Lagrange's polynomial passes through (2)

x	0	2
y	−3	1

then $\int_0^2 y\,dx$ is equal to ....

(A) − 1 (B) − 2 (C) − 1 (D) 2

25. If Lagrange's polynomial passes through (2)

x	0	1
y	1	2

then $\int_0^1 y\,dx$ is equal to ....

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) $\frac{1}{2}$ (D) 3

26. The set of points (x_i, y_i), i = 0, 1, 2, 3, are (0, 1), (1, 2), (2, 7), (3, 25) then the value of Δy_0 is .... (1)

(A) 1 (B) − 6 (C) 6 (D) − 12

27. For $f(x) = x^2$, h = 2, second forward difference $\Delta^2 f(x)$ is given by .... (2)

(A) 6 (B) 12 (C) 4 (D) 8

28. For $f(x) = x^2$, h = 1, $\Delta\nabla\, f(x)$ is given by .... (2)

(A) −2 (B) 1 (C) 2 (D) −1

29. For $f(x) = x^2$, h = 1, $\delta^2 f(x)$ is given by .... (2)

(A) 2 (B) 0 (C) $2x^2$ (D) 3x

30. For $f(x) = x^2$, h = 1, $\mu^2\, f(x)$ is given by .... (2)

(A) $x^2 + 1$ (B) $x^2 - \frac{1}{2}$ (C) x^2 (D) $x^2 + \frac{1}{2}$

31. The set of points (x_i, y_i), i = 0, 1, 2, 3 are (0, 1), (1, 1), (2, 7), (3, 25) then the value of $\Delta^3 y_0$ is .... (2)

(A) 12 (B) 6 (C) −6 (D) −12

32. The set of points (x_i, y_i), i = 0, 1, 2, 3 are (0, 1), (1, 1), (2, 7), (3, 25) then the value of $\Delta^2 y_0$ is .... (2)

(A) 12 (B) 18 (C) 6 (D) −12

33. The set of points (x_i, y_i), i = 0, 1, 2, 3 are (0, 1), (1, 1), (2, 7), (3, 25) then the value of $\Delta^2 y_1$ is .... (2)

(A) 12 (B) 18 (C) 6 (D) −12

34. The set of points (x_i, y_i), i = 0, 1, 2, 3 are (0, 1), (1, 1), (2, 7), (3, 25) then the value of $\Delta^2 y_1$ is .... (2)

(A) 18 (B) 12 (C) 6 (D) −12

35. If a curve passing through (0, −3), (2, 1) is given by $y = y_0 + u\Delta y_0$ and h = 2 then y at x = 1 is equal to ....
(Given : $x = x_0 + uh$) …… (2)

(A) 1 (B) 0 (C) −1 (D) 2

36. If a curve passing through (0, 0), (2, 4), (4, 8) is given by $y = y_0 + u\Delta y_0$ then y at x = 1 is equal to (Given : $x = x_0 + uh$) .... (2)

(A) 1 (B) 0 (C) −1 (D) 2

37. If a curve passing through (1, 1), (2, 4), (3, 7) is given by $y = y_0 + u\Delta y_0$ then y at x = 1.5 is equal to (Given : $x = x_0 + uh$) .... (2)

(A) 2.5 (B) 1.5 (C) 1 (D) 2

38. Given that (2)

x	4	6	8	10
y	1	3	8	16

using Newton's forward formula, the value of y when x = 5, is, ...

(Use : $\Delta y_0 = 2$, $\Delta^2 y_0 = 3$, $\Delta^3 y_0 = 0$ and $x = x_0 + uh$)

(A) 2 (B) 1.625 (C) 1.15 (D) 2.37

39. The pressure p of wind corresponding to velocity v is given by the following data

v	10	20	30	40
p	1	2	4	7

using Newton's forward formula, the value of p when v = 25 is, ... (2)

(Use : $\Delta v_0 = 1$, $\Delta^2 v_0 = 1$, $\Delta^3 v_0 = 0$ and $v = v_0 + uh$)

(A) 3 (B) 3.5 (C) 3.75 (D) 2.875

40. Given that (2)

x	2	4	6	8
y	2	3	8	19

using Newton's backward formula, the value of y when x = 7 is, ...

