CYCLES IN $kx + c$ FUNCTIONS

First Last, First Last, First Last, First Last, and First Last

Abstract. This paper treats cycles in $kx + c$ functions.

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Fundamentals short and sweet

$C_{k,c}$ cycle	We consider the function $f_{k,c}(x)$ given by equation 1. A cycle $C_{k,c}$ is the sequence $(v_1, v_2, \ldots, v_n)$ of distinct positive integer, where $f_{k,c}(v_1) = v_2$ and $f_{k,c}(v_2) = v_3$ and so forth and finally $f_{k,c}(v_n) = v_1$.
Primitive cycle	If all members of a cycle share a same common divisor greater than one, then this cycle is referred to as a *non-primitve* cycle, otherwise it is a *primitve* cycle.
Non-reduced word	Let $C_{k,c}$ be a cycle having U odd and D even members. The non-reduced word describing this cycle is a word of length $U + D$ over the alphabet $\{u, d\}$, which has a u at those positions, where an odd member and a d where an even member is located in the cycle.
Parity vector	Analoguously to the non-reduced word, the (binary) parity vector of a cycle $C_{k,c} = (v_1, v_2, \ldots, v_n)$ has $n = U + D$ entries. It has a 1 at position i, if v_i is odd, and otherwise 0. We will consider a non-reduced word, for example *uuududuuuddudd* synonymous with the parity vector $(1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0)$ or even simpler with the binary sequence (binary word) 11101011100100.
Interrelated cycles	Two cycles are called *interrelated* if they have the same length and if they both have an equal amount of odd members, which means their non-reduced words contain an equal number of u and v. Analoguously their parity vectors have the same number of zeros and ones.

1. Introduction

Starting point of our considerations is the function:

$$f_{k,c}(x) = \begin{cases} {}^{kx+c}/_{2} & 2 \nmid x \\ {}^{x}/_{2} & \text{otherwise} \end{cases} \tag{1}$$

Let S be a set containing two elements u and d, which are bijective functions over $\mathbb{Q}$:

$$u(x) = {}^{k \cdot x + c}/_{2} \qquad d(x) = {}^{x}/_{2} \tag{2}$$

Let a binary operation be the left-to-right composition of functions $u \circ d$, where $u \circ d(x) = d(u(x))$. S^* is the monoid, which is freely generated by S. The identity element is the identity function $id_{\mathbb{Q}} = e$. We call e an *empty string*. S^* consists of all expressions (strings) that can be concatenated from the generators u and d. Every string can be written in precisely one way as product of factors u and d and natural exponents $e_i > 0$:

$$e, u^{e_1}, d^{e_1}, u^{e_1}d^{e_2}, d^{e_1}u^{e_2}, u^{e_1}d^{e_2}u^{e_3}, d^{e_1}u^{e_2}d^{e_3}, \ldots$$

These uniquely written products are called *reduced words* over S. Using exponents $u_i, d_i > 0$, we construct strings $s_i = u^{u_i}d^{d_i}$ and concatenate these to a larger string:

$$s_1 s_2 \cdots s_n = u^{u_1}d^{d_1}u^{u_2}d^{d_2} \cdots u^{u_n}d^{d_n}$$

Note that each string s_i is a reduced word, since $u_i, d_i > 0$. Let us evaluate this (large) string by inputting a natural number v_1. If the result is again v_1 then we obtain a cycle:

$$u^{u_1}d^{d_1}u^{u_2}d^{d_2} \cdots u^{u_n}d^{d_n}(v_1) = d^{d_n}(u^{u_n}(\cdots d^{d_2}(u^{u_2}(d^{d_1}(u^{u_1}(v_1)))))) = v_1$$

We write the sums briefly as $U = u_1 + \ldots + u_n$ and $D = d_1 + \ldots + d_n$. The cycle contains $U + D$ elements. We summarize this fact to the following definition 1.1:

Definition 1.1. *A cycle consists of $U + D$ elements, where $U = u_1 + \ldots + u_n$ is the number of its odd members and $D = d_1 + \ldots + d_n$ the number of its even members.*

Moreover we define $A = a_1 + \ldots + a_n$ with

$$a_i = 2^{\sum_{j=1}^{i-1} u_j + d_j} \cdot (k^{u_i} - 2^{u_i}) \cdot k^{\sum_{j=i+1}^{n} u_j}$$

Theorem 1.2 calculates the smallest member of the cycle $C_{k,c}$, which in line with definition 1.1 consists of U odd and D even members [1]:

