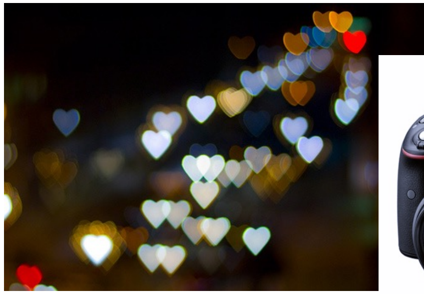
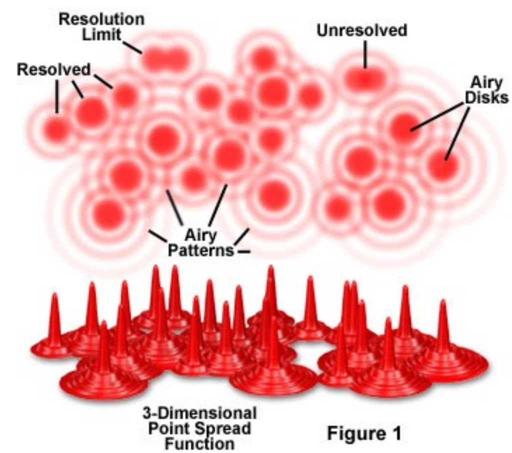
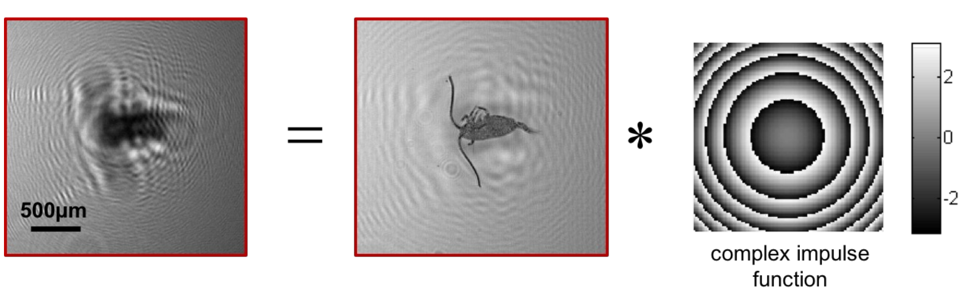


Introduction to Inverse Problem in Imaging

EC 522 Computational Optical Imaging

Lei Tian





Mathematical tools & road map

- Vector space (IIP Appx A)
  - Key idea: think about the imaging signals as a *vector*
- Linear operator (IIP Appx B)
  - Key idea: think about imaging process as a linear transformation, i.e. a linear *operator*
  - Later, we use perform discretization and convert the operator into a *matrix*

IIP: Introduction to inverse problems in imaging Appx A

Vector space

- » Describes the possible "object / image" element

- » Need to describe each element, also the distance/relationship between pairs of elements

Vector space

To define a vector space, we need:

- A set of vectors V
  - These can be finite-dimensional vectors, sequences, functions, etc.
- A field of scalars
  - Real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$
- Vector addition: produces a vector from two vectors
- Scalar multiplication: produces a vector from a scalar and a vector

More details: see
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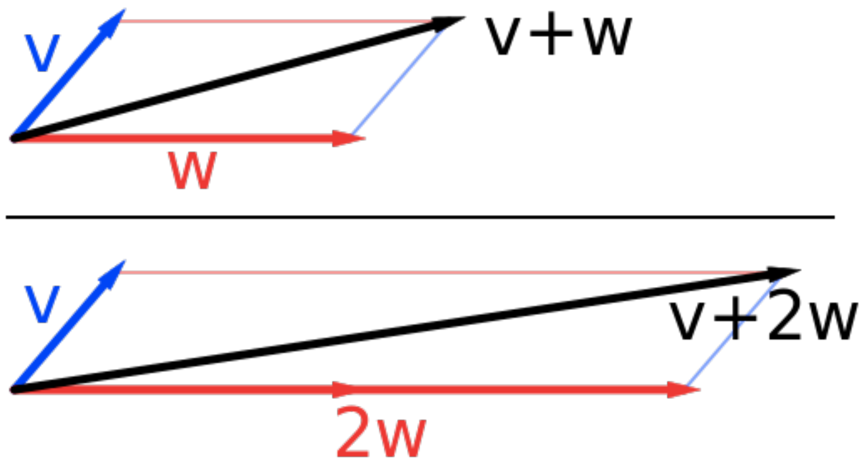
Vector space

» A collection of objects (vectors)

» Can be added together and multiplied by scalars.



