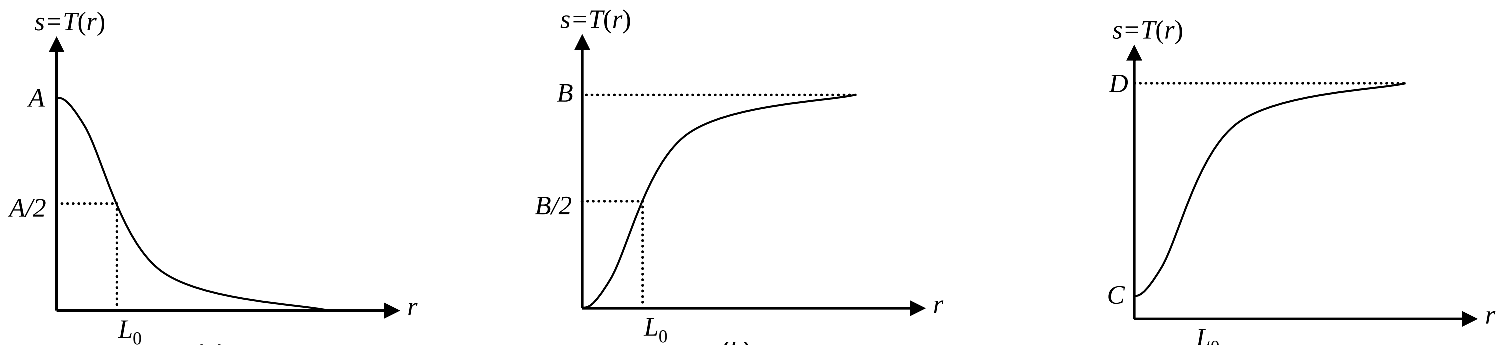
Image Enhancement in the Spatial Domain

- **Example 1(PR3.1):** *Exponentials of the form $e^{-\alpha r^2}$, α a positive constant, are useful for constructing smooth gray-level transformation functions. Construct the transformation functions having the general shapes shown in the following figures. The constants shown are input parameters, and your proposed transformations must include them in their specifications.*



(a) (b) (c)

(a) ***General form of the function:*** $s = T(r) = Ae^{-\alpha r^2}$

In Figure (a): $Ae^{-\alpha L_0{}^2} = A/2$ **solving for α :** $-\alpha L_0{}^2 = \ln(0.5) = -0.693$

$$\alpha = 0.693 / L_0^2$$

Then:

$$\boxed{s = T(r) = Ae^{-\frac{0.693}{L_0^2} r^2}}$$

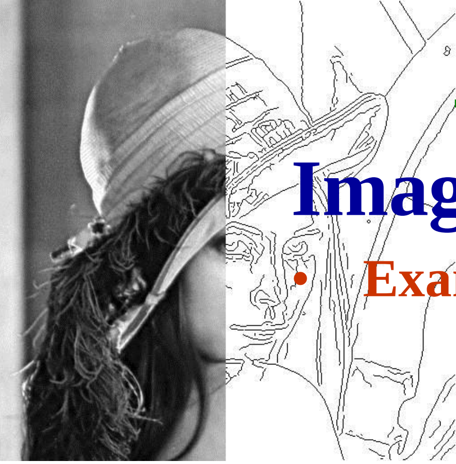
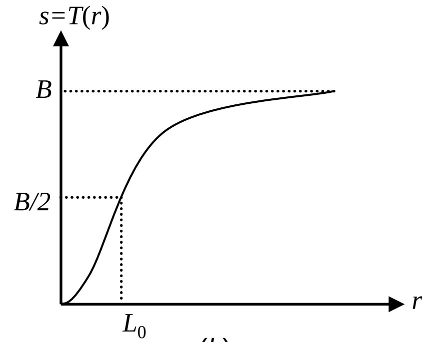
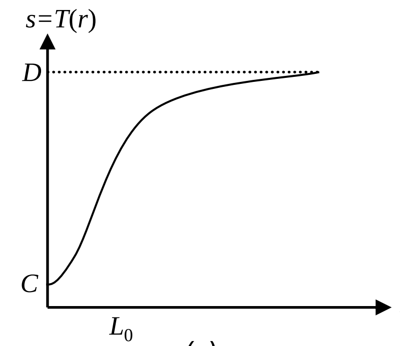


Image Enhancement in the Spatial Domain

- **Example 1(PR3.1):**



(b)



(c)

(b) General form of the function: $$s = T(r) = B - Be^{-\alpha r^2} = B(1 - e^{-\alpha r^2})$$

In Figure (b): $B(1 - e^{-\alpha L_0^2}) = B/2$ **Then:** $-\alpha L_0{}^2 = \ln(0.5) = -0.693$

$$\alpha = 0.693 / L_0^2$$

$$\boxed{s = T(r) = B(1 - e^{-\frac{0.693}{L_0^2} r^2})}$$

(c) General form of the function: $$\boxed{s = T(r) = (D - C)(1 - e^{-\frac{0.693}{L_0^2} r^2}) + C}$$

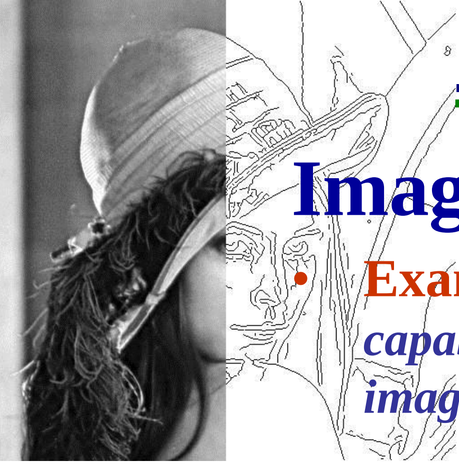


Image Enhancement in the Spatial Domain

- **Example 2(PR3.3):** ***Propose a set of gray-level-slicing transformations capable of producing all the individual bit planes of an 8-bit monochrome image.***

Consider **mod(x,y)** ***to be the modular division resulting the remainder of the integer division x/y.***

For Bit plane 7 (MSB): the following function can be written

$$T(r)=\begin{cases} 255 & for\ \ 2^7 \le r \\ 0 & otherwise \end{cases}$$

For Bit plane 6:

$$T(r)=\begin{cases} 255 & for\ \ 2^6 \le \mathrm{mod}(r,2^7) \\ 0 & otherwise \end{cases}$$

