Lecture 5: Model-Free Prediction

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Outline

Reading: (Sutton & Barto Oct 2015) Chapters 5, 6, and 7 on prediction

Model-Free Reinforcement Learning

- Last lecture:
  - Planning by dynamic programming
  - Solve a *known* MDP
- This lecture:
  - Model-free prediction
  - Estimate the value function of an *unknown* MDP
- Next lecture:
  - Model-free control
  - Optimise the value function of an *unknown* MDP

Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is *model-free*: no knowledge of MDP transitions / rewards
- MC learns from *complete* episodes: no bootstrapping
- MC uses the simplest possible idea: value = mean return
- Caveat: can only apply MC to *episodic* MDPs
  - All episodes must terminate

Monte-Carlo for Prediction and Control

- MC can be used for prediction:
  - Input: Episodes of experience $\{S_1, A_1, R_2, ..., S_T\}$ generated by following policy π in given MDP
  - or: Episodes of experience $\{S_1, R_2, ..., S_T\}$ generated by MRP
  - Output: Value function V^π
- Or for control:
  - Input: Episodes of experience $\{S_1, A_1, R_2, ..., S_T\}$ in given MDP
  - Output: Optimal value function V^*
  - Output: Optimal policy π^*
- We will focus on *prediction* this lecture; *control* next lecture

Monte-Carlo Policy Evaluation

- Goal: learn V^π from episodes of experience under policy π

$$S_1, A_1, R_2, ..., S_k \sim \pi$$

- Recall that the *return* is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$V^\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

- Monte-Carlo policy evaluation uses *empirical mean* return instead of *expected* return

First-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- The *first* time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Accumulate total return $M(s) \leftarrow M(s) + G_t$
- Value is estimated by mean return $V(s) = M(s)/N(s)$
- By law of large numbers, $V(s) \rightarrow V^{\pi}(s)$ as $N(s) \rightarrow \infty$

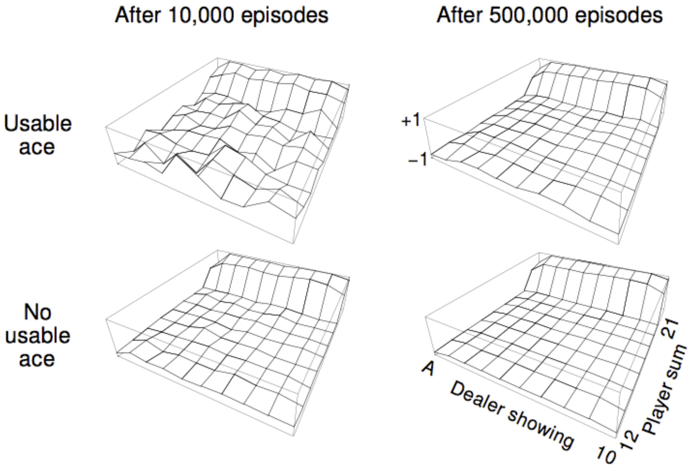
Every-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- *Every* time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Accumulate total return $M(s) \leftarrow M(s) + G_t$
- Value is estimated by mean return $V(s) = M(s)/N(s)$
- Again, $V(s) \rightarrow V^{\pi}(s)$ as $N(s) \rightarrow \infty$

Blackjack Example

- States (200 of them):
  - Current sum (12-21)
  - Dealer's showing card (ace-10)
  - Do I have a "useable" ace? (yes-no)
- Action stick: Stop receiving cards (and terminate)
- Action twist: Take another card (random, no replacement)
- Reward for stick:
  - $+1$ if sum of cards $>$ sum of dealer cards
  - 0 if sum of cards $=$ sum of dealer cards
  - -1 if sum of cards $<$ sum of dealer cards
- Reward for twist:
  - -1 if sum of cards > 21 (and terminate)
  - 0 otherwise
- Transitions: automatically twist if sum of cards < 12

