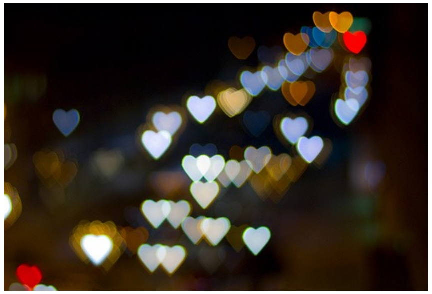
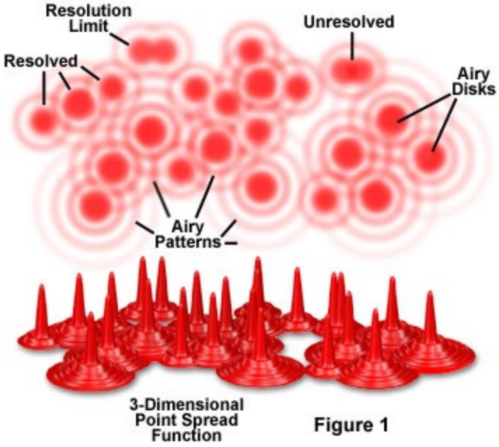
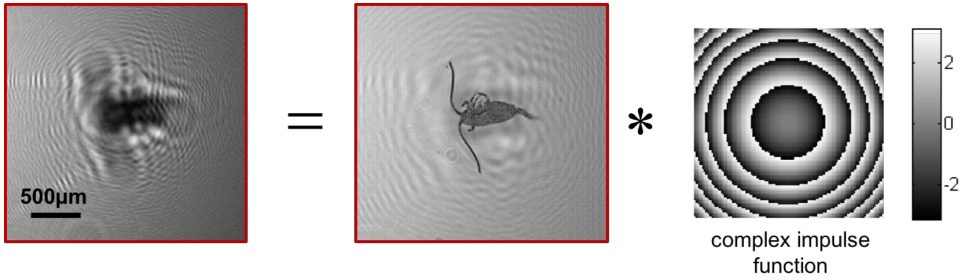


Introduction to Inverse Problem in Imaging

EC 522 Computational Optical Imaging

Lei Tian





Admins

- HW 2 is posted
  - Due 2/21 (Wednesday; after Presidents’ day break)

Mathematical tools & road map

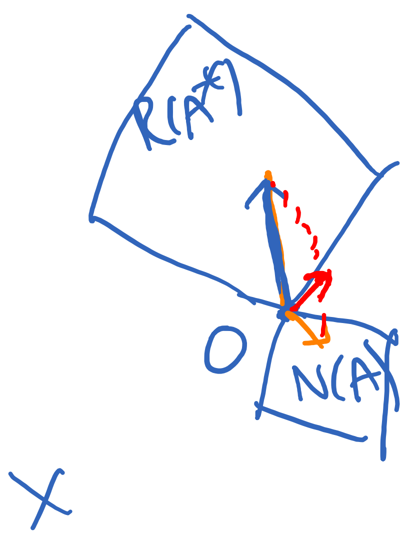
- Vector space (IIP Appx A)
  - Key idea: think about the imaging signals as a *vector*
- Linear operator (IIP Appx B)
  - Key idea: think about imaging process as a linear transformation, i.e. a linear *operator*
  - Later, we will perform discretization and convert the operator into a *matrix*

