THE ROLE OF ENGEL EXPANSIONS IN COLLATZ SEQUENCES

First Last and First Last

Abstract. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. Presently, there are scarcely investigations to treat the problem from the angle of the question "which are the corner cases the Collatz Sequences?". We pursue this question and to this end examine ascending continued fractions – the so called Engel expansions. We demonstrate that Engel expansions form worst case sequences $v_1, v_2, \dots, v_n, v_{n+1}$ maximizing v_{n+1} and maximizing the product $(1 + {}^1\!/_{3v_1})(1 + {}^1\!/_{3v_2}) \cdots (1 + {}^1\!/_{3v_n})(1 + {}^1\!/_{3v_{n+1}})$.

1. Introduction

The Collatz conjecture is a well-known number theory problem and is the subject of numerous publications. An overview is provided by Lagarias [1]. Therefore, our description of the topic will be brief. The mathematician Lothar Collatz introduced a function $g : \mathbb{N} \to \mathbb{N}$ as follows:

$$g(x) = \begin{cases} 3x + 1 & 2 \nmid x \\ x/2 & \text{otherwise} \end{cases} \tag{1}$$

In the following, we only consider compressed Collatz sequences that solely contain the odd members, such as described by Bruckman [2], who used the more convenient function that opts out all even integers:

$$f(x) = (3x + 1) \cdot 2^{-\alpha(x)}, \text{where} \quad 2^{\alpha(x)} \parallel (3x + 1) \tag{2}$$

Note that $\alpha(x)$ is the largest possible exponent for which $2^{\alpha(x)}$ exactly divides $3x + 1$. Especially for prime powers, one often says p^α *divides* the integer x *exactly*, denoted as $p^\alpha \parallel x$, if p^α is the greatest power of the prime p that divides x.

A (compressed) Collatz sequence $v_1, v_2, \dots, v_n, v_{n+1}$ allowed at most one division by 2 between two successive members. Dividing only once between two successive members,

2010 *Mathematics Subject Classification.* 37P99.
Key words and phrases. Engel Expansions, Collatz Sequences.

maximizes v_{n+1}. Such a sequence forms the following ascending continued fraction (cf. also [3, p. 11]):

$$v_{n+1} = \frac{3\frac{3\frac{3\frac{3v_1+1}{2}+1}{2}+1}{2}+1}{2} \cdots = \frac{3^n v_1 + \sum_{i=0}^{n-1} 3^i 2^{n-1-i}}{2^n} = \frac{3^n(v_1+1) - 2^n}{2^n} \tag{3}$$

Example 1.1. *A concrete example for such a sequence is $v_1 = 31$, $v_2 = 47$, $v_3 = 71$, $v_4 = 107$, $v_5 = 161$. And, to follow that example, we can calculate v_5 in a straightforward way:*

$$v_5 = v_{n+1} = \frac{3^4(31+1) - 2^4}{2^4} = 161$$

Besides, by choosing a starting number $v_1 = 2^{n+1} - 1$, we are able to infinitely generate sequences each forming an ascending continued fraction. As per equation 3 the last member in this sequence is the odd number $v_{n+1} = 3^n \cdot 2 - 1$.

Remark 1.2. Ascending variants of a continued fraction, such as used in equation 3, shall not be confused with continued fractions as treated in [4], [5], [6]. Ascending continued fractions used in our case correspond to the so-called "Engel Expansions" [7].

As illustrated below, we can formulate the ascending continued fractions in a generalized fashion, whereas the analogy to 3 is given by $b_1 = b_2 = b_3 = b_4 = 2$ and $a_1 = 3^0$, $a_2 = 3^1$, $a_3 = 3^2$ and $a_4 = 3^3 + 3^4 v_1$:

$$\frac{a_1 + \frac{a_2 + \frac{a_3 + \frac{a_4}{b_4}}{b_3}}{b_2}}{b_1} \cdots = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \frac{a_4}{b_1 b_2 b_3 b_4} + \cdots$$