Theorem 1.2. *The smallest number v_1 belonging to a cycle $C_{k,c}$ having U odd and D even members is:*

$$v_1 = \frac{c \cdot A}{(k-2)(2^{U+D} - k^U)}$$

Example 1.3. *We consider a cycle $C_{3,11}$ that has $U + D = 8 + 6 = 14$ elements and choose $(u_1, u_2, u_3, u_4) = (3, 1, 3, 1)$ and $(d_1, d_2, d_3, d_4) = (1, 1, 2, 2)$. Its smalles element is $v_1 = 13$ and we obtain all elements by evaluating the strings: $v_2 = u(v_1)$, $v_3 = u(v_2)$, $v_4 = u(v_3)$ and $v_5 = d(v_4)$ and so forth. It applies:*

$$uuud \circ ud \circ uuudd \circ udd(v_1) = u^3d \circ ud \circ u^3d^2 \circ ud^2(v_1) = s_1 \circ s_2 \circ s_3 \circ s_4(v_1) = v_1$$

This cycle is $(v_1, v_2, v_3, \ldots, v_{14}) = (13, 25, 43, 70, 35, 58, 29, 49, 79, 124, 62, 31, 52, 26)$. *We calculate* v_1 *directly as follows:*

$$v_1 = \frac{11 \cdot 11609}{(3-2)(2^{8+6} - 3^8)} = \frac{11 \cdot 11609}{9823} = 13$$

In this case $11609 = A = a_1 + a_2 + a_3 + a_4 = 4617 + 1296 + 3648 + 2048$*:*

$$\begin{array}{llll}
a_1 = 2^0 & (3^3 - 2^3) & 3^{1+3+1} & = 4617 \\
a_2 = 2^{3+1} & (3^1 - 2^1) & 3^{3+1} & = 1296 \\
a_3 = 2^{3+1+1+1} & (3^3 - 2^3) & 3^1 & = 3648 \\
a_4 = 2^{3+1+1+1+3+2} & (3^1 - 2^1) & 3^0 & = 2048
\end{array}$$

2. Conditions for cycles

A positive integer k is called a *Crandall number*, if there exists a cycle $C_{k,1}$ and the following very fundamental theorem 2.1 is well known, see [2], [3]:

Theorem 2.1. *Every Wieferich number is a Crandall number. In other words, if* k *is a Wieferich number, then a cycle* $C_{k,1}$ *exists.*

In conformity with definition 1.1, let us consider a cycle $C_{k,c}$ consisting of U odd integers and D even integers. The theorem 2.2 specifies the following cycle restriction:

Theorem 2.2. *A cycle only exists if the inequality* $2^{U+D} - k^U > 0$ *holds.*

The following theorem details the condition for the existence of a cycle [4]:

Theorem 2.3. *A cycle* $C_{k,c}$ *only exists if the integer* c *and the difference* $2^{U+D} - k^U$ *are not coprime:* $\gcd(c, 2^{U+D} - k^U) > 1$.

For a sequence $(v_1, v_2, \ldots, v_n)$ of numbers, let us define a (binary) parity vector consisting of n elements, which has a 1 at position i, if v_i is odd, and otherwise 0. This vector corresponds to the non-reduced word over the alphabet $\{u, d\}$ as introduced in section 1. This word has a u at each position which in the vector is occupied by a 1, and it has a v at a position at which in the vector is a 0.

Let $0 \leq x_1 < x_2 < \ldots < x_U <\leq U - 1$ be all positions (the indexing is zero-based) in the parity vector occupied by 1 or equivalently all positions in the word $s \in S^*$ at which there is a u. We can detail theorem 2.3 as follows by theorem 2.4:

Theorem 2.4. *A cycle* $C_{k,c}$ *only exists if the divisibility* $2^{U+D} - k^U \mid c \cdot z(s)$ *holds, where* $z(s) = \sum_{i=1}^{U} = 3^{U-i} 2^{x_i}$.

Example 2.5. *We refer to* $C_{3,11} = (13, 25, 43, 70, 35, 58, 29, 49, 79, 124, 62, 31, 52, 26)$ *again. The corresponding parity vector is* $(1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0)$ *and the non-reduced word is uuududuuuddudd. The indices are* $(x_1, \ldots, x_8) = (0, 1, 2, 4, 6, 7, 8, 11)$ *and therefore* $z(8) = 3^7 2^0 + 3^6 2^1 + 3^5 2^2 + 3^4 2^4 + 3^3 2^6 + 3^2 2^7 + 3^1 2^8 + 3^0 2^{11} = 11609$.

Correctly it applies that $2^{8+6} - 3^8 \mid 11 \cdot 11609$*, more specifically it is* $9.823 \mid 127.699$ *and* $9.823 \cdot 13 = 127.699$.

