


THE ROLE OF ENGEL EXPANSIONS IN COLLATZ SEQUENCES

 First Last and First Last

ABSTRACT. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. Presently, there are scarcely investigations to treat the problem from the angle of the question "which are the corner cases the Collatz Sequences?". We pursue this question and to this end examine ascending continued fractions – the so called Engel expansions. We demonstrate that Engel expansions form worst case sequences $v_1, v_2, \dots, v_n, v_{n+1}$ maximizing v_{n+1} and maximizing the product $(1 + 1/3v_1)(1 + 1/3v_2) \cdots (1 + 1/3v_n)(1 + 1/3v_{n+1})$.

1. INTRODUCTION

The Collatz conjecture is a well-known number theory problem and is the subject of numerous publications. An overview is provided by Lagarias [1]. Therefore, our description of the topic will be brief. The mathematician Lothar Collatz introduced a function $g : \mathbb{N} \rightarrow \mathbb{N}$ as follows:

$$(1) \quad g(x) = \begin{cases} 3x + 1 & 2 \nmid x \\ x/2 & \text{otherwise} \end{cases}$$

In the following, we only consider compressed Collatz sequences that solely contain the odd members, such as described by Bruckman [2], who used the more convenient function that opts out all even integers:

$$(2) \quad f(x) = (3x + 1) \cdot 2^{-\alpha(x)}, \text{ where } 2^{\alpha(x)} \parallel (3x + 1)$$

Note that $\alpha(x)$ is the largest possible exponent for which $2^{\alpha(x)}$ exactly divides $3x + 1$. Especially for prime powers, one often says p^α *divides* the integer x *exactly*, denoted as $p^\alpha \parallel x$, if p^α is the greatest power of the prime p that divides x .

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Definition 1.1. A (compressed) Collatz sequence $v_1, v_2, \dots, v_n, v_{n+1}$ ends when one of the following two conditions is reached:

1. $v_{n+1} = 1$
2. $v_{n+1} \in \{v_1, v_2, v_3, \dots, v_n\}$

When the first condition applies, the Collatz conjecture is true for a specific sequence. If the second condition is fulfilled, the sequence has led to a cycle.

A Collatz sequence $v_1, v_2, \dots, v_n, v_{n+1}$ allowed at most one division by 2 between two successive members. Dividing only once between two successive members, maximizes v_{n+1} . Such a sequence forms the following ascending continued fraction (cf. also [3, p. 11]):

$$(3) \quad v_{n+1} = \frac{3 \frac{3v_1 + 1}{2} + 1}{2} \dots = \frac{3^n v_1 + \sum_{i=0}^{n-1} 3^i 2^{n-1-i}}{2^n} = \frac{3^n(v_1 + 1) - 2^n}{2^n}$$

Example 1.2. A concrete example for such a sequence is $v_1 = 31$, $v_2 = 47$, $v_3 = 71$, $v_4 = 107$, $v_5 = 161$. And, to follow that example, we can calculate v_5 in a straightforward way:

$$v_5 = v_{n+1} = \frac{3^4(31 + 1) - 2^4}{2^4} = 161$$

Besides, by choosing a starting number $v_1 = 2^{n+1} - 1$, we are able to infinitely generate sequences each forming an ascending continued fraction. As per equation 3 the last member in this sequence is the odd number $v_{n+1} = 3^n \cdot 2 - 1$.

Remark 1.3. Ascending variants of a continued fraction, such as used in equation 3, shall not be confused with continued fractions as treated in [4], [5], [6]. Ascending continued fractions used in our case correspond to the so-called "Engel Expansions" [7].

