Boston University
Department of Electrical and Computer Engineering
EC522 Computational Optical Imaging
Homework No. 2

Issued: Monday, Feb. 5, 2024 Due: 11:59 pm Wednesday, Feb. 21, 2024

Problem 1: Example of geometric optics model based LSI imaging – Defocus and Depth of focus in Imaging

Depth of focus (DOF) is an important imaging metric, which measures the ability to image or filter out the objects that are away from the focal plane. This problem will explore this concept and its quantification from the computational standpoint.

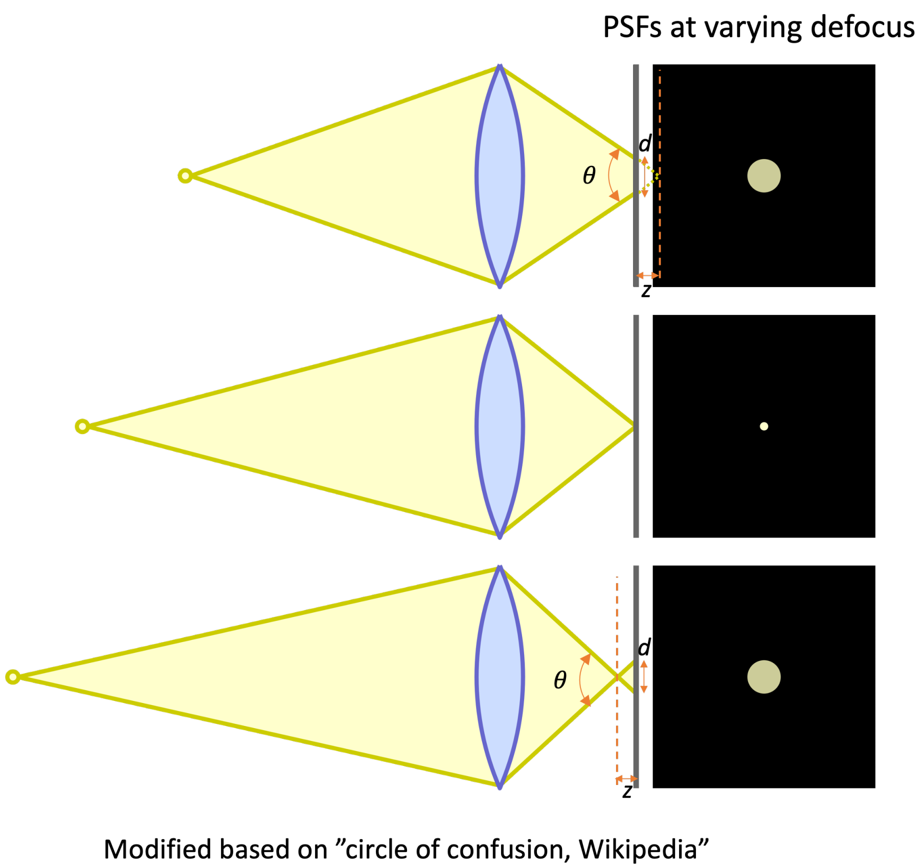

Figure 1: Circle of confusion due to defocus.

In an ideal camera, the defocus effect can be approximated by an LSI model governed by the convolution model. As such, the output image g captured by the camera is related to the input object's intensity distribution f by

$$g(x, y) = f(x, y) * h(x, y), \tag{1}$$

where $*$ denote the convolution, the defocus point-spread function (PSF) h is characterized by the "circle of confusion" that can be modeled using a geometric optics model, as illustrated in Fig. 1. A point source on the left is imaged onto the camera sensor on the right by a lens. Depending on the axial location of the point source, it can either form a "sharp" point image on the camera or a "defocused" image that resembles a circle – hence the term "circle of confusion".

For simplicity, one can assume a point source can always be focused to a point image by a lens (i.e. a focus). However, the camera may not be placed at the correct plane, which results in "defocus". As seen in the geometric relation depicted in Fig. 1, the larger the camera's displacement z from the actual focus is, the larger the defocus circular PSF is, corresponding to more severe blur in the captured image. For more detailed treatment, one may refer to `https://en.wikipedia.org/wiki/Circle_of_confusion`.

Based on the geometrical relation illustrated in Fig. 1, the size of the defocus PSF d can be approximated by

$$d = \theta z, \tag{2}$$

where z is the defocus distance, and θ is the angle of acceptance of the lens, which is an important quantity of an imaging system / camera, and is related to the lens size and the focal length. [Often times, the measure of θ is by the numerical aperture (NA), $\mathrm{NA} = \sin(\theta/2)$, or the f-number $\mathrm{f}/\# = 1/\theta$ of the camera lens].

In the following, we will formulate the forward model of this imaging problem using the tools learned in the lectures.

(a) Forward model of linear shift invariant (LSI) system in its **operator form**

(1) Consider a **1D system** in which the defocus PSF becomes a "line of confusion". Construct the forward model that relates the output intensity image g with the input object f given the camera's displacement z, and the angle of acceptance θ. (**Hint:** 1D line-shape PSF can be modeled as a rectangular function.)

Since the system is LSI, the forward operator A is a convolution operator,

$$(Af)(x) = f(x) * h(x), \tag{3}$$

where h is the 1D PSF. Here, considering the defocus resulting in a PSF corresponds to a "line of confusion", with a length of $d = \theta z$, the PSF h can be represented as a rectangular function, as

$$h(x) = \mathrm{rect}\left(\frac{x}{\theta z}\right), \tag{4}$$

(2) Find the range space, null space, as well as the spectral representation of the forward operator, the adjoint operator.

The spectral representation of the forward operator is

$$(Af)(x) = \int F(u)H(u)\exp(i2\pi xu)du, \tag{5}$$

where $F(u)$ is the Fourier transform (FT) of the object, $H(u)$ is the transfer function (TF), and is

$$H(u) = \mathrm{sinc}(\theta z u). \tag{6}$$

The range and null space can be defined according to the form of $H(u)$. $H(u_N) = 0$ when $u_N = m/\theta z$, where $m = \pm 1, \pm 2, \cdots$.

The range of A, $R(A)$ is defined by all the 1D functions that do not contain any frequency components at u_N.

The null space of A, $N(A)$ is defined by the 1D functions that only contain a discrete set of frequencies at u_N.

The adjoint operator is

$$\begin{aligned}(A^*g)(x) &= g(x) * h^*(-x)\\ &= \int G(u)H^*(u)\exp(i2\pi xu)du\\ &= \int G(u)H(u)\exp(i2\pi xu)du,\end{aligned} \tag{7}$$

where $G(u)$ is the FT of $g(x)$, where $H^*(u) = H(u)$ according to Eq. (6).

The inverse operator does not exist since $H(u_N) = 0$ for a set of infinite points (it has a non-empty null space).

(b) Based on the analysis in (a) and what we learned about **null space**, explain the following observations in practice:

(3) Why the larger the defocus distance z is, the "harder" it is to recover the in-focus object given only a single defocused image.

(4) Why the larger the NA is, the harder it is to recover the in-focus object given only a single defocused image.

Since the null space of A is defined by the set of frequencies at $u_N = m/\theta z$, where $m = \pm 1, \pm 2, \cdots$. By increasing z and/or θ, the spacing between these frequencies become narrower. Now consider a bandlimited object signal (e.g. with a bandwidth Ω), the narrower the spacing between u_N means that more of these frequencies will be within the bandlimit, which in turn results in more frequency components of the object being "invisible" to the measurement (i.e. in the null space of A).

Problem 2: Example of wave optics model based LSI imaging – Digital holography

Holography is a 3D imaging technique, in the sense that it allows recreate the 3D scene (optically or digitally) from its single 2D measurement. In this problem, we will explore the general idea of in-line (Gabor) holography for 2D imaging and understand the unique feature about holography using the tools we have learned so far.

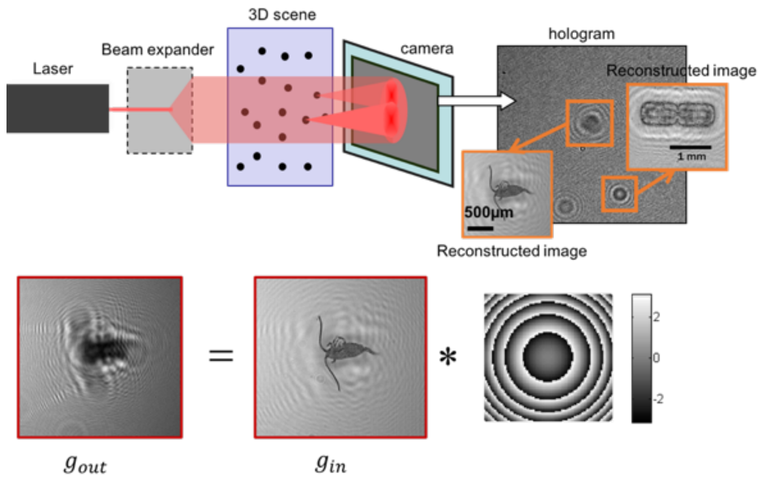

Figure 2: In-line digital holography.

A schematic of the in-line holography is shown in Fig. 2. To record a hologram, a coherent light source (e.g. laser) is used to illuminate the 3D scene. Accordingly, the formation of the hologram needs to be modeled using the wave optics model (as opposed to the geometric optics model which does not account for the effect of interference). The hologram (i.e. the intensity image captured by the camera) is the result from the interference between the unperturbed illumination (i.e. the reference beam) and the light diffracted from the 3D object.

Using a wave optics model, the formation of the hologram from a 2D object at a depth z can be approximated using the following linear shift invariant (LSI) model

$$g_{\text{out}}(x, y) = g_{\text{in}}(x, y; z) * h(x, y; z), \tag{8}$$

where $*$ denote the 2D convolution, g_{out} is the signal term of interest contained in the hologram measurement, g_{in} is the object signal and is a complex valued function, and h is the point spread function (PSF) and is also a complex valued function. The form of h can be found by the free-space propagation and wave diffraction theory, which has the following approximated form,

$$h(x, y; z) = \frac{1}{i\lambda z} \exp\left\{ ik\frac{x^2 + y^2}{2z} \right\}, \tag{9}$$

and the corresponding transfer function (i.e. the 2D Fourier transform of the PSF

$h(x, y; z)$ at a given depth z):

$$H(u, v; z) = \exp\left\{-i\pi\lambda z(u^2 + v^2)\right\}, \tag{10}$$

where $k = 2\pi/\lambda$ is a constant (i.e. the wavenumber), λ is the wavelength of the laser, x, y denote the lateral coordinates and z denotes the axial direction along which the laser propagates from, and u, v denote the spatial frequency coordinates, according to the following 2D Fourier transform definition

$$H(u, v) = \iint h(x, y) \exp\{-i2\pi(ux + vy)\} dxdy. \tag{11}$$

In the following, we will formulate the forward model of this imaging problem using the tools learned in the lectures.

(1) Construct the forward model in the linear operator form.

Since the system is LSI, the forward operator A is a convolution operator,

$$(Af)(x, y) = f(x, y) * h(x, y), \tag{12}$$

where $*$ is the 2D convolution, and h is the 2D PSF, defined by

$$h(x, y) = \frac{1}{i\lambda z} \exp\left\{ik\frac{x^2 + y^2}{2z}\right\}. \tag{13}$$

(2) Find the range, null space, spectral representation and adjoint of the LSI system in the linear operator form.

The spectral representation of the forward operator is

$$(Af)(x, y) = \int F(u, v)H(u, v) \exp(i2\pi(ux + vy))dudv, \tag{14}$$

where $F(u, v)$ is the 2D Fourier transform (FT) of the object, $H(u, v)$ is the transfer function (TF), and is defined by Eq. (10).

The range and null space can be defined according to the form of $H(u, v)$. In particular, there is *no solution* for the condition $H(u, v) = 0$. As a result, 1) The range of A, $R(A)$ contains all 2D complex functions. 2) The null space of A, $N(A) = \{\mathbf{0}\}$.

The adjoint operator is

$$\begin{aligned}(A^*g)(x, y) &= g(x, y) * h^*(-x, -y) \\ &= \int G(u, v)H^*(u, v) \exp(i2\pi(ux + vy))dudv\end{aligned} \tag{15}$$

where $G(u,v)$ is the 2D FT of $g(x,y)$, where

$$H^*(u,v) = \exp\left\{i\pi\lambda z(u^2+v^2)\right\}, \tag{16}$$

according to Eq. (10).

Since A is invertible (based on our null space analysis), we can also find the inverse operator A^{-1}. Recall that the inverse operator is akin to the "direct deconvolution" process as

$$(A^{-1}g)(x,y) = \int \frac{G(u,v)}{H(u,v)} \exp(i2\pi(ux+vy))dudv. \tag{17}$$

Interestingly, we can find

$$H^*(u,v) = \frac{1}{H(u,v)}, \tag{18}$$

according to Eq. (10) and (16). This means that $A^{-1} = A^*$! In other words, in this special case, the ajoint operator performs exactly the inverse (i.e. direct deconvolution), making holography a very special imaging modality.

(3) Based on what we learned about **null space**, explain why it is "easier" to recover an object that this is defocused (i.e. $z \neq 0$) using holography, as compared to a standard camera (as in Problem 1).

It is "easier" to recover a defocused object using holography because the forward model has a trivial null space, making A easily invitable.