CYCLES IN $kx + c$ FUNCTIONS

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Abstract. This paper treats cycles in $kx + c$ functions.

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Fundamentals short and sweet

$C_{k,c}$ cycle	We consider the function $f_{k,c}(x)$ given by equation 1. A cycle $C_{k,c}$ is the sequence $(v_1, v_2, \ldots, v_n)$ of distinct positive integer, where $f_{k,c}(v_1) = v_2$ and $f_{k,c}(v_2) = v_3$ and so forth and finally $f_{k,c}(v_n) = v_1$.
Primitive cycle	If all members of a cycle share a same common divisor greater than one, then this cycle is referred to as a *non-primitve* cycle, otherwise it is a *primitve* cycle.
Non-reduced word	Let $C_{k,c}$ be a cycle having U odd and D even members. The non-reduced word describing this cycle is a word of length $U + D$ over the alphabet $\{u, d\}$, which has a u at those positions, where an odd member and a d where an even member is located in the cycle.
Parity vector	Analoguously to the non-reduced word, the binary parity vector of a cycle $C_{k,c} = (v_1, v_2, \ldots, v_n)$ has $n = U + D$ entries. It has a 1 at position i, if v_i is odd, and otherwise 0. For instance, we consider the non-reduced word *uuududuuuddudd* synonymous to the parity vector $(1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0)$ or even simpler to the binary sequence (binary word) 11101011100100.
Interrelated cycles	Two cycles are called *interrelated* if they have the same length and if they both have an equal amount of odd members, which means their non-reduced words contain an equal number of u and v. Analoguously their parity vectors have the same number of zeros and ones.

1. Introduction

Starting point of our considerations is the function:

$$f_{k,c}(x) = \begin{cases} {}^{kx+c}/_2 & 2 \nmid x \\ {}^{x}/_2 & \text{otherwise} \end{cases} \tag{1}$$

Let S be a set containing two elements u and d, which are bijective functions over $\mathbb{Q}$:

$$u(x) = {}^{k \cdot x + c}/_2 \qquad\qquad d(x) = {}^{x}/_2 \tag{2}$$

Let a binary operation be the left-to-right composition of functions $u \circ d$, where $u \circ d(x) = d(u(x))$. S^* is the monoid, which is freely generated by S. The identity element is the identity function $id_{\mathbb{Q}} = e$. We call e an *empty string*. S^* consists of all expressions (strings) that can be concatenated from the generators u and d. Every string can be written in precisely one way as product of factors u and d and natural exponents $e_i > 0$:

$$e, u^{e_1}, d^{e_1}, u^{e_1}d^{e_2}, d^{e_1}u^{e_2}, u^{e_1}d^{e_2}u^{e_3}, d^{e_1}u^{e_2}d^{e_3}, \ldots$$

These uniquely written products are called *reduced words* over S. Using exponents $u_i, d_i > 0$, we construct strings $s_i = u^{u_i}d^{d_i}$ and concatenate these to a larger string:

$$s_1 s_2 \cdots s_n = u^{u_1}d^{d_1}u^{u_2}d^{d_2} \cdots u^{u_n}d^{d_n}$$

Note that each string s_i is a reduced word, since $u_i, d_i > 0$. Let us evaluate this (large) string by inputting a natural number v_1. If the result is again v_1 then we obtain a cycle:

$$u^{u_1}d^{d_1}u^{u_2}d^{d_2} \cdots u^{u_n}d^{d_n}(v_1) = d^{d_n}(u^{u_n}(\cdots d^{d_2}(u^{u_2}(d^{d_1}(u^{u_1}(v_1)))))) = v_1$$

We write the sums briefly as $U = u_1 + \ldots + u_n$ and $D = d_1 + \ldots + d_n$. The cycle contains $U + D$ elements. We summarize this fact to the following definition 1.1:

Definition 1.1. *A cycle consists of $U + D$ elements, where $U = u_1 + \ldots + u_n$ is the number of its odd members and $D = d_1 + \ldots + d_n$ the number of its even members.*

Moreover we define $A = a_1 + \ldots + a_n$ with

$$a_i = 2^{\sum_{j=1}^{i-1} u_j + d_j} \cdot (k^{u_i} - 2^{u_i}) \cdot k^{\sum_{j=i+1}^{n} u_j}$$