As illustrated below, we can formulate the ascending continued fractions in a generalized fashion, whereas the analogy to 3 is given by $b_1 = b_2 = b_3 = b_4 = 2$ and $a_1 = 3^0$, $a_2 = 3^1$, $a_3 = 3^2$ and $a_4 = 3^3 + 3^4 v_1$:

$$\frac{a_1 + \frac{a_2 + \frac{a_3 + \frac{a_4}{b_4}}{b_3}}{b_2}}{b_1} \cdots = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \frac{a_4}{b_1 b_2 b_3 b_4} + \cdots$$

The generalized form of equation 3 may be used to compute any of the above-named ascending continued fraction that has $a_i = k^{i-1}$, $b_i = b$ for $i \in \mathbb{N}$ and $a_n = k^{n-1} + k^n v_1$:

$$(4) \quad v_{n+1} = \frac{k^n(kv_1 - bv_1 + 1) - b^n}{b^n(k - b)}$$

Table 1 provides for $k = 1, 3, 5, 7$ each formula that calculates v_{n+1} of an Engel expansion along with some example sequences. We obtained these formulas by inserting $b = 2$ and $k = 1, 3, 5, 7$ into equation 4.

k	equation for v_{n+1}	example sequence	resulting v_{n+1}
1	$v_{n+1} = \frac{v_1 - 1 + 2^n}{2^n}$	513, 257, 129, 65, 33, 17, 9, 5, 3	$v_9 = \frac{513 - 1 + 2^8}{2^8} = 3$
3	$v_{n+1} = \frac{3^n(v_1 + 1) - 2^n}{2^n}$	127, 191, 287, 431, 647, 971, 1457	$v_7 = \frac{3^6(127 + 1) - 2^6}{2^6} = 1457$
5	$v_{n+1} = \frac{5^n(3v_1 + 1) - 2^n}{3 \cdot 2^n}$	85, 213, 533, 1333, 3333, 8333	$v_6 = \frac{5^5(3 \cdot 85 + 1) - 2^5}{3 \cdot 2^5} = 8333$
7	$v_{n+1} = \frac{7^n(5v_1 + 1) - 2^n}{5 \cdot 2^n}$	51, 179, 627, 2195, 7683	$v_5 = \frac{7^4(5 \cdot 51 + 1) - 2^4}{5 \cdot 2^4} = 7683$

TABLE 1. Some exemplary Engel expansions for $b = 2$ and $k = 1, 3, 5, 7$

2. INCLUDE MORE DIVISIONS BY TWO INTO AN ENGEL EXPANSION

For calculating the largest possible v_{n+1} , we considered so far Engel expansions which contain only n division by two within a Collatz sequence of $n + 1$ members. In the following we include m additional divisions by two and thus a total of $m + n$ divisions.

Our starting point again is a Collatz sequence v_1, v_2, \dots, v_{n+1} consisting of $n + 1$ members. We have a total of $m + n$ divisions by two that are distributed over n positions:

$$\alpha = n + m = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

The number of positive integer solutions of the diophantine equation above is given by the following binomial coefficient, which is commonly used to calculate combinations with repetition, see [8, p. 54]:

$$\binom{n + (\alpha - n) - 1}{\alpha - n} = \binom{n + m - 1}{m}$$

Each of these solutions describes a possible way to distribute the $m + n$ divisions by two across n positions within a Collatz sequence containing $n + 1$ members. But how do we distribute these divisions in such a way that v_{n+1} becomes maximum? This is precisely the case if as many divisions as possible are performed right at the beginning, which is illustrated in appendix 7.1. The Engel expansion of this case (that maximizes the value of v_{n+1}) provides the following formula for calculating the last sequence member v_{n+1} :

$$(5) \quad v_{n+1} = \left(\frac{k}{2}\right)^{n-1} \left(\frac{kv_1 + 1}{2^{\alpha_1}} + \frac{1}{k-2}\right) - \frac{1}{k-2} = \left(\frac{k}{2}\right)^{n-1} \left(v_2 + \frac{1}{k-2}\right) - \frac{1}{k-2}$$