For Bit plane 5:

$$T(r)=\begin{cases} 255 & for\ \ 2^5 \le \mathrm{mod}(r,2^6) \\ 0 & otherwise \end{cases}$$

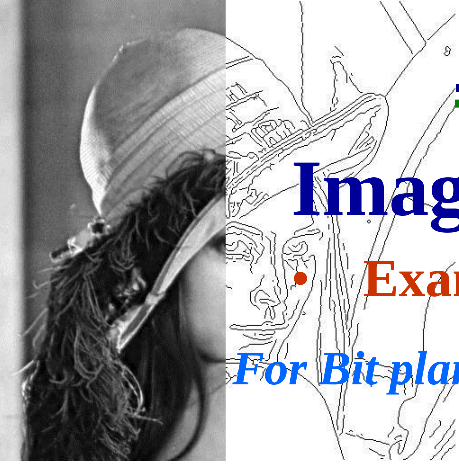


Image Enhancement in the Spatial Domain

- **Example 2(PR3.3):**

For Bit plane 4:

$$T(r) = \begin{cases} 255 & \mathit{for}\ \ 2^4 \le \mathrm{mod}(r, 2^5) \\ 0 & \mathit{otherwise} \end{cases}$$

For Bit plane 3:

$$T(r) = \begin{cases} 255 & \mathit{for}\ \ 2^3 \le \mathrm{mod}(r, 2^4) \\ 0 & \mathit{otherwise} \end{cases}$$

For Bit plane 2:

$$T(r) = \begin{cases} 255 & \mathit{for}\ \ 2^2 \le \mathrm{mod}(r, 2^3) \\ 0 & \mathit{otherwise} \end{cases}$$

For Bit plane 1:

$$T(r) = \begin{cases} 255 & \mathit{for}\ \ 2^1 \le \mathrm{mod}(r, 2^2) \\ 0 & \mathit{otherwise} \end{cases}$$

For Bit plane 0:

$$T(r) = \begin{cases} 255 & \mathit{for}\ \ 2^0 \le \mathrm{mod}(r, 2^1) \\ 0 & \mathit{otherwise} \end{cases}$$

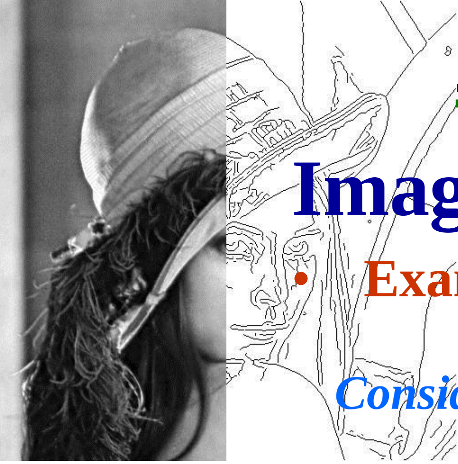
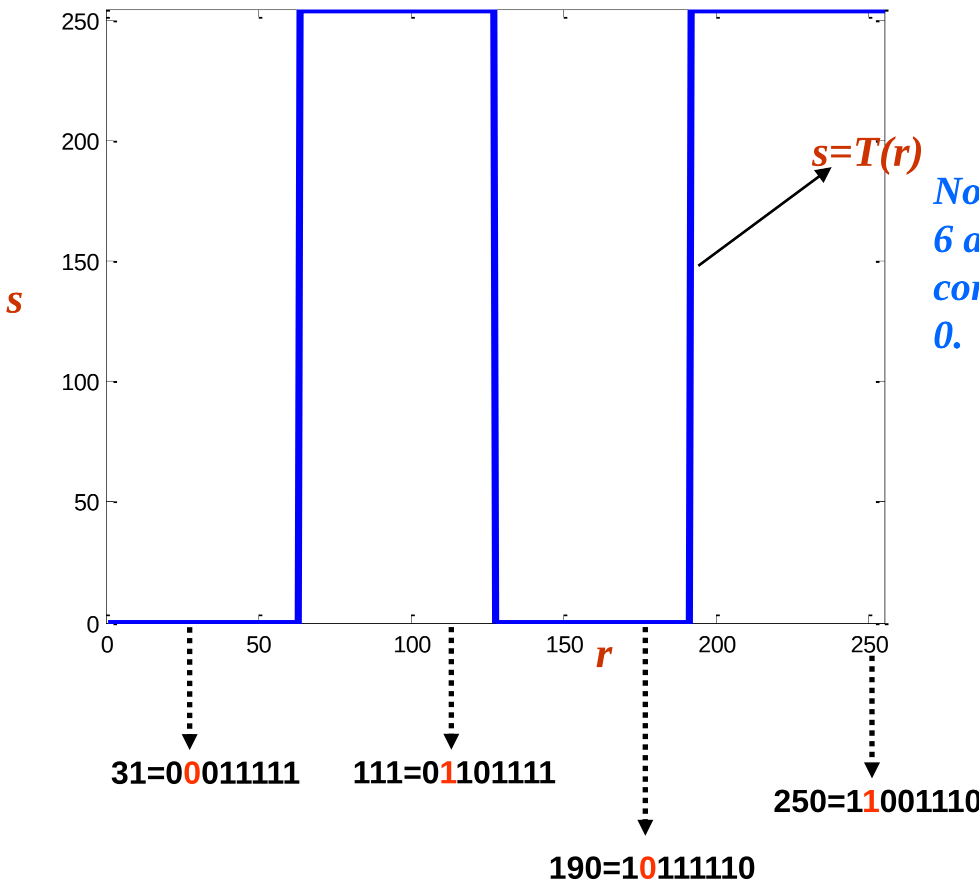


Image Enhancement in the Spatial Domain

- **Example 2(PR3.3):**

Consider Bit plane 6:

$$T(r) = \begin{cases} 255 & \text{for} \quad 2^6 \leq \mathrm{mod}(r, 2^7) \\ 0 & \text{otherwise} \end{cases}$$



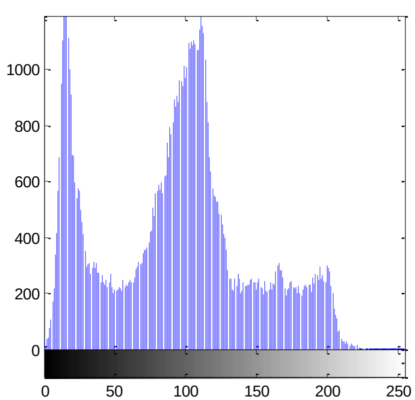
Note that all the 1s corresponding to bit 6 are scaled to 255 and all the bits corresponding to 0s are scaled down to 0.

