Analysis of Divide-Conquer Algorithms

The typical recurrence describing the complexity of a divide and conquer algorithm would look like:

$$T(N) = aT(N/b) + f(N)$$

where

- N/b is the size of each subproblem
- a is the number of subproblems and
- $f(N)$ is the cost of dividing up the problem into subproblems and combining the solutions.

Usually, dividing into subproblems is quite direct and the ingenuity is in stitching together the solutions.

Closest Pair

Given a collection

$$P = \{(x_1, y_1), (x_2, y_2) \ldots (x_N, y_N)\}$$

of points on the plane, find the pair that is closest to each other.

$$dist((x_i, y_i), (x_j, y_j)) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

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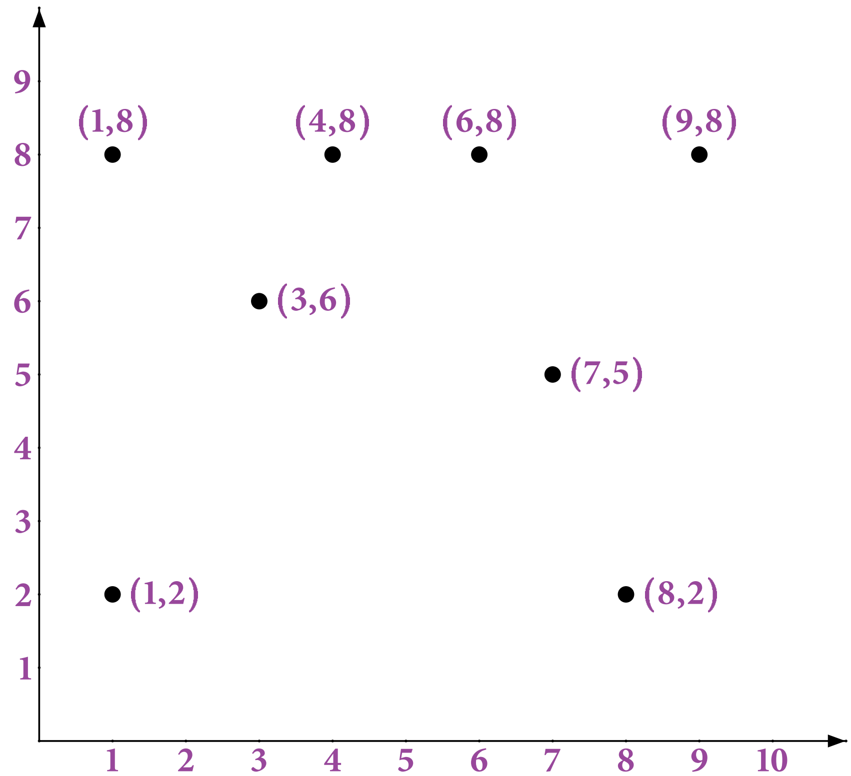
The obvious algorithm that considers all possible pairs and finds the minimum takes $O(N^2)$ time.