Example 2.1. We choose the sequence for $k = 5$ starting at $v_1 = 67$ continuing with $v_2 = 21$, $v_3 = 53$, $v_4 = 133$, $v_5 = 333$, $v_6 = 833$ and $v_7 = 2083$. The last sequence member can be calculated by equation 5 directly as follows:

$$v_7 = \left(\frac{5}{2}\right)^{6-1} \left(21 + \frac{1}{5-2}\right) - \frac{1}{5-2} = 2083$$

Moreover theorem 2.2 specifies the limitation of the effect, which the division permutations have on the last sequence member v_{n+1} . As demonstrated in appendix 7.1, the maximum impact on v_{n+1} of permuting division by two in a given Collatz sequence can be calculated by the difference between the highest and lowest possible value of v_{n+1} .

Theorem 2.2. Let $v_1, v_2, \dots, v_n, v_{n+1}$ be a sequence in which a total of $n + m$ divisions by two took place. No matter how these divisions are permuted, i.e. performed sooner or later, the last member v_{n+1} can differ at most by the following product:

$$\left(\frac{3^{n-1}}{2^{n-1}} - 1\right) \left(1 - \frac{1}{2^m}\right)$$

3. SUM OF RECIPROCATED COLLATZ MEMBERS

A product $\prod(1+a_n)$ with positive terms a_n is convergent if the series $\sum a_n$ converges, see Knopp [9, p. 220]. A similar statement provides Murphy [10], who write the factors in the form $c_n = 1+a_n$ and explains that if $\prod c_n$ is convergent then $c_n \rightarrow 1$ and therefore if $\prod(1+a_n)$ is convergent then $a_n \rightarrow 0$.

We write the sum of reciprocated Collatz members as $1/kv_1 + 1/kv_2 + \dots + 1/kv_n + 1/kv_{n+1}$. In order to formulate this sum independently from the successive members v_2, v_3, \dots , we substitute these as follows:

$$\begin{aligned}
 v_1 &= v_1 \\
 v_2 &= \frac{kv_1 + 1}{2^{\alpha_1}} \\
 v_3 &= \frac{k^2v_1 + k + 2^{\alpha_1}}{2^{\alpha_1+\alpha_2}} \\
 v_4 &= \frac{k^3v_1 + k^2 + k \cdot 2^{\alpha_1} + 2^{\alpha_1+\alpha_2}}{2^{\alpha_1+\alpha_2+\alpha_3}} \\
 &\vdots \\
 v_{n+1} &= \frac{k^n v_1 + \sum_{j=1}^n k^{j-1} 2^{\alpha_1+\dots+\alpha_n-\sum_{l>n-j} \alpha_l}}{2^{\alpha_1+\dots+\alpha_n}}
 \end{aligned}
 \tag{6}$$

The sum of the reciprocal Collatz sequence members can be expressed as a term that only depends from v_1 and from the number of divisions by two $\alpha_1, \alpha_2, \alpha_3, \dots$ between two successive members:

$$\sum_{i=1}^{n+1} \frac{1}{kv_i} = \frac{1}{k} \left(\frac{1}{v_1} + \sum_{i=1}^n \frac{1}{v_{i+1}} \right) = \frac{1}{k} \left(\frac{1}{v_1} + \sum_{i=1}^n \frac{2^{\alpha_1+\dots+\alpha_i}}{k^i v_1 + \sum_{j=1}^i k^{j-1} 2^{\alpha_1+\dots+\alpha_n-\sum_{l>i-j} \alpha_l}} \right)$$

