THE ROLE OF ENGEL EXPANSIONS IN COLLATZ SEQUENCES

First Last and First Last

Abstract. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. Presently, there are scarcely investigations to treat the problem from the angle of the question "which are the corner cases the Collatz Sequences?". We pursue this question and to this end examine ascending continued fractions – the so called Engel expansions. We demonstrate that Engel expansions form worst case sequences $v_1, v_2, \ldots, v_n, v_{n+1}$ maximizing v_{n+1} and maximizing the product $(1 + {}^1\!/_{3v_1})(1 + {}^1\!/_{3v_2}) \cdots (1 + {}^1\!/_{3v_n})(1 + {}^1\!/_{3v_{n+1}})$

1. Introduction

The Collatz conjecture is a well-known number theory problem and is the subject of numerous publications. An overview is provided by Lagarias [1]. Therefore, our description of the topic will be brief. The mathematician Lothar Collatz introduced a function $g : \mathbb{N} \to \mathbb{N}$ as follows:

$$g(x) = \begin{cases} 3x+1 & 2 \nmid x \\ x/2 & \text{otherwise} \end{cases} \tag{1}$$

In the following, we only consider compressed Collatz sequences that solely contain the odd members, such as described for example by Bruckman [2], who used the more convenient function that opts out all even integers:

$$f(x) = (3x+1) \cdot 2^{-\alpha(x)}, \text{where} \quad 2^{\alpha(x)} \parallel (3x+1) \tag{2}$$

Note that $\alpha(x)$ is the largest possible exponent for which $2^{\alpha(x)}$ exactly divides $3x+1$. Especially for prime powers, one often says p^α *divides* the integer x *exactly*, denoted as $p^\alpha \parallel x$, if p^α is the greatest power of the prime p that divides x.

A (compressed) Collatz sequence $v_1, v_2, \ldots, v_n, v_{n+1}$ allowed at most one division by 2 between two successive members. Dividing only once between two successive members, maximizes v_{n+1}. Such a sequence forms the following ascending continued fraction (cf. also [3, p. 11]):

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$$v_{n+1} = \frac{3\frac{3\frac{3\frac{3v_1+1}{2}+1}{2}+1}{2}+1}{2} \cdots = \frac{3^n v_1 + \sum_{i=0}^{n-1} 3^i 2^{n-1-i}}{2^n} = \frac{3^n(v_1+1)-2^n}{2^n} \tag{3}$$

The sum of the products of the powers of three and two, contained within the above term, can be simplified to the difference $3^n - 2^n$ by converting the sum expression into the form $(x-1)(1+x+x^2+\cdots+x^{n-2}+x^{n-1}) = x^n - 1$ as follows:

$$\frac{2^n}{2^n}(3-2)\sum_{i=0}^{n-1} 3^i 2^{n-1-i} = \frac{2^n}{\cancel{2^{n-1}}} \cdot \frac{3-2}{2} \sum_{i=0}^{n-1} 3^i 2^{\cancel{n-1}-i} = 2^n\left(\frac{3}{2}-1\right)\sum_{i=0}^{n-1}\left(\frac{3}{2}\right)^i = 2^n\left(\left(\frac{3}{2}\right)^n - 1\right)$$

Example 1.1. *A concrete example for such a sequence is $v_1 = 31$, $v_2 = 47$, $v_3 = 71$, $v_4 = 107$, $v_5 = 161$. And, to follow that example, we can calculate v_5 in a straightforward way:*

$$v_5 = v_{n+1} = \frac{3^4(31+1)-2^4}{2^4} = 161$$

Besides, by choosing a starting number $v_1 = 2^{n+1} - 1$, we are able to infinitely generate sequences each forming an ascending continued fraction. As per equation 3 the last member in this sequence is the odd number $v_{n+1} = 3^n \cdot 2 - 1$.

Remark 1.2. Ascending variants of a continued fraction, such as used in equation 3, shall not be confused with continued fractions as treated for example in [4], [5], [6]. These ascending continued fractions correspond to the so-called "Engel Expansions" [7].