Consider the list of points

$$(1,2),\ (8,2),\ (7,5),\ (9,8),\ (6,8),\ (3,6),\ (4,8),\ (1,8)$$



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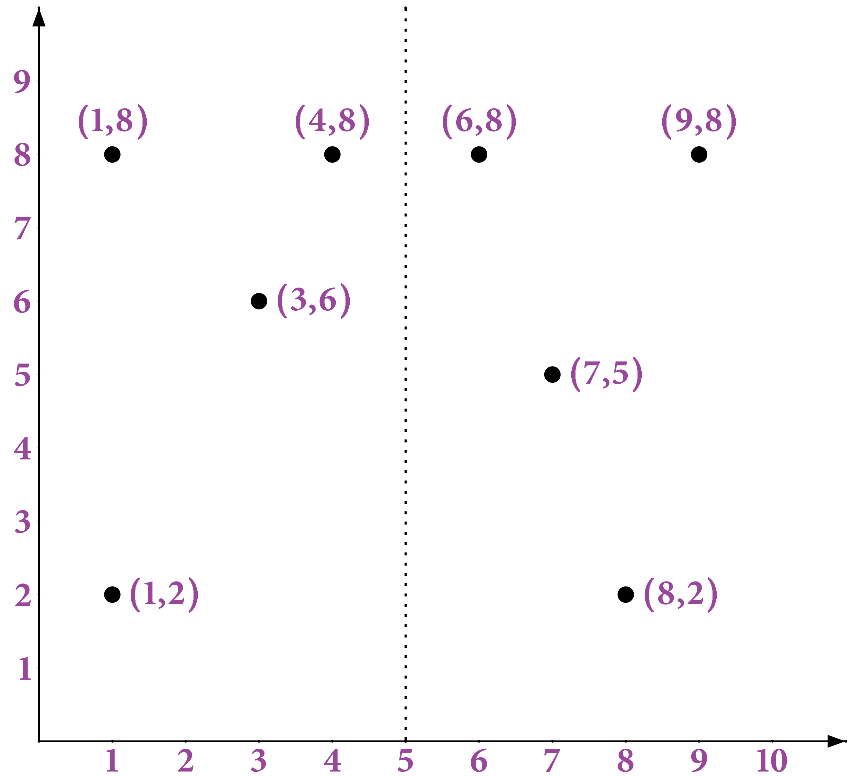
Two Subproblems



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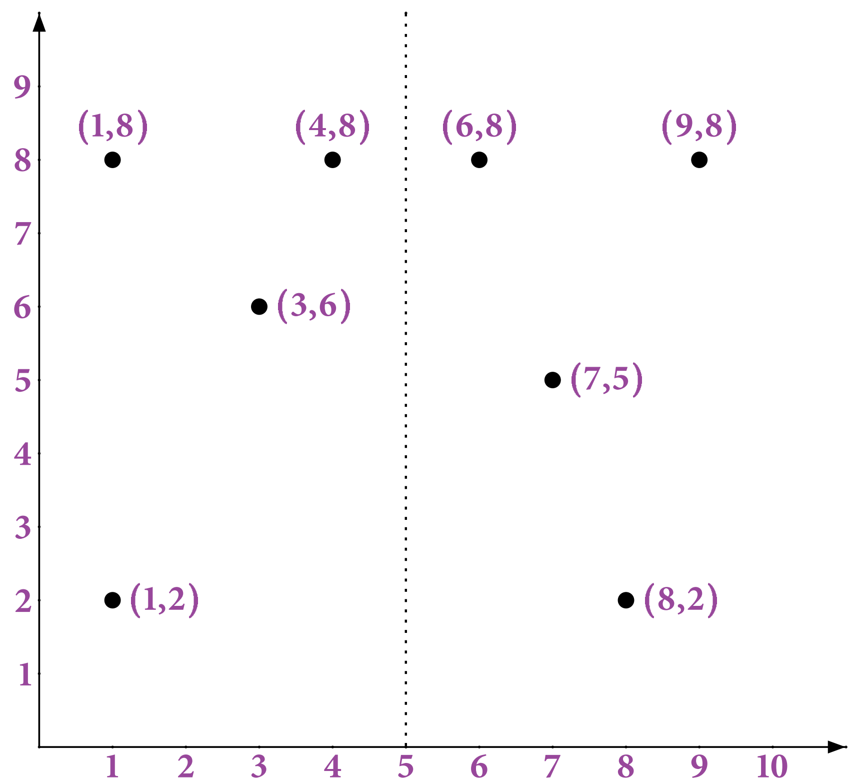
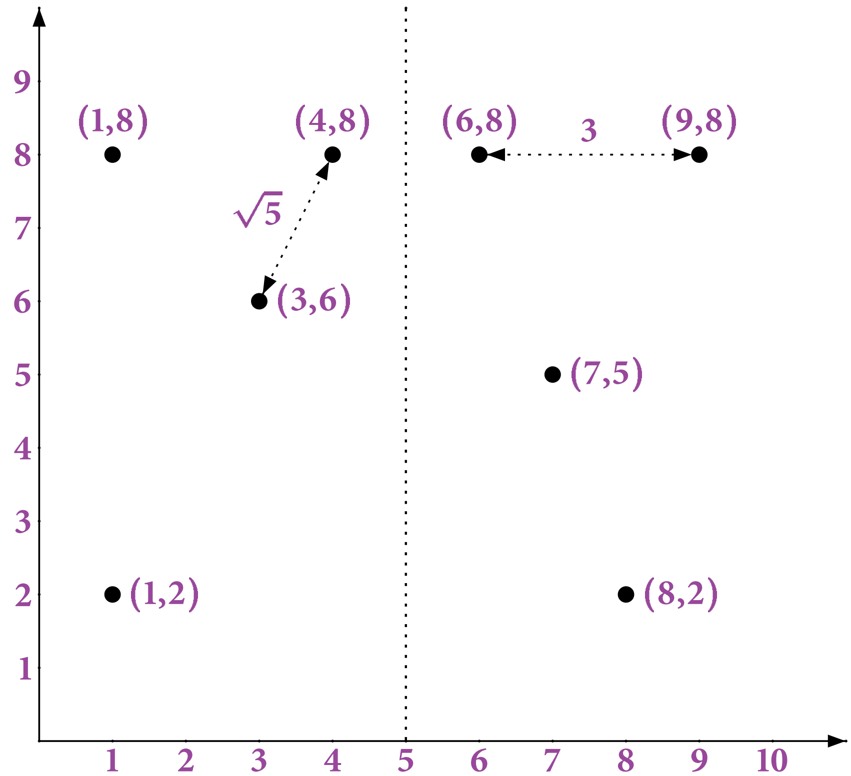


