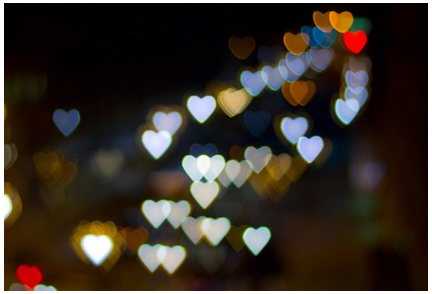
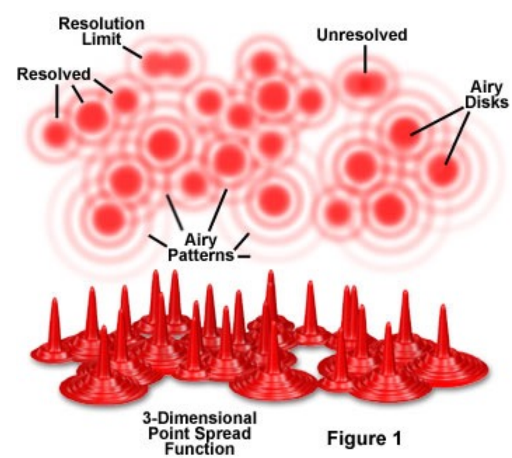


Introduction to Inverse Problem in Imaging

EC 522 Computational Optical Imaging

Lei Tian

complex impulse function



Admins

- HW 2 is posted
  - Due 2/21 (Wednesday; after Presidents’ day break)

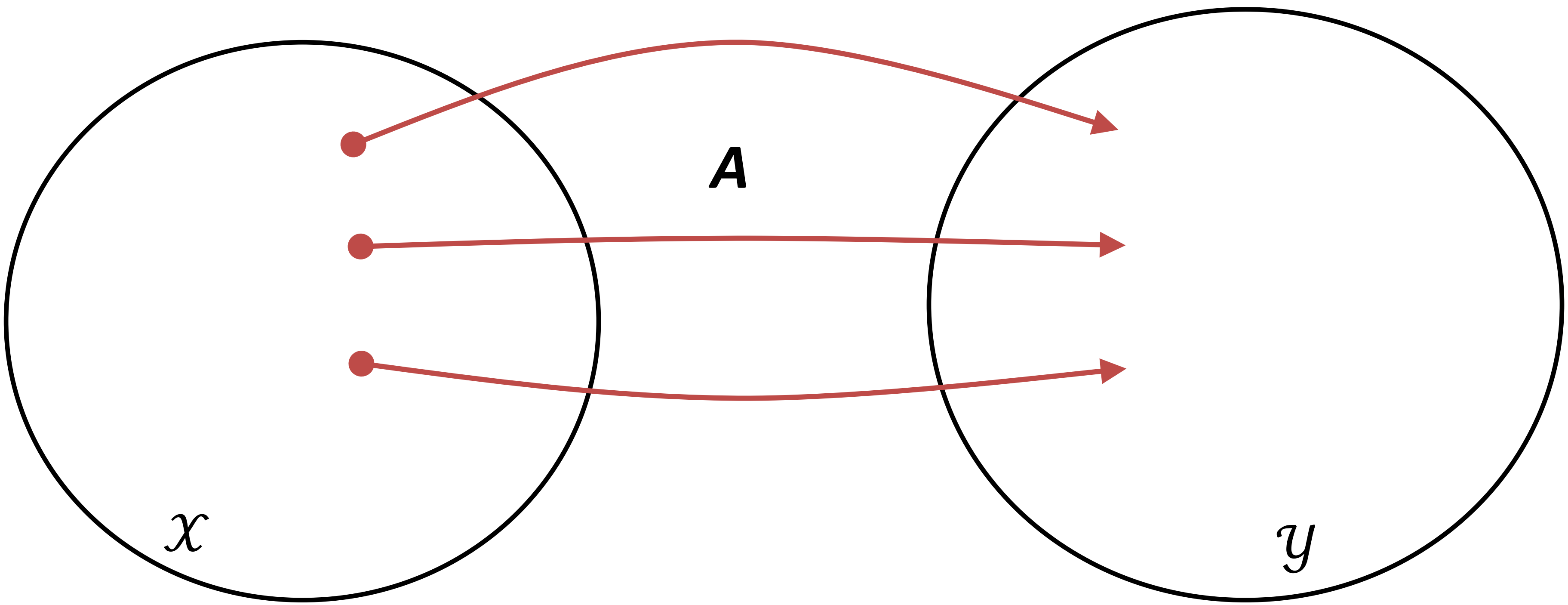
Mathematical tools & road map

- » Vector space (IIP Appx A)
  - » Key idea: think about the imaging signals as a ***vector***
- » Linear operator (IIP Appx B)
  - » Key idea: think about imaging process as a linear transformation, i.e. a linear ***operator***
  - » Later, we will perform discretization and convert the operator into a ***matrix***