Blackjack Value Function after Monte-Carlo Learning



Policy: stick if sum of cards ≥ 20, otherwise twist

Incremental Mean

The mean $\mu_1, \mu_2, ...$ of a sequence $x_1, x_2, ...$ can be computed incrementally,

$$\begin{aligned}
\mu_k &= \frac{1}{k}\sum_{j=1}^{k} x_j \\
&= \frac{1}{k}\left(x_k + \sum_{j=1}^{k-1} x_j\right) \\
&= \frac{1}{k}\left(x_k + (k-1)\mu_{k-1}\right) \\
&= \mu_{k-1} + \frac{1}{k}\left(x_k - \mu_{k-1}\right)
\end{aligned}$$

Incremental Monte-Carlo Updates

- Update $V(s)$ incrementally after episode $S_1, A_1, R_2, ..., S_T$
- For each state S_t with return G_t

$$N(S_t) \leftarrow N(S_t) + 1$$

$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)}\left(G_t - V(S_t)\right)$$

- In non-stationary problems, it is useful to track a running mean, i.e. forget old episodes.

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

Temporal-Difference Learning

- TD methods learn directly from experience
- TD is *model-free*: no knowledge of MDP transitions / rewards
- TD also learns from *incomplete* episodes, by *bootstrapping*
- TD updates a guess towards a guess

MC and TD

- Goal: learn V^π online from experience under policy π
- Incremental every-visit Monte-Carlo
  - Update value $V(S_t)$ towards actual return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$

- Simplest temporal-difference learning algorithm: TD(0)
  - Update value $V(S_t)$ towards *estimated* return $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$

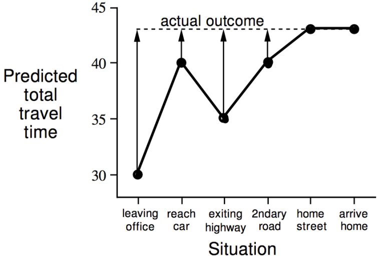
  - $R_{t+1} + \gamma V(S_{t+1})$ is called the *TD target*
  - $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ is called the *TD error*

Driving Home Example

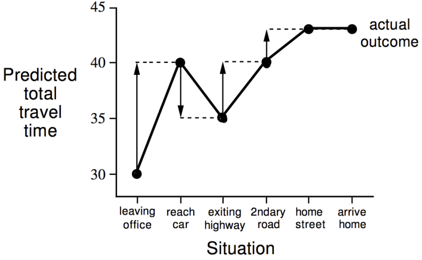
State	Elapsed Time (minutes)	Predicted Time to Go	Predicted Total Time
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

Driving Home Example: MC vs. TD

Changes recommended by Monte Carlo methods (α=1)



Changes recommended by TD methods (α=1)



Advantages and Disadvantages of MC vs. TD

- TD can learn *before* knowing the final outcome
  - TD can learn online after every step
  - MC must wait until end of episode before return is known
- TD can learn *without* the final outcome
  - TD can learn from incomplete sequences
  - MC can only learn from complete sequences
  - TD works in continuing (non-terminating) environments
  - MC only works for episodic (terminating) environments

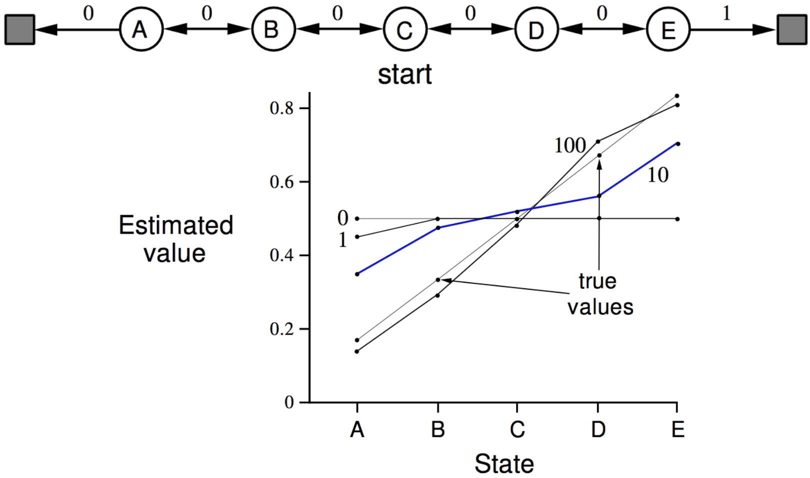
Bias/Variance Trade-Off

- Return $G_t = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T$ is an *unbiased* estimate of $V^\pi(S_t)$
- TD target $R_{t+1} + \gamma V(S_{t+1})$ is a *biased* estimate of $V^\pi(S_t)$
  - Unless $V(S_{t+1}) = V^\pi(S_{t+1})$
- But the TD target is much lower variance:
  - Return depends on *many* random actions, transitions, rewards
  - TD target depends on *one* random action, transition, reward