The generalized form of equation 3 may be used to compute any of the above-named ascending continued fraction that has $a_i = k^{i-1}$, $b_i = b$ for $i \in \mathbb{N}$ and $a_n = k^{n-1} + k^n v_1$:

$$v_{n+1} = \frac{k^n(kv_1 - bv_1 + 1) - b^n}{b^n(k-b)} \tag{4}$$

k	**Engel expansion formula**	**example sequence**	**resulting v_{n+1}**
1	$v_{n+1} = \frac{v_1 - 1 + 2^n}{2^n}$	$17, 9, 5, 3$	$\frac{4}{3}$
3	$1 + \frac{1}{5v_1} + \frac{5v_1+1}{15v_1v_2}\left(1 - \left(\frac{2}{5}\right)^{n-1}\right)$	$v_1 = 1, v_2 = 3, n = 2$	$\frac{32}{25}$
5	$1 + \frac{1}{7v_1} + \frac{7v_1+1}{35v_1v_2}\left(1 - \left(\frac{2}{7}\right)^{n-1}\right)$	$v_1 = 1, n = 1$	$\frac{8}{7}$
7	$1 + \frac{1}{7v_1} + \frac{7v_1+1}{35v_1v_2}\left(1 - \left(\frac{2}{7}\right)^{n-1}\right)$	$v_1 = 1, n = 1$	$\frac{8}{7}$

TABLE 1. Exemplary Engel expansions for $b = 2$ and $k = 1, 3, 5, 7$

2. Include more divisions by two into an Engel expansion

For calculating the largest possible v_{n+1}, we considered so far Engel expansions which contain only n division by two within a Collatz sequence of $n+1$ memebers. In the following we include m additional divisions by two and thus a total of $m+n$ divisions.

Let us take a look at two corner cases:

- the one where we do the additional m divisions by 2 at the end and
- the one where we do these additional divisions at the very beginning.

The first case is our starting point to examine how the swapping a division by two affects the node v_{n+1}. For this, let us compare the Engel expansion where we devide by 2^m afterwards with one where we divide by 2 in the penultimate step and by 2^{m-1} in last step. One can immediately recognize the following inequality with a mere look:

$$\frac{1+\dfrac{3+\dfrac{3^2+\dfrac{3^3+3^4v_1}{2}}{2}}{2}}{2\cdot 2^m} < \frac{1+\dfrac{3+\dfrac{3^2+\dfrac{3^3+3^4v_1}{2}}{2}}{2\cdot \mathbf{2}}}{2\cdot 2^{m-1}}$$

To put it simply, in the expansion on the right side of the above-shown inequality we perform one division by two a little bit earlier as we do it in the expansion on the left side of the expansion. Almost all summands of both expansions cancel out each other:

$$\frac{1}{2\cdot 2^m}+\cancel{\frac{3}{2^2\cdot 2^m}+\frac{3^2}{2^3\cdot 2^m}+\frac{3^3+3^4v_1}{2^4\cdot 2^m}} < \frac{1}{2\cdot 2^{m-1}}+\cancel{\frac{3}{2^2\cdot \mathbf{2}\cdot 2^{m-1}}+\frac{3^2}{2^3\cdot \mathbf{2}\cdot 2^{m-1}}\frac{3^3+3^4v_1}{2^4\cdot \mathbf{2}\cdot 2^{m-1}}}$$

The second case deals with Engel expansions where we perform that additional m divisions by two as early as possible. The resulting value decreases, when we make a division by two later:

$$\frac{1+\dfrac{3+\dfrac{3^2+\dfrac{3^3+3^4v_1}{2\cdot 2^{m-1}}}{2\cdot \mathbf{2}}}{2}}{2} < \frac{1+\dfrac{3+\dfrac{3^2+\dfrac{3^3+3^4v_1}{2\cdot 2^m}}{2}}{2}}{2}$$

Also here almost all summands of both Engel expansions, they cancel each other out:

$$\cancel{\frac{1}{2}} + \cancel{\frac{3}{2^2}} + \frac{3^2}{2^3 \cdot \mathbf{2}} + \cancel{\frac{3^3 + 3^4 v_1}{2^4 \cdot \mathbf{2} \cdot 2^{m-1}}} < \cancel{\frac{1}{2}} + \cancel{\frac{3}{2^2}} + \frac{3^2}{2^3} + \cancel{\frac{3^3 + 3^4 v_1}{2^4 \cdot 2^m}}$$