IIP: Introduction to inverse problems in imaging Appx A

Linear operator

Linear operator



- Linear operator A maps the input to the output:
  - $A: \mathcal{X} \rightarrow \mathcal{Y}$

Linear operator

- Linear operator A: $\mathcal{X} \rightarrow \mathcal{Y}$ satisfies
  - $A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A(f_1) + \alpha_2 A(f_2)$, for any complex numbers α_1 and α_2
    - Additivity: $A(f_1 + f_2) = A(f_1) + A(f_2)$
    - Scalability: $A(\alpha f) = \alpha A(f)$
  - Simplified notation as
    - $Af = g$

Linear operator: example

- *Fourier transform* is linear operator
  - $\mathcal{F}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{F}(f_1) + \alpha_2 \mathcal{F}(f_2)$

- CTFT: $\mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$
- DFT: $\mathbb{C}^N \to \mathbb{C}^N$

Fourier transform can be treated as linear operator!

Range space*

» The ***Range space*** of a linear operator A: $\mathcal{R}(A)$

» The set of all elements $g \in \mathcal{Y}$ from $Af = g$

$$\mathcal{R}(A) = \{g = Af \in \mathcal{Y}, f \in \mathcal{X}\}$$

» $\mathcal{R}(A)$ is a subspace in $\mathcal{Y}$

* plays a central role in inverse problem

Null space *

» The **null space** of a linear operator A: $\mathcal{N}(A)$

» The set of all elements $f \in \mathcal{X}$ such that $Af = 0$

$$\mathcal{N}(A) = \{f \in \mathcal{X}, Af = 0\}$$

» $\mathcal{N}(A)$ is a subspace in $\mathcal{X}$

* plays a central role in inverse problem

LSI system and Convolution operator

Convolution operator

- Convolution operator *A*
  - $$(Af)(x) = K(x) * f(x) = \int K(x - x')f(x')\mathrm{d}x'$$
- Spectral representation
  - $$(Af)(x) = \int \widetilde{K}(u)\, \tilde{f}(u) e^{i2\pi xu} \mathrm{d}u$$
- Properties: linear & bounded

Why?

Convolution operator

» Properties: bounded

» Proof:

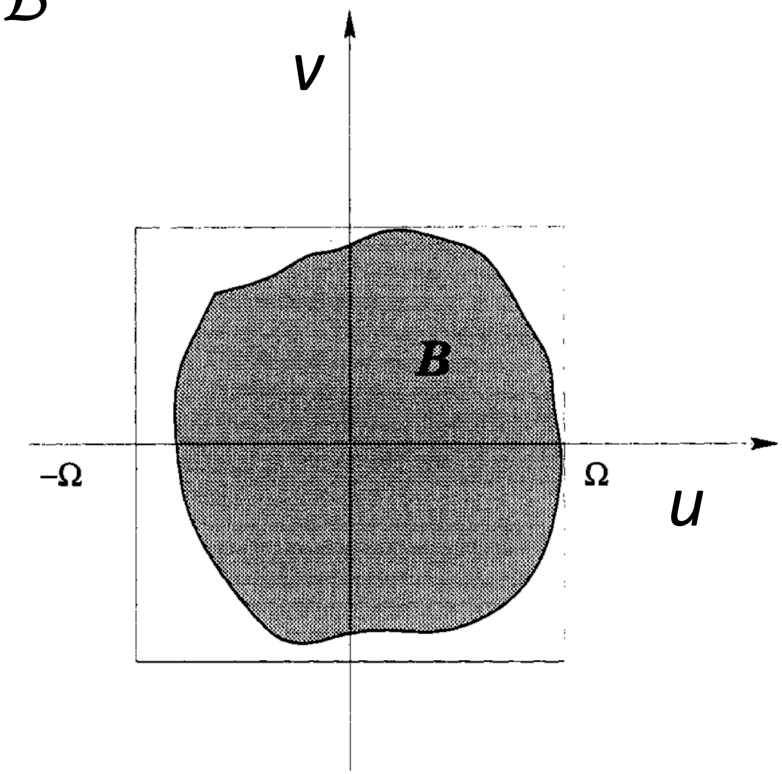
  » Existence of FT

$$\hat{K}_{max} = \max_{\omega} |\hat{K}(\omega)|$$

$$\Longrightarrow \quad \|Af\| \leq \hat{K}_{max} \|f\|.$$

Bandlimited convolution operator/system

» Frequency Support (Band): $\mathcal{B}$



» Bandlimited system

» $\widetilde{K}(u) = 0$, for $u \notin \mathcal{B}$

Range of a convolution operator

» Range of convolution operator *A*, $\mathcal{R}(A)$

» $\mathcal{R}(A)$contains all the bandlimited functions with a band that is contained in $\mathcal{B}$:

» $\mathcal{R}(A) \subset \{g \in L^2 | \tilde{g}(u) = 0, u \notin \mathcal{B}\}$

