


# BINARY P-ADIC INTEGERS IN COLLATZ SEQUENCES

 First Last and First Last

ABSTRACT. The Collatz conjecture is a number theoretical problem, which has puzzled countless researchers using myriad approaches. We describe an approach from the perspective of 2-adic (binary) algebra.

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**Fundamentals short and sweet**

unit	An element $a$ of a ring $R$ is called a "unit" (an invertible element) if there exist an element $b$ such that $ab = 1$ [1, p. 24]. Units are elements with inverses with respect to multiplication in the ring. Let $F$ be a field, then an element $a$ of $F$ is a non-unit iff $a = 0$ . The sum of any two non-units in $F$ is again a non-unit in $F$ .
unitary ring	A unitary ring is a ring with a multiplicative identity $1$ (which differs from the additive identity $1 \neq 0$ ) such that $1a = a = a1$ for all elements $a$ of the ring.
Ideal	Let $(R, +, \cdot)$ be a commutative unitary ring. Then the subset $I \subseteq R$ is called an ideal of $R$ if $(I, +)$ is a commutative group and if $xI \subseteq I$ for all $x \in R$ , see [2, p. 66-67].
quot. ring	Using an ideal of a ring $I \subseteq R$ , we may define an equivalence relation $\sim$ on $R$ by $a \sim b$ iff $a - b$ is in $I$ [3, p. 69]. The equivalence class of $a$ in $R$ is given by $[a] = a + I := \{a + r   r \in I\}$ for $r \in R$ and referred to as "residue class of $a$ modulo $I$ ", see [4, p. 122], [3, p. 70]. The set of all these equivalence classes becomes the quotient ring (residue class ring) modulo the ideal $I$ , denoted by $R/I$ .
compl. residue system	Let $I \subseteq R$ be an ideal and $[a]$ the residue classes of $a$ modulo $I$ , which means that $a + I = b + I$ when $a \equiv b \pmod{I}$ or respectively $a - b \in I$ [3, p. 70]. $R$ is the disjoint union of the different residue classes $a$ modulo $I$ . A subset $M \subseteq R$ , which contains exactly one element from each of these residue classes, is called a complete residue system of $R$ modulo $I$ , see [3, p. 70].
$[a]_n$	The residue class (also termed congruence class) of the integers for a modulus $n$ is the set $[a]_n = \{a + kn   k \in \mathbb{Z}\}$ and sometimes denoted by $\bar{a}_n$ or by $a + n\mathbb{Z}$ , see [2, p. 15], [4, p. 122], [5, p. 25].
$\mathbb{Z}/n\mathbb{Z}$	The set of all residue classes $[a]_n$ is called the ring of integers modulo $n$ and denoted by $\mathbb{Z}/n\mathbb{Z} = \{[a]_n   a \in \mathbb{Z}\}$ and trivially $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$ and for all $n \neq 0$ we have $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \dots, [n-1]\}$ , see [2, p. 15], [5, p. 25].