Image Enhancement in the Spatial Domain

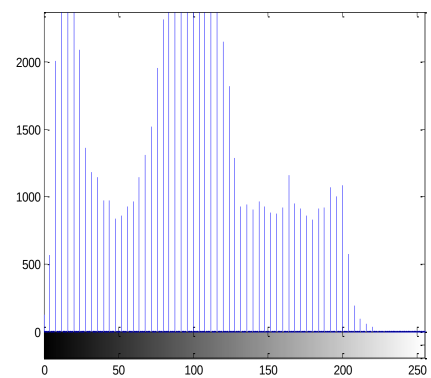
- **Example 3(PR3.4):** *a) What effect would setting to zero the lower-order bit planes have on the histogram of an image in general?*
- *b) What would be the effect on the histogram if we set to zero the higher-order bit planes instead?*

a) Removing the low order bit planes would mean the loss of some high frequency details. Furthermore the image histogram will be more sparse as compared with the all 8-bit plane case.

This is because, there will be no component representing intermediate pixel values such as 1,2,3,4, 5, 6,7 and 9,10,11,12,13,14,15 etc. Instead there will be 0 and 8 and 16 etc. This would cause the height some of the remaining histogram peaks to increase in general. Typically, less variability in gray level values will reduce contrast.



All the bits included



2 of the LSBs removed

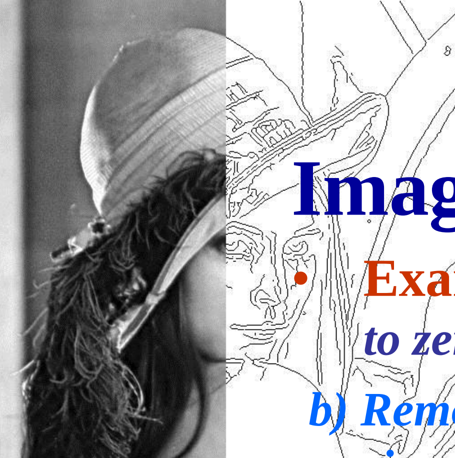
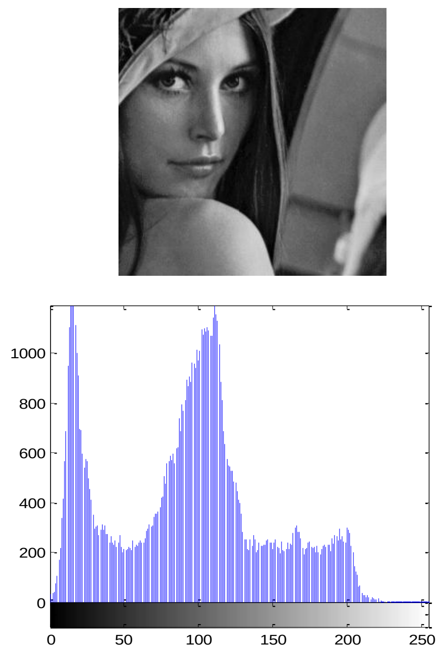


Image Enhancement in the Spatial Domain

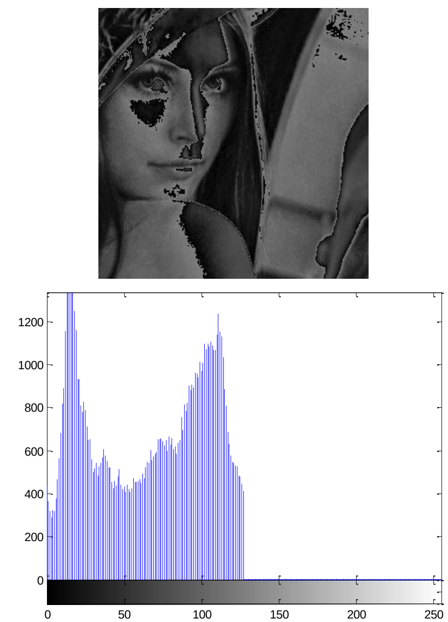
- **Example 3(PR3.4):** ***b) What would be the effect on the histogram if we set to zero the higher-order bit planes instead?***

b) Removing the high order bit planes would mean the loss of some very important DC components away from the image.

The meaning of this is that the image is much darker and a lot of the low frequency components will be lost.



All the bits included



MSB removed

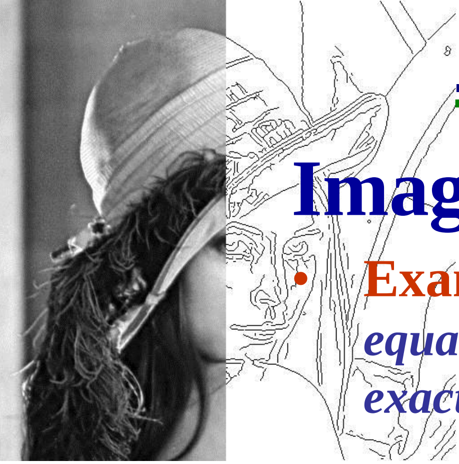


Image Enhancement in the Spatial Domain

- **Example 4(PR3.6):** *Suppose that a digital image is subjected to histogram equalization . Show that a second pass of histogram equalization will produce exactly the same result as the first pass?*

*Let **n** be the total number of pixels and let n_{r_j} be the number of pixels in the input image with intensity value r_j . Then, the histogram equalization transformation is:*

$$s_k = T(r_k) = \sum_{j=0}^{k} \frac{n_{r_j}}{n} = \frac{1}{n} \sum_{j=0}^{k} n_{r_j}$$

Since every pixel (and no others) with value r_k is mapped to value s_k, it follows that $n_{s_k} = n_{r_k}$. A second pass of histogram equalization would produce values v_k according to the transformation:

$$v_k = T(s_k) = \sum_{j=0}^{k} \frac{n_{s_j}}{n}, \ \ but \ \ n_{s_j} = n_{r_j}, \ \ then$$

$$v_k = T(s_k) = \sum_{j=0}^{k} \frac{n_{r_j}}{n} = s_k$$

Which shows that a second pass of histogram equalization would yield the same result as the first pass.