IIP: Introduction to inverse problems in imaging Appx A

Linear operator

Linear operator

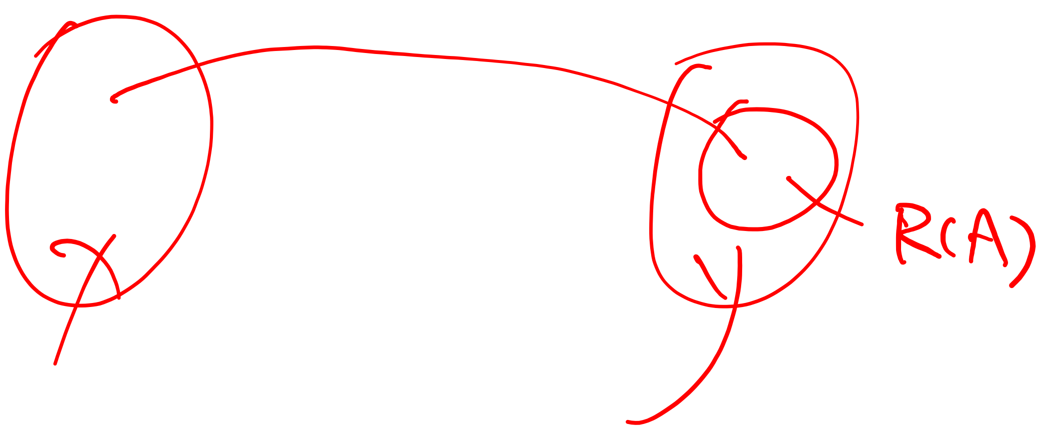
- Linear operator A: $\mathcal{X} \to \mathcal{Y}$ satisfies
  - $A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A(f_1) + \alpha_2 A(f_2)$, for any complex numbers α_1 and α_2
    - Additivity: $A(f_1 + f_2) = A(f_1) + A(f_2)$
    - Scalability: $A(\alpha f) = \alpha A(f)$



Range space*

- The ***Range space*** of a linear operator A: $\mathcal{R}(A)$
  - The set of all elements $g \in \mathcal{Y}$ from $Af = g$

