Boston University
Department of Electrical and Computer Engineering
EC522 Computational Optical Imaging
Homework No. 3

Issued: Wednesday, Feb. 21, 2024 Due: 11:59 pm Monday, Mar. 4, 2024

Problem 1: Example of geometric optics model based LSI imaging – Defocus and Depth of focus in Imaging – Cont'd

Recall in HW 2, we discussed the following problem (in blue).

Depth of focus (DOF) is an important imaging metric, which measures the ability to image or filter out the objects that are away from the focal plane. This problem will explore this concept and its quantification from the computational standpoint.

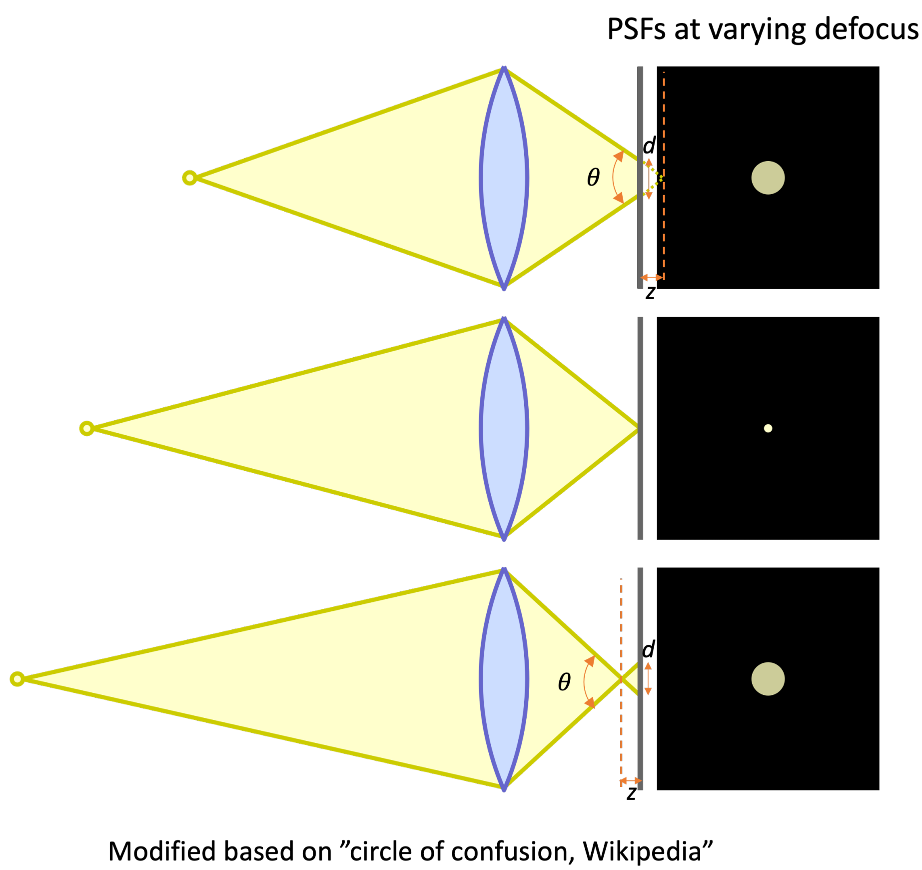

Figure 1: Circle of confusion due to defocus.

In an ideal camera, the defocus effect can be approximated by an LSI model governed by the convolution model. As such, the output image g captured by the camera is related to the input object's intensity distribution f by

$$g(x, y) = f(x, y) * h(x, y), \tag{1}$$

where $*$ denote the convolution, the defocus point-spread function (PSF) h is characterized by the "circle of confusion" that can be modeled using a geometric optics model, as illustrated in Fig. 1. A point source on the left is imaged onto the camera sensor on the right by a lens. Depending on the axial location of the point source, it can either form a "sharp" point image on the camera or a "defocused" image that resembles a circle – hence the term "circle of confusion".