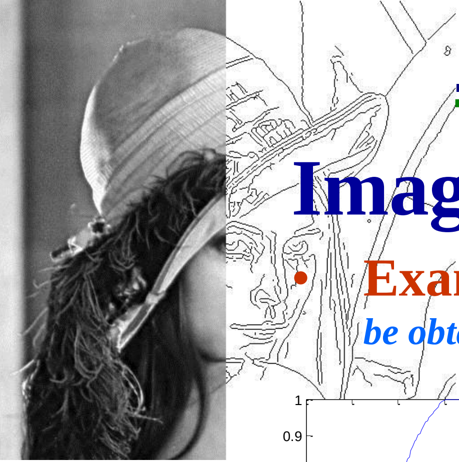
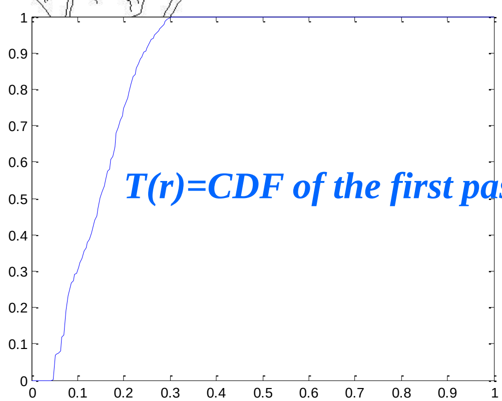
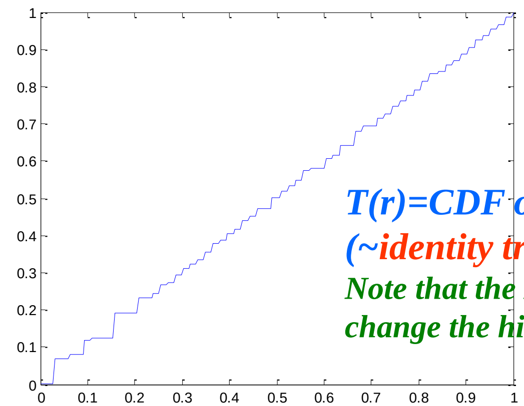


Image Enhancement in the Spatial Domain

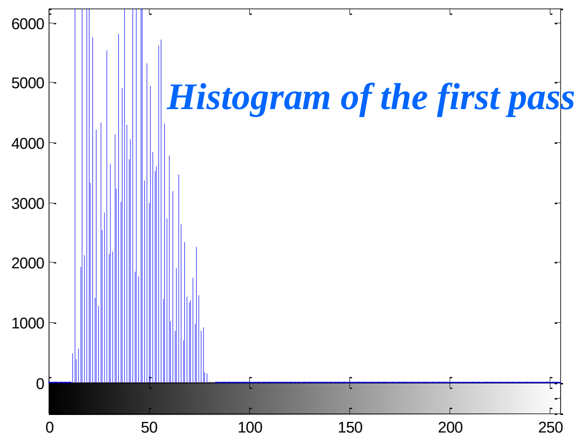
- **Example 4(PR3.6):** *Given an image, the following histograms and CDF (T(r) graphs can be obtained.*



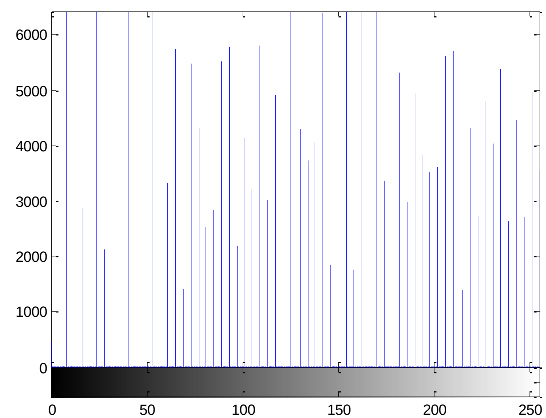
T(r)=CDF of the first pass



T(r)=CDF of the 2nd pass
(~identity transformation)
Note that the Identity Transformation does not change the histogram of the input image



Histogram of the first pass

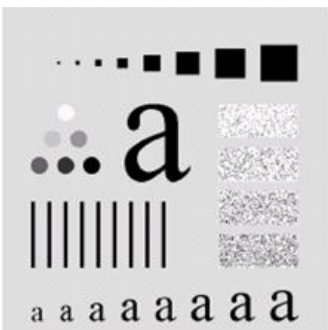


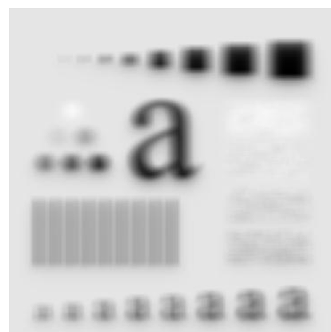
Histogram of the 2nd pass
(~uniformly distributed)

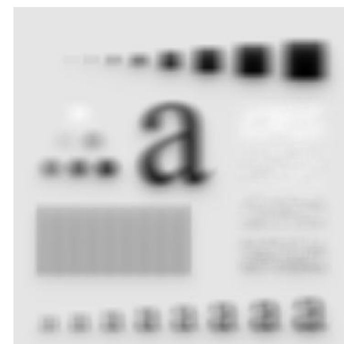
Prepared By: *Dr. Hasan Demirel, PhD*

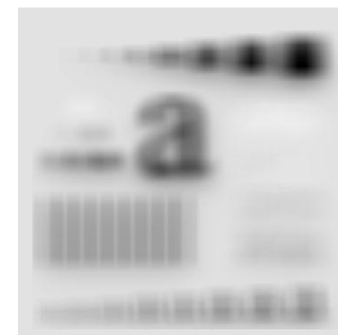
Image Enhancement in the Spatial Domain

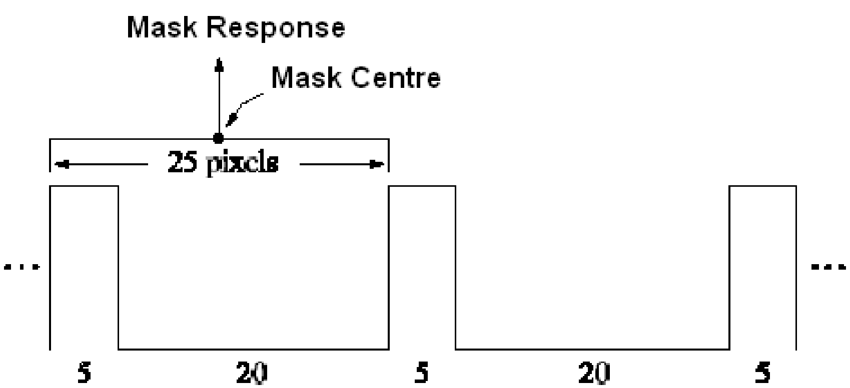
- **Example 6(PR3.22):** *The three images shown below were blurred using the square averaging masks of sizes n=23, 25 and 45 respectively. The vertical bars on the lower part of (a) and (c) are blurred, but clear separation exists between them. However, the bars have merged in in image (b), in spite of the fact that the mask that produced the image is significantly smaller than the mask produced image (c) .Explain this.*
- *(Note that the vertical bars are 5 pixels wide and 20 pixels apart.)*


