Lecture 12
Modulation and Sampling

- The Fourier transform of the product of two signals
- Modulation of a signal with a sinusoid
- Sampling with an impulse train
- The sampling theorem

Convolution and the Fourier transform

suppose $f(t)$, $g(t)$ have Fourier transforms $F(\omega)$, $G(\omega)$

the **convolution** $y = f * g$ of f and g is given by

$$y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$$

(we integrate from $-\infty$ to ∞ because $f(t)$ and $g(t)$ are not necessarily zero for negative t)

from the table of Fourier transform properties:

$$Y(\omega) = F(\omega)G(\omega)$$

$i.e.$, convolution in the time domain corresponds to multiplication in the frequency domain

Multiplication and the Fourier transform

the Fourier transform of the product

$$y(t) = f(t)g(t)$$

is given by

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)G(\omega-\lambda)d\lambda \\ Y &= \frac{1}{2\pi}(F * G) \end{aligned}$$

i.e., multiplication in the time domain corresponds to convolution in the frequency domain

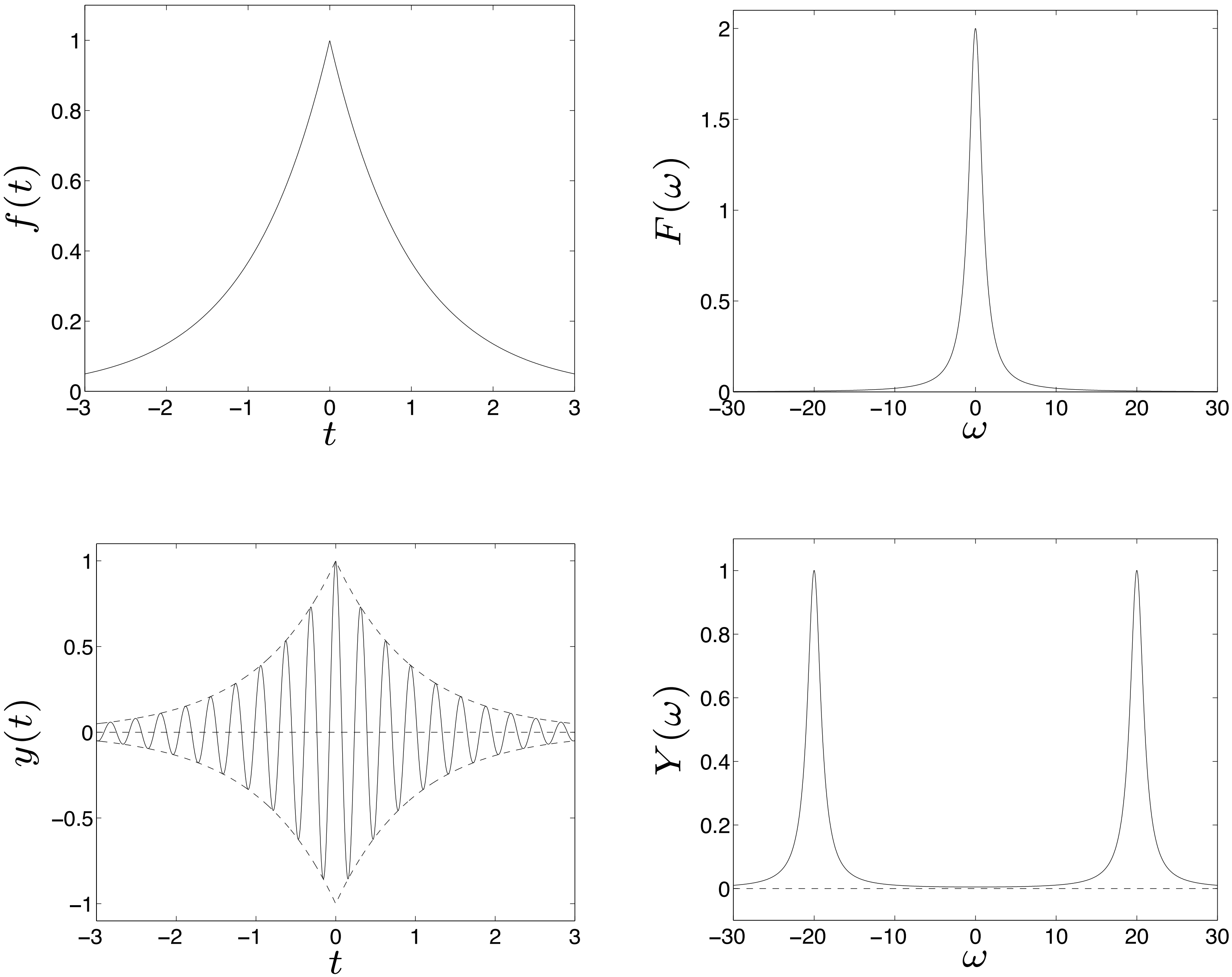
example:

$$f(t) = e^{-|t|}, \quad F(\omega) = \frac{2}{1+\omega^2}$$

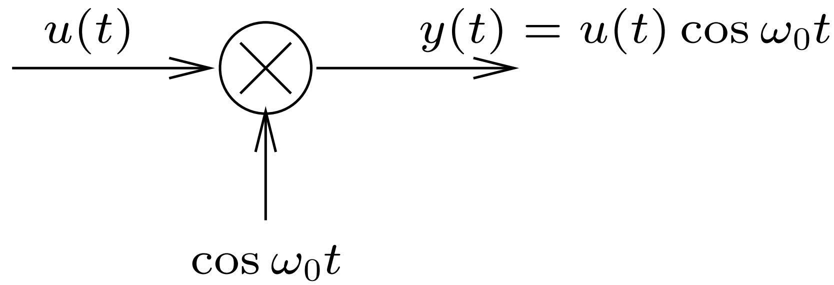
$$g(t) = \cos 20t, \quad G(\omega) = \pi\delta(\omega - 20) + \pi\delta(\omega + 20)$$

the Fourier transform of $y(t) = e^{-|t|}\cos 20t$ is given by

$$\begin{aligned}
Y(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)G(\omega-\lambda)\,d\lambda \\
&= \frac{1}{2}\int_{-\infty}^{\infty} F(\lambda)\delta(\omega-\lambda-20)\,d\lambda + \frac{1}{2}\int_{-\infty}^{\infty} F(\lambda)\delta(\omega-\lambda+20)\,d\lambda \\
&= \frac{1}{2}F(\omega-20) + \frac{1}{2}F(\omega+20) \\
&= \frac{1}{1+(\omega-20)^2} + \frac{1}{1+(\omega+20)^2}
\end{aligned}$$


f(t)
t
F(ω)
ω
y(t)
t
Y(ω)
ω

Sinusoidal amplitude modulation (AM)