For simplicity, one can assume a point source can always be focused to a point image by a lens (i.e. a focus). However, the camera may not be placed at the correct plane, which results in "defocus". As seen in the geometric relation depicted in Fig. 1, the larger the camera's displacement z from the actual focus is, the larger the defocus circular PSF is, corresponding to more severe blur in the captured image. For more detailed treatment, one may refer to `https://en.wikipedia.org/wiki/Circle_of_confusion`.

Based on the geometrical relation illustrated in Fig. 1, the size of the defocus PSF d can be approximated by

$$d = \theta z, \tag{2}$$

where z is the defocus distance, and θ is the angle of acceptance of the lens, which is an important quantity of an imaging system / camera, and is related to the lens size and the focal length. [Often times, the measure of θ is by the numerical aperture (NA), $\mathrm{NA} = \sin(\theta/2)$, or the f-number $f/\# = 1/\theta$ of the camera lens].

(1) Consider a **1D system** in which the defocus PSF becomes a "line of confusion". Construct the *convolution matrix* $\mathbf{A}$ that relates the (vectorized) output intensity image $\mathbf{g}$ with the (vectorized) input object $\mathbf{f}$ given the (vectorized) defocus PSF $\mathbf{h}$ (set by the camera's displacement z, and the angle of acceptance θ. (**Hint:** the 1D line-shape PSF can be modeled as a discrete rectangular signal).

Consider only bandlimited signals with finite signal length, the LSI system can be fully described by the circular (circulant) convolution matrix $\mathbf{H}$, given that the sampling has been performed satisfying the Nyquist sampling requirement,

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \mathbf{h} * \mathbf{f}, \tag{3}$$

where the column vectors $\mathbf{g}$ and $\mathbf{f}$ are discrete samples from the object function f and the image function g, respectively. The column vector $\mathbf{h}$ represents the discrete PSF, and is the 1st column vector in $\mathbf{H}$. The elements in $\mathbf{H}$ satisfies

$$\mathbf{H}_{mn} = \mathbf{h}_{m-n}, \tag{4}$$

where m and n are the column and row indices, respectively. Assuming using ideal impulse sampling for both object and the image functions, the elements of $\mathbf{h}$ are samples from the continuous PSF function h, as

$$\mathbf{h}_q = h(q\Delta), \tag{5}$$

where Δ is the sampling rate, and q is the signal index. Relating to the definition of h in (a), we see that

$$\mathbf{h}_q = \mathrm{Rect}\left(\frac{q}{[\theta z/\Delta]}\right), \tag{6}$$

where Rect$(\cdot)$ represents the discrete rectangular signal; the signal length L is set by $L = [\theta z/\Delta]$, in which $[p]$ denotes the largest integer that is less than p.

(2) Formulate the *spectral representation* of the *forward* matrix $\mathbf{A}$, the *inverse* matrix $\mathbf{A}^{-1}$ (if exist), and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.

The key point here is that a circulant matrix can be fully represented by its eigenvalue decomposition (EVD), in which the eigenvectors are the column vectors of the discrete Fourier transform (DFT) matrix $\mathbf{W}$, and the eigenvalues corresponds to the DFT of the discrete PSF.

$$\mathbf{H} = \frac{1}{N}\mathbf{W}^H \mathrm{diag}(\hat{\mathbf{h}})\mathbf{W}, \tag{7}$$

where $\hat{\mathbf{h}}$ denotes the DFT of $\mathbf{h}$ and represents to the discrete TF, and diag$(\hat{\mathbf{h}})$ forms a diagonal matrix with the diagonal elements defined by the elements of vector $\hat{\mathbf{h}}$.