Another restriction given by theorem 2.6:

Theorem 2.6. *The number of cycles* $C_{k,c}$ *is alsway less than or equal to the number of cycles* $C_{k,a \cdot c}$ *where* a *is an odd number.*

3. Boundary features of cycles

Halbeisen and Hungerbühler [5] introduced a boundary feature as function $M(n, U)$, where n is the cycle length and U the number of odd members in that cycle. We will take this function as a basis for further considerations.

Let n be the length of a cycle and U the number of odd members in that cycle. Moreover, let $S_{n,U}$ denote the set of all binary words (sequences) of length n containing exactly U ones and otherwise only zeros. This set contains exactly $\binom{n}{U}$ words – exactly the number of ways in which we may select U elements out of n total where the order is irrelevant. In the example given by table 1, the elements of the set $S_{5,3}$ are all listed in the first column.

The left shift function $\lambda_n : S^* \to S^*$ rotates a binary word of length n by one element to the left, for example $\lambda_5(uuudd) = uuddu$ or $\lambda_5(11100) = 11001$, see [5]. The second column of table 1 contains all words that result from the binary word $s \in S^*$ (given by the first column) left shifted up to n times: $\lambda_5^1(s), \ldots, \lambda_5^5(s)$. In generalized terms, this set is denoted as $\sigma(s) = \{\lambda_n^i(s) : 1 \leq i \leq n\}$.

The third column of table 1 contains the corresponding values $z(\lambda_5^1(s)), \ldots, z(\lambda_5^5(s))$ remembering that $z : S^* \to \mathbb{N}$ is the function, which we defined by theorem 2.4. The last column contains the minimum of these values. Finally, the largest of all these minima is $M(5, 3)$ or generally, see [5]:

$$M(n, U) = \max_{s \in S_{n,U}} \{ \min_{t \in \sigma(s)} z(t) \} \tag{3}$$

Additionally to Halbeisens and Hungerbühlers boundary feature $M(n, U)$ we introduce another boundary feature as a function $N(n, U)$. Let $r = \gcd(n, U)$, the function $N(n, U)$ is defined as follows:

$$N(n,U) = 2 \cdot M(n,U) - \sum_{i=0}^{r-1} 2^{i \cdot n/r} 3^{U-1-i \cdot U/r} \tag{4}$$

	word s	**set of left shifted words $\sigma(s)$**	$\{z(t) : t \in \sigma(s)\}$	$\min_{t \in \sigma(s)} z(t)$
1	11100	$11100, 11001, 10011, 00111, 01110$	$19, 31, 49, 76, 38$	19
2	11010	$11010, 10101, 01011, 10110, 01101$	$23, 37, 58, 29, 46$	23
3	11001	$11001, 10011, 00111, 01110, 11100$	$31, 49, 76, 38, 19$	19
4	10110	$10110, 01101, 11010, 10101, 01011$	$29, 46, 23, 37, 58$	23
5	10101	$10101, 01011, 10110, 01101, 11010$	$37, 58, 29, 46, 23$	23
6	10011	$10011, 00111, 01110, 11100, 11001$	$49, 76, 38, 19, 31$	19
7	01110	$01110, 11100, 11001, 10011, 00111$	$38, 19, 31, 49, 76$	19
8	01101	$01101, 11010, 10101, 01011, 10110$	$46, 23, 37, 58, 29$	23
9	01011	$01011, 10110, 01101, 11010, 10101$	$58, 29, 46, 23, 37$	23
10	00111	$00111, 01110, 11100, 11001, 10011$	$76, 38, 19, 31, 49$	19
	The largest of all minimum z values is M(n,U) = M(5,3) =			23

Table 1. Calculation of $M(5,3)$

4. Constructing one cycle from another

text about interrelation of cycles comes here
11609,22325,38399,62510,31255,51794,25897,43757,70547,110732,55366,27683,46436,23218
And the reduced form (divided by 19*47):
13,25,43,70,35,58,29,49,79,124,62,31,52,26

References

[1] A. Gupta. On cycles of generalized collatz sequences. *ArXiv Mathematics e-prints*, 2020.
[2] R. E. Crandall. On the "3x+1" problem. *Mathematics of Computation*, 32(144):1281–1292, 1978.
[3] Z. Franco and C. Pomerance. On a conjecture of crandall concerning the "qx+1" problem. *Mathematics of Computation*, 64(211):1333–1336, 1995.
[4] D. Cox. The 3n+1 problem: A probabilistic approach. *Journal of Integer Sequences*, 15(5):1–11, 2012.

[5] L. Halbeisen and N. Hungerbuehler. Optimal bounds for the length of rational collatz cycles. *Acta Arithmetica*, 78(3):227–239, 1997.

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