Theorem 1.2 calculates the smallest member of the cycle $C_{k,c}$, which in line with definition 1.1 consists of U odd and D even members [1]:

Theorem 1.2. *The smallest number v_1 belonging to a cycle $C_{k,c}$ having U odd and D even members is:*

$$v_1 = \frac{c \cdot A}{(k-2)(2^{U+D} - k^U)}$$

Example 1.3. *We consider a cycle $C_{3,11}$ that has $U + D = 8 + 6 = 14$ elements and choose $(u_1, u_2, u_3, u_4) = (3, 1, 3, 1)$ and $(d_1, d_2, d_3, d_4) = (1, 1, 2, 2)$. Its smalles element is $v_1 = 13$ and we obtain all elements by evaluating the strings: $v_2 = u(v_1)$, $v_3 = u(v_2)$, $v_4 = u(v_3)$ and $v_5 = d(v_4)$ and so forth. It applies:*

$$uuud \circ ud \circ uuudd \circ udd(v_1) = u^3d \circ ud \circ u^3d^2 \circ ud^2(v_1) = s_1 \circ s_2 \circ s_3 \circ s_4(v_1) = v_1$$

This cycle is $(v_1, v_2, v_3, \ldots, v_{14}) = (13, 25, 43, 70, 35, 58, 29, 49, 79, 124, 62, 31, 52, 26)$. *We calculate* v_1 *directly as follows:*

$$v_1 = \frac{11 \cdot 11609}{(3-2)(2^{8+6} - 3^8)} = \frac{11 \cdot 11609}{9823} = 13$$

In this case $11609 = A = a_1 + a_2 + a_3 + a_4 = 4617 + 1296 + 3648 + 2048$*:*

$$\begin{array}{llll} a_1 = 2^0 & (3^3 - 2^3) & 3^{1+3+1} & = 4617 \\ a_2 = 2^{3+1} & (3^1 - 2^1) & 3^{3+1} & = 1296 \\ a_3 = 2^{3+1+1+1} & (3^3 - 2^3) & 3^1 & = 3648 \\ a_4 = 2^{3+1+1+1+3+2} & (3^1 - 2^1) & 3^0 & = 2048 \end{array}$$

Theorem 1.4. *The maximum odd element in cycle occurs immediately before the maximum even element.*

2. Conditions for cycles

A positive integer k is called a *Crandall number*, if there exists a cycle $C_{k,1}$ and the following very fundamental theorem 2.1 is well known, see [2], [3]:

Theorem 2.1. *Every Wieferich number is a Crandall number. In other words, if* k *is a Wieferich number, then a cycle* $C_{k,1}$ *exists.*

In conformity with definition 1.1, let us consider a cycle $C_{k,c} = (v_1, v_2, \ldots, v_n)$ consisting of U odd integers and D even integers. We define a binary parity vector (it is synonymous to a binary sequence or binary non-reduced word) consisting of $n = U + D$ elements, which has a 1 at position i, if v_i is odd, and otherwise 0. This vector corresponds to the non-reduced word over the alphabet $\{u, d\}$ as introduced in section 1 having a u at each position which in the vector is occupied by a 1, and a d at a position at which in the vector is a 0. The theorem 2.2 specifies several cycle restrictions:

Theorem 2.2. *For a cycle* $C_{k,c}$ *with* U *odd and* D *even members applies:*

(a) A cycle only exists if the inequality $2^{U+D} - k^U > 0$ *holds.*

(b) The condition for the existence of a cycle can be detailed as follows [4]: A cycle $C_{k,c}$ *only exists if the integer* c *and the difference* $2^{U+D} - k^U$ *are not coprime:* $\gcd(c, 2^{U+D} - k^U) > 1$.

(c) Let $0 \leq x_1 < x_2 < \ldots < x_U <\leq U - 1$ *be all positions (the indexing is zero-based) in the parity vector occupied by* 1 *or equivalently all positions in the word* $s \in S^*$ *at which there is a letter* u*. A cycle* $C_{k,c}$ *only exists if the divisibility* $2^{U+D} - k^U \mid c \cdot z(s)$ *holds, where* $z(s) = \sum_{i=1}^{U} = 3^{U-i} 2^{x_i}$.

