


BINARY P-ADIC INTEGERS IN COLLATZ SEQUENCES

 First Last and First Last

ABSTRACT. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. We describe an approach from the perspective of 2-adic (binary) algebra.

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Fundamentals short and sweet

unit	An element a of a ring R is called a "unit" (an invertible element) if there exist an element b such that $ab = 1$ [1, p. 24]. Units are elements with inverses with respect to multiplication in the ring. Let F be a field, then an element a of F is a non-unit iff $a = 0$. The sum of any two non-units in F is again a non-unit in F .
unitary ring	A unitary ring is a ring with a multiplicative identity 1 (which differs from the additive identity $1 \neq 0$) such that $1a = a = a1$ for all elements a of the ring.
Ideal	Let $(R, +, \cdot)$ be a commutative unitary ring. Then the subset $I \subseteq R$ is called an ideal of R if $(I, +)$ is a commutative group and if $xI \subseteq I$ for all $x \in R$, see [2, p. 66-67].
quot. ring	Using an ideal of a ring $I \subseteq R$, we may define an equivalence relation \sim on R by $a \sim b$ iff $a - b$ is in I [3, p. 69]. The equivalence class of a in R is given by $[a] = a + I := \{a + r r \in I\}$ for $r \in R$ and referred to as "residue class of a modulo I ", see [4, p. 120], [3, p. 70]. The set of all these equivalence classes becomes the quotient ring (residue class ring) modulo the ideal I , denoted by R/I .
compl. residue system	Let $I \subseteq R$ be an ideal and $[a]$ the residue classes of a modulo I , which means that $a + I = b + I$ when $a \equiv b \pmod{I}$ or respectively $a - b \in I$ [3, p. 70]. R is the disjoint union of the different residue classes a modulo I . A subset $M \subseteq R$, which contains exactly one element from each of these residue classes, is called a complete residue system of R modulo I , see [3, p. 70].
$[a]_n$	The residue class (also termed congruence class) of the integers for a modulus n is the set $[a]_n = \{a + kn k \in \mathbb{Z}\}$ and sometimes denoted by \bar{a}_n or by $a + n\mathbb{Z}$, see [2, p. 15], [4, p. 120], [5, p. 25].
$\mathbb{Z}/n\mathbb{Z}$	The set of all residue classes $[a]_n$ is called the ring of integers modulo n and denoted by $\mathbb{Z}/n\mathbb{Z} = \{[a]_n a \in \mathbb{Z}\}$ and trivially $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$ and for all $n \neq 0$ we have $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \dots, [n-1]\}$, see [2, p. 15], [5, p. 25].

direct prod.	If R_1, R_2, \dots, R_n are rings, the cartesian product $R_1 \times R_2 \times \dots \times R_n$ forms the set of all ordered n -tuples (r_1, r_2, \dots, r_n) , where $r_i \in R_i$. The addition and multiplication of these n -tuples is defined "coordinatewise" by components. The resulting ring is called a "direct product" of the original rings R_i [2, p. 51], [6, p. 169].
princip. ideal	A "principle ideal" is an ideal in a ring R which is generated by a single element a of R through multiplication by every element of R . There are some rings in which every ideal is a principle ideal, so-called "principle ideal rings" [2, p. 68].
max. ideal	A proper Ideal M of a ring R is called "maximal ideal" of R if there is no other proper ideal N of R properly containing M [6, p. 247], [1, p. 37]. A Note on "proper containment": If R is any set, then R is the improper subset of R . Any other subset $N \neq R$ is a proper subset of R and denoted by $N \subset R$ or $N \subsetneq R$ [6, p. 2].
prime ideal	Let a and b are two elements of R and P a proper ideal such that their product ab is an element of P . P is called a prime ideal if at least one of a and b belongs to P , in other words from $ab \in P$ and $a \notin P$ always follows $b \in P$ [1, p. 9].
max. prime ideal	A proper prime ideal P is said to be a "maximal prime ideal" of the ring R , if there is no other proper prime ideal containing P [1, p. 23].
local ring	A commutative ring R is called a local ring if it has a unique maximal ideal M [7, p. 522].
Noeth. ring	A ring R is called "Noetherian" when in R the maximal condition for ideals is satisfied, in other words if every ideal I of R is finitely generated, that is, if we can find a finite set a_1, a_2, \dots, a_n of elements, such that $I = Ra_1 + Ra_2 + \dots + Ra_n$ [1, p. 19, 101].
semi- local ring	A semi-local ring is a Noetherian ring which has only a finite number of maximal ideals [1, p. 107].