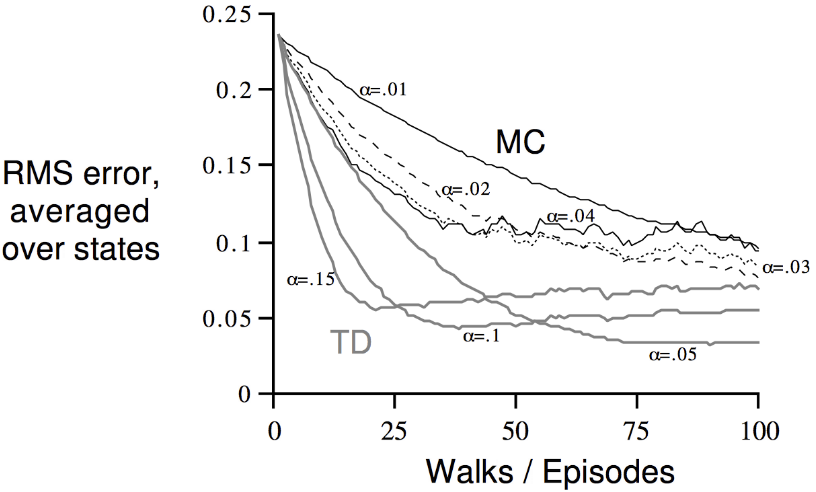
Advantages and Disadvantages of MC vs. TD (2)

- MC has high variance, zero bias
  - Good convergence properties
  - (even with function approximation)
  - Not sensitive to initial value
  - Very simple to understand and use
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD(0) converges to $V^{\pi}(s)$
  - (but not always with function approximation)
  - More sensitive to initial value

Random Walk Example



Random Walk: MC vs. TD



Batch MC and TD

- MC and TD converge: $V(s) \rightarrow V^{\pi}(s)$ as experience $\rightarrow \infty$ and $\alpha \rightarrow 0$
- But what about batch solution for finite experience?

$$s_1^1, a_1^1, r_2^1, ..., s_{T_1}^1$$
$$\vdots$$
$$s_1^K, a_1^K, r_2^K, ..., s_{T_K}^K$$

  - e.g. Repeatedly sample episode $k \in [1, K]$
  - Apply MC or TD(0) to episode k

AB Example

Two states A, B; no discounting; 8 episodes of experience

$\mathrm{A}, 0, \mathrm{B}, 0$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

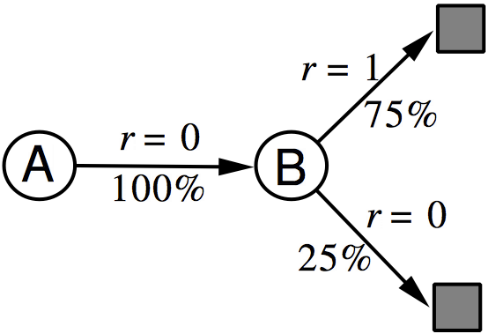
$\mathrm{B}, 0$

What is $V(A), V(B)$?

AB Example

Two states A, B; no discounting; 8 episodes of experience

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0



What is $V(A), V(B)$?

Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns

$$\sum_{k=1}^{K} \sum_{t=1}^{T_k} \left(v_t^k - V(s_t^k)\right)^2$$

  - In the AB example, $V(A) = 0$
- TD(0) converges to solution of max likelihood Markov model
  - Solution to the MDP $\langle \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma \rangle$ that best fits the data

$$\hat{\mathcal{P}}_{s,s'}^{a} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k, s_{t+1}^k = s, a, s')$$

$$\hat{\mathcal{R}}_{s}^{a} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k = s, a) r_t^k$$

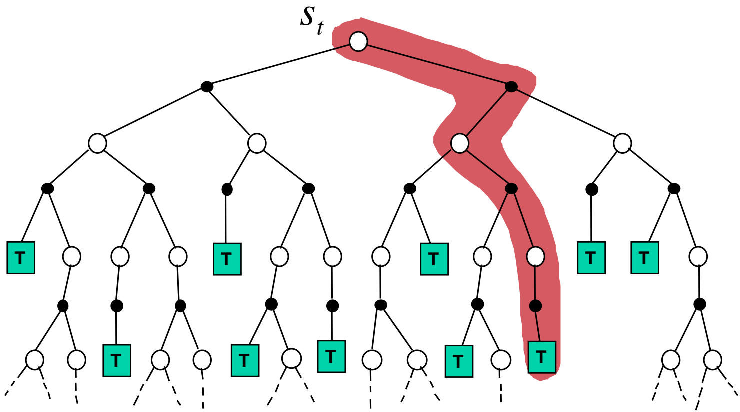
  - In the AB example, $V(A) = 0.75$

Advantages and Disadvantages of MC vs. TD (3)

- TD exploits Markov property
  - Usually more efficient in Markov environments
- MC does not exploit Markov property
  - Usually more accurate in non-Markov environments

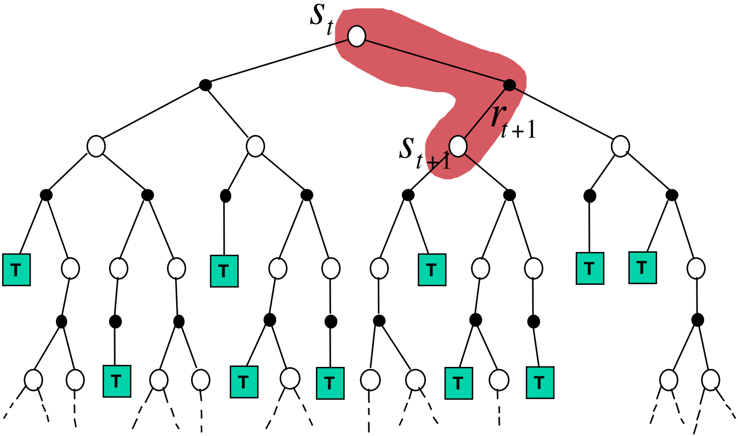
Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$



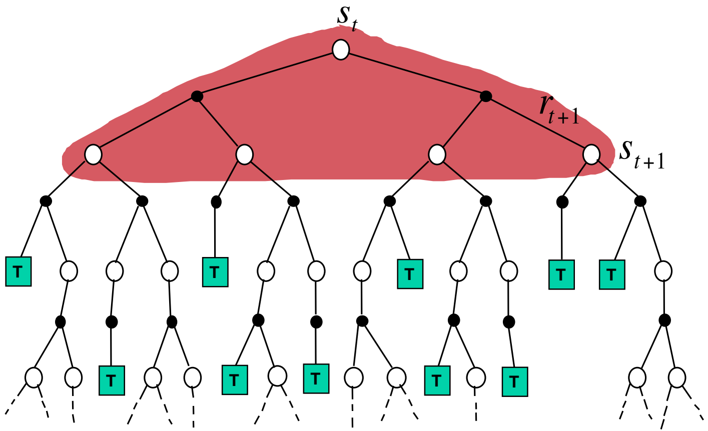
Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$



Dynamic Programming Backup

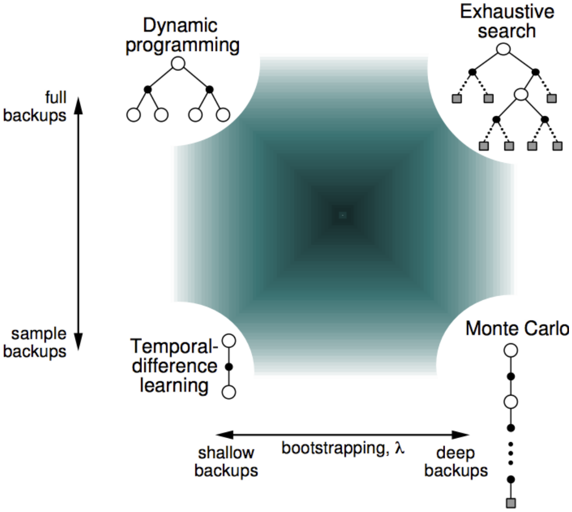
$$V(S_t) \leftarrow \mathbb{E}_\pi \left[R_{t+1} + \gamma V(S_t)\right]$$