(d) The number of cycles $C_{k,c}$ *is alsway less than or equal to the number of cycles* $C_{k,a \cdot c}$ *where* a *is an odd number.*

Example 2.3. *We refer to* $C_{3,11} = (13, 25, 43, 70, 35, 58, 29, 49, 79, 124, 62, 31, 52, 26)$ *again. The corresponding parity vector is* $(1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0)$ *and the non-reduced word is uuududuuuddudd. The indices are* $(x_1, \ldots, x_8) = (0, 1, 2, 4, 6, 7, 8, 11)$ *and therefore* $z(8) = 3^7 2^0 + 3^6 2^1 + 3^5 2^2 + 3^4 2^4 + 3^3 2^6 + 3^2 2^7 + 3^1 2^8 + 3^0 2^{11} = 11609$.

Correctly it applies that $2^{8+6} - 3^8 \mid 11 \cdot 11609$*, more specifically it is* $9.823 \mid 127.699$ *and* $9.823 \cdot 13 = 127.699$.

3. Boundary features of cycles

Halbeisen and Hungerbühler [5] introduced a boundary feature as function $M(n, U)$, where n is the cycle length and U the number of odd members in that cycle. We will take this function as a basis for further considerations.

Let n be the length of a cycle and U the number of odd members in that cycle. Moreover, let $S_{n,U}$ denote the set of all binary words (sequences) of length n containing exactly U ones and otherwise only zeros. This set contains exactly $\binom{n}{U}$ words – exactly the number of ways in which we may select U elements out of n total where the order is irrelevant. In the example given by table 1, the elements of the set $S_{5,2}$ are all listed in the first column.

The left shift function $\lambda : S^* \to S^*$ rotates a binary word of length n by one element to the left, for example $\lambda(uuddd) = udddu$ or $\lambda(11000) = 10001$, see [5]. The second column of table 1 contains all words that result from the binary word $s \in S^*$ (given by the first column) left shifted up to n times: $\lambda^1(s), \ldots, \lambda^5(s)$. In generalized terms, this set is denoted as $\sigma(s) = \{\lambda^i(s) : 1 \leq i \leq n\}$.

The third column of table 1 contains the corresponding values $z(\lambda^1(s)), \ldots, z(\lambda^5(s))$ remembering that $z : S^* \to \mathbb{N}$ is the function, which we defined by theorem **??**. The last column contains the minimum of these values. Finally, the largest of all these minima is $M(5, 2)$ or generally, see [5]:

$$M(n, U) = \max_{s \in S_{n,U}} \{ \min_{t \in \sigma(s)} z(t) \} \tag{3}$$

Additionally to Halbeisens and Hungerbühlers boundary feature $M(n, U)$ we introduce another boundary feature as a function $N(n, U)$. Let $r = \gcd(n, U)$, the function $N(n, U)$ is defined as follows:

$$N(n, U) = 2 \cdot M(n, U) - \sum_{i=0}^{r-1} 2^{i \cdot n/r} 3^{U-1-i \cdot U/r} \tag{4}$$

Example 3.1. *We choose a cycle $C_{k,c} = C_{3,23}$ of length $n = 5$ having $U = 2$ odd members, where $c = 2^n - 3^U = 2^5 - 3^2 = 23$. Let us choose the binary words* 11000 *and* 10100 *and calculate the smallest member of the corresponding cycle in each case (using theorem 1.2).*

In the first case, namely 11000 *synonymous with $uuddd = u^2d^3 = u^{u_1}d^{d_1}$ we obtain $v_1 = {}^{c\cdot A}/_{(k-2)(2^{U+D}-k^U)} = {}^{23\cdot 5}/_{(3-2)(2^{2+3}-3^2)} = 5$. The resulting cycle is $(5, 19, 40, 20, 10)$ which is given by the first row and third column in table 1.*

In the second case, 10100 *that is synonymous with $ududd = u^1d^1u^1d^2 = u^{u_1}d^{d_1}u^{u_2}d^{d_2}$ we obtain $v_1 = {}^{23\cdot 7}/_{(3-2)(2^{2+3}-3^2)} = 7$. The resulting cycle is $(7, 22, 11, 28, 14)$ which is given by the second row and third column in table 1.*