Examples: a collections of images can form a vector space!

Images with 256 x 256 pixels with real-valued pixel values

u_1

u_2

…

u_N

The vector space should also contain these images

u_1+u_2

u_2+2u_3

…

$u_1-u_2+u_3$

What is the dimensionality of this vector space?
Any subspace you can find in this vector space?

Any subspace you can find in this vector space?

Vector space

» Describes the possible “object / image” element

» Need to describe each element, also the distance/relationship between pairs of elements

To define geometry in a vector space: Inner product

Inner product provides a measure of angles and orientation

Definition (Inner product)

- An inner product for V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying
  1. Distributivity: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
  2. Linearity in the first argument: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
  3. Hermitian symmetry: $\langle x, y \rangle^* = \langle y, x \rangle$
  4. Positive definiteness: $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$

Note: $\langle x, \alpha y \rangle = \alpha^* \langle x, y \rangle$

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Inner product: Examples

Examples

- On $\mathbb{C}^N$: $\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^* = y^* x$

- On $\mathbb{C}^{\mathbb{R}}$: $\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt$

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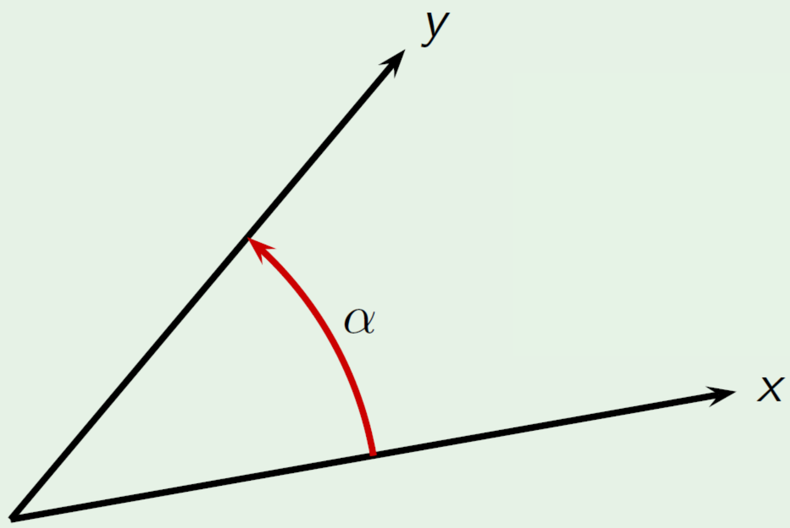
Inner product: examples

Inner product in $\mathbb{R}^2$



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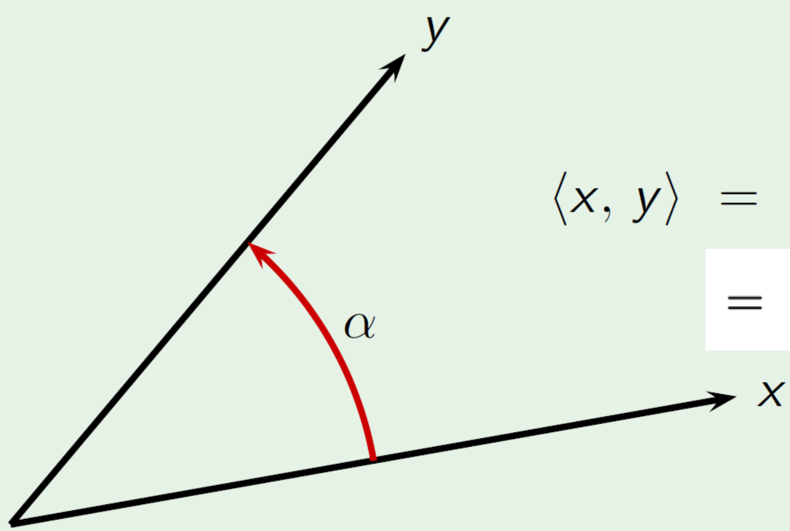
Inner product: examples

Inner product in $\mathbb{R}^2$

$$\langle x, \, y \rangle \; = \; x_0 y_0 + x_1 y_1$$



$$\langle x, \, y \rangle \; = \; \sqrt{(x_0^2 + x_1^2)(y_0^2 + y_1^2)} \, \cos \alpha$$

$$= \quad \|x\| \, \|y\| \cos \alpha$$

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FT as inner products!

