Probability and Statistics

Assignment -2 [2018 Sep]

1. Suppose X, Y are independent discrete random variables with $P(X = 1) = P(X = -1) = P(Y = 1) = P(Y = -1) = 0.5$. Let $Z = XY$. Show that X, Z are independent and Y, Z are independent. But X, Y, Z are NOT independent.

2. Let X, Y be discrete real valued random variables such that

$$\begin{aligned} P(X = 0, Y = 0) = \tfrac{3}{16}, P(X = 0, Y = 1) = \tfrac{3}{16}, \\ P(X = 1, Y = 0) = \tfrac{5}{16}, P(X = 1, Y = 1) = \tfrac{5}{16} \end{aligned}$$

   Show that X, Y are independent. Let $W = X + Y$. Are W and Y independent?

3. Let X be a $[0, \infty)$ valued continuous random variable with density function f. Show that

$$P(X > \lambda) \leq \frac{1}{\lambda} E(X).$$

4. Let X be a real valued continuous random variable with density function f. Let

$$\mu = \int a f(a) da$$

   be finite and

$$\sigma^2 = \int (a - \mu)^2 f(a) da$$

   be finite. Show that

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

   (Hint: consider the function $g(x) = \begin{cases} (x-\mu)^2 & \text{if } |x| \geq t \\ 0 & \text{if } |x| < t \end{cases}$. Note $E(g(X) \leq t^2 \sigma^2$.)

5. Let X and Y be independent random variables with Normal distribution, with mean 0 and variance 1. Let $-1 < \alpha < 1$ and $W = \alpha X + \sqrt{(1 - \alpha^2)} Y$, $Z = \alpha Y - \sqrt{(1 - \alpha^2)} X$.

   (a) Show that W and Z are independent random variables with Normal distribution, with mean 0 and variance 1.

   (b) Show that (X, W) has bivariate normal distribution. Obtain the correlation between X, W.

   (c) Show that (X, Y, W, Z) has multivariate normal distribution. Obtain its mean vector and variance covariance matrix.

6. Suppose (X, Y) has bivariate normal distribution with each X and Y having mean 0 and variance 1. Suppose the correlation between X, Y is α. Let $W = \frac{1}{\sqrt{(1-\alpha)}}(Y - \alpha X)$. Show that X and W are independent random variables with Normal distribution, with mean 0 and variance 1.