THE ROLE OF ENGEL EXPANSIONS IN COLLATZ SEQUENCES

ⓘ First Last and First Last

ABSTRACT. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. Presently, there are scarcely investigations to treat the problem from the angle of the question "which are the corner cases the Collatz Sequences?". We pursue this question and to this end examine ascending continued fractions – the so called Engel expansions. We demonstrate that Engel expansions form worst case sequences $v_1, v_2, \ldots, v_n, v_{n+1}$ maximizing v_{n+1} and maximizing the product $(1 + 1/3v_1)(1 + 1/3v_2) \cdots (1 + 1/3v_n)(1 + 1/3v_{n+1})$

1. INTRODUCTION

The Collatz conjecture is a well-known number theory problem and is the subject of numerous publications. An overview is provided by Lagarias [1]. Therefore, our description of the topic will be brief. The mathematician Lothar Collatz introduced a function $g : \mathbb{N} \to \mathbb{N}$ as follows:

$$(1) \qquad g(x) = \begin{cases} 3x + 1 & 2 \nmid x \\ x/2 & \text{otherwise} \end{cases}$$

In the following, we only consider compressed Collatz sequences that solely contain the odd members, such as described by Bruckman [2], who used the more convenient function that opts out all even integers:

$$(2) \qquad f(x) = (3x + 1) \cdot 2^{-\alpha(x)}, \text{where} \quad 2^{\alpha(x)} \parallel (3x + 1)$$

Note that $\alpha(x)$ is the largest possible exponent for which $2^{\alpha(x)}$ exactly divides $3x+1$. Especially for prime powers, one often says p^α *divides* the integer x *exactly*, denoted as $p^\alpha \parallel x$, if p^α is the greatest power of the prime p that divides x.

A (compressed) Collatz sequence $v_1, v_2, \ldots, v_n, v_{n+1}$ allowed at most one division by 2 between two successive members. Dividing only once between two successive members,

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maximizes v_{n+1}. Such a sequence forms the following ascending continued fraction (cf. also [3, p. 11]):

$$(3) \quad v_{n+1} = \cfrac{3\cfrac{3\cfrac{3\cfrac{3v_1+1}{2}+1}{2}+1}{2}+1}{2}\cdots = \frac{3^n v_1 + \sum_{i=0}^{n-1} 3^i 2^{n-1-i}}{2^n} = \frac{3^n(v_1+1)-2^n}{2^n}$$

Example 1.1. *A concrete example for such a sequence is $v_1 = 31$, $v_2 = 47$, $v_3 = 71$, $v_4 = 107$, $v_5 = 161$. And, to follow that example, we can calculate v_5 in a straightforward way:*

$$v_5 = v_{n+1} = \frac{3^4(31+1)-2^4}{2^4} = 161$$

Besides, by choosing a starting number $v_1 = 2^{n+1}-1$, we are able to infinitely generate sequences each forming an ascending continued fraction. As per equation 3 the last member in this sequence is the odd number $v_{n+1} = 3^n \cdot 2 - 1$.

Remark 1.2. Ascending variants of a continued fraction, such as used in equation 3, shall not be confused with continued fractions as treated in [4], [5], [6]. Ascending continued fractions used in our case correspond to the so-called "Engel Expansions" [7].

As illustrated below, we can formulate the ascending continued fractions in a generalized fashion, whereas the analogy to 3 is given by $b_1 = b_2 = b_3 = b_4 = 2$ and $a_1 = 3^0$, $a_2 = 3^1$, $a_3 = 3^2$ and $a_4 = 3^3 + 3^4 v_1$:

$$\cfrac{a_1 + \cfrac{a_2 + \cfrac{a_3 + \cfrac{a_4}{b_4}}{b_3}}{b_2}}{b_1}\cdots = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \frac{a_4}{b_1 b_2 b_3 b_4} + \cdots$$

The generalized form of equation 3 may be used to compute any of the above-named ascending continued fraction that has $a_i = k^{i-1}$, $b_i = b$ for $i \in \mathbb{N}$ and $a_n = k^{n-1} + k^n v_1$:

$$(4) \qquad\qquad v_{n+1} = \frac{k^n(kv_1 - bv_1 + 1) - b^n}{b^n(k-b)}$$

2. INCLUDE MORE DIVISIONS BY TWO INTO AN ENGEL EXPANSION

For calculating the largest possible v_{n+1}, we considered so far Engel expansions which contain only n division by two within a Collatz sequence of $n+1$ memebers. In the following we include m additional divisions by two and thus a total of $m + n$ divisions.

Let us take a look at two corner cases:
- the one where we do the additional m divisions by 2 at the end and
- the one where we do these additional divisions at the very beginning.