$$\mathcal{R}(A) = \{g = Af \in \mathcal{Y}, f \in \mathcal{X}\}$$

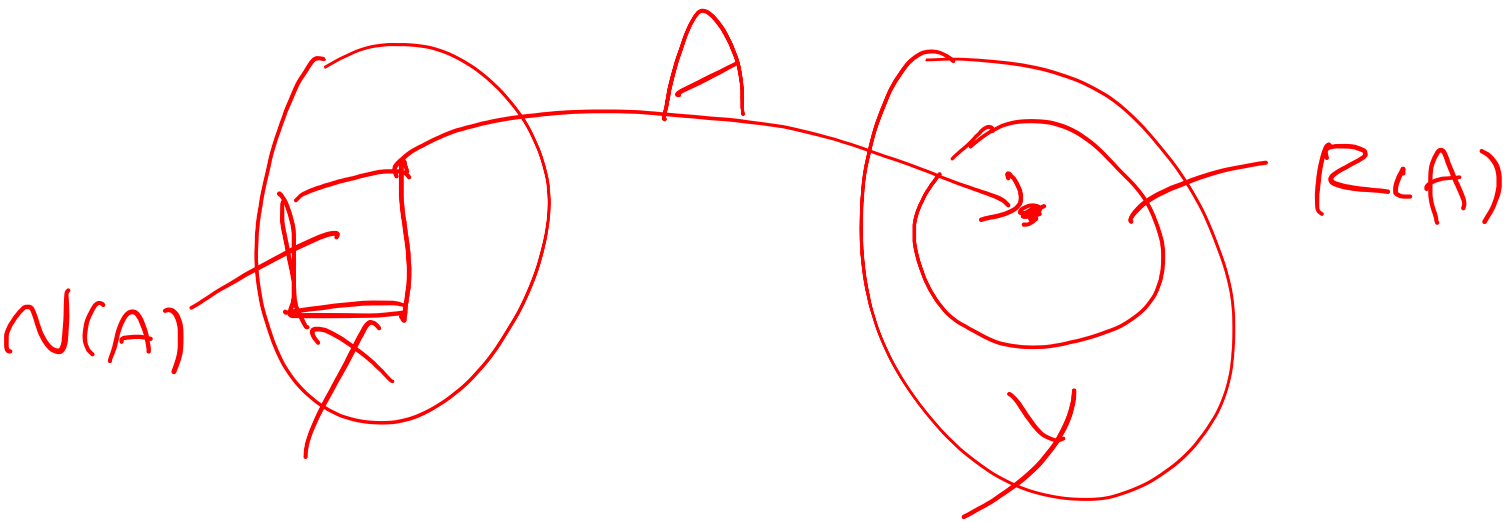


Null space *

» The **null space** of a linear operator A: $\mathcal{N}(A)$

» The set of all elements $f \in \mathcal{X}$ such that $Af = 0$

$$\mathcal{N}(A) = \{f \in \mathcal{X}, Af = 0\}$$

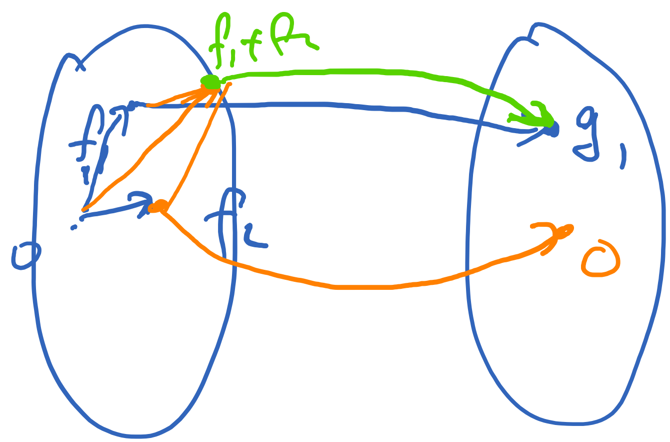


Implication of Null space

$$A(f_1 + f_2) \overset{\text{linear.}}{=} A(f_1) + A(f_2)$$

$$f_1 \notin N(A) \qquad = \underbrace{A(f_1)}_{g_1 \neq 0}$$

$$f_2 \in N(A)$$



Adjoint operator *

» The ***adjoint*** operator ***A*** * **(or** ***A***$^{\mathbf{H}}$**)** of a linear and bounded operator *A*

  » $A^*: \mathcal{Y} \rightarrow \mathcal{X}$ is the adjoint of $A: \mathcal{X} \rightarrow \mathcal{Y}$, when

$$\langle Ax, y\rangle_{\mathcal{Y}} = \langle x, A^* y\rangle_{\mathcal{X}} \text{ for every } x \in \mathcal{X}, y \in \mathcal{Y}$$

A^*

$Ax = g \neq y$

» Generalization of the **Hermitian transpose** (complex conjugate transpose) of a matrix

$\neq x$

$\langle g, y\rangle_{\mathcal{Y}} = \langle x, A^* g\rangle$

x y

Hermitian / adjoint – can be used interchangeably
https://en.wikipedia.org/wiki/Hermitian_adjoint

Example: DFT

$$f \xrightarrow{A = F} g$$

DFT:

$$\boxed{g[m] = \sum_{n=0}^{N-1} f[n]\, e^{-i 2\pi \frac{m \cdot n}{N}}}$$

adjoint

$$\langle Ax, y \rangle_y = \langle x, A^* y \rangle_x$$

Inv. DFT.

$$\langle \underbrace{F}_{= P} \cdot x, y \rangle_y \overset{?}{=} \underbrace{\langle x, \overbrace{F^{-1} y}^{= q} \rangle_x}_{= x^* \cdot q = \sum_{n=0}^{N-1} x^*[n]\, q[n]}$$

$$\langle p, y \rangle_y = p^* \cdot y = \begin{bmatrix} p^*[0] & p^*[1] & \cdots \end{bmatrix} \begin{bmatrix} y[0] \\ y[1] \\ \vdots \end{bmatrix} = \sum_{m=0}^{N-1} p^*[m] \cdot y[m]$$

$$\langle P, y \rangle_y = \sum_{m=0}^{N-1} p^*(m) \cdot y[m]$$

$$= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}mn} \right)^* y[m]$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x^*[n] \, e^{j 2\pi \frac{m \cdot n}{N}} \cdot y[m]$$

$$F^{-1} \cdot y[n] = \ell[n]$$

Example: deconvolution

$$Af(x) = \int_{-\infty}^{\infty} f(x') \cdot h(x-x') = g(x)$$

$$\Updownarrow$$

spectral reprentation

$$Af(x) = \int_{-\infty}^{\infty} \tilde{F}(u) \cdot \tilde{H}(u)\, e^{i2\pi u \cdot x}\, du$$

$$\langle Af, g \rangle = \langle f, A^* g \rangle. \quad A^* g(x) = \int_{-\infty}^{\infty} \tilde{G}(u)\, \tilde{H}^*(u)\, e^{i2\pi ux}\, du$$

Adjoint of a convolution operator

» Adjoint of convolution operator A^*

» $(A^*g)(x) = K^*(-x) * g(x)$

$$= \int K^*(x' - x) g(x') \mathrm{d}x'$$

» Spectral representation

» $(A^*g)(x) = \int \widetilde{K}^*(u)\, \tilde{g}(u) e^{i2\pi xu} \mathrm{d}u$

Proof?