Original image


(a)


(b)


(c)



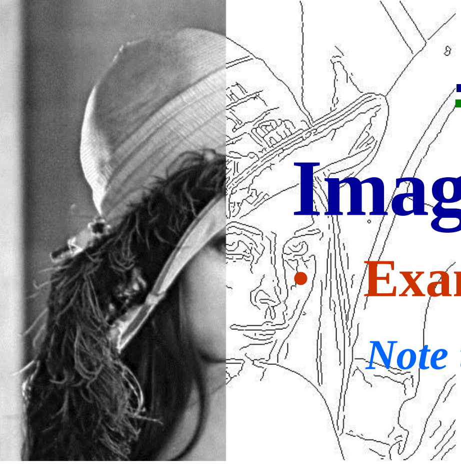
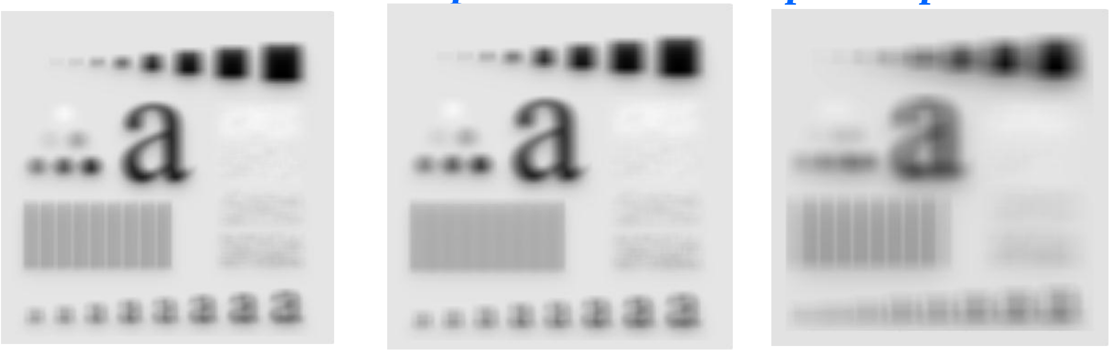


Image Enhancement in the Spatial Domain

- **Example 6(PR3.22):**

Note that the vertical bars are 5 pixels wide and 20 pixels apart.



(a) **(b)** **(c)**

The reason why the mask with size n=25 producing merged uniform region around the bars is because of the sizes of the bars and the separation of the bars in the horizontal direction. The width of each bar is 5 pixels and each bar is separated by 20 pixels.

In such an environment as the mask moves in the horizontal direction there will be 5 black and 20 light gray pixels in each row at a time. This will provide the same average value for each pixel in the region producing a merged uniform gray level.

However this will not be the same in 23 and 45 pixel masks!!

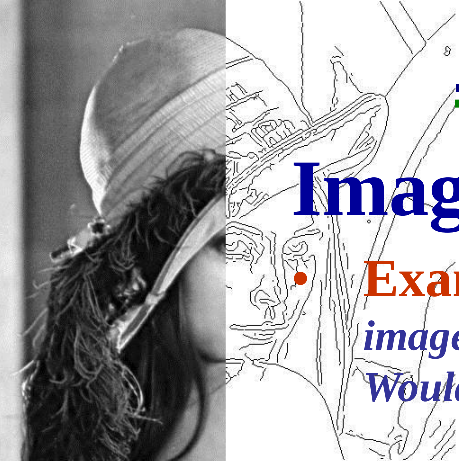


Image Enhancement in the Spatial Domain

- **Example 7(PR3.24):** ***In a given application an averaging mask is applied to input images to reduce noise, and then a Laplacian mask is applied to enhance small details. Would the result be the same if the order of these operations were reversed?***

Laplacian operation and averaging can be expressed by the following 3x3 masks:

$$Laplacian \rightarrow g(x,y) = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)$$

$$Averaging \rightarrow h(x,y) = \frac{1}{9}\sum_{i=1}^{9} f_i$$

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

Averaging Mask

0	1	0
1	-4	1
0	1	0

Laplacian Mask

Both of the two operators are multiplying the pixels in a 3x3 neighborhood with constant numbers and perform the addition. Therefore, these operations are linear operations.

The order of two linear operations does not matter. The result would be same in any order.

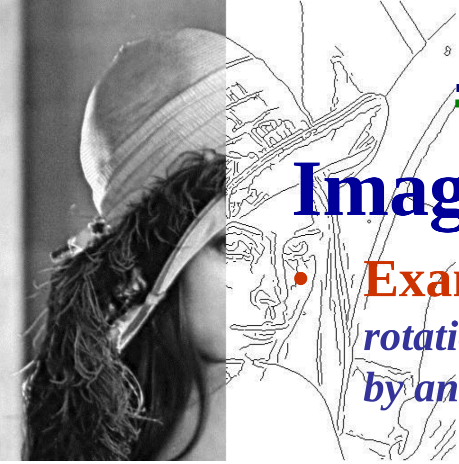


Image Enhancement in the Frequency Domain

- **Example 8(PR3.25) :** *Show that the Laplacian operation is isotropic (invariant to rotation). You will need the following equations relating coordinates after axis rotation by an angle θ.*

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

Laplacian operator is defined as:
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For the rotated Laplacian operator:
$$\nabla^2 f = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}.$$

Given that:
$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta$$

If we show that the right sides of the first 2 equations are equal than the Laplacian operation is rotation invariant.