The first case is our starting point to examine how the swapping a division by two affects the node v_{n+1}. For this, let us compare the Engel expansion where we devide by 2^m afterwards with one where we divide by 2 in the penultimate step and by 2^{m-1} in last step. One can immediately recognize the following inequality with a mere look:

$$\frac{1 + \dfrac{3 + \dfrac{3^2 + \dfrac{3^3 + 3^4 v_1}{2}}{2}}{2}}{2 \cdot 2^m} < \frac{1 + \dfrac{3 + \dfrac{3^2 + \dfrac{3^3 + 3^4 v_1}{2}}{2 \cdot \mathbf{\color{red}2}}}{2}}{2 \cdot 2^{m-1}}$$

To put it simply, in the expansion on the right side of the above-shown inequality we perform one division by two a little bit earlier as we do it in the expansion on the left side of the expansion. Almost all summands of both expansions cancel out each other:

$$\frac{1}{2 \cdot 2^m} + \frac{3}{2^2 \cdot 2^m} + \frac{3^2}{2^3 \cdot 2^m} + \frac{3^3 + 3^4 v_1}{2^4 \cdot 2^m} < \frac{1}{2 \cdot 2^{m-1}} + \frac{3}{2^2 \cdot \mathbf{\color{red}2} \cdot 2^{m-1}} + \frac{3^2}{2^3 \cdot \mathbf{\color{red}2} \cdot 2^{m-1}} \frac{3^3 + 3^4 v_1}{2^4 \cdot \mathbf{\color{red}2} \cdot 2^{m-1}}$$

The second case deals with Engel expansions where we perform that additional m divisions by two as early as possible. The resulting value decreases, when we make a division by two later:

$$\frac{1 + \dfrac{3 + \dfrac{3^2 + \dfrac{3^3 + 3^4 v_1}{2 \cdot 2^{m-1}}}{2 \cdot \mathbf{\color{red}2}}}{2}}{2} < \frac{1 + \dfrac{3 + \dfrac{3^2 + \dfrac{3^3 + 3^4 v_1}{2 \cdot 2^m}}{2}}{2}}{2}$$

Also here almost all summands of both Engel expansions, they cancel each other out:

$$\frac{1}{2} + \frac{3}{2^2} + \frac{3^2}{2^3 \cdot \mathbf{2}} + \frac{3^3 + 3^4 v_1}{2^4 \cdot \mathbf{2} \cdot 2^{m-1}} < \frac{1}{2} + \frac{3}{2^2} + \frac{3^2}{2^3} + \frac{3^3 + 3^4 v_1}{2^4 \cdot 2^m}$$

While the first case minimizes the value of the node v_{n+1}, the second case maximizes it. The difference between the maximum and the minimum is given by the following equation:

$$\frac{3^{n-1}\left(\frac{3v_1+1}{2\cdot 2^m} + 1\right) - 2^{n-1}}{2^{n-1}} - \frac{3^n(v_1+1) - 2^n}{2^{n+m}} = \left(\frac{3^{n-1}}{2^{n-1}} - 1\right)\left(1 - \frac{1}{2^m}\right)$$

This has the consequence that for a given sequence consisting of $n+1$ members, between which a total of $n+m$ divisions have taken place, the permutation of these divisions has a very limited effect on the node v_{n+1} as described by theorem 2.1.

Theorem 2.1. *Let $v_1, v_2, \ldots, v_n, v_{n+1}$ be a sequence in which a total of $n+m$ divisions by two took place. No matter how these divisions are permuted, i.e. performed sooner or later, the last member v_{n+1} can differ at most by the following product:*

$$\left(\frac{3^{n-1}}{2^{n-1}} - 1\right)\left(1 - \frac{1}{2^m}\right)$$

3. Sum of reciprocated Collatz members

A product $\prod(1+a_n)$ with positive terms a_n is convergent if the series $\sum a_n$ converges, see Knopp [8, p. 220]. A similar statement provides Murphy [9], who write the factors in the form $c_n = 1 + a_n$ and explains that if $\prod c_n$ is convergent then $c_n \to 1$ and therefore if $\prod(1+a_n)$ is convergent then $a_n \to 0$.