Side note:

- I found it is easier to work with $u = \omega/2\pi$ in FT and IFT, the textbook uses ω.
- Throughout the lecture, we will use the definition in the u-space.

Geometric relation between null space and range space

orthogonal complement

$$\mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}$$
$$\mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$$

$\forall x \in \mathcal{X}$

$x = x_R + x_N$

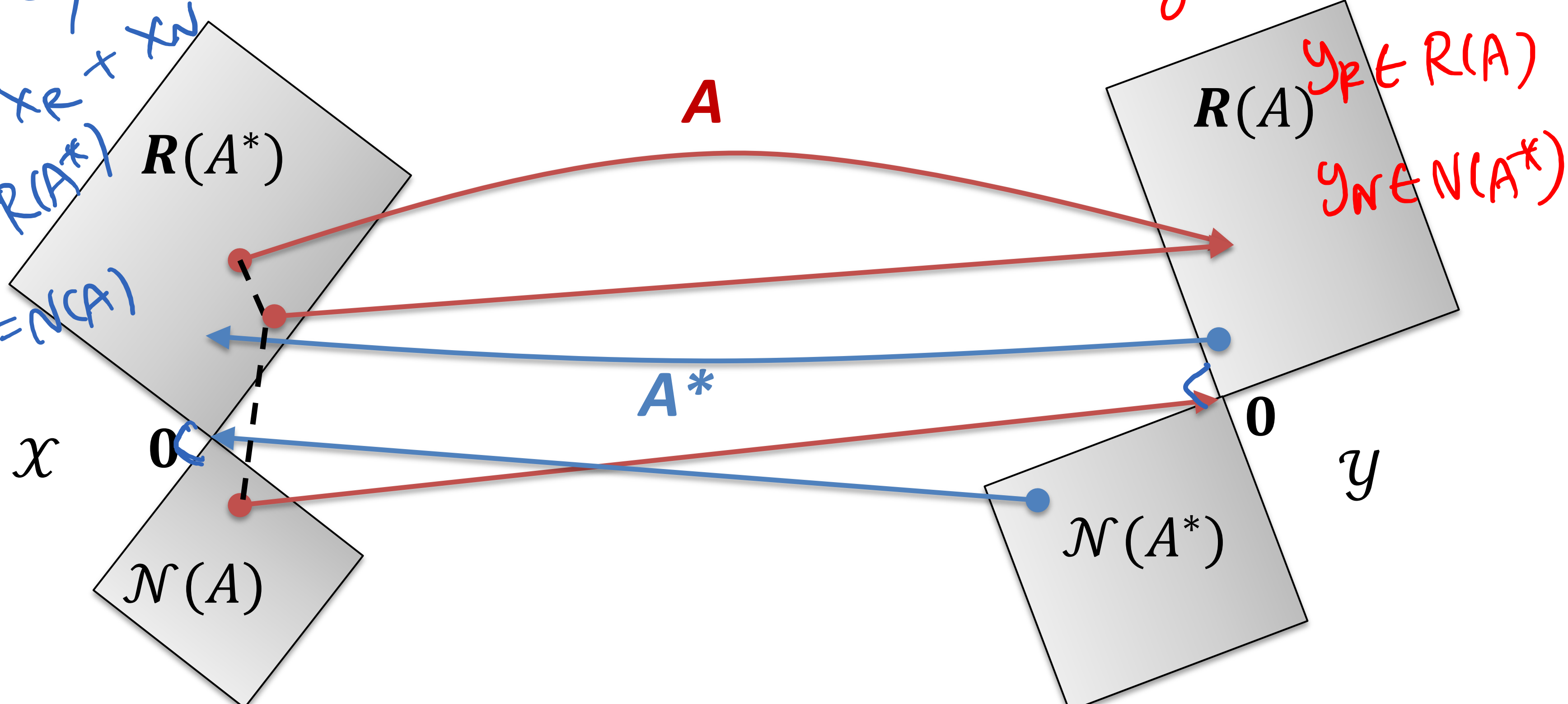
$x_R \in R(A^*)$

$x_N = N(A)$

$y = y_R + y_N$

$y_R \in R(A)$

$y_N \in N(A^*)$



Example: Relation between range and null space of a convolution operator

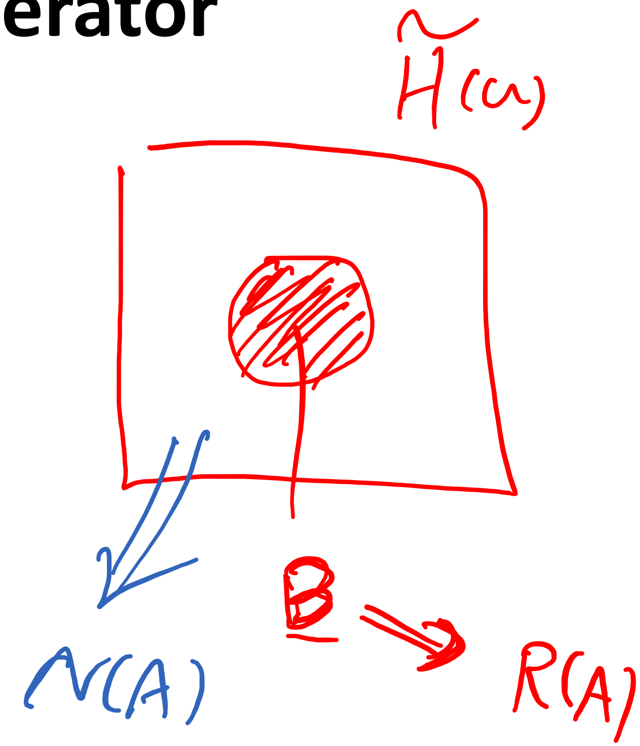
A is a convolution operator

» $f_1 \in \mathcal{R}(A)$

» $f_2 \in \mathcal{N}(A)$

» $f_1 \perp f_2$

and $$\mathcal{N}(A) = \mathcal{R}(A)^{\perp}$$



Why?

Relation between range and null space of a convolution operator

A is a convolution operator

» $f_1 \in \mathcal{R}(A)$ Only contain frequency component $u \in \mathcal{B}$

» $f_2 \in \mathcal{N}(A)$ Only contain frequency component $u \notin \mathcal{B}$

» $f_1 \perp f_2$

and $\mathcal{N}(A) = \mathcal{R}(A)^{\perp}$

Self-adjoint operator.

» If $A = A^*$, A is self-adjoint or Hermitian