The Range of *A* is set by $\mathcal{B}$

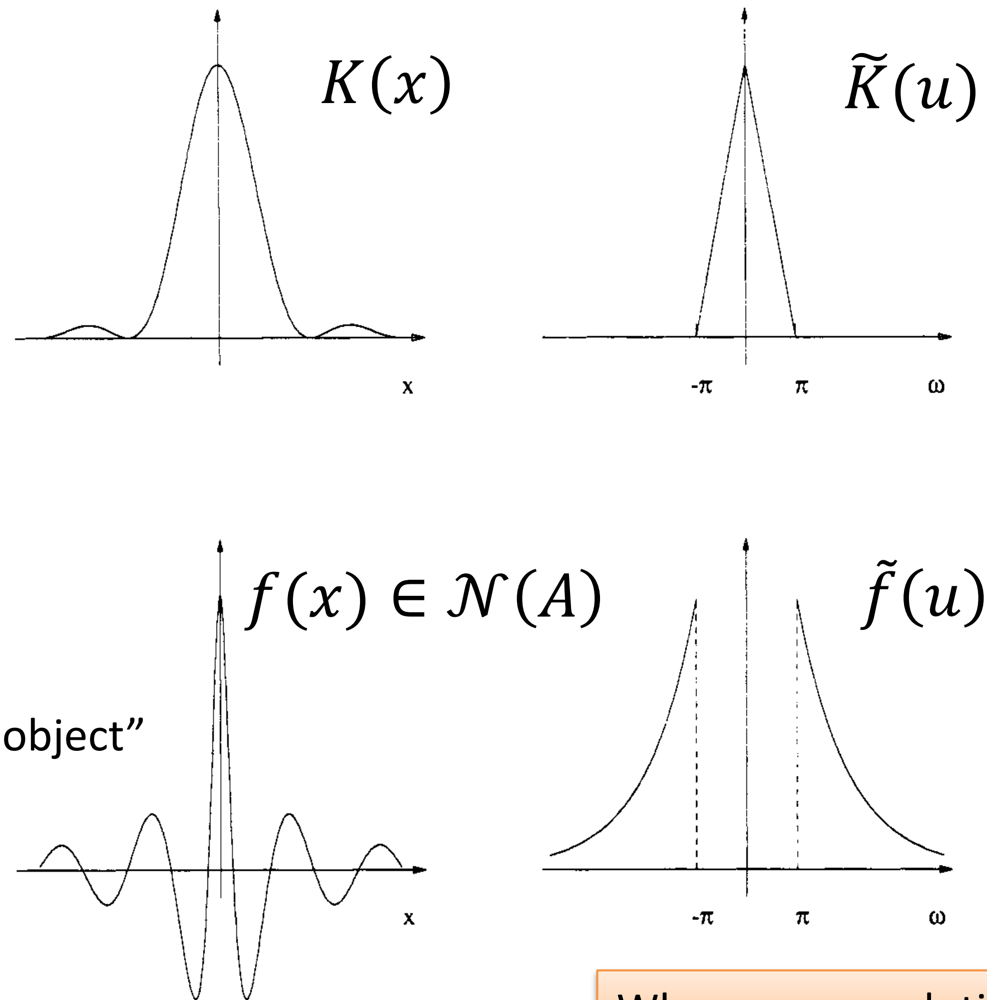
Side note:

- I found it is easier to work with $u = \omega/2\pi$ in FT and IFT, the textbook uses ω.
- Throughout the lecture, we will use the definition in the *u*-space.

Null space of a convolution operator

- Null space of convolution operator $A, \mathcal{N}(A)$
  - $\mathcal{N}(A)$ contains all the functions that satisfy
    - $\mathcal{N}(A) \supset \{f \in L^2 | \tilde{f}(u) = 0, u \in \mathcal{B}\}$
  - "invisible object"?

Example: Null space of a convolution operator



“invisible object”

Why super-resolution is hard?

Introduction to inverse problems in imaging Chap 2

Linear operator (cont'd)

- Linear operator *A* is ***bounded*, if**
  - there exists a constant M such that, for every *f*

$$\|Af\|_{\mathcal{Y}} \le M \|f\|_{\mathcal{X}}$$

- The operator ***norm*** of *A:*

$$\|A\| = \sup_{f \in \mathcal{X}} \frac{\|Af\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}}$$

supremum = the least upper bound

- The operator *A* is ***continuous**, if*
  - $\|f_n - f\|_{\mathcal{X}} \to 0$ then $\|Af_n - Af\|_{\mathcal{Y}} \to 0$
  - Iff *A* is bounded

Inverse operator

» A bounded linear operator A: $\mathcal{X} \rightarrow \mathcal{Y}$ is *invertible* if there exists a bounded linear operator B: $\mathcal{Y} \rightarrow \mathcal{X}$ such that

$$BAx = x \text{ for every } x \in \mathcal{X}, \text{ and}$$

$$ABy = y \text{ for every } y \in \mathcal{Y}$$

» *Theoretical useful, in practice, inverse rarely exist*

Inverse of a convolution operator

- If $\widetilde{K}(u)$ has a support coincides with the whole frequency space

Non-zero everywhere except for isolated point

- The inverse operator A^{-1}
  - $(A^{-1}g)(x) = \frac{1}{2\pi} \int \frac{\tilde{g}(u)}{\widetilde{K}(u)}\ e^{i2\pi xu} \mathrm{d}u$
  - *Not* bounded if $\widetilde{K}(u)$ contains zeros.