Bootstrapping and Sampling

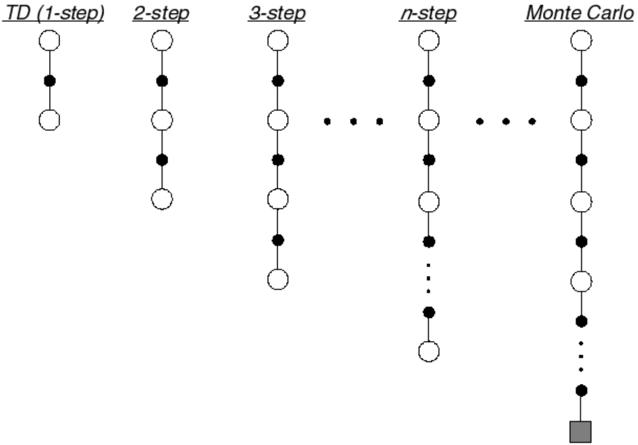
- Bootstrapping: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps
- Sampling: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

Unified View of Reinforcement Learning



n-Step Prediction

- Let TD target look n steps into the future



n-Step Return

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$\begin{array}{lll} n = 1 & (TD) & G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\ n = 2 & & G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\ \vdots & & \vdots \\ n = \infty & (MC) & G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T \end{array}$$

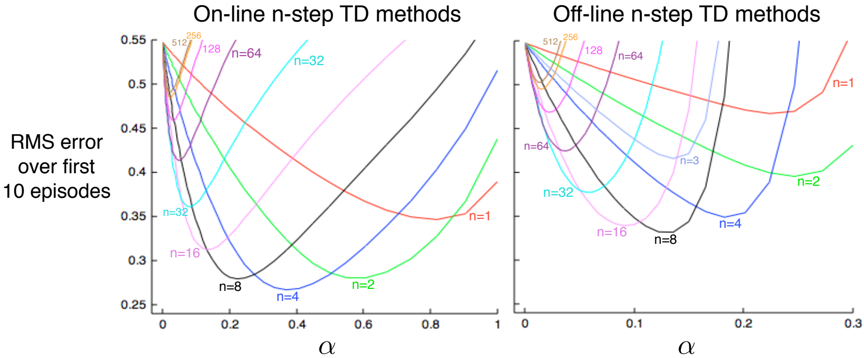
- Define the n-step return

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- n-step temporal-difference learning

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^{(n)} - V(S_t) \right)$$

Large Random Walk Example

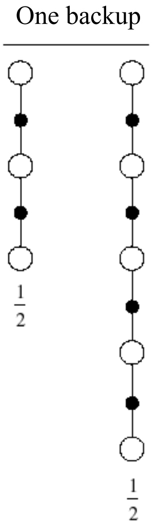


Averaging n-Step Returns

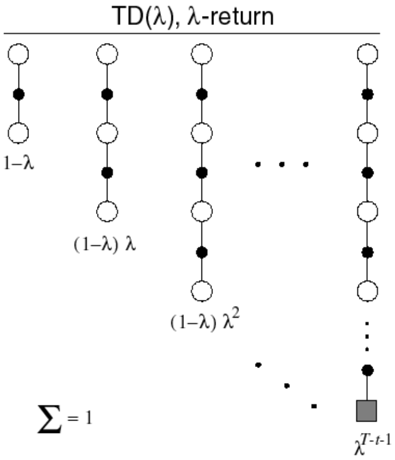
- We can average n-step returns over different n
- e.g. average the 2-step and 4-step returns

$$\frac{1}{2} G^{(2)} + \frac{1}{2} G^{(4)}$$

- Combines information from two different time-steps
- Can we efficiently combine information from all time-steps?



