SE/EC 524/674 Optimization Theory and Methods

Yannis Paschalidis
yannisp@bu.edu, http://sites.bu.edu/paschalidis/

Department of Electrical and Computer Engineering,
Division of Systems Engineering,
Department of Biomedical Engineering,
Faculty of Computing & Data Sciences, and
Hariri Institute for Computing and Computational Science and Engineering,
Boston University

Lecture 1: Outline

1. Administrative stuff.
2. Some History.
3. LP flavors.

History of Optimization

Fermat, 1638; Newton, 1670

$$\min\ f(x) \qquad x\text{: scalar}$$

$$\frac{df(x)}{dx} = 0$$

Euler, 1755

$$\min\ f(x_1, \ldots, x_n)$$

$$\nabla f(\mathbf{x}) = 0$$

Lagrange, 1797

$$\begin{aligned} &\min\ f(x_1, \ldots, x_n) \\ &\text{s.t. } g_k(x_1, \ldots, x_n) = 0, \qquad k = 1, \ldots, m. \end{aligned}$$

Euler, Lagrange Problems in infinite dimensions ($n \to \infty$), calculus of variations.

Linear Programming (LP)

$$\begin{aligned} \text{minimize} \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\ & 2x_1 + x_2 \geq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

in **vector notation**

$$\mathbf{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

LP History

George Dantzig, 1947 Simplex method.

Fourier, 1826 Method for solving system of linear inequalities.

de la Vallée Poussin simplex-like method for objective function with absolute values.

Kantorovich, Koopmans, 1930s Formulations and solution method.

von Neumann, 1928 game theory, duality.

Farkas, Minkowski, Carathéodory, 1870-1930 Foundations.

1950s Applications.

1960s Large Scale Optimization.

1970s Complexity theory.

Khachyan, 1979 The ellipsoid algorithm.

Karmakar, 1984 Interior point algorithms.

Applications of LP

- Transportation (WW II, air traffic control, crew scheduling, etc.)
- Telecommunications (routing, scheduling, resource allocation)
- Manufacturing (production planning, scheduling, resource allocation)
- Medicine, Computational Biology (metabolic networks, protein side-chain packing).
- Engineering.
- Typesetting (TEX, LATEX)

Possible solution outcomes

1. There exists a **unique optimal solution**.
2. There exist **multiple optimal solutions** (their set being either bounded or unbounded).
3. Optimal cost is $-\infty$ and no feasible solution is optimal (**unbounded problem**).
4. Feasible set is empty (**infeasible problem**).

Various LP Flavors

(General LP)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{a} \\ & \mathbf{B}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{D}\mathbf{x} = \mathbf{d} \\ & x_i \geq 0, \quad i \in I \\ & x_j \leq 0, \quad j \in J. \end{aligned}$$

reduces to

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned}$$

(Standard form LP)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Remark

Every LP can be written in standard form.