Adjoint operator *

- The ***adjoint*** operator ***A** (or *A*^H^)** of a linear and bounded operator *A*
  - $A^*: \mathcal{Y} \rightarrow \mathcal{X}$ is the adjoint of $A: \mathcal{X} \rightarrow \mathcal{Y}$, when

$$\langle Ax, y\rangle_{\mathcal{Y}} = \langle x, A^* y\rangle_{\mathcal{X}} \text{ for every } x \in \mathcal{X}, y \in \mathcal{Y}$$

- If $A = A^*$, A is self-adjoint or Hermitian
- Generalization of the **Hermitian transpose** (complex conjugate transpose) of a matrix

*** important in inverse problem**

Exercise: Adjoint operator

What is the adjoint of Fourier transform operator?

Example: adjoint for LSI system

Adjoint of a convolution operator

» Definition of Adjoint operator:

$$\langle Ax, y \rangle_{\mathcal{Y}} = \langle x, A^* y \rangle_{\mathcal{X}}$$

» Use definition of the convolution operator

$$\begin{aligned}
\langle Ax, y \rangle_{\mathcal{Y}} &= \frac{1}{(2\pi)^q} \int \hat{K}(\omega) \hat{f}(\omega) \hat{g}^*(\omega) d\omega \\
&= \frac{1}{(2\pi)^q} \int \hat{f}(\omega) \left[\hat{K}^*(\omega) \hat{g}(\omega) \right]^* d\omega \\
&= (f, A^* g)
\end{aligned}$$

$$\Longrightarrow \quad (A^* g)(\boldsymbol{x}) = \frac{1}{(2\pi)^q} \int \hat{K}^*(\omega) \hat{g}(\omega) e^{i\boldsymbol{x}\cdot\omega} d\omega.$$

Properties of Adjoint operator

» The adjoint A* is unique

» (A*)* = A

» The operators AA* and A*A are self-adjoint

» $\|A^*\| = \|A\|$

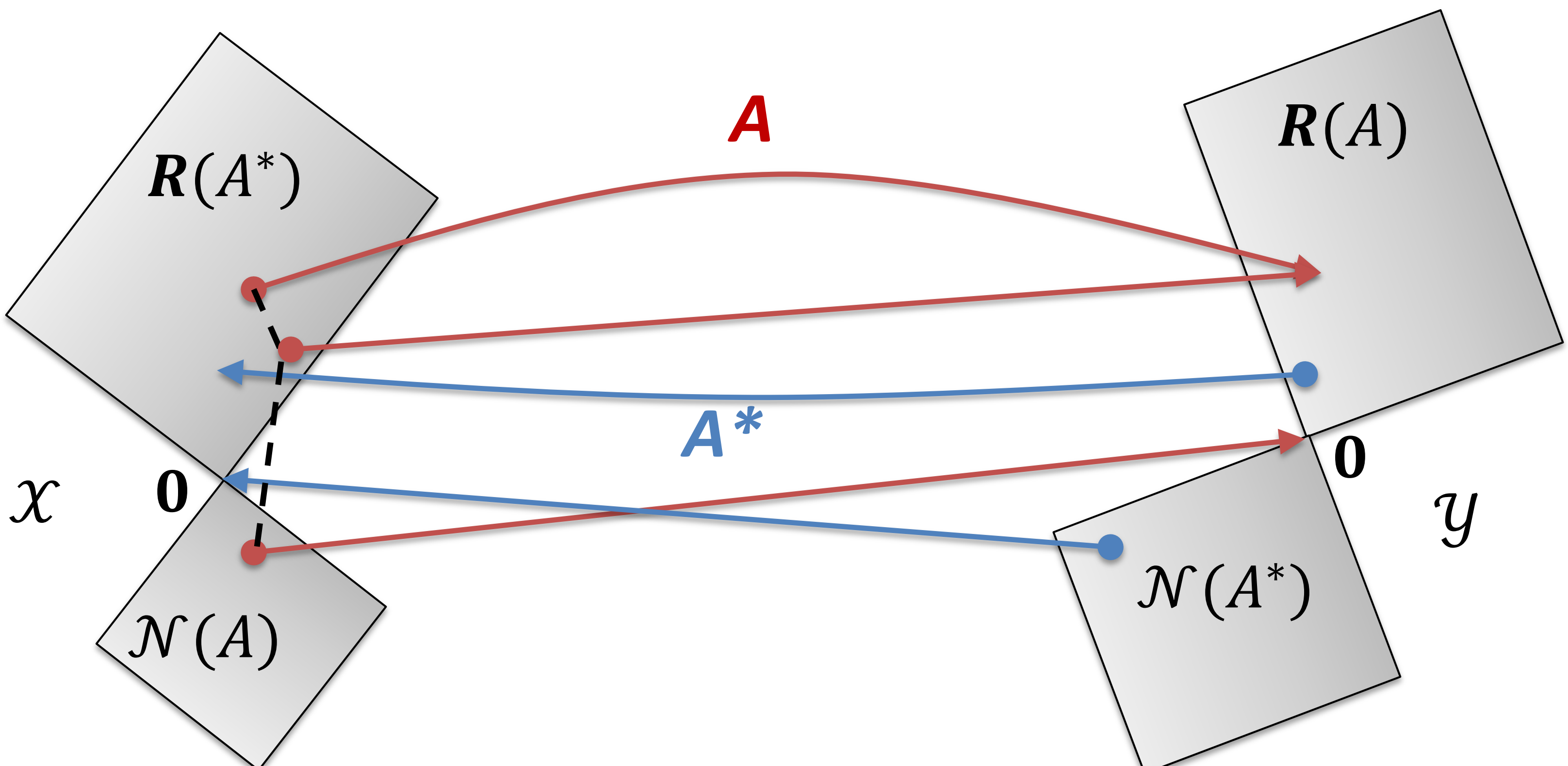
» If A is invertible, $(A^{-1})^*=(A^*)^{-1}$