(Use : $\nabla y_3 = 9$, $\nabla^2 y_3 = 4$, $\nabla^3 y_3 = 0$ and $x = x_n + uh$)

(A) 13 (B) 11 (C) 14 (D) 12

41. Given that (2)

x	4	6	8	10
y	1	3	8	16

using Newton's backward formula, the value of y when x = 9 is ....

(Use : $\nabla y_n = 8$, $\nabla^2 y_n = 3$, $\nabla^3 y_n = 0$ and $x = x_n + uh$)

(A) 10.125 (B) 11.125 (C) 12.125 (D) 13.125

42. Find $\frac{dy}{dx}$ at x = 0 for the tabulated data .... (2)

x	0	1	2	3
y	1	1	15	40

(Given : $\Delta y_0 = 0$, $\Delta^2 y_0 = 14$, $\Delta^3 y_0 = -3$)

(A) −3 (B) −2 (C) −8 (D) −7

43. Find $\frac{dy}{dx}$ at x = 0 for the tabulated data .... (2)

x	0	1	2	3
y	4	8	15	7

(Given : $\Delta y_0 = 4$, $\Delta^2 y_0 = 3$, $\Delta^3 y_0 = -18$)

(A) $-\frac{3}{2}$ (B) $-\frac{7}{2}$ (C) $-\frac{5}{2}$ (D) $-\frac{1}{2}$

44. Find $\frac{dy}{dx}$ at x = 3 for the tabulated data .... (2)

x	0	1	2	5
y	1	1	15	40

(Given : $\nabla y_3 = 25$, $\nabla^2 y_3 = 11$, $\nabla^3 y_3 = -3$)

(A) 29.5 (B) 29 (C) 24.5 (D) 24

45. Find $\frac{dy}{dx}$ at x = 3 for the tabulated data .... (2)

x	0	1	2	3
y	4	8	15	7

(Given : $\nabla y_3 = -8$, $\nabla^2 y_3 = -15$, $\nabla^3 y_3 = -18$)

(A) $-\frac{15}{2}$ (B) $-\frac{43}{2}$ (C) $-\frac{13}{2}$ (D) -21

ANSWERS

1. (D)	2. (C)	3. (B)	4. (A)	5. (C)	6. (A)	7. (B)	8. (D)
9. (D)	10. (B)	11. (C)	12. (A)	13. (C)	14. (D)	15. (A)	16. (B)
17. (A)	18. (B)	19. (D)	20. (C)	21. (C)	22. (D)	23. (A)	24. (B)
25. (B)	26. (A)	27. (D)	28. (C)	29. (A)	30. (D)	31. (D)	32. (C)
33. (A)	34. (B)	35. (C)	36. (D)	37. (A)	38. (B)	39. (D)	40. (C)
41. (D)	42. (C)	43. (D)	44. (A)	45. (B)			

Type III : Trapezoidal Rule, Simpson's $\frac{1}{3}^{rd}$ and $\frac{3}{8}^{th}$ Rule : **Marks**

1. Trapezoidal rule for evaluating $\int_{x_0}^{x_n} f(x)\,dx$ where f(x) takes the values $y_0, y_1, y_2, \ldots, y_n$ for equally spaced values $x_0, x_1, x_2, \ldots, x_n$ with step size h is .... (1)

(A) $\frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})]$ (B) $\frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \ldots) + 2(y_3 + y_4 + \ldots)]$

(C) $\frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$ (D) $\frac{1}{2}[(y_0 + y_n) + (y_1 + y_2 + \ldots + y_{n-1})]$

2. Simpson's $\frac{1}{3}^{rd}$ rule for evaluating $\int_{x_0}^{x_n} f(x)\,dx$ where f(x) takes the values $y_0, y_1, y_2, \ldots y_n$ for equally spaced values $x_0, x_1, x_2, \ldots, x_n$ with step size h is .... (1)

(A) $\frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})]$ (B) $\frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \ldots) + 2(y_3 + y_6 + \ldots)]$

(C) $\frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$ (D) $\frac{1}{2}[(y_0 + y_n) + (y_1 + y_2 + \ldots + y_{n-1})]$