Solve each subproblem.

Solutions to Subproblems



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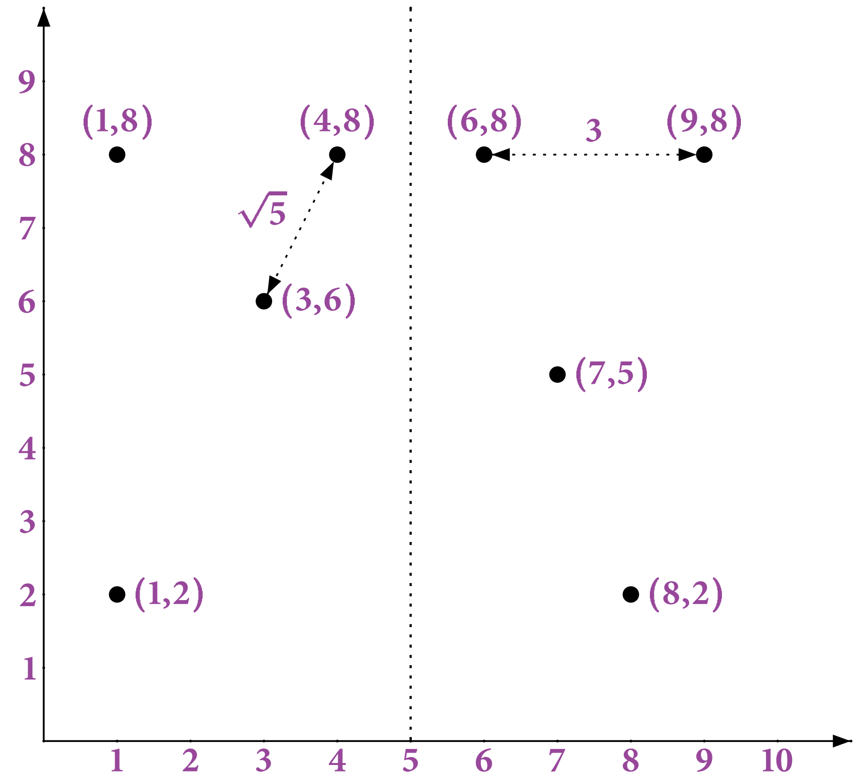

How do we put together the solutions?

Composing the Solutions

What is the difficulty?