» $(A+B)^*=A^*+B^*$ $\boxed{\text{why}}$

» $(BA)^*=A^*B^*$ $\boxed{\text{why}}$

Geometric relation between null space and range space

$$\boxed{\begin{aligned}\mathcal{N}(A) &= \mathcal{R}(A^*)^\perp \\ \mathcal{N}(A^*) &= \mathcal{R}(A)^\perp\end{aligned}}$$



Relation between null space and range

» Proof

Proof B.1. We prove the first relation (B.8). From the definition (B.6) of the adjoint operator it follows that, if $f \in \mathcal{N}(A)$, i.e. $Af = 0$, then the l.h.s. of the equation (B.6) is zero for any g, so that f is orthogonal to all elements A^*g, i.e. to $\mathcal{R}(A^*)$. This result implies that $\mathcal{N}(A)$ is contained in $\mathcal{R}(A^*)^\perp$. On the other hand, if $f \in \mathcal{R}(A^*)^\perp$, then the r.h.s. of equation (B.6) is zero for any g and therefore Af is orthogonal to all elements of $\mathcal{Y}$. But this implies $Af = 0$, i.e. $f \in \mathcal{N}(A)$. It follows that $\mathcal{R}(A^*)^\perp$ is contained in $\mathcal{N}(A)$. By combining the two inclusions, we get $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp$. The second relation in (B.8) is obtained from the first one by simply exchanging A with A^*. $\square$.

Example: Relation between range and null space of a convolution operator

A is a convolution operator

- $f_1 \in \mathcal{R}(A)$
- $f_2 \in \mathcal{N}(A)$

- $f_1 \perp f_2$

and $$\mathcal{N}(A) = \mathcal{R}(A)^{\perp}$$

Why?

Relation between range and null space of a convolution operator

A is a convolution operator

- » $f_1 \in \mathcal{R}(A)$ Only contain frequency component $u \in \mathcal{B}$
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Adjoint of a convolution operator

- Adjoint of convolution operator A^*
  - $$(A^*g)(x) = K^*(-x) * g(x) = \int K^*(x' - x) g(x') \mathrm{d}x'$$

- Spectral representation
  - $$(A^*g)(x) = \int \widetilde{K}^*(u)\ \tilde{g}(u) e^{i2\pi xu} \mathrm{d}u$$

Proof?

Side note:

- I found it is easier to work with $u = \omega/2\pi$ in FT and IFT, the textbook uses ω.
- Throughout the lecture, we will use the definition in the u-space.

Unitary operator

- Preserve geometry (lengths and angles) when mapping one vector space to another
- A bounded linear operator $A\colon \mathcal{X} \to \mathcal{Y}$ is unitary, when
  - A is invertible
  - A preserves inner product

$$\langle f, h\rangle_{\mathcal{X}} = \langle Af, Ah\rangle_{\mathcal{Y}}, \text{ for every } f, h \in \mathcal{X}$$

Foundations of Signal processing, Chap. 2, pg 47-50

Unitary operator

» A is unitary if and only if

$$A^{-1} = A^{*} \text{ or } A^{*}A = I$$

» If A is unitary, then $\|Ax\|^2 = \|x\|^2$

Eigenvector and eigenvalue of a linear operator

- An **eigenvector** of a linear operator *A: H→ H* is a nonzero vector $v \in H$, such that
$$Av = \lambda v$$
- $\lambda \in \mathbb{C}$ is the **eigenvalue**.