4. THE PRODUCT OF RECIPROCATED COLLATZ MEMBERS INCREMENTED BY ONE

In a similar way to deduce the sum of reciprocal vertices depending only on v_1 as performed in 3, we evolve the formula for the product of reciprocated Collatz members (incremented by one):

$$(8) \quad \prod_{i=1}^{n+1} \left(1 + \frac{1}{kv_i}\right) = 1 + \frac{2^{\alpha_1+\dots+\alpha_n} + k \cdot 2^{\alpha_1+\dots+\alpha_{n-1}} + \dots + k^{n-1} \cdot 2^{\alpha_1} + k^n}{k^{n+1}v_1}$$

$$(9) \quad = 1 + \frac{2^{\alpha_1+\dots+\alpha_n} + k \cdot \sum_{j=1}^i k^{j-1} 2^{\alpha_1+\dots+\alpha_n - \sum_{l>i-j} \alpha_l}}{k^{n+1}v_1}$$

$$(10) \quad = \frac{2^{\alpha_1+\dots+\alpha_n} (1 + kv_{n+1})}{k^{n+1}v_1}$$

We inserted the sum used in equation 7 into the above-given equation 8 and then obtained equation 9. Now let us divide this product by the last factor in order to retrieve the product which iterates to n instead of $n+1$:

$$(11) \quad \prod_{i=1}^n \left(1 + \frac{1}{kv_i}\right) = \frac{\prod_{i=1}^{n+1} \left(1 + \frac{1}{kv_i}\right)}{\frac{kv_{n+1}+1}{kv_{n+1}}} = \frac{2^{\alpha_1+\dots+\alpha_n} v_{n+1}}{k^n v_1}$$

The above-shown equation 11 becomes simplified, when we replaced the numerator by equation 10. The question which sequence maximizes its last member v_{n+1} ties into the question: Which sequence maximizes the product? The product formula 11 does not depend from all vertices v_1, v_2, \dots, v_n , it depends only from $2^\alpha = 2^{\alpha_1+\dots+\alpha_n}$, from the first sequence member v_1 and the final one v_{n+1} .

5. MAXIMIZING THE PRODUCT OF RECIPROCATED COLLATZ MEMBERS

Consider a Collatz sequence containing n elements and starting at a given integer v_1 . The corresponding product given by equation 11 becomes largest if we

- maximize the last member v_{n+1} ,
- choose a smallest possible v_1 ,
- choose an α as large as possible and
- keep n to a minimum (which is contrary to maximizing α)

It is therefore a maximization problem over several variables. In the following we will reflect on how to approach a solution to the problem.

Minimizing v_1 : This is obvious and tricially achievable by iterating v_1 through small integers, starting at $v_1 = 1$.

Maximizing α at lowest n : Maximizing α while keeping n as minimal as possible means that we must choose a preferably short Collatz sequence, but one in which as many divisions by two as possible occurred. The value n depends directly on α .

Maximizing v_{n+1} : If we fix v_1, α, n then this maximum occurs when the sequence is an Engel expansion, id est when we run the most divisions by two at the beginning. Consequently, the exponent alpha (the total number of divisions by two) is the sum of a large α_1 and the remaining alpha values which are all one:

$$\alpha = n + m = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha_1 + 1 + \cdots + 1 = \alpha_1 + n - 1$$

The product of reciprocated Collatz members (incremented by one) for such an Engel expansion is given by the following equation:

$$(12) \quad \prod_{i=1}^n \left(1 + \frac{1}{kv_i}\right) = 1 + \frac{1}{kv_1} + \frac{2^{\alpha_1}}{k(k-2)v_1} \left(1 - \left(\frac{2}{k}\right)^{n-1}\right)$$

$$(13) \quad = 1 + \frac{1}{kv_1} + \frac{kv_1 + 1}{k(k-2)v_1v_2} \left(1 - \left(\frac{2}{k}\right)^{n-1}\right)$$