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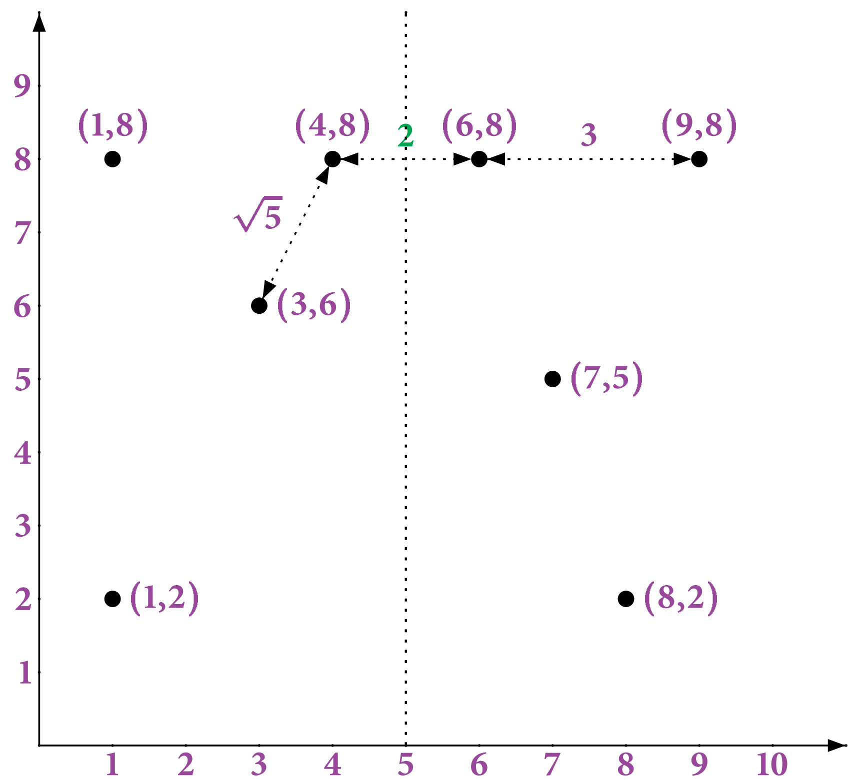
The closest pair of points can span across the division.



An observation

Let d_l and d_r be the answers to the two subproblems. Let d be the minimum of these two values.

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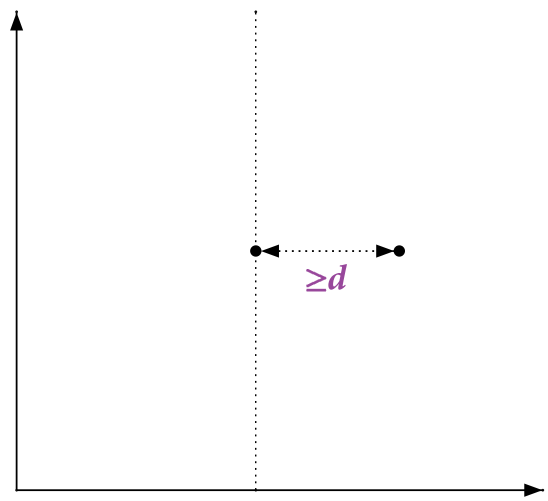
Observation: If (x, y) and (x', y') are from across the division and the distance between them is less than d then $|x - m| < d$ and $|x' - m| < d$.



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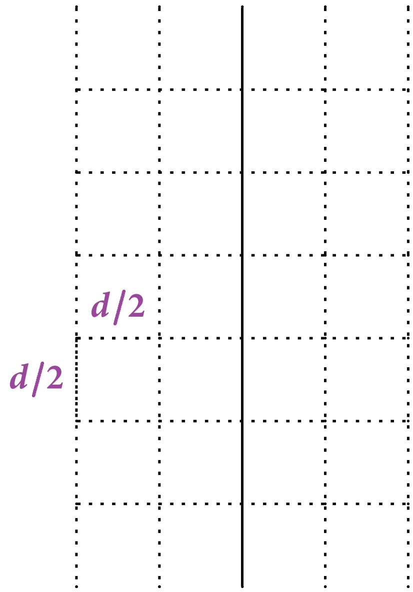
S might contain all the points!