3. Simpson's $\frac{3}{8}^{th}$ rule for evaluating $\int_{x_0}^{x_n} f(x)\,dx$ where f(x) takes the values $y_0, y_1, y_2, \ldots, y_n$ for equally spaced values $x_0, x_1, x_2, \ldots, x_n$ with step size h is .... (1)

(A) $\frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})]$ (B) $\frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots) + 2(y_3 + y_4 + \ldots)]$

(C) $\frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$ (D) $\frac{1}{2}[(y_0 + y_n) + (y_1 + y_2 + \ldots + y_{n-1})]$

4. A curve passes through the set of points (2)

x	0	1	2	3
y	1	3	7	13

Value of $\int_0^3 y\,dx$ by trapezoidal rule is given by ....

(A) 21 (B) 15 (C) 19 (D) 17

5. Value of π obtained by evaluating the integral $\int_0^1 \frac{1}{1+x^2}\,dx$, using Trapezoidal rule with $h = \frac{1}{2}$ is given by ....

$\left(\text{Given : } \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}\right)$ (2)

(A) 3.15 (B) 3.2 (C) 3.1 (D) 3.3

6. A curve passes through the set of points (2)

x	0	1	2	3	4
y	1	2.72	7.39	20.09	54.6

Value of $\int_0^4 y\,dx$ by trapezoidal rule is given by ……

(A) 58 (B) 54.95 (C) 53.87 (D) 55.60

7. The value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta$ evaluated by trapezoidal rule taking $h = \frac{\pi}{4}$ is given by .... (2)

(Given : $\cos\frac{\pi}{4} = 0.8409$)

(A) 1.8981 (B) 1.0526 (C) 2.55 (D) 2.9898

8. For the tabulated data,

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	0	0.5	0.866	1

Evaluation of $\int_0^{\pi/2} \sin\theta\,d\theta$ using trapezoidal rule gives .... (2)

(A) 0.977 (B) 0.988 (C) 0.966 (D) 0.954

9. Value of $\log_e 2$ obtained by evaluating the integral $\int_0^1 \frac{1}{1+x}\,dx$, using Simpson's $\frac{1}{3}^{rd}$ rule with $h = \frac{1}{2}$ is given by ……

$\left(\text{Given : } \int_0^1 \frac{1}{1+x}\,dx = \log_e 2\right)$ (2)

(A) 0.5934 (B) 0.6560 (C) 0.6944 (D) 0.6140

10. The value of $\int_1^2 \frac{1}{x}\,dx$ evaluated by Simpson's $\frac{1}{3}^{rd}$ rule taking h = 0.5 is given by .... (2)

(A) 0.6612 (B) 0.6842 (C) 0.6742 (D) 0.6944

11. The value of $\int_0^{0.8} e^{-x^2}\, dx$ evaluated by Simpson's $\frac{1}{3}^{rd}$ rule taking h = 0.2 is given by .... (2)

Given :

x	0	0.2	0.4	0.6	0.8
e^{-x^2}	1	0.960	0.852	0.697	0.527

(A) 0.5878 (B) 0.6577 (C) 0.4354 (D) 0.5345

12. The value of $\int_0^{\pi/2} \left(\frac{\sin x}{x}\right) dx$ evaluated by Simpson's $\frac{1}{3}^{rd}$ rule taking $h = \frac{\pi}{8}$ is given by .... (2)

Given :

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$\frac{\sin x}{x}$	1	0.974	0.900	0.784	0.636

(A) 1.3700 (B) 1.7300 (C) 1.500 (D) 1.4300

13. Value of $\log_e 7$ obtained by evaluating the integral $\int_0^6 \frac{1}{1+x}\, dx$, using Simpson's $\frac{3}{8}^{th}$ rule with h = 2 is given by ....

$\left(\text{Given : } \int_0^6 \frac{1}{1+x}\, dx = \log_e 7\right)$ (2)

(A) 1.931 (B) 2.124 (C) 1.912 (D) 2.057

14. The table below shows the temperature f(t) as function of time (2)

Time, t	1	2	3	4
Temperature, f(t)	81	75	80	83

Using Simpson's $\frac{3}{8}^{th}$ rule the value of $\int_1^4 f(t)\, dt$ is ....