We write the sum of reciprocated Collatz members as $1/kv_1 + 1/kv_2 + \ldots + 1/kv_n + 1/kv_{n+1}$. In order to formulate this sum independently from the successive members $v_2, v_3, \ldots$, we substitute these as follows:

$$v_1 = v_1$$

$$v_2 = \frac{kv_1 + 1}{2^{\alpha_1}}$$

$$v_3 = \frac{k^2 v_1 + k + 2^{\alpha_1}}{2^{\alpha_1 + \alpha_2}}$$

(5)
$$v_4 = \frac{k^3 v_1 + k^2 + k \cdot 2^{\alpha_1} + 2^{\alpha_1 + \alpha_2}}{2^{\alpha_1 + \alpha_2 + \alpha_3}}$$

$$\vdots$$

(6)
$$v_{n+1} = \frac{k^n v_1 + \sum_{j=1}^{n} k^{j-1} 2^{\alpha_1 + \ldots + \alpha_n - \sum_{l > n-j} \alpha_l}}{2^{\alpha_1 + \ldots + \alpha_n}}$$

The sum of the reciprocal Collatz sequence members can be expressed as a term that only depends from v_1 and from the number of dvisions by two $\alpha_1, \alpha_2, \alpha_3, \ldots$ between two successive members:

$$\sum_{i=1}^{n+1} \frac{1}{kv_i} = \frac{1}{k} \left(\frac{1}{v_1} + \sum_{i=1}^{n} \frac{1}{v_{i+1}} \right) = \frac{1}{k} \left(\frac{1}{v_1} + \sum_{i=1}^{n} \frac{2^{\alpha_1 + \ldots + \alpha_i}}{k^i v_1 + \sum_{j=1}^{i} k^{j-1} 2^{\alpha_1 + \ldots + \alpha_n - \sum_{l > i-j} \alpha_l}} \right)$$

4. The product of reciprocated Collatz members incremented by one

In a similar way to deduce the sum of reciprocal vertices depending only on v_1 as performed in 3, we evolve the formula for the product of reciprocated Collatz members (incremented by one):

(7)
$$\prod_{i=1}^{n+1} \left(1 + \frac{1}{kv_i} \right) = 1 + \frac{2^{\alpha_1 + \ldots + \alpha_n} + k \cdot 2^{\alpha_1 + \ldots + \alpha_{n-1}} + \ldots + k^{n-1} \cdot 2^{\alpha_1} + k^n}{k^{n+1} v_1}$$

(8)
$$= 1 + \frac{2^{\alpha_1 + \ldots + \alpha_n} + k \cdot \sum_{j=1}^{i} k^{j-1} 2^{\alpha_1 + \ldots + \alpha_n - \sum_{l > i-j} \alpha_l}}{k^{n+1} v_1}$$

(9)
$$= \frac{2^{\alpha_1 + \ldots + \alpha_n} \left(1 + kv_{n+1} \right)}{k^{n+1} v_1}$$

We inserted the sum used in equation 6 into the above-given equation 7 and then obtained equation 8. Now let us divide this product by the last factor in order to retrieve the product which iterates to n instead of $n + 1$:

$$(10) \qquad \prod_{i=1}^{n}\left(1+\frac{1}{kv_i}\right) = \frac{\prod_{i=1}^{n+1}\left(1+\frac{1}{kv_i}\right)}{\frac{kv_{n+1}+1}{kv_{n+1}}} = \frac{2^{\alpha_1+\ldots+\alpha_n}v_{n+1}}{k^n v_1}$$

The above-shown equation 10 becomes simplified, when we replaced the numerator by equation 9. The question which sequence maximizes the target node v_{n+1} ties into the question: Which sequence maximizes the product? The product formula 10 does not depend from all vertices $v_1, v_2, \ldots, v_n$, it depends only from $2^\alpha = 2^{\alpha_1+\ldots+\alpha_n}$, from the starting node v_1 and the target node v_{n+1}. In order to maximize this product, one needs to maximize the target node v_{n+1}, which exactly the Engel expansion does. Hence, for a given v_1, the Engel expansion is the worst case sequence maximizing the product of reciprocated Collatz sequence members $\prod_{i=1}^{n}(1 + 1/kv_i)$.

Let us take a closer look at this product and use the ascending continued fractions (Engel expansions) for examining it. The exciting question is: Does this product have a limit value even in the case where between successive Collatz sequence members v_i and v_{i+1} a division by two has been performed only once? We insert $n+1 = i$ and $b = 2$ into equation 4 to get a formula for v_i, which we insert into the product and obtain:

$$(11) \qquad \prod_{i=1}^{n}\left(1+\frac{1}{kv_i}\right) = \frac{1}{(k-2)v_1} - \frac{1}{(k-2)v_1}\left(\frac{2}{k}\right)^n + 1$$

The special case $k = 3$ produces the following product of reciprocated Collatz members belonging to an Engel expansion (that sequence maximizing v_{n+1}):

$$(12) \qquad \prod_{i=1}^{n}\left(1+\frac{1}{3v_i}\right) = \frac{1}{v_1} - \frac{1}{v_1}\left(\frac{2}{3}\right)^n + 1$$

This product is limited, or to be more specific, it cannot exceed the value $4/3$. The largest possible value of this product can be received by setting $v_1 = n = 1$. In this case the product returns $1 - 2/3 + 1$ and the value of n cannot be larger than one, since with $v_1 = 1$ we already reached the end of the Collatz sequence. Inserting any odd value greater than one into v_1 leads to a smaller product, no matter how large you choose n. For instance, when setting $v_1 = 3$ the product converges (for n to infinity) from below towards $4/3$. The larger you choose v_1, the smaller becomes the product.