λ-return



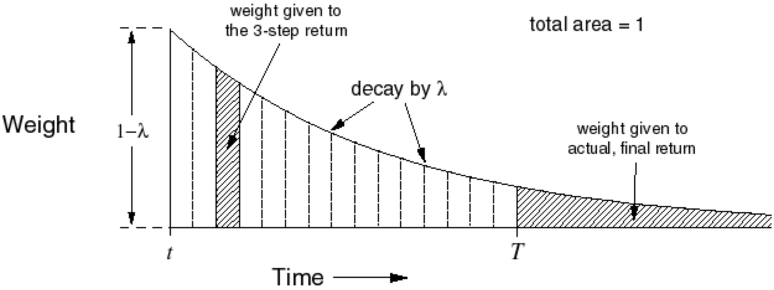
- The *λ-return* G_t^λ combines all n-step returns $G_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

- Forward-view TD(λ)

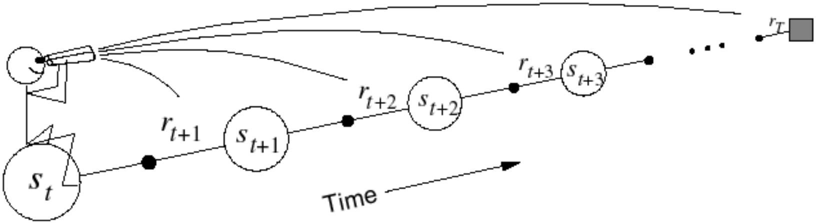
$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$$

TD(λ) Weighting Function



$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

Forward-view TD(λ)

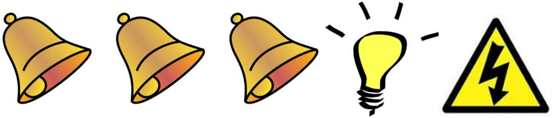


- Update value function towards the λ-return
- Forward-view looks into the future to compute G_t^{λ}
- Like MC, can only be computed from complete episodes

Backward View TD(λ)

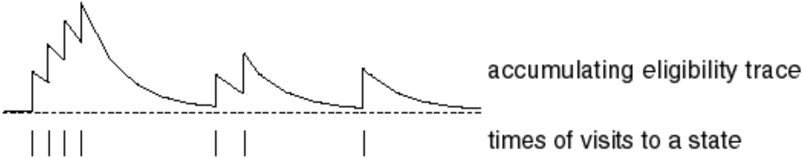
- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

Eligibility Traces



- Credit assignment problem: did bell or light cause shock?
- Frequency heuristic: assign credit to most frequent states
- Recency heuristic: assign credit to most recent states
- *Eligibility traces* combine both heuristics

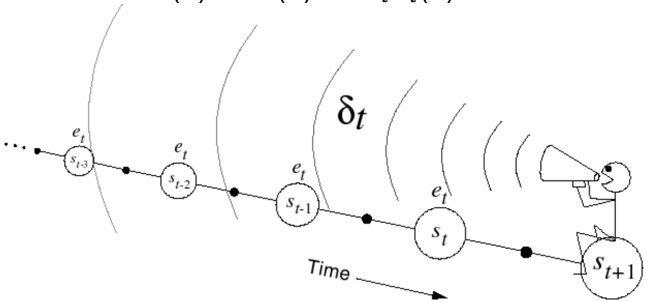
$$E_0(s) = 0$$
$$E_t(s) = \gamma \lambda E_{t-1}(s) + \mathbf{1}(s = S_t)$$



Backward View TD(λ)

- Keep an eligibility trace for every state s
- Update value $V(s)$ for every state s
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$
$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$



Backward-View TD(λ) Algorithm

Initialize $V(s)$ arbitrarily (but set to 0 if s is terminal)
Repeat (for each episode):
    Initialize $E(s) = 0$, for all $s \in \mathcal{S}$
    Initialize S
    Repeat (for each step of episode):
        $A \leftarrow$ action given by π for S
        Take action A, observe reward, R, and next state, S'
        $\delta \leftarrow R + \gamma V(S') - V(S)$
        $E(S) \leftarrow E(S) + 1$ (accumulating traces)
        or $E(S) \leftarrow (1 - \alpha)E(S) + 1$ (dutch traces)
        or $E(S) \leftarrow 1$ (replacing traces)
        For all $s \in \mathcal{S}$:
            $V(s) \leftarrow V(s) + \alpha \delta E(s)$
            $E(s) \leftarrow \gamma \lambda E(s)$
        $S \leftarrow S'$
    until S is terminal