We start with,
$$\begin{aligned}\frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \\ &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta.\end{aligned}$$

Image Enhancement in the Frequency Domain

- **Example 8(PR3.25):**

$$x = x' \cos\theta - y' \sin\theta \quad \text{and} \quad y = x' \sin\theta + y' \cos\theta$$

Taking the partial derivative of this expression again with respect to x' yields:

$$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\sin\theta\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta.$$

Repeat the same operation for y' :

$$\begin{aligned}\frac{\partial f}{\partial y'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} \\ &= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta.\end{aligned}$$

Taking the partial derivative of this expression again with respect to y' yields:

$$\frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2}\sin^2\theta - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cos\theta\sin\theta - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta$$

Adding the 2 expressions for the second derivatives:

$$\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Both sides are equal, Hence Laplacian is rotational invariant.

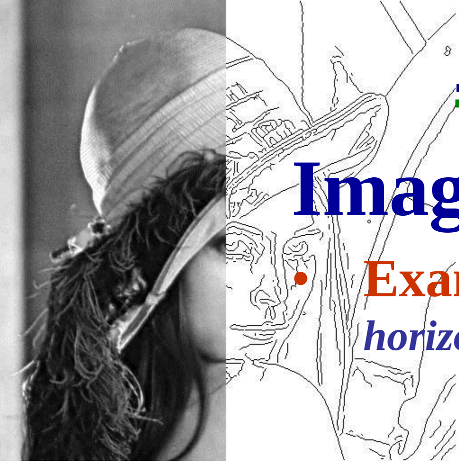
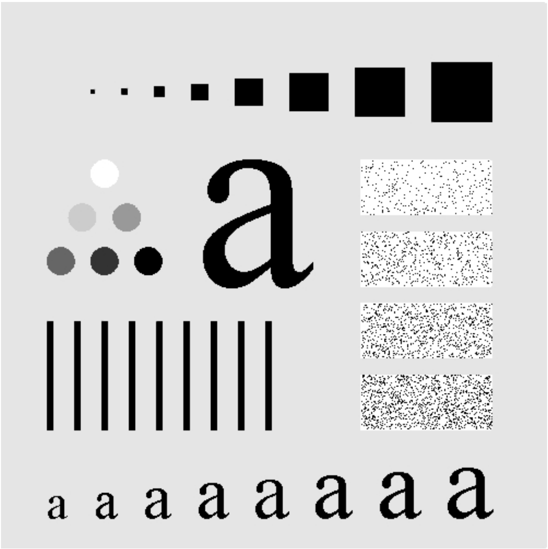
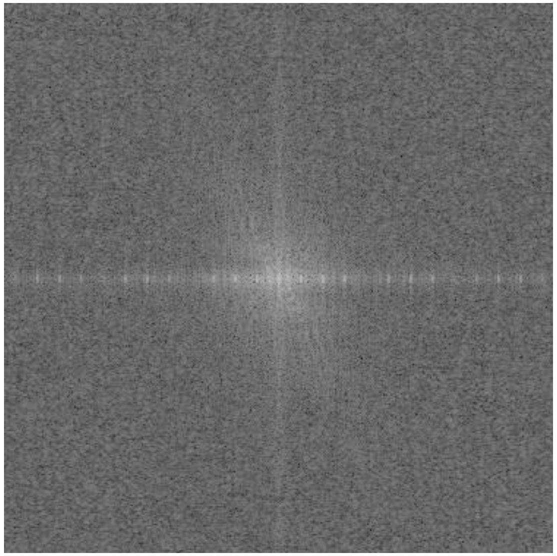


Image Enhancement in the Spatial Domain

- **Example 9(PR4.7):** ***What is the source of nearly periodic bright points in the horizontal axis of the spectrum in the following figure.***





The nearly periodic bright points in the frequency spectrum corresponds to the periodic bars as well as the repeating boxes, letters and circles in the horizontal direction.

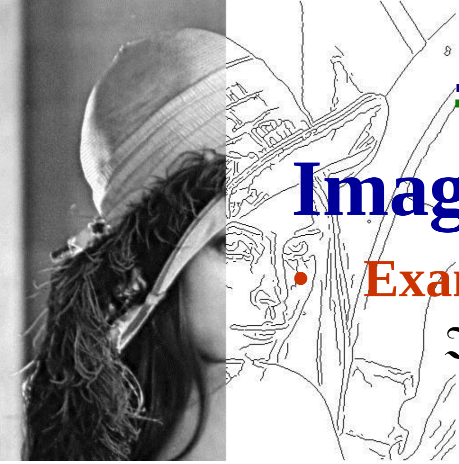


Image Enhancement in the Spatial Domain

- **Example 10(PR4.6) -a):** ***Prove the validity of the following Equation.***

$$\Im[f(x,y)(-1)^{x+y}] = F(u-M/2, v-N/2) \quad , \Im[.]\, denotes\ the\ Fourier\ Transform$$

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$F(u-M/2, v-N/2) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi((u-M/2)x/M+(v-N/2)y/N)}$$

$$= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi((ux/M-1/2x)+(vy/N-1/2y))}$$

$$e^{-j2\pi((ux/M-1/2x)+(vy/N-1/2y))} = e^{j\pi(x+y)}e^{-j2\pi(ux/M+vy/N)} = (-1)^{x+y}e^{-j2\pi(ux/M+vy/N)}$$

$$e^{j\pi(x+y)} = (-1)^{x+y} \quad \rightarrow e^{j\pi(x+y)} = cos(\pi(x+y)) + \cancel{j\, sin(\pi(x+y))}$$

always zero

1 or -1 depending on the addition of x+y. If (x+y) is even then 1, otherwise -1.

$$= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)(-1)^{x+y}e^{-j2\pi(ux/M+vy/N)}$$

Image Enhancement in the Spatial Domain

- **Example 10(PR4.6) -b):** *Prove the validity of the following 2 Equations.*

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$$

$$\Im[f(x,y)e^{j2\pi(u_0x/M+v_0y/N)}] = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[f(x,y)e^{j2\pi(u_0x/M+v_0y/N)}\right]e^{-j2\pi(ux/M+vy/N)}$$

$$= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi([u-u_0]x/M+[v-v_0]y/N)}$$

$$= F(u-u_0, v-v_0)$$

Similarly, it can be shown that : $\Im[F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}] = f(x-x_0, y-y_0)$