As illustrated below, we can formulate the ascending continued fractions in a generalized fashion, whereas the analogy to 3 is given by $b_1 = b_2 = b_3 = b_4 = 2$ and $a_1 = 3^0$, $a_2 = 3^1$, $a_3 = 3^2$ and $a_4 = 3^3 + 3^4 v_1$:

$$\frac{a_1 + \frac{a_2 + \frac{a_3 + \frac{a_4}{b_4}}{b_3}}{b_2}}{b_1} \cdots = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \frac{a_4}{b_1 b_2 b_3 b_4} + \cdots$$

The generalized form of equation 3 may be used to compute any of the above-named ascending continued fraction that has $a_i = k^{i-1}$, $b_i = b$ for $i \in \mathbb{N}$ and $a_n = k^{n-1} + k^n v_1$:

$$v_{n+1} = \frac{k^n(kv_1 - bv_1 + 1) - b^n}{b^n(k-b)} \tag{4}$$

2. Include more divisions by two into an Engel expansion

For calculating the largest possible v_{n+1}, we considered so far Engel expansions which contain only n division by two within a Collatz sequence of $n+1$ memebers. In the following we include m additional divisions by two and thus a total of $m+n$ divisions. We look at two corner cases:

- the one where we do the additional m divisions by 2 at the end and
- the one where we do these additional divisions at the very beginning.

The first case is our starting point to examine how the swapping a division by two affects the node v_{n+1}. For this, let us compare the Engel expansion where we devide by 2^m afterwards with one where we divide by 2 in the penultimate step and by 2^{m-1} in last step. One can immediately recognize the following inequality with a mere look:

$$\frac{1+\frac{3+\frac{3^2+\frac{3^3+3^4v_1}{2}}{2}}{2}}{2\cdot 2^m} < \frac{1+\frac{3+\frac{3^2+\frac{3^3+3^4v_1}{2}}{2}}{2\cdot \mathbf{2}}}{2\cdot 2^{m-1}}$$

To put it simply, in the expansion on the right side of the above-shown inequality we perform one division by two a little bit earlier as we do it in the expansion on the left side of the expansion. Almost all summands of both expansions cancel out each other:

$$\frac{1}{2\cdot 2^m} + \cancel{\frac{3}{2^2\cdot 2^m} + \frac{3^2}{2^3\cdot 2^m} + \frac{3^3+3^4v_1}{2^4\cdot 2^m}} < \frac{1}{2\cdot 2^{m-1}} + \cancel{\frac{3}{2^2\cdot \mathbf{2}\cdot 2^{m-1}} + \frac{3^2}{2^3\cdot \mathbf{2}\cdot 2^{m-1}} \frac{3^3+3^4v_1}{2^4\cdot \mathbf{2}\cdot 2^{m-1}}}$$

The second case deals with Engel expansions where we perform tha additional m division by two as early as possible. The resulting value decreases, when we make a division by two later:

$$\frac{1+\frac{3+\frac{3^2+\frac{3^3+3^4v_1}{2\cdot 2^{m-1}}}{2\cdot \mathbf{2}}}{2}}{2} < \frac{1+\frac{3+\frac{3^2+\frac{3^3+3^4v_1}{2\cdot 2^m}}{2}}{2}}{2}$$

Also here almost all summands of both Engel expansions, they cancel each other out:

$$\frac{1}{\cancel{2}}\cancel{+}\frac{\cancel{3}}{2^2}+\frac{3^2}{2^3\cdot \mathbf{2}}+\frac{3^3+\cancel{3^4v_1}}{\cancel{2^4\cdot\mathbf{2}}\cdot 2^{m-1}} < \frac{1}{\cancel{2}}\cancel{+}\frac{\cancel{3}}{2^2}+\frac{3^2}{2^3}+\frac{3^3+\cancel{3^4v_1}}{\cancel{2^4}\cdot 2^m}$$