While the first case minimizes the value of v_{n+1}, the second case maximizes it. The difference between the maximum and the minimum is given by the following equation:

$$\frac{3^{n-1}\left(\frac{3v_1+1}{2\cdot 2^m}+1\right)-2^{n-1}}{2^{n-1}} - \frac{3^n\left(v_1+1\right)-2^n}{2^{n+m}} = \left(\frac{3^{n-1}}{2^{n-1}}-1\right)\left(1-\frac{1}{2^m}\right)$$

This has the consequence that for a given sequence consisting of $n+1$ members, between which a total of $n+m$ divisions have taken place, the permutation of these divisions has a limited effect on the node v_{n+1} as described by theorem 2.1.

Theorem 2.1. *Let $v_1, v_2, \ldots, v_n, v_{n+1}$ be a sequence in which a total of $n+m$ divisions by two took place. No matter how these divisions are permuted, i.e. performed sooner or later, the last member v_{n+1} can differ at most by the following product:*

$$\left(\frac{3^{n-1}}{2^{n-1}}-1\right)\left(1-\frac{1}{2^m}\right)$$

The Engel expansion of the second case (that maximizes the value of v_{n+1}) provides the following formula for calculating the sequence member v_{n+1}:

$$v_{n+1} = \left(\frac{k}{2}\right)^{n-1}\left(\frac{kv_1+1}{2^{\alpha_1}}+\frac{1}{k-2}\right)-\frac{1}{k-2} = \left(\frac{k}{2}\right)^{n-1}\left(v_2+\frac{1}{k-2}\right)-\frac{1}{k-2} \tag{5}$$

Let us refer to example 1.1 using the sequence starting at $v_1 = 31$, $v_2 = 47$ and so forth. We calculate v_5 directly as follows:

$$v_5 = \left(\frac{3}{2}\right)^{4-1}\left(47+\frac{1}{3-2}\right)-\frac{1}{3-2} = 161$$

Example 2.2. *We choose the sequence for $k=5$ starting at $v_1 = 67$ continuing with $v_2 = 21$, $v_3 = 53$, $v_4 = 133$, $v_5 = 333$, $v_6 = 833$ and $v_7 = 2083$. The last sequence member can be calculated by equation 5 directly as follows:*

$$v_7 = \left(\frac{5}{2}\right)^{6-1}\left(21+\frac{1}{5-2}\right)-\frac{1}{5-2} = 2083$$

3. Sum of reciprocated Collatz members

A product $\prod(1+a_n)$ with positive terms a_n is convergent if the series $\sum a_n$ converges, see Knopp [8, p. 220]. A similar statement provides Murphy [9], who write the factors in the form $c_n = 1 + a_n$ and explains that if $\prod c_n$ is convergent then $c_n \to 1$ and therefore if $\prod(1 + a_n)$ is convergent then $a_n \to 0$.

We write the sum of reciprocated Collatz members as $1/kv_1 + 1/kv_2 + \ldots + 1/kv_n + 1/kv_{n+1}$. In order to formulate this sum independently from the successive members $v_2, v_3, \ldots$, we substitute these as follows:

$$
\begin{aligned}
v_1 &= v_1 \\
v_2 &= \frac{kv_1 + 1}{2^{\alpha_1}} \\
v_3 &= \frac{k^2 v_1 + k + 2^{\alpha_1}}{2^{\alpha_1+\alpha_2}} \\
v_4 &= \frac{k^3 v_1 + k^2 + k \cdot 2^{\alpha_1} + 2^{\alpha_1+\alpha_2}}{2^{\alpha_1+\alpha_2+\alpha_3}} \qquad (6) \\
&\vdots \\
v_{n+1} &= \frac{k^n v_1 + \sum_{j=1}^{n} k^{j-1} 2^{\alpha_1+\ldots+\alpha_n - \sum_{l>n-j} \alpha_l}}{2^{\alpha_1+\ldots+\alpha_n}} \qquad (7)
\end{aligned}
$$