The spectral representation is

$$\mathbf{Hf} = \sum_{n=0}^{N-1} \hat{\mathbf{h}}_n(\mathbf{f}\cdot\mathbf{w}_n)\mathbf{w}_n, \tag{8}$$

where $\mathbf{w}_n$ is the nth column vector of $\mathbf{W}^H$. From the pure matrix point of view, the meaning of the spectral representation is that the column vector $\mathbf{f}$ is first projected on the linear space defined by the set of basis $\mathbf{w}_n$. The output vector from the linear system $\mathbf{Hf}$ is in the linear space spanned by the set of basis $\mathbf{w}_n$, with the coefficients defined by $\hat{\mathbf{h}}_n(\mathbf{f}\cdot\mathbf{w}_n)$.

The discrete TF, $\hat{\mathbf{h}}$ of this problem is defined by the (N-point) DFT of the discrete rectangular function of length L,

$$\hat{\mathbf{h}}(n) = \mathrm{DFT}\left\{\mathrm{Rect}\left(\frac{q}{L}\right)\right\} = \begin{cases} L, & \text{for } n = 0. \\ \frac{\sin(\pi L n/N)}{\sin(\pi n/N)}, & \text{for } n = 1, 2, \cdots, N-1. \end{cases} \tag{9}$$

A key difference between the continuous TF and the discrete TF is illustrated in this example. Specifically, the continuous TF goes to zero at the set of frequencies at u_N. This is *not* the case for the discrete TF. This is because $\hat{\mathbf{h}}(n)$ goes to zero only if the following relation can be satisfied

$$\pi Ln/N = m\pi, \quad \text{where } m = \pm 1, \pm 2, \cdots, \tag{10}$$

which requires solutions exist for $n = mN/L$.

The adjoint is simply the Hermitian of the matrix $\mathbf{H}$,

$$(\mathbf{H}^*)_{mn} = (\mathbf{H}^*)_{nm} \tag{11}$$

The spectral representation of the adjoint is

$$\mathbf{H}^*\mathbf{g} = \sum_{n=0}^{N-1} \hat{\mathbf{h}}_n^*(\mathbf{g} \cdot \mathbf{w}_n)\mathbf{w}_n, \tag{12}$$

which shows that the adjoint has a discrete TF that is simply the complex conjugate of the discrete TF of the forward model.

The inverse matrix does exist for many cases when the relation $n = mN/L$ cannot be satisfied. It can be written in its spectral representation as

$$\mathbf{H}^{-1}\mathbf{g} = \sum_{n=0}^{N-1} \frac{1}{\hat{\mathbf{h}}}(\mathbf{g} \cdot \mathbf{w}_n)\mathbf{w}_n, \tag{13}$$

where $\frac{1}{\hat{\mathbf{h}}}$ denotes taking the element-wise inverse of the vector $\hat{\mathbf{h}}$.

This problem shows that by simply discretizing the forward problem of a LSI system, the null space of the forward operator can be completely removed! This in fact is a general observation for many linear imaging systems. But does this mean the problem become easier to solve? The intuition tells us the answer should be NO. We will see why this is indeed the case when analyzing the corresponding inverse problem.

(3) Formulate the *range space* and the *null space* of the *forward* matrix $\mathbf{A}$ and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.

Since $\widehat{\mathbf{h}}$ never goes to zero, i.e. the DFT transfer function is always non-zero, the null space of $\mathbf{H}$ is trivial, containing only $\mathbf{0}$. For the same reason, the range of $\mathbf{H}$ is defined by the entire linear space spanned by all the column vectors of $\mathbf{W}s$, which covers the entire DFT frequencies. The same argument can be made for $\mathbf{H}^*$.

Problem 2: Example of wave optics model based LSI imaging – Digital holography – Cont'd

Recall in HW 2, we discussed the following problem (in blue).

Holography is a 3D imaging technique, in the sense that it allows recreate the 3D scene (optically or digitally) from its single 2D measurement. In this problem, we will explore the general idea of in-line (Gabor) holography for 2D imaging and understand the unique feature about holography using the tools we have learned so far.