While the first case minimizes the value of the node v_{n+1}, the second case maximizes it. The difference between the maximum and the minimum is given by the following equation:

$$\begin{aligned}
&\frac{3^{n-1}\left(\frac{3v_1+1}{2\cdot 2^m}+1\right)-2^{n-1}}{2^{n-1}}-\frac{3^n\left(v_1+1\right)-2^n}{2^{n+m}}\\
=&\frac{3^{n-1}\cdot\left(3v_1+1+2^{m+1}\right)-2^{n-1}\cdot 2^{m+1}-3^n\left(v_1+1\right)+2^n}{2^{m+1}\cdot 2^{n-1}}\\
=&\frac{3^{n-1}+3^{n-1}\cdot 2^{m+1}-2^{n+m}-3^n+2^n}{2^{n+m}}=\frac{3^{n-1}-3\cdot 3^{n-1}+3^{n-1}\cdot 2^{m+1}-2^{n+m}+2^n}{2^{n+m}}\\
=&\frac{-2\cdot 3^{n-1}+3^{n-1}\cdot 2^{m+1}-2^{n+m}+2^n}{2^{n+m}}=\frac{\left(2\cdot 3^{n-1}-2^n\right)\left(2^m-1\right)}{2^n\cdot 2^m}\\
=&\left(\frac{3^{n-1}}{2^{n-1}}-1\right)\left(1-\frac{1}{2^m}\right)
\end{aligned}$$

This has the consequence that for a given sequence consisting of $n+1$ members, between which a total of $n+m$ divisions have taken place, the permutation of these divisions has a very limited effect on the node v_{n+1} as described by theorem 2.1.

Theorem 2.1. *Let $v_{n+1}, v_n, \ldots, v_2, v_1$ be a sequence in which a total of $n+m$ divisions took place (a path in which a total of $n+m$ edges has been contracted). No matter how these divisions are permuted, i.e. performed sooner or later, the node v_{n+1} can differ at most by the following product:*

$$\left(\frac{3^{n-1}}{2^{n-1}}-1\right)\left(1-\frac{1}{2^m}\right)$$

3. The product in the condition for alpha's upper limit

Let us take a closer look at the product contained in condition **??** for the case $k = 3$ and use the ascending continued fractions for examining this product. The exciting question is, does this product have a limit value even in the case where we only contract a single edge between successive nodes? Setting accordingly the sequence, which maximizes v_{n+1}, into the product expressed by condition **??**, we obtain a product that is limited, or to be more specific, which in the worst case $v_1 = 1$ converges (for n to infinity) towards 2:

$$\prod_{i=1}^{n}\left(1+\frac{1}{3v_i}\right)=\prod_{i=1}^{n}\frac{3^i(v_1+1)-2^i}{3^i(v_1+1)-3*2^{i-1}}=\frac{1}{v_1}-\frac{1}{v_1}\left(\frac{2}{3}\right)^n+1 \tag{5}$$

The above-illustrated last forming step, simplifies this product significantly into an expression waiving a product formulation. A detailed breakdown including all intermediate steps of this simplification is shown in the appendix **??**. The correctness of this simplification can be proven inductively too, which we detail in appendix **??**. The most important and the most interesting aspect of this result is, that the above simplified term cannot exceed the value 2, whatever you choose to insert into n or into v_1:

$$\frac{1}{v_1}-\frac{1}{v_1}\left(\frac{2}{3}\right)^{n+1}+1<2$$

Since, as shown above, the product cannot exceed the value 2, the logarithmic product expression in the condition **??** cannot exceed the value one and this condition becomes a consistently true statement:

$$n\log_2 3-\lfloor n\log_2 3\rfloor<2-1$$

Thus, for $k = 3$ the condition **??** for alphas's upper limit is met for all sequences that maximize v_{n+1}.

4. Include additional divisions into the product

How does the product, contained in condition **??**, look like if we include the additional m divisions into the Engel expansion as per section 2? To answer this question, we consider the sequence $v_{n+1}, v_n, v_{n-1}, \ldots, v_2, v_1$ and we set $v_2 = \frac{3v_1+1}{2\cdot 2^m}$. Then reusing the continued fraction given by equation 3, we obtain:

$$v_{n+1}=\frac{3\frac{3\frac{3\frac{3v_1+1}{2\cdot 2^m}+1}{2}+1}{2}+1}{2}\cdots=\frac{3\frac{3\frac{3v_2+1}{2}+1}{2}+1}{2}\cdots=\frac{3^{n-1}(v_2+1)-2^{n-1}}{2^{n-1}} \tag{6}$$

$$=\frac{3^{n-1}(\frac{3v_1+1}{2\cdot 2^m}+1)-2^{n-1}}{2^{n-1}}=\frac{3^n v_1+3^{n-1}+3^{n-1}2^{m+1}}{2^{m+n}}-1$$