Example: LSI system

Eigenvector/eigenvalue of self-adjoint operator

» All eigenvalues are *real*

» Eigenvectors corresponding to distinct eigenvalues are *orthogonal*

Definite linear operator

- Consider operator $A: H \to H$
- Positive semidefinite: $A \geq 0$, when
  - $(Ax, x) \geq 0$, for all $x \in H$
- Positive definite: $A > 0$, when
  - $(Ax, x) > 0$, for all $x \in H$

A summary of
LSI system and Convolution operator

Convolution operator

» Convolution operator *A*

» $(Af)(x) = K(x) * f(x)$
$= \int K(x - x')f(x')\mathrm{d}x'$

» Spectral representation

» $(Af)(x) = \int \widetilde{K}(u)\, \tilde{f}(u) e^{i2\pi xu} \mathrm{d}u$

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Why?

Convolution operator

- Properties: bounded
- Proof:
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$$\hat{K}_{max} = \max_{\omega} |\hat{K}(\omega)|$$

$$\longrightarrow \quad \|Af\| \leq \hat{K}_{max} \|f\|.$$

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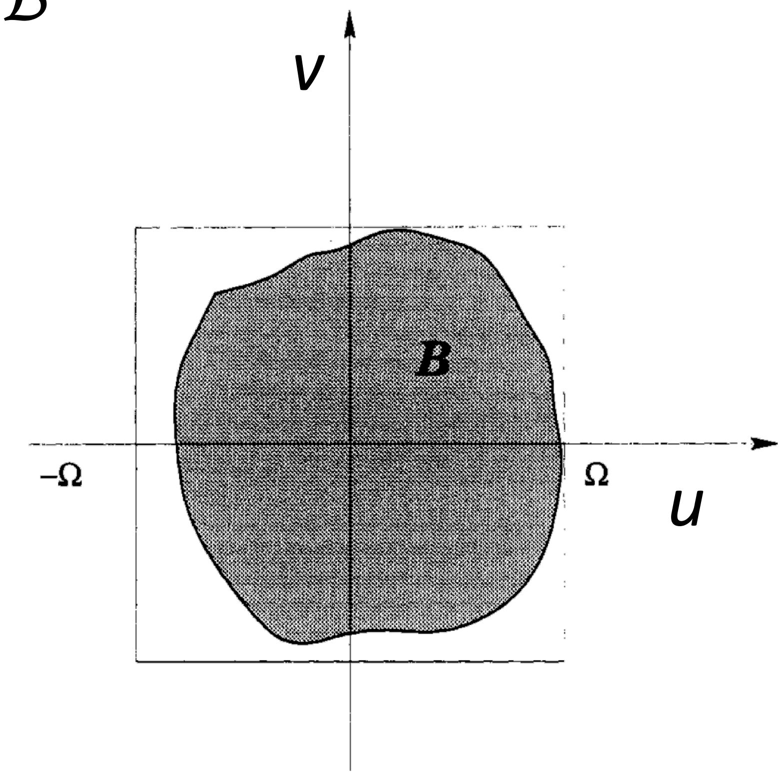
» Bandlimited system

» $\widetilde{K}(u) = 0$, for $u \notin \mathcal{B}$



Introduction to inverse problems in imaging Chap 2

Range of a convolution operator

» Range of convolution operator A, $\mathcal{R}(A)$

» $\mathcal{R}(A)$contains all the bandlimited functions with a band that is contained in $\mathcal{B}$:

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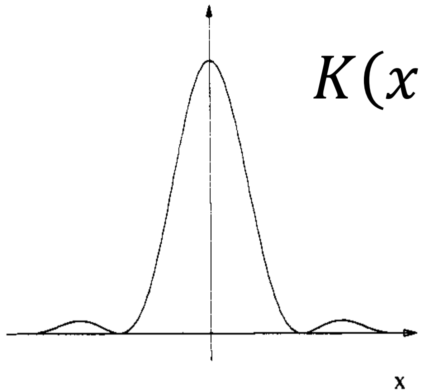
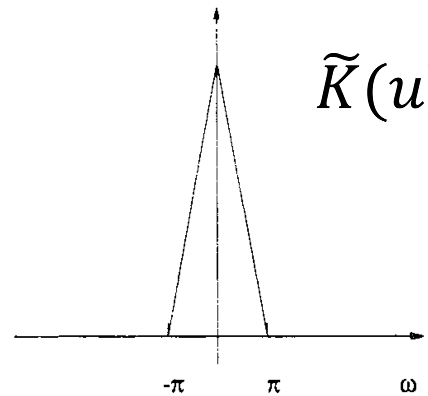
Example: Null space of a convolution operator