(A) 215.87 (B) 240.87 (C) 235.87 (D) 225.87

15. Speeds of moving object at different times are recorded as (2)

t (hrs)	0	1	2	3
v (km/hr)	20	40	45	30

Using Simpson's $\frac{3}{8}^{th}$ rule, distance travelled in 3 hours is given by ....

(A) 116.5 km (B) 114.375 km (C) 118.525 km (D) 120.125 km

ANSWERS

1. (A)	2. (C)	3. (B)	4. (D)	5. (C)	6. (A)	7. (B)	8. (A)
9. (C)	10. (D)	11. (B)	12. (A)	13. (D)	14. (C)	15. (B)	

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Ch. 10 Numerical Solution of Ordinary Differential EQuation

Type I : Numerical Solution of Ordinary Differential Equations : **Marks**

1. Given equation is $\frac{dy}{dx} = f(x, y)$ with initial condition $x = x_0$, $y = y_0$ and h is step size. Euler's formula to calculate y_1 at $x = x_0 + h$ is given by .... (1)

(A) $y_1 = y_0 + hf(x_0, y_0)$ (B) $y_1 = y_0 + hf(x_1, y_1)$ (C) $y_1 = y_1 + hf(x_0, y_0)$ (D) $y_1 = hf(x_0, y_0)$

2. Given equation is $\frac{dy}{dx} = f(x, y)$ with initial condition $x = x_0$, $y = y_0$ and h is step size. Modified Euler's formula to calculate $y_1^{(1)}$ at $x = x_0 + h$ is given by .... (1)

(A) $y_0 + h[f(x_0, y_0) + f(x_1, y_1)]$ (B) $y_0 + \frac{h}{4}[f(x_0, y_0) + f(x_1, y_1)]$

(C) $y_0 + \frac{h}{3}[f(x_0, y_0) + f(x_1, y_1)]$ (D) $y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

3. In Runge-Kutta method of fourth order, k_1, k_2, k_3, k_4 are calculated then $y = y_0 + k$. Formula for k_2 is .... (1)

(A) $\frac{h}{2} f(x_0, y_0)$ (B) $hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$ (C) $\frac{h}{2} f\left(x_0 + \frac{h}{3}, y_0 + \frac{k_1}{3}\right)$ (D) $\frac{h}{3} f(x_0 + h, y_0 + k_1)$

4. In Runge-Kutta method of fourth order, k_1, k_2, k_3, k_4 are calculated then $y = y_0 + k$. Formula for k_3 is .... (1)

(A) $\frac{h}{3} f(x_0 + h, y_0 + k_2)$ (B) $\frac{h}{2} f\left(x_0 + \frac{h}{3}, y_0 + \frac{k_2}{3}\right)$ (C) $hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$ (D) $\frac{h}{2} f(x_0, y_0)$

5. In Runge-Kutta method of fourth order, k_1, k_2, k_3, k_4 are calculated then $y = y_0 + k$. Formula for k_4 is .... (1)

(A) $\frac{h}{3} f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right)$ (B) $\frac{h}{2} f(x_3, y_3)$ (C) $\frac{h}{2} f(x_0, y_0)$ (D) $hf(x_0 + h, y_0 + k_3)$

6. In Runge-Kutta method of fourth order, k_1, k_2, k_3, k_4 are calculated then $y = y_0 + k$. k is calculated from .... (1)

(A) $k = \frac{1}{4}(k_1 + k_2 + k_3 + k_4)$ (B) $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

(C) $k = \frac{1}{8}(k_1 + 2k_2 + k_3 + k_4$ (D) $k = \frac{1}{10}(k_1 + 2k_2 + 2k_3 + k_4)$

7. Differential equation $\frac{dy}{dx} = x + y$, with $y(0) = 0$, $h = 0.2$ is to be solved using Euler's method. The value of y at $x = 0.4$ is given by .... (2)

(A) 0.4 (B) 0 (C) 0.04 (D) 0.2

8. Differential equation $\frac{dy}{dx} = x^2 + y^2$, with $y(1) = 2.3$, $h = 0.1$ is to be solved using Euler's method. The value of y at $x = 1.1$ is given by .... (2)