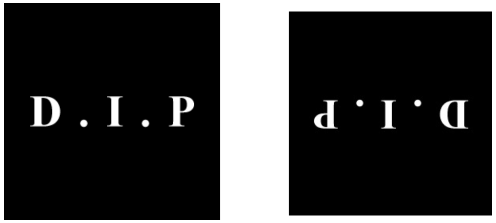
This is the Translation property of the 2D Fourier transform:

When x_0=u_0=M/2 and x_0=u_0=N/2, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$

$$f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$$

Image Enhancement in the Spatial Domain

- **Example 11(PR4.9):** *Consider the images shown below. The image on the right is obtained by (a) multiplying the image on the left by $(-1)^{x+y}$; (b) computing the DFT; (c) taking the complex conjugate of the transform; (d) computing the inverse DFT; and (e) multiplying the real part of the result by $(-1)^{x+y}$. Explain (mathematically) why the image on the right appears as it does.*



The complex conjugate simply changes j to –j in the inverse transform, so the image on the right is given by:

$$\Im[F(u,v)^*] = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{-j2\pi(ux/M+vy/N)}$$

$$= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(u(-x)/M+v(-y)/N)}$$

$$= f(-x,-y)$$

Which simply mirrors f(x,y) about the origin, thus producing the image on the right

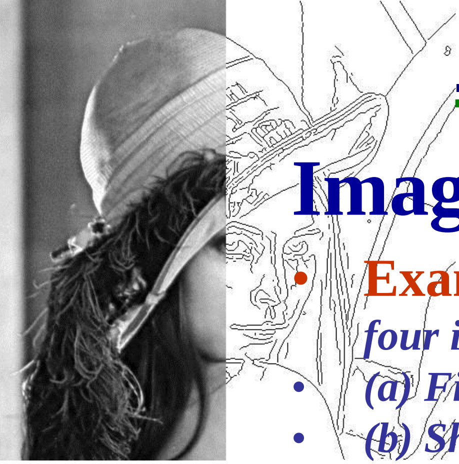


Image Enhancement in the Spatial Domain

- **Example 12(PR4.14):** *Suppose that you form a low pass filter that averages the four immediate neighbors of a point (x,y), but excludes the point itself.*
- *(a) Find the equivalent filter H(u,v) in the frequency domain.*
- *(b) Show that H(u,v) is a lowpass filter.*

a) The spatial average is given by:

$$g(x,y)=\frac{1}{4}\left[f(x,y+1)+f(x+1,y)+f(x-1,y)+f(x,y-1)\right]$$

Then, using the following property:

$$\boxed{f(x-x_0,y-y_0)\Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}}$$

$$\begin{aligned}G(u,v) &= \frac{1}{4}\left[e^{j2\pi v/N}+e^{j2\pi u/M}+e^{-j2\pi u/M}+e^{-j2\pi v/N}\right]F(u,v)\\ &= H(u,v)F(u,v),\end{aligned}$$

Where the H(u,v) is the filter function. We get the following transfer function:

$$H(u,v)=\frac{1}{2}\left[\cos(2\pi u/M)+\cos(2\pi v/N)\right]$$

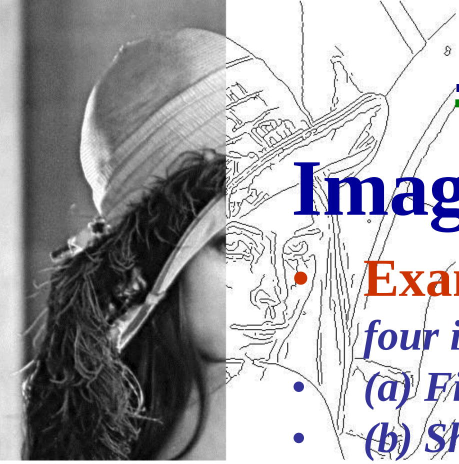


Image Enhancement in the Spatial Domain

- **Example 12(PR4.14):** ***Suppose that you form a low pass filter that averages the four immediate neighbors of a point (x,y), but excludes the point itself.***
- ***(a) Find the equivalent filter H(u,v) in the frequency domain.***
- ***(b) Show that H(u,v) is a lowpass filter.***

a) The H(u,v) Filter function can be centered by:

$$H(u,v) = \frac{1}{2}\left[\cos(2\pi[u - M/2)/M) + \cos(2\pi[v - N/2]/N)\right]$$

b) Consider one variable for convenience. As u ranges from 0 to M, the value of cos(2π[u-M/2]/M) starts at -1, peaks at 1 when u = M/2 (the center of the filter) and then decreases to -1 again when u = M. Thus, we see that the amplitude of the filter decreases as a function of distance from the origin of the centered filter, which is the characteristic of a lowpass filter.

A similar argument is easily carried out when considering both variables simultaneously.

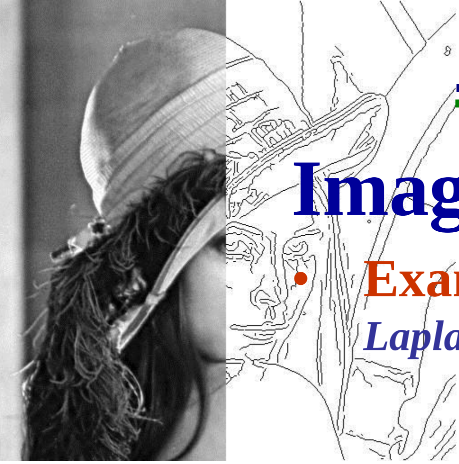


Image Enhancement in the Spatial Domain

- **Example 8-PR4.19:** ***Derive the frequency domain filter that corresponds to the Laplacian operator in the spatial domain.***

0	1	0
1	–4	1
0	1	0

Consider theLaplacian mask given. Then,

$$g(x,y) = [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)] - 4f(x,y).$$

$$G(u,v) = H(u,v)F(u,v)$$

Where

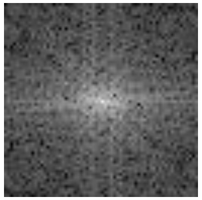
$$\begin{aligned} H(u,v) &= \left[e^{j2\pi u/M} + e^{-j2\pi u/M} + e^{j2\pi v/N} + e^{-j2\pi v/N} - 4\right] \\ &= 2\left[\cos(2\pi u/M) + \cos(2\pi v/N) - 2\right]. \end{aligned}$$

