Introduction to Programming in Python

Duration: 120 Minutes

1. Write a python function `closest` which takes 3 arguments: a non-empty list of integers `L` and two integers `x` and `y` and returns the element `a` of `L` such that $|y - (x + a)|$ is minimized. What is the complexity of your program. *(5 marks)*

2. Write a python function `better` which takes 3 arguments: the first argument is a list of pairs of integers `L`, the second and third arguments `x` and `y` are integers. It returns true if for each entry of the form `(x,a)` in `L` there is some entry `(y,b)` in `L` such that $a \leq b$. Otherwise it returns false. Here are some examples:

```
better([(1,2),(-3,4),(3,-5),(2,4),(1,3),(2,1)],2,1) = False
better([(1,2),(-3,4),(3,-5),(2,4),(1,3),(2,1)],1,2) = True
better([(1,2),(-3,4),(3,-5),(2,4),(1,3),(2,1)],5,2) = True
better([(1,2),(-3,4),(3,-5),(2,4),(1,3),(2,1)],2,5) = False
```

(5 marks)

3. Describe a python function `braid` which takes three lists `I`,`F` and `S` as input and outputs a single list. The list `I` is a sequence of positive integers. It merges the lists `F` and `S` as follows: the first `I[0]` entries of the answer are from `F`, the next `I[1]` entries are from `S`, the next `I[2]` entries are from `F` and so on. Here are some examples:

```
braid([1,0,1,2],['a','b','c'],[13,14,15,16] = ['a','b',13,14]
braid([1,2,3,2],['a','b','c'],[13,14,15,16] = ['a',13,14]
braid([1,2,2,3],['a','b','c'],[13,14,15,16] = ['a',13,14,'b','c']
braid([1,0,2,2],['a','b','c'],[13,14,15,16] = ['a','b','c',13,14]
braid([],['a','b','c'],[13,14,15,16] = []
```

(5 marks)

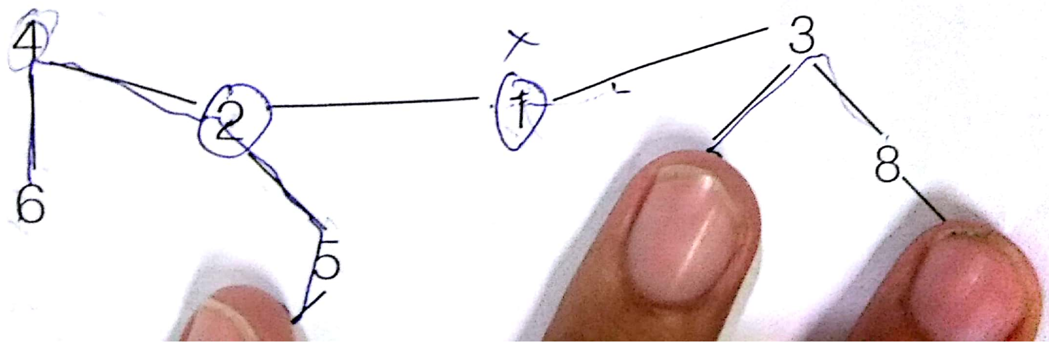
4. An (undirected) graph $G = (V, E)$ is said to be bi-partite, if we can write V as the disjoint union of two non-empty sets V_1 and V_2 so that no edge connects two vertices in V_1 or two vertices in V_2. That is, all edges have one end point in V_1 and the other in V_2. Note that for a given bi-partite graph there may be more than one way to choose V_1 and V_2. For example, if the graph has no edges then any way of dividing V into two non-empty sets will work.

   Prove that a graph is bi-partite if and only if it does not contain any odd length cycles. *(5 marks)*

5. Describe a python function `divide` which takes graph G (you may assume that it is given by its adjacency sets as done in all the functions in class) as input and outputs two sets V_1 and V_2 giving a bi-partition of G if G is bi-partite. It should output `None` if it is not bi-partite. *(5 marks)*

6. Write a python function `largest` which takes arguments, a undirected graph `G` given by adjacency sets and identify of a vertex in the graph v. You are guaranteed that the given graph is a tree. The function should return the size of the largest set of vertices `S` such that there is path between every pair of vertices in `S` that does not go through `v`. Consider the following graph:

largest(G, 1)



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Mid-Sem examination, September 2017

Probability and Statistics - I

Answer any 4 questions. Each question carries 25 marks. **Time: 2 hours.**

✓1. (i) Let X be a random variable with Binomial distribution with parameters n, p. Obtain the mean and variance of X.

(ii) Let Y be a random variable taking values in $E = \{0, 1, 2, \ldots\}$ with $P(Y \geq m) > 0$ for all $m \in E$. Suppose

$$P(X \geq m + n \mid X \geq m) = P(X \geq n).$$

Find the distribution of X.

✓2. Let X, Y be independent random prfssn variables with parameters α and β respectively. Let $Z = X+Y$. Find the conditional distribution of X given $Z = m$, i.e. compute

$$P(X = k \mid Z = m). \quad \mathrm{Bin}\left(m, \frac{\alpha}{\alpha+\beta}\right)$$

✓3. Let $X_1, X_2, \ldots, X_n \ldots$ be i.i.d. random variables with $E(X^2) < \infty$. Show that

(i) $Var(X_1 + X_2 + \ldots + X_n) = nVar(X_1)$.

(ii) $\lim_{n\to\infty} P(|\frac{1}{n}(X_1 + X_2 + \ldots + X_n) - E(X)| > \epsilon) = 0$ for all $\epsilon > 0$.

4. Let X be a random variable with density f such that $E(X^2) < \infty$. Suppose $f(x) = f(-x)$ for all x. Let U be a random variable independent of X with $P(U = 1) = P(U = -1) = 0.5$. Let $Z = U|X|$. Show that

(i) Z has the same distribution as X.

(ii) Correlation between X, Z is 0.

(iii) $P(X = Z) > 0$ and hence that X, Z are not independent.

✓5. Let X and Y be independent random variables, both having normal distribution with mean 0 and variance 1. Let $U = X + Y$ and $V = X - Y$. Show that U and V are independent random variables with normal distribution. Find mean and variances of U, V. Give reasons and state any result you use.

$$E(X^2) = \sum_{x=0}^{n} x^2 \cdot \frac{\lfloor n}{\lfloor x \cdot \lfloor n-x} \cdot p^x \cdot q^{n-x}$$

$$E(X^2) - E(X)$$

$$+ (E(X))^2 + E(X)$$

$$= \sum_{x=1}^{n} x^2 \cdot \frac{\lfloor n}{\lfloor x \cdot \lfloor n-x} \cdot p^x \cdot q^{n-x}$$

$$\sum_{x=1}^{n} x \cdot \frac{\lfloor n}{\lfloor x-1 \; \lfloor n-x} \cdot p^x \cdot q^{n-x}$$

$$\binom{m}{k} \frac{\alpha^k \beta^{m-k}}{(\alpha+\beta)^m}$$

$$P(X=\alpha \cap Z=\beta) = P(X=\alpha) \cdot P(Z=\beta)$$

$$P(X=\alpha, Z=\beta) \quad P(X=\alpha)$$

$$\frac{e^{-\alpha}\alpha^k}{k!} \cdot \frac{e^{-\beta}\beta^{m-k}}{(m-k)!}$$

$$\frac{m!\,P(X=k,\, Y=m-k)}{P(X=k)P(Y=m-k)} \over P(Z=m)$$