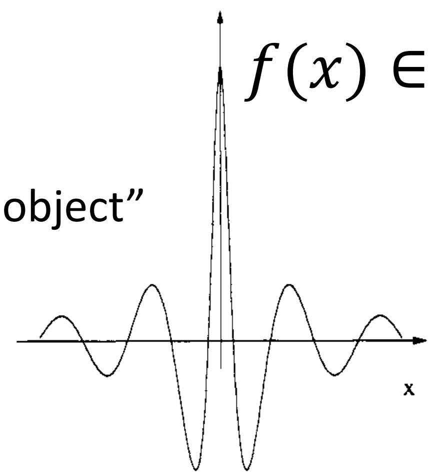
$K(x)$

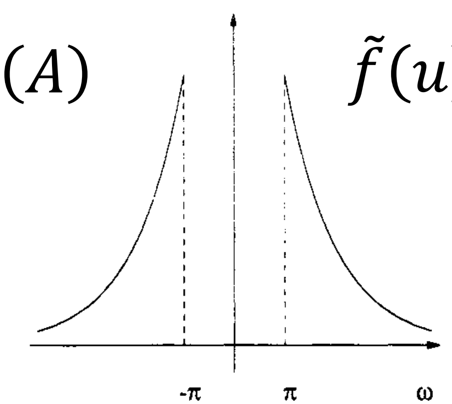


$\tilde{K}(u)$

$f(x) \in \mathcal{N}(A)$

"invisible object"





$\tilde{f}(u)$

Why super-resolution is hard?

Introduction to inverse problems in imaging Chap 2

Relation between range and null space of a convolution operator

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$$\Longrightarrow \quad (A^*g)(x) = \frac{1}{(2\pi)^q}\int \hat{K}^*(\omega)\hat{g}(\omega)e^{ix\cdot\omega}d\omega.$$

Inverse of a convolution operator

» If $\widetilde{K}(u)$ has a support coincides with the whole frequency space

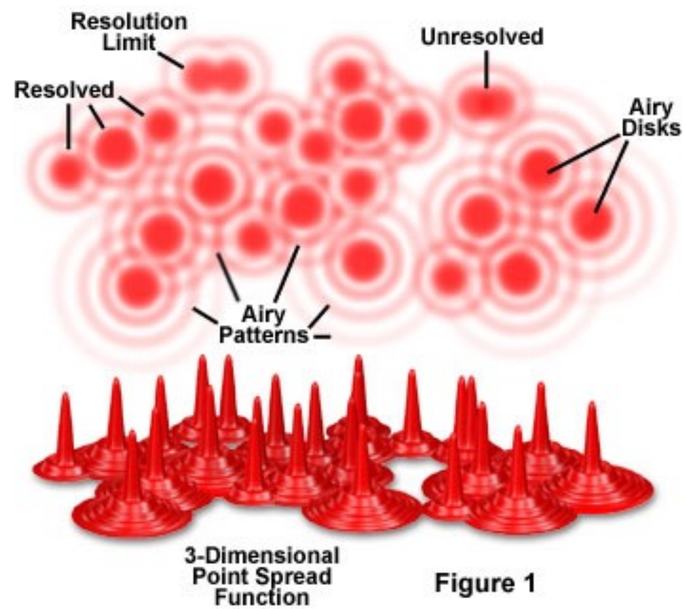
Non-zero everywhere except for isolated point

» The inverse operator A^{-1}

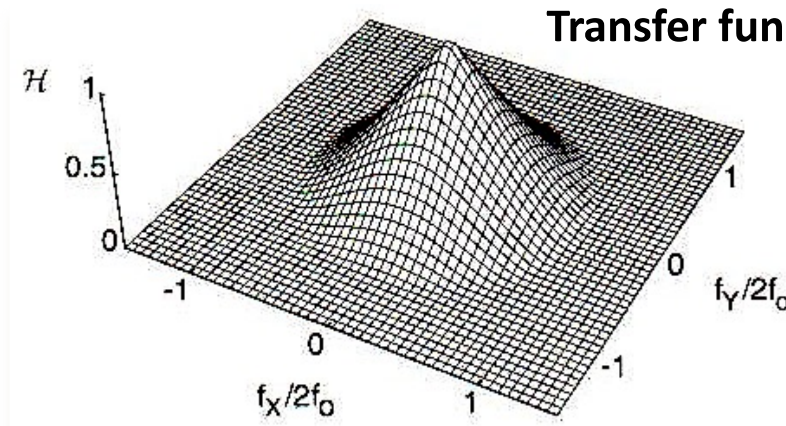
» $(A^{-1}g)(x) = \frac{1}{2\pi}\int \frac{\tilde{g}(u)}{\widetilde{K}(u)}\, e^{i2\pi xu}\mathrm{d}u$

» *Not* bounded if $\widetilde{K}(u)$ contains zeros.