Table 1 exhibits how $M(n, U)$ is calculated, which in our concrete case is $M(5, 2) = 7$. Additionally we calculate $N(5, 2) = 2 \cdot M(5, 2) - 2^0 3^{2-1-0} = 14 - 3 = 11$.

	word s	**set $\sigma(s)$ of left shifted words**	**$\{z(t) : t \in \sigma(s)\}$**	**$\min_{t\in\sigma(s)} z(t)$**
1	11000	11000, 10001, 00011, 00110, 01100	5, 19, 40, 20, 10	5
2	10100	10100, 01001, 10010, 00101, 01010	7, 22, 11, 28, 14	7
3	10010	10010, 00101, 01010, 10100, 01001	11, 28, 14, 7, 22	7
4	10001	10001, 00011, 00110, 01100, 11000	19, 40, 20, 10, 5	5
5	01100	01100, 11000, 10001, 00011, 00110	10, 5, 19, 40, 20	5
6	01010	01010, 10100, 01001, 10010, 00101	14, 7, 22, 11, 28	7
7	01001	01001, 10010, 00101, 01010, 10100	22, 11, 28, 14, 7	7
8	00110	00110, 01100, 11000, 10001, 00011	20, 10, 5, 19, 40	5
9	00101	00101, 01010, 10100, 01001, 10010	28, 14, 7, 22, 11	7
10	00011	00011, 00110, 01100, 11000, 10001	40, 20, 10, 5, 19	5
	The largest of all minimum z values is $M(n, U) = M(5, 2) =$			7

TABLE 1. Calculation of $M(5, 2)$

4. CONSTRUCTING ONE CYCLE FROM ANOTHER

Cycles may interrelate, which means they have the same length and an equal amount of odd members. We refer to example 3.1 and consider the cycle $C_{3,23} = (5, 19, 40, 20, 10)$. A cycle, which interrelates to $C_{3,23}$ is for example $C_{3,69} = (15, 57, 120, 60, 30)$.

If we go back to example 1.3, then we can provide two interrelated cycles as well. For $k = 3$ and $n = U + D = 8 + 6 = 14$ we obtain $c = 2^n - k^U = 2^{14} - 3^8 = 9823$ and the cycle

$C_{3,9823} = (11609, 22325, 38399, 62510, 31255, 51794, 25897, 43757, 70547, 110732, 55366, 27683, 46436, 23218)$.

When we divide the parameter c and all cycle members by 893, then we obtain the reduced interrelated cycle $C_{3,11} = (13, 25, 43, 70, 35, 58, 29, 49, 79, 124, 62, 31, 52, 26)$.

Theorem 4.1. *Let $C_{k,c}$ be a cycle of length $n = U + D$ that has U odd and D even members, where $c = 2^n - k^U$. It always applies that $M(n, U)$ is greater than the smallest member and $N(n, U)$ is less than the largest odd member of this cycle.*

If c is divisable by an odd integer a, then for the (reduced) interrelated cycle $C_{k,c/a}$ it applies that ${}^{M(n,U)}/_a$ is greater than the smallest member and ${}^{N(n,U)}/_a$ is less than the largest odd member of this reduced cycle.

Example 4.2. *Starting point for us are again Cycles $C_{3,c}$. For $n = 14$ and $U = 8$ we have $c = 2^{14} - 3^8 = 9823$ and $M(14, 8) = 21109$ and $N(14, 8) = 36575$. The cycle's $C_{3,9823}$ smallest member is 11609 and its largest odd member is 70547. It applies that $11609 \leq M(14, 8)$ and $N(14, 8) \leq 70547$.*

Since $c = 9823 = 893 \cdot 11$ we are able to reduce the cycle $C_{3,9823}$ via division of c and all its cycle members by 893, which leads to the cycle $C_{3,11}$ having the smallest member 13 and largest odd member 79. The inequalities hold: $13 \leq {}^{M(14,8)}/_{893}$ and ${}^{N(14,8)}/_{893} \leq 79$, since $13 \leq 23.638$ and $40.957 \leq 79$.

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