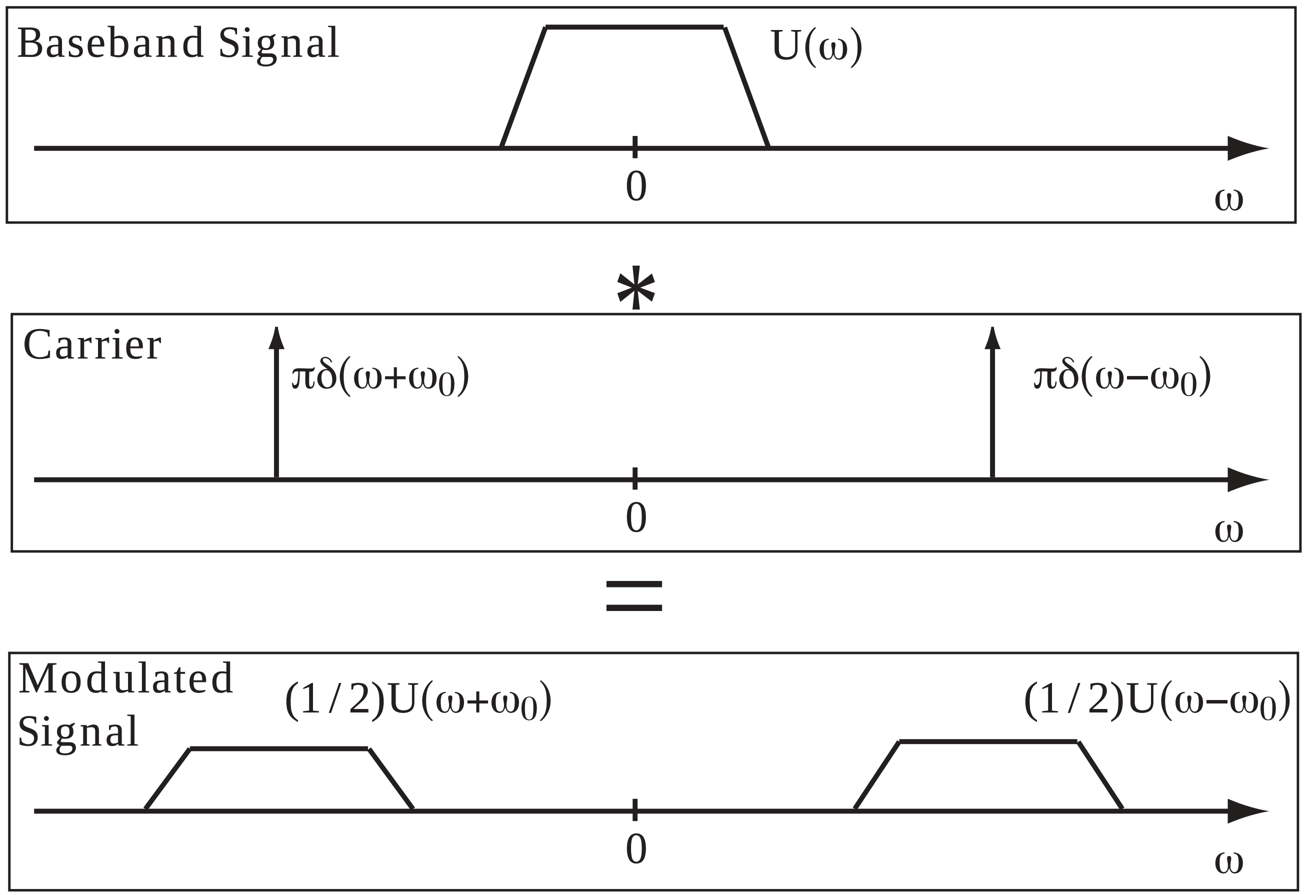


Fourier transform of y

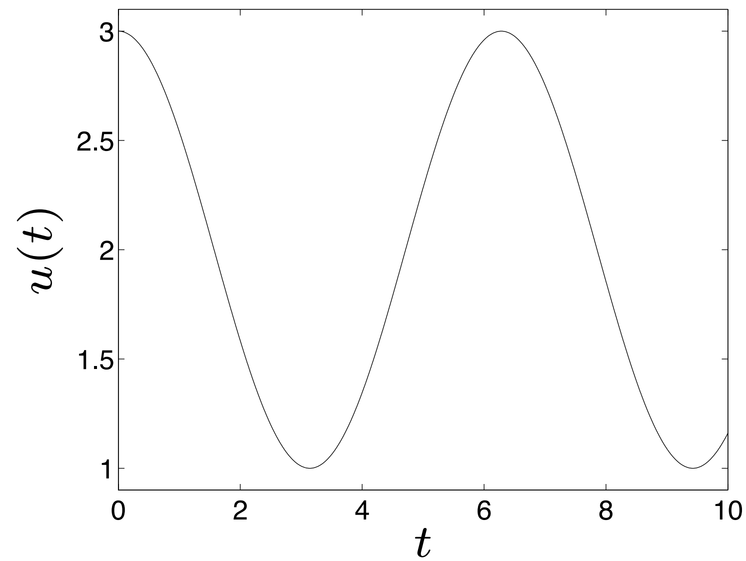
$$
\begin{aligned}
Y(\omega) &= \frac{1}{2\pi}U(\omega) * (\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)) \\
&= \frac{1}{2}U(\omega-\omega_0) + \frac{1}{2}U(\omega+\omega_0)
\end{aligned}
$$

- $\cos\omega_0 t$ is the *carrier signal*
- $y(t)$ is the modulated signal
- the Fourier transform of the modulated signal is the Fourier transform of the input signal, shifted by $\pm\omega_0$