Let us divide up the region between $X = m - d$ and $X = m + d$ into squares of size $d/2$



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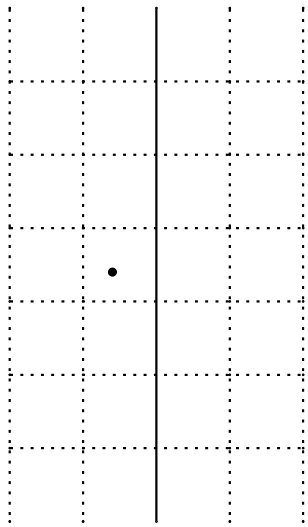
- Each square lies entirely on one side of the partition.
- The farthest points in a square are separated by the diagonal. The length of the diagonal is:

$$\sqrt{2}.d/2 = \frac{d}{\sqrt{2}} < d$$

If there are two in any one square then d cannot be the distance between the closest pair of points in the subproblems.

Candidate pairs



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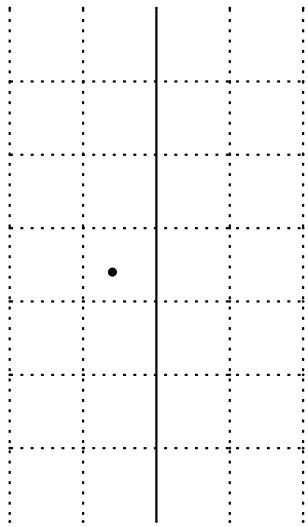
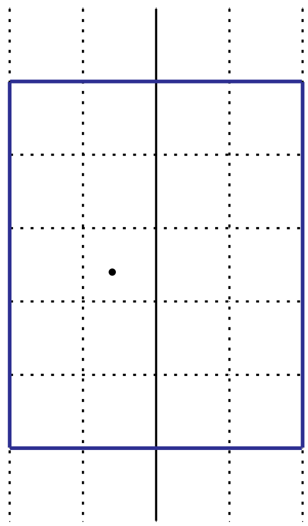
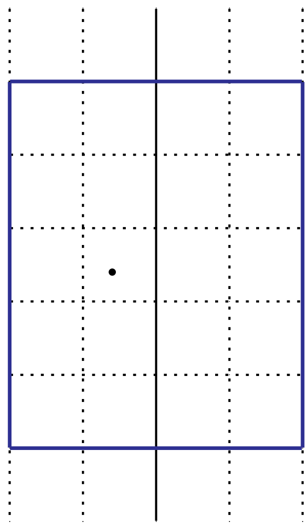


Where can points (in S) closer than d lie?

Candidate pairs



Only within the marked rectangle.



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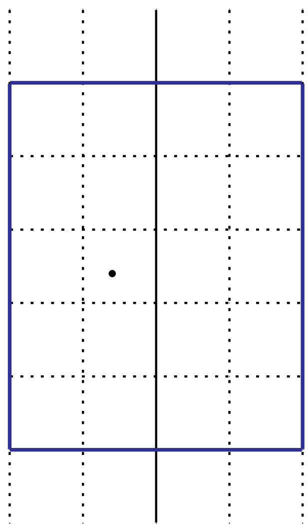
Thus, there are at the most **19** points whose distance from the given point is less than d

How to find the candidate pairs?



Sort the points in S based on the Y-coordinate. Then, a candidate pair cannot be separated by more than 11 in the list!

We can check all candidate pairs within S using at the most $\mathbf{11.|S|}$ calculations.

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5. Sort the points in S along the Y-axis. ($O(|S|log(|S|))$)
6. Consider all pairs within distance 11 in this list and compute the minimum. ($O(|S|)$)

$$T(N) = 2T(N/2) + c.Nlog(N)$$

Reducing the complexity

Start with two lists P_x and P_y listing the points sorted along X-axis and along Y-axis respectively.

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