TD(λ) and TD(0)

- When $\lambda = 0$, only current state is updated

$$E_t(s) = \mathbf{1}(s = S_t)$$
$$V(s) \leftarrow V(s) + \alpha\delta_t E_t(s)$$

- This is exactly equivalent to TD(0) update

$$V(S_t) \leftarrow V(S_t) + \alpha\delta_t$$

MC and TD(1)

- Consider an episode where s is visited once at time-step k,
- TD(1) eligibility trace discounts time since visit,

$$E_t(s) = \gamma E_{t-1}(s) + \mathbf{1}(S_t = s)$$
$$= \begin{cases} 0 & \text{if } t < k \\ \gamma^{t-k} & \text{if } t \geq k \end{cases}$$

- TD(1) updates accumulate error *online*

$$\sum_{t=1}^{T} \alpha \delta_t E_t(s) = \alpha \sum_{t=k}^{T} \gamma^{t-k} \delta_t = \alpha \left(G_k - V(S_k) \right)$$

Telescoping in TD(1)

When $\lambda = 1$, sum of TD errors telescopes into MC error,

$$
\begin{aligned}
& \delta_t + \gamma \delta_{t+1} + \gamma^2 \delta_{t+2} + ... + \gamma^{T-1} \delta_{T-1} \\
= & R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \\
+ & \gamma R_{t+2} + \gamma^2 V(S_{t+2}) - \gamma V(S_{t+1}) \\
+ & \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3}) - \gamma^2 V(S_{t+2}) \\
& \quad \vdots \\
+ & \gamma^{T-1} R_T + \gamma^T V(S_T) - \gamma^{T-1} V(S_{T-1}) \\
= & R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} ... + \gamma^{T-1} R_T - V(S_t) \\
= & G_t - V(S_t)
\end{aligned}
$$

TD(λ) and TD(1)

- TD(1) is roughly equivalent to every-visit Monte-Carlo
- Error is accumulated online, step-by-step
- If value function is only updated offline at end of episode
- Then total update is exactly the same as MC

Forwards and Backwards TD(λ)

- Consider an episode where s is visited once at time-step k,
- TD(λ) eligibility trace discounts time since visit,

$$\begin{aligned} E_t(s) &= \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s) \\ &= \begin{cases} 0 & \text{if } t < k \\ (\gamma\lambda)^{t-k} & \text{if } t \geq k \end{cases} \end{aligned}$$

- Backward TD(λ) updates accumulate error *online*

$$\sum_{t=1}^{T} \alpha\delta_t E_t(s) = \alpha \sum_{t=k}^{T} (\gamma\lambda)^{t-k}\delta_t = \alpha \left(G_k^\lambda - V(S_k) \right)$$

- By end of episode it accumulates total error for λ-return
- For multiple visits to s, $E_t(s)$ accumulates many errors

Offline Equivalence of Forward and Backward TD

Offline updates

- Updates are accumulated within episode
- but applied in batch at the end of episode

Theorem

The sum of offline updates is identical for forward-view and backward-view TD(λ)

$$\sum_{t=1}^{T} \alpha \delta_t E_t(s) = \sum_{t=1}^{T} \alpha \left(G_t^{\lambda} - V(S_t) \right) \mathbf{1}(S_t = s)$$

Onine Equivalence of Forward and Backward TD

Online updates

- TD(λ) updates are applied online at each step within episode
- Forward and backward-view TD(λ) are slightly different
- NEW: True online TD(λ) achieves perfect equivalence
- By using a dutch eligibility trace and an extra small correction
- Sutton and van Seijen, ICML 2014
- Generalizations: van Hasselt and Sutton, Arxiv 2015

Summary of Forward and Backward TD(λ)

	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
Forward view	TD(0)	Forward TD(λ)	MC