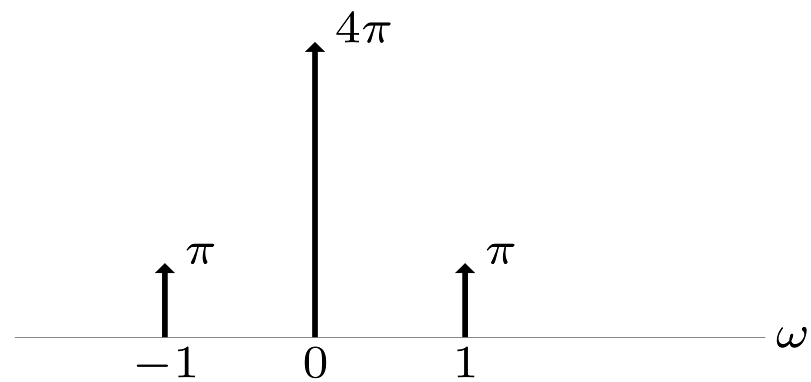
Sinusoidal amplitude modulation

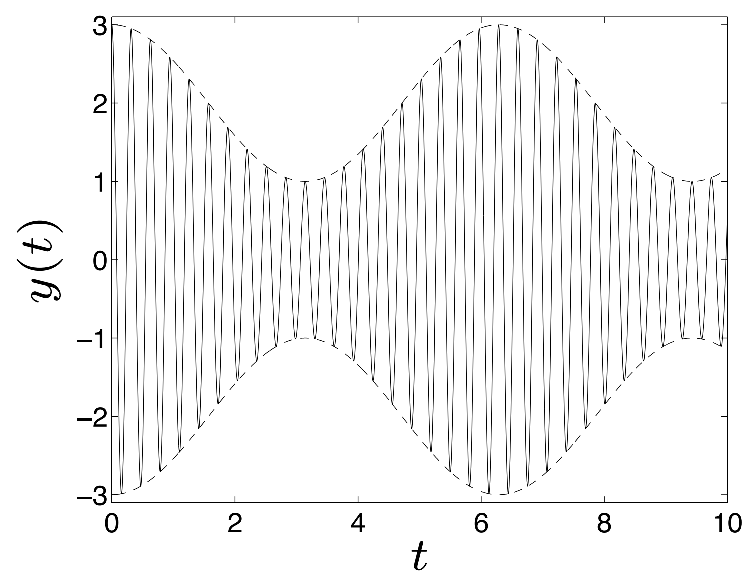


example: $u(t) = 2 + \cos t$, $\omega_0 = 20$

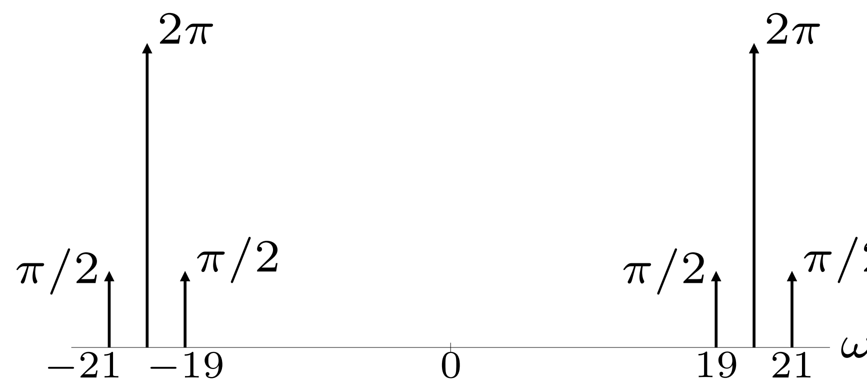


$$U(\omega) = 4\pi\delta(\omega) + \pi\delta(\omega - 1) + \pi\delta(\omega + 1)$$