eg:

$$Af(x) = \int_{-\infty}^{\infty} \tilde{F}(u) \cdot \tilde{H}(u)\, e^{i 2\pi ux}\, du$$

if $\tilde{H}(u)$ is real-valued

$$A^* g(x) = \int_{-\infty}^{\infty} \tilde{G}(u)\, \tilde{H}^*(u) \cdot e^{i2\pi ux}\, du$$

$\Rightarrow A = A^*$

Properties of Adjoint operator

» The adjoint A^* is unique

$\Rightarrow$ » $(A^*)^* = A$

» The operators AA^* and A^*A are self-adjoint

» If A is invertible, $(A^{-1})^*=(A^*)^{-1}$

$\Rightarrow$ » $(A+B)^* = A^*+B^*$

$\Rightarrow$ » $(BA)^* = A^*B^*$

$(AA^*)^* = (A^*)^* \cdot A^*$

$= A \cdot A^*$

$(BA)^* \cdot g = A^* B^* \cdot g$

Unitary operator

identity

» *A* is unitary if and only if

$$A^{-1} = A^* \text{ or } \underbrace{A^*A = I}$$

e.g. F.T.

» If A is unitary, then $\|Ax\|^2 = \|x\|^2$

Foundations of Signal processing, Chap. 2, pg 47-50

Unitary operator

» Preserve geometry (lengths and angles) when mapping one vector space to another

» A bounded linear operator $A \colon \mathcal{X} \to \mathcal{Y}$ is unitary, when

  » A is invertible

  » A preserves inner product

$$\langle f, h \rangle_{\mathcal{X}} = \langle Af, Ah \rangle_{\mathcal{Y}}, \text{ for every } f, h \in \mathcal{X}$$

~ generalized Parseval's Thm.

