# Chapter 5
# PERTURBATION AND DUALITY

An optimization problem that turns out to be infeasible or unbounded prompts us to go back and examine the underlying model assumptions. Even when a solution is obtained, we typically would like to assess the validity of the model. Is the solution insensitive to changes in model parameters? This is important because the exact values of parameters may not be known and one would like to avoid being misled by a solution obtained using incorrect values. Thus, it's rarely enough to address an application by formulating a model, solving the resulting optimization problem and presenting the solution as *the* answer. One would need to confirm that the model is suitable and this can, at least in part, be achieved by considering a *family of optimization problems* constructed by perturbing parameters of concern. The resulting sensitivity analysis uncovers troubling situations with unstable solutions and indicates better formulations.

Embedding an actual problem of interest within a family of problems is also a primary path to optimality conditions as well as computationally attractive, alternative problems, which under ideal circumstances, and when properly tuned, may even furnish the minimum value of the actual problem. The tuning of these alternative problems turns out to be intimately tied to finding multipliers in optimality conditions and thus emerges as a main component of several optimization algorithms. In fact, the tuning amounts to solving certain *dual* optimization problems. We'll now turn to the opportunities afforded by this broad perspective.

## 5.A Rockafellians

Suppose that we've formulated an optimization model to address a particular application. The model involves parameters with values that are somewhat unsettled. The parameters might represent the assumed budget of resources available, the probabilities of various outcomes, weights in an additive objective function or targets in goal optimization. A preliminary study would involve the solution of the resulting problem with nominal values for the parameters. However, a more comprehensive approach considers different values of the parameters and their effect on minima and minimizers in an effort to validate



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the model. To formalize this thinking, let's introduce a function that fully represents the actual objective function as well as parametric perturbations.

**Definition 5.1** (Rockafellian). For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$, we say that $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a *Rockafellian* with *anchor* at $\bar{u} \in \mathbb{R}^m$ if

$$f(\bar{u}, x) = f_0(x) \quad \forall x \in \mathbb{R}^n.$$

Suppose that $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ represents the actual problem of interest. Its minimization might bring forth a "good" decision, but when examined in isolation $f_0$ fails to identify the effect of changing parameter values. An associated Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ explicitly specifies the dependence on $m$ parameters and defines the *family of problems*

$$\Big\{ \operatorname*{minimize}_{x \in \mathbb{R}^n} f(u, x), \quad u \in \mathbb{R}^m \Big\}.$$

The minimum values $p(u) = \inf f(u, \cdot)$, the sets of minimizers $P(u) = \operatorname{argmin} f(u, \cdot)$ and other quantities can then be examined as they vary with the *perturbation vector* $u \in \mathbb{R}^m$. For example, in goal optimization as laid out by 4.6, we may face the problem

$$\operatorname*{minimize}_{x \in \mathbb{R}^n} f_0(x) = \sum\nolimits_{i=1}^m \theta_i \max\big\{0, f_i(x) - \bar{u}_i\big\},$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ models a quantity of interest and $\theta_i$ and $\bar{u}_i$ specify the corresponding weight and target. The exact targets might be under discussion and $\bar{u} = (\bar{u}_1, \dots, \bar{u}_m)$ is just a tentative choice. It would be prudent to consider a range of possible target values. Thus, we may adopt the Rockafellian with

$$f(u, x) = \sum\nolimits_{i=1}^m \theta_i \max\big\{0, f_i(x) - u_i\big\},$$

which then has anchor at $\bar{u}$.

In budgeting, we may seek to minimize $f_0(x)$ subject to the constraint that $g(x)$ shouldn't exceed a budget $\bar{u}$. With a focus on the budget level, we could consider a Rockafellian with

$$f(u, x) = f_0(x) + \iota_{(-\infty,0]}\big(g(x) + u\big)$$

and thus capture sensitivity of the solution to budgetary perturbations.

We typically assume that all perturbation vectors $u \in \mathbb{R}^m$ are of interest and not only those in some set $U \subset \mathbb{R}^m$ such as the nonnegative vectors. This simplifies the development, but refinements follow by nearly identical arguments to those detailed below.

By considering a Rockafellian, we can carry out *sensitivity analysis* and answer the fundamental question: Would we make a significantly different decision (or assessment or prediction) if the parameters were perturbed? If the answer is yes, then it would be prudent to reevaluate model assumptions and parameter values. At the minimum, any recommendation derived from the analysis should be qualified as being sensitive to assumptions.

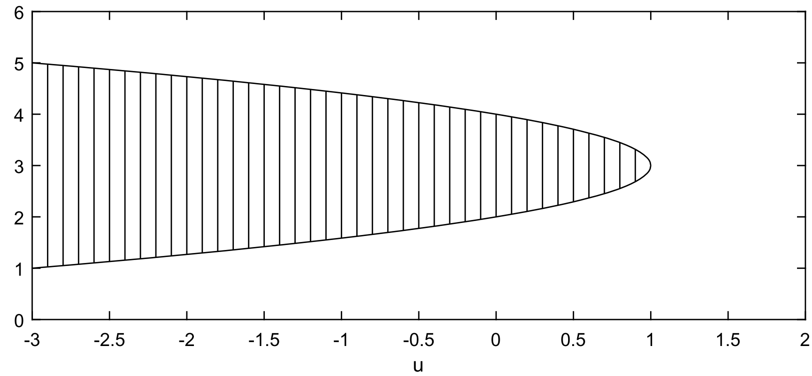


Fig. 5.1: The feasible set $C(u)$ for varying $u$ in 5.2.

There are many different Rockafellians that can be associated with a minimization problem as they represent different ways the problem can be changed. This offers much flexibility in modeling and computations. The choice of a particular Rockafellian reflects concerns beyond getting a solution and in some way completes the formulation of the problem. These concerns could be about the implications of shifting, restructuring and removing constraints. But by no means are the choice of Rockafellians limited to those affecting only the constraints. One might consider perturbations that alter the rewards associated with some individual or a particular combination of decisions; the goal optimization example above illustrates a possibility. When dealing with stochastic optimization problems, a Rockafellian might be selected to reflect dependence of the solutions on the distribution of the random components, for example.

**Example 5.2** (perturbation). The problem of minimizing $x^2 + 1$ subject to $(x-2)(x-4)+1 \leq 0$ can be associated with a Rockafellian defined by

$$f(u,x) = x^2 + 1 + \iota_{(-\infty,0]}\big(g(u,x)\big) \quad \text{and} \quad g(u,x) = (x-2)(x-4) + u,$$

with anchor at $\bar{u} = 1$. The minimum value $p(u) = \inf f(u,\cdot)$ doesn't vary continuously in $u$ at 1. Among the numerous Rockafellians that might be associated with the problem, this particular one highlights the sensitivity to changes on the right-hand side of the constraint.

**Detail.** Since the feasible set $C(u) = \operatorname{dom} f(u,\cdot)$ is given by the constraint $(x-2)(x-4) + u \leq 0$, we obtain as seen in Figure 5.1:

$$C(u) = \begin{cases} \big[\, 3 - \sqrt{1-u},\ \ 3 + \sqrt{1-u}\,\big] & \text{if } u \leq 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

As functions of $u$, the minimizers and minimum values become

$$\operatorname{argmin} f(u,\cdot) = \begin{cases} \{0\} & \text{if } u < -8 \\ \{3 - \sqrt{1-u}\} & \text{if } -8 \le u \le 1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$p(u) = \inf f(u,\cdot) = \begin{cases} 1 & \text{if } u < -8 \\ 11 - u - 6\sqrt{1-u} & \text{if } -8 \le u \le 1 \\ \infty & \text{otherwise.} \end{cases}$$

At every $u < 1$, $p$ is continuous. However, $p(1) = 10$ and $p(u) = \infty$ for $u > 1$, which makes $p$ discontinuous at 1. By looking at the level-sets $\{p \le \alpha\}$, we realize that they're closed for all $\alpha \in \mathbb{R}$. Thus, $p$ is lsc by 4.8. □

A Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ together with the epigraphical approximation theory of §4.C provide the basis for sensitivity analysis. In particular, they offer a means to examine continuity properties of the *inf-projection*

$$u \mapsto p(u) = \inf f(u,\cdot),$$

which we refer to as the *min-value function*, and related properties for the set-valued mapping $u \mapsto P(u) = \operatorname{argmin} f(u,\cdot)$ representing minimizers. However, a cautionary lesson is furnished by the example $f(u,x) = \max\{-1, ux\}$, which defines a convex function in $x$ for any $u \in \mathbb{R}$. As indicated by Figure 5.2, $f(1/\nu,\cdot) \xrightarrow{e} f(0,\cdot)$ but $p(1/\nu) = -1$ for all $\nu \in \mathbb{N}$ and certainly fails to converge to $p(0) = 0$. The trouble in this example is that minimizers from $\operatorname{argmin} f(1/\nu,\cdot)$ don't have a cluster point as assumed in 4.14. This pathological situation is eliminated under tightness.

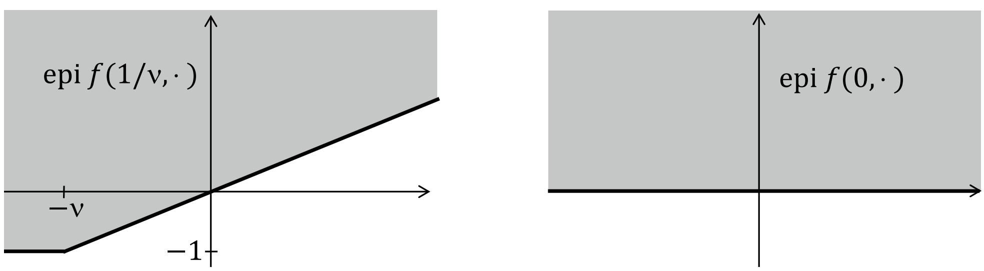


Fig. 5.2: The minimum value of $f(1/\nu,\cdot)$ doesn't converge to that of $f(0,\cdot)$ even though $f(1/\nu,\cdot)$ epi-converges to $f(0,\cdot)$.

**Definition 5.3** (tightness). The functions $\{f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ are *tight* if for all $\varepsilon > 0$, there are compact $B_\varepsilon \subset \mathbb{R}^n$ and $\nu_\varepsilon \in \mathbb{N}$ such that

$$\inf_{B_\varepsilon} f^\nu \le \inf f^\nu + \varepsilon \quad \forall \nu \ge \nu_\varepsilon.$$

The functions *epi-converge tightly* if in addition to being tight they also epi-converge to some function.

The restriction to some compact set $B_\varepsilon$ in the definition produces the same or a higher minimum value. Tightness is about limiting the increase to an arbitrarily small $\varepsilon$ by choosing $B_\varepsilon$ large enough. For a single function, this is always possible if its infimum isn't $-\infty$. For a collection of functions to be tight, the set $B_\varepsilon$ needs to work for all but a finite number of the functions. In the discussion prior to the definition, tightness fails because for any compact set $B$, $\inf_B f(1/\nu, \cdot\,) \geq -1/2$ for sufficiently large $\nu$ and $\inf f(1/\nu, \cdot\,) = -1$.

A collection $\{f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ is tight if $\cup_{\nu\in\mathbb{N}} \operatorname{dom} f^\nu$ is contained in a bounded set. However, many other possibilities exist. For example, if there's a compact set $B \subset \mathbb{R}^n$ such that $B \cap \operatorname{argmin} f^\nu$ is nonempty for all $\nu$, then the condition of the definition also holds.

In the context of a function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with values $f(u, x)$, tightness of a collection $\{f(u^\nu, \cdot\,), \nu \in \mathbb{N}\}$ is closely related to whether the function is *level-bounded in $x$ locally uniformly in $u$* by which we mean the following property:

$$\begin{aligned}&\forall \bar{u} \in \mathbb{R}^m \text{ and } \alpha \in \mathbb{R} \quad \exists \varepsilon > 0 \text{ and a bounded set } B \subset \mathbb{R}^n\\ &\text{such that } \big\{f(u, \cdot\,) \leq \alpha\big\} \subset B \;\; \forall u \in \mathbb{B}(\bar{u}, \varepsilon).\end{aligned}$$

Informally, the property amounts to having, for each $\bar{u}$ and $\alpha$, a bounded level-set $\{f(\bar{u}, \cdot\,) \leq \alpha\}$ with the bound remaining valid under perturbation around $\bar{u}$.

**Proposition 5.4** (tightness from uniform level-boundedness). *For a function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with values $f(u, x)$, suppose that $f$ is level-bounded in $x$ locally uniformly in $u$. If $\{u^\nu \in \mathbb{R}^m, \nu \in \mathbb{N}\}$ has a limit and $\{\inf f(u^\nu, \cdot\,), \nu \in \mathbb{N}\}$ is bounded from above, then $\{f(u^\nu, \cdot\,), \nu \in \mathbb{N}\}$ is tight.*

**Proof.** Let $\varepsilon \in (0, \infty)$, $\bar{u}$ be the limit of $\{u^\nu, \nu \in \mathbb{N}\}$ and $\alpha \in \mathbb{R}$ be an upper bound on $\{\inf f(u^\nu, \cdot\,), \nu \in \mathbb{N}\}$. Since $f$ is level-bounded in $x$ locally uniformly in $u$, there's $\delta > 0$ and a bounded set $B \subset \mathbb{R}^n$ such that

$$\big\{f(u, \cdot\,) \leq \alpha + \varepsilon\big\} \subset B \quad \forall u \in \mathbb{B}(\bar{u}, \delta).$$

Let $\nu_\varepsilon$ be such that $u^\nu \in \mathbb{B}(\bar{u}, \delta)$ for all $\nu \geq \nu_\varepsilon$. Set $B_\varepsilon = \operatorname{cl} B$, which is compact. Then, for $\nu \geq \nu_\varepsilon$,

$$\big\{f(u^\nu, \cdot\,) \leq \alpha + \varepsilon\big\} \subset B_\varepsilon.$$

Fix $\nu \geq \nu_\varepsilon$. First, suppose that $\inf f(u^\nu, \cdot\,) \in \mathbb{R}$. Then, there's $x^\nu$ such that

$$f(u^\nu, x^\nu) \leq \inf f(u^\nu, \cdot\,) + \varepsilon \leq \alpha + \varepsilon.$$

Consequently,

$$\inf\nolimits_{B_\varepsilon} f(u^\nu, \cdot\,) \leq f(u^\nu, x^\nu) \leq \inf f(u^\nu, \cdot\,) + \varepsilon.$$

Second, suppose that $\inf f(u^\nu, \cdot\,) = -\infty$. Let $\mu \leq \alpha + \varepsilon$. Then, there's $x^\nu$ such that $f(u^\nu, x^\nu) \leq \mu \leq \alpha + \varepsilon$. Thus,

$$\inf\nolimits_{B_\varepsilon} f(u^\nu, \cdot\,) \leq f(u^\nu, x^\nu) \leq \mu.$$

Since $\mu$ is arbitrary, we've established that $\inf_{B_\varepsilon} f(u^\nu, \cdot) = -\infty$. This means that

$$\inf_{B_\varepsilon} f(u^\nu, \cdot) \leq \inf f(u^\nu, \cdot) + \varepsilon$$

holds in both cases. $\square$

The upper bound on $\{\inf f(u^\nu, \cdot), \nu \in \mathbb{N}\}$ in the proposition rules out $f : \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}$ with $f(u, x) = 1/|u|$ if $x = 1/|u|$ and $f(u, x) = \infty$ otherwise, which is level-bounded in $x$ locally uniformly in $u$ but $\{f(1/\nu, \cdot), \nu \in \mathbb{N}\}$ isn't tight; see Figure 5.3.

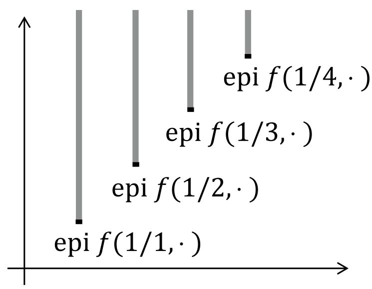


Fig. 5.3: Example of a function $f$ that's level-bounded in $x$ locally uniformly in $u$ but fails to produce a tight collection of functions.

With the refinement of tightness, we obtain the following consequences of epi-convergence, which also furnish the proof of 4.14.

**Theorem 5.5** (consequences of epi-convergence). *Suppose that $f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}$ epi-converges to a proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then, the following hold for any $\varepsilon \in [0, \infty)$:*

(a) $\limsup(\inf f^\nu) \leq \inf f$.
(b) *If $\varepsilon^\nu \in [0, \infty) \to \varepsilon$, then*

$$\text{LimOut}\left(\varepsilon^\nu\text{-argmin} f^\nu\right) \subset \varepsilon\text{-argmin} f.$$

(c) *If $\{x^\nu \in \varepsilon^\nu\text{-argmin} f^\nu, \nu \in N\}$ converges for some $N \in \mathcal{N}_\infty^\#$ and $\varepsilon^\nu \to 0$, then*

$$\lim_{\nu \in N}(\inf f^\nu) = \inf f.$$

(d) $\inf f^\nu \to \inf f > -\infty \iff \{f^\nu, \nu \in \mathbb{N}\}$ *is tight.*
(e) $\inf f^\nu \to \inf f \implies \exists\, \varepsilon^\nu \in [0, \infty) \to \varepsilon$ *such that*

$$\text{LimInn}\left(\varepsilon^\nu\text{-argmin} f^\nu\right) \supset \varepsilon\text{-argmin} f.$$

**Proof.** We leverage the characterization 4.15 of epi-convergence. For (a), suppose first that $\inf f$ is finite and let $\gamma \in (0, \infty)$. There are $x \in \mathbb{R}^n$ such that $f(x) \leq \inf f + \gamma$ and also, by 4.15(b), $x^\nu \to x$ such that $\limsup f^\nu(x^\nu) \leq f(x)$. Thus,

$$\limsup(\inf f^\nu) \leq \limsup f^\nu(x^\nu) \leq f(x) \leq \inf f + \gamma.$$

Second, suppose that $\inf f = -\infty$. Then, there are $x \in \mathbb{R}^n$ such that $f(x) \le -\gamma$ and also, by 4.15(b), $x^\nu \to x$ such that $\limsup f^\nu(x^\nu) \le f(x)$. Thus,

$$\limsup (\inf f^\nu) \le \limsup f^\nu(x^\nu) \le f(x) \le -\gamma.$$

Since $\gamma$ is arbitrary, $\limsup (\inf f^\nu) \le \inf f$ holds in both cases.

For (b), suppose that $\bar{x} \in \mathrm{LimOut}(\varepsilon^\nu\text{-}\mathrm{argmin}\, f^\nu)$. Then, by the definition of outer limits, there are $N \in \mathcal{N}_\infty^\#$ and $x^\nu \in \varepsilon^\nu\text{-}\mathrm{argmin}\, f^\nu \to_N \bar{x}$. Thus,

$$\limsup_{\nu \in N} f^\nu(x^\nu) \le \limsup_{\nu \in N}(\inf f^\nu + \varepsilon^\nu) \le \inf f + \varepsilon,$$

where the last inequality follows by (a). In view of 4.15(a), this implies that

$$f(\bar{x}) \le \liminf_{\nu \in N} f^\nu(x^\nu) \le \limsup_{\nu \in N} f^\nu(x^\nu) \le \inf f + \varepsilon.$$

Since $f$ is proper, $\inf f < \infty$ and then $\bar{x} \in \mathrm{dom}\, f$. Thus, $\bar{x} \in \varepsilon\text{-}\mathrm{argmin}\, f$.

For (c), let $\bar{x}$ be the limit of $\{x^\nu, \nu \in N\}$. By (b), $\bar{x} \in \mathrm{argmin}\, f$. Then, 4.15(a) implies that

$$\liminf_{\nu \in N}(\inf f^\nu + \varepsilon^\nu) \ge \liminf_{\nu \in N} f^\nu(x^\nu) \ge f(\bar{x}) = \inf f.$$

Since $\limsup(\inf f^\nu) \le \inf f$ holds from (a), the conclusion follows.

For (d), suppose that $\inf f^\nu \to \inf f > -\infty$ and let $\gamma > 0$. Then, there are $\nu_1 \in \mathbb{N}$ such that $\inf f \le \inf f^\nu + \gamma/3$ for all $\nu \ge \nu_1$ and also $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \le \inf f + \gamma/3$. By 4.15(b), there are $x^\nu \to \bar{x}$ and $\nu_2 \ge \nu_1$ such that $f^\nu(x^\nu) \le f(\bar{x}) + \gamma/3$ for all $\nu \ge \nu_2$. Let $B$ be a compact set containing $\{x^\nu, \nu \in \mathbb{N}\}$. Thus, for $\nu \ge \nu_2$,

$$\inf_B f^\nu \le f^\nu(x^\nu) \le f(\bar{x}) + \tfrac{1}{3}\gamma \le \inf f + \tfrac{2}{3}\gamma \le \inf f^\nu + \gamma.$$

We've shown that $\nu_2$ and $B$ furnish the required index and set in the definition of tightness; see 5.3.

For the converse, we first rule out the possibility $\inf f = -\infty$. If this were the case, then $\inf f^\nu \to -\infty$ by (a) and, for some compact set $B \subset \mathbb{R}^n$, $\inf_B f^\nu \to -\infty$ because of tightness. We must then also have $\{x^\nu \in B, \nu \in \mathbb{N}\}$ such that $f^\nu(x^\nu) \to -\infty$. Since $B$ is compact, there are $N \in \mathcal{N}_\infty^\#$ and $\bar{x}$ such that $x^\nu \to_N \bar{x}$. By 4.15(a),

$$\liminf_{\nu \in N} f^\nu(x^\nu) \ge f(\bar{x}) > -\infty$$

because $f$ is proper. This contradicts $f^\nu(x^\nu) \to -\infty$ and thus $\inf f > -\infty$.

Second, we show that for any compact set $B \subset \mathbb{R}^n$, there's $\bar{\nu}$ such that $\{\inf_B f^\nu, \nu \ge \bar{\nu}\}$ is bounded from below. For the sake of contradiction, suppose that there's $N \in \mathcal{N}_\infty^\#$ such that $\inf_B f^\nu \to_N -\infty$. Since $B$ is compact, this implies the existence of $\{x^\nu \in B, \nu \in N\}$ with $f^\nu(x^\nu) \to_N -\infty$, another subsequence $N' \subset N$ and a limit $\bar{x}$ of $\{x^\nu, \nu \in N'\}$. By 4.15(a),

$$\liminf_{\nu \in N'} f^\nu(x^\nu) \ge f(\bar{x}) > -\infty$$

because $f$ is proper. This contradicts $f^\nu(x^\nu) \to_N -\infty$.