FT as inner products!

» Fourier transform:

» CTFT: $X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$

$= \langle x(t), e^{j\omega t} \rangle$

- On $\mathbb{C}^{\mathbb{R}}$: $\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt$

» DFT: $X(k) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$, $\mathrm{k} = 0, \dots, \mathrm{N}-1$

$= \left\langle x[n], e^{j\frac{2\pi}{N}kn} \right\rangle$

- On $\mathbb{C}^N$: $\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^* = y^* x$

Fourier transform can be treated as inner product!

A special geometry: Orthogonality

- A pair of vectors x, y are **orthogonal**, $x \perp y$, if $\langle x, y\rangle = 0$

- Let S be a set of vectors
  - S is orthogonal when all $x, y \in S, x \neq y$, we have $x \perp y$
  - S is orthonormal when it is orthogonal and for all $x \in S$, $\langle x, x\rangle = 1$
  - A vector x is orthogonal to S when $x \perp s$ for all $s \in S$, *written* $x \perp S$
  - S_0 and S_1 are orthogonal when every $s_0 \in S_0$ is orthogonal to S_1, written $S_0 \perp S_1$

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Orthogonality in FT

Orthogonality & Orthogonal complement

- The pairs of elements f, h are **orthogonal**
  - If $\langle f, h \rangle = 0$
  - Written as $f \perp h$

- **Orthogonal complement**
  - S is a subset of elements of $\mathcal{X}$
  - The *orthogonal complement* of S , denoted by $S^{\perp}$
  - The set of all functions/vectors of $\mathcal{X}$ which are orthogonal to **<u>all</u>** functions of S

Example

Example

Norm: a measure of length / size

- This length of f is called the *norm* of f
  - Denoted by $\|f\|$
  - A common definition $\|f\| = \langle f, f \rangle^{1/2}$
  - Not all norms are defined by an inner product
- Properties
  - Positive definite: $\|f\| \geq 0$, $\|f\| = 0$ if and only if f=0
  - Positive scalability: $\|\alpha f\| = |\alpha| \|f\|$
  - Triangle inequality: $\|f + g\| \leq \|f\| + \|g\|$ with equality if and only if $f = \alpha g$ (in parallel)

Norm: example

Examples

L$_2$-norm

- On $\mathbb{C}^N$: $\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{n=0}^{N-1} |x_n|^2\right)^{1/2}$

- On $\mathbb{C}^{\mathbb{R}}$: $\|x\| = \sqrt{\langle x, x\rangle} = \left(\int_{-\infty}^{\infty} |x(t)|^2\, dt\right)^{1/2}$

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Orthogonality and norm

Pythagorean theorem

- $x \perp y \quad \Rightarrow \quad \|x+y\|^2 \;=\; \|x\|^2 + \|y\|^2$
- $\{x_k\}_{k\in K}$ orthogonal $\quad \Rightarrow \quad \left\|\sum_{k\in K} x_k\right\|^2 = \sum_{k\in K} \|x_k\|^2$

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Orthogonality and norm

Properties

- Cauchy–Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|$$

» Why?

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Orthogonality and norm

Properties

- Cauchy–Schwarz inequality

$$|\langle x, y\rangle| \;\leq\; \|x\|\,\|y\|$$

» Why?

  » $\langle x, y\rangle = \|x\|\|y\|\cos\alpha$

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Other examples of norms commonly used in computational imaging algorithms

$$\text{On } \mathbb{C}^{\mathbb{Z}}: \ \|x\|_p \ = \ \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{1/p}, \quad p \in [1, \infty)$$