5. Include additional divisions into the product

How does this product looks like if we include the additional m divisions into the Engel expansion as per section 2? To answer this question, we consider the sequence $v_1, v_2, \ldots, v_n, v_{n+1}$ and we set $v_2 = {}^{3v_1+1}/_{2 \cdot 2^m}$. Then reusing the continued fraction given by equation 3, we obtain:

(13)
$$
v_{n+1} = \cfrac{3\cfrac{3\cfrac{3\cfrac{3v_1+1}{2 \cdot 2^m}+1}{2}+1}{2}+1}{2} \cdots = \cfrac{3\cfrac{3v_2+1}{2}+1}{2} \cdots = \frac{3^{n-1}(v_2+1)-2^{n-1}}{2^{n-1}}
$$

$$
= \frac{3^{n-1}\left(\frac{3v_1+1}{2 \cdot 2^m}+1\right)-2^{n-1}}{2^{n-1}} = \frac{3^n v_1 + 3^{n-1} + 3^{n-1}2^{m+1}}{2^{m+n}} - 1
$$

The product will be calculated by using equation 12:

(14)
$$
\prod_{i=1}^{n}\left(1+\frac{1}{3v_i}\right) = \left(1+\frac{1}{3v_1}\right) \cdot \prod_{i=2}^{n}\left(1+\frac{1}{3v_i}\right) = \left(1+\frac{1}{3v_1}\right) \cdot \prod_{i=1}^{n-1}\left(1+\frac{1}{3v_{i+1}}\right)
$$

$$
= \left(1+\frac{1}{3v_1}\right) \cdot \left(\frac{1}{v_2}-\frac{1}{v_2}\left(\frac{2}{3}\right)^{n-1}+1\right)
$$

Finally substituting $v_2 = {}^{3v_1+1}/_{2 \cdot 2^m}$ into equation 14 leads to the simplified formula of the product:

(15)
$$
\prod_{i=1}^{n}\left(1+\frac{1}{3v_i}\right) = \left(1+\frac{1}{3v_1}\right) \cdot \frac{1-\left(\frac{2}{3}\right)^{n-1}+v_2}{v_2} = \frac{1+2^{m+1}}{3v_1} - \frac{2^m}{v_1}\left(\frac{2}{3}\right)^n + 1
$$

Note that the total number of divisions by two is $n+m = \alpha = \alpha_1 + \ldots + \alpha_n$ and the number of divisions in the first step is $m+1 = \alpha_1$.

Example 5.1. *An example provides the sequence $v_1 = 661$, $v_2 = 31$, $v_3 = 47$, $v_4 = 71$, $v_5 = 107$. When we input $v_1 = 661$ with $m = 5$ and $n = 4$ into equation 13 we retrieve the value of v_5:*

$$
v_5 = v_{n+1} = \frac{3^4 \cdot 661 + 3^3 + 3^3 \cdot 2^6}{2^9} - 1 = 107
$$

In this sequence five ($m = 5 = \alpha_1 - 1$) additional divisions by two took place in the first step using $v_1 = 661$:

$$\frac{3 \cdot 661 - 1}{2 \cdot 2^5} = v_2 = 31$$

We now verify the formula for the product by taking this particular example. To this end we input $v_1 = 661$ together with $m = 5$ and $n = 4$ into equation 15:

$$\left(1 + \frac{1}{3 \cdot 661}\right)\left(1 + \frac{1}{3 \cdot 31}\right)\left(1 + \frac{1}{3 \cdot 47}\right)\left(1 + \frac{1}{3 \cdot 71}\right) = \frac{1 + 2^6}{3 \cdot 661} - \frac{2^5}{661}\left(\frac{2}{3}\right)^4 + 1$$

$$= 1.023215853$$

6. Condition for a limited growth of the Engel expansion

Let us look now into the question of what condition must be met to prevent a greater growth than a decline in Collatz sequences. Specifically we consider an Engel expansion comprising $n + 1$ sequence members that include m additional divisions by two at the beginning. The last member v_{n+1} in such a sequence can be calulated by formula 13. In order to restrict the growth of this sequence, we require that the last member has to be smaller than the first one. For this we define the condition $v_{n+1} < v_1$:

$$\frac{3^n v_1 + 3^{n-1} + 3^{n-1} 2^{m+1}}{2^{m+n}} - 1 < v_1$$

Reshaping this inequality leads to the following condition:

$$(16) \qquad \frac{3^{n-1}\left(2^{m+1} - 2\right)}{2^{m+n} - 3^n} - 1 < v_1$$

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