Third, for a compact set $B$, we establish that $\liminf(\inf_B f^\nu) \geq \inf_B f$. We can assume without loss of generality that $B$ is nonempty because otherwise the statement holds trivially. Let $\alpha = \liminf(\inf_B f^\nu)$. If $\alpha = \infty$, the claim holds trivially. The case $\alpha = -\infty$ is ruled out by the prior paragraph. For $\alpha \in \mathbb{R}$, the prior paragraph ensures that $\inf_B f^\nu > -\infty$ for sufficiently large $\nu$. For such $\nu$, there's $x^\nu \in B$ such that $f^\nu(x^\nu) \leq \inf_B f^\nu + \nu^{-1}$. Consequently, there exist also $\bar{x} \in B$ and $N \in \mathcal{N}_\infty^\#$ such that $x^\nu \underset{N}{\to} \bar{x}$. By 4.15(a), $\liminf_{\nu\in N} f^\nu(x^\nu) \geq f(\bar{x})$ and then

$$\liminf_{\nu\in N}\big(\inf_B f^\nu\big) \geq \liminf_{\nu\in N}\big(f^\nu(x^\nu) - \nu^{-1}\big) \geq f(\bar{x}) \geq \inf_B f.$$

To remove the restriction to $N$, suppose for the sake of contradiction that $\alpha < \inf_B f$. Since $\alpha \in \mathbb{R}$, there's $N' \in \mathcal{N}_\infty^\#$ such that $\{\inf_B f^\nu, \nu \in N'\}$ has $\alpha$ as limit. We can then repeat the arguments above for this subsequence and conclude that

$$\liminf_{\nu\in N''}\big(\inf_B f^\nu\big) \geq \inf_B f \;\text{ for some subsequence } N'' \subset N'.$$

But, $\alpha$ is also the limit of $\{\inf_B f^\nu, \nu \in N''\}$ and this contradicts the assumption $\alpha < \inf_B f$.

Fourth, let $\varepsilon > 0$ and $B_\varepsilon$ be the corresponding compact set according to 5.3. Then, by tightness and the previous paragraph,

$$\liminf\big(\inf f^\nu\big) + \varepsilon \geq \liminf\big(\inf_{B_\varepsilon} f^\nu\big) \geq \inf_{B_\varepsilon} f \geq \inf f.$$

Since $\varepsilon$ is arbitrary, we've established that $\liminf(\inf f^\nu) \geq \inf f$ and then also $\inf f^\nu \to \inf f$ by (a).

For (e), let $\bar{x} \in \varepsilon\text{-}\mathrm{argmin} f$. Then, $f(\bar{x}) < \infty$. By 4.15(b), there's $x^\nu \to \bar{x}$ such that $\limsup f^\nu(x^\nu) \leq f(\bar{x})$. First, suppose that $\inf f \in \mathbb{R}$. Then,

$$\begin{aligned} f^\nu(x^\nu) - \inf f^\nu &= f^\nu(x^\nu) - f(\bar{x}) + f(\bar{x}) - \inf f + \inf f - \inf f^\nu \\ &\leq f^\nu(x^\nu) - f(\bar{x}) + \varepsilon + \inf f - \inf f^\nu. \end{aligned}$$

We've shown that $x^\nu \in \varepsilon^\nu\text{-}\mathrm{argmin} f^\nu$, where

$$\varepsilon^\nu = \varepsilon + \max\big\{0, f^\nu(x^\nu) - f(\bar{x}) + \inf f - \inf f^\nu\big\}.$$

Thus, $\bar{x} \in \mathrm{LimInn}(\varepsilon^\nu\text{-}\mathrm{argmin} f^\nu)$. Since $\varepsilon^\nu \to \varepsilon$, the assertion holds. Second, suppose that $\inf f = -\infty$. Then, $f(\bar{x}) = -\infty$, but that contradicts the fact that $f$ is proper. □

While $\mathrm{LimOut}(\mathrm{argmin} f^\nu) \subset \mathrm{argmin} f$ holds by item (b) of the theorem, the inclusion can be strict as illustrated in Figure 5.4, where $f(x) = 0$ and $f^\nu(x) = x^2/\nu$ if $x \in [-1, 1]$ and $f(x) = f^\nu(x) = \infty$ otherwise. Then, $\mathrm{argmin} f^\nu = \{0\}$ for all $\nu$ but $\mathrm{argmin} f = [-1, 1]$. Still, item (e) shows that if the tolerances $\varepsilon^\nu$ in the approximating problems vanish sufficiently slowly, then

$$\varepsilon^\nu\text{-}\mathrm{argmin} f^\nu \xrightarrow{s} \mathrm{argmin} f.$$

In Figure 5.4, one can select $\varepsilon^\nu = 1/\nu$ because then $\varepsilon^\nu$-argmin $f^\nu = [-1, 1]$, which actually coincides with argmin $f$ regardless of $\nu$.

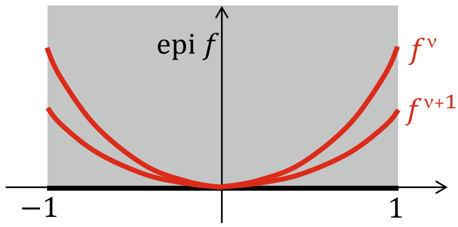


Fig. 5.4: Set-convergence of near-minimizers of $f^\nu$ to argmin $f$.

Before applying the theorem in the context of Rockafellians, we refine our terminology regarding semicontinuity. We say that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *lsc at* $\bar{x}$ if

$$x^\nu \to \bar{x} \quad \Longrightarrow \quad \liminf f(x^\nu) \geq f(\bar{x}). \tag{5.1}$$

This holds for all $\bar{x} \in \mathbb{R}^n$ if and only if $f$ is lsc by [105, Lemma 1.7]. Thus, this sequential condition supplements the other characterizations in 4.8. Likewise, $f$ is *upper semicontinuous (usc) at* $\bar{x}$ if $-f$ is lsc at $\bar{x}$ or, equivalently,

$$x^\nu \to \bar{x} \quad \Longrightarrow \quad \limsup f(x^\nu) \leq f(\bar{x}).$$

It's *usc* if this holds for all $\bar{x} \in \mathbb{R}^n$. Consequently, $f$ is continuous at $\bar{x}$ if and only if it's both lsc and usc at $\bar{x}$.

We can now state the main result about stability of minima and minimizers under perturbations as defined by a Rockafellian. Typically, we would like the min-value function to be continuous. Then small changes in the perturbation vector imply small changes in the minimum value. Sensitivity analysis often revolves around checking whether this indeed is the case. In the absence of continuity, our stipulation of the minimum "cost" might be highly sensitive to small changes in modeling assumptions. For example, if the min-value function is only lsc at $\bar{u}$, then the minimum cost could become much higher under a minor perturbation away from $\bar{u}$, possibly resulting in an unwelcome surprise. A min-value function that's usc at $\bar{u}$ may jump down under perturbation away from $\bar{u}$, which is more desirable as it represents unexpected improvements in the minimum value.

**Theorem 5.6** (stability). *For a proper function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$, let*

$$p(u) = \inf f(u, \cdot\,), \qquad P(u) = \operatorname{argmin} f(u, \cdot\,), \qquad P_\varepsilon(u) = \varepsilon\text{-}\operatorname{argmin} f(u, \cdot\,)$$

*with $u \in \mathbb{R}^m$ and $\varepsilon \in (0, \infty)$. Given $\bar{u} \in \mathbb{R}^m$, the following hold:*

(a) *$p$ is lsc at $\bar{u}$ when $f$ is lsc and, for any $u^\nu \to \bar{u}$, $\{f(u^\nu, \cdot\,), \nu \in \mathbb{N}\}$ is tight.*
(b) *$p$ is usc at $\bar{u}$ when, for any $u^\nu \to \bar{u}$ and $\bar{x} \in \mathbb{R}^n$, there's $x^\nu \to \bar{x}$ such that*

$$\limsup f(u^\nu, x^\nu) \leq f(\bar{u}, \bar{x}),$$

*which in particular holds when $f$ is usc.*

(c) *$p$ is continuous at $\bar{u}$ and, for $\varepsilon \in (0, \infty)$,*

$$\bigcup_{u^\nu \to \bar{u}} \operatorname{LimOut} P(u^\nu) \subset P(\bar{u}) \quad \textit{and} \quad \bigcap_{u^\nu \to \bar{u}} \operatorname{LimInn} P_\varepsilon(u^\nu) \supset P(\bar{u})$$

*when $f(\bar{u}, \cdot\,)$ is proper and, for any $u^\nu \to \bar{u}$, $f(u^\nu, \cdot\,) \xrightarrow{e} f(\bar{u}, \cdot\,)$ tightly. These requirements hold if $P(\bar{u}) \neq \emptyset$, $f$ is lsc, $f(\cdot\,, x)$ is continuous for all $x$ and $f$ is level-bounded in $x$ locally uniformly in $u$.*

**Proof.** For (a), let $u^\nu \to \bar{u}$. With $f(u^\nu, \cdot\,)$ and $f(\bar{u}, \cdot\,)$ in the roles of $f^\nu$ and $f$, respectively, we can repeat the proof of 5.5, part (d), steps 2–4, and conclude that $\liminf p(u^\nu) \geq p(\bar{u})$. A closer examination shows that the requirement of $f(\bar{u}, \cdot\,)$ being proper can be relaxed to $f(\bar{u}, x) > -\infty$ for all $x$, which holds because $f$ is proper. Moreover, $f(u^\nu, \cdot\,) \xrightarrow{e} f(\bar{u}, \cdot\,)$ can be relaxed to $\liminf f(u^\nu, x^\nu) \geq f(\bar{u}, x)$ for $x^\nu \to x$ and this holds by lsc of $f$.

For (b), one can follow the arguments in the proof of 5.5(a). Part (c) is a direct consequence of 5.5, with the sufficient condition seen as follows: If there's $\bar{x} \in \operatorname{argmin} f(\bar{u}, \cdot\,)$, then $f(\bar{u}, \bar{x}) < \infty$ and $f(\bar{u}, \cdot\,)$ is proper. The characterization of epi-convergence in 4.15 and the continuity assumptions establish that $f(u^\nu, \cdot\,) \xrightarrow{e} f(\bar{u}, \cdot\,)$. Then, 5.5(a) ensures that $\limsup p(u^\nu) \leq p(\bar{u}) = f(\bar{u}, \bar{x}) < \infty$, which in turn establishes tightness via 5.4. □

In 5.2, $f$ is proper and lsc. For $u^\nu \to \bar{u}$, $\{C(u^\nu), \nu \in \mathbb{N}\}$ is bounded so that $\{f(u^\nu, \cdot\,), \nu \in \mathbb{N}\}$ is tight. Thus, 5.6(a) establishes that $p$ is lsc in that example.

**Example 5.7** (regularization as perturbation). In data analytics (cf. 2.5, 2.38) and approaches related to the proximal point method (§1.H), a continuous objective function $f_0 : \mathbb{R}^n \to \mathbb{R}$ is often augmented with a regularization term of the form $\theta \|x\|_1$ or $\frac{1}{2\lambda}\|x - \bar{x}\|_2^2$. The weight ($\theta$ or $1/\lambda$) associated with the term is an important modeling choice and we examine its perturbation. Specifically, let's consider the problem

$$\underset{x \in C}{\text{minimize}}\; f_0(x) + |\bar{u}| r(x),$$

where $r : \mathbb{R}^n \to [0, \infty)$ is a continuous function representing regularization and $C \subset \mathbb{R}^n$ is nonempty and closed. We use the absolute value of the perturbation parameter to ensure nonnegativity. A Rockafellian for the problem is given by

$$f(u, x) = \iota_C(x) + f_0(x) + |u| r(x).$$

Under the assumption that $\iota_C(x) + f_0(x) \to \infty$ when $\|x\|_2 \to \infty$, the stability theorem 5.6(c) applies with $\bar{u} = 0$ and we conclude that the minimum value and minimizers of $f(u, \cdot\,)$ change "continuously" as $u$ shifts from a small number to 0 as specified in that theorem.

**Detail.** Since $\iota_C + f_0$ is lsc, level-bounded and proper, 4.9 ensures that it has a minimizer. This level-boundedness implies that $f$ is level-bounded in $x$ locally uniformly in $u$. Since $f$ is lsc and $f(\cdot\,, x)$ is continuous for all $x$, the sufficient condition in 5.6(c) holds. □

**Example 5.8** (perturbation of inequalities). For lsc $f_0, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, q$, let's consider the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f_0(x) + \iota_{(-\infty,0]^q}\big(G(x)\big),$$

where $G(x) = (g_1(x), \dots, g_q(x))$ and the Rockafellian given by

$$f(u, x) = f_0(x) + \iota_{(-\infty,0]^q}\big(G(x) + u\big).$$

The min-value function given by $p(u) = \inf f(u, \cdot\,)$ is lsc under a tightness assumption, but isn't necessarily continuous at 0. Thus, the minimum value obtained from solving the actual problem should be presented with the qualification that it could easily be much higher if the constraints are changed slightly.

**Detail.** To show that $f$ is lsc, let $u^\nu \to \bar{u}$, $x^\nu \to \bar{x}$ and $\alpha \in \mathbb{R}$ such that $f(u^\nu, x^\nu) \le \alpha$ for all $\nu$. Then, $f_0(x^\nu) \le \alpha$ and $g_i(x^\nu) + u^\nu \le 0$ for all $i$, which imply that $f_0(\bar{x}) \le \alpha$ and $g_i(\bar{x}) + \bar{u} \le 0$ as well by the lsc property. Thus, $f(\bar{u}, \bar{x}) \le \alpha$ and $f$ is lsc by 4.8. We can then bring in the stability theorem 5.6(a) to conclude that $p$ is lsc as long as the required tightness assumption is satisfied. For instance, we achieve this when the feasible sets $\{x \mid G(x) + u^\nu \le 0\}$ are uniformly bounded as $\{u^\nu, \nu \in \mathbb{N}\}$ converges.

Continuity of the min-value function $p$ is a much more delicate issue as already brought forth by 5.2, which is of the present form with $f_0(x) = x^2 + 1$ and $g_1(x) = (x-2)(x-4)+1$. Since $p$ is lsc in that case as argued below the stability theorem 5.6, let's examine usc at 0 and the assumption in 5.6(b). We realize that, for $u^\nu \searrow 0$ and $\bar{x} = 3$, there's no $x^\nu \to \bar{x}$ such that $\limsup f(u^\nu, x^\nu) \le f(0, \bar{x})$; we always have $f(u^\nu, x^\nu) = \infty$ and the assumption in 5.6(b) fails. However, the difficulty is unique to $u = 0$. The min-value function is continuous at $u \neq 0$. □

**Exercise 5.9** (change in risk averseness). For $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ and a nonempty compact set $C \subset \mathbb{R}^n$, consider the superquantile-risk minimization problem (3.3) and its minimum value as a function of the parameter $\alpha$. (A high $\alpha$ implies a cautious approach that attempts to avoid decisions with high "cost." This is referred to as having high risk averseness. In practice, it's often difficult to determine the right level of risk averseness.) Identify a Rockafellian that represents changes in $\alpha$. Under the assumption that $f(\xi, \cdot\,)$ is continuous for all $\xi \in \Xi$, show that the min-value function defined by the Rockafellian is continuous at any $\bar{\alpha} \in (0, 1)$.

**Guide.** Express $\alpha = e^u/(1 + e^u)$ and define a Rockafellian of the form

$$\varphi\big(u, (x, \gamma)\big) = \iota_C(x) + \gamma + (1 + e^u)\sum\nolimits_{\xi \in \Xi} p_\xi \max\big\{0, f(\xi, x) - \gamma\big\}.$$

Confirm that it's level-bounded in $(x, \gamma)$ locally uniformly in $u$ and leverage the stability theorem 5.6. □

## 5.B Quantitative Stability

Although continuity of a min-value function conveys a sense of stability for the family of problems defined by a Rockafellian, it doesn't quantify *how much* the minimum value changes under a particular perturbation. We'll now make a step in that direction and compute subgradients of min-value functions, which represent first-order estimates of this change. En route to formulae for such subgradients, we'll discover that a Rockafellian not only defines a family of alternative problems but also an optimality condition, with an associated multiplier vector, for any one of the individual problems. Thus, embedding a problem within a family via a Rockafellian has benefits beyond sensitivity analysis and emerges as a main approach to constructing optimality conditions and computational procedures.

Let's start by looking at optimality conditions for the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$. The condition $0 \in \partial f_0(x)$ is necessary for a local minimizer of $f_0$ by the Fermat rule 4.73. A Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ for the problem, with anchor at $\bar{u}$, has $f(\bar{u}, x) = f_0(x)$ for all $x \in \mathbb{R}^n$ by definition, but also gives rise to the alternative optimality condition

$$(y, 0) \in \partial f(\bar{u}, x)$$

for the actual problem under a qualification as stated in the next theorem. The auxiliary vector $y \in \mathbb{R}^m$ is associated with the perturbation vector $u$ and emerges as a key quantity in sensitivity analysis. While this introduces additional unknowns compared to $0 \in \partial f_0(x)$, it's often easier to solve for $(x, y)$ in the alternative condition than finding just $x$ in the original one. We refer to $y$ as a *multiplier vector*.

**Theorem 5.10** (Rockafellar condition for optimality). *For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$, suppose that $\bar{x} \in \mathbb{R}^n$ is a local minimizer, $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper lsc Rockafellian with anchor at $\bar{u} \in \mathbb{R}^m$ and the following qualification holds:*

$$(y, 0) \in \partial^\infty f(\bar{u}, \bar{x}) \implies y = 0. \tag{5.2}$$

*Then,*

$$\exists \bar{y} \in \mathbb{R}^m \ \text{ such that } \ (\bar{y}, 0) \in \partial f(\bar{u}, \bar{x}).$$

*This condition is sufficient for $\bar{x}$ to be a (global) minimizer of $f_0$ when $f$ is epi-regular at $(\bar{u}, \bar{x})$ and $f_0$ is convex.*

**Proof.** Since $f_0(x) = f(\bar{u}, x) = f(F(x))$ with $F(x) = (\bar{u}, x)$, we can apply the chain rule 4.64 to obtain

$$\partial f_0(\bar{x}) \subset \nabla F(\bar{x})^\top \partial f(\bar{u}, \bar{x}) \tag{5.3}$$

as long as the qualification (4.14) holds. Since $\nabla F(\bar{x})$ is simply the $(m + n) \times n$-matrix consisting of the $m \times n$ zero matrix stacked on top of the $n \times n$ identity matrix, the right-hand side in this inclusion amounts to those $v \in \mathbb{R}^n$ such that $(y, v) \in \partial f(\bar{u}, \bar{x})$ for some $y \in \mathbb{R}^m$. The Fermat rule 4.73 ensures that $0 \in \partial f_0(\bar{x})$ and, when combined with (5.3), also the first conclusion. Still, it remains to verify (4.14), which now reads:

$$w \in \partial^\infty f(\bar{u}, \bar{x}) \ \text{ and } \ \nabla F(\bar{x})^\top w = 0 \implies w = 0.$$

Let's say that $w = (y, v)$ with $y \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Then, $\nabla F(\bar{x})^\top w = v$ so the requirement that $v = 0$ holds trivially. The additional requirement that $y = 0$ is satisfied due to (5.2).

When $f$ is epi-regular at $(\bar{u}, \bar{x})$, the chain rule 4.64 yields equality in (5.3). By the optimality condition 2.19 for convex functions, the second conclusion follows. □

A vast number of possibilities emerge from the theorem, with the Rockafellian now possibly being chosen based on computational concerns more than those of sensitivity analysis. The perspective brings a broader view of multipliers beyond a specific constraint structure and the associated variational geometry discussed in §4.J. In fact, every Rockafellian defines an optimality condition for the actual problem expressed by a multiplier vector, at least as long as the qualification (5.2) holds. However, two examples show that the multipliers emerging from the Rockafellar condition 5.10 are very much in tune with those in earlier developments.

**Example 5.11** (connection with equality constraints). For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$, suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper lsc Rockafellian with anchor at $\bar{u} \in \mathbb{R}^m$. Then, the problem can be reformulated as

$$\mathop{\text{minimize}}_{u \in \mathbb{R}^m, x \in \mathbb{R}^n} f(u, x) \ \text{ subject to } \ f_i(u, x) = \bar{u}_i - u_i = 0, \ \ i = 1, \dots, m.$$

The multipliers associated with these equality constraints in the sense of 4.76 coincide with $y$ emerging from the Rockafellar condition 5.10.

**Detail.** When $f$ is smooth, the reformulation has the optimality condition

$$\nabla f(u, x) + \sum\nolimits_{i=1}^{m} y_i \nabla f_i(u, x) = 0$$

by 4.76. This simplifies to $y = \nabla_u f(u, x)$ and $\nabla_x f(u, x) = 0$, which together with the feasibility condition $\bar{u} = u$, produce $(y, 0) = \nabla f(\bar{u}, x)$. Thus, the multiplier vector $y$ from 4.76 coincides with the multiplier vector in the Rockafellar condition 5.10.

Without smoothness, a similar connection holds. The equality constraints form a set $C \subset \mathbb{R}^m \times \mathbb{R}^n$ with normal cone (cf. 4.49)

$$N_C(\bar{u}, x) = \left\{ \sum\nolimits_{i=1}^{m} y_i \nabla f_i(\bar{u}, x) \;\middle|\; y \in \mathbb{R}^m \right\} = \mathbb{R}^m \times \{0\}^n.$$

Assuming that the sum rule 4.67 applies at $(\bar{u}, x)$, the Fermat rule 4.73 then produces the optimality condition

$$0 \in \partial f(\bar{u}, x) + N_C(\bar{u}, x) = \partial f(\bar{u}, x) + \left(\mathbb{R}^m \times \{0\}^n\right)$$

for the reformulation. This is equivalent to having $(y, 0) \in \partial f(\bar{u}, x)$ for some $y \in \mathbb{R}^m$ and we've recovered the Rockafellar condition 5.10. □

**Example 5.12** (multipliers for max-functions). For smooth $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, the problem

$$\mathop{\text{minimize}}_{x \in \mathbb{R}^n} \ \max\nolimits_{i=1,\dots,m} f_i(x)$$

can be associated with the Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ given by

$$f(u,x) = h\big(F(x) + u\big),$$

where $h(z) = \max\{z_1, \dots, z_m\}$ and $F(x) = (f_1(x), \dots, f_m(x))$. The actual problem is recovered by setting $u = 0$, which then is the anchor of this Rockafellian. For example, the perturbation vector $u$ may represent changes to the targets in goal optimization; cf. 4.6. The multipliers arising from the Rockafellar condition 5.10 coincide with those of 4.81.