p-norm or L_p-norm

$$\|x\|_\infty \ = \ \sup_{n\in\mathbb{Z}} |x_n|$$

infinity-norm

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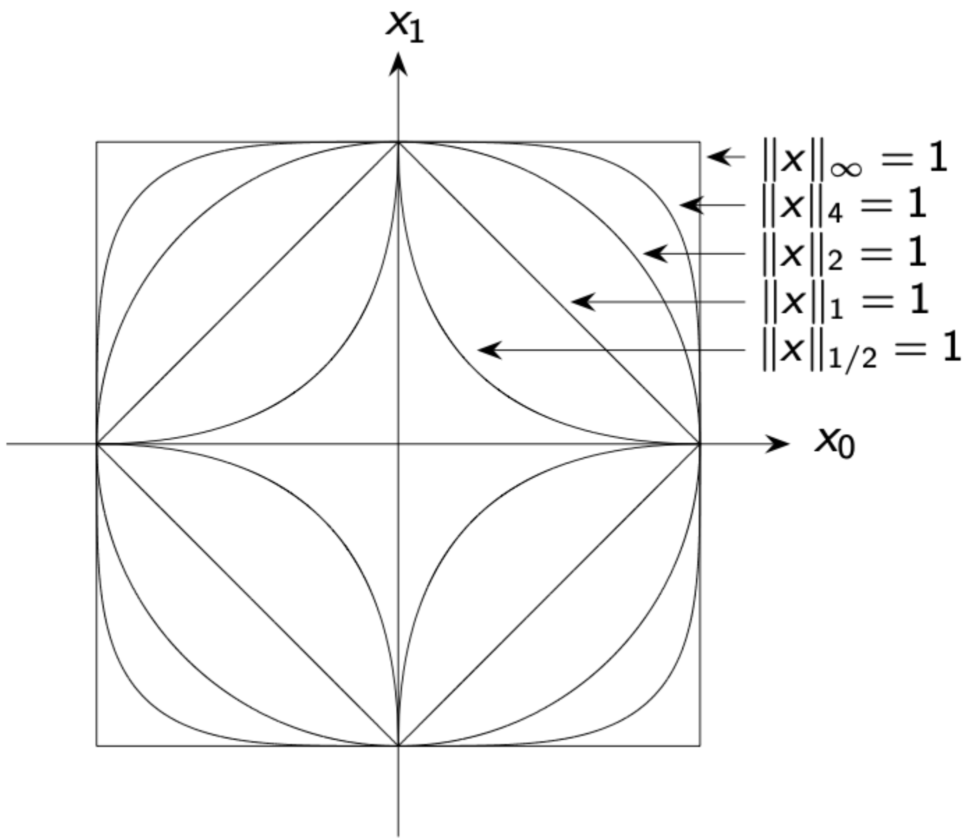
Geometry of Lp-norm

» "Unit ball" visualization



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Distance

» The ***distance*** of the element *f* from the element *h*

$$d(f, h) = \|f - h\|$$

» Properties:

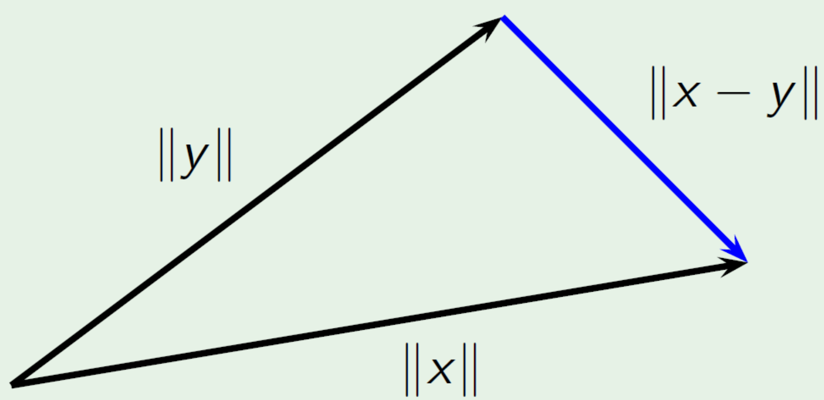
- $d(f, h) \geq 0$; $d(f, h) = 0$, if and only if $f = h$;
- $d(f, h) = d(h, f)$;
- $d(f, h) \leq d(f, g) + d(g, h)$ for any g (triangle inequality)

» Useful when describing the convergence of inversion algorithms

Norm and distance

Norm and distance in $\mathbb{R}^2$

$$
\begin{aligned}
\|x\| &= \sqrt{\langle x,\, x\rangle} = \sqrt{x_0^2 + x_1^2} \\
\|y\| &= \sqrt{\langle y,\, y\rangle} = \sqrt{y_0^2 + y_1^2} \\
\|x - y\| &= \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}
\end{aligned}
$$



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Decomposition and basis

- Basis $V = \{v_k\}_{k\in\{1,2,\cdot n\}} \subset U$
  - Linearly independent: a set of ***n*** ***orthonormal*** functions $v_1, v_2, \ldots, v_n$
  - V is complete and the linear space U = span(V)
    - The **dimension** is *n*
  - i.e. satisfying the conditions:

$$\langle v_i, v_j \rangle = \delta_{ij}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Linear decomposition

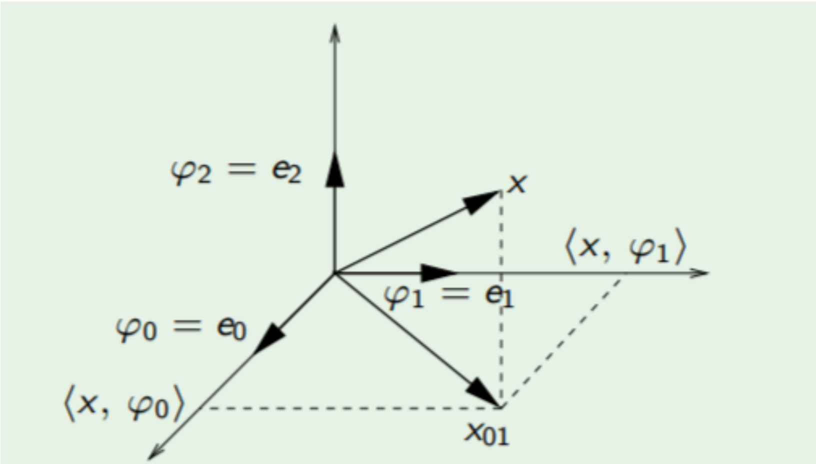
- Linear decomposition:
  - For any $f \in U, f = \sum_{k=1}^{n} c_k v_k$
- The coefficient is ***unique***: $c_k = \langle f, v_k \rangle$

- The norm is $\|f\|^2 = \sum_{k=1}^{n} |c_k|^2$

Example: decomposition and basis

Example

- The standard basis for $\mathbb{R}^N$
  $$e_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \quad k = 0, \ldots, N-1$$
  any $x \in \mathbb{R}^N$, $\quad x = \sum_{k=0}^{N-1} x_k e_k$



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Decomposition and basis in matrix form

Decomposition and basis

- Linear decomposition:
  - For any $f \in U, f = \sum_{k=1}^{n} c_k v_k$
- The coefficient is ***unique***: $c_k = \langle f, v_k \rangle$

- Putting in a compact matrix form:
  - $c = V^* f$
    - $c = [c_1, c_2, \cdots, c_n]^T$
    - $V = [v_1, v_2, \cdots, v_n]$
    - V^*: Hermitian = complex conjugate transpose of matrix V
  - $f = \sum_{k=1}^{n} \langle f, v_k \rangle v_k = Vc = VV^* f$

Why?

Decomposition and basis

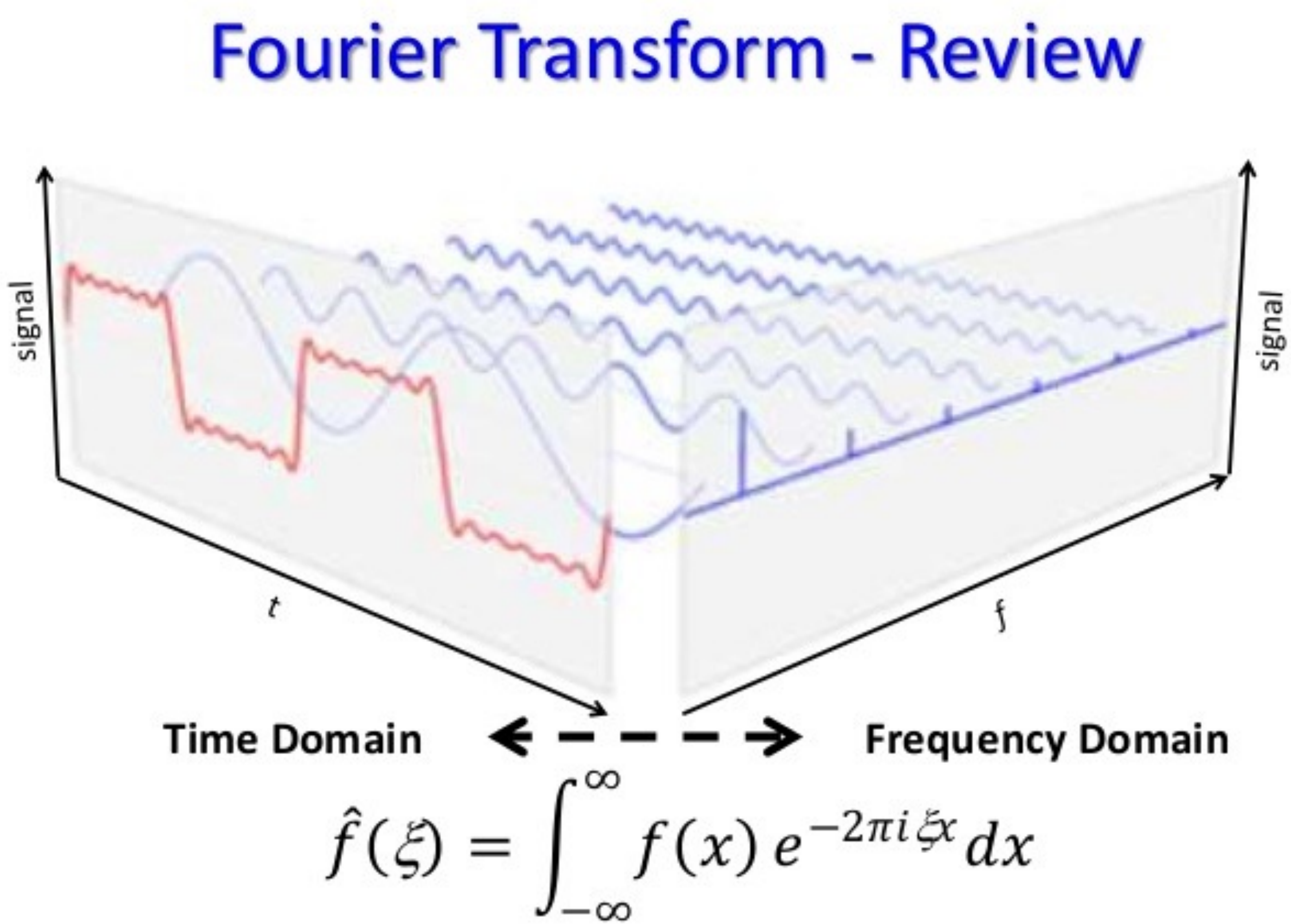
- The norm is $\|f\|^2 = \sum_{k=1}^{n} |c_k|^2$

- In general… Parseval's equalities
  - $\|f\|^2 = \sum_{k=1}^{n} |\langle f, v_k \rangle|^2 = \|V^* f\|^2 = \|c\|^2$

Meaning of this?

  - $\langle f, g \rangle = \langle V^* f, V^* g \rangle = \langle c, d \rangle$
    - Where $c_k = \langle f, v_k \rangle$, $d_k = \langle g, v_k \rangle$

Decomposition and bases: FT



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Decomposition and bases: DFT

» Discrete Fourier Transform (DFT):

» Orthonormal basis: $\left\{e^{j\frac{2\pi k}{N}n}, k = 0, \dots, N-1\right\}$, satisfying

$$\left\langle e^{j\frac{2\pi k}{N}n}, e^{j\frac{2\pi l}{N}n} \right\rangle = \delta_{k,l}$$

» Decomposition:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k}{N}n}$$

» Coefficient determined by:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \left\langle x[n], e^{j\frac{2\pi k}{N}n} \right\rangle$$

Fourier transform can be treated as basis decomposition!

Change of basis

- Basis is not unique
- How are the expansion coefficients in two orthonormal bases are related?
- $f = \Phi\alpha = \Psi\beta$
- $\beta = \Psi^* f = \Psi^* \Phi\alpha = \mathrm{C}_{\Phi,\Psi}\alpha$
- In the matrix form

$$C_{\Phi,\Psi} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \psi_{-1}\rangle & \langle \varphi_0, \psi_{-1}\rangle & \langle \varphi_1, \psi_{-1}\rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_0\rangle & \boxed{\langle \varphi_0, \psi_0\rangle} & \langle \varphi_1, \psi_0\rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_1\rangle & \langle \varphi_0, \psi_1\rangle & \langle \varphi_1, \psi_1\rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$