The product will be calculated by using equation 5:

$$\prod_{i=1}^{n}\left(1+\frac{1}{3v_i}\right) = \left(1+\frac{1}{3v_1}\right)\cdot\prod_{i=2}^{n}\left(1+\frac{1}{3v_i}\right) = \left(1+\frac{1}{3v_1}\right)\cdot\prod_{i=1}^{n-1}\left(1+\frac{1}{3v_{i+1}}\right) = \left(1+\frac{1}{3v_1}\right)\cdot\left(\frac{1}{v_2}-\frac{1}{v_2}\left(\frac{2}{3}\right)^{n-1}+1\right) \tag{7}$$

Finally substituting $v_2 = \frac{3v_1+1}{2\cdot 2^m}$ into equation 7 leads to the simplified formula of the product:

$$\prod_{i=1}^{n}\left(1+\frac{1}{3v_i}\right) = \left(1+\frac{1}{3v_1}\right)\cdot\frac{1-\left(\frac{2}{3}\right)^{n-1}+v_2}{v_2} = \frac{1+2^{m+1}}{3v_1}-\frac{2^m}{v_1}\left(\frac{2}{3}\right)^{n}+1 \tag{8}$$

Example 4.1. *An example provides the sequence $v_1 = 661$, $v_2 = 31$, $v_3 = 47$, $v_4 = 71$, $v_5 = 107$. When we input $v_1 = 661$ with $m = 5$ and $n = 4$ into equation 6 we retrieve the value of v_5:*

$$v_5 = v_{n+1} = \frac{3^4\cdot 661+3^3+3^3\cdot 2^6}{2^9}-1 = 107$$

In this sequence five ($m = 5$) additional divisions by two took place in the first step using v_1:

$$\frac{3\cdot 661-1}{2\cdot 2^5} = v_2 = 31$$

Let us now verify the formula for the product by taking this particular example. To this end we input $v_1 = 661$ together with $m = 5$ and $n = 4$ into equation 8:

$$\left(1+\frac{1}{3\cdot 661}\right)\left(1+\frac{1}{3\cdot 31}\right)\left(1+\frac{1}{3\cdot 47}\right)\left(1+\frac{1}{3\cdot 71}\right) = \frac{1+2^6}{3\cdot 661}-\frac{2^5}{661}\left(\frac{2}{3}\right)^4+1 = 1.023215853$$

5. Condition for a limited growth of the Engel expansion

Let us look now into the question of what condition must be met to prevent a greater growth than a decline in Collatz sequences. Specifically we consider an Engel expansion comprising $n+1$ sequence members that include m additional divisions by two at the beginning. The last member v_{n+1} in such a sequence can be calulated by formula 6. In order to restrict the growth of this sequence, we require that the last member has to be smaller than the first one. For this we define the condition $v_{n+1} < v_1$:

$$\frac{3^n v_1 + 3^{n-1} + 3^{n-1}2^{m+1}}{2^{m+n}} - 1 < v_1$$

By transforming this inequality, which is thoroughly described in the appendix **??** step by step, we obtain the condition:

$$\frac{3^{n-1}\left(2^{m+1}-2\right)}{2^{m+n}-3^n} - 1 < v_1 \tag{9}$$

6. Engel expansions maximize the product

The question which sequence maximizes the target node v_{n+1} ties into the question which sequence maximizes the product in the condition for cycle-alpha's upper limit given by equation **??**. The product formula that do not depend from all vertices $v_1, v_2, \ldots v_n$ has been evolved in appendix **??**. This formla depends only from 2^α, from the starting node v_1 and the target node v_{n+1}:

$$\prod_{i=1}^{n}\left(1+\frac{1}{kv_i}\right) = \frac{2^{\alpha_1+\ldots+\alpha_n}v_{n+1}}{k^n v_1}$$

In order to maximize this product, one needs to maximize the target node v_{n+1}, which exactly the Engel expansion does. Hence, for a given v_1, the Engel expansion is the worst case sequence maximizing the product in the condition for cycle-alpha's upper limit.

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