**Detail.** We compute $\partial f(u,x)$ by the chain rule 4.64 and let $\hat{F}(u,x) = F(x) + u$ so that $f(u,x) = h(\hat{F}(x,u))$. Thus, $\nabla \hat{F}(u,x) = (I, \nabla F(x))$ is an $m \times (m+n)$-matrix, where $I$ is the $m \times m$ identity matrix. Since $h$ is epi-regular at every point by virtue of being convex and real-valued,

$$\partial f(u,x) = \nabla \hat{F}(u,x)^\top \partial h\big(\hat{F}(u,x)\big),$$

where the subgradients of $h$ at a point $\bar{z} = (\bar{z}_1, \dots, \bar{z}_m)$ are given in 4.66 as $y \in \mathbb{R}^m$ satisfying $\sum_{i=1}^m y_i = 1$, $y_i \geq 0$ if $\bar{z}_i = h(\bar{z})$ and $y_i = 0$ otherwise. The qualification (4.14) holds because the only horizon subgradient of $h$ at any point is 0 by 4.65. Hence,

$$(\tilde{y}, 0) \in \partial f(u,x) \iff \begin{bmatrix} \tilde{y} \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ \nabla F(x)^\top \end{bmatrix} y \;\text{ for some }\; y \in \mathbb{R}^m \;\text{ with }\; \sum_{i=1}^m y_i = 1;$$
$$y_i \geq 0 \;\text{ if }\; f_i(x) + u_i = h\big(F(x) + u\big), \quad y_i = 0 \;\text{ otherwise.}$$

Certainly, $\tilde{y} = y$ and we recover the optimality condition 4.81 if $u = 0$. □

Let's now return to sensitivity analysis and show that the multiplier vectors emerging from the Rockafellar condition 5.10 are essentially the subgradients of the corresponding min-value function. Consequently, passing from a minimization problem in $x$ via a Rockafellian to a family of problems and an optimality condition involving a pair $(x,y)$ isn't only computationally advantageous but also furnishes important insight.

**Theorem 5.13** (subgradients of min-value function). *For a proper lsc function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$, with $f(u,x)$ level-bounded in $x$ locally uniformly in $u$, let*

$$p(u) = \inf f(u, \cdot\,) \quad \textit{and} \quad P(u) = \operatorname{argmin} f(u, \cdot\,) \;\; \forall u \in \mathbb{R}^m.$$

*Then, at any $\bar{u} \in \operatorname{dom} p$,*

$$\partial p(\bar{u}) \subset \bigcup_{\bar{x} \in P(\bar{u})} \big\{ y \in \mathbb{R}^m \;\big|\; (y, 0) \in \partial f(\bar{u}, \bar{x}) \big\}.$$

*If $f$ is convex, then $p$ is convex, the inclusion holds with equality and the sets in the union coincide.*

**Proof.** For $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}$ with $F(u,x,\alpha) = (u,\alpha)$, we've that[1] $\operatorname{epi} p = F(\operatorname{epi} f)$. To see this, note that $(u,\alpha) \in \operatorname{epi} p$ implies $\inf f(u,\cdot\,) \leq \alpha < \infty$. Since $f(u,\cdot\,)$ is lsc with bounded level-sets and $p(u) < \infty$, there's $x \in \operatorname{argmin} f(u,\cdot\,)$ by 4.9 and $f(u,x) \leq \alpha$. If $(u,\alpha) \in F(\operatorname{epi} f)$, then $p(u) \leq f(u,x) \leq \alpha < \infty$ for some $x$.

[1] For $C \subset \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m$, we recall that $F(C) = \{F(x) \mid x \in C\}$.

For $\bar{u} \in \operatorname{dom} p$, $p(\bar{u}) \in \mathbb{R}$ because $\operatorname{argmin} f(\bar{u}, \cdot)$ is nonempty by 4.9 and $f$ is proper. By definition, $v \in \partial p(\bar{u})$ if and only if $(v, -1) \in N_{\operatorname{epi} p}(\bar{u}, p(\bar{u}))$ and our goal becomes to determine $N_{F(\operatorname{epi} f)}(\bar{u}, p(\bar{u}))$. In this regard, we can leverage [105, Theorem 6.43], which states that for closed $C \subset \mathbb{R}^n$, smooth $G : \mathbb{R}^n \to \mathbb{R}^m$ and $\bar{u} \in G(C)$, one has

$$N_{G(C)}(\bar{u}) \subset \bigcup_{\bar{x} \in G^{-1}(\bar{u}) \cap C} \left\{ y \in \mathbb{R}^m \,\middle|\, \nabla G(\bar{x})^\top y \in N_C(\bar{x}) \right\}$$

provided that $G^{-1}(\mathbb{B}(\bar{u}, \varepsilon)) \cap C$ is bounded for some $\varepsilon > 0$. Applying this fact with $G = F$ and $C = \operatorname{epi} f$, we obtain for $(\bar{u}, \bar{\alpha}) \in F(\operatorname{epi} f)$ that

$$N_{F(\operatorname{epi} f)}(\bar{u}, \bar{\alpha}) \subset \bigcup_{(u, x, \beta)} \left\{ (y, \gamma) \in \mathbb{R}^m \times \mathbb{R} \,\middle|\, (y, 0, \gamma) \in N_{\operatorname{epi} f}(u, x, \beta) \right\}$$

because $\nabla F(u, x, \alpha)$ is the $(m + 1) \times (m + n + 1)$-matrix with an $m \times m$ identity matrix in the upper left corner, 1 in the bottom right corner and 0 elsewhere. The union is computed over $(u, x, \beta) \in F^{-1}(\bar{u}, \bar{\alpha}) \cap \operatorname{epi} f$, but this reduces to having $x$ with $(\bar{u}, x, \bar{\alpha}) \in \operatorname{epi} f$. When $\bar{\alpha} = p(\bar{u})$, this reduces further to $x \in P(\bar{u})$. Consequently,

$$N_{\operatorname{epi} p}\big(\bar{u}, p(\bar{u})\big) \subset \bigcup_{\bar{x} \in P(\bar{u})} \left\{ (y, \gamma) \in \mathbb{R}^m \times \mathbb{R} \,\middle|\, (y, 0, \gamma) \in N_{\operatorname{epi} f}\big(\bar{u}, \bar{x}, p(\bar{u})\big) \right\}.$$

The boundedness assumption required in [105, Theorem 6.43] holds because $f$ is level-bounded in $x$ locally uniformly in $u$. We can now conclude that

$$\begin{aligned} y \in \partial p(\bar{u}) &\iff (y, -1) \in N_{\operatorname{epi} p}\big(\bar{u}, p(\bar{u})\big) \implies (y, 0, -1) \in N_{\operatorname{epi} f}\big(\bar{u}, \bar{x}, p(\bar{u})\big) \\ &\iff (y, 0) \in \partial f(\bar{u}, \bar{x}) \;\text{ for some }\; \bar{x} \in P(\bar{u}). \end{aligned}$$

This proves the first assertion. Under convexity, $p$ is convex by 1.21 and [105, Theorem 6.43] furnishes a refinement that ensures equality in the above inclusion and that the union can be dropped. □

We can view $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ in the theorem as a Rockafellian with anchor at $\bar{u}$ for the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$, where $f_0(x) = f(\bar{u}, x)$ for all $x \in \mathbb{R}^n$. Then, $p(\bar{u})$ is the minimum value of the problem and $\partial p(\bar{u})$ estimates the effect of perturbation on the minimum value. The theorem goes beyond the continuity properties in the stability theorem 5.6 by quantifying the rate of change. Under the assumption that the algorithm employed to solve the problem furnishes both a minimizer of $f_0$ and associated multipliers in the sense of the Rockafellar condition, we're now able to quickly estimate the effect of a particular perturbation by invoking the theorem. More significantly, however, we can identify a poorly formulated problem. If the multiplier vector $y$ is relatively large in magnitude, then the minimum value tends to change substantially even under small changes to $u$ away from $\bar{u}$ and this calls into question the suitability of the formulation. One can also examine the different components of $y$ and see what aspect of a perturbation is more influential.

**Example 5.14** (perturbation of inequalities). Let's return to 5.2 and the Rockafellian given by

$$f(u,x) = x^2 + 1 + \iota_{(-\infty,0]}(g(u,x)), \quad \text{with} \quad g(u,x) = (x-2)(x-4) + u.$$

We can utilize 5.13 to estimate how the min-value function $p$ changes near $u = 0$.

**Detail.** Since $f$ is proper and lsc, we can apply the Rockafellar condition 5.10 to the actual problem of minimizing $f(0,\cdot\,)$. By 4.58 and the chain rule 4.64,

$$\partial f(u,x) = (0,\ 2x) + \nabla g(u,x) N_{(-\infty,0]}\big(g(u,x)\big) = (0,\ 2x) + (1,\ 2x-6)Y(u,x),$$

for $(u,x) \in \operatorname{dom} f$, where $Y(u,x) = [0,\infty)$ if $g(u,x) = 0$ and $Y(u,x) = \{0\}$ otherwise. We note that the qualification (4.14) holds because

$$\nabla g(u,x)y = (1,\ 2x-6)y = (0,0) \implies y = 0.$$

Thus, $(y,0) \in \partial f(0,x)$ if and only if $0 = 2x + y(2x-6)$ for some $y \in Y(0,x)$. We see immediately that $(x^\star, y^\star) = (2,2)$ is the unique solution of the condition. Since $f$ is lsc and convex, it's epi-regular at all points of finiteness. The qualification (5.2) also hold, which can be established by utilizing 4.65. We then conclude that $x^\star$ is the unique minimizer of the actual problem by the Rockafellar condition 5.10. (This could equally well have been obtained from the KKT condition 4.77.) Moveover, $\partial p(0) = \{y^\star\}$ by 5.13. Thus, $p$ is actually differentiable at 0 with gradient 2. A small perturbation $u$ is then expected to change the minimum value with approximately $2u$. Since $p$ is convex by 1.21, it follows via the subgradient inequality 2.17 that

$$p(u) \geq p(0) + 2(u-0) = 5 + 2u \quad \forall u \in \mathbb{R}.$$

The right-hand side furnishes a locally accurate approximation of $p$ near 0; see 5.2 as well as Figure 5.5.

While $u = 1$ is still in $\operatorname{dom} p$, $p$ has a vertical slope at that point and $\partial p(1) = \emptyset$. This can be discovered through 5.13 as well because there's no $y$ with $(y,0) \in \partial f(1,\bar{x})$ when $\bar{x} = 3$, the minimizer of $f(1,\cdot\,)$; see also 5.8. □

**Exercise 5.15** (perturbation in projections). For a given $\bar{x} \in \mathbb{R}^n$, consider the projection problem of minimizing $\frac{1}{2}\|x - \bar{x}\|_2^2$ subject to $Ax = b$, where $A$ is an $m \times n$-matrix with rank $m$. Use 5.13 to estimate the effect of changing $\bar{x}$ on the minimum value.

**Guide.** Adopt the Rockafellian $f(u,x) = \iota_X(x) + \frac{1}{2}\|x - (\bar{x}+u)\|_2^2$, with $X = \{x \mid Ax = b\}$, and compute its subgradients and then use the Rockafellar condition 5.10 to compute minimizers. This resembles the calculations in 2.40. Then, invoke 5.13. □

The subgradient expressions in 5.13 are far-reaching but don't explicitly bring forward any structure that might be present in a model. As an example, let's consider a composite function that covers many practical situations. In the process, we'll confirm that the multipliers arising in Chap. 4 from the variational geometry of normal cones indeed correspond to those of the Rockafellar condition 5.10 under the choice of a particular Rockafellian. This generalizes the connections in 5.11 and 5.12.

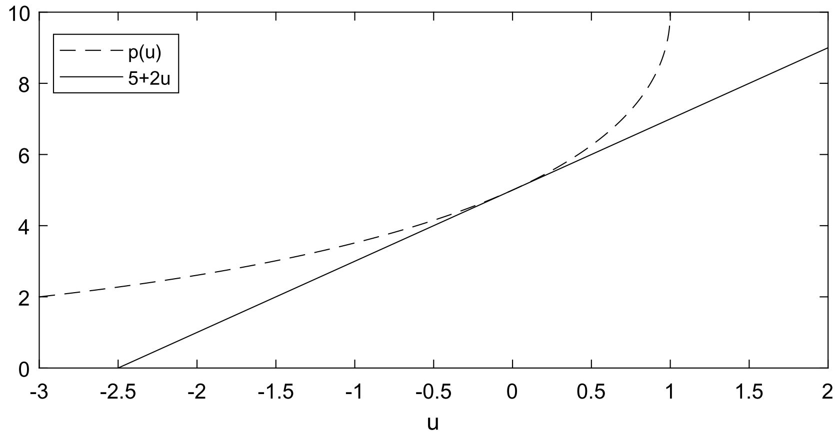


Fig. 5.5: Min-value function $p$ and its estimate in 5.14.

**Proposition 5.16** (multiplier vectors as subgradients). *For smooth $f_0 : \mathbb{R}^n \to \mathbb{R}$, smooth $F : \mathbb{R}^n \to \mathbb{R}^m$, proper, lsc and convex $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ and nonempty closed $X \subset \mathbb{R}^n$, consider the problem*

$$\operatorname*{minimize}_{x\in X} f_0(x) + h\big(F(x)\big)$$

*and the associated Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by*

$$f(u,x) = \iota_X(x) + f_0(x) + h\big(F(x) + u\big).$$

*Suppose that $f(u,x)$ is level-bounded in $x$ locally uniformly in $u$. Let $p(u) = \inf f(u,\cdot)$, $P(u) = \operatorname{argmin} f(u,\cdot)$ and, for $x \in P(u)$,*

$$Y(u,x) = \big\{y \in \partial h\big(F(x)+u\big) \;\big|\; -\nabla f_0(x) - \nabla F(x)^\top y \in N_X(x)\big\}.$$

*Then, for $\bar{u} \in \operatorname{dom} p$, one has*

$$\partial p(\bar{u}) \subset \bigcup_{\bar{x}\in P(\bar{u})} Y(\bar{u},\bar{x}).$$

**Proof.** Suppose that $\bar{y} \in \partial p(\bar{u})$. Then, by 5.13, there's $\bar{x} \in P(\bar{u})$ such that $(\bar{y},0) \in \partial f(\bar{u},\bar{x})$. We derive an outer approximation of this set of subgradients. Let

$$g(x, z_0, z_1, \dots, z_m) = \iota_X(x) + z_0 + h(z_1,\dots,z_m)$$
$$G(u,x) = \big(x, f_0(x), f_1(x)+u_1, \dots, f_m(x)+u_m\big),$$

where $f_1(x),\dots,f_m(x)$ are the components of $F(x)$. Then, $f(u,x) = g(G(u,x))$ and the chain rule 4.64 leads to

$$\partial f(\bar{u},\bar{x}) \subset \nabla G(\bar{u},\bar{x})^\top \partial g\big(G(\bar{u},\bar{x})\big)$$

because $g$ is proper and lsc; we verify the qualification (4.14) below. Via 4.63,

$$\partial g(x, z_0, z_1, \dots, z_m) = N_X(x) \times \{1\} \times \partial h(z_1, \dots, z_m) \ \ \forall (x, z_0, z_1, \dots, z_m) \in \operatorname{dom} g.$$

Moreover, $\nabla G(u, x)$ is the $(n + 1 + m) \times (m + n)$-matrix with the $m \times m$ identity matrix in the bottom left corner, the $n \times n$ identity matrix in the upper right corner, $\nabla f_0(x)$, viewed as a row vector, stacked on top of $\nabla F(x)$ in the bottom right corner and 0 elsewhere. Thus,

$$\nabla G(\bar{u}, \bar{x})^\top \begin{bmatrix} w \\ y_0 \\ y \end{bmatrix} = \begin{bmatrix} y \\ w + \nabla f_0(\bar{x}) y_0 + \nabla F(\bar{x})^\top y \end{bmatrix}. \tag{5.4}$$

Then, $(\bar{y}, 0)$ must satisfy $\bar{y} = y$ and $0 = w + \nabla f_0(\bar{x}) y_0 + \nabla F(\bar{x})^\top y$ for some

$$w \in N_X(\bar{x}), \quad y_0 = 1, \quad y \in \partial h\big(F(\bar{x}) + \bar{u}\big).$$

These expressions simplify to

$$-\nabla f_0(\bar{x}) - \nabla F(\bar{x})^\top \bar{y} \in N_X(\bar{x}) \quad \text{and} \quad \bar{y} \in \partial h\big(F(\bar{x}) + \bar{u}\big).$$

Consequently, $\bar{y} \in Y(\bar{u}, \bar{x})$. It remains to verify the qualification (4.14). By 4.63 and 4.65,

$$\partial^\infty g\big(G(\bar{u}, \bar{x})\big) \subset N_X(\bar{x}) \times \{0\} \times N_{\operatorname{dom} h}\big(F(\bar{x}) + \bar{u}\big).$$

If $(w, y_0, y) \in \partial^\infty g(G(\bar{u}, \bar{x}))$ and this vector makes (5.4) vanish, then $(w, y_0, y) = (0, 0, 0)$ and the qualification holds. □

In the broad context of minimization of a composite function, the proposition shows that under mild assumptions the min-value function associated with a typical Rockafellian has all its subgradients being multiplier vectors associated with some minimizer of the actual problem. This fact holds even when the actual problem is nonconvex and nonsmooth.

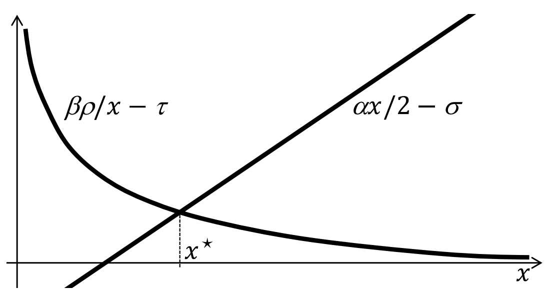


Fig. 5.6: Objective function to determine an economic order quantity.

**Example 5.17** (economic order quantity in inventory management). A store manager needs to determine how many units to order of a given product each time the product runs out. The number of units sold per year is $\rho > 0$, which means that an order quantity of $x$ units results in $\rho/x$ orders per year. Suppose that each one of these orders has a fixed cost of $\beta > 0$. (The cost of the units doesn't factor in here because the total number of units ordered across the whole year is always $\rho$.) The manager also faces an inventory cost of $\alpha > 0$ per unit and year. With an order quantity $x$, the average inventory is $x/2$ so the annual inventory cost becomes $\alpha x/2$. The manager would like to determine an order quantity $x$ such that both the ordering cost $\beta\rho/x$ and the inventory cost $\alpha x/2$ are low. Let's adopt the goal optimization model (cf. 4.6)

$$\operatorname*{minimize}_{x \in X \subset \mathbb{R}} \ \max\bigl\{\beta\rho/x - \tau,\ \tfrac{1}{2}\alpha x - \sigma\bigr\},$$

where $\tau, \sigma$ are the goals for the ordering cost and the inventory cost, respectively; see Figure 5.6. What's the effect of changing the goals?

**Detail.** For perturbation $u \in \mathbb{R}^2$ of the goals, let's adopt the Rockafellian given by

$$f(u, x) = \iota_X(x) + h\bigl(F(x) + u\bigr),$$

where $h(z) = \max\{z_1, z_2\}$ and

$$F(x) = \bigl(f_1(x), f_2(x)\bigr) = \bigl(\beta\rho/x - \tau, \tfrac{1}{2}\alpha x - \sigma\bigr).$$

Then, $f$ has anchor at $\bar{u} = 0$ and the actual model corresponds to minimizing $f(0, \cdot)$. Assuming that $X$ is a nonempty closed subset of the positive numbers, then the assumptions of 5.16 are satisfied as long as $f_1$ is extended in a smooth manner from being defined on the positive numbers to all of $\mathbb{R}$. (This is only a technical issue as we don't consider order quantities below one anyways.) Since the subgradients of $h$ are known from 4.66, we obtain in the notation of 5.16 that

$$\begin{aligned} Y(u, x) = \Bigl\{y \in \mathbb{R}^2 \Bigm| \beta\rho y_1 x^{-2} - \tfrac{1}{2}\alpha y_2 \in N_X(x) \\ y_1 + y_2 = 1;\ y_i \geq 0 \ \text{ if } f_i(x) + u_i = f(u, x),\ y_i = 0 \ \text{ otherwise};\ i = 1, 2\Bigr\} \end{aligned}$$

for $x \in P(u)$. In general, one would employ an algorithm to compute a minimizer $x^\star$ of the model at hand under $\bar{u}$, which hopefully also computes the corresponding multipliers in $Y(\bar{u}, x^\star)$. If we assume that $\sigma = \tau = 0$ and $X$ isn't active at a minimizer, then we obtain the unique minimizer analytically by solving $f_1(x) = f_2(x)$, which produces

$$x^\star = \sqrt{2\beta\rho/\alpha}, \quad \text{with } Y(0, x^\star) = \bigl\{\bigl(\tfrac{1}{2}, \tfrac{1}{2}\bigr)\bigr\}.$$

Then, by 5.16, the minimum value function $p(u) = \inf f(u, \cdot)$ is differentiable at $u = 0$ with $\nabla p(0) = (1/2, 1/2)$. This provides the insight that if the current goal of zero ordering cost is changed to a small positive number $\tau$, which corresponds to setting $u = (-\tau, 0)$, then the change in minimum value is approximately $\langle \nabla p(0), u\rangle = -\tau/2$. The negative value is reasonable as raising the goal value reduces the shortfall. □

## 5.C Lagrangians and Dual Problems

In addition to their role in optimality conditions and sensitivity analysis, Rockafellians for a minimization problem furnish a principal path to constructing *problem relaxations*. These are alternative problems with minimum values no higher than that of the actual one. Relaxations are computationally useful and, when properly tuned, may even reproduce the minimum value of the actual problem. The process of tuning the relaxations is in itself an optimization problem, which we refer to as a *dual problem*. We'll eventually see that dual problems furnish yet another way of computing multipliers.