The sum of the reciprocal Collatz sequence members can be expressed as a term that only depends from v_1 and from the number of dvisions by two $\alpha_1, \alpha_2, \alpha_3, \ldots$ between two successive members:

$$\sum_{i=1}^{n+1} \frac{1}{kv_i} = \frac{1}{k}\left(\frac{1}{v_1} + \sum_{i=1}^{n} \frac{1}{v_{i+1}}\right) = \frac{1}{k}\left(\frac{1}{v_1} + \sum_{i=1}^{n} \frac{2^{\alpha_1+\ldots+\alpha_i}}{k^i v_1 + \sum_{j=1}^{i} k^{j-1} 2^{\alpha_1+\ldots+\alpha_n - \sum_{l>i-j} \alpha_l}}\right)$$

4. The product of reciprocated Collatz members incremented by one

In a similar way to deduce the sum of reciprocal vertices depending only on v_1 as performed in 3, we evolve the formula for the product of reciprocated Collatz members (incremented by one):

$$\prod_{i=1}^{n+1}\left(1+\frac{1}{kv_i}\right) = 1+\frac{2^{\alpha_1+\ldots+\alpha_n}+k\cdot 2^{\alpha_1+\ldots+\alpha_{n-1}}+\ldots+k^{n-1}\cdot 2^{\alpha_1}+k^n}{k^{n+1}v_1} \tag{8}$$

$$= 1+\frac{2^{\alpha_1+\ldots+\alpha_n}+k\cdot\sum_{j=1}^{i}k^{j-1}2^{\alpha_1+\ldots+\alpha_n-\sum_{l>i-j}\alpha_l}}{k^{n+1}v_1} \tag{9}$$

$$= \frac{2^{\alpha_1+\ldots+\alpha_n}\left(1+kv_{n+1}\right)}{k^{n+1}v_1} \tag{10}$$

We inserted the sum used in equation 7 into the above-given equation 8 and then obtained equation 9. Now let us divide this product by the last factor in order to retrieve the product which iterates to n instead of $n+1$:

$$\prod_{i=1}^{n}\left(1+\frac{1}{kv_i}\right) = \frac{\prod_{i=1}^{n+1}\left(1+\frac{1}{kv_i}\right)}{\frac{kv_{n+1}+1}{kv_{n+1}}} = \frac{2^{\alpha_1+\ldots+\alpha_n}v_{n+1}}{k^n v_1} \tag{11}$$

The above-shown equation 11 becomes simplified, when we replaced the numerator by equation 10. The question which sequence maximizes its last member v_{n+1} ties into the question: Which sequence maximizes the product? The product formula 11 does not depend from all vertices $v_1, v_2, \ldots, v_n$, it depends only from $2^\alpha = 2^{\alpha_1+\ldots+\alpha_n}$, from the first sequence member v_1 and the final one v_{n+1}.

Consider a Collatz sequence containing n elements and starting at a given integer v_1. The corresponding product given by equation 11 becomes as large as possible if we maximize the last member v_{n+1}. This maximum occurs when the sequence is an Engel expansion, id est when we run the most divisions by two at the beginning. Consequently, the exponent alpha (the total number of divisions by two) is the sum of a large α_1 and the remaining alpha values which are all one:

$$\alpha = n+m = \alpha_1+\alpha_2+\ldots+\alpha_n = \alpha_1+1+\ldots+1 = \alpha_1+n-1$$

The product of reciprocated Collatz members (incremented by one) for an Engel expansion is given by the following equation:

$$\prod_{i=1}^{n}\left(1+\frac{1}{kv_i}\right) = 1+\frac{1}{kv_1}+\frac{2^{\alpha_1}}{k(k-2)v_1}\left(1-\left(\frac{2}{k}\right)^{n-1}\right) \tag{12}$$

$$= 1+\frac{1}{kv_1}+\frac{kv_1+1}{k(k-2)v_1v_2}\left(1-\left(\frac{2}{k}\right)^{n-1}\right) \tag{13}$$