Example of convolution operator: microscopes

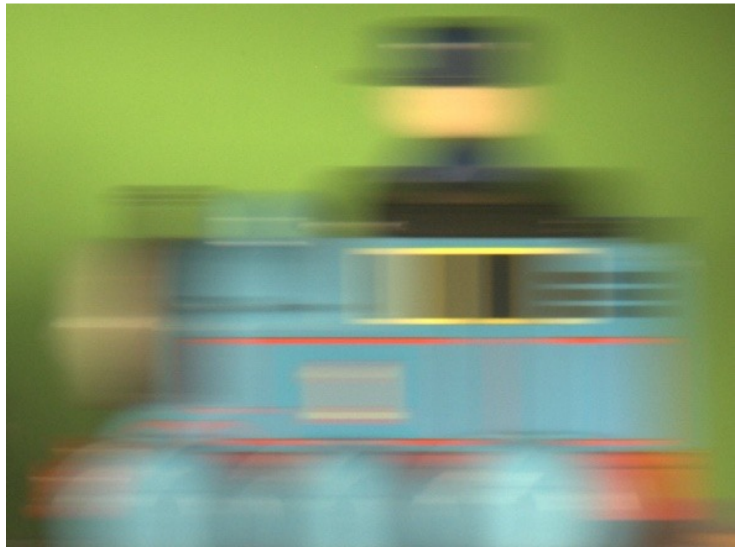


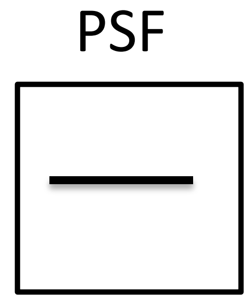
Transfer function



- » Range and null space?
- » Adjoint operator?
- » Inverse operator?

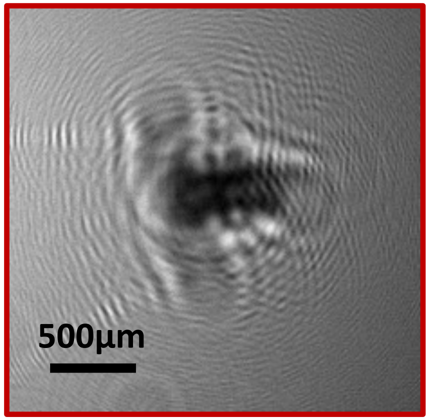
Example: motion blur





- What are Object space $\mathcal{X}$ and image space $\mathcal{Y}$?
- What is the operator ***A***? linear?
- Find an element in the null space?
  - What's the implication of this?

Example of Shift-invariant system: holography

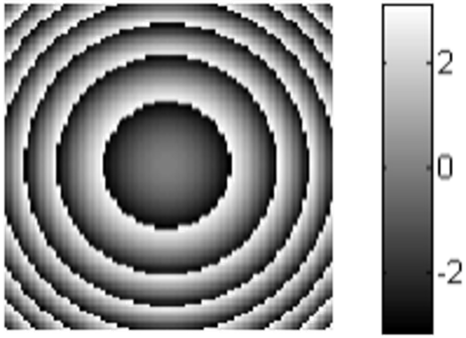


g_{out}

$=$

g_{in}

$*$



$$h(x,y) \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{(x^2+y^2)}{2z}}$$

complex PSF

Transfer function $H(u,v) = e^{i2\pi z/\lambda} \exp\{-i\lambda z(u^2+v^2)\}$

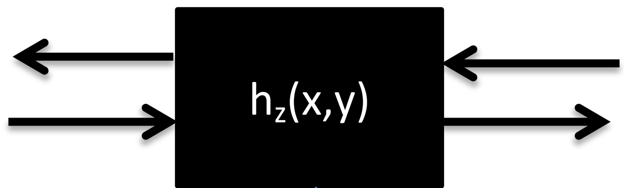
- » Range and null space?
- » Adjoint operator?
- » Inverse operator?

HW 2 P2

Application: back-propagation using adjoint operator = inverse operator!?

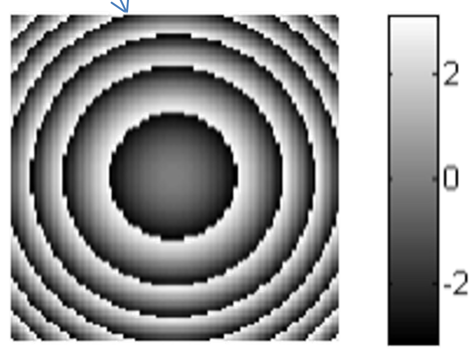
just make z→-z? Why?

$$g(x,y) = A(x,y)\,\mathrm{e}^{i\varphi(x,y)}$$



$$g'(x',y') = g(x,y) \otimes h_z(x,y)$$

H(x,y) has complex point spread function (PSF) and transfer function



$$h_z(x,y) \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{\left(x^2+y^2\right)}{2z}}$$

Transfer function $H(u,v) = e^{i2\pi z/\lambda} \exp\{-i\lambda z(u^2+v^2)\}$