Boston University
Department of Electrical and Computer Engineering
EC522 Computational Optical Imaging
Homework No. 3

Issued: Wednesday, Feb. 21, 2024 Due: 11:59 pm Monday, Mar. 4, 2024

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**Problem 1: Example of geometric optics model based LSI imaging – Defocus and Depth of focus in Imaging – Cont'd**

Recall in HW 2, we discussed the following problem (in blue).

Depth of focus (DOF) is an important imaging metric, which measures the ability to image or filter out the objects that are away from the focal plane. This problem will explore this concept and its quantification from the computational standpoint.

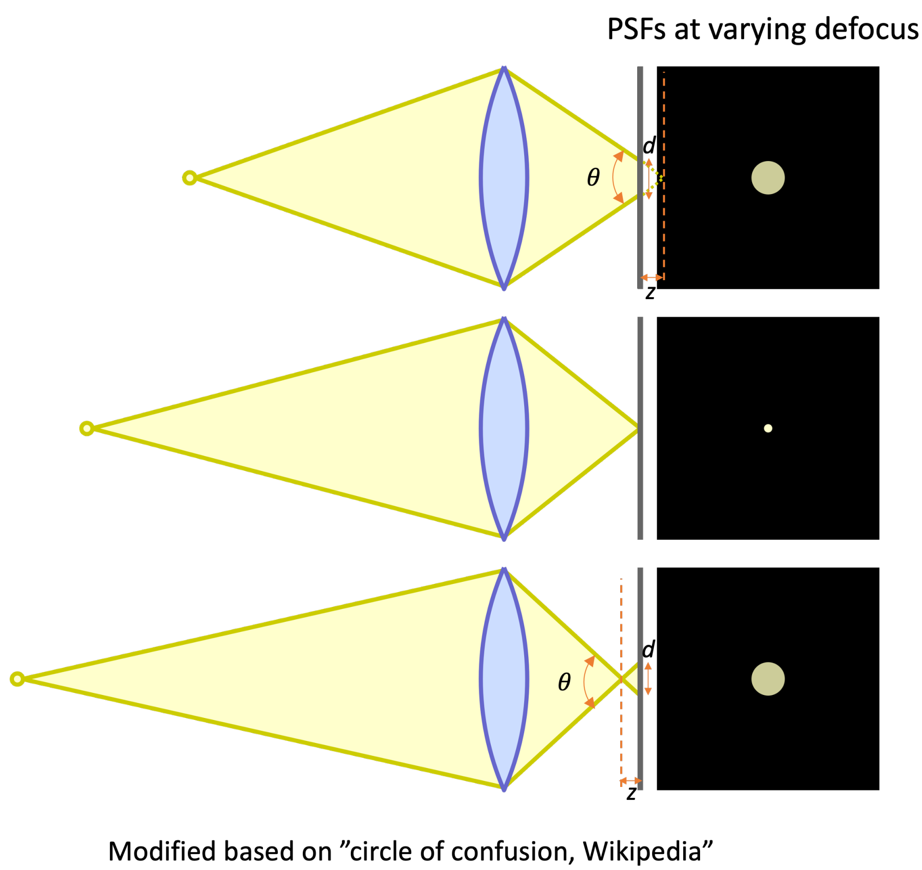


Figure 1: Circle of confusion due to defocus.

In an ideal camera, the defocus effect can be approximated by an LSI model governed by the convolution model. As such, the output image $g$ captured by the camera is related to the input object's intensity distribution $f$ by

$$g(x, y) = f(x, y) * h(x, y), \tag{1}$$

where $*$ denote the convolution, the defocus point-spread function (PSF) $h$ is characterized by the "circle of confusion" that can be modeled using a geometric optics model, as illustrated in Fig. 1. A point source on the left is imaged onto the camera sensor on the right by a lens. Depending on the axial location of the point source, it can either form a "sharp" point image on the camera or a "defocused" image that resembles a circle – hence the term "circle of confusion".

For simplicity, one can assume a point source can always be focused to a point image by a lens (i.e. a focus). However, the camera may not be placed at the correct plane, which results in "defocus". As seen in the geometric relation depicted in Fig. 1, the larger the camera's displacement $z$ from the actual focus is, the larger the defocus circular PSF is, corresponding to more severe blur in the captured image. For more detailed treatment, one may refer to `https://en.wikipedia.org/wiki/Circle_of_confusion`.

Based on the geometrical relation illustrated in Fig. 1, the size of the defocus PSF $d$ can be approximated by

$$d = \theta z, \tag{2}$$

where $z$ is the defocus distance, and $\theta$ is the angle of acceptance of the lens, which is an important quantity of an imaging system / camera, and is related to the lens size and the focal length. [Often times, the measure of $\theta$ is by the numerical aperture (NA), $\mathrm{NA} = \sin(\theta/2)$, or the f-number $\mathrm{f}/\# = 1/\theta$ of the camera lens].