direct prod.	If $R_1, R_2, \dots, R_n$ are rings, the cartesian product $R_1 \times R_2 \times \dots \times R_n$ forms the set of all ordered $n$ -tuples $(r_1, r_2, \dots, r_n)$ , where $r_i \in R_i$ . The addition and multiplication of these $n$ -tuples is defined "coordinatewise" by components. The resulting ring is called a "direct product" of the original rings $R_i$ [2, p. 51], [6, p. 169].
prod. of ideals	Let $I, J$ be two ideals of a ring $R$ . Their product $IJ$ is defined as the set of all finite sums $a_1b_1 + \dots + a_nb_n$ in which $n \geq 0$ and $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in J$ , see [7, p. 87].
power of an ideal	Let $I$ be an ideal in $R$ . The $n$ -th power of $I$ , denoted by $I^n$ , is the $n$ -times product of the ideal $I$ with itself. $I^n$ contains sums of elements of the form $a_1a_2 \dots a_n$ where $a_1, a_2, \dots, a_n \in I$ and the products refer to the multiplication defined in $R$ . Consequently, $I^{n+1} \subseteq I^n$ . Note that the product and power of ideals should not be confused with the direct product of rings.
filtra- tion	Let $R$ be a ring. A sequence of ideals $I_0, I_1, I_2, \dots$ is said to be a "filtration" on $R$ if $I_0 = R$ and for each integer $j \geq 0$ applies that $I_j \supseteq I_{j+1}$ and if $I_j I_k \subseteq I_{j+k}$ , see [8, p. 269].
princip. ideal	A "principle ideal" is an ideal in a ring $R$ which is generated by a single element $a$ of $R$ through multiplication by every element of $R$ . There are some rings in which every ideal is a principle ideal, so-called "principle ideal rings" [2, p. 68].
max. ideal	A proper Ideal $M$ of a ring $R$ is called "maximal ideal" of $R$ if there is no other proper ideal $N$ of $R$ properly containing $M$ [6, p. 247], [1, p. 37]. A Note on "proper containment": If $R$ is any set, then $R$ is the improper subset of $R$ . Any other subset $N \neq R$ is a proper subset of $R$ and denoted by $N \subset R$ or $N \subsetneq R$ [6, p. 2].
prime ideal	Let $a$ and $b$ are two elements of $R$ and $P$ a proper ideal such that their product $ab$ is an element of $P$ . $P$ is called a prime ideal if at least one of $a$ and $b$ belongs to $P$ , in other words from $ab \in P$ and $a \notin P$ always follows $b \in P$ [1, p. 9].
max. prime ideal	A proper prime ideal $P$ is said to be a "maximal prime ideal" of the ring $R$ , if there is no other proper prime ideal containing $P$ [1, p. 23].
local ring	A commutative ring $R$ is called a local ring if it has a unique maximal ideal $M$ [9, p. 522].

Noeth. ring	A ring $R$ is called "Noetherian" when in $R$ the maximal condition for ideals is satisfied, in other words if every ideal $I$ of $R$ is finitely generated, that is, if we can find a finite set $a_1, a_2, \dots, a_n$ of elements, such that $I = Ra_1 + Ra_2 + \dots + Ra_n$ [1, p. 19, 101].
semi- local ring	A semi-local ring is a Noetherian ring which has only a finite number of maximal ideals [1, p. 107].
zero seq.	A zero sequence is a sequence, which converges towards 0 [7, p. 154]. Given the context of ideal theory, let $R$ be a ring and $I$ an ideal. In the ring $R^{\mathbb{N}} = \prod_{n \in \mathbb{N}} R$ , which is the repeated direct product of $R$ with itself, a sequence $(x_i)_{i \in \mathbb{N}}$ is called a zero sequence if for every $s \in \mathbb{N}$ there exist a $N \in \mathbb{N}$ (depending on $s$ ) such that $x_n \in I^s$ for all $n > N$ .
Cauchy seq. in $\mathbb{Q}, \mathbb{R}$	A sequence $(x_i)_{i \in \mathbb{N}}$ in $\mathbb{Q}$ or $\mathbb{R}$ is a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer $N$ such that $ x_n - x_m  < \epsilon$ for all $n, m \geq N$ , see [7, p. 153], [10, p. 24], [11, p. 10].
Cauchy seq. in a ring	Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of elements in $R^{\mathbb{N}}$ , the repeated direct product of a ring with itself, and $I$ an ideal in $R$ . This sequence is a Cauchy sequence if for every $s \in \mathbb{N}$ there exist a $N \in \mathbb{N}$ such that $x_n - x_m \in I^s$ for all $n, m > N$ .
Cauchy seq. in a local ring	Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of elements in a local ring $R$ and $M$ is the maximal ideal of $R$ . This sequence is a Cauchy sequence if, given any $s \in \mathbb{N}$ , we can always find an integer $N$ such that $x_n - x_m \in M^s$ whenever $n > m > N$ , see [1, p. 63, 85]. It is a Cauchy sequence iff $x_n - x_{n-1} \rightarrow 0$ as $n \rightarrow \infty$ [1, p. 85].
compl. of a ring	Let $R$ be a ring, $I$ an ideal, $I_{ZS}$ the ideal of all zero sequences in $R^{\mathbb{N}}$ , and $S_{CS}$ the subring of $R^{\mathbb{N}}$ containing all Cauchy sequences. The quotient ring $\hat{R}_I := S_{CS}/I_{ZS}$ is called the completion of $R$ with respect to $I$ . $S_{CS}/I_{ZS}$ is the residue class ring of $S_{CS}$ modulo $I$ .
concor. ext.	Let $R, S$ be local rings. If a sequence of elements of $S$ is a Cauchy sequence in $S$ iff it is a Cauchy sequence in $R$ , then we say that $R$ is a "concordant extension" of $S$ [1, p. 87]. When $R, S$ are semi-local rings $R \subseteq S$ , $R$ is said to be a "concordant extension" of $S$ if a sequence $(s_n)$ of elements in $S$ is regular in $S$ iff $(s_n)$ is regular in $R$ [12].

