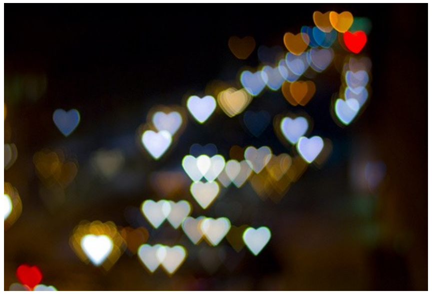
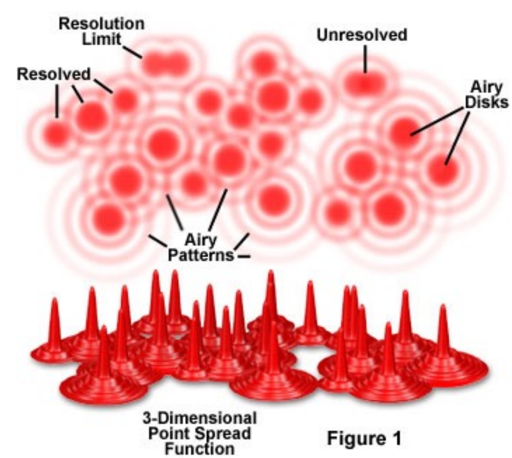


Introduction to Inverse Problem in Imaging

EC 522 Computational Optical Imaging

Lei Tian



complex impulse function

Admins

- HW 2 is posted
  - Due 2/21 (Wednesday; after Presidents' day break)

Mathematical tools & road map

» Vector space (IIP Appx A)

  » Key idea: think about the imaging signals as a *vector*

» Linear operator (IIP Appx B)

  » Key idea: think about imaging process as a linear transformation, i.e. a linear *operator*

  » Later, we will perform discretization and convert the operator into a *matrix*

IIP: Introduction to inverse problems in imaging Appx A

Linear operator

Linear operator

- Linear operator A: $\mathcal{X} \to \mathcal{Y}$ satisfies
  - $A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A(f_1) + \alpha_2 A(f_2)$, for any complex numbers α_1 and α_2
    - Additivity: $A(f_1 + f_2) = A(f_1) + A(f_2)$
    - Scalability: $A(\alpha f) = \alpha A(f)$

Range space*

- The ***Range space*** of a linear operator A: $\mathcal{R}(A)$
  - The set of all elements $g \in \mathcal{Y}$ from $Af = g$

$$\mathcal{R}(A) = \{g = Af \in \mathcal{Y}, f \in \mathcal{X}\}$$

Null space *

» The **null space** of a linear operator A: $\mathcal{N}(A)$

» The set of all elements $f \in \mathcal{X}$ such that $Af = 0$

$$\mathcal{N}(A) = \{f \in \mathcal{X}, Af = 0\}$$

Implication of Null space

Adjoint operator *

- The ***adjoint*** operator ***A* (or A^H)*** of a linear and bounded operator A
  - $A^*: \mathcal{Y} \rightarrow \mathcal{X}$ is the adjoint of $A: \mathcal{X} \rightarrow \mathcal{Y}$, when

$$\langle Ax, y \rangle_{\mathcal{Y}} = \langle x, A^* y \rangle_{\mathcal{X}} \text{ for every } x \in \mathcal{X}, y \in \mathcal{Y}$$

- Generalization of the **Hermitian transpose** (complex conjugate transpose) of a matrix

Hermitian / adjoint – can be used interchangeably
https://en.wikipedia.org/wiki/Hermitian_adjoint

Example: DFT

Example: deconvolution

Adjoint of a convolution operator

- Adjoint of convolution operator A^*
  - $(A^*g)(x) = K^*(-x) * g(x)$
    $$= \int K^*(x' - x) g(x') \mathrm{d}x'$$

- Spectral representation
  - $(A^*g)(x) = \int \widetilde{K}^*(u)\, \tilde{g}(u) e^{i2\pi xu} \mathrm{d}u$

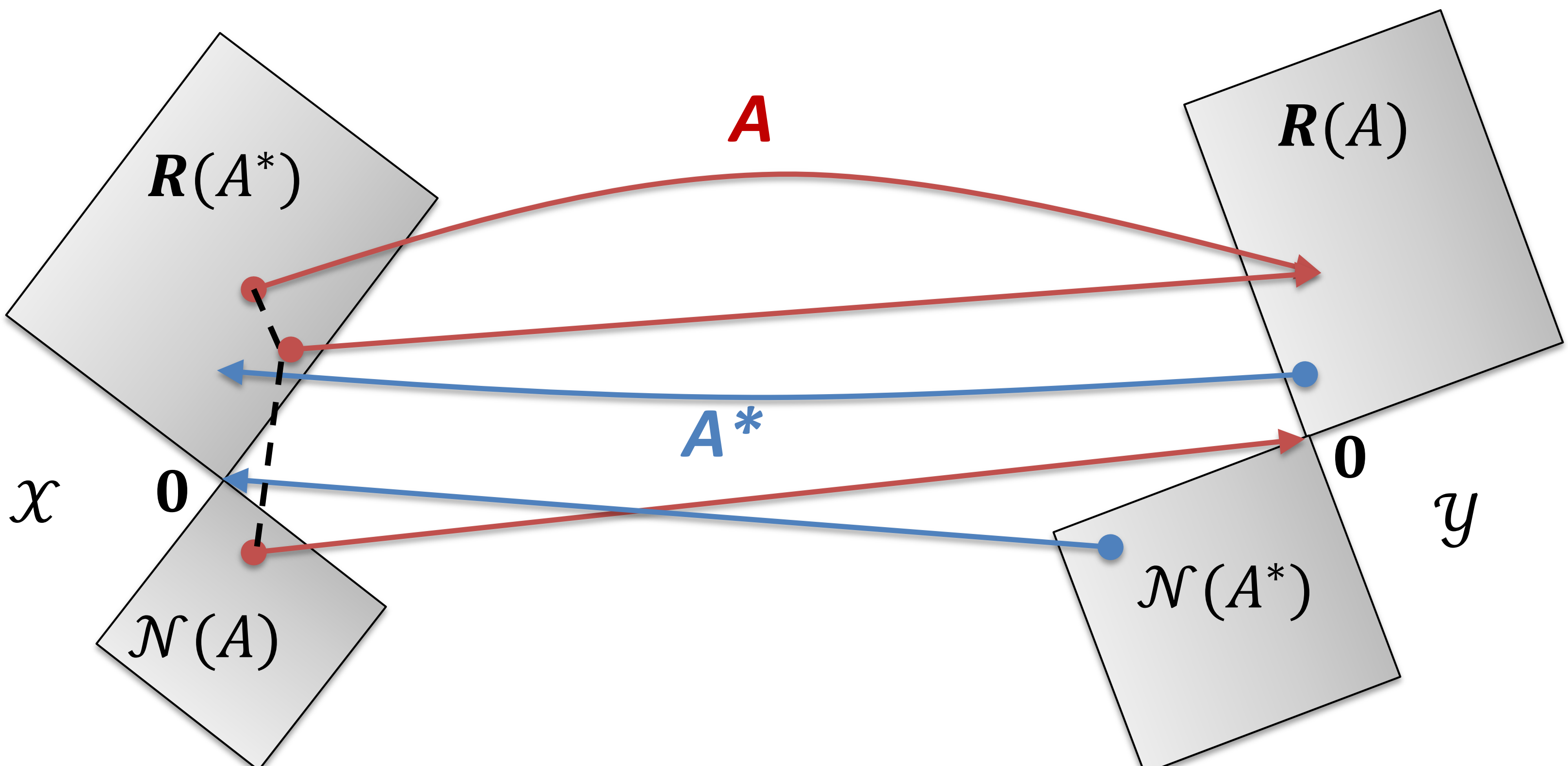
Proof?

Side note:

- I found it is easier to work with $u = \omega/2\pi$ in FT and IFT, the textbook uses ω.
- Throughout the lecture, we will use the definition in the u-space.

Geometric relation between null space and range space

$$\mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}$$
$$\mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$$



Example: Relation between range and null space of a convolution operator

A is a convolution operator

- » $f_1 \in \mathcal{R}(A)$
- » $f_2 \in \mathcal{N}(A)$

- » $f_1 \perp f_2$

and $\mathcal{N}(A) = \mathcal{R}(A)^{\perp}$

Why?

Relation between range and null space of a convolution operator

A is a convolution operator

- » $f_1 \in \mathcal{R}(A)$ Only contain frequency component $u \in \mathcal{B}$
- » $f_2 \in \mathcal{N}(A)$ Only contain frequency component $u \notin \mathcal{B}$

- » $f_1 \perp f_2$

and $$\mathcal{N}(A) = \mathcal{R}(A)^{\perp}$$

Self-adjoint

» If $A = A^*$, A is self-adjoint or Hermitian

Example

Properties of Adjoint operator

» The adjoint A^* is unique

» $(A^*)^* = A$

» The operators AA^* and A^*A are self-adjoint

» If A is invertible, $(A^{-1})^*=(A^*)^{-1}$

» $(A+B)^* = A^*+B^*$

» $(BA)^* = A^*B^*$

Unitary operator

» A is unitary if and only if

$$A^{-1} = A^* \text{ or } A^*A = I$$

» If A is unitary, then $\|Ax\|^2 = \|x\|^2$

Foundations of Signal processing, Chap. 2, pg 47-50

Unitary operator

- Preserve geometry (lengths and angles) when mapping one vector space to another
- A bounded linear operator $A: \mathcal{X} \rightarrow \mathcal{Y}$ is unitary, when
  - A is invertible
  - A preserves inner product

$$\langle f, h \rangle_{\mathcal{X}} = \langle Af, Ah \rangle_{\mathcal{Y}}, \text{ for every } f, h \in \mathcal{X}$$

Foundations of Signal processing, Chap. 2, pg 47-50

Eigenvector and eigenvalue of a linear operator

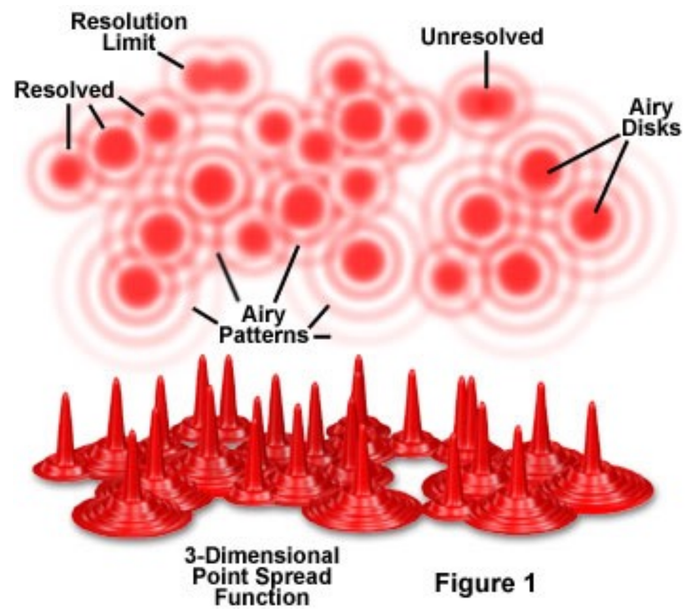
» An **eigenvector** of a linear operator *A: H*→ *H* is a nonzero vector $v \in H$, such that

$$Av = \lambda v$$

» $\lambda \in \mathbb{C}$ is the **eigenvalue**.

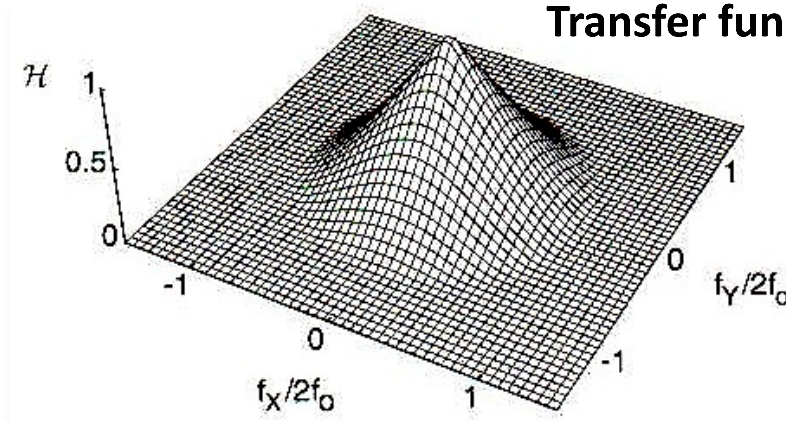
Example: convolution operator

Example of convolution operator: microscopes

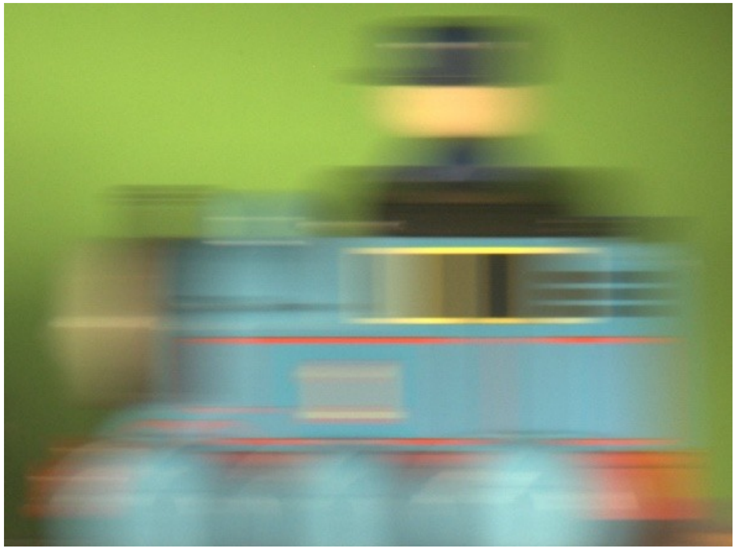


» Range and null space?
» Adjoint operator?
» Inverse operator?

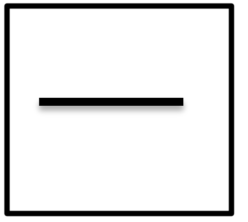
Transfer function



Example: motion blur

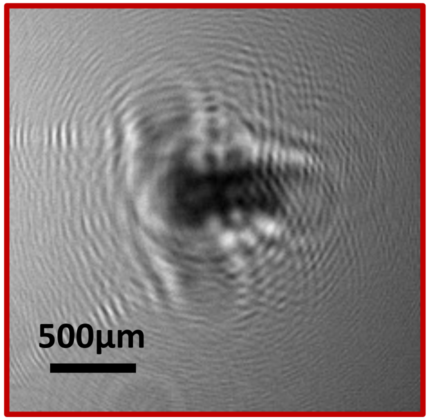


PSF



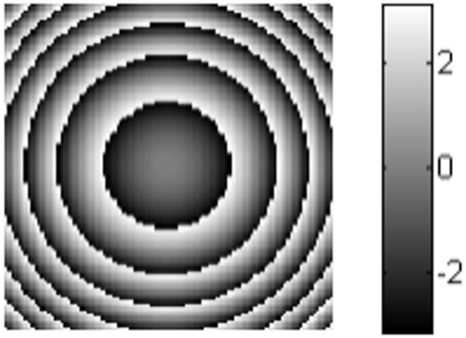
- What are Object space $\mathcal{X}$ and image space $\mathcal{Y}$?
- What is the operator $\boldsymbol{A}$? linear?
- Find an element in the null space?
  - What's the implication of this?

Example of Shift-invariant system: holography



$$=$$

$$*$$



g_{out} $\qquad$ g_{in}

$$h(x,y) \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{\left(x^2+y^2\right)}{2z}}$$

complex PSF

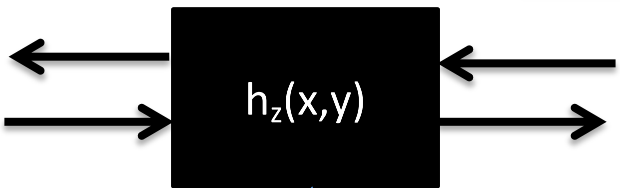
Transfer function $\quad H(u,v) = e^{i2\pi z/\lambda} \exp\{-i\lambda z(u^2+v^2)\}$

- Range and null space?
- Adjoint operator?
- Inverse operator?

Application: back-propagation using adjoint operator = inverse operator!?

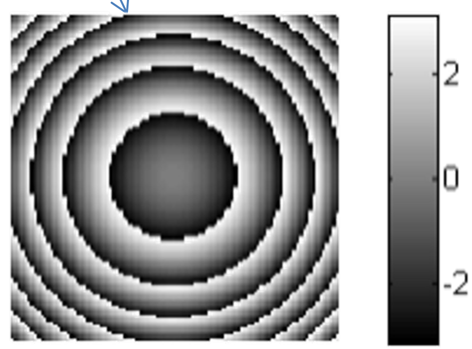
just make z→-z? Why?

$$g(x,y) = A(x,y)\,\mathrm{e}^{i\varphi(x,y)}$$



$$g'(x',y') = g(x,y) \otimes h_z(x,y)$$

H(x,y) has complex point spread function (PSF) and transfer function



$$h_z(x,y) \approx \frac{e^{ikz}}{i\lambda z} e^{ik\frac{(x^2+y^2)}{2z}}$$

Transfer function $H(u,v) = e^{i2\pi z/\lambda} \exp\{-i\lambda z(u^2+v^2)\}$

Mathematical tools & road map

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IIP: Introduction to inverse problems in imaging Appx A

From continuous to discrete model

Fully discrete LSI model

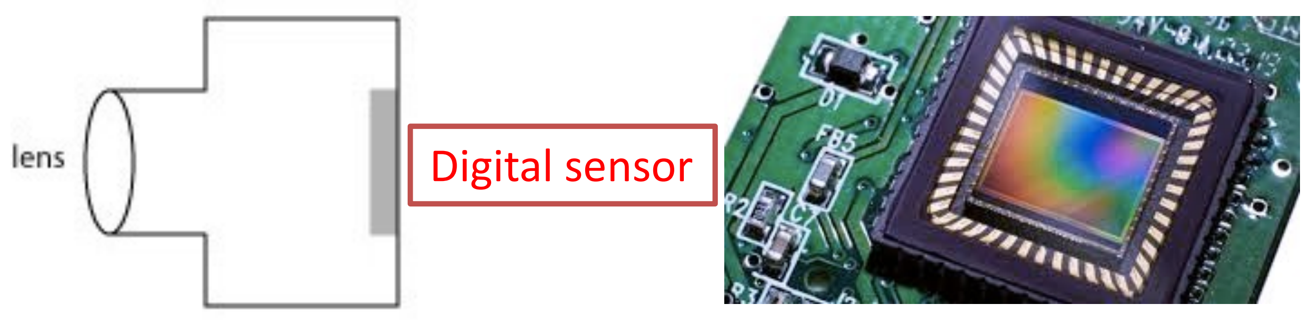
- **Sampling & Discretization**
  - **Sampling of the image signal**
  - Discretize the object signal
- What about the LSI system?
  - How to discretize the linear operator?
- “Transfer function” of discrete LSI system?
  - Eigenvalues of the imaging matrix

Sampling

Why do we need sampling

» Lenses and filters are analog optical processors

» Camera digitizes (samples) optical field

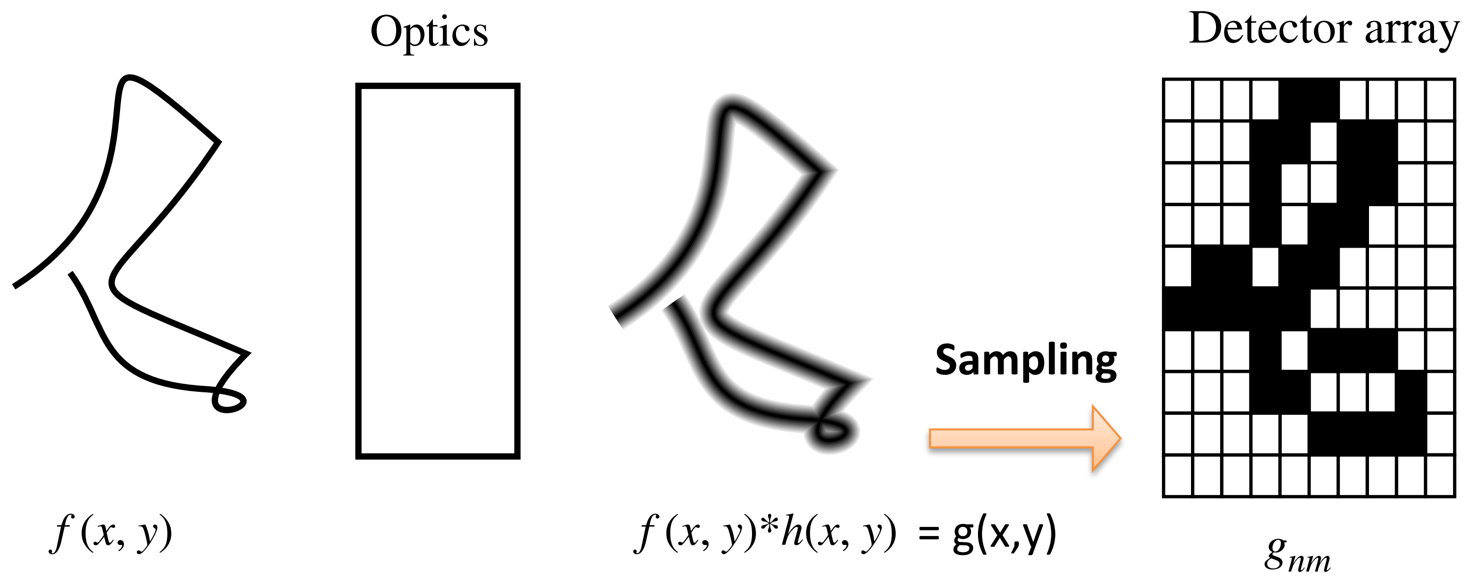


Optical pre-processing is analog(continuous)

Digital post-processing is digital(discrete)

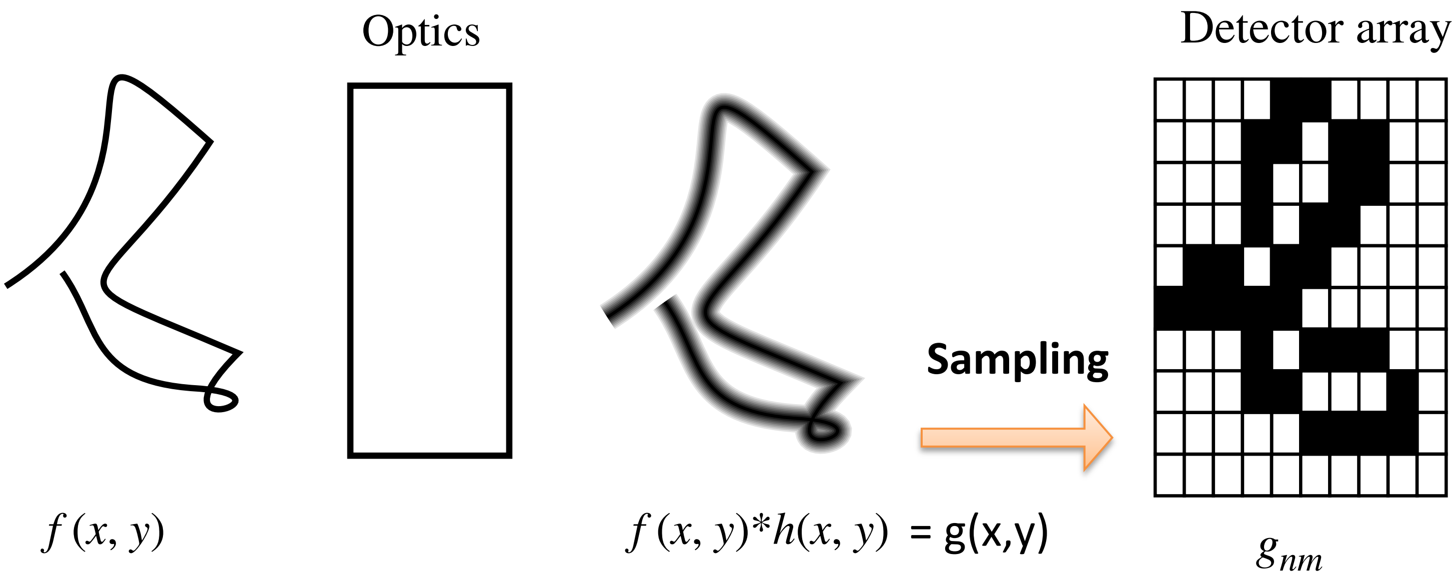
How sampling works on a typical optical detector?

Sampling by optical detectors



D. Brady, Optical Imaging and Spectroscopy, Chap 7.2.

Sampling by optical detectors



Pixel size: X * Y, spacing Δ * Δ

Pixel sampling function

$$g_{nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-X/2}^{X/2}\int_{-Y/2}^{Y/2} f(x,y)h(x'-x, y'-y) \times \boxed{p(x'-n\Delta, y'-m\Delta)}\, dx'\, dy'\, dx\, dy$$

How to describe square pixels?

Similar to zero-order hold sampling!

D. Brady, Optical Imaging and Spectroscopy, Chap 7.2.

Effect of pixel sampling: pixel transfer function

$$g_{nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-X/2}^{X/2}\int_{-Y/2}^{Y/2} f(x, y)h(x' - x, y' - y)$$
$$\times \boxed{p(x' - n\Delta, y' - m\Delta)}\, dx'\, dy'\, dx\, dy$$

Pixel sampling function

$\mathcal{F}$

$$\hat{g}(u, v) = \hat{f}(u, v)\underbrace{\hat{h}(u, v)\boxed{\hat{p}(u, v)}}$$

Pixel transfer function (PTF)

System transfer function (STF) combines optical TF and pixel TF!

$\mathcal{F}^{-1}$

$$g_{nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i u n\Delta} e^{2\pi i v m\Delta} \hat{f}(u, v)\hat{h}(u, v)\hat{p}(u, v)\, du\, dv$$

How different is the “rect-sampling” as compared to the “impulse sampling"?

D. Brady, Optical Imaging and Spectroscopy, Chap 7.2.

Sampling

- Practically
  - Sampling relates the continuous world to discrete world

- Mathematically
  - Sampling can also be treated by linear decomposition!

Linear decomposition and sampling

» Denote the (real-valued) m^{th} pixel sampling function: $p_m(x)$

  » In practice, can assume the pixel sampling function forms an orthonormal basis: $\{p_m, m = 0, \ldots, N-1\}$, satisfying

$$\langle p_m, p_n \rangle = \delta_{m,n}$$

» The pixel reading from the m^{th} pixel is

  » $g_m = \int g(x) p_m(x) \mathrm{d}x = (g, p_m)$

  » which can be treated as the inner product between *g* and mth basis p_m

» The optical detector takes *N* discrete samples from a continuous object $g(x)$ to produce a vector (an image)

$$\begin{aligned}\mathbf{g} &= [g_0, g_1, \cdots, g_{N-1}]^T \\ &= [(g, p_0), (g, p_1), \cdots, (g, p_{N-1})]^T\end{aligned}$$

» In other words, the object is linearly decomposed as

$$g(x) = \sum_{m=0}^{N-1} g_m p_m$$

Semi-Discrete mapping

» Mapping from continuous object to discrete measurement

» Recall the continuous linear forward model

$$g = Af = \int h(x, x')f(x')\mathrm{d}x'$$

» $g_m = (Af, p_m) = (f, A^* p_m)$

Definition of Adjoint operator

Physical meaning?

"impulse response" from the detector point of view & reciprocity

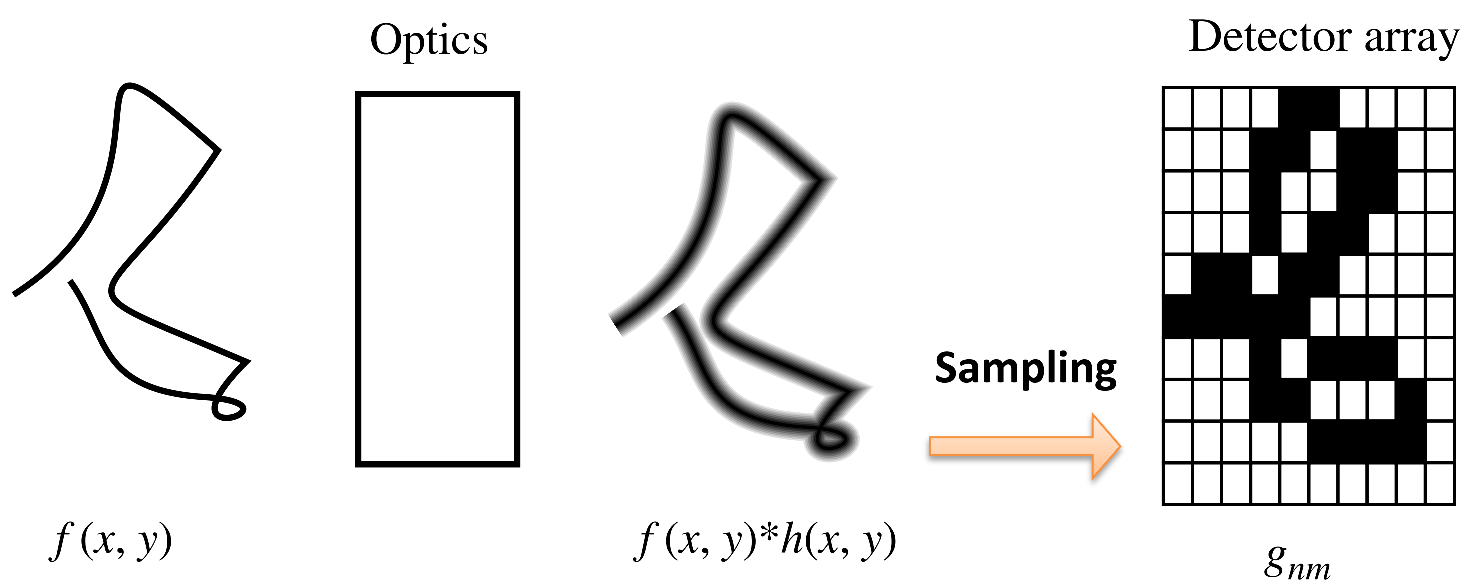
» Define $\phi_m(x') = (A^* p_m)(x') = \int h^*(x', x) p_m(x) \mathrm{d}x$

Proof?

» Semi-discrete mapping A_M:

$$g_m = (A_M f)_m = (f, \phi_m)$$

Semi-discrete mapping



- » Measurement g_{nm} is discrete
- » Description of object $f(x,y)$ is continuous
- » Not convenient for computation
- » Mostly useful for theoretical purpose
- » Next, fully discretized model
  - » most widely used approach for both computation and analysis!

Fully discrete LSI model

- **Sampling & Discretization**
  - Sampling of the image signal
  - **Discretize the object signal**
- What about the LSI system?
  - How to discretize the linear operator?
- "Transfer function" of discrete LSI system?
  - Eigenvalues of the imaging matrix

Discretization/decomposition of object function

» Represent continuous function $f(x)$ by a vector

$$\mathbf{f} = [f_0, f_1, \cdots, f_{N-1}]^T$$

» Assume a set of orthonormal basis functions $\{\psi_0, \psi_1, \cdots, \psi_{N-1}\}$

$$f(x) = \sum_{n=0}^{N-1} f_n \psi_n(x)$$

Idea is similar to sampling! → sampling requirement needs to be satisfied

Introduction to inverse problems in imaging Chap 9.1

Fully discrete LSI model

- **Sampling & Discretization**
  - Sampling of the image signal
  - Discretize the object signal
- **What about the LSI system?**
  - **How to discretize the linear operator?**
- “Transfer function” of discrete LSI system?
  - Eigenvalues of the imaging matrix

Fully discrete linear forward model

» $g_m = (f, \phi_m) = (\sum_{n=0}^{N-1} f_n \psi_n, \phi_m)$

$$= \sum_{n=0}^{N-1} (\psi_n, \phi_m) f_n$$

» Discrete forward model

$$\mathbf{g} = \mathbf{A}\mathbf{f}$$

» $\mathbf{A}_{mn} = (\psi_n, \phi_m) = (\psi_n, A^* p_m) = \boxed{(A\psi_n, p_m)}$

Proof?

Physical interpretation?

The result of a point measurement is Impulse response of A and the then "blurred" by the pixel

Most widely used: fully discrete LSI model with ideal (impulse) sampling

» Assume ideal impulse sampling on both object and detector

$$p_m = \delta(x - x_m)$$

» Standard basis

$$\psi_n(x) = \delta(x - x_n)$$

» LSI model

$$h(x, x') = h(x - x')$$

» What is $\mathbf{A}$?

$$\boxed{\mathbf{A}_{mn} = h(x_m - x_n)}$$

What does A matrix look like?