Example 5.1. An example for $k = 3$ provides the sequence $v_1 = 661$, $v_2 = 31$, $v_3 = 47$, and $v_4 = 71$. In this case $\alpha_1 = 6 = m + 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 1$. We now calculate the product of reciprocated Collatz sequence members by inserting $v_1 = 661$ and $v_2 = 31$ together with $k = 3$ and $n = 4$ into equation 13:

$$\begin{aligned} \prod_{i=1}^4 \left(1 + \frac{1}{3v_i}\right) &= \left(1 + \frac{1}{3 \cdot 661}\right) \left(1 + \frac{1}{3 \cdot 31}\right) \left(1 + \frac{1}{3 \cdot 47}\right) \left(1 + \frac{1}{3 \cdot 71}\right) \\ &= 1 + \frac{1}{3 \cdot 661} + \frac{3 \cdot 661 + 1}{3 \cdot (3-2) \cdot 661 \cdot 31} \left(1 - \left(\frac{2}{3}\right)^{4-1}\right) \\ &= 1.0232158532713247 \end{aligned}$$

Maximizing the product of reciprocated Collatz sequence members $\prod_{i=1}^n (1 + 1/kv_i)$ requires us to maximize the equation 12 or 13. This maximization is achieved by choosing v_1 as small as possible (which we already know). Additionally we need to select a largest possible α_1 or a lowest possible v_2 (which is clear as well). Lastly, this maximization also requires the choice of a value n as large as possible. Why n large as possible? As mentioned before, n is directly dependent on α , and specifically $\alpha = \alpha_1 + n - 1$. In other words, maximizing α is equivalent to maximizing n (turning one of both setting screws is enough). To find the largest possible product, we iterate the variables v_1 and n independently through integers $1, 2, \dots$ and calculate the product for each combination of the variable assignment using equation 13.

k	product formula	maximum case	resulting product
1	$1 + \frac{1}{v_1} + \frac{v_1+1}{-v_1v_2} (1 - 2^{n-1})$	$v_1 = 1, n = 1$	2
3	$1 + \frac{1}{3v_1} + \frac{3v_1+1}{3v_1v_2} \left(1 - \left(\frac{2}{3}\right)^{n-1}\right)$	$v_1 = 1, n = 1$ $v_1 = 3, v_2 = 5, n = \infty$	$\frac{4}{3}$
5	$1 + \frac{1}{5v_1} + \frac{5v_1+1}{15v_1v_2} \left(1 - \left(\frac{2}{5}\right)^{n-1}\right)$	$v_1 = 1, v_2 = 3, n = 2$	$\frac{32}{25}$
7	$1 + \frac{1}{7v_1} + \frac{7v_1+1}{35v_1v_2} \left(1 - \left(\frac{2}{7}\right)^{n-1}\right)$	$v_1 = 1, n = 1$	$\frac{8}{7}$

TABLE 2. Formulas that calculate the Engel expansion's product for $k = 1, 3, 5, 7$

Table 2 provides for $k = 1, 3, 5, 7$ the formula for calculating the Engel expansion's product. This table furthermore indicates the maximum cases along with the corresponding results for each product. Recall that we are not allowed to choose an arbitrarily large n if we set $v_1 = 1$ because of the halting conditions given by definition 1.1. The case $v_1 = 3, v_2 = 5, n = \infty$ for $k = 3$ disregards the halting condition, the value n runs towards infinity and thus much further than allowed – nevertheless, the product value converges towards $4/3$. Note that for $n = 1$ it does not matter what value v_2 takes – it has no effect on the resulting product value.

Now one may say that there exist some Collatz sequences, which are not an Engel expansions but their product might be larger than the one resulting from the corresponding Engel expansion, for example:

$$v$$

6. CONDITION FOR A LIMITED GROWTH OF THE ENGEL EXPANSION

Let us look now into the question of what condition must be met to prevent a greater growth than a decline in Collatz sequences. Specifically we consider an Engel expansion comprising $n + 1$ sequence members that include m additional divisions by two at the beginning. The last member v_{n+1} in such a sequence can be calculated by formula ???. In order to restrict the growth of this sequence, we require that the last member has to be smaller than the first one. For this we define the condition $v_{n+1} < v_1$:

$$\frac{3^n v_1 + 3^{n-1} + 3^{n-1} 2^{m+1}}{2^{m+n}} - 1 < v_1$$

Reshaping this inequality leads to the following condition:

$$(14) \quad \frac{3^{n-1}(2^{m+1} - 2)}{2^{m+n} - 3^n} - 1 < v_1$$

7. APPENDIX

7.1. Permuting divisions by two. In order to illustrate how permuting divisions by two affect the last sequence member v_{n+1} , we take a look at two corner cases:

- the one where we do the additional m divisions by 2 at the end and
- the one where we do these additional divisions at the very beginning.