compl. of a local ring	Let $S$ be a local ring. A local ring $R$ will be called a completion of $S$ if $R$ is a concordant extension of $S$ and $R$ is complete and if every element of $R$ is the limit of a sequence of elements of $S$ . Each local ring has a completion [1, p. 92].
compl. local ring	A local ring $R$ is called "complete" if every Cauchy sequence composed of elements of $R$ has a limit in $R$ [1, p. 85], [13, p. 184].
$p$ -adic val. for $\mathbb{Z}$	Fix a prime number $p$ in $\mathbb{Z}$ . The $p$ -adic valuation of a nonzero integer $n = r \cdot p^{v_p(n)}$ is the highest exponent $v_p(n)$ such that $p^{v_p(n)}$ divides $n$ (we say $p^{v_p(n)}$ divides $n$ "exactly"). Hence $p$ and $r$ are coprime. If $n, p$ are coprime then $v_p(n) = 0$ , and by convention $v_p(0) = \infty$ , see [14].
$p$ -adic val. for $\mathbb{Q}$	The $p$ -adic valuation can be extended to the field of rational numbers. Let $x = n \cdot s^{-1}$ be a rational number, then $v_p(x) = v_p(n) - v_p(s)$ . Any nonzero rational number $x$ can be uniquely represented as $x = rp^{v_p(x)}s^{-1}$ , where $r, s \in \mathbb{Z}$ , $s > 0$ , and $\gcd(r, s) = \gcd(r, p) = \gcd(s, p) = 1$ , see [7, p. 154], [15].
$p$ -adic norm	Let $x$ be any number in $\mathbb{Q}$ , for which we already know that it can be written as $x = rp^{v_p(x)}s^{-1}$ , where $p$ is a prime number, $s > 0$ and $r$ are integers not divisible by $p$ . The $p$ -adic norm of $x$ is defined by $ x _p = p^{-v_p(x)}$ for $x \neq 0$ , and $ 0 _p = 0$ , see [14], [7, p. 154], [16].
$p$ -adic dist.	Let $x, y \in \mathbb{Q}$ . The $p$ -adic distance between $x$ and $y$ is defined by $d_p(x, y) =  x - y _p$ , see [7, p. 155].
$\mathbb{Q}_p$	The field $\mathbb{Q}_p$ of $p$ -adic numbers is the set of equivalence classes of Cauchy sequences [11, p. 10]. The elements of $\mathbb{Q}_p$ , the so-called $p$ -adic numbers, are equivalence classes of Cauchy sequences $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{Q}$ with respect to the equivalence relation $(a_n) \sim (b_n)$ if $(a_n - b_n)$ is a $p$ -adic zero sequence, see [7, p. 159]. Furthermore $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the $p$ -adic distance $d_p$ [7, p. 159].
$\mathbb{Z}_p$	The ring $\mathbb{Z}_p$ of $p$ -adic integers is the completion of $\mathbb{Z}$ with respect to the $p$ -adic norm. That is, $\mathbb{Z}_p$ is the set of all equivalence classes of Cauchy sequences $(a_n)$ where $(a_n)$ and $(b_n)$ are equivalent if $\lim_{n \rightarrow \infty}  a_n - b_n _p = 0$ , see [17]. $\mathbb{Z}_p$ is a local ring whose maximal ideal is the principal ideal $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p :  x _p < 1\}$ , see [18, p. 74].

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