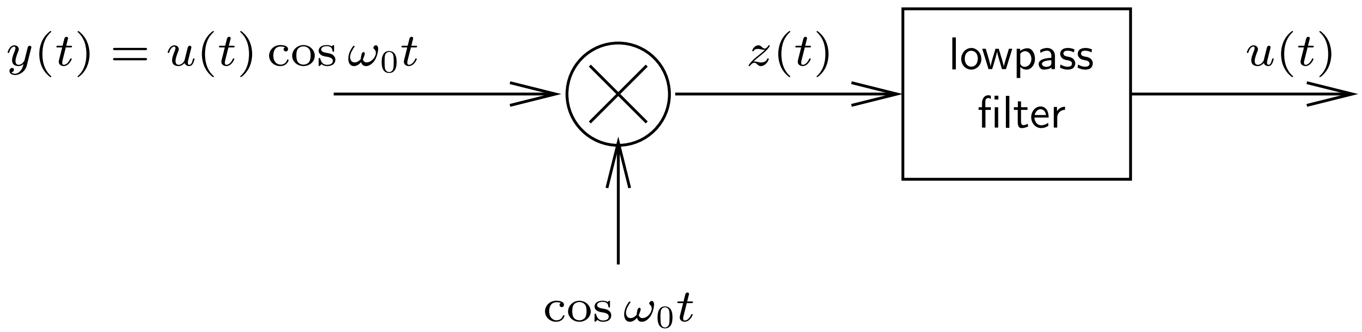




$$Y(\omega) = \frac{1}{2}U(\omega - 20) + \frac{1}{2}U(\omega + 20)$$



demodulation

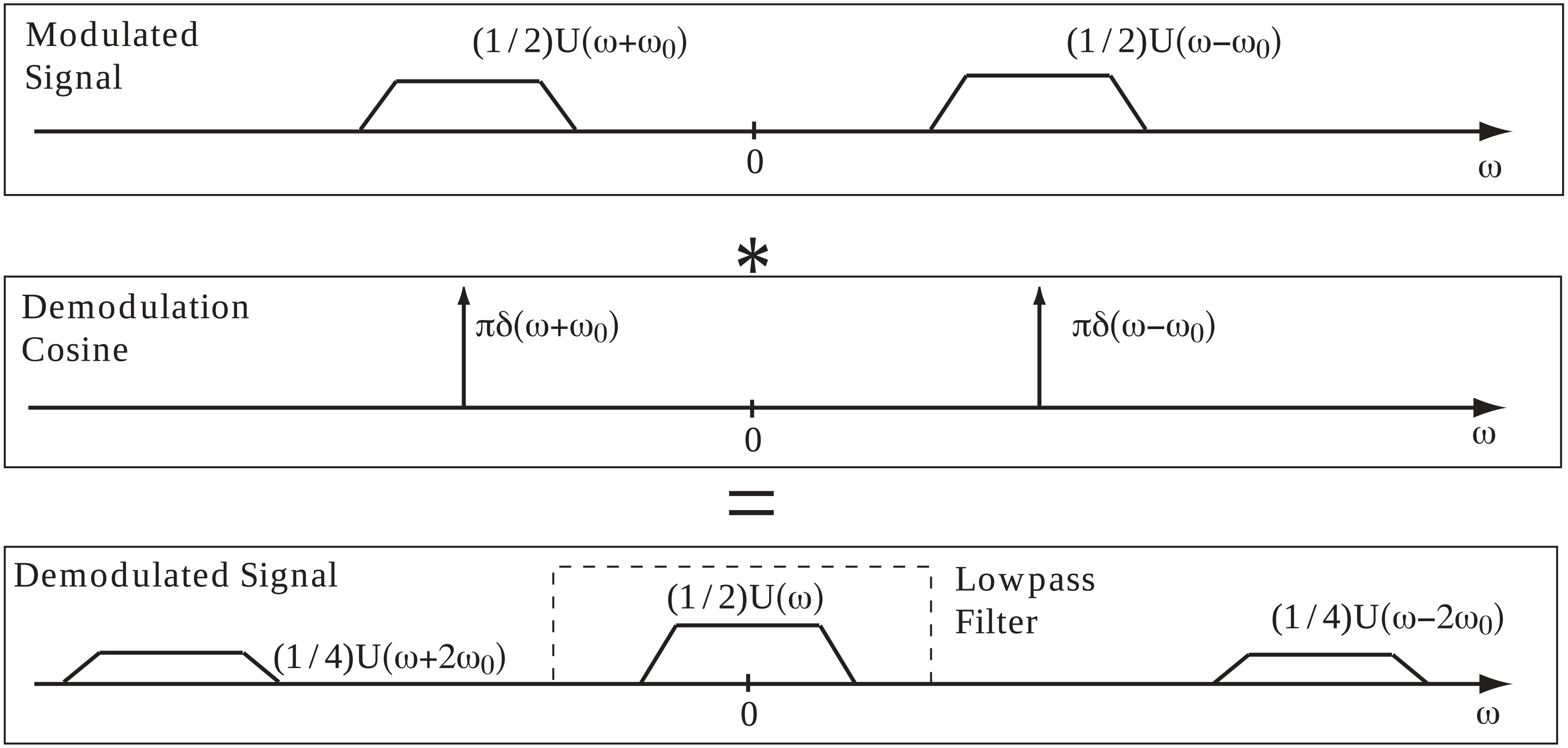


Fourier transform of y and z:

$$
\begin{aligned}
Y(\omega) &= \frac{1}{2}U(\omega-\omega_0) + \frac{1}{2}U(\omega+\omega_0) \\
Z(\omega) &= \frac{1}{2\pi}Y(\omega) * (\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)) \\
&= \frac{1}{2}Y(\omega-\omega_0) + \frac{1}{2}Y(\omega+\omega_0) \\
&= \frac{1}{4}U(\omega-2\omega_0) + \frac{1}{2}U(\omega) + \frac{1}{4}U(\omega+2\omega_0)
\end{aligned}
$$

if U is bandlimited, we can eliminate the 1st and 3rd term by lowpass filtering

Sinusoidal amplitude demodulation



Application

Suppose for example that $u(t)$ is an audio signal (frequency range $10\text{Hz} - 20\text{kHz}$)

We rather not transmit u directly using electromagnetic waves:

- the wavelength is several 100 km, so we'd need very large antennas
- we'd be able to transmit only one signal at a time
- the Navy communicates with submerged submarines in this band

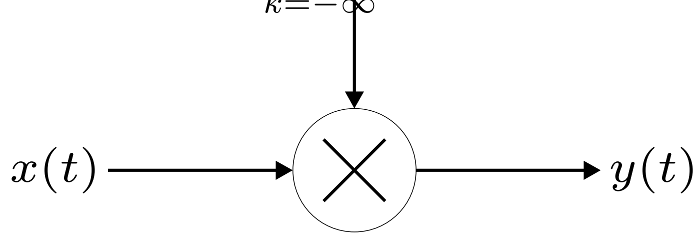
Modulating the signal with a carrier signal with frequency 500 kHz to 5 GHz:

- allows us to transmit and receive the signal
- allows us to transmit many signals simultaneously (frequency division multiplexing)

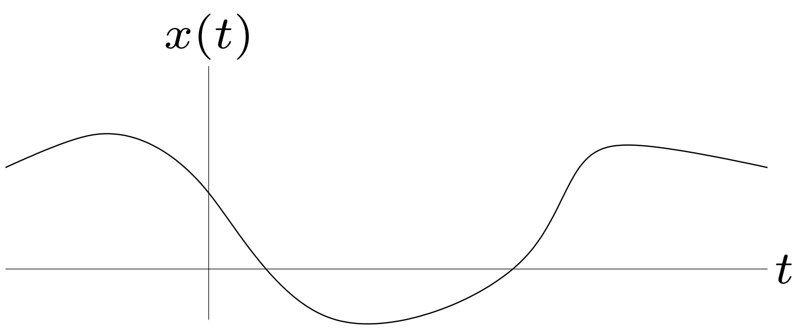
Sampling with an impulse train

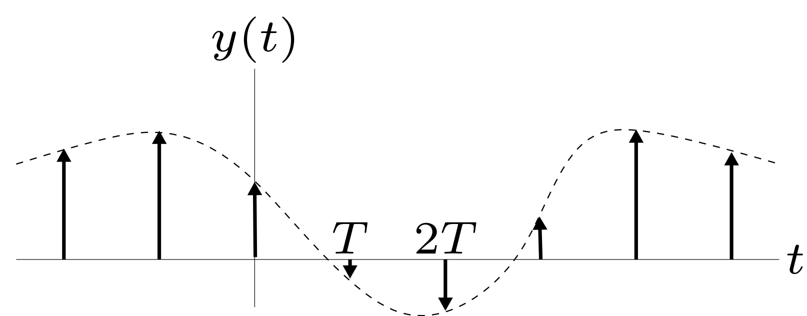
Multiply a signal $x(t)$ with a unit impulse train with period T

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



Sampled signal: $y(t) = x(t) \displaystyle\sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$