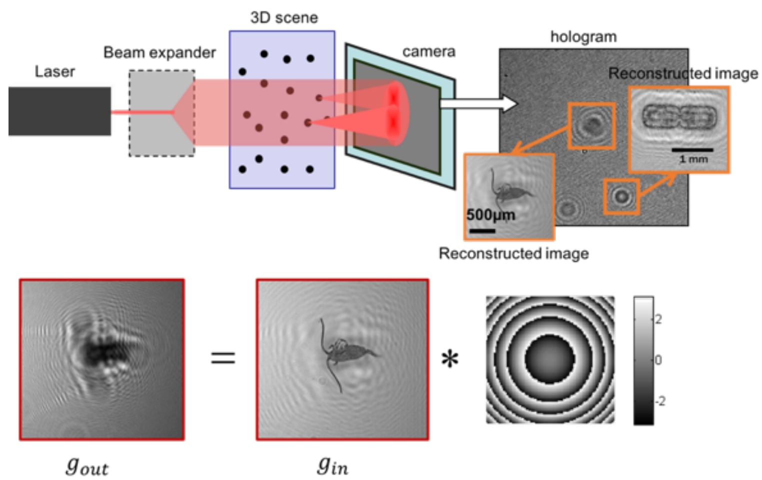

Figure 2: In-line digital holography.

A schematic of the in-line holography is shown in Fig. 2. To record a hologram, a coherent light source (e.g. laser) is used to illuminate the 3D scene. Accordingly, the formation of the hologram needs to be modeled using the wave optics model (as opposed to the geometric optics model which does not account for the effect of interference). The hologram (i.e. the intensity image captured by the camera) is the result from the interference between the unperturbed illumination (i.e. the reference beam) and the light diffracted from the 3D object.

Using a wave optics model, the formation of the hologram from a 2D object at a depth z can be approximated using the following linear shift invariant (LSI) model

$$g_{\text{out}}(x, y) = g_{\text{in}}(x, y; z) * h(x, y; z), \tag{14}$$

where $*$ denote the 2D convolution, g_{out} is the signal term of interest contained in the hologram measurement, g_{in} is the object signal and is a complex valued function, and h is the point spread function (PSF) and is also a complex valued function. The form of h can be found by the free-space propagation and wave diffraction theory, which has the following approximated form,

$$h(x, y; z) = \frac{1}{i\lambda z} \exp\left\{ik\frac{x^2 + y^2}{2z}\right\}, \tag{15}$$

and the corresponding transfer function (i.e. the 2D Fourier transform of the PSF $h(x, y; z)$ at a given depth z):

$$H(u, v; z) = \exp\left\{-i\pi\lambda z(u^2 + v^2)\right\}, \tag{16}$$

where $k = 2\pi/\lambda$ is a constant (i.e. the wavenumber), λ is the wavelength of the laser, x, y denote the lateral coordinates and z denotes the axial direction along which the laser propagates from, and u, v denote the spatial frequency coordinates, according to the following 2D Fourier transform definition

$$H(u, v) = \iint h(x, y) \exp\{-i2\pi(ux + vy)\} dx dy. \tag{17}$$

(1) Construct the *convolution matrix* $\mathbf{A}$ that relates the (vectorized) output intensity image $\mathbf{g}$ with the (vectorized) input object $\mathbf{f}$ given the (vectorized) PSF $\mathbf{h}$.

Since the underlying problem is 2D, let us denote the discrete samples from the 2D object, PSF, and image functions as 2D matrices, $\mathbf{f}', \mathbf{h}'$, and $\mathbf{g}'$, respectively. To formulate the matrix relation, we also need to define their corresponding vectorized form (which are obtained by lexicographically stacking the 2D matrices into column vectors) as $\mathbf{f}, \mathbf{h}$, and $\mathbf{g}$, respectively. This vectorization process is commonly denoted as $\mathbf{f} = \text{vec}(\mathbf{f}')$, $\mathbf{h} = \text{vec}(\mathbf{h}')$, $\mathbf{g} = \text{vec}(\mathbf{g}')$.