Example 4.1. *An example for $k = 3$ provides the sequence $v_1 = 661$, $v_2 = 31$, $v_3 = 47$, and $v_4 = 71$. In this case $\alpha_1 = 6 = m + 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 1$. We now calcultae the product of reciprocated Collatz sequence members by inserting $v_1 = 661$ and $v_2 = 31$ together with $k = 3$ and $n = 4$ into equation 13:*

$$\begin{aligned}\prod_{i=1}^{4}\left(1+\frac{1}{3v_i}\right) &= \left(1+\frac{1}{3\cdot 661}\right)\left(1+\frac{1}{3\cdot 31}\right)\left(1+\frac{1}{3\cdot 47}\right)\left(1+\frac{1}{3\cdot 71}\right)\\ &= 1+\frac{1}{3\cdot 661}+\frac{3\cdot 661+1}{3\cdot(3-2)\cdot 661\cdot 31}\left(1-\left(\frac{2}{3}\right)^{4-1}\right)\\ &= 1.0232158532713247\end{aligned}$$

Maximizing the product of reciprocated Collatz sequence members $\prod_{i=1}^{n}(1 + {}^1\!/_{kv_i})$ requires us to maximize the equation 13.

k	**product formula**	**maximum case**	**resulting product**
3	$1+\frac{1}{3v_1}+\frac{3v_1+1}{3v_1v_2}\left(1-\left(\frac{2}{3}\right)^{n-1}\right)$	$v_1 = 1, n = 1$	$\frac{4}{3}$
5	$1+\frac{1}{5v_1}+\frac{5v_1+1}{15v_1v_2}\left(1-\left(\frac{2}{5}\right)^{n-1}\right)$	$v_1 = 1, v_2 = 3, n = 2$	$\frac{32}{25}$
7	$1+\frac{1}{7v_1}+\frac{7v_1+1}{35v_1v_2}\left(1-\left(\frac{2}{7}\right)^{n-1}\right)$	$v_1 = 1, n = 1$	$\frac{8}{7}$

TABLE 2. Formulas that calculate the Engel expansion's product for $k = 3, 5, 7$

5. Condition for a limited growth of the Engel expansion

Let us look now into the question of what condition must be met to prevent a greater growth than a decline in Collatz sequences. Specifically we consider an Engel expansion comprising $n + 1$ sequence members that include m additional divisions by two at the beginning. The last member v_{n+1} in such a sequence can be calulated by formula **??**. In order to restrict the growth of this sequence, we require that the last member has to be smaller than the first one. For this we define the condition $v_{n+1} < v_1$:

$$\frac{3^n v_1 + 3^{n-1} + 3^{n-1}2^{m+1}}{2^{m+n}} - 1 < v_1$$

Reshaping this inequality leads to the following condition:

$$\frac{3^{n-1}\left(2^{m+1}-2\right)}{2^{m+n}-3^n} - 1 < v_1 \tag{14}$$

References

[1] J. C. Lagarias. *The Ultimate Challenge: The 3x+1 Problem.* American Mathematical Society, Providence, RI, 2010.
[2] P. S. Bruckman. Retracted article: A proof of the collatz conjecture. *International Journal of Mathematical Education in Science and Technology*, 39(3):403–407, 2008.
[3] T.M.M. Laarhoven. The 3n + 1 conjecture, 7 2009.
[4] C. G. Moore. *Introduction to Continued Fractions.* National Council of Teachers of Mathematics, 1964.
[5] D. Hensley. *Continued Fractions.* World Scientific Publishing, 2006.
[6] J. Borwe, A. van der Poorten, J. Shallit, and W. Zudlin. *Neverending Fractions: An Introduction to Continued Fractions.* Cambridge University Press, 2014.
[7] C. Kraaikamp and J. Wu. On a new continued fraction expansion with non-decreasing partial quotients. *Monatshefte für Mathematik*, 143:285–298, 2004.
[8] K. Knopp. *Theorie und Anwendung der Unendlichen Reihen.* Springer, 2 edition, 1924.
[9] T. Murphy. 2006 course 4281: Prime numbers, 2006.

First Lastname, Graduate School of Mathematics, XYZ University, City, Adresszusatz, ZIP, Germany
Email address: first.last@university.de

First Lastname, Graduate School of Mathematics, XYZ University, City, Adresszusatz, ZIP, Germany
Email address: first.last@university.de