Eigenvector and eigenvalue of a linear operator

» An **eigenvector** of a linear operator $A: H \rightarrow H$ is a nonzero vector $v \in H$, such that

$$\boxed{Av = \lambda v}$$

» $\lambda \in \mathbb{C}$ is the **eigenvalue**.

Example: convolution operator

PSF $h(x) \leftrightarrow$ Transfer function (TF) $\hat{H}(u)$

input

$$\boxed{f(x) = e^{i2\pi u x} = v}$$

$$\boxed{Av = \lambda v}$$

$$\overset{v}{Af(x)} = \int f(x') \cdot h(x-x')\, dx'$$

$$= \int h(x') \cdot f(x-x') \cdot dx'$$

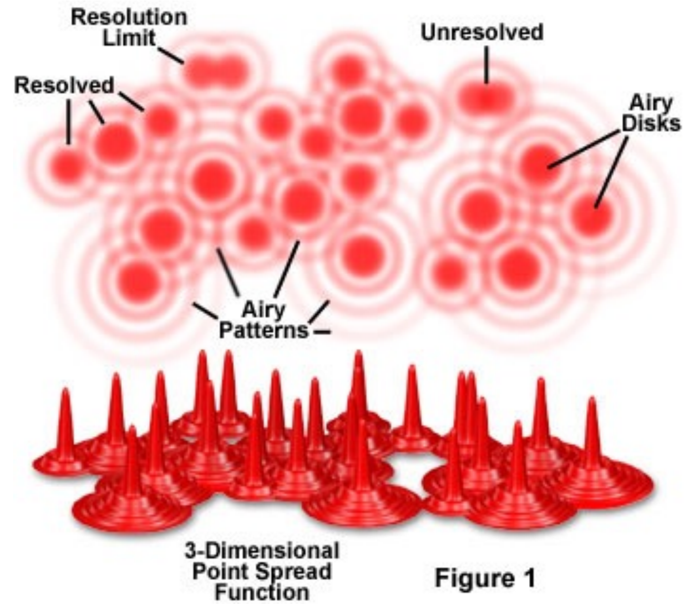
$$= \underline{\int h(x') e^{i2\pi u_0 (x-x')} dx'}$$