zero seq.	A zero sequence is a sequence, which converges towards 0 [8, p. 154]. Given the context of ideal theory, let R be a ring and I an ideal. In the ring $R^{\mathbb{N}} = \prod_{n \in \mathbb{N}} R$, which is the repeated direct product of R with itself, a sequence $(x_i)_{i \in \mathbb{N}}$ is called a zero sequence if for every $s \in \mathbb{N}$ there exist a $N \in \mathbb{N}$ (depending on s) such that $x_n \in I^s$ for all $n > N$.
Cauchy seq. in \mathbb{Q}, \mathbb{R}	A sequence $(x_i)_{i \in \mathbb{N}}$ in \mathbb{Q} or \mathbb{R} is a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer N such that $ x_n - x_m < \epsilon$ for all $n, m \geq N$, see [8, p. 153], [9, p. 24], [15, p. 10].
Cauchy seq. in a ring	Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of elements in $R^{\mathbb{N}}$, the repeated direct product of a ring with itself, and I an ideal in R . This sequence is a Cauchy sequence if for every $s \in \mathbb{N}$ there exist a $N \in \mathbb{N}$ such that $x_n - x_m \in I^s$ for all $n, m > N$.
Cauchy seq. in a local ring	Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of elements in a local ring R and M is the maximal ideal of R . This sequence is a Cauchy sequence if, given any $s \in \mathbb{N}$, we can always find an integer N such that $x_n - x_m \in M^s$ whenever $n > m > N$, see [1, p. 63, 85]. It is a Cauchy sequence iff $x_n - x_{n-1} \rightarrow 0$ as $n \rightarrow \infty$ [1, p. 85].
compl. of a ring	Let R be a ring, I an ideal, I_{ZS} the ideal of all zero sequences in $R^{\mathbb{N}}$, and S_{CS} the subring of $R^{\mathbb{N}}$ containing all Cauchy sequences. The quotient ring $\hat{R}_I := S_{CS}/I_{ZS}$ is called the completion of R with respect to I . S_{CS}/I_{ZS} is the residue class ring of S_{CS} modulo I .
concor. ext.	Let R, S be local rings. If a sequence of elements of S is a Cauchy sequence in S iff it is a Cauchy sequence in R , then we say that R is a "concordant extension" of S [1, p. 87]. When R, S are semi-local rings $R \subseteq S$, R is said to be a "concordant extension" of S if a sequence (s_n) of elements in S is regular in S iff (s_n) is regular in R [10].
compl. of a local ring	Let S be a local ring. A local ring R will be called a completion of S if R is a concordant extension of S and R is complete and if every element of R is the limit of a sequence of elements of S . Each local ring has a completion [1, p. 92].
compl. local ring	A local ring R is called "complete" if every Cauchy sequence composed of elements of R has a limit in R [1, p. 85], [11, p. 184].