The first case is our starting point to examine how the swapping a division by two affects the sequence member v_{n+1} . For this, let us compare the Engel expansion where we divide by 2^m afterwards with one where we divide by 2 in the penultimate step and by 2^{m-1} in last step. One can immediately recognize the following inequality with a mere look:

$$\frac{1 + \frac{3 + \frac{3^2 + \frac{3^3 + 3^4 v_1}{2}}{2}}{2}}{2 \cdot 2^m} < \frac{1 + \frac{3 + \frac{3^2 + \frac{3^3 + 3^4 v_1}{2}}{2}}{2 \cdot \mathbf{2}}}{2 \cdot 2^{m-1}}$$

To put it simply, in the expansion on the right side of the above-shown inequality we perform one division by two a little bit earlier as we do it in the expansion on the left side of the expansion. Almost all summands of both expansions cancel out each other:

$$\frac{1}{2 \cdot 2^m} + \frac{3}{\cancel{2^2 \cdot 2^m}} + \frac{3^2}{2^3 \cdot 2^m} + \frac{3^3 + 3^4 v_1}{2^4 \cdot 2^m} < \frac{1}{2 \cdot 2^{m-1}} + \frac{3}{\cancel{2^2 \cdot \mathbf{2} \cdot 2^{m-1}}} + \frac{3^2}{2^3 \cdot \mathbf{2} \cdot 2^{m-1}} + \frac{3^3 + 3^4 v_1}{2^4 \cdot \mathbf{2} \cdot 2^{m-1}}$$

The second case deals with Engel expansions where we perform that additional m divisions by two as early as possible. The resulting value v_{n+1} decreases, when we make a division by two later:

$$\frac{1 + \frac{3 + \frac{3^2 + \frac{3^3 + 3^4 v_1}{2 \cdot 2^{m-1}}}{2 \cdot \mathbf{2}}}{2}}{2} < \frac{1 + \frac{3 + \frac{3^2 + \frac{3^3 + 3^4 v_1}{2 \cdot 2^m}}{2}}{2}}{2}$$

Also here almost all summands of both Engel expansions, they cancel each other out:

$$\frac{1}{\cancel{2}} + \frac{\cancel{3}}{2^2} + \frac{3^2}{2^3 \cdot \mathbf{2}} + \frac{3^3 + 3^4 v_1}{\cancel{2^4 \cdot \mathbf{2} \cdot 2^{m-1}}} < \frac{1}{\cancel{2}} + \frac{\cancel{3}}{2^2} + \frac{3^2}{2^3} + \frac{3^3 + 3^4 v_1}{\cancel{2^4 \cdot 2^m}}$$

While the first case minimizes the value of v_{n+1} , the second case maximizes it. The difference between the maximum and the minimum is given by the following equation:

$$\frac{3^{n-1} \left(\frac{3v_1+1}{2 \cdot 2^m} + 1 \right) - 2^{n-1}}{2^{n-1}} - \frac{3^n (v_1 + 1) - 2^n}{2^{n+m}} = \left(\frac{3^{n-1}}{2^{n-1}} - 1 \right) \left(1 - \frac{1}{2^m} \right)$$

This has the consequence that for a given sequence consisting of $n + 1$ members, between which a total of $n + m$ divisions have taken place, the permutations of these divisions have a limited effect on the last sequence member v_{n+1} as described by theorem 2.2.

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