Consider only bandlimited signals with finite signal length, the LSI system can be fully described by the block circular convolution matrix $\mathbf{H}$. To simplify the notation and without loss of generality, we will also assume that the same sampling rate Δ is used along both x and y and the dimension of $\mathbf{f}'$ is $N \times N$.

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \mathbf{h}' * \mathbf{f}', \tag{18}$$

where

$$\mathbf{h}'_{mn} = \frac{1}{i\lambda z} \exp\left\{ik\frac{(m^2 + n^2)\Delta^2}{2z}\right\} \tag{19}$$

(2) Formulate the *spectral representation* of the *forward* matrix $\mathbf{A}$, the *inverse* matrix $\mathbf{A}^{-1}$ (if exist), and the *adjoint/Hermitian* matrix $\mathbf{A}^*$.

The key point here is that a block circulant matrix can be fully represented by its eigenvalue decomposition (EVD), in which the eigenvectors are the column vectors of the 2D discrete Fourier transform (DFT) matrix $\mathbf{W}_{2D}$, and the eigenvalues corresponds to the 2D DFT of the discrete PSF.

$$\mathbf{H} = \frac{1}{N^2}\mathbf{W}_{2D}^{H}\mathrm{diag}(\mathrm{vec}(\widehat{\mathbf{h}'}))\mathbf{W}_{2D}, \tag{20}$$

where $\widehat{\mathbf{h}'}$ denotes the 2D DFT of $\mathbf{h}'$, representing to the 2D discrete TF,

$$\widehat{\mathbf{h}'}_{mn} = \exp\left\{-i\pi\lambda z\frac{m^2+n^2}{(\Delta N)^2}\right\} \tag{21}$$

The spectral representation of $\mathbf{H}$ is

$$\mathbf{Hf} = \frac{1}{N^2}\mathbf{W}_{2D}^{H}\mathrm{diag}(\mathrm{vec}(\widehat{\mathbf{h}'}))\mathbf{W}_{2D}\mathbf{f}. \tag{22}$$

The meaning of the spectral representation is that the output of a 2D LSI can be computed by 1) computing the 2D DFT of $\mathbf{f}$; 2) filtering (i.e. element-wise multiplication) by the 2D discrete TF $\widehat{\mathbf{h}'}$; 3) taking the 2D inverse DFT.

The adjoint is defined by

$$\mathbf{H}^{*}\mathbf{g} = \frac{1}{N^2}\mathbf{W}_{2D}^{H}\mathrm{diag}\left(\mathrm{vec}((\widehat{\mathbf{h}'})^{*})\right)\mathbf{W}_{2D}\mathbf{g}. \tag{23}$$

Since $\widehat{\mathbf{h}'}$ never goes to zero, the inverse does exist, and is defined by

$$\mathbf{H}^{-1}\mathbf{g} = \frac{1}{N^2}\mathbf{W}_{2D}^{H}\mathrm{diag}\left(\frac{1}{\mathrm{vec}(\widehat{\mathbf{h}'})}\right)\mathbf{W}_{2D}\mathbf{g}. \tag{24}$$

One can even show that, for this specific problem, $\mathbf{H}^{-1}$ is approximately (subject to small numerical errors) equal to $\mathbf{H}^{*}$.

(3) Formulate the *range space* and the *null space* of the *forward* matrix $\mathbf{A}$ and the *adjoint/Hermitian* matrix $\mathbf{A}^{*}$.

Since $\widehat{\mathbf{h}'}$ never goes to zero, i.e. the DFT transfer function is always non-zero, the null space of $\mathbf{H}$ is trivial, containing only $\mathbf{0}$. For the same reason, the range of $\mathbf{H}$ is defined by the entire linear space spanned by all the column vectors of $\mathbf{W}s$, which covers the entire DFT frequencies. The same argument can be made for $\mathbf{H}^{*}$.