1. Wrapping up the main results

We stated that for a Collatz sequence $v_1, v_2, \ldots, v_{n+1}$ and the corresponding product $\beta = \beta_1\beta_2 \cdots \beta_n = (1 + {}^1\!/_{3v_1})(1 + {}^1\!/_{3v_2}) \cdots (1 + {}^1\!/_{3v_n})$ the following equation holds:

$$v_{n+1} = \frac{3^n v_1 \beta}{2^\alpha} \tag{1}$$

Note that $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ is the total number of divisions by two that have been performed within this sequence starting from v_1 and ending with v_{n+1}.

Moreover we stated that the maximum possible number of division by two in such a sequence is given by equation 2.

$$\hat{\alpha} = \lfloor n \log_2 3 + \log_2 v_1 \rfloor + 1 \tag{2}$$

2. Problem Statement

Now the following argument was raised: Let v_i be a member of a Collatz sequence, for example $v_i = 17$. An *overflow point* is the next (nearest) power of two above v_i, in this case the overflow point is 32. Theoretically, the overflow point can move higher than the next power of two above 17 (due to multiplication by 3).

When considering a Collatz sequence starting at v_1 and ending with $v_{n+1} = 1$ and introducing a variable δ that represents the accumulation of "+1" we would obtain from equation 1:

$$1 = v_{n+1} = \frac{3^n v_1 \beta}{2^\alpha} = \frac{3^n v_1 + \delta}{2^\alpha}$$

The raised concern is now that nothing may prevent δ to grow larger than $3^n v_1$ possibly leading to $\beta > 2$, since $3^n v_1 \beta = 3^n v_1 + \delta$. Having $\beta > 2$, a beta larger than two would imply for $2^\alpha = 3^n v_1 \beta$ and thus for $\alpha = n \log_2 3 + \log_2 v_1 + \log_2 \beta$ that $\log_2 \beta > 1$ violating the inequality given by equation 2.

We can calculate δ directly using the following sum, see equation A.2 in appendinx of [3, p. 36]:

$$\delta = \sum_{j=1}^{n} 3^{j-1} 2^{\alpha_1 + \ldots + \alpha_n - \sum_{l > n-j} \alpha_l} \tag{3}$$

An example is the sequence $(v_1, v_2, v_3, v_4, v_5) = (37, 7, 11, 17, 13)$ where $v_1 = 37$, $n = 4$ and $v_{n+1} = v_5 = 13$. The beta is $\beta = (1 + {}^1\!/_{111})(1 + {}^1\!/_{21})(1 + {}^1\!/_{33})(1 + {}^1\!/_{51}) = {}^{3328}\!/_{2997}$. The alpha is $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4 + 1 + 1 + 2 = 8$ and finally the delta is $\delta = 3^0 \cdot 2^{\alpha_1+\alpha_2+\alpha_3} + 3^1 \cdot 2^{\alpha_1+\alpha_2} + 3^2 \cdot 2^{\alpha_1} + 3^3 \cdot 2^0 = 3^0 \cdot 2^6 + 3^1 \cdot 2^5 + 3^2 \cdot 2^4 + 3^3 \cdot 2^0 = 331$.

Indeed it appies

$$v_{n+1} = v_5 = \frac{3^4 \cdot 37 \cdot {}^{3328}/_{2997}}{2^8} = \frac{3328}{2^8} = \frac{3^4 \cdot 37 + 331}{2^8} = \frac{2997 + 331}{2^8} = 13$$

Halbeisen and Hungerbühler [4] introduced a function φ, which we can use to describe the δ. This function φ takes a binary number (binary string) s of length $l(s)$ as input and produces an integer output as follows:

$$\varphi(s) = \sum_{j=1}^{l(s)} s_j 3^{s_{j+1}+\ldots+s_{l(s)}} 2^{j-1} \tag{4}$$

Let us take for example the binary string $s = s_1s_2s_3s_4s_5s_6s_7 = 1000111 = 71$ as input for the function φ, which will yield the delta from our example $\delta = \varphi(1000111) = 331$:

$$\begin{array}{llllll}
 & s_1 \cdot 3^{s_2+s_3+s_4+s_5+s_6+s_7} 2^0 & & 1 \cdot 3^{0+0+0+1+1+1} 2^0 & & 1 \cdot 3^3 2^0 \\
+ & s_2 \cdot 3^{s_3+s_4+s_5+s_6+s_7} 2^1 & + & 0 \cdot 3^{0+0+1+1+1} 2^1 & + & 0 \cdot 3^3 2^1 \\
+ & s_3 \cdot 3^{s_4+s_5+s_6+s_7} 2^2 & + & 0 \cdot 3^{0+1+1+1} 2^2 & + & 0 \cdot 3^3 2^2 \\
+ & s_4 \cdot 3^{s_5+s_6+s_7} 2^3 & + & 0 \cdot 3^{1+1+1} 2^3 & + & 0 \cdot 3^3 2^3 \\
+ & s_5 \cdot 3^{s_6+s_7} 2^4 & + & 1 \cdot 3^{1+1} 2^4 & + & 1 \cdot 3^2 2^4 \\
+ & s_6 \cdot 3^{s_7} 2^5 & + & 1 \cdot 3^1 2^5 & + & 1 \cdot 3^1 2^5 \\
+ & s_7 \cdot 3^0 2^6 & + & 1 \cdot 3^0 2^6 & + & 1 \cdot 3^0 2^6 \\
= & 331 & = & 331 & = & 331
\end{array}$$

Note: $71 = v_1 + v_2 + v_3 + v_4 - 1$.

We have to proove that δ cannot exceed $3^n v_1$.

Halbeisen and Hungerbühler proved that for two distinct binary strings $s = s_1s_2\ldots s_l$ and $t = t_1t_2\ldots t_l$, which have the same Hamming weight, it applies [4]:

Theorem 1. *If* $\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} t_i$ *for all* $k \in \{1, \ldots, l\}$ *then* $\varphi(s) > \varphi(t)$.

References

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[3] E. Sultanow, C. Koch, and S. Cox. Collatz sequences in the light of graph theory. Technical report, University of Potsdam, 2020.

[4] Hungerbühler N. Halbeisen, L. Optimal bounds for the length of rational collatz cycles. *Acta Arithmetica*, 78(3):227–239, 1997.