p -adic val. for \mathbb{Z}	Fix a prime number p in \mathbb{Z} . The p -adic valuation of a nonzero integer $n = r \cdot p^{v_p(n)}$ is the highest exponent $v_p(n)$ such that $p^{v_p(n)}$ divides n (we say $p^{v_p(n)}$ divides n "exactly"). Hence p and r are coprime. If n, p are coprime then $v_p(n) = 0$, and by convention $v_p(0) = \infty$, see [12].
p -adic val. for \mathbb{Q}	The p -adic valuation can be extended to the field of rational numbers. Let $x = n \cdot s^{-1}$ be a rational number, then $v_p(x) = v_p(n) - v_p(s)$. Any nonzero rational number x can be uniquely represented as $x = rp^{v_p(x)}s^{-1}$, where $r, s \in \mathbb{Z}$, $s > 0$, and $\gcd(r, s) = \gcd(r, p) = \gcd(s, p) = 1$, see [8, p. 154], [13].
p -adic norm	Let x be any number in \mathbb{Q} , for which we already know that it can be written as $x = rp^{v_p(x)}s^{-1}$, where p is a prime number, $s > 0$ and r are integers not divisible by p . The p -adic norm of x is defined by $ x _p = p^{-v_p(x)}$ for $x \neq 0$, and $ 0 _p = 0$, see [12], [8, p. 154], [14].
p -adic dist.	Let $x, y \in \mathbb{Q}$. The p -adic distance between x and y is defined by $d_p(x, y) = x - y _p$, see [8, p. 155].
\mathbb{Q}_p	The field \mathbb{Q}_p of p -adic numbers is the set of equivalence classes of Cauchy sequences [15, p. 10]. The elements of \mathbb{Q}_p , the so-called p -adic numbers, are equivalence classes of Cauchy sequences $(a_n)_{n \in \mathbb{N}}$ in \mathbb{Q} with respect to the equivalence relation $(a_n) \sim (b_n)$ if $(a_n - b_n)$ is a p -adic zero sequence, see [8, p. 159]. Furthermore \mathbb{Q}_p is the completion of \mathbb{Q} with respect to the p -adic distance d_p [8, p. 159].
\mathbb{Z}_p	The ring \mathbb{Z}_p of p -adic integers is the completion of \mathbb{Z} with respect to the p -adic norm. That is, \mathbb{Z}_p is the set of all equivalence classes of Cauchy sequences (a_n) where (a_n) and (b_n) are equivalent if $\lim_{n \rightarrow \infty} a_n - b_n _p = 0$, see [16]. \mathbb{Z}_p is a local ring whose maximal ideal is the principal ideal $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : x _p < 1\}$, see [17, p. 74].

For $k = 1$, let's say that $\frac{v_1\beta}{2^\alpha} = \frac{v_1+\delta}{2^\alpha} = 1$ it is clear that an overflow is provoked by δ the accumulation of "+1", and it occurs before this δ reaches v_1 since v_1 is already larger than half of its next power of 2 (the one it will overflow to). So it is clear that $\delta < v_1$ and therefore $\beta < 2$

For $k = 3$, however, the multiplication by 3 make it possible that δ grows larger than v_1

Let's say $\frac{3^n v_1 \beta}{2^\alpha} = \frac{3^n v_1 + \delta}{2^\alpha} = 1$

Imagine we are at an intermediate step $v_i = 17$ (10001) and we already have some "+1 accumulation" $\delta = 13$ (1101). $\delta < v_i$, the sum (30 or 11110) still bellow overflow point (power of 2 just above 17 which is 32) and in the case of $k = 1$, δ would grow up to overflow with the guarantee it will stay smaller than v_1 since the overflow point don't move.

With the $k = 3$ case, the overflow point can move higher than the next power of 2 above 17 (due to multiplication by 3). If you multiply by 3, you get $v_{i+1} = 51$ (110011) and $\delta = 39$ (100111), but as you can see, the overflow point is not above the main term anymore (the sum is already larger than that power of 2), and does not prevent δ to grow larger than the v_i 's with accumulated "+1". In which case you can end up with $\beta > 2$ and therefore 3.8 would not be true anymore.

Ok, we prove our formula for alpha inductively.

$$\hat{\alpha}(n) = \lfloor n \cdot \log_2 3 + \log_2 v_1 \rfloor + 1$$

For the base case $n = 1$ we have an always true statement:

$$\hat{\alpha}(1) = \lfloor \log_2 3 + \log_2 v_1 \rfloor + 1$$

Now we need to show that an arbitrary n induces an always a true statement for $n + 1$:

$$\hat{\alpha}(n + 1) = \lfloor (n + 1) \cdot \log_2 3 + \log_2 v_1 \rfloor + 1$$

In order to show that this statement is true, we set

$$\frac{2^{\hat{\alpha}(n+1)}}{2^{\hat{\alpha}(n)}} = \frac{2^{\lfloor (n+1) \cdot \log_2 3 + \log_2 v_1 \rfloor}}{2^{\lfloor n \cdot \log_2 3 + \log_2 v_1 \rfloor}} = \begin{cases} 2 & \text{if } n \text{ even} \\ 4 & \text{otherwise} \end{cases}$$

This finally means that $\hat{\alpha}(n + 1) = 2 * \hat{\alpha}(n)$ for all even n and $\hat{\alpha}(n + 1) = 4 * \hat{\alpha}(n)$ for all odd n .

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