$$\cancel{e^{i2\pi u_0 x}}\ H(u_0)$$

$$'' \quad ''$$

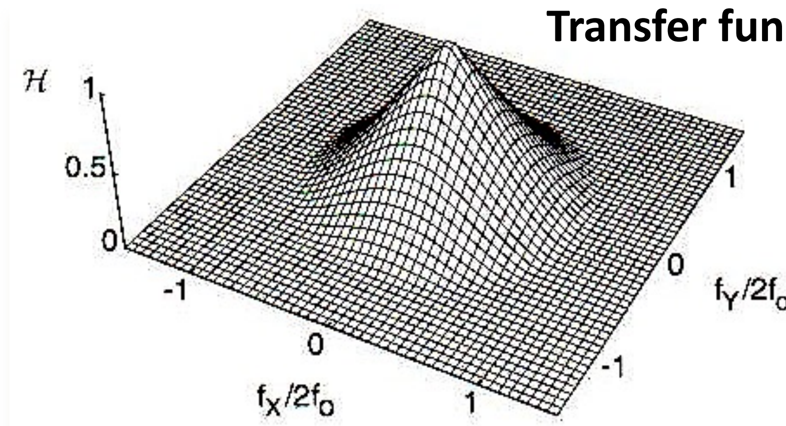
$$H(u_0) \cdot v$$

Example of convolution operator: microscopes

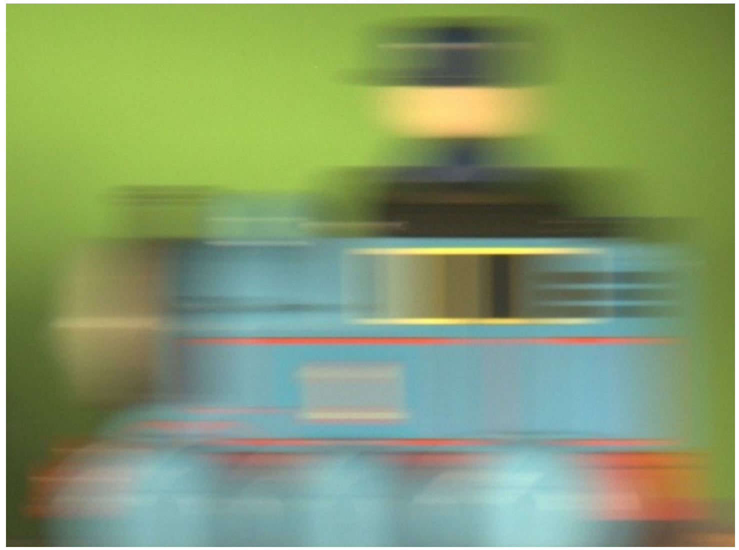


- » Range and null space?
- » Adjoint operator?
- » Inverse operator?

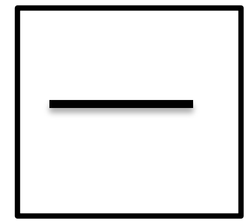
Transfer function



Example: motion blur

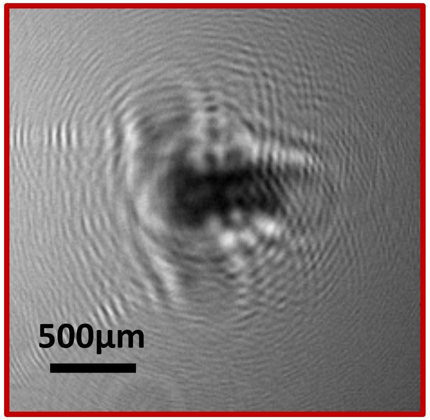


PSF



- What are Object space $\mathcal{X}$ and image space $\mathcal{Y}$?
- What is the operator $\boldsymbol{A}$? linear?
- Find an element in the null space?
  - What's the implication of this?

Example of Shift-invariant system: holography

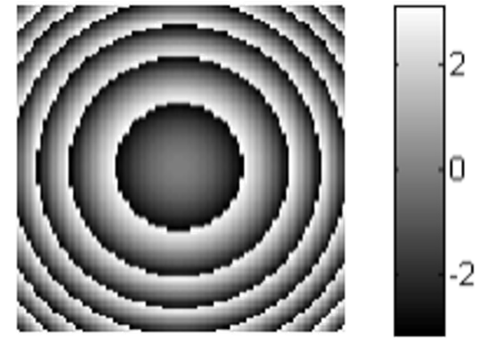


g_{out}

=

g_{in}

*



$$h(x,y) \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{\left(x^2+y^2\right)}{2z}}$$

complex PSF

Transfer function $H(u,v) = e^{i2\pi z/\lambda} \exp\{-i\lambda z(u^2+v^2)\}$

- » Range and null space?
- » Adjoint operator?
- » Inverse operator?

Application: back-propagation using adjoint operator = inverse operator!?

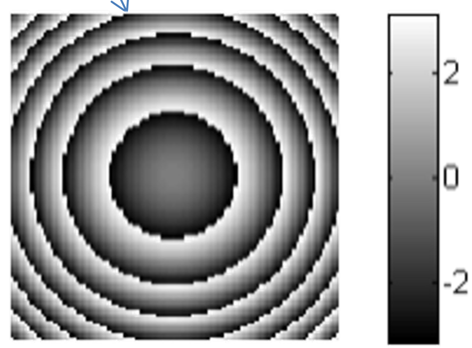
just make z→-z? Why?

$$g(x,y) = A(x,y)\,\mathrm{e}^{i\varphi(x,y)}$$

$h_z(x,y)$

$$g'(x',y') = g(x,y) \otimes h_z(x,y)$$

H(x,y) has complex point spread function (PSF) and transfer function



$$h_z(x,y) \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{\left(x^2+y^2\right)}{2z}}$$

Transfer function $H(u,v) = e^{i2\pi z/\lambda} \exp\{-i\lambda z(u^2+v^2)\}$