Probability and Statistics

Assignment -1 [2018 Aug]

1. Let X, Y be discrete real valued random variables such that

$$P(X=0,Y=0)=\tfrac{3}{16}, P(X=0,Y=1)=\tfrac{1}{16},$$
$$P(X=1,Y=0)=\tfrac{7}{16}, P(X=1,Y=1)=\tfrac{5}{16}$$

   and let $Z=X+Y$. Find the distributions of X, Y and Z and compute $E(X)$, $E(Y)$, E(Z), $E(XY)$, $E(XZ)$ and $E(YZ)$. Obtain

$$P(X=1|Z=1).$$

2. Let X, Y be discrete real valued random variables taking values in $I=\{0,1,2,3,\dots,10\}$ such that $P(X=i,Y=j)=P(X=i)P(Y=j)$, $0\le i\le 10$, $0\le j\le 10$.
   Show from first principles that for any functions f,g on integers

$$E[f(X)g(X)]=E[f(X)]E[g(X)]$$

3. Let X have binomial distribution with parameters n and p. Show that $E(X)=np$ and $Var(X)=npq$. Taking $n=200$ and $p=0.01$, obtain a upper bound on $P(X\ge 6)$ using Tchebychev's inequality. Also evaluate the exact probability in question using R. Write an R-code to estimate the probability using simulation size 1000,000 and compare the answer with exact value.

4. Let X have Poisson distribution with parameter λ Show that $E(X)=\lambda$ and $Var(X)=\lambda$. Taking $\lambda=2$, obtain a upper bound on $P(X\ge 6)$ using Tchebychev's inequality. Also evaluate the exact probability in question using R. Write an R-code to estimate the probability using simulation size 1000,000 and compare the answer with exact value.

5. Let X_1,X_2 be independent random variables with normal distribution, both with mean 0 and variance 1. Find the (joint) characteristic function of $\mathbf{X}=(X_1,X_2)$:

$$\phi(t_1,t_2)=E[\exp\{it_1X_1+it_2X_2\}].$$

   Let $Z_1=\frac{1}{\sqrt{2}}(X_1+X_2)$ and $Z_2=\frac{1}{\sqrt{2}}(X_1-X_2)$. Find the (joint) characteristic function of $\mathbf{Z}=(Z_1,Z_2)$. Show that Z_1 and Z_2 are independent.

6. Let X_1,X_2 be independent random variables with normal distribution, both with mean 0 and variance 1. Let $Y_1=X_1$ and $Y_2=\theta X_1+\sqrt{(1-\theta^2)}X_2$ where $-1<\theta<1$. Find the characteristic function of (Y_1,Y_2):

$$\phi(t_1,t_2)=E[\exp\{it_1Y_1+it_2Y_2\}].$$

7. Let X be a random variable with exponential distribution with parameter λ. For what values of t does the mgf

$$m(t)=E[\exp\{tX\}]$$

   exists and is finite? Obtain an expression for the same.