(A) 3.389 (B) 2.929 (C) 0.629 (D) 1.523

9. Differential equation $\frac{dy}{dx} = x - y^2$, with $y(0) = 1$, $h = 0.1$ is to be solved using Euler's Method. If $y(0.1) = 0.9$ then y at $x = 0.2$ is given by …… (2)

(A) 1.892 (B) 0.289 (C) 0.829 (D) 0.991

10. Equation $\frac{dy}{dx} = 1 + xy$ with $y(0) = 1$, $h = 0.1$ is to be solved using Euler's method Given $y(0.1) =1.1$, then y at $x = 0.2$ is .... (2)

(A) 1.222 (B) 1.211 (C) 1.232 (D) 1.192

11. Tabulated solution of the equation $\frac{dy}{dx} = 1 + x$ with $y(0) = 1$, $h = 0.1$ using Euler's method given by .... (2)

(A)

x	0	0.1	0.2
y	1	1.1	1.21

(B)

x	0	0.1	0.2
y	1	1.09	2.12

(C)

x	0	0.1	0.2
y	1	1.25	1.5

(D)

x	0	0.1	0.2
y	1	1.2	1.3

12. Given equation is $\frac{dy}{dx} = \frac{y - x}{y + x}$, with initial conditions $x = 0$, $y = 1$ and step size $h = 0.2$. By Euler's formula y_1 at $x = 0.2$ is equal to 1.2. First approximation $y_1^{(1)}$ at $x = 0.2$ calculated by modified Euler's formula is given by .... (2)

(A) 1.3428 (B) 0.3428 (C) 1.0714 (D) 1.1714

13. Given equation is $\frac{dy}{dx} = \frac{x + y}{x}$ with initial condition $x = 2$, $y = 2$ and step size $h = 0.5$. By Euler's formula y_1 at $x = 2.5$ is equal to 3. First approximation $y_1^{(1)}$ at $x = 2.5$ calculated by modified Euler's formula is given by .... (2)

(A) 1.375 (B) 4.5 (C) 3.05 (D) 3.375

14. Given equation is $\frac{dy}{dx} = \frac{1}{x + y}$ with initial condition $x = 0$, $y = 2$ and step size $h = 0.2$. By Euler's formula y_1 at $x = 0.2$ is equal to 2.1. First approximation $y_1^{(1)}$ at $x = 0.2$ calculated by modified Euler's formula is given by .... (2)

(A) 2.0869 (B) 2.0935 (C) 2.057 (D) 2.075

15. Given equation is $\frac{dy}{dx} = x + y$ with initial condition $x = 0$, $y = 1$ and step size $h = 0.2$. By Euler's formula y_1 at $x = 0.2$ is equal to 1.2. First approximation $y_1^{(1)}$ at $x = 0.2$ calculated by modified Euler's formula is given by .... (2)

(A) 1.24 (B) 1.26 (C) 1.22 (D) 1.28

16. Given equation is $\frac{dy}{dx} = x + y^2$ with initial condition $x = 0$, $y = 1$ and step size $h = 0.2$, k_1 as defined in Runge-Kutta method is given by …… (2)

(A) 0.1 (B) 0.4 (C) 0.3 (D) 0.2

17. Given equation is $\frac{dy}{dx} = \frac{1}{x + y}$ with initial condition $x = 0$, $y = 1$ and step size $h = 0.2$. In Runge-Kutta method k_1, k_2, k_3, k_4 are calculated and are given by $k_1 = 0.2$, $k_2 = 0.167$, $k_3 = 0.169$, $k_4 = 0.1461$. y at $x = 0.2$ is given by .... (2)

(A) 1.1697 (B) 1.4231 (C) 1.3522 (D) 1.5291

18. Given equation is $\frac{dy}{dx} = x^2 + y^2$ with initial condition $y(1) = 1.5$ and step size $h = 0.1$, k_1 is calculated as 0.325, k_2 is given by using Runge-Kutta method is .... (2)