(1) Consider a **1D system** in which the defocus PSF becomes a "line of confusion". Construct the *convolution matrix* $\mathbf{A}$ that relates the (vectorized) output intensity image $\mathbf{g}$ with the (vectorized) input object $\mathbf{f}$ given the (vectorized) defocus PSF $\mathbf{h}$ (set by the camera's displacement $z$, and the angle of acceptance $\theta$. (**Hint:** the 1D line-shape PSF can be modeled as a discrete rectangular signal).

(2) Formulate the *spectral representation* of the *forward* matrix $\mathbf{A}$, the *inverse* matrix $\mathbf{A}^{-1}$ (if exist), and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.

(3) Formulate the *range space* and the *null space* of the *forward* matrix $\mathbf{A}$ and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.

**Problem 2: Example of wave optics model based LSI imaging – Digital holography – Cont'd**

Recall in HW 2, we discussed the following problem (in blue).

Holography is a 3D imaging technique, in the sense that it allows recreate the 3D scene (optically or digitally) from its single 2D measurement. In this problem, we will explore the general idea of in-line (Gabor) holography for 2D imaging and understand the unique feature about holography using the tools we have learned so far.

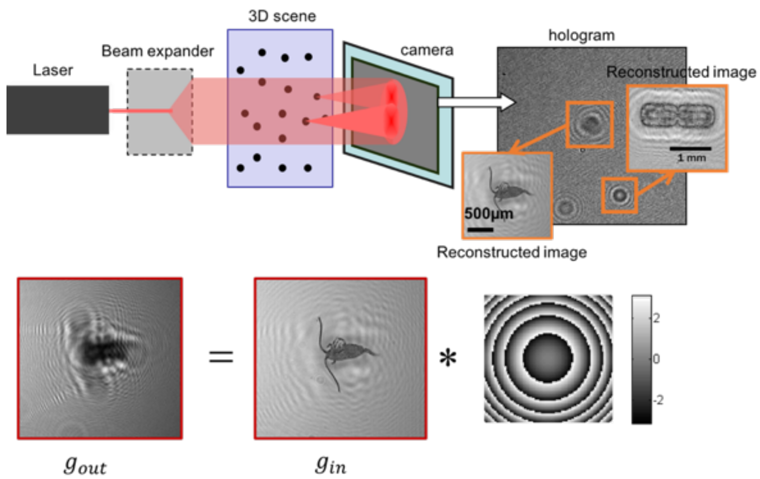


Figure 2: In-line digital holography.

A schematic of the in-line holography is shown in Fig. 2. To record a hologram, a coherent light source (e.g. laser) is used to illuminate the 3D scene. Accordingly, the formation of the hologram needs to be modeled using the wave optics model (as opposed to the geometric optics model which does not account for the effect of interference). The hologram (i.e. the intensity image captured by the camera) is the result from the interference between the unperturbed illumination (i.e. the reference beam) and the light diffracted from the 3D object.

Using a wave optics model, the formation of the hologram from a 2D object at a depth $z$ can be approximated using the following linear shift invariant (LSI) model

$$g_{\text{out}}(x, y) = g_{\text{in}}(x, y; z) * h(x, y; z), \tag{3}$$

where $*$ denote the 2D convolution, $g_{\text{out}}$ is the signal term of interest contained in the hologram measurement, $g_{\text{in}}$ is the object signal and is a complex valued function, and $h$ is the point spread function (PSF) and is also a complex valued function. The form of $h$ can be found by the free-space propagation and wave diffraction theory, which has the following approximated form,

$$h(x, y; z) = \frac{1}{i\lambda z} \exp\left\{ik\frac{x^2 + y^2}{2z}\right\}, \tag{4}$$

and the corresponding transfer function (i.e. the 2D Fourier transform of the PSF $h(x, y; z)$ at a given depth $z$):

$$H(u, v; z) = \exp\left\{-i\pi\lambda z(u^2 + v^2)\right\}, \tag{5}$$

where $k = 2\pi/\lambda$ is a constant (i.e. the wavenumber), $\lambda$ is the wavelength of the laser, $x, y$ denote the lateral coordinates and $z$ denotes the axial direction along which the laser propagates from, and $u, v$ denote the spatial frequency coordinates, according to the following 2D Fourier transform definition

$$H(u, v) = \iint h(x, y) \exp\{-i2\pi(ux + vy)\}dxdy. \tag{6}$$

(1) Construct the *convolution matrix* $\mathbf{A}$ that relates the (vectorized) output intensity image $\mathbf{g}$ with the (vectorized) input object $\mathbf{f}$ given the (vectorized) PSF $\mathbf{h}$.

(2) Formulate the *spectral representation* of the *forward* matrix $\mathbf{A}$, the *inverse* matrix $\mathbf{A}^{-1}$ (if exist), and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.

(3) Formulate the *range space* and the *null space* of the *forward* matrix $\mathbf{A}$ and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.