For $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$, let's consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f_0(x)$$

and an associated Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with anchor at 0. (The focus on 0 instead of a more general $\bar{u}$ promotes symmetry below, without much loss of generality because one can always shift the perturbation vector by redefining $f$.) The Rockafellian defines a family of problems, among which

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f(0, x)$$

is the actual problem of interest because $f(0, x) = f_0(x)$ for all $x$. The choice of Rockafellian could be dictated by a concern about certain model parameters as part of a sensitivity analysis, but just as well by the need for constructing tractable relaxations of the actual problem and this is the goal now.

We can't do worse if permitted to *optimize* the perturbation vector together with $x$. Thus, the problem of minimizing $f(u, x)$ over $(u, x) \in \mathbb{R}^m \times \mathbb{R}^n$ is a relaxation of the actual problem. Regardless of $y \in \mathbb{R}^m$, this is also the case for minimizing $f(u, x) - \langle y, u\rangle$ over $(u, x) \in \mathbb{R}^m \times \mathbb{R}^n$ as $u = 0$ remains a possibility. Consequently, for any $y \in \mathbb{R}^m$,

$$\begin{aligned}\inf_{x\in\mathbb{R}^n} f_0(x) &= \inf_{x\in\mathbb{R}^n} f(0, x) \\ &\geq \inf_{(u,x)\in\mathbb{R}^m\times\mathbb{R}^n} \big\{f(u, x) - \langle y, u\rangle\big\} = \inf_{x\in\mathbb{R}^n} l(x, y),\end{aligned} \tag{5.5}$$

where we use the short-hand notation

$$l(x, y) = \inf_{u\in\mathbb{R}^m} \big\{f(u, x) - \langle y, u\rangle\big\}. \tag{5.6}$$

The function $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ given by this formula is the *Lagrangian* of $f$.

A benefit from bringing in a vector $y$ is that we now can tune the relaxation by selecting $y$ appropriately. We interpret $y_i$ as the "price" associated with setting $u_i$ nonzero so that $-y_i u_i$ is the additional cost for such $u_i$ to be accrued on top of $f(u, x)$. Then, $\inf_x l(x, y)$ is the lowest possible cost that can be achieved when one is permitted to "buy" perturbations at prices stipulated by $y$. The notation $y$ for this price vector isn't arbitrary; deep connections with multipliers emerge below. However, we start by examining Lagrangians arising from typical Rockafellians.

**Example 5.18** (Lagrangian for equalities and inequalities). For $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \dots, m$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, q$, let's consider the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\, f_0(x) \;\text{ subject to }\; f_i(x) = 0,\; i = 1, \dots, m;\; g_i(x) \le 0,\; i = 1, \dots, q.$$

Among many possibilities, a Rockafellian is obtained by perturbing the right-hand side of the constraints. While such perturbations can be viewed from the angle of sensitivity analysis as changing "budgetary thresholds," they can also be introduced purely for the purpose of developing optimality conditions and/or algorithms. Specifically, let $D = \{0\}^m \times (-\infty, 0]^q$ and

$$F(x) = \big(f_1(x), \dots, f_m(x), g_1(x), \dots, g_q(x)\big).$$

A Rockafellian $f : \mathbb{R}^{m+q} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ for the problem is defined by

$$f(u, x) = f_0(x) + \iota_D\big(F(x) + u\big).$$

The actual problem is then to minimizing $f(0, \cdot\,)$ and the Lagrangian has

$$l(x, y) = \begin{cases} f_0(x) + \big\langle F(x), y \big\rangle & \text{if } y_{m+1}, \dots, y_{m+q} \ge 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Moreover, the actual problem is equivalently stated as

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; \sup\nolimits_{y \in \mathbb{R}^{m+q}} l(x, y).$$

**Detail.** With $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{m+q}$, the Rockafellian $f$ produces the Lagrangian given by

$$l(x, y) = \inf\nolimits_{u \in \mathbb{R}^{m+q}} \big\{ f_0(x) + \iota_D\big(F(x) + u\big) - \langle y, u \rangle \big\}.$$

If there's $y_i < 0$ for some $i \in \{m+1, \dots, m+q\}$, then we can select $u_j = -f_j(x)$ for all $j \in \{1, \dots, m\}$ and $u_j = -g_j(x)$ for all $j \in \{m+1, \dots, m+q\} \setminus \{i\}$ so that $\iota_D(F(x) + u)$ remains 0 as $u_i \to -\infty$. But, then

$$f_0(x) + \iota_D\big(F(x) + u\big) - \langle y, u \rangle \to -\infty$$

and $l(x, y) = -\infty$.

If $y_i \ge 0$ for all $i \in \{m+1, \dots, m+q\}$, then $\bar{u}$, with components $\bar{u}_j = -f_j(x)$ for all $j \in \{1, \dots, m\}$ and $\bar{u}_j = -g_j(x)$ for all $j \in \{m+1, \dots, m+q\}$, solves

$$\underset{u \in \mathbb{R}^{m+q}}{\text{minimize}}\, f_0(x) + \iota_D\big(F(x) + u\big) - \langle y, u \rangle$$

and this results in

$$l(x, y) = f_0(x) - \langle y, \bar{u} \rangle = f_0(x) + \big\langle y, F(x) \big\rangle.$$

With the expression for $l$ established, we also see that

$$\sup_{y\in\mathbb{R}^{m+q}} l(x,y) = \sup\left\{f_0(x) + \left\langle F(x), y\right\rangle \,\middle|\, y_{m+1},\dots,y_{m+q} \geq 0\right\} = f(0,x)$$

because if $f_i(x) = 0$ for all $i = 1,\dots,m$ and $g_i(x) \leq 0$ for all $i = 1,\dots,q$, then $y = 0$ is a maximizer. If $g_i(x) > 0$ for some $i$, then $y_i \to \infty$ brings the supremum to infinity. If $f_i(x) \neq 0$ for some $i$, then $y_i \to \infty$ or $-\infty$ brings the supremum to infinity again. □

**Exercise 5.19** (Lagrangian). In the setting of 5.18 with $f_0(x) = \exp(x)$ and $g_1(x) = -x$, plot the Lagrangian as a function of $x$ for many different $y \in \mathbb{R}$ and compare it with $f_0 + \iota_{[0,\infty)}$. Confirm that (5.5) holds.

**Example 5.20** (Lagrangian for max-function). For $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,m$, let's consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \max_{i=1,\dots,m} f_i(x)$$

and the associated Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ expressed by

$$f(u,x) = h\big(F(x) + u\big),$$

where $F(x) = (f_1(x),\dots,f_m(x))$ and $h(z) = \max\{z_1,\dots,z_m\}$. We recover the actual problem by minimizing $f(0,\cdot\,)$. The corresponding Lagrangian has

$$l(x,y) = \begin{cases} \big\langle F(x), y\big\rangle & \text{if } y \geq 0,\ \sum_{i=1}^m y_i = 1\\ -\infty & \text{otherwise.}\end{cases}$$

Moreover, the actual problem is equivalently stated as

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \sup_{y\in\mathbb{R}^m} l(x,y).$$

**Detail.** With $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the Rockafellian $f$ leads to the Lagrangian given by

$$l(x,y) = \inf_{u\in\mathbb{R}^m}\left\{h\big(F(x)+u\big) - \langle y,u\rangle\right\}.$$

If $y_i < 0$ for some $i$, then we can select $u_j = 0$ for all $j \in \{1,\dots,m\}\setminus\{i\}$ and let $u_i \to -\infty$, which implies that $h(F(x)+u) - \langle y,u\rangle \to -\infty$ and $l(x,y) = -\infty$.

If $y_i \geq 0$ for all $i$ and $\sum_{i=1}^m y_i = \beta \neq 1$, then $u_i = \alpha$ for all $i$ gives that

$$h\big(F(x)+u\big) - \langle y,u\rangle = \max_{i=1,\dots,m}\left\{f_i(x) + \alpha\right\} - \alpha\beta = \max_{i=1,\dots,m} f_i(x) + \alpha(1-\beta).$$

If $\beta > 1$, then $\alpha \to \infty$ makes the previous term approach $-\infty$ and likewise for $\beta < 1$ with $\alpha \to -\infty$. In either case, $l(x,y) = -\infty$.

If $y_i \geq 0$ for all $i$ and $\sum_{i=1}^m y_i = 1$, then $\bar{u} = (-f_1(x),\dots,-f_m(x))$ solves

$$\underset{u\in\mathbb{R}^m}{\text{minimize}}\, h\big(F(x)+u\big) - \langle y,u\rangle$$

because increasing $u_i$ above $\bar{u}_i$ isn't beneficial due to $y_i \le 1$. Likewise, reducing $u_i$ below $\bar{u}_i$ only benefits $h(F(x)+u)$ if all $u_1, \dots, u_m$ are reduced, but then $-\langle y, u\rangle$ grows a corresponding amount because $\sum_{i=1}^m y_i = 1$. The minimizer $\bar{u}$ then produces the formula for $l(x, y)$.

We see that

$$\sup_{y\in\mathbb{R}^m} l(x, y) = \max_{i=1,\dots,m} f_i(x) = f(0, x) \;\; \forall x \in \mathbb{R}^n$$

because, generally, $\max\{\langle z, y\rangle \mid y \ge 0, \sum_{i=1}^m y_i = 1\} = \max_{i=1,\dots,m} z_i$. □

The Rockafellians in these examples lead to explicit expressions for the corresponding Lagrangians. This is especially useful if they were to be employed within an algorithm for the purpose of computing lower bounds on the minimum values of the actual problems via (5.5). Still, one shouldn't feel confined to these choices; opportunities for innovative constructions abound.

The examples show that a problem, after being enriched with perturbations as defined by a Rockafellian, can be restated in terms of a Lagrangian: The actual problem can be viewed as minimizing the worst-case Lagrangian. This holds much beyond these examples. Before treating more general problems, however, we introduce a central concept from convex analysis that's hidden in the definition of Lagrangians.

**Definition 5.21** (conjugate function). For $h : \mathbb{R}^m \to \overline{\mathbb{R}}$, the function $h^* : \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by

$$h^*(v) = \sup_{u\in\mathbb{R}^m} \big\{\langle v, u\rangle - h(u)\big\}$$

is the *conjugate* of $h$. The mapping of a function $h$ into its conjugate is referred to as the *Legendre-Fenchel transform*.

For a Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and its Lagrangian $l$ defined by (5.6), we see that

$$-l(\bar{x}, \cdot\,) \;\text{ is the conjugate of }\; f(\cdot\,, \bar{x})$$

regardless of $\bar{x} \in \mathbb{R}^n$. From this angle, a conjugate function is an economical way of expressing a Lagrangian associated with a particular Rockafellian. Geometrically, the conjugate of a function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ defines the affine functions with epigraphs containing the epigraph of $h$. From Figure 5.7, we see specifically that

$$\begin{aligned}
&(v, \beta) \in \operatorname{epi} h^* \iff \sup_{u\in\mathbb{R}^m} \big\{\langle v, u\rangle - h(u)\big\} \le \beta \qquad (5.7)\\
&\iff \langle v, u\rangle - h(u) \le \beta \;\; \forall u \in \mathbb{R}^m \iff \langle v, u\rangle - \beta \le h(u) \;\; \forall u \in \mathbb{R}^m.
\end{aligned}$$

Thus, $\operatorname{epi} h \subset \operatorname{epi}(\langle v, \cdot\,\rangle - \beta)$ as illustrated in the figure, where in fact

$$h(u) = \begin{cases} \frac{1}{2}u^2 & \text{if } u \le 0 \\ 0 & \text{if } u \in (0, 1] \\ \infty & \text{otherwise} \end{cases} \qquad\qquad h^*(v) = \begin{cases} \frac{1}{2}v^2 & \text{if } v \le 0 \\ v & \text{otherwise.} \end{cases}$$

The expression for $h^*$ follows directly from the definition.

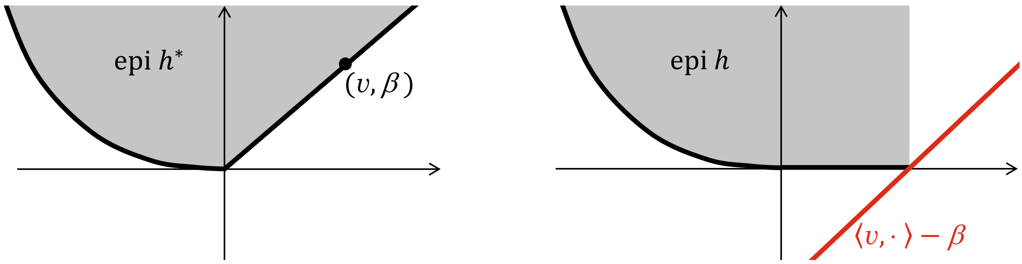


Fig. 5.7: A convex function and its conjugate.

**Exercise 5.22** (conjugate of quadratic function). Determine the conjugate of the quadratic function given by

$$h(u) = \langle c, u \rangle + \tfrac{1}{2}\langle u, Qu \rangle,$$

where $Q$ is a symmetric positive definite $m \times m$-matrix. Specialize the resulting formula for the conjugate in the case of $c = 0$ and $Q = I$, the identity matrix.

**Guide.** Write $h^*(v) = -\inf_{u \in \mathbb{R}^m}\{h(u) - \langle u, v \rangle\}$ and determine a minimizer using the optimality condition 2.19. □

One might hope that applying the Legendre-Fenchel transform twice returns the original function. As stated next, this is indeed the case under convexity with minor exceptions.

**Theorem 5.23** (Fenchel-Moreau). *For $h : \mathbb{R}^m \to \overline{\mathbb{R}}$, one has*

$$(h^*)^*(u) \leq h(u) \;\; \forall u \in \mathbb{R}^m,$$

*with equality holding when $h$ is proper, lsc and convex and then $h^*$ is also proper, lsc and convex.*

**Proof.** Let $\bar{u} \in \mathbb{R}^m$. We first establish that $(h^*)^*(\bar{u}) \leq h(\bar{u})$ or, equivalently, that

$$\langle v, \bar{u} \rangle - h^*(v) \leq h(\bar{u}) \;\; \forall v \in \mathbb{R}^m.$$

Let $\bar{v} \in \mathbb{R}^m$ be arbitrary. Thus, it suffices to show that

$$\langle \bar{v}, \bar{u} \rangle - h^*(\bar{v}) \leq h(\bar{u}). \tag{5.8}$$

If $h^*(\bar{v}) = \infty$, then the relation holds trivially. If $h^*(\bar{v}) = -\infty$, then for any $\beta \in \mathbb{R}$, $(\bar{v}, \beta) \in \operatorname{epi} h^*$ so that $\langle \bar{v}, \bar{u} \rangle - \beta \leq h(\bar{u})$ by (5.7). Thus, $h(\bar{u}) = \infty$ and the relation holds again. If $h^*(\bar{v}) \in \mathbb{R}$, then $(\bar{v}, h^*(\bar{v})) \in \operatorname{epi} h^*$ so that (5.8) holds by (5.7).

Second, suppose that $\bar{u} \in \operatorname{int}(\operatorname{dom} h)$. Since $h$ is proper and convex, $h(\bar{u})$ is finite and by 2.25 there's a subgradient $\bar{v} \in \partial h(\bar{u})$. By the optimality condition 2.19, $\bar{u} \in \operatorname{argmin}\{h - \langle \bar{v}, \cdot \rangle\} = \operatorname{argmax}\{\langle \bar{v}, \cdot \rangle - h\}$. Thus,

$$h^*(\bar{v}) = \sup_{u \in \mathbb{R}^m} \big\{\langle \bar{v}, u \rangle - h(u)\big\} = \langle \bar{v}, \bar{u} \rangle - h(\bar{u}).$$

Using this fact, we obtain that

$$(h^*)^*(\bar{u}) = \sup_{v\in\mathbb{R}^m} \big\{\langle \bar{u}, v\rangle - h^*(v)\big\} \geq \langle \bar{u}, \bar{v}\rangle - h^*(\bar{v}) = \langle \bar{u}, \bar{v}\rangle - \langle \bar{u}, \bar{v}\rangle + h(\bar{u}).$$

Thus, $(h^*)^*(\bar{u}) = h(\bar{u})$ when also invoking the first assertion. A similar argument takes care of $\bar{u} \notin \mathrm{int}(\mathrm{dom}\, h)$; see [105, Theorems 8.13, 11.1] for further details. □

The theorem allows us to establish that the actual objective function of interest is recovered by maximizing the Lagrangian as experienced in the above examples.

**Proposition 5.24** (reformulation in terms of a Lagrangian). *For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a proper Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with anchor at 0, suppose that $f(\cdot, x)$ is lsc and convex for all $x \in \mathbb{R}^n$. If $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is the corresponding Lagrangian defined by (5.6), then*

$$f_0(x) = f(0, x) = \sup_{y\in\mathbb{R}^m} l(x, y) \quad \forall x \in \mathbb{R}^n.$$

**Proof.** Let $\bar{x} \in \mathbb{R}^n$ be fixed. By definition,

$$-l(\bar{x}, y) = \sup_{u\in\mathbb{R}^m} \big\{\langle y, u\rangle - f(u, \bar{x})\big\} \quad \forall y \in \mathbb{R}^m$$

so $-l(\bar{x}, \cdot\,)$ is the conjugate of $f(\cdot\,, \bar{x})$. If $f(\cdot\,, \bar{x})$ is proper, then the Fenchel-Moreau theorem 5.23 applies and

$$f(u, \bar{x}) = \sup_{y\in\mathbb{R}^m} \big\{\langle u, y\rangle + l(\bar{x}, y)\big\}.$$

In particular, $f(0, \bar{x}) = \sup_{y\in\mathbb{R}^m} l(\bar{x}, y)$. If $f(u, \bar{x}) = \infty$ for all $u$, then trivially $l(\bar{x}, y) = \infty$ for all $y$ so $\sup_{y\in\mathbb{R}^m} l(\bar{x}, y) = \infty$ and the claim holds again. □

Thus far, a Lagrangian has served in two capacities: Under mild assumptions on the underlying Rockafellian as laid out in 5.24, its maximization with respect to $y$ recovers the actual objective function and, without any assumptions, its minimization with respect to $x$ furnishes a lower bound on the minimum value of the actual problem via (5.5). This maximization and minimization define

$$\forall x \in \mathbb{R}^n : \quad \varphi(x) = \sup_{y\in\mathbb{R}^m} l(x, y)$$
$$\forall y \in \mathbb{R}^m : \quad \psi(y) = \inf_{x\in\mathbb{R}^n} l(x, y).$$

The function $\varphi$ is simply an equivalent expression for the actual objective function under the assumptions of 5.24 so its minimization coincides in that case with the actual problem. The function $\psi$ quantifies the lower bound in (5.5) so its maximization produces the *best possible* lower bound. We think of this maximization as an effort to tune the relaxations produced by the chosen Rockafellian. A Lagrangian therefore sets up a pair of optimization problems with profound connections.

Specifically, a Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ associated with the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and an anchor at 0 defines via its Lagrangian $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, given by (5.6), a *pair* of *primal* and *dual problems*:

| | | | |
|---|---|---|---|
| (primal) | $\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \varphi(x),$ | where | $\varphi(x) = \sup_{y\in\mathbb{R}^m} l(x,y)$ |
| (dual) | $\underset{y\in\mathbb{R}^m}{\text{maximize}}\ \psi(y),$ | where | $\psi(y) = \inf_{x\in\mathbb{R}^n} l(x,y)$ |

The name "primal problem" is motivated by its equivalence with the actual problem of interest under the assumptions of 5.24 because then

$$f_0(x) = f(0,x) = \varphi(x) \quad \forall x \in \mathbb{R}^n.$$

In parallel, we refer to $x$ and $y$ as *primal* and *dual variables*, respectively, and $\varphi$ and $\psi$ as *primal* and *dual objective functions*, respectively.

**Theorem 5.25** (weak duality). *For a Lagrangian $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, regardless of its expression, and its primal and dual problems, one has*

$$\inf_{x\in\mathbb{R}^n} \varphi(x) \geq \sup_{y\in\mathbb{R}^m} \psi(y) \geq \psi(\bar{y}) \quad \forall \bar{y} \in \mathbb{R}^m.$$

**Proof.** For $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$,

$$\varphi(\bar{x}) = \sup_{y\in\mathbb{R}^m} l(\bar{x},y) \geq l(\bar{x},\bar{y}) \geq \inf_{x\in\mathbb{R}^n} l(x,\bar{y}) = \psi(\bar{y}).$$

Thus, $\inf_{x\in\mathbb{R}^n} \varphi(x) \geq \psi(\bar{y})$ and the conclusion follows since $\bar{y}$ is arbitrary. □

The theorem establishes that a dual problem is always a relaxation of its primal problem, which reinforces the fact from (5.5) that the minimum value of a Lagrangian bounds from below the minimum value of the actual problem. This can be exploited algorithmically to address difficult optimization problems as we'll see in the next section. Before we enter into details, however, let's consider some common examples of primal and dual problems.

**Example 5.26** (dual linear problems). For $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and an $m \times n$-matrix $A$, the linear optimization problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \langle c,x\rangle \ \text{ subject to } \ Ax = b,\ x \geq 0$$

coincides with the primal problem produced by the Rockafellian expressed as

$$f(u,x) = \langle c,x\rangle + \iota_{[0,\infty)^n}(x) + \iota_{\{0\}^m}(b - Ax + u),$$

which then defines the Lagrangian given by

$$l(x,y) = \begin{cases} \langle c,x\rangle - \langle y, Ax-b\rangle & \text{if } x \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

The corresponding dual problem is

$$\underset{y \in \mathbb{R}^m}{\text{maximize}} \ \langle b, y \rangle \ \text{ subject to } \ A^\top y \leq c.$$

**Detail.** In an argument similar to those in 5.18, we see that

$$l(x, y) = \inf_{u \in \mathbb{R}^m} \big\{ f(u, x) - \langle y, u \rangle \big\} = \begin{cases} \langle b, y \rangle + \langle c - A^\top y, x \rangle & \text{if } x \geq 0 \\ \infty & \text{otherwise,} \end{cases}$$

which is equivalent to the asserted expression. From 5.24, $f(0, x) = \sup_{y \in \mathbb{R}^m} l(x, y)$, but this can be seen directly as well. Thus, the actual problem coincides with the primal problem produced by the Rockafellian via this Lagrangian. The dual objective function has

$$\psi(y) = \inf_{x \in \mathbb{R}^n} l(x, y) = \begin{cases} \langle b, y \rangle & \text{if } c - A^\top y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

and this yields the asserted dual problem.

The process is symmetric. If we start with the problem of minimizing $\langle -b, y \rangle$ subject to $A^\top y \leq c$ and adopting the Rockafellian

$$\tilde{f}(u, y) = \langle -b, y \rangle + \iota_{(-\infty, 0]^n}(A^\top y - c + u),$$

then we obtain the Lagrangian

$$\tilde{l}(y, x) = \begin{cases} \langle -c, x \rangle + \big\langle Ax - b, y \big\rangle & \text{if } x \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

and

$$\tilde{\psi}(x) = \inf_{y \in \mathbb{R}^m} \tilde{l}(y, x) = \begin{cases} \langle -c, x \rangle & \text{if } Ax = b, \ x \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem to that of minimizing $\langle -b, y \rangle$ subject to $A^\top y \leq c$ is therefore maximizing $\langle -c, x \rangle$ subject to $Ax = b$ and $x \geq 0$.

The above choice of Rockafellian has the advantage that it leads to an explicit expression for the dual problem. However, there are many alternatives. For example, one might only be concerned about changes to one of the equality constraints. This leads to a different Lagrangian and, in turn, another dual problem. Specifically, let's consider $Ax = b$ *and* $\langle a, x \rangle = \alpha$, the latter being of concern. This leads to the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \langle c, x \rangle \ \text{ subject to } \ Ax = b, \ \langle a, x \rangle = \alpha, \ x \geq 0,$$

which coincides with the primal problem produced by an alternative Rockafellian expressed as

$$f(u, x) = \langle c, x \rangle + \iota_{[0, \infty)^n}(x) + \iota_{\{0\}^m}(b - Ax) + \iota_{\{0\}}\big(\alpha - \langle a, x \rangle + u\big).$$

Now, only one of the equality constraints is perturbed. This Rockafellian produces an alternative Lagrangian given by

$$l(x,y) = \inf_{u\in\mathbb{R}} \big\{ f(u,x) - yu \big\} = \begin{cases} \alpha y + \langle c - ay, x\rangle & \text{if } Ax = b,\ x \geq 0 \\ \infty & \text{otherwise,} \end{cases}$$

which defines a dual problem of maximizing $\psi(y) = \inf_x l(x,y)$ over $y \in \mathbb{R}$. While this dual problem has only a single variable, its objective function isn't generally available in an explicit form; for each $y$, one needs to solve a linear optimization problem to obtain $\psi(y)$. Still, insight as well as computational benefits may emerge from this approach. □

**Example 5.27** (dual quadratic problems). For $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, an $m \times n$-matrix $A$ and a symmetric positive definite $n \times n$-matrix $Q$, the quadratic optimization problem

$$\operatorname*{minimize}_{x\in\mathbb{R}^n} \ \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle \ \text{ subject to } \ Ax = b,\ x \geq 0$$

coincides with the primal problem produced by the Rockafellian given as

$$f(u,v,x) = \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + \iota_{\{0\}^m}\big(b - Ax + u\big) + \iota_{(-\infty,0]^n}(-x + v),$$

which then defines the Lagrangian expressed as

$$l(x,y,z) = \begin{cases} \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + \langle b - Ax, y\rangle + \langle -x, z\rangle & \text{if } z \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The corresponding dual problem is

$$\operatorname*{maximize}_{y\in\mathbb{R}^m, z\in\mathbb{R}^n} \ \alpha + \langle \tilde{b}, y\rangle + \langle Q^{-1}c, z\rangle - \tfrac{1}{2}\big\langle Q^{-1}(A^\top y + z), A^\top y + z\big\rangle \ \text{ subject to } \ z \geq 0,$$

where $\alpha = -\frac{1}{2}\langle c, Q^{-1}c\rangle$ and $\tilde{b} = b + AQ^{-1}c$, which is a quadratic problem.

**Detail.** The quadratic problem fits the framework of 5.18. From the discussion there, the actual problem coincides with the primal problem produced by this Lagrangian. Suppose that $z \geq 0$. Since $Q$ is positive definite and symmetric, it follows that $l(\cdot, y, z)$ is strictly convex by 1.24. The optimality condition 2.19 establishes that

$$\bar{x} = Q^{-1}(A^\top y + z - c)$$

is the minimizer of $l(\cdot, y, z)$ because the inverse $Q^{-1}$ exists and, in fact, is also positive definite and symmetric. The dual objective function has $\psi(y,z) = l(\bar{x}, y, z)$, which simplifies to the stated expression after some algebra. Since $Q^{-1}$ is positive definite, it has a square root $R$ such that $Q^{-1} = R^\top R$. The dual objective function can then be expressed as

$$\alpha + \langle \tilde{b}, y\rangle + \big\langle Q^{-1}c, z\big\rangle - \tfrac{1}{2}\big\|R(A^\top y + z)\big\|_2^2,$$

which defines a concave function in $(y,z)$; cf. 1.18(c). □

These examples give dual problems with concave objective functions. This is always the case: For a Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$, the dual objective function $\psi$ has

$$-\psi(y) = -\inf_{x,u} \big\{ f(u,x) - \langle y,u\rangle \big\} = \sup_{x,u} \big\{ \langle y,u\rangle - f(u,x) \big\}.$$

Thus, $-\psi$ is convex by 1.18(a). Interestingly, regardless of convexity in a primal problem and in any of the underlying functions, the dual objective function $\psi$ is concave and thus its maximization can be achieved by minimizing the convex function $-\psi$. Although the dual problem may only provide a lower bound on the minimum value of the actual problem at hand, it could be more tractable due to this property.

A versatile proposition for problems with a composite structure, which includes the earlier examples, is central to the later developments. The resulting Lagrangian and corresponding dual problem are the foundations for several algorithms.

**Proposition 5.28** (Lagrangian for composite function). *For $f_0 : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^m$ and proper, lsc and convex $h : \mathbb{R}^m \to \overline{\mathbb{R}}$, consider the problem*

$$\operatorname*{minimize}_{x \in X \subset \mathbb{R}^n} f_0(x) + h\big(F(x)\big).$$

*The Rockafellian given by*

$$f(u,x) = \iota_X(x) + f_0(x) + h\big(F(x) + u\big)$$

*recovers the actual problem as minimizing $f(0,\cdot\,)$ and produces a Lagrangian with*

$$l(x,y) = \iota_X(x) + f_0(x) + \big\langle F(x), y \big\rangle - h^*(y).$$

*Moreover, the actual problem is equivalently stated as*

$$\operatorname*{minimize}_{x \in \mathbb{R}^n} \ \sup_{y \in \mathbb{R}^m} l(x,y).$$

**Proof.** By (5.6), the Lagrangian takes the form

$$\begin{aligned} l(x,y) &= \inf_{u \in \mathbb{R}^m} \big\{ \iota_X(x) + f_0(x) + h\big(F(x)+u\big) - \langle y,u\rangle \big\} \\ &= \iota_X(x) + f_0(x) - \sup_{u \in \mathbb{R}^m} \big\{ \langle y,u\rangle - h\big(F(x)+u\big) \big\}. \end{aligned}$$

For $a \in \mathbb{R}^m$, we deduce from its definition 5.21 that the conjugate of $u \mapsto h(u+a)$ is the function $v \mapsto h^*(v) - \langle a,v\rangle$. Thus, the above sup-expression equals $h^*(y) - \langle F(x), y\rangle$ and we obtain the formula for $l$. If $f$ is proper, then 5.24 applies because $f(\cdot, x)$ is lsc and convex for all $x$; see in part 1.18(c). In that case, we obtain

$$f(0,x) = \sup_{y \in \mathbb{R}^m} l(x,y).$$

If $f$ isn't proper, then it must have $f(u,x) = \infty$ for all $u,x$. Thus, $l(x,y) = \infty$ for all $x,y$ and the same expression holds for $f(0,x)$ in this case as well. $\square$

We recall the convention $\infty - \infty = \infty$. So, in the proposition, $l(x, y) = \infty$ for $x \notin X$ even if $h^*(y) = \infty$. The expression for the Lagrangian is made concrete by explicit formulae for conjugate functions. Let's record some common cases.

**Example 5.29** (conjugate functions). For $U \subset \mathbb{R}^m$, one has

$$h(u) = \iota_U(u) \quad \Longrightarrow \quad h^*(v) = \sup_{u \in U} \langle u, v \rangle,$$

which, if $U$ is a cone, specializes to

$$h(u) = \iota_U(u) \quad \Longrightarrow \quad h^*(v) = \iota_{\mathrm{pol}\, U}(v),$$

where $\mathrm{pol}\, U = \{v \mid \langle u, v \rangle \leq 0 \ \ \forall u \in U\}$ is the polar to $U$; see (2.11) and Figure 2.26.

For any $V \subset \mathbb{R}^m$, one has

$$h(u) = \sup_{v \in V} \langle u, v \rangle \quad \Longrightarrow \quad h^*(v) = \iota_{\mathrm{cl\, con}\, V}(v)$$

and then $h^*(v) = \iota_V(v)$ when $V$ is closed and convex.

Moreover,

$$h(u) = \max\{u_1, \dots, u_m\} \quad \Longrightarrow \quad h^*(v) = \begin{cases} 0 & \text{if } v \geq 0,\ \sum_{i=1}^m v_i = 1 \\ \infty & \text{otherwise.} \end{cases}$$

**Detail.** The definition of conjugate functions immediately leads to the formulae in the case of $h = \iota_U$. For $h = \sup_{v \in V} \langle \cdot, v \rangle$, we refer to [105, Theorem 8.24]. The last claim is implicitly derived in 5.20. Additional formulae are given in [105, Section 11.F]. □

**Exercise 5.30** (logarithm). (a) For $f(x) = -\ln x$ if $x > 0$ and $f(x) = \infty$ otherwise, show that $f^*(y) = -\ln(-y) - 1$ if $y < 0$ and $f^*(y) = \infty$ otherwise. (b) For $f(x) = x \ln x$ if $x > 0$, $f(0) = 0$ and $f(x) = \infty$ otherwise, show that $f^*(y) = \exp(y - 1)$.

**Exercise 5.31** (exponential). For $f(x) = \exp(x)$, show that $f^*(y) = y \ln y - y$ if $y > 0$, $f^*(0) = 0$ and $f^*(y) = \infty$ otherwise.

**Exercise 5.32** (inverse). For $f(x) = 1/x$ if $x > 0$ and $f(x) = \infty$ otherwise, show that $f^*(y) = -2\sqrt{-y}$ if $y \leq 0$ and $f^*(y) = \infty$ otherwise.

In addition to being helpful in expressing Lagrangians as in the context of 5.28, conjugate functions actually underpin dual problems even more fundamentally. For any Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and the corresponding Lagrangian $l$, the dual objective function has

$$\begin{aligned} \psi(y) &= \inf_x l(x, y) = \inf_{u,x} \big\{ f(u, x) - \langle u, y \rangle \big\} \\ &= -\sup_{u,x} \big\{ \langle u, y \rangle + \langle x, 0 \rangle - f(u, x) \big\} = -f^*(y, 0). \end{aligned}$$

Consequently, the dual objective function is immediately available from any explicit formula for the conjugate of $f$ that might be available and this holds regardless of the actual problem and its perturbation as expressed by $f$.

**Exercise 5.33** (linear constraints). For $f_0 : \mathbb{R}^n \to \mathbb{R}$, $b \in \mathbb{R}^m$, $d \in \mathbb{R}^q$ as well as $m \times n$ and $q \times n$ matrices $A$ and $D$, consider the problem of minimizing $f_0(x)$ subject to $Ax = b$ and $Dx \leq d$ and the Rockafellian $f : \mathbb{R}^{m+q} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$f(u, v, x) = f_0(x) + \iota_{\{0\}^m}(Ax - b + u) + \iota_{(-\infty,0]^q}(Dx - d + v).$$

Show that the corresponding dual objective function has

$$\psi(y, z) = \begin{cases} -\langle b, y\rangle - \langle d, z\rangle - f_0^*\big(-A^\top y - D^\top z\big) & \text{if } z \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

**Guide.** The Rockafellian fits the framework of 5.18, which then furnishes the expression for the corresponding Lagrangian. □

## 5.D Lagrangian Relaxation

It's important to be able to assess the quality of a candidate solution to a problem and thereby determine if further calculations are needed. Optimality conditions serve this purpose, but dual problems also offer possibilities that are often viable in practice. For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a candidate solution $\bar{x}$, the *optimality gap* of $\bar{x}$ is given by

$$f_0(\bar{x}) - \inf f_0.$$

A relatively small optimality gap indicates that $\bar{x}$ is rather good in the sense that the associated "cost" isn't much higher than the minimum value. This could convince a decision-maker to adopt $\bar{x}$, making further efforts to find an even better solution superfluous. An optimality gap is measured in the units of the objective function and is thus well understood by a decision-maker in most cases. The meaning of an optimality gap of \$1 million is clear: We're potentially leaving this amount on the table, which might be small or large depending on the circumstances.

Since there are often application-dependent heuristic algorithms for finding a candidate solution $\bar{x}$ of a difficult problem, the challenging part in calculating the optimality gap is to estimate $\inf f_0$. Typically, we prefer a lower bound on $\inf f_0$ as it produces a conservative estimate of the optimality gap and this is where a dual problem can be brought in.

The steps are as follows: First, express the actual objective function $f_0$ via a Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with an anchor at 0 so that $f(0, x) = f_0(x)$ for all $x \in \mathbb{R}^n$. Second, form the corresponding Lagrangian $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ by (5.6). Third, using any $\bar{y} \in \mathbb{R}^m$, obtain a lower bound

$$\psi(\bar{y}) = \inf_{x \in \mathbb{R}^n} l(x, \bar{y}) \leq \inf f(0, \cdot\,) = \inf f_0$$

by minimizing the Lagrangian; see (5.5). These steps are referred to as *Lagrangian relaxation* and are especially productive when the actual problem is difficult, but $l(\cdot, y)$

is relatively simple to minimize. The best possible lower bound is achieved by tuning $y$, i.e., solving the corresponding dual problem to obtain

$$\sup_{y\in\mathbb{R}^m} \psi(y) \leq \inf f_0.$$

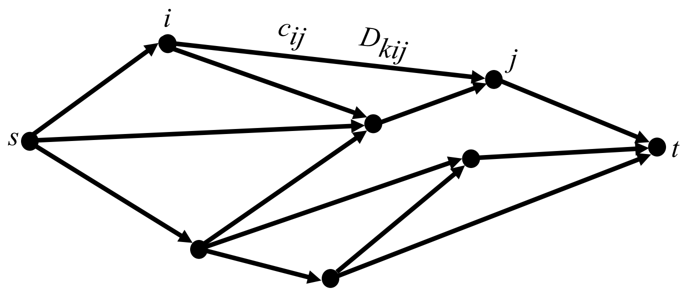


Fig. 5.8: Directed graph with vertices and edges.

**Example 5.34** (constrained shortest path problem). Let $(V, E)$ be a directed graph with vertex set $V$ and edge set $E$. Each edge $(i, j) \in E$ connects distinct vertices $i, j \in V$, and it possesses length $c_{ij} \in [0, \infty)$ and weights $D_{kij} \in [0, \infty)$ for $k = 1, \dots, q$; see Figure 5.8. A directed $s$-$t$ path is an ordered set of edges of the form $\{(s, i_1), (i_1, i_2), \dots, (i_{\nu-1}, t)\}$ for some $\nu \in \mathbb{N}$. Given two distinct vertices $s, t \in V$, the *shortest-path problem* seeks to determine a directed $s$-$t$ path such that the sum of the edge lengths along the path is minimized. This is a well-studied problem that can be solved efficiently using specialized algorithms; see [1, Chapters 4 and 5].

For nonnegative $d_k, k = 1, \dots, q$, the task becomes significantly harder if the sum of the weights $D_{kij}$ along the path can't exceed $d_k$ for each $k$. This is the *constrained shortest-path problem*, which can be addressed by Lagrangian relaxation. In routing of a drone through a discretized three-dimensional airspace, the weights $D_{1ij}$ might represent fuel consumption along edge $(i, j)$, which can't exceed a capacity $d_1$. Figure 5.9 illustrates a route satisfying such a fuel constraint while minimizing exposure to enemy radars expressed by $c_{ij}$; cf. [110].

**Detail.** The constrained shortest-path problem is formulated as follows. Suppose that $m$ is the number of vertices in $V$ and $n$ is the number of edges in $E$. Let $A$ denote the $m \times n$-vertex-edge incidence matrix such that if edge $e = (i, j)$, then $A_{ie} = 1$, $A_{je} = -1$ and $A_{i'e} = 0$ for any $i' \neq i, j$. Also, let $b_s = 1$, $b_t = -1$ and $b_i = 0$ for $i \in V \setminus \{s, t\}$ and collect them in the vector $b$. For each $k = 1, \dots, q$, we place the edge weights $\{D_{kij}, (i, j) \in E\}$ in the vector $D_k$. The $q \times n$-matrix $D$ has $D_k$ as its $k$th row. Let $d = (d_1, \dots, d_q)$. With $c$ being the vector of $\{c_{ij}, (i, j) \in E\}$, the constrained shortest-path problem may then be formulated as (cf. [1, p. 599])

$$\text{minimize } \langle c, x\rangle \quad \text{subject to } Ax = b,\ Dx \leq d,\ x \in \{0, 1\}^n.$$

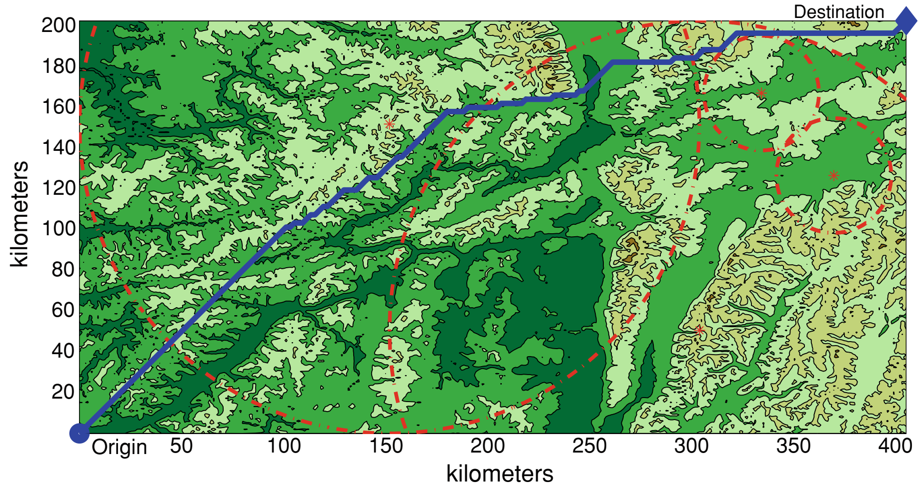


Fig. 5.9: Route for a drone through a three-dimensional airspace to a destination (blue line) that minimizes exposure to enemy radars (red circles) while satisfying a fuel constraint. Altitude changes to leverage terrain masking aren't shown.

A point $\bar{x} \in \{0,1\}^n$ satisfying $Ax = b$ corresponds to an $s$-$t$ path with $\bar{x}_{ij} = 1$ if edge $(i,j)$ is on the path and $\bar{x}_{ij} = 0$ otherwise. We assume there's at least one such path. Then, $\langle c, \bar{x}\rangle$ gives the length of the path and $\langle D_k, \bar{x}\rangle$ the $k$th weight of the path.

In the absence of the weight-constraint $Dx \le d$, the model reduces to a shortest path problem and this opens up an opportunity for applying Lagrangian relaxation via 5.28. Let

$$X = \big\{x \in \{0,1\}^n \;\big|\; Ax = b\big\},$$

which is nonempty by assumption, and consider the Rockafellian given by

$$f(u,x) = \iota_X(x) + \langle c, x\rangle + \iota_{(-\infty,0]^q}(Dx - d + u).$$

The actual problem corresponds to minimizing $f(0,\cdot\,)$ over $\mathbb{R}^n$. These definitions fit the setting of 5.28 and

$$l(x,y) = \iota_X(x) + \langle c, x\rangle + \langle Dx - d, y\rangle - \iota_Y(y), \quad \text{where } Y = \{y \in \mathbb{R}^q \mid y \ge 0\}$$

by 5.29. Thus, the chosen Rockafellian results in a Lagrangian without the constraint $Dx \le d$. The corresponding dual problem is

$$\underset{y \in \mathbb{R}^q}{\text{maximize}}\, \psi(y) = \inf_{x \in \mathbb{R}^n} l(x,y).$$

For $y \ge 0$, which are the only values of concern because $\psi(y) = -\infty$ otherwise, we obtain explicitly that

$$\psi(y) = -\langle d, y\rangle + \inf_{x \in X}\langle c + D^\top y, x\rangle.$$

The minimization occurring here is nothing but a shortest-path problem on the directed graph, but with edge lengths changed from $c_{ij}$ to $c_{ij} + \sum_{k=1}^{q} D_{kij} y_k$, which can be solved efficiently using specialized algorithms. The resulting minimum value, modified by $\langle d, y \rangle$, yields $\psi(y)$, a lower bound on the minimum value of the constrained shortest-path problem as seen from (5.5) and the introductory discussion in this section. The lower bound can be used to assess the optimality gap for any candidate path, for example, obtained by a greedy search or an enumeration algorithm; see [23]. □

For a Lagrangian $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ and the corresponding dual objective function $\psi$, it would certainly be desirable to obtain the best possible lower bound $\sup_{y \in \mathbb{R}^m} \psi(y)$. Since $\psi$ is concave and typically possesses a simple domain, the subgradient method of §2.I is often viable for this purpose. Returning to the familiar minimization setting, the task at hand is to

$$\underset{y \in \mathbb{R}^m}{\text{minimize}} \ \tilde{\psi}(y) = -\psi(y) = \sup_{x \in \mathbb{R}^n} -l(x, y).$$

The subgradient method would need a description of $\operatorname{dom} \tilde{\psi}$ and a way to compute subgradients of the convex function $\tilde{\psi}$. We develop general formulae for this in §6.E, but some cases are approachable with the existing machinery.

**Example 5.35** (constrained shortest path problem; cont.). Solving the dual problem in 5.34 reduces to

$$\underset{y \in Y}{\text{minimize}} \ \tilde{\psi}(y) = \sup_{x \in X} \big\{ -\langle c, x \rangle - \langle Dx - d, y \rangle \big\}.$$

Since $X$ contains only a finite number of points, subgradients of $\tilde{\psi}$ can be obtained from 4.66 for any $\bar{y} \geq 0$:

$$v \in \partial \tilde{\psi}(\bar{y}) \iff v = \sum\nolimits_{x \in X^\star(\bar{y})} z_x (d - Dx) \ \text{ for some } \ z_x \geq 0 \ \text{ with } \ \sum\nolimits_{x \in X^\star(\bar{y})} z_x = 1,$$

where

$$X^\star(\bar{y}) = \operatorname{argmin}_{x \in X} \big\{ \langle c, x \rangle + \langle Dx - d, \bar{y} \rangle \big\}.$$

Consequently, we obtain a subgradient $v$ of $\tilde{\psi}$ at $\bar{y}$ by computing a minimizer $\bar{x} \in X^\star(\bar{y})$, which can be achieved by a shortest-path algorithm (see [1, Chapters 4 and 5]), and then setting $v = d - D\bar{x}$. Moreover, $\operatorname{dom} \tilde{\psi} = Y = [0, \infty)^q$ is easily projected onto and this makes the subgradient method viable.

## 5.E Saddle Points

In the setting of equality and inequality constraints, we saw already in §4.K that a Lagrangian provides an efficient way of expressing optimality conditions and we'll now extend this idea to many more problems. The resulting insight connects multipliers with dual variables and also leads to an interpretation of a Lagrangian as the pay-off function in a game between two players.

**Proposition 5.36** (optimality condition in Lagrangian form). *For smooth $f_0 : \mathbb{R}^n \to \mathbb{R}$, smooth $F : \mathbb{R}^n \to \mathbb{R}^m$, nonempty closed $X \subset \mathbb{R}^n$ and proper, lsc and convex $h : \mathbb{R}^m \to \overline{\mathbb{R}}$, consider the problem*

$$\underset{x \in X}{\text{minimize}}\ f_0(x) + h\big(F(x)\big),$$

*the Rockafellian defined as*

$$f(u, x) = \iota_X(x) + f_0(x) + h\big(F(x) + u\big)$$

*and the associated Lagrangian given by*

$$l(x, y) = \iota_X(x) + f_0(x) + \big\langle F(x), y \big\rangle - h^*(y).$$

*Then, the optimality condition 4.75 for the problem, i.e.,*

$$\bar{y} \in \partial h\big(F(\bar{x})\big) \qquad -\nabla f_0(\bar{x}) - \nabla F(\bar{x})^\top \bar{y} \in N_X(\bar{x}), \tag{5.10}$$

*is equivalently expressed as*

$$0 \in \partial_x l(\bar{x}, \bar{y}) \qquad 0 \in \partial_y(-l)(\bar{x}, \bar{y}). \tag{5.11}$$

*If $l(\cdot, y)$ is convex for all $y \in \mathbb{R}^m$, then these conditions can also be stated as*

$$\bar{x} \in \operatorname{argmin} l(\cdot, \bar{y}) \qquad \bar{y} \in \operatorname{argmax} l(\bar{x}, \cdot).$$

Before proving the result, we observe that (5.11) appears in §4.K for the special case $X = \mathbb{R}^n$ and $h = \iota_{\{0\}^m}$. Then, $0 \in \partial_x l(\bar{x}, \bar{y})$ becomes $0 = \nabla f_0(\bar{x}) + \nabla F(\bar{x})^\top \bar{y}$ and $0 \in \partial_y(-l)(\bar{x}, \bar{y})$ reduces to $0 = F(\bar{x})$ because $h^*$ is identically equal to 0. Thus, (5.11) simply states that $\nabla l(\bar{x}, \bar{y}) = 0$.

The proposition confirms the connection between multipliers and dual variables: They're two names for the same object! The multiplier interpretation of $\bar{y}$ in the proposition is expressed in (5.10) and stems from the normal cones to $\operatorname{epi} h$ as they define the subgradients of $h$. The dual variable perspective of (5.11) comes from the Lagrangian and its role in defining a dual problem.

The proof of the proposition is supported by the following general fact.

**Proposition 5.37** (inversion rule for subgradients). *For a proper, lsc and convex function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ and its conjugate $h^* : \mathbb{R}^m \to \overline{\mathbb{R}}$, one has*

$$v \in \partial h(u) \iff u \in \partial h^*(v) \iff h(u) + h^*(v) = \langle u, v \rangle.$$

**Proof.** Suppose that $\bar{u} \in \partial h^*(\bar{v})$. By the convexity of $h^*$, the optimality condition 2.19 and 4.58(c), this means that

$$\bar{v} \in \operatorname{argmin}\big\{h^* - \langle \bar{u}, \cdot \rangle\big\} = \operatorname{argmax}\big\{\langle \bar{u}, \cdot \rangle - h^*\big\}.$$

Thus, in view of the Fenchel-Moreau theorem 5.23, which applies because $h$ is proper, lsc and convex, one has

$$h(\bar{u}) = (h^*)^*(\bar{u}) = \sup_{v\in\mathbb{R}^m}\big\{\langle\bar{u}, v\rangle - h^*(v)\big\} = \langle\bar{u}, \bar{v}\rangle - h^*(\bar{v}).$$

Thus, we've proven "$\Longrightarrow$" in the second equivalence. Since

$$\sup_{u\in\mathbb{R}^m}\big\{\langle\bar{v}, u\rangle - h(u)\big\} = h^*(\bar{v}) = \langle\bar{v}, \bar{u}\rangle - h(\bar{u})$$

by the just established equality, we've that $\bar{u}$ achieves this maximum, i.e.,

$$\bar{u} \in \operatorname{argmax}\big\{\langle\bar{v}, \cdot\,\rangle - h\big\} = \operatorname{argmin}\big\{h - \langle\bar{v}, \cdot\,\rangle\big\}.$$

This in turn implies $\bar{v} \in \partial h(\bar{u})$ by the optimality condition 2.19 together with 4.58(c). We've established that "$\Longleftarrow$" holds in the first equivalence.

Suppose that $\bar{v} \in \partial h(\bar{u})$. Since $h^*$ is proper, lsc and convex by the Fenchel-Moreau theorem 5.23, the above arguments can be repeated with the roles of $h$ and $h^*$ reversed. This establishes "$\Longrightarrow$" in the first equivalence. Suppose that $h(\bar{u}) + h^*(\bar{v}) = \langle\bar{u}, \bar{v}\rangle$ so that $h^*(\bar{v}) = \langle\bar{u}, \bar{v}\rangle - h(\bar{u})$. Then, $\bar{u}$ must attain the maximum in the definition of $h^*(\bar{v})$. Again, this implies that $\bar{v} \in \partial h(\bar{u})$ and the conclusion follows. □

**Proof of 5.36.** The Lagrangian follows by 5.28. For $\bar{x} \in X$ and $\bar{y} \in \operatorname{dom} h^*$, 4.56 and 4.58 yield

$$\partial_x l(\bar{x}, \bar{y}) = N_X(\bar{x}) + \nabla f_0(\bar{x}) + \nabla F(\bar{x})^\top \bar{y}$$

so that $0 \in \partial_x l(\bar{x}, \bar{y})$ corresponds to $-\nabla f_0(\bar{x}) - \nabla F(\bar{x})^\top \bar{y} \in N_X(\bar{x})$. Similarly,

$$\partial_y(-l)(\bar{x}, \bar{y}) = -F(\bar{x}) + \partial h^*(\bar{y}).$$

Then, $0 \in \partial_y(-l)(\bar{x}, \bar{y})$ amounts to having $F(\bar{x}) \in \partial h^*(\bar{y})$ and, equivalently by 5.37, $\bar{y} \in \partial h(F(\bar{x}))$. The requirement $\bar{x} \in X$ is a necessity for $N_X(\bar{x})$ to contain a vector and likewise for $\partial_x l(\bar{x}, \bar{y})$; recall $l(\bar{x}, \bar{y})$ needs to be finite. Moreover, $\bar{y} \in \operatorname{dom} h^*$ is a necessity for $-l(\bar{x}, \bar{y})$ to be finite. Under the convexity assumption, the optimality condition 2.19 produces the final claim because $-l(\bar{x}, \cdot\,)$ is convex; see the discussion after 5.27. □

A function with two arguments that's convex in the first one and concave in the second one, which we refer to as a *convex-concave function*, can be visualized as having a graph that's shaped like a saddle. In view of 5.36, a Lagrangian with this property identifies a solution $\bar{x}$ and a corresponding multiplier vector $\bar{y}$ through minimization in its first argument and maximization in its second one. The corresponding $(\bar{x}, \bar{y})$ is an example of a saddle point.

For a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ (which could be a Lagrangian), we say that $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a *saddle point* of $g$ when

$$\boxed{\bar{x} \in \operatorname{argmin} g(\cdot\,, \bar{y}) \qquad \bar{y} \in \operatorname{argmax} g(\bar{x}, \cdot\,)}$$

A Lagrangian that's convex in its first argument is a convex-concave function because it's always concave in the second argument; see the discussion after 5.27. Consequently, for such Lagrangians, $(\bar{x}, \bar{y})$ satisfies the optimality condition (5.10) if and only if it's a saddle point by 5.36.

The saddle point condition can equivalently be expressed as

$$g(x, \bar{y}) \geq g(\bar{x}, \bar{y}) \geq g(\bar{x}, y) \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m,$$

which highlights the connection with game theory. Suppose that $g$ is the pay-off in a game between two non-cooperating players that proceeds as follows: Player 1 selects $x \in \mathbb{R}^n$ and Player 2 selects $y \in \mathbb{R}^m$, the choices are revealed simultaneously, and Player 1 pays $g(x, y)$ to Player 2; a negative amount indicates that Player 1 gets money. A *solution to this game* is a saddle point $(\bar{x}, \bar{y})$ of $g$. This is meaningful because then neither player has any incentive to deviate from their respective strategies $\bar{x}$ and $\bar{y}$. In selecting $\bar{x}$, Player 1 guarantees that the amount paid to Player 2 won't exceed $g(\bar{x}, \bar{y})$, even if Player 2 knew in advance that $\bar{x}$ would be chosen. This results from the right-most inequality above. At the same time, in selecting $\bar{y}$, Player 2 guarantees that the amount received from Player 1 won't fall short of $g(\bar{x}, \bar{y})$, regardless of whether Player 1 acts with knowledge of this choice or not, as seen from the left-most inequality above. The task of finding a solution to the game is an example from the broader class of *variational problems*.

**Example 5.38** (two-person zero-sum game). Two players, Minnie and Maximilian, choose strategies $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, respectively, to play repeatedly a game with pay-off:

$$-\langle y, Ax \rangle \text{ for Minnie} \qquad \langle y, Ax \rangle \text{ for Maximilian,}$$

where $A$ is an $m \times n$-matrix to which one refers as the *pay-off matrix*.

**Detail.** In a repeated game, it becomes important not to be predictable. A way of accounting for this is to assume that the strategies $x$ and $y$ assign probabilities to the choice of a specific column (Minnie) and row (Maximilian) of $A$; i.e., Minnie chooses to play column $j$ with probability $x_j$, whereas Maximilian plays row $i$ with probability $y_i$. Thus, in these terms, Minnie has to find a strategy

$$x \in X = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum\nolimits_{j=1}^n x_j = 1, \; x \geq 0 \right\}$$

that on average maximizes her returns and Maximilian has to select a strategy

$$y \in Y = \left\{ y \in \mathbb{R}^m \;\middle|\; \sum\nolimits_{i=1}^m y_i = 1, \; y \geq 0 \right\}$$

that maximizes his average returns. Since the payment to Maximilian, on average, is exactly $\langle y, Ax \rangle$, we've the pay-off as indicated initially. Moreover, a solution of the game is a saddle point of the convex-concave function $g : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ given by

$$g(x, y) = \langle y, Ax \rangle + \iota_X(x) - \iota_Y(y).$$

Although, $g$ didn't arise in the context of a minimization problem, we can make the connection with 5.36 as follows: With $X$, $Y$ and $A$ as defined by the game, set $f_0$ to the zero function, $F(x) = Ax$ and $h(z) = \max\{z_1, \dots, z_m\}$ in that proposition, invoke 5.29 and we obtain a Lagrangian of the form $\langle y, Ax\rangle + \iota_X(x) - \iota_Y(y)$, which coincides with the convex-concave function $g$ in the game. This perspective offers the possibility of solving the game by finding a solution to the optimality condition (5.10). Tracing further back, this optimality condition stems from the problem

$$\underset{x\in X}{\text{minimize}}\, h\big(F(x)\big) = \max\big\{\langle A_1, x\rangle, \dots, \langle A_m, x\rangle\big\},$$

where $A_i$ is the $i$th row of $A$. But, this can be achieved by solving the linear optimization problem

$$\underset{x\in X, \alpha\in\mathbb{R}}{\text{minimize}}\, \alpha \;\; \text{subject to} \;\; \langle A_i, x\rangle \le \alpha, \;\; i = 1, \dots, m.$$

An application of the interior-point method yields a minimizer $x^\star$ as well as the corresponding multiplier vector $y^\star$, which then satisfies (5.10); see §2.G. Regardless of how $x^\star$ is obtained, it's always possible to utilize (5.10) to recover $y^\star$. Another algorithmic strategy is laid in §5.J.

We note that there's symmetry in this game and our focus on Minnie over Maximilian is arbitrary. One could equally well have taken Maximillian's perspective, with his decision then becoming "primal" and that of Minnie being a multiplier vector. □

**Exercise 5.39** (penny-nickel-dime game). Suppose that in a game of two players each has a penny, a nickel and a dime. The game consists of each player selecting one of these coins and displaying it. If the sum of the cents on the two displayed coins is an odd number, then Player 1 wins Player 2's coin, but if the sum is even, Player 2 wins Player 1's coin. Set up a pay-off matrix $A$ for this game and solve it as indicated in 5.38 by determining strategies $x \in X$ and $y \in Y$ for Players 1 and 2, respectively.

**Guide.** The solution of the game is $x^\star = (1/2, 0, 1/2)$ and $y^\star = (10/11, 0, 1/11)$. Interestingly, neither player should ever play nickel. Also, $\langle y^\star, Ax^\star\rangle = 0$, which means the game is fair: neither player gets ahead in the long run if they follow these strategies. Since $(x^\star, y^\star)$ is a saddle point, any other strategy can't be better. For example, if Player 2 changes to $\bar{y} = (1/3, 1/3, 1/3)$, then $\langle \bar{y}, Ax^\star\rangle = -2/3$ and Player 1 makes approximately 66 cents per 100 games. □

We next turn the attention to the connection between saddle points of a Lagrangian and the primal and dual problems defined by the Lagrangian.

**Theorem 5.40** (saddle points). *For a proper function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$, with $f(\cdot, x)$ lsc and convex for all $x \in \mathbb{R}^n$, and the corresponding Lagrangian $l$ given by (5.6), consider the primal and dual objective functions defined by*

$$\varphi(x) = \sup\nolimits_{y\in\mathbb{R}^m} l(x, y) \quad \textit{and} \quad \psi(y) = \inf\nolimits_{x\in\mathbb{R}^n} l(x, y).$$

*Then, we've the following relations:*

$$\left.\begin{array}{l}\bar{x} \in \operatorname{argmin} \varphi \\ \bar{y} \in \operatorname{argmax} \psi \\ \inf \varphi = \sup \psi\end{array}\right\} \iff (\bar{x}, \bar{y}) \textit{ is a saddle point of } l \iff \varphi(\bar{x}) = \psi(\bar{y}) = l(\bar{x}, \bar{y})$$

$$\implies 0 \in \partial_x l(\bar{x}, \bar{y}) \quad \textit{and} \quad 0 \in \partial_y(-l)(\bar{x}, \bar{y}).$$

*If $l(\cdot, y)$ is convex for all $y \in \mathbb{R}^m$, then the converse of the last implication also holds, with all these conditions being equivalent to*

$$(\bar{y}, 0) \in \partial f(0, \bar{x}).$$

**Proof.** The two equivalences hold trivially by the definition of a saddle point. A saddle point $(\bar{x}, \bar{y})$ of $l$ satisfies by definition $\bar{x} \in \operatorname{argmin} l(\cdot, \bar{y})$ and $\bar{y} \in \operatorname{argmax} l(\bar{x}, \cdot)$, which in turn implies that $0 \in \partial_x l(\bar{x}, \bar{y})$ and $0 \in \partial_y(-l)(\bar{x}, \bar{y})$ by the Fermat rule 4.73. If $l(\cdot, \bar{y})$ is convex, then the converse holds by optimality condition 2.19; recall that $-l(\bar{x}, \cdot)$ is convex.

For the final assertion, suppose that $(\bar{x}, \bar{y})$ satisfies $0 \in \partial_x l(\bar{x}, \bar{y})$ and $0 \in \partial_y(-l)(\bar{x}, \bar{y})$. The first inclusion implies that $l(x, \bar{y}) \geq l(\bar{x}, \bar{y})$ for all $x \in \mathbb{R}^n$. Moreover, the existence of a subgradient for $l(\cdot, \bar{y})$ at $\bar{x}$ ensures that $l(\bar{x}, \bar{y})$ is finite. Thus, $\varphi(\bar{x})$ as well as $f(0, \bar{x})$ are finite; see 5.24. We then have that $f(\cdot, \bar{x})$ is proper, lsc and convex. By the Fenchel-Moreau theorem 5.23, $-l(\bar{x}, \cdot)$ is proper, lsc and convex as well and 5.37 applies. In particular, $0 \in \partial_y(-l)(\bar{x}, \bar{y})$ implies that $f(0, \bar{x}) = l(\bar{x}, \bar{y})$. Collecting these facts, one obtains

$$l(x, \bar{y}) = \inf_u \left\{f(u, x) - \langle \bar{y}, u\rangle\right\} \geq l(\bar{x}, \bar{y}) = f(0, \bar{x}) \ \ \forall x \in \mathbb{R}^n$$

and then also

$$f(u, x) \geq f(0, \bar{x}) + \langle \bar{y}, u - 0\rangle + \langle 0, x - \bar{x}\rangle \ \ \forall u \in \mathbb{R}^m, x \in \mathbb{R}^n. \tag{5.12}$$

This means that $(\bar{y}, 0) \in \partial f(0, \bar{x})$ by the subgradient inequality 2.17.

For the converse, suppose that $(\bar{y}, 0) \in \partial f(0, \bar{x})$. Again by 2.17, (5.12) holds. But, this means that

$$f(u, \bar{x}) \geq f(0, \bar{x}) + \langle \bar{y}, u - 0\rangle \ \ \forall u \in \mathbb{R}^m$$

so that $\bar{y} \in \partial_u f(0, \bar{x})$. Since $f(0, \bar{x})$ is finite, $f(\cdot, \bar{x})$ is proper, lsc and convex. Its conjugate is $-l(\bar{x}, \cdot)$. Thus, by 5.37, $\bar{y} \in \partial_u f(0, \bar{x})$ ensures that $0 \in \partial_y(-l)(\bar{x}, \bar{y})$ and $f(0, \bar{x}) = l(\bar{x}, \bar{y})$. From (5.12),

$$\inf_u \left\{f(u, x) - \langle \bar{y}, u\rangle\right\} \geq f(0, \bar{x}) = l(\bar{x}, \bar{y}) \ \ \forall x \in \mathbb{R}^n.$$

On the left-hand side, we've $l(x, \bar{y})$ and the right-hand side can be written as $l(\bar{x}, \bar{y}) + \langle 0, x - \bar{x}\rangle$. By the subgradient inequality 2.17, this means that $0 \in \partial_x l(\bar{x}, \bar{y})$. □

The implications of this theorem are manifold, especially in view of the equivalence between the primal problem and an actual problem of interest; see 5.24. From a numerical point of view, it suggests the possibility of minimizing a function $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ by constructing a Rockafellian with anchor at 0, which then plays the role of $f$ in the theorem, and then find a saddle point of the associated Lagrangian. A minimization problem can therefore essentially always be reformulated as a game between a "primal player" and a "dual player," and this fact doesn't rely on convexity in the actual problem. Alternatively, one could solve the corresponding dual problem to obtain $\bar{y}$ and then somehow recover a primal solution, which might be immediately available as seen next.

**Example 5.41** (primal-dual relation for linear problems). From 5.26, we recall that

$$\begin{aligned} &\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \langle c, x\rangle \\ &\text{subject to } Ax = b \\ &\qquad\qquad\quad x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} &\underset{y\in\mathbb{R}^m}{\text{maximize}}\ \langle b, y\rangle \\ &\text{subject to } A^\top y \leq c \end{aligned}$$

are primal-dual pairs under the particular Rockafellian considered there. The primal problem has a minimizer if and only if the dual problem has a maximizer, in which case their optimal values coincide. Moreover[2], one has

$$\begin{aligned} \text{primal problem unbounded} &\implies \text{dual problem infeasible} \\ \text{primal problem infeasible} &\implies \text{dual problem infeasible or unbounded} \\ \text{dual problem unbounded} &\implies \text{primal problem infeasible} \\ \text{dual problem infeasible} &\implies \text{primal problem infeasible or unbounded.} \end{aligned}$$

**Detail.** By 2.45, a minimizer of the primal problem is characterized by

$$\begin{aligned} &\forall j = 1,\dots,n: \quad z_j \geq 0 \text{ if } x_j = 0;\ \ z_j = 0 \text{ otherwise} \\ &-c = A^\top y - z, \ \ Ax = b, \ \ x \geq 0. \end{aligned}$$

After eliminating $z$ and flipping the sign of $y$, we obtain that

$$\begin{aligned} &\forall j = 1,\dots,n: \quad \text{if } x_j > 0, \text{ then } \langle A^j, y\rangle = c_j \\ &A^\top y \leq c, \ \ Ax = b, \ \ x \geq 0, \end{aligned} \tag{5.13}$$

where $A^j$ is the $j$th column of $A$. Again by 2.45, a maximizer of the dual problem or, equivalently, a minimizer of $\langle -b, y\rangle$ subject to $A^\top y \leq c$, is characterized by

$$\begin{aligned} &\forall j = 1,\dots,n: \quad x_j \geq 0 \text{ if } \langle A^j, y\rangle = c_j;\ \ x_j = 0 \text{ otherwise} \\ &b = Ax, \ \ A^\top y \leq c, \end{aligned}$$

where we pretentiously label the multiplier vector by $x$. We see that $(\bar{x}, \bar{y})$ satisfies these conditions if and only if the pair also satisfies (5.13). In view of 2.45, this means that $\bar{x}$ is

[2] The problem of minimizing a function $f$ is unbounded if $\inf f = -\infty$ (cf. §4.A) and the problem of maximizing $f$ is unbounded if $\sup f = \infty$.

a minimizer of the primal problem with corresponding multiplier vector $\bar{y}$ if and only if $\bar{y}$ is a maximizer of the dual problem with corresponding multiplier vector $\bar{x}$. In such a case,

$$\langle c, \bar{x}\rangle = \sum\nolimits_{j=1}^{n} c_j \bar{x}_j = \sum\nolimits_{j=1}^{n} \langle A^j, \bar{y}\rangle \bar{x}_j = \sum\nolimits_{i=1}^{m} \langle A_i, \bar{x}\rangle \bar{y}_i = \langle b, \bar{y}\rangle,$$

where $A_i$ is the $i$th row of $A$. Thus, the assertion about equal objective function values holds. The symmetry between the primal and dual problems implies that one can apply the interior-point method (§2.G) to the dual problem and then recover a primal solution by simply recording the corresponding multipliers.

If the primal problem is unbounded, then the dual problem is infeasible by weak duality 5.25. Likewise, if the dual problem is unbounded, then the primal problem must be infeasible. If one of the problems is infeasible, then the other one is either infeasible or unbounded; it can't have a solution as that would have implied one for the first problem as well. A linear optimization problem either has a minimizer, is infeasible or is unbounded. The case in Figure 4.3(middle) can't occur; see [105, Corollary 11.16]. Consequently, we've exhausted the possibilities.

An example of when both the primal and the dual problems are infeasible is furnished by

$$\operatorname*{minimize}_{x\in\mathbb{R}^4} \; -x_1 - x_2 \;\text{ subject to }\; x_1 - x_2 - x_3 = 1, \; -x_1 + x_2 - x_4 = 1, \; x \geq 0$$

$$\operatorname*{maximize}_{y\in\mathbb{R}^2} \; y_1 + y_2 \;\text{ subject to }\; y_1 - y_2 \leq -1, \; -y_1 + y_2 \leq -1, \; -y_1 \leq 0, \; -y_2 \leq 0.$$

Thus, the present choice of Rockafellian doesn't generally satisfy the desired property that the resulting dual problem has a maximum value reasonably near the minimum value of the primal problem. In this case, the difference between the two values is infinity. The next section discusses "ideal" Rockafellians that produce a difference of 0.

We can examine the various equivalences in the saddle point theorem 5.40. For example, if $\bar{x}$ minimizes the primal problem, $\bar{y}$ maximizes the dual problem and we also have the same optimal values, then the Lagrangian $l$ from 5.26 has

$$l(\bar{x}, \bar{y}) = \langle c, \bar{x}\rangle - \langle \bar{y}, A\bar{x} - b\rangle = \langle c, \bar{x}\rangle$$

because $\bar{x}$ is feasible in the primal problem. This confirms the right-most equivalence in 5.40. The other facts can be illustrated similarly. □

In a solution strategy for minimizing a function $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ based on constructing a Rockafellian, determining the Lagrangian $l$, maximizing the resulting dual objective function $\psi$ to obtain $\bar{y}$ and, finally, minimizing $l(\cdot, \bar{y})$ to gain $\bar{x}$, we always have that

$$\varphi(\bar{x}) \geq \psi(\bar{y}) = \inf l(\cdot, \bar{y}) = l(\bar{x}, \bar{y})$$

by weak duality 5.25, where $\varphi$ is the primal objective function. Thus, if $\varphi(\bar{x})$ turns out to be equal to $l(\bar{x}, \bar{y})$, then for $x \in \mathbb{R}^n$ one has

$$\varphi(x) = \sup l(x, \cdot) \geq l(x, \bar{y}) \geq l(\bar{x}, \bar{y}) = \varphi(\bar{x})$$

and $\bar{x}$ must be a minimizer of the primal problem; see also the saddle point theorem 5.40. In turn, this means that $\bar{x}$ is a minimizer of $f_0$ as well by 5.24, which we assume applies. Regardless of whether this works out perfectly, we obtain an upper bound on the optimality gap for $\bar{x}$ by the expression $f_0(\bar{x}) - l(\bar{x}, \bar{y})$; see (5.5). Since the dual problem and the Lagrangian problem may both be simpler than the actual one, this could be a viable strategy.

In the more specific case of 5.28, one has

$$\bar{x} \in \operatorname{argmin} l(\cdot, \bar{y}) \quad \text{and} \quad \bar{y} \in \partial h\big(F(\bar{x})\big) \quad \Longrightarrow \quad \bar{x} \in \operatorname{argmin} \iota_X + f_0 + h \circ F.$$

This follows by an application of 5.37:

$$\begin{aligned} \iota_X(\bar{x}) + f_0(\bar{x}) + h\big(F(\bar{x})\big) &= \iota_X(\bar{x}) + f_0(\bar{x}) + \big\langle F(\bar{x}), \bar{y}\big\rangle - h^*(\bar{y}) \\ &= \inf l(\cdot, \bar{y}) \le \inf_x \sup_y l(x, y) = \inf \iota_X + f_0 + h \circ F. \end{aligned}$$

Thus, after minimizing $l(\cdot, \bar{y})$ and obtaining $\bar{x}$, if it turns out that $\bar{y} \in \partial h(F(\bar{x}))$, then $\bar{x}$ solves the actual problem.

**Example 5.42** (two-person zero-sum game; cont.). Let's return to the game in 5.38 and view it from the vantage point of the saddle point theorem 5.40. With $A_i, i = 1, \dots, m$, being the rows and $A^j, j = 1, \dots, n$, being the columns of the pay-off matrix, a solution of the game is obtained by having

$$\begin{aligned} \text{Minnie solve} &\quad \underset{x \in X}{\text{minimize}} \;\; \max_{i=1,\dots,m} \langle A_i, x\rangle \\ \text{Maximilian solve} &\quad \underset{y \in Y}{\text{maximize}} \;\; \min_{j=1,\dots,n} \langle A^j, y\rangle. \end{aligned}$$

Since both $X$ and $Y$ are compact, it follows by 4.9 that there are minimizers for these problems and there's a solution of the game regardless of the pay-off matrix.

**Detail.** Let $\varphi(x) = \sup_{y \in \mathbb{R}^m} l(x, y)$ and $\psi(y) = \inf_{x \in \mathbb{R}^n} l(x, y)$, with

$$l(x, y) = \langle y, Ax\rangle + \iota_X(x) - \iota_Y(y).$$

As argued in 5.38, Minnie's problem corresponds to the primal problem of minimizing $\varphi$. A parallel argument to the one carried out there establishes that Maximilian's problem is equivalent to maximizing $\psi$. Thus, Minnie and Maximilian solve the primal and dual problems in the saddle point theorem 5.40. The optimality conditions $0 \in \partial_x l(\bar{x}, \bar{y})$ and $0 \in \partial_y(-l)(\bar{x}, \bar{y})$, which specialize to $-A^\top \bar{y} \in N_X(\bar{x})$ and $A\bar{x} \in N_Y(\bar{y})$, respectively, then characterize a solution to the game. This can be used to verify the solution of the penny-nickel-dime game in 5.39. □

## 5.F Strong Duality

Fundamental to a dual problem is its lower bounding property that we refer to as weak duality; cf. 5.25. However, the practical usefulness of the property diminishes if the resulting bound is much below the minimum value of the primal problem. For linear optimization problems, with a particular Rockafellian, the bound is tight by 5.41: The maximum value of the dual problem reaches the whole way up to the minimum value of the primal problem when the latter isn't infinite.

Given a Lagrangian $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, the primal-dual pair

$$\left\{ \underset{x\in\mathbb{R}^n}{\text{minimize}}\ \varphi(x), \qquad \underset{y\in\mathbb{R}^m}{\text{maximize}}\ \psi(y) \right\},$$

with $\varphi(x) = \sup_{y\in\mathbb{R}^m} l(x, y)$ and $\psi(y) = \inf_{x\in\mathbb{R}^n} l(x, y)$, possesses *strong duality* when

$$\inf \varphi = \sup \psi.$$

In addition to the linear optimization setting, this best-case scenario takes place when the Lagrangian has a saddle point; see the saddle point theorem 5.40. However, it isn't automatic and we may have a *duality gap* given by

$$\inf \varphi - \sup \psi.$$

**Example 5.43** (failure of strong duality). For the problem of minimizing $x^3$ subject to $x \geq 0$ and the Rockafellian given by

$$f(u, x) = x^3 + \iota_{(-\infty,0]}(-x + u),$$

we obtain a Lagrangian of the form

$$l(x, y) = \begin{cases} x^3 - xy & \text{if } y \geq 0 \\ -\infty & \text{otherwise;} \end{cases}$$

see 5.18. The dual objective function $\psi(y) = -\infty$ for all $y \in \mathbb{R}$, while the minimum value of the corresponding primal problem is 0 so the duality gap is $\infty$.

**Detail.** In this case, the Lagrangian isn't convex in its first argument. However, strong duality may fail even under convexity. Consider the problem

$$\underset{x\in\mathbb{R}^2}{\text{minimize}}\ e^{-x_1} \ \text{ subject to }\ g(x) \leq 0, \ \text{ where }\ g(x) = \begin{cases} x_1^2/x_2 & \text{if } x_2 > 0 \\ \infty & \text{otherwise,} \end{cases}$$

and a Rockafellian of the form

$$f(u, x) = \begin{cases} e^{-x_1} & \text{if } g(x) + u \leq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Similar to 5.18, this produces a Lagrangian with

$$l(x,y) = \begin{cases} e^{-x_1} + yg(x) & \text{if } x \in \operatorname{dom} g, y \geq 0 \\ \infty & \text{if } x \notin \operatorname{dom} g \\ -\infty & \text{otherwise.} \end{cases}$$

Consequently, the dual objective function $\psi(y) = 0$ if $y \geq 0$, but $\psi(y) = -\infty$ otherwise. The maximum value of the dual problem is therefore 0. The actual problem coincides with the primal problem of minimizing $\varphi(x) = \sup_y l(x,y)$, which has minimum value of 1. Thus, the duality gap equals 1 even though $l(\cdot, y)$ is convex regardless of $y \in \mathbb{R}$. □

Despite these discouraging examples, there's a large class of problems beyond linear optimization problems for which strong duality holds as we see next.

For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a proper Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with anchor at 0, suppose that $f(\cdot, x)$ is lsc and convex for all $x \in \mathbb{R}^n$. We recall from 5.24 that in terms of the corresponding Lagrangian $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, given by (5.6), the primal objective function has

$$\varphi(x) = \sup_{y \in \mathbb{R}^m} l(x,y) = f(0,x) = f_0(x).$$

Let $p(u) = \inf f(u, \cdot)$ so then $p(0) = \inf f_0 = \inf \varphi$. In contrast, by the definition of conjugates in 5.21,

$$\begin{aligned} (p^*)^*(u) &= \sup_y \{\langle u,y\rangle - p^*(y)\} = \sup_y \Big\{\langle u,y\rangle - \sup_{\bar{u}} \{\langle \bar{u},y\rangle - p(\bar{u})\}\Big\} \\ &= \sup_y \Big\{\langle u,y\rangle + \inf_{\bar{u}} \{p(\bar{u}) - \langle \bar{u},y\rangle\}\Big\} \\ &= \sup_y \Big\{\langle u,y\rangle + \inf_{x,\bar{u}} \{f(\bar{u},x) - \langle \bar{u},y\rangle\}\Big\} \\ &= \sup_y \{\langle u,y\rangle + \inf_x l(x,y)\} = \sup_y \{\langle u,y\rangle + \psi(y)\} \end{aligned} \tag{5.14}$$

so that $(p^*)^*(0) = \sup \psi$. From this perspective, the question of strong duality boils down to whether $p(0) = (p^*)^*(0)$. We know from the Fenchel-Moreau theorem 5.23 that $p(0) \geq (p^*)^*(0)$ so we immediately obtain an alternative proof of weak duality 5.25. But, 5.23 also specifies a sufficient condition for strong duality: $p$ is proper, lsc and convex, which then implies that $p(u) = (p^*)^*(u)$ for all $u$ including 0.

Let's see how these requirements translate when the Rockafellian $f$ is proper and lsc. The stability theorem 5.6(a) ensures that $p$ is lsc if a tightness condition holds. In particular, $p$ is lsc if $f(u,x)$ is level-bounded in $x$ locally uniformly in $u$; see 5.4 for details. The same level-bounded condition ensures that $p$ is proper as well. By 1.21, $p$ is convex when $f$ is convex.

In the composite setting with smooth convex $f_0 : \mathbb{R}^n \to \mathbb{R}$, affine $F : \mathbb{R}^n \to \mathbb{R}^m$, nonempty, closed and convex $X \subset \mathbb{R}^n$ and proper, lsc and convex $h : \mathbb{R}^m \to \overline{\mathbb{R}}$, the Rockafellian given by

$$f(u,x) = \iota_X(x) + f_0(x) + h\big(F(x) + u\big)$$

is proper, lsc and convex; see 1.18. The assumption on $F$ can be relaxed to having each component function being convex when $h = \iota_{(-\infty,0]^m}$. In either case, $p$ is then convex by 1.21. One can build on this via some tightness assumption to confirm that $p$ is also proper and lsc. The following result summarizes key insights, with proof and more details available in [105, Theorem 11.39].

**Theorem 5.44** (strong duality). *For a proper, lsc and convex function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with corresponding Lagrangian $l$ given by (5.6), the primal and dual problems*

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \varphi(x) = \sup_{y\in\mathbb{R}^m} l(x,y) \qquad \underset{y\in\mathbb{R}^m}{\text{maximize}}\ \psi(y) = \inf_{x\in\mathbb{R}^n} l(x,y)$$

*satisfy strong duality, i.e.,*

$$\inf \varphi = \sup \psi,$$

*provided that* $0 \in \operatorname{int}(\operatorname{dom} p)$*, where $p$ is the min-value function given by $p(u) = \inf f(u,\cdot)$.*

*If in addition $p(0) > -\infty$, then*

$$\partial p(0) = \operatorname{argmax} \psi,$$

*which must be a nonempty and bounded set.*

In 5.14, $\operatorname{dom} p = (-\infty, 1]$ so we certainly have $0 \in \operatorname{int}(\operatorname{dom} p)$ and strong duality holds; see Figure 5.5. Moreover, $p(0) = 5$ and $\partial p(0) = \{2\}$, which imply that the dual problem has 2 as its unique maximizer, with 5 as maximum value. We can determine all of this based on 5.44 without having a detailed formula for the dual problem. The formula for $\partial p(0)$ supplements 5.13, but most significantly it highlights the profound role played by a dual problem. Under the conditions of the theorem, solving the dual problem furnishes both the minimum value of the actual problem as well as its sensitivity to perturbations as defined by a Rockafellian.

**Exercise 5.45** (primal-dual pairs). For $f_0(x) = 2x_1^2 + x_2^2 + 3x_3^2$ and $g(x) = -2x_1 - 3x_2 - x_3 + 1$, consider the problem of minimizing $f_0(x)$ subject to $g(x) \leq 0$ and the Rockafellian defined by $f(u,x) = f_0(x) + \iota_{(-\infty,0]}(g(x) + u)$. Determine the corresponding Lagrangian as well as the primal and dual problems. Solve the problems and verify strong duality. Check the assumptions in the strong duality theorem 5.44.

**Guide.** The actual problem fits the setting of 5.28, with the conjugate following from 5.29, and this produces a Lagrangian $l$. Obtain a minimizer of $l(\cdot, y)$ for each $y$ analytically using the optimality condition 2.19 and derive a formula for the dual objective function. To apply 5.44, one can show that the feasible set remains nonempty under small changes to the right-hand side. □

The requirement $0 \in \operatorname{int}(\operatorname{dom} p)$ in the strong duality theorem 5.44 is assured when $f(u,\cdot)$ approximates $f(0,\cdot)$ in a certain sense for $u$ near 0. This is formalized in the next statement, which highlights the role of suitable approximations to ensure strong duality.

**Proposition 5.46** (strong duality from epigraphical inner limits). *For a proper, lsc and convex function $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$, suppose that*

$$\operatorname{LimInn}\big(\operatorname{epi} f(u^\nu,\cdot)\big) \supset \operatorname{epi} f(0,\cdot) \ \ \textit{whenever} \ \ u^\nu \to 0.$$

*Let the primal and dual objective functions $\varphi$ and $\psi$ be constructed from $f$ as in 5.44 and $p$ be the min-value function given by $p(u) = \inf f(u, \cdot)$.*

*If $0 \in \operatorname{dom} p$, then strong duality holds, i.e., $\inf \varphi = \sup \psi$.*

**Proof.** Our goal is to show that the assumption implies $0 \in \operatorname{int}(\operatorname{dom} p)$ because then the conclusion follows from 5.44. For the sake of contradiction, suppose that $0 \notin \operatorname{int}(\operatorname{dom} p)$. Then, there's $\bar{u}^\nu \to 0$ such that $p(\bar{u}^\nu) = \infty$. Let $u^\nu \to 0$. We see from the proof of the characterization 4.15 of epi-convergence that

$$\operatorname{LimInn}\big(\operatorname{epi} f(u^\nu, \cdot)\big) \supset \operatorname{epi} f(0, \cdot)$$

is equivalent to 4.15(b), i.e., for every $x$, there's $x^\nu \to x$ such that $\limsup f(u^\nu, x^\nu) \leq f(0, x)$. Thus, the assumptions of 5.6(b) hold and $p$ is usc at 0. Since $p(\bar{u}^\nu) = \infty$, this implies that $p(0) = \infty$, which contradicts $0 \in \operatorname{dom} p$. $\square$

The conclusion of the proposition isn't surprising. The strong duality theorem 5.44 requires that the min-value function doesn't jump to infinity as the perturbation vector $u$ departs from 0 and this can be avoided when the family of functions $\{f(u, \cdot), u \in \mathbb{R}^m\}$ defining the perturbations are epigraphically well behaved. After all, epi-convergence is the key ingredient for the minima of functions to converge to the minimum value of a limiting function. This insight provides guidance for selecting Rockafellians in the first place: the epigraphical behavior together with convexity are keys to achieving strong duality.

In a more specific setting, strong duality is guaranteed by ensuring that the constraint functions leave some "slack," which often is easily verified.

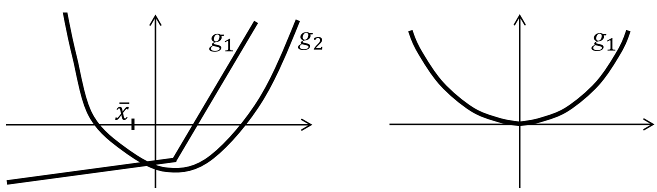


Fig. 5.10: Examples of when the Slater condition holds (left) and fails (right).

**Example 5.47** (Slater constraint qualification). For smooth convex functions $f_0, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, q$ and the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f_0(x) \ \text{ subject to } \ g_i(x) \leq 0, \ i = 1, \ldots, q,$$

let's consider the Rockafellian given by

$$f(u, x) = f_0(x) + \iota_{(-\infty,0]^q}\big(G(x) + u\big), \ \text{ with } G(x) = \big(g_1(x), \ldots, g_q(x)\big).$$

The resulting primal and dual problems (as defined via the Lagrangian given by 5.18) satisfy strong duality provided that the following *Slater constraint qualification* holds:

$$\exists \bar{x} \;\text{ such that }\; g_i(\bar{x}) < 0, \quad i = 1, \ldots, q.$$

Figure 5.10(left) illustrates a situation, with two constraints, that satisfies the qualification. If a constraint function doesn't reach below 0 as would be the case with $g_1(x) = x^2$, then the qualification fails; see Figure 5.10(right).

**Detail.** Let $p(u) = \inf f(u, \cdot\,)$. Under the Slater constraint qualification, there exist $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$ such that $g_i(\bar{x}) + u_i \le 0$ when $|u_i| \le \delta$ for all $i$. Consequently, $p(u) \le f_0(\bar{x}) \in \mathbb{R}$ when $\|u\|_\infty \le \delta$, which means that $0 \in \operatorname{int}(\operatorname{dom} p)$ and strong duality holds by 5.44.

The Slater constraint qualification implies the Mangasarian-Fromovitz constraint qualification in (4.10) at every feasible point for this class of problems. To see this, let $\hat{x}$ be feasible, i.e., $g_i(\hat{x}) \le 0$ for $i = 1, \ldots, q$, and let the multipliers $\{z_i, i \in \mathbb{A}(\hat{x})\}$ satisfy

$$\sum\nolimits_{i \in \mathbb{A}(\hat{x})} z_i \nabla g_i(\hat{x}) = 0 \quad \text{and} \quad z_i \ge 0 \;\; \forall i \in \mathbb{A}(\hat{x}),$$

where $\mathbb{A}(\hat{x}) = \{i \mid g_i(\hat{x}) = 0\}$. Our goal is to show that $z_i = 0$ for $i \in \mathbb{A}(\hat{x})$. Let $\bar{x}$ be a point that verifies the Slater constraint qualification. By the gradient inequality 1.22, one has

$$0 > g_i(\bar{x}) - g_i(\hat{x}) \ge \big\langle \nabla g_i(\hat{x}), \bar{x} - \hat{x} \big\rangle$$

for $i \in \mathbb{A}(\hat{x})$. Then,

$$0 = \Big\langle \sum\nolimits_{i \in \mathbb{A}(\hat{x})} z_i \nabla g_i(\hat{x}),\; \bar{x} - \hat{x} \Big\rangle = \sum\nolimits_{i \in \mathbb{A}(\hat{x})} z_i \big\langle \nabla g_i(\hat{x}), \bar{x} - \hat{x} \big\rangle.$$

Since the inner products on the right-hand side evaluate to negative numbers and $z_i \ge 0$, we must have $z_i = 0$ for all $i \in \mathbb{A}(\hat{x})$. □

**Example 5.48** (strong duality without Slater constraint qualification). In the setting of 5.47, the Slater constraint qualification is sufficient for strong duality but by no means necessary. For example, consider the problem of minimizing $x$ subject to $x^2 \le 0$. The Slater constraint qualification fails, but strong duality holds.

**Detail.** The setting involves the Rockafellian given by

$$f(u, x) = x + \iota_{(-\infty, 0]}(x^2 + u).$$

We obtain directly that

$$p(u) = \inf f(u, \cdot\,) = \begin{cases} -\sqrt{-u} & \text{for } u \le 0 \\ \infty & \text{otherwise} \end{cases}$$

so the requirement $0 \in \operatorname{int}(\operatorname{dom} p)$ of the strong duality theorem 5.44 doesn't hold. The corresponding Lagrangian has

$$l(x, y) = \begin{cases} x + yx^2 & \text{for } y \ge 0 \\ -\infty & \text{otherwise} \end{cases}$$

by 5.18 and the dual objective function has

$$\psi(y) = \begin{cases} -1/(4y) & \text{for } y > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Consequently, $p(0) = \sup \psi = 0$. □

An approach to settle border cases of the kind described in this example is to again return to an epigraphical analysis of the underlying Rockafellian.

**Theorem 5.49** (strong duality from epi-convergence). *For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a proper Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with anchor at 0, suppose that there are $u^\nu \to 0$ and $y^\nu \in \mathbb{R}^m$ such that*

(a) $f(u^\nu, \cdot\,) \xrightarrow{e} f(0, \cdot\,)$ *tightly*
(b) $\liminf \langle y^\nu, u^\nu \rangle \leq 0$
(c) $\inf f(u^\nu, \cdot\,) = \sup \psi^\nu = \psi^\nu(y^\nu)$,

*where*

$$\psi^\nu(y) = \inf_x l^\nu(x, y) \;\; \text{and} \;\; l^\nu(x, y) = \inf_u \big\{ f(u^\nu + u, x) - \langle y, u \rangle \big\}.$$

*Let $\psi$ be the dual objective function produced by $f$ via (5.6). Then,*

$$\inf f_0 = \sup \psi > -\infty.$$

**Proof.** Let $p$ be the min-value function defined by $p(u) = \inf f(u, \cdot\,)$. Since $f$ is proper, $f(0, x) > -\infty$ for all $x \in \mathbb{R}^n$. Thus, $f(0, \cdot\,)$ is proper when $0 \in \operatorname{dom} p$ and we can apply 5.5(d) to establish that $p(u^\nu) \to p(0) > -\infty$. When $0 \notin \operatorname{dom} p$, we can modify the arguments in the proof of 5.5 as follows.

For any compact set $B \subset \mathbb{R}^n$, $\inf_B f(u^\nu, \cdot\,) \to \infty$. To see this, let's assume for the sake of contradiction, that there are $N \in \mathcal{N}_\infty^\#$ and $\alpha \in \mathbb{R}$ such that $\inf_B f(u^\nu, \cdot\,) \leq \alpha$ for $\nu \in N$. Since $B$ is compact, this implies the existence of $\{x^\nu \in B, \nu \in N\}$ with $f(u^\nu, x^\nu) \leq \alpha + 1$, a further subsequence $N' \subset N$ and a limit $\bar{x}$ of $\{x^\nu, \nu \in N'\}$. By the (partial) characterization of epi-convergence in 4.15(a),

$$\liminf_{\nu \in N'} f(u^\nu, x^\nu) \geq f(0, \bar{x}) = \infty$$

because $\inf f(0, \cdot\,) = \infty$. This contradicts that $f(u^\nu, x^\nu) \leq \alpha + 1$ for $\nu \in N$.

Next, let $\varepsilon > 0$. By the tightness assumption, there's a compact set $B_\varepsilon$ such that

$$\inf f(u^\nu, \cdot\,) + \varepsilon \geq \inf_{B_\varepsilon} f(u^\nu, \cdot\,)$$

for sufficiently large $\nu$; see 5.3. Since the right-hand side tends to infinity as just established, $p(u^\nu) \to \infty$. Thus, $p(u^\nu) \to p(0)$ even when $0 \notin \operatorname{dom} p$.

Let $l$ be the Lagrangian of $f$ defined by (5.6). Note that

$$l^\nu(x, y) = \inf_w \big\{ f(w, x) - \langle y, w - u^\nu \rangle \big\} = l(x, y) + \langle y, u^\nu \rangle.$$

Thus, $\psi^\nu(y) = \psi(y) + \langle y, u^\nu \rangle$. This fact together with assumption (c) result in

$$p(u^\nu) = \sup \psi^\nu = \psi^\nu(y^\nu) = \psi(y^\nu) + \langle y^\nu, u^\nu \rangle \leq \sup \psi + \langle y^\nu, u^\nu \rangle.$$

Consequently,

$$p(0) = \liminf p(u^\nu) \leq \liminf \big( \sup \psi + \langle y^\nu, u^\nu \rangle \big) \leq \sup \psi.$$

By also invoking the lower bound (5.5) and the fact that $f_0 = f(0, \cdot)$, the conclusion follows. □

The theorem reveals the following insight: If we can construct a Rockafellian that leads to a sequence of perturbed functions epi-converging tightly to the actual objective function and these perturbed functions on their own are associated with a strong duality property, then the resulting dual problem indeed reproduces the minimum value of the actual problem provided that assumption (b) also holds.

Since $u^\nu \to 0$, assumption (b) certainly holds when $\{y^\nu, \nu \in \mathbb{N}\}$ is bounded. One can view $\psi^\nu$ as a dual objective function produced by the Rockafellian of the form $f^\nu(u, x) = f(u^\nu + u, x)$. The vector $y^\nu$ then solves the dual problem associated with $f^\nu$. Thus, it's plausible that $\{y^\nu, \nu \in \mathbb{N}\}$ could be bounded.

In some cases, assumption (b) is automatic even when $\{y^\nu, \nu \in \mathbb{N}\}$ is unbounded. For example, if

$$f(u, x) = \hat{f}_0(x) + \iota_{(-\infty,0]^m}\big(F(x) + u\big)$$

for $\hat{f}_0 : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^m$, both smooth, then

$$\psi^\nu(y) = \inf_{x \in \mathbb{R}^n} \hat{f}_0(x) + \big\langle F(x) + u^\nu, y \big\rangle - \iota_{[0,\infty)^m}(y)$$

by 5.28 and 5.29. Thus, a maximizer $y^\nu$ of $\psi^\nu$ is necessarily nonnegative. We can then choose $u^\nu \leq 0$ so that $\langle y^\nu, u^\nu \rangle \leq 0$ and $f(u^\nu, \cdot) \xrightarrow{e} f(0, \cdot)$, which can be seen by working directly from the characterization 4.15. (Tightness must be checked separately, but holds for instance when $\hat{f}_0(x) \to \infty$ as $\|x\|_2 \to \infty$.)

The theorem isn't restricted to any particular type of Rockafellian and may even go beyond the setting of 5.24. Still, in the convex case, several aspects simplify.

**Corollary 5.50** *For the problem of minimizing $f_0 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a proper, lsc and convex Rockafellian $f : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ with anchor at 0, suppose that there's $u^\nu \in \operatorname{int}(\operatorname{dom} p) \to 0$ such that $f(u^\nu, \cdot) \xrightarrow{e} f(0, \cdot)$ tightly, where $p$ is the min-value function given by $p(u) = \inf f(u, \cdot)$. Let $\psi$ be the dual objective function produced by $f$ via (5.6).*

*If $\inf f_0 < \infty$, then*

$$\inf f_0 = \sup \psi$$

*and this value is finite.*

**Proof.** Since $0 \in \operatorname{dom} p$ because $\inf f_0 < \infty$, the initial argument in the proof of 5.49 establishes that $p(u^\nu) \to p(0) \in \mathbb{R}$ and then $p(u^\nu) > -\infty$ for sufficiently large $\nu$. Thus, the strong duality theorem 5.44 applies to the Rockafellian $f^\nu : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f^\nu(u,x) = f(u^\nu + u, x),$$

with a corresponding min-value function in the form

$$p^\nu(u) = \inf f^\nu(u,\cdot\,) = p(u^\nu + u).$$

In particular, we note that $0 \in \mathrm{int}(\mathrm{dom}\, p^\nu)$ because $u^\nu \in \mathrm{int}(\mathrm{dom}\, p)$ and, for sufficiently large $\nu$, $p^\nu(0) > -\infty$ because $p(u^\nu) > -\infty$. From these facts as well as 5.24, we realize that $p^\nu(0) = \sup \psi^\nu$ and that there's

$$y^\nu \in \mathrm{argmax}\, \psi^\nu = \partial p^\nu(0) = \partial p(u^\nu);$$

see 5.49 for the definition of $\psi^\nu$. Consequently, $p(u^\nu) = \sup \psi^\nu = \psi^\nu(y^\nu)$. The min-value function $p$ is convex by 1.21 so the subgradient inequality 2.17 establishes that

$$p(2u^\nu) - p(u^\nu) \geq \langle y^\nu, u^\nu \rangle.$$

Both terms on the left-hand side tend to the same real number. Thus, $\limsup \langle y^\nu, u^\nu \rangle \leq 0$. All the assumptions of 5.49 then hold and the conclusion follows. □

**Example 5.51** (strong duality without Slater constraint qualification; cont.). The corollary confirms the strong duality in 5.48 even though the Slater constraint qualification fails.

**Detail.** In this case, the Rockafellian, given by $f(u,x) = x + \iota_{(-\infty,0]}(x^2 + u)$, is proper, lsc and convex. Moreover, $p(u) = -\sqrt{-u}$ for $u \leq 0$ and $p(u) = \infty$ otherwise. Thus, one can take $u^\nu = -1/\nu$ in the corollary and then $f(u^\nu,\cdot\,) \xrightarrow{e} f(0,\cdot\,)$; the epi-convergence is actually tight since $\mathrm{dom}\, f(u^\nu,\cdot\,) \subset \mathrm{dom}\, f(u^1,\cdot\,)$. □

## 5.G Reformulations

Strong duality presents numerous possibilities for reformulations. Even for difficult problems without easily obtainable strong duality, there might be parts of the formulation that can be reworked using duality. We'll illustrate the possibilities with two examples involving an infinite number of constraints.

**Example 5.52** (linear constraints under uncertainty). For $f_0 : \mathbb{R}^n \to \mathbb{R}$, $X \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, consider the problem

$$\underset{x \in X}{\mathrm{minimize}}\; f_0(x) \;\; \text{subject to} \;\; \langle a, x \rangle \leq \alpha$$

and suppose that there's uncertainty about the vector $a$. As in Chap. 3, we could have viewed $a$ as a random vector with a probability distribution. However, when we lack data from which to build a distribution or when we seek guaranteed feasibility for a range of values of $a$, a reasonable alternative is to adopt the following simple model of uncertainty. Let $\bar{a} \in \mathbb{R}^n$ be a nominal value of $a$, $s \in \mathbb{R}^n$ be a nonnegative scaling vector and

$$\left\{a \in \mathbb{R}^n \;\middle|\; a_j = \bar{a}_j + s_j\xi_j \;\; \forall \xi \in \Xi,\; j = 1, \dots, n\right\}$$

be the set of considered values of $a$, where

$$\Xi = \left\{\xi \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} |\xi_j| \le \beta,\; |\xi_j| \le 1,\;\; j = 1, \dots, n\right\}$$

for some $\beta \ge 0$. Thus, any value of $a_j$ between $\bar{a}_j - s_j$ and $\bar{a}_j + s_j$ might be considered, but this is further restricted by the size of $\beta$. In particular, if $\beta < n$, then not all components of $a$ can be at their extreme values.

We guarantee a solution satisfying $\langle a, x\rangle \le \alpha$ for every considered value of $a$ by imposing the constraints

$$\big\langle a(\xi), x\big\rangle \le \alpha \;\; \forall \xi \in \Xi, \quad \text{where } \; a(\xi) = (\bar{a}_1 + s_1\xi_1, \dots, \bar{a}_n + s_n\xi_n).$$

The apparent implementation challenge associated with the now infinite number of constraints can be overcome using strong duality.

**Detail.** The infinite collection of constraints is equivalent to the single constraint

$$\sup\nolimits_{\xi \in \Xi} \big\langle a(\xi), x\big\rangle \le \alpha,$$

where the left-hand side can be expressed by

$$\begin{aligned} \sup\nolimits_{\xi \in \Xi} \big\langle a(\xi), x\big\rangle &= \sum_{j=1}^{n} \bar{a}_j x_j + \sup\nolimits_{\xi \in \Xi} \sum_{j=1}^{n} s_j x_j \xi_j \\ &= \langle \bar{a}, x\rangle + \sup\left\{\sum_{j=1}^{n} s_j |x_j| \xi_j \;\middle|\; \sum_{j=1}^{n} \xi_j \le \beta,\;\; 0 \le \xi_j \le 1,\;\; j = 1, \dots, n\right\}. \end{aligned}$$

The change to only nonnegative $\xi_j$ is permitted due to the fact that $s_j \ge 0$. Let's fix $x \in \mathbb{R}^n$. The last supremum is the maximum value of a linear problem; a maximizer exists by 4.9. Moreover, by strong duality from 5.41 this can be computed just as well by solving the corresponding minimization problem, which is referred to as the primal problem in 5.41. Specifically, the supremum is achieved by

$$\underset{\xi \in \mathbb{R}^n}{\text{maximize}}\; \langle b, \xi\rangle \;\; \text{subject to} \;\; A^\top \xi \le c, \tag{5.15}$$

where

$$b = \big(s_1|x_1|, \dots, s_n|x_n|\big), \qquad c = (\beta, 1, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{1+2n}$$

and $A$ is an $n \times (1 + 2n)$-matrix with ones in the first column, the $n \times n$ identity matrix occupies the next $n$ columns and the negative of the $n \times n$ identity matrix fills out the rest. By 5.41, the primal problem is then

$$\underset{\eta \in \mathbb{R}^{1+2n}}{\text{minimize}}\; \langle c, \eta\rangle \;\; \text{subject to} \;\; A\eta = b,\; \eta \ge 0.$$

In more detail, this becomes

$$\underset{\eta \geq 0}{\text{minimize}}\ \beta\eta_0 + \sum\nolimits_{j=1}^{n} \eta_j \ \text{ subject to }\ \eta_0 + \eta_j - \eta_{n+j} = s_j|x_j|, \quad j = 1, \dots, n.$$

We can eliminate $\eta_{n+1}, \dots, \eta_{2n}$ and obtain the equivalent formulation

$$\underset{\eta_0, \dots, \eta_n \geq 0}{\text{minimize}}\ \beta\eta_0 + \sum\nolimits_{j=1}^{n} \eta_j \ \text{ subject to }\ \eta_0 + \eta_j \geq s_j|x_j|, \quad j = 1, \dots, n.$$

Since the minimum value here matches the corresponding maximum value in (5.15) by 5.41, we obtain that

$$\begin{aligned}
&\sup\nolimits_{\xi \in \Xi} \langle a(\xi), x \rangle \\
&= \langle \bar{a}, x \rangle + \inf\Big\{ \beta\eta_0 + \sum\nolimits_{j=1}^{n} \eta_j \,\Big|\, \eta_0 \geq 0,\ \eta_0 + \eta_j \geq s_j|x_j|,\ \eta_j \geq 0,\ j = 1, \dots, n \Big\}.
\end{aligned}$$

Thus, finding $x$ such that $\sup_{\xi \in \Xi} \langle a(\xi), x \rangle \leq \alpha$ is equivalent to finding $x$ and $\eta_0, \dots, \eta_n$ such that

$$\langle \bar{a}, x \rangle + \beta\eta_0 + \sum\nolimits_{j=1}^{n} \eta_j \leq \alpha, \quad \eta_0 \geq 0, \quad \eta_0 + \eta_j \geq s_j|x_j|, \quad \eta_j \geq 0, \quad j = 1, \dots, n.$$

The only remaining nonlinear parts are the $|x_j|$-terms, but these can be reformulated by doubling the number of constraints because $s_j \geq 0$. The problem of minimizing $f_0(x)$ subject to $\langle a(\xi), x \rangle \leq \alpha$ for all $\xi \in \Xi$ becomes then

$$\begin{aligned}
\underset{x \in \mathbb{R}^n, \eta \in \mathbb{R}^{1+n}}{\text{minimize}}\ f_0(x) \ \text{ subject to }\ & \langle \bar{a}, x \rangle + \beta\eta_0 + \sum\nolimits_{j=1}^{n} \eta_j \leq \alpha \\
& \eta_0 + \eta_j - s_j x_j \geq 0, \quad j = 1, \dots, n \\
& \eta_0 + \eta_j + s_j x_j \geq 0, \quad j = 1, \dots, n \\
& x \in X, \quad \eta_0, \eta_j \geq 0, \quad j = 1, \dots, n.
\end{aligned}$$

The infinite collection of constraints in $n$ variables is now expressed by $2 + 3n$ constraints using $1 + 2n$ variables. The reformulation remains valid for arbitrary $f_0$ and $X$ as they didn't enter the derivation. □

**Example 5.53** (maximum-flow interdiction). In a directed graph $(V, E)$ with vertex set $V$ and edge set $E$, there are two special vertices $s \neq t$ identified: $s$ is the source vertex and $t$ is the sink vertex. A network operator would like to maximize the flow of a commodity from $s$ to $t$ across the edges which have finite capacities. In the *maximum-flow network-interdiction problem*, an interdictor wishes to minimize that maximum flow by interdicting (destroying) edges. Using strong duality, the problem can be formulated as a reasonably tractable minimization problem.

**Detail.** Each edge $e \in E$ has a capacity $u_e > 0$ and interdiction cost $r_e > 0$; see Figure 5.11. The interdiction cost is the amount of some resource necessary to destroy edge $e$, i.e., reduce that edge's capacity to 0. We also define the outgoing and incoming edges from vertex $i$ as

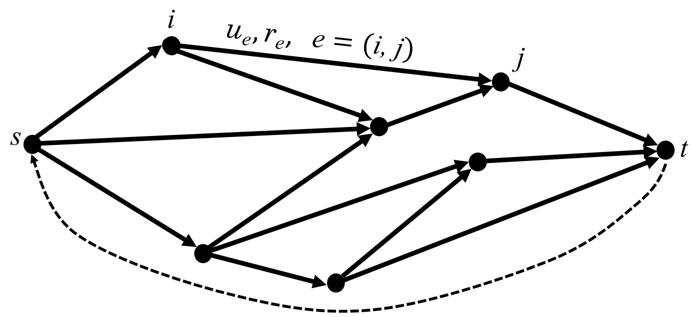


Fig. 5.11: Directed graph for the maximum-flow network-interdiction problem.

$$E_i^+ = \{(i, j) \in E \mid j \in V\} \quad \text{and} \quad E_i^- = \{(j, i) \in E \mid j \in V\},$$

respectively. Let $x_e = 1$ if edge $e$ is interdicted, and let $x_e = 0$ otherwise. The variable $y_e$ denotes the flow on edge $e \in E$ and $y_a$ the flow on an artificial "return edge" $a = (t, s) \notin E$; see Figure 5.11. Let $n$ be the number of edges in $E$ and $y = (y_e, e \in E)$. Then, for a given interdiction plan $x$, the maximum flow through the network is

$$f(x) = \sup_{y_a, y} \Big\{ y_a \;\Big|\; \sum_{e \in E_i^+} y_e - \sum_{e \in E_i^-} y_e = \delta_i y_a \;\; \forall i \in V, \\ 0 \le y_e \le u_e(1 - x_e) \;\; \forall e \in E \Big\},$$

with $\delta_s = 1$, $\delta_t = -1$ and $\delta_i = 0$ for all $i \in V \setminus \{s, t\}$; cf. [1, Chapters 6-7]. The equality constraints ensure flow balance at each vertex. The last set of inequalities forces the effective capacity of an edge to 0 if $x_e = 1$ and otherwise leaves it at the nominal $u_e$. Equivalently,

$$f(x) = \sup_{y_a, y} \Big\{ y_a - \sum_{e \in E} x_e y_e \;\Big|\; \sum_{e \in E_i^+} y_e - \sum_{e \in E_i^-} y_e = \delta_i y_a \;\; \forall i \in V, \\ 0 \le y_e \le u_e \;\; \forall e \in E \Big\}.$$

The *maximum-flow network-interdiction problem* then takes the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\; f(x) \;\text{ subject to }\; \sum_{e \in E} r_e x_e \le \alpha, \quad x \in \{0, 1\}^n,$$

where $\alpha$ is the amount of interdiction resource available. In contrast to earlier examples with an objective function given as the maximum over a *finite* number of functions, we here face an uncountable number. Since minimizing $f$ is equivalent to minimizing a scalar $\tau$ subject to the constraint $f(x) \le \tau$, the maximum-flow network-interdiction problem can be viewed as resulting in an infinite number of constraints. However, using strong duality we can reformulate the problem in terms of a finite number of linear equalities and inequalities.

Let's fix $x \geq 0$. We see that $-f(x)$ is the minimum value of the linear optimization problem

$$\underset{y_a, y_1, \ldots, y_n}{\text{minimize}} \; -y_a + \sum\nolimits_{e \in E} x_e y_e \;\; \text{subject to} \;\; \sum\nolimits_{e \in E_i^+} y_e - \sum\nolimits_{e \in E_i^-} y_e = \delta_i y_a \;\; \forall i \in V$$
$$0 \leq y_e \leq u_e \;\; \forall e \in E,$$

which always has a minimizer by 4.9. We plan to use the strong duality in 5.41 and just need to place the minimization problem in the right form by eliminating upper bounds on the variables. This leads to the reformulation

$$\underset{y_a, y_1, \ldots, y_{2n}}{\text{minimize}} \; -y_a + \sum\nolimits_{e \in E} x_e y_e \;\; \text{subject to} \;\; \sum\nolimits_{e \in E_i^+} y_e - \sum\nolimits_{e \in E_i^-} y_e = \delta_i y_a \;\; \forall i \in V$$
$$y_a \geq 0, \;\; y_e + y_{n+e} = u_e, \;\; y_e \geq 0, \;\; y_{n+e} \geq 0 \;\; \forall e \in E,$$

where the addition of $y_a \geq 0$ doesn't change the minimum value. Let's redefine $y$ by setting $y = (y_a, y_1, \ldots, y_{2n})$. The reformulation is of the form minimizing $\langle c, y \rangle$ subject to $Ay = b$ and $y \geq 0$, where $