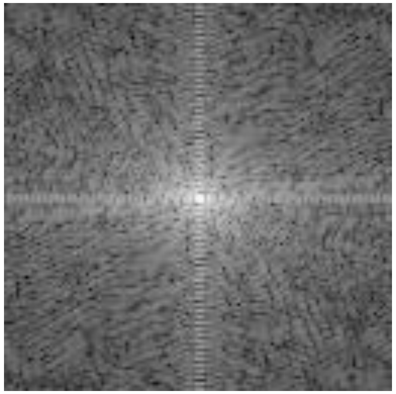
The H(u,v) Filter function can be centered by:

$$H(u,v) = 2\left[\cos(2\pi\left[u - M/2\right]/M) + \cos(2\pi\left[v - N/2\right]/N) - 2\right]$$

Image Enhancement in the Spatial Domain

- **Example 8-PR4.22:** ***The two fourier spectra shown are of the same image. The spectrum of the left corresponds to the original image, and the spectrum on the right was obtained after the image was padded with zeros.***
- ***(a) Explain the difference of the overall contrast.***
- ***(b) Explain the significant increase in signal strength along the vertical and the horizontal axes of the spectrum shown on the right.***





(a) Padding with zero increases the size but reduces the average gray level of image. The average gray level of the padded image is less than the original image. F(0,0) in the padded image is less than F(0,0) of the original image. All the others away from the origin are less in the padded image than the original image. This produces a narrower range of values hence a lower contrast spectrum in the padded image.

(b) Padding with 0's introduces significant discontinuities at the borders of the original image. This process introduces strong horizontal and vertical edges where the image ends abruptly and then continues with o values. These sharp transitions correspond to the strength of the spectrum along the horizontal and vertical axes.

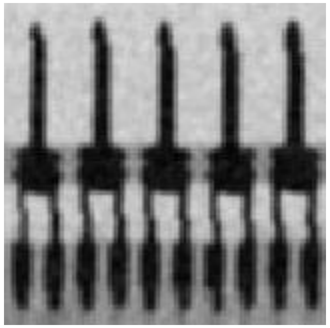
Image Restoration

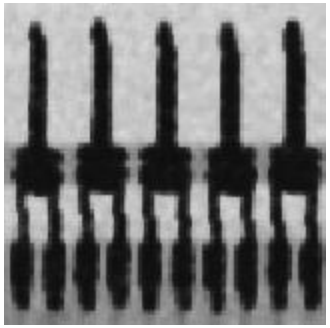
Restoration in the presence of Noise: Only-Spatial Filtering

- **Example 1-PR5.10:** ***Given the two subimages below. The sub image on the left is the result of using arithmetic mean filter of size 3x3. The other subimage is the result of using the geometric mean filter of the same size.***

a) ***Why the subimage obtained with geometric filtering is less blurred. Hint you can start your analysis with 1-D step edge profile of the image.***

b) ***Explain why the black components on the right image are thicker.***





a) ***Lets consider the mathematical expressions of the arithmetic and geometric mean filters:***

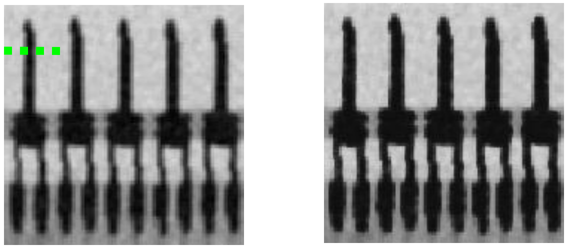
$$\hat{f}(x,y)=\frac{1}{mn}\sum_{s,t\in S_{xy}} g(s,t) \qquad \hat{f}(x,y)=\left[\prod_{s,t\in S_{xy}} g(s,t)\right]^{\frac{1}{mn}}$$

Prepared By: *Dr. Hasan Demirel, PhD*

Image Restoration

Restoration in the presence of Noise: Only-Spatial Filtering

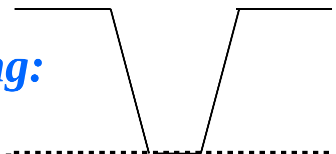
- **Example 1-PR5.10 :**



a) If we take a rough estimate of 1-D Step function profile before the filtering:

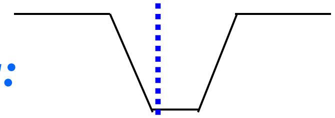
Arithmetic mean filtering:

Before filtering:



6 6 6 6 0 0 6 6 6 6

After filtering:

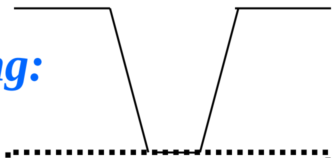


6 6 6 4 2 2 4 6 6 6

$$\hat{f}(x,y)=\frac{1}{mn}\sum_{s,t\in S_{xy}} g(s,t)$$

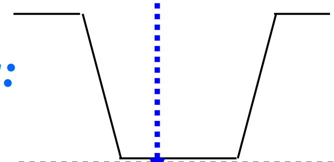
Geometric mean filtering:

Before filtering:



6 6 6 6 0 0 6 6 6 6

After filtering:



6 6 6 0 0 0 0 6 6 6

$$\hat{f}(x,y)=\left[\prod_{s,t\in S_{xy}} g(s,t)\right]^{\frac{1}{mn}}$$

Prepared By: Dr. Hasan Demirel, PhD

Image Restoration

Restoration in the presence of Noise: Only-Spatial Filtering

- **Example 1-PR5.10 :**

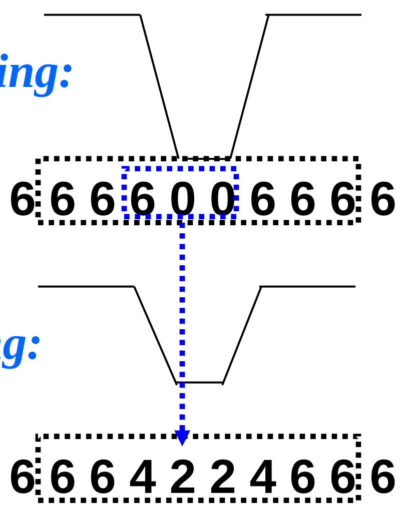
a) *The resulting 1-D profiles clearly indicates that the arithmetic mean filter produces smoother/blurred transition and*

b) *The geometric filtering increases the thickness of the black components.*

Arithmetic mean filtering:

Before filtering:

After filtering:



Geometric mean filtering:

Before filtering:

After filtering: