BINARY P-ADIC INTEGERS IN COLLATZ SEQUENCES

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Abstract. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. We describe an approach from the perspective of 2-adic (binary) algebra.

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Fundamentals short and sweet

unit	An element a of a ring R is called a "unit" (an invertible element) if there exist an element b such that $ab = 1$ [1, p. 24]. Units are elements with inverses with respect to multiplication in the ring. Let F be a field, then an element a of F is a non-unit iff $a = 0$. The sum of any two non-units in F is again a non-unit in F.
unitary ring	A unitary ring is a ring with a multiplicative identity 1 (which differs from the additive identity $1 \neq 0$) such that $1a = a = a1$ for all elements a of the ring.
Ideal	Let $(R, +, \cdot)$ be a commutative unitary ring. Then the subset $I \subseteq R$ is called an ideal of R if $(I, +)$ is a commutative group and if $xI \subseteq I$ for all $x \in R$, see [2, p. 66-67].
quot. ring	Using an ideal of a ring $I \subseteq R$, we may define an equivalence relation $\sim$ on R by $a \sim b$ iff $a - b$ is in I [3, p. 69]. The equivalence class of a in R is given by $[a] = a + I := \{a + r \mid r \in I\}$ for $r \in R$ and referred to as "residue class of a modulo I", see [4, p. 120], [3, p. 70]. The set of all these equivalence classes becomes the quotient ring (residue class ring) modulo the ideal I, denoted by R/I.
compl. residue system	Let $I \subseteq R$ be an ideal and $[a]$ the residue classes of a modulo I, which means that $a + I = b + I$ when $a \equiv b \mod I$ or respectively $a - b \in I$ [3, p. 70]. R is the disjoint union of the different residue classes a modulo I. A subset $M \subseteq R$, which contains exactly one element from each of these residue classes, is called a complete residue system of R modulo I, see [3, p. 70].
$[a]_n$	The residue class (also termed congruence class) of the integers for a modulus n is the set $[a]_n = \{a + kn \mid k \in \mathbb{Z}\}$ and sometimes denoted by $\bar{a}_n$ or by $a + n\mathbb{Z}$, see [2, p. 15], [4, p. 120], [5, p. 25].
$\mathbb{Z}/n\mathbb{Z}$	The set of all residue classes $[a]_n$ is called the ring of integers modulo n and denoted by $\mathbb{Z}/n\mathbb{Z} = \{[a]_n \mid a \in \mathbb{Z}\}$ and trivially $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$ and for all $n \neq 0$ we have $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$, see [2, p. 15], [5, p. 25].

direct prod.	If $R_1, R_2, \ldots, R_n$ are rings, the cartesian product $R_1 \times R_2 \times \ldots \times R_n$ forms the set of all ordered n-tuples $(r_1, r_2, \ldots, r_n)$, where $r_i \in R_i$. The addition and multiplication of these n-tuples is defined "coordinatewise" by components. The resulting ring is called a "direct product" of the original rings R_i [2, p. 51], [6, p. 169].
prod. of ideals	Let I, J be two ideals of a ring R. Their product IJ is defined as the set of all finite sums $a_1b_1 + \ldots + a_nb_n$ in which $n \geq 0$ and $a_1, \ldots, a_n \in I$ and $b_1, \ldots, b_n \in J$, see [7, p. 87].
power of an ideal	Let I be an ideal in R. The n-th power of I, denoted by I^n, is the n-times product of the ideal I with itself. I^n contains sums of elements of the form $a_1a_2 \cdots a_n$ where $a_1, a_2, \ldots, a_n \in I$ and the products refer to the multiplication defined in R. Consequently, $I^{n+1} \subseteq I^n$. Note that the product and power of ideals should not be confused with the direct product of rings.
filtration	Let R be a ring. A sequence of ideals $I_0, I_1, I_2, \ldots$ is said to be a "filtration" on R if $I_0 = R$ and for each integer $j \geq 0$ applies that $I_j \supseteq I_{j+1}$ and if $I_jI_k \subseteq I_{j+k}$, see [8, p. 269].
princip. ideal	A "principle ideal" is an ideal in a ring R which is generated by a single element a of R through multiplication by every element of R. There are some rings in which every ideal is a principle ideal, so-called "principle ideal rings" [2, p. 68].
max. ideal	A proper Ideal M of a ring R is called "maximal ideal" of R if there is no other proper ideal N of R properly containing M [6, p. 247], [1, p. 37]. A Note on "proper containment": If R is any set, then R is the improper subset of R. Any other subset $N \neq R$ is a proper subset of R and denoted by $N \subset R$ or $N \subsetneq R$ [6, p. 2].
prime ideal	Let a and b are two elements of R and P a proper ideal such that their product ab is an element of P. P is called a prime ideal if at least one of a and b belongs to P, in other words from $ab \in P$ and $a \notin P$ always follows $b \in P$ [1, p. 9].
max. prime ideal	A proper prime ideal P is said to be a "maximal prime ideal" of the ring R, if there is no other proper prime ideal containing P [1, p. 23].
local ring	A commutative ring R is called a local ring if it has a unique maximal ideal M [9, p. 522].

Noeth. ring	A ring R is called "Noetherian" when in R the maximal condition for ideals is satisfied, in other words if every ideal I of R is finitely generated, that is, if we can find a finite set $a_1, a_2, \ldots, a_n$ of elements, such that $I = Ra_1 + Ra_2 + \ldots + Ra_n$ [1, p. 19, 101].
semi-local ring	A semi-local ring is a Noetherian ring which has only a finite number of maximal ideals [1, p. 107].
zero seq.	A zero sequence is a sequence, which converges towards 0 [7, p. 154]. Given the context of ideal theory, let R be a ring and I an ideal. In the ring $R^{\mathbb{N}} = \prod_{n\in\mathbb{N}} R$, which is the repeated direct product of R with itself, a sequence $(x_i)_{i\in\mathbb{N}}$ is called a zero sequence if for every $s \in \mathbb{N}$ there exist a $N \in \mathbb{N}$ (depending on s) such that $x_n \in I^s$ for all $n > N$.
Cauchy seq. in $\mathbb{Q}$, $\mathbb{R}$	A sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{Q}$ or $\mathbb{R}$ is a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer N such that $\lvert x_n - x_m \rvert < \epsilon$ for all $n, m \geq N$, see [7, p. 153], [10, p. 24], [11, p. 10].
Cauchy seq. in a ring	Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of elements in $R^{\mathbb{N}}$, the repeated direct product of a ring with itself, and I an ideal in R. This sequence is a Cauchy sequence if for every $s \in \mathbb{N}$ there exist a $N \in \mathbb{N}$ such that $x_n - x_m \in I^s$ for all $n, m > N$.
Cauchy seq. in a local ring	Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of elements in a local ring R and M is the maximal ideal of R. This sequence is a Cauchy sequence if, given any $s \in \mathbb{N}$, we can always find an integer N such that $x_n - x_m \in M^s$ whenever $n > m > N$, see [1, p. 63, 85]. It is a Cauchy sequence iff $x_n - x_{n-1} \to 0$ as $n \to \infty$ [1, p. 85].
compl. of a ring	Let R be a ring, I an indeal, I_{ZS} the ideal of all zero sequences in $R^{\mathbb{N}}$, and S_{CS} the subring of $R^{\mathbb{N}}$ containing all Cauchy sequences. The quotient ring $\hat{R}_I := S_{CS}/I_{ZS}$ is called the completion of R with respect to I. S_{CS}/I_{ZS} is the residue class ring of S_{CS} modulo I.
concor. ext.	Let R, S be local rings. If a sequence of elements of S is a Cauchy sequence in S iff it is a Cauchy sequence in R, then we say that R is a "concordant extension" of S [1, p. 87]. When R, S are semi-local rings $R \subseteq S$, R is said to be a "concordant extension" of S if a sequence (s_n) of elements in S is regular in S iff (s_n) is regular in R [12].

