Boston University
Department of Electrical and Computer Engineering
EC522 Computational Optical Imaging
Homework No. 5

Issued: Monday, Mar. 18 — Due: 11:59 pm Monday, Apr. 1

Deblurring from multiple defocused measurements

This problem will continue our discussions on solving the inverse problem from defocused measurement. In HW 4, we consider the case where we are only given a single measurement. In HW 5, we will consider the recovery of a single object by taking *multiple* defocused measurements.

As an illustration, let us consider two defocused measurement, as described by

$$g_1 = f * h_1; \tag{1}$$
$$g_2 = f * h_2; \tag{2}$$

The following Matlab files are provided.
a) The object, f, is in the mat-file `I1.mat`.
b) We will consider two defocus distances, including $z_1 = 0.1$mm and $z_2 = -0.2$mm. The corresponding PSFs, h_1 and h_2 are provided in `psf1.mat` and `psf2.mat`, respectively.

First, derive your results for the following problems.
(1) Rewrite the two measurements into a single linear model in the $\mathbf{y} = \mathbf{A}\mathbf{x}$ form.

The forward model can be derived by taking the following steps:

1. rewrite $g_1 = f * h_1$ into $\mathbf{g}_1 = \mathbf{A}_1\mathbf{f}$
2. rewrite $g_2 = f * h_2$ into $\mathbf{g}_2 = \mathbf{A}_2\mathbf{f}$
3. combine the two as
$$\mathbf{g} = \begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix} = \mathbf{A}\mathbf{f} = \begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}\mathbf{f}. \tag{3}$$

In addition, since both $\mathbf{A}_1$ and $\mathbf{A}_2$ describe LSI system, they can be written in their eigenvalue decomposition (EVD) as $\mathbf{A}_i = \frac{1}{N^2}\mathbf{W}_{\text{2D}}^H \text{diag}(\widehat{\mathbf{h}}_i)\mathbf{W}_{\text{2D}}, i = 1, 2$.

(2) following the derivation procedure demonstrated in the lectures, derive the least-squares (LS) solution of this problem.

The LS solution satisfies
$$\mathbf{A}^H\mathbf{A}\mathbf{f}_{\text{LS}} = \mathbf{A}^H\mathbf{g}, \tag{4}$$

where

$$
\begin{aligned}
\mathbf{A}^H\mathbf{g} &= \left[\mathbf{A}_1^H \mathbf{A}_2^H\right]\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix} = \mathbf{A}_1^H\mathbf{g}_1 + \mathbf{A}_2^H\mathbf{g}_2 \\
&= \frac{1}{N^2}\mathbf{W}_{2\mathrm{D}}^H \mathrm{diag}(\widehat{\mathbf{h}_1}^*)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_1 + \frac{1}{N^2}\mathbf{W}_{2\mathrm{D}}^H \mathrm{diag}(\widehat{\mathbf{h}_2}^*)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_2 \\
&= \frac{1}{N^2}\mathbf{W}_{2\mathrm{D}}^H \left(\mathrm{diag}(\widehat{\mathbf{h}_1}^*)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_1 + \mathrm{diag}(\widehat{\mathbf{h}_2}^*)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_2\right)
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
\mathbf{A}^H\mathbf{A}\mathbf{f}_{\mathrm{LS}} &= \left[\mathbf{A}_1^H \mathbf{A}_2^H\right]\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}\mathbf{f}_{\mathrm{LS}} = (\mathbf{A}_1^H\mathbf{A}_1 + \mathbf{A}_2^H\mathbf{A}_2)\mathbf{f}_{\mathrm{LS}} \\
&= \frac{1}{N^2}\mathbf{W}_{2\mathrm{D}}^H(\mathrm{diag}(|\widehat{\mathbf{h}_1}|^2 + |\widehat{\mathbf{h}_2}|^2)\mathbf{W}_{2\mathrm{D}}\mathbf{f}_{\mathrm{LS}}.
\end{aligned}
\tag{6}
$$

Thus,

$$
\mathbf{f}_{\mathrm{LS}} = \frac{1}{N^2}\mathbf{W}_{2\mathrm{D}}^H\left[\mathrm{diag}\left(\frac{\widehat{\mathbf{h}_1}^*}{|\widehat{\mathbf{h}_1}|^2 + |\widehat{\mathbf{h}_2}|^2}\right)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_1 + \mathrm{diag}\left(\frac{\widehat{\mathbf{h}_2}^*}{|\widehat{\mathbf{h}_1}|^2 + |\widehat{\mathbf{h}_2}|^2}\right)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_2\right] \tag{7}
$$

(3) Derive the Tikhonov regularization solution of this problem.

The Tikhonov regularization solution is

$$
\mathbf{f}_\mu = (\mathbf{A}^H\mathbf{A} + \mu\mathbf{I})^{-1}\mathbf{A}^H\mathbf{g}, \tag{8}
$$

following the same derivation as in (B2), we can find

$$
\mathbf{f}_\mu = \frac{1}{N^2}\mathbf{W}_{2\mathrm{D}}^H\left[\mathrm{diag}\left(\frac{\widehat{\mathbf{h}_1}^*}{|\widehat{\mathbf{h}_1}|^2 + |\widehat{\mathbf{h}_2}|^2 + \mu}\right)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_1 + \mathrm{diag}\left(\frac{\widehat{\mathbf{h}_2}^*}{|\widehat{\mathbf{h}_1}|^2 + |\widehat{\mathbf{h}_2}|^2 + \mu}\right)\mathbf{W}_{2\mathrm{D}}\mathbf{g}_2\right] \tag{9}
$$

Next, write Matlab scripts to complete the following questions. Submit both your scripts as well as the output results.

(4) Simulate the output images for the two cases, assuming no noise is present.

(5) Consider both measurements suffer from the same level of white Gaussian noise (WGN) for all pixels. For $n_{\mathrm{std}} = 1, 10$. Simulate the corresponding noisy images.

(6) At each noise level, find the "best" deblurred images in each case.

(7) How does the noise level affect the reconstruction quality? Qualitatively explain the artifacts you see in the reconstruction. How does the noise level affect the optimal value of Tikhonov regularization parameter?

In general, one should observe that multiple measurements can improve reconstruction quality. This can be understood intuitively by the effect of averaging. More rigorously, as seen in both LS and TR solutions, the denominator of the inversion filter becomes $|\widehat{\mathbf{h}_1}|^2 + |\widehat{\mathbf{h}_2}|^2$ as compared to $|\widehat{\mathbf{h}}_i|^2$ in the single measurement case. In general, the summation removes the near-zero transfer functions since the locations of zeros in $\widehat{\mathbf{h}_1}$ and $\widehat{\mathbf{h}_2}$ are not the same, which implies that the effective condition number of the combined system is better (smaller). We will see the rigorous definition of the condition number of this case in the next lecture.