(A) 0.37554 (B) 0.35791 (C) 04252 (D) 0.38664

ANSWERS

1. (A)	2. (D)	3. (B)	4. (C)	5. (D)	6. (B)	7. (C)	8. (B)	9. (C)
10. (B)	11. (C)	12. (D)	13. (C)	14. (B)	15. (A)	16. (D)	17. (A)	18. (D)

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MODEL QUESTION PAPER - I

Engineering Mathematics - III (2019 Pattern)

Sem - II: Second Year Computer Engineering/Information Technology

Phase I : In Semester Examination (ISE)

Time : 1 Hrs. **Max. Marks : 30**

NB : Attempt Q. 1 or Q. 2, Q. 3 or Q. 4.

1. (a) Solve (any two) : (10)

(i) $(D^2 - 4D + 4)\, y = e^{2x} + \cos 2x$

(ii) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1 + e^x}$ (By method of variation of parameters)

(iii) $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = x^2 \sin (\log x)$.

(b) Solve the simultaneous linear differential equations (05)

$(D + 2)\, x - 3\, y = t$

$(D + 2)\, y - 3\, x = e^{2t}$

OR

2. (a) Solve (any two) : (10)

(i) $(D^3 + 2D + 1)\, y = e^{-x} \log x$

(ii) $(D^2 + 4)\, y = \sec 2x$ (By method of variation of parameters)

(iii) $(3x + 2)^2 \dfrac{d^2y}{dx^2} + 3\,(3x + 2) \dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

(b) Solve : $\dfrac{dx}{3z - 4y} = \dfrac{dy}{4x - 2z} = \dfrac{dz}{2y - 3x}$

3. (a) Using Fourier Integral representation, show that $\displaystyle\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x$, where > 0. (10)

(b) Attempt (any one): (05)

(i) Find z-transform of $f(k) = k\, 5^k,\ k \geq 0$

(ii) Find the inverse z-transform of $\dfrac{1}{(z - 1)\,(z - 2)}$, $|z| > 2$.

(c) Solve the difference $12\, f(k + 2) - 7f(k + 1) + f(k) = 0,\ k \geq 0,\ f(0) = 0,\ f(1) = 3$. (05)

OR

4. (a) Solve (any one) : (05)

(i) Find z-transform of $f(k) = 2^k \cos (3k + 2),\ k \geq 0$

(ii) Find the inverse z-transform of $F(z) = \dfrac{10z}{(z - 1)\,(z - 2)}$ (by inversion integral method)

(b) Find the Fourier sine-transform of the function: $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \\ x, & x > 2 \end{cases}$ (05)

(c) Solve the integral equation $\displaystyle\int_0^\infty f(x) \cos \lambda x\, dx = e^{-\lambda},\ \lambda > 0$. (05)

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MODEL QUESTION PAPER - II

Engineering Mathematics - III (2019 Course)

Sem - II: Second Year Computer Engineering / Information Technology

Phase II : End Semester Examination (ESE)

Time : $2\frac{1}{2}$ Hrs. **Max. Marks : 70**

NB : Attempt Q. 1 or Q. 2, Q. 3 or Q. 4, Q. 5 or Q. 6 Q. 7 or Q. 8.

1. (a) Runs scored in 10 matches of current IPL season by two batsmen A and B are tabulated as under (06)

Batsman A	46	34	52	78	65	81	26	46	19	47
Batsman B	59	25	81	47	73	78	42	35	42	10

Decide who is better batsman and who is more consistent.

(b) Fit a straight line of the form $y = mx + c$ to the following data, by using the method of least squares. (06)

x	0	1	2	3	4	5	6	7
y	–5	–3	–1	1	3	5	7	9

(c) Given: $n = 6$, $\sum (x - 18.5) = -3$, $\sum (y - 50) = 20$, $\sum (x - 18.5)^2 = 19$, $\sum (y - 50)^2 = 850$, $\sum (x - 18.5)(y - 50) = -120$.

Calculate coefficient of correlation. (06)

OR

2. (a) The first four moments of a distribution about the value 5 are 2, 20, 40 and 50. From the given information obtain the first four central moments, mean, standard deviation and coefficient of skewness and kurtosis. (06)

(b) Following is the data given for values of X and Y. Fit a second-degree polynomial of the type are $ax^2 + bx + c$ where a, b, c are constants. (06)

x	–3	–2	–1	0	1	2	3
y	12	4	1	2	7	15	30

(c) If the two lines of regression are $9x + y - \lambda = 0$ and $4x + y = \mu$ and the means of x and y are 2 and –3 respectively, find the values of $\bar{x}$, $\bar{y}$ and the coefficient of correlation between x and y. (06)

3. (a) A, B play a game of alternate tossing a coin, one who gets head first wins the game. Find the probability of B winning the game if A has a strat. (05)

(b) A coin is so biased that appearance of head is twice likely as that of tail. If a throw is made 6 times, find the probability that at least 2 heads will appear. (06)

(c) In a certain factory turning out razor blades, there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective and two defective blades, in a consignment of 10,000 packets. (06)

OR

4. (a) If random variable X takes the values X = 1, 2, 3 with corresponding probabilities $\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$. Find E(x) and E($x^2$). (05)

(b) In a certain examination test, 2000 students appeared in a subject of statistics. Average marks obtained were 50% with standard deviation 5%. How many students do you expect to obtain more than 60% of marks, supposing that marks distributed normally? (06)

[**Given :** A (z = 2) = 0.4772]

(c) Among 64 offspring's of a certain cross between guinea pigs 34 were red, 10 were black and 20 were white. According to a genetic model, these numbers should be in the ratio 9 : 3 : 4. Are the data consistent with the model at 5% level ? [**Given :** χ_2^2 : 0.05 = 5.991] (06)

5. (a) Find the real rood of the equation $x^3 + 2x - 5 = 0$ by applying Newton-Raphson method at the end of fifth iteration. (06)

(b) Solve $3x - 1 - \cos x = 0$ by using the method of successive approximations correct to three decimal places. (06)

(c) Solve by Gauss-Seidel method, the system of equations (06)

$27x_1 + 2x_2 + 3x_3 = 11$

$-2x_1 + 35x_2 + 4x_3 = 15$

$3x_1 + 2x_2 + 55x_3 = 21$

OR

6. (a) Solve the following system of equations by Gauss elimination method: (06)

$10x + 2y + 3z = 9$

$2x + 20y - 2z = -44$

$-2x + 3y + 10z = 22$

(b) Solve by Jacobi's iteration method the following system of equations: (06)

$20x_1 + x_2 - 2x_3 = 17$

$3x_1 + 20x_2 - x_3 = -18$

$2x_1 - 3x_2 + 20x_3 = 25$

(c) Use Regula-Falsi method to find a real root of the equation $e^x - 4x = 0$ correct to three decimal places. (06)

7. (a) Find interpolating polynomial passing through following tabulated data (06)

x	0	0.1	0.2	0.3	0.4	0.5
y	1	1.046	1.094	1.44	1.297	1.252

Estimate the values of y at x = 0.05 and $\frac{dy}{dx}$ at x = 0.5

(b) Evaluate $\int_0^3 \frac{dx}{1 + x}$ with 7 ordinates by using Simpson's $\frac{3^{th}}{8}$ rule and hence calculate log 2. (05)

(c) Solve the equation $\frac{dy}{dx} = 1 + xy$, $x_0 = 0$, $y_0 = 1$ to find y at x = 0.1 and x = 0.2 correct to three decimal places using modified Euler's method taking h = 0.1. (06)

OR

8. (a) Use Euler's modified to find the value of y satisfying the equation $\frac{dy}{dx} = \log(x + y)$, y (1) = 2, for x = 1.2 and x = 1.4 correct to three decimal places by taking h = 0.2. (06)

(b) Use Runge-Kutta method of fourtht order to solve $\frac{dy}{dx} = \frac{1}{x + y}$, $x_0 = 0$, $y_0 = 1$ to find y at x = 0.4 correct to three decimal places taking h = 0.2. (06)

(c) Given (05)

x	1	1.2	1.3	1.4
$\sqrt{\mathbf{x}}$	1	1.095	1.140	1.183

Find $\sqrt{1.1}$ using Lagrange's interpolation.

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NOTE

NOTE