compl. of a local ring	Let S be a local ring. A local ring R will be called a completion of S if R is a concordant extension of S and R is complete and if every element of R is the limit of a sequence of elements of S. Each local ring has a completion [1, p. 92].
compl. local ring	A local ring R is called "complete" if every Cauchy sequence composed of elements of R has a limit in R [1, p. 85], [13, p. 184].
p-adic val. for $\mathbb{Z}$	Fix a prime number p in $\mathbb{Z}$. The p-adic valuation of a nonzero integer $n = r \cdot p^{v_p(n)}$ is the highest exponent $v_p(n)$ such that $p^{v_p(n)}$ divides n (we say $p^{v_p(n)}$ divides n "exactly"). Hence p and r are coprime. If n, p are coprime then $v_p(n) = 0$, and by convention $v_p(0) = \infty$, see [14].
p-adic val. for $\mathbb{Q}$	The p-adic valuation can be extended to the field of rational numbers. Let $x = n \cdot s^{-1}$ be a rational number, then $v_p(x) = v_p(n) - v_p(s)$. Any nonzero rational number x can be uniquely represented as $x = rp^{v_p(x)}s^{-1}$, where $r, s \in \mathbb{Z}$, $s > 0$, and $\gcd(r, s) = \gcd(r, p) = \gcd(s, p) = 1$, see [7, p. 154], [15].
p-adic norm	Let x be any number in $\mathbb{Q}$, for which we already know that it can be written as $x = rp^{v_p(x)}s^{-1}$, where p is a prime number, $s > 0$ and r are integers not divisible by p. The p-adic norm of x is defined by $\lvert x\rvert_p = p^{-v_p(x)}$ for $x \neq 0$, and $\lvert 0\rvert_p = 0$, see [14], [7, p. 154], [16].
p-adic dist.	Let $x, y \in \mathbb{Q}$. The p-adic distance between x and y is defined by $d_p(x, y) = \lvert x - y\rvert_p$, see [7, p. 155].
$\mathbb{Q}_p$	The field $\mathbb{Q}_p$ of p-adic numbers is the set of equivalence classes of Cauchy sequences [11, p. 10]. The elements of $\mathbb{Q}_p$, the so-called p-adic numbers, are eqivalence classes of Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ with respect to the equivalence relation $(a_n) \sim (b_n)$ if $(a_n - b_n)$ is a p-adic zero sequence, see [7, p. 159]. Furthermore $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic distance d_p [7, p. 159].
$\mathbb{Z}_p$	The ring $\mathbb{Z}_p$ of p-adic integers is the completion of $\mathbb{Z}$ with respect to the p-adic norm. That is, $\mathbb{Z}_p$ is the set of all equivalence classes of Cauchy sequences (a_n) where (a_n) and (b_n) are equivalent if $\lim_{n\to\infty} \lvert a_n - b_n\rvert_p = 0$, see [17]. $\mathbb{Z}_p$ is a local ring whose maximal ideal is the principal ideal $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \lvert x\rvert_p < 1\}$, see [18, p. 74].

References

[1] D. G. Northcott. *Ideal Theory*, volume 42 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Cambridge University Press, Cambridge, United Kingdom, 1953.
[2] J. Wolfart. *Einführung in die Zahlentheorie und Algebra*. Vieweg+Teubner, Wiesbaden, Germany, 2 edition, 2011.
[3] R. Schulze-Pillot. *Einführung in Algebra und Zahlentheorie*. Springer, Berlin, Germany, 3 edition, 2015.
[4] M. Schubert. *Mathematik für Informatiker*. Vieweg+Teubner, Wiesbaden, Germany, 2009.
[5] S. Müller-Stach and J. Piontkowski. *Elementare und algebraische Zahlentheorie*. Vieweg+Teubner, Wiesbaden, Germany, 2 edition, 2011.
[6] J. B. Fraleigh. *A First Course in Abstract Algebra*. Pearson, Harlow, United Kingdom, 7 edition, 2014.
[7] A. Schmidt. *Einführung in die algebraische Zahlentheorie*. Springer, Berlin, Germany, 2007.
[8] T. G. Lucas. Integrality properties in rings with zero divisors. In *Ideal Theoretic Methods in Commutative Algebra*, volume 220 of *lecture notes in pure and applied mathematics*. CRC, Boca Raton, FL, 2001.
[9] J. J. Rotman. *A First Course in Abstract Algebra*. Prentice Hall, Upper Saddle River, NJ, 3 edition, 2005.
[10] N. J. Higham. *The Princeton Companion to Applied Mathematics*. Princeton University Press, Princeton, NJ, 2015.
[11] N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. Springer, New York, NY, 2 edition, 1984.
[12] E. H. Batho. Some remarks on non-commutative extensions of local rings. *Nagoya Mathematical Journal*, 14.
[13] G. Kemper. *A Course in Commutative Algebra*. Springer, Heidelberg, Germany, 2011.
[14] T. C. Herwig. The p-adic completion of q and hensel's lemma. Technical report, Department of Mathematics, University of Chicago, 2011.
[15] E. W. Weisstein. "p-adic number." from mathworld–a wolfram web resource.
[16] E. W. Weisstein. "p-adic norm." from mathworld–a wolfram web resource.
[17] A. Gupta. The p-adic integers, analytically and algebraically. Technical report, Department of Mathematics, University of Chicago, 2018.
[18] F. Q. Gouvêa. *p-adic Numbers*. Springer, Cham, Switzerland, 3 edition, 2020.

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