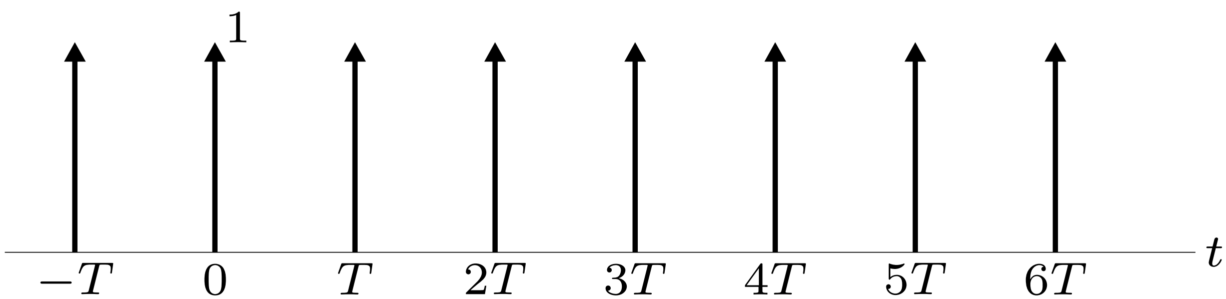




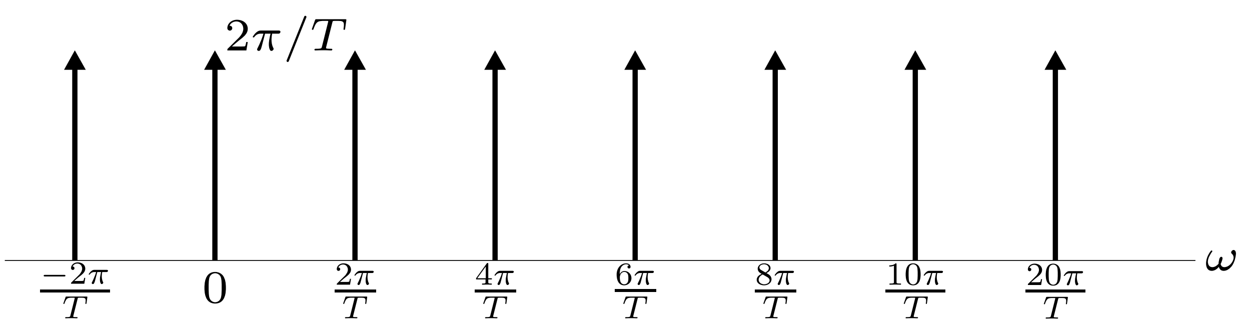
(a train of impulses with magnitude . . . , $x(-T)$, $x(0)$, $x(T)$, $x(2T)$, . . .)

The Fourier transform of an impulse train

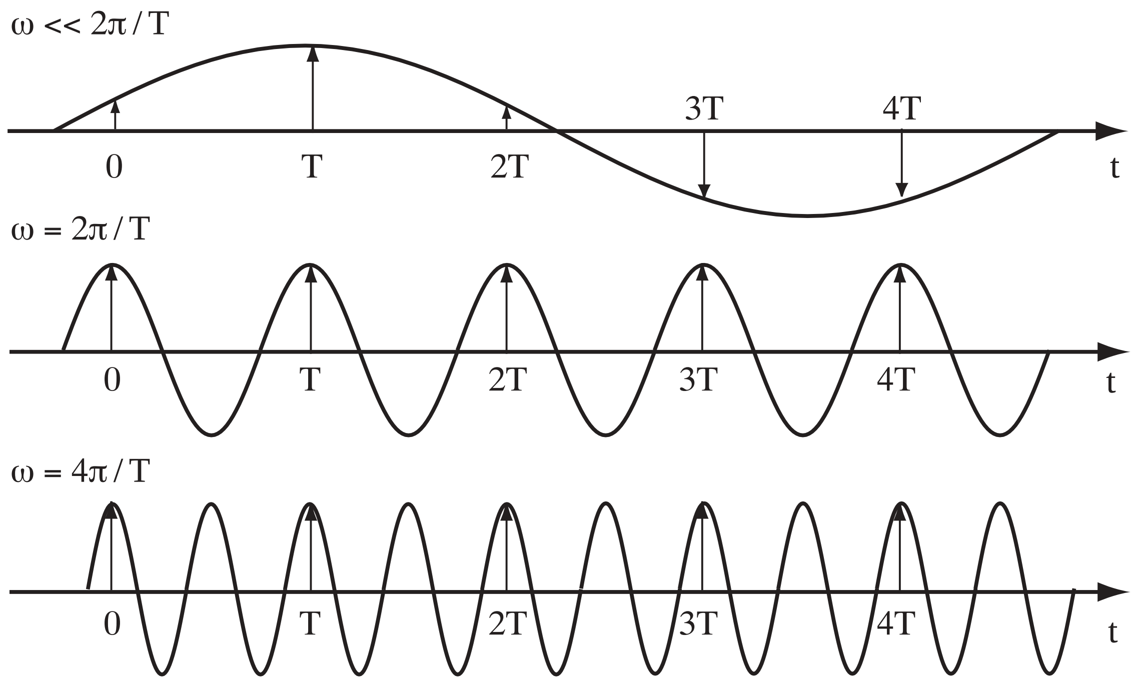
train of unit impulses with period T: $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$



Fourier transform (from table): $P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$



Consequences of Sampling



- Frequencies well below the sampling rate ($\omega << 2\pi/T$) are "sampled" in the sense we expect.
- Frequencies at multiples of the sampling rate ($\omega = 2\pi n/T$) look like they are constant. We can't tell them from DC. These frequencies "alias" as DC.

Frequency domain interpretation of sampling

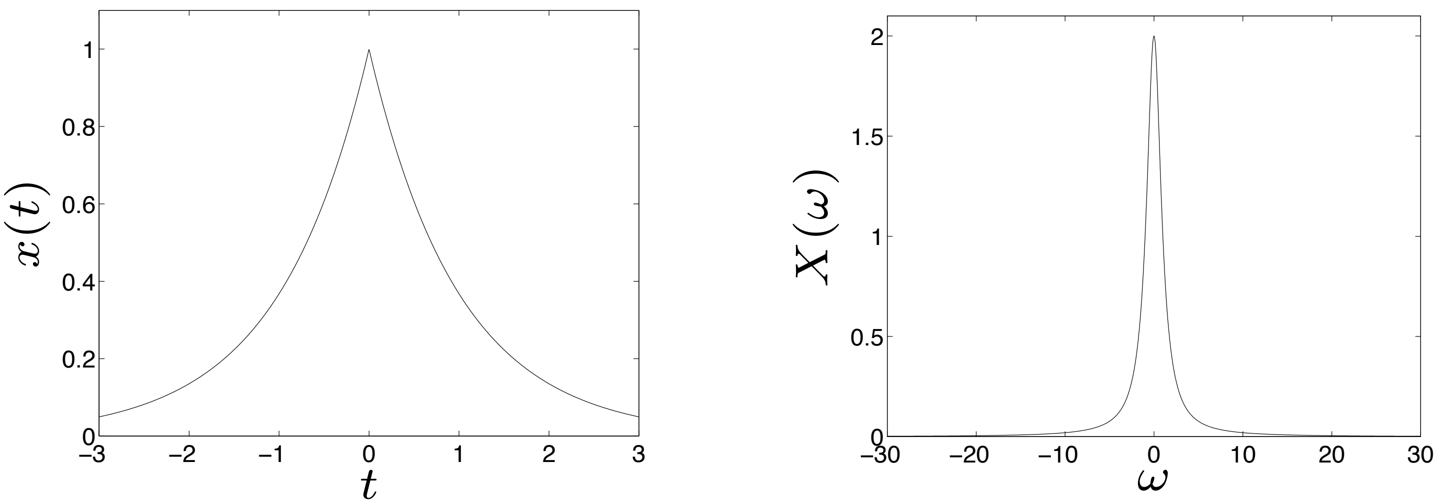
The Fourier transform of the sampled signal is

$$Y = \frac{1}{2\pi}(X * P),$$

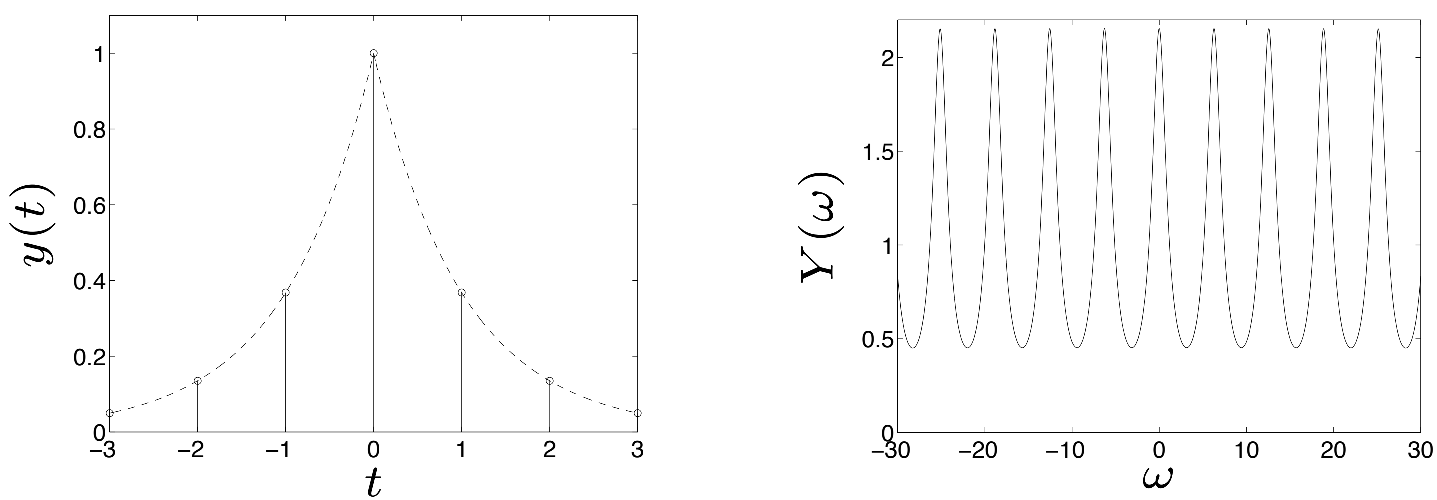
$i.e.$, the convolution of X with $P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$

$$\begin{aligned}
Y(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)P(\omega-\lambda)\, d\lambda \\
&= \frac{1}{T}\int_{-\infty}^{\infty} X(\lambda)\left(\sum_{k=-\infty}^{\infty} \delta(\omega-\lambda-\frac{2\pi k}{T})\right) d\lambda \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda-\frac{2\pi k}{T})\, d\lambda \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega-\frac{2\pi k}{T})
\end{aligned}$$

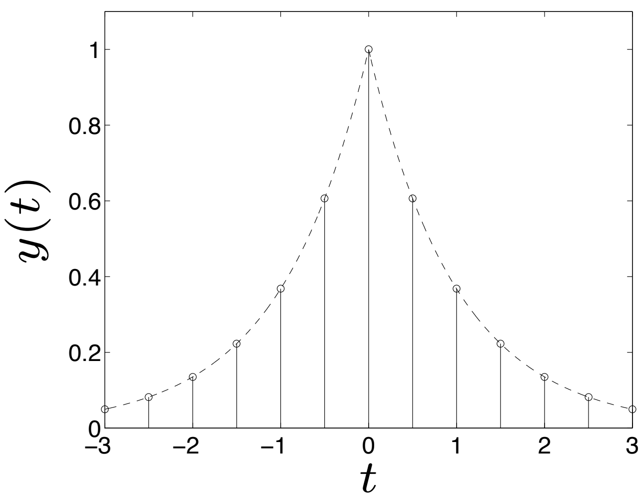
example: sample $x(t) = e^{-|t|}$ at different rates

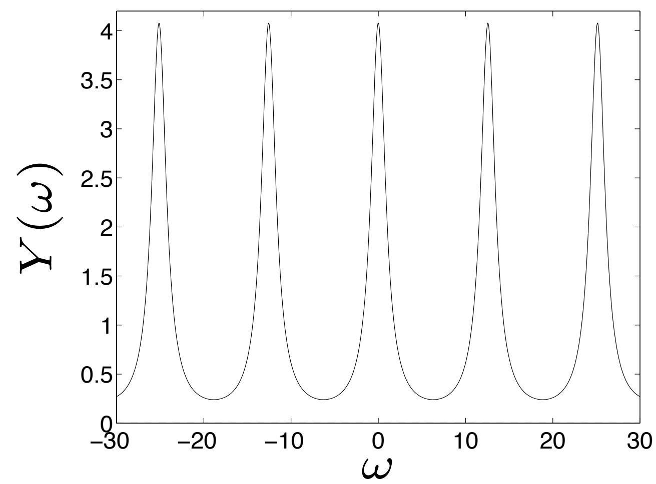


x sampled with $T = 1$ $(2\pi/T = 6.3)$

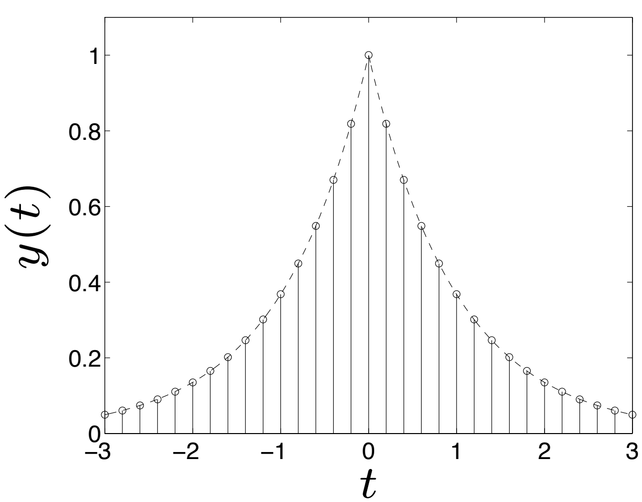


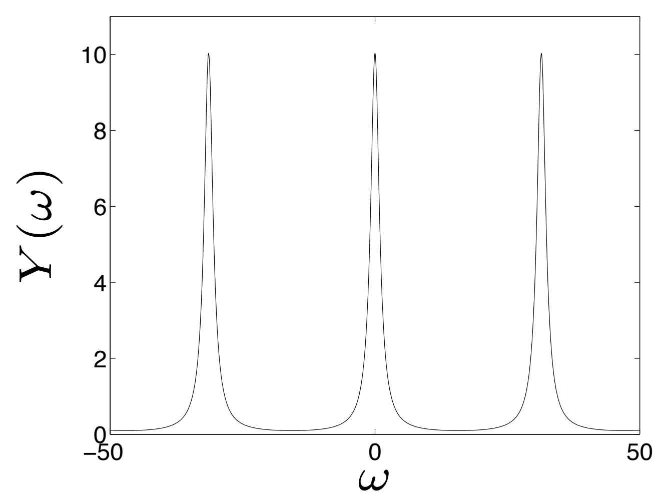
x sampled with $T = 0.5$ $(2\pi/T = 12.6)$





x sampled with $T = 0.2$ $(2\pi/T = 31.4)$

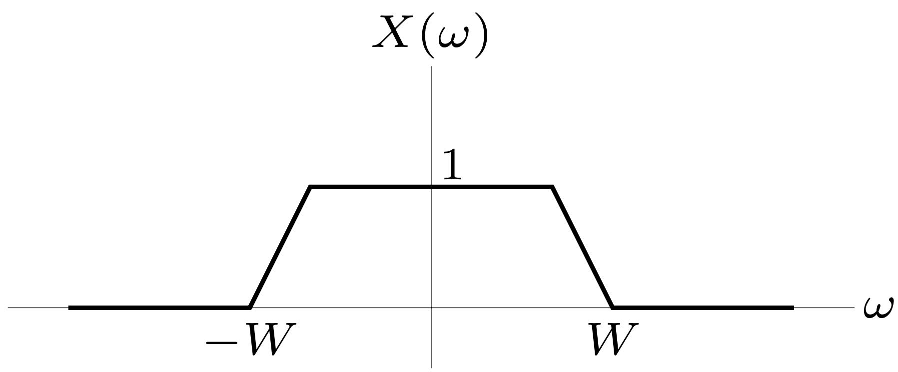




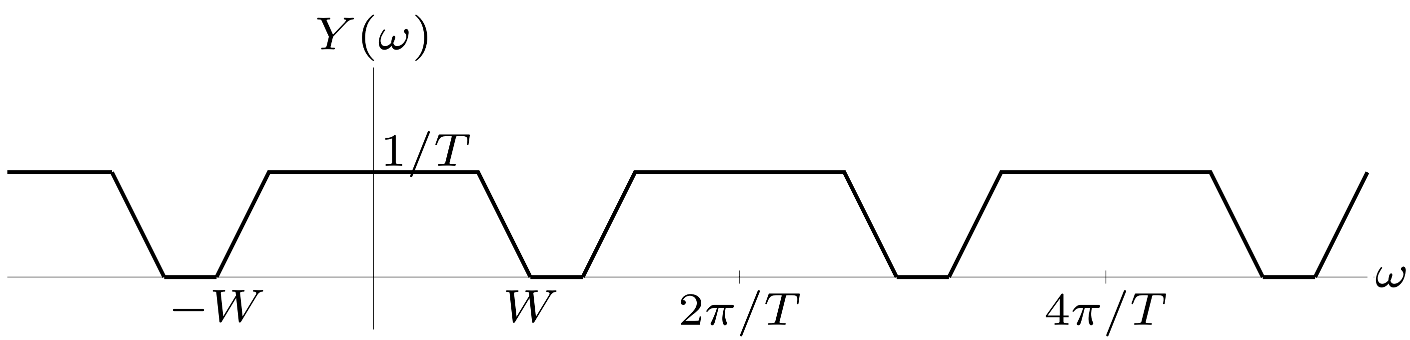
The sampling theorem

can we recover the original signal x from the sampled signal y ?

example: a *band-limited* signal x (with bandwidth W)



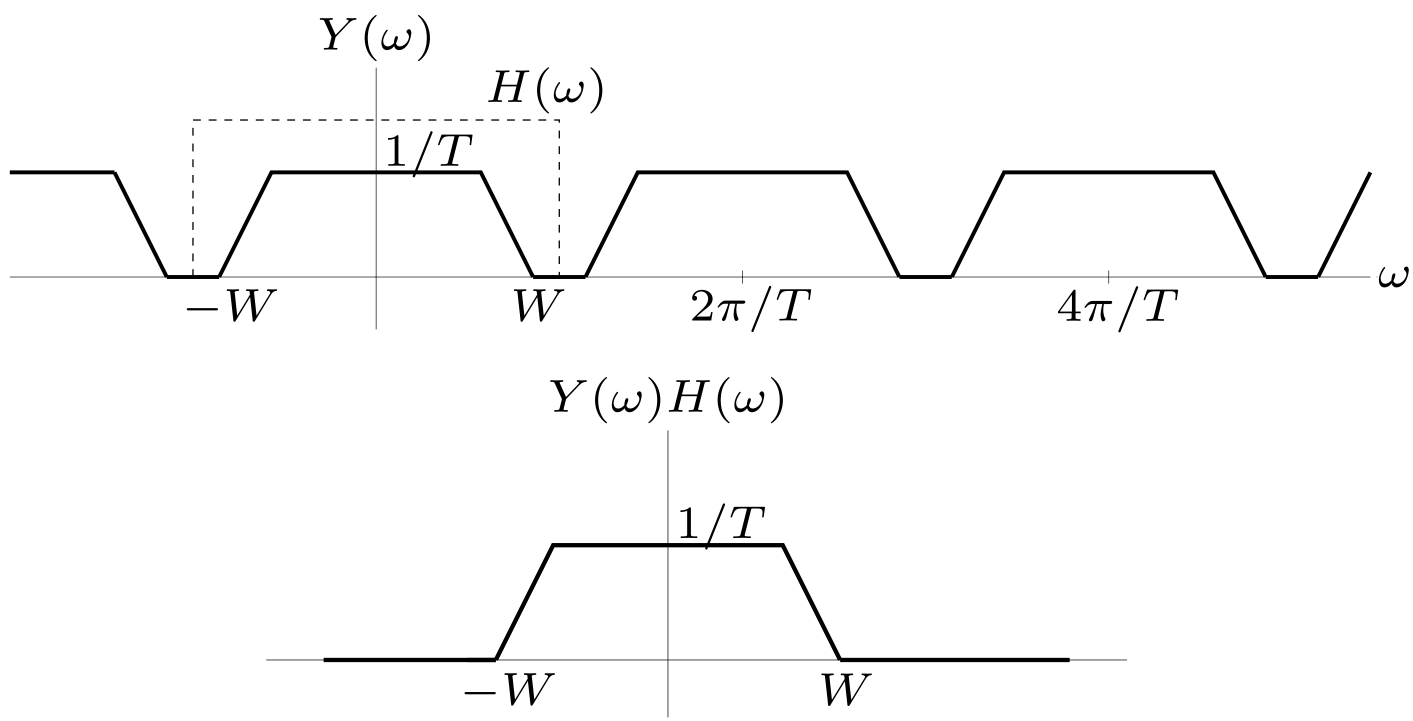
Fourier transform of $y(t) = \sum_k x(kT)\delta(t - kT)$:



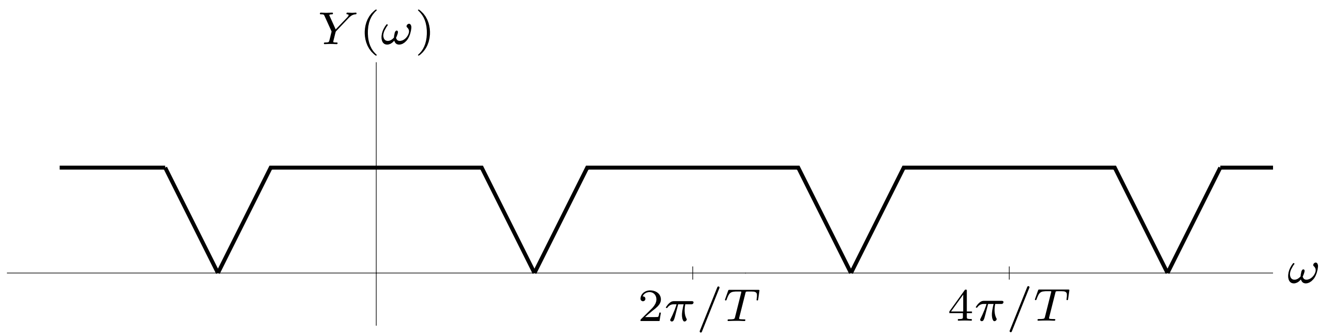
suppose we filter y through an ideal lowpass filter with cutoff frequency ω_c, *i.e.*, we multiply $Y(\omega)$ with

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases}$$

if $W \leq \omega_c \leq 2\pi/T - W$, then the result is $H(\omega)Y(\omega) = X(\omega)/T$, *i.e.*, we recover X exactly

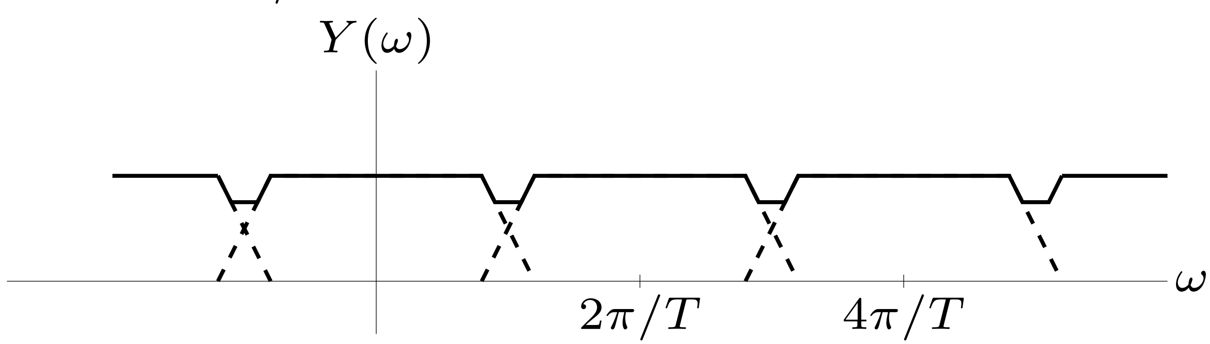


same signal, sampled with $T = \pi/W$



we can still recover $X(\omega)$ perfectly by lowpass filtering with $\omega_c = W$

sample with $T > \pi/W$



$X(\omega)$ cannot be recovered from $Y(\omega)$ by lowpass filtering

the sampling theorem

suppose x is a *band-limited* signal with bandwidth W, $i.e.$,

$$X(\omega) = 0 \text{ for } |\omega| > W$$

and we sample at a rate $1/T$

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$$

then we can recover x from y if $T \leq \pi/W$

- the sampling rate must be at least $1/T = W/\pi$ samples per second (W/π is called the *Nyquist rate*)
- the distortion introduced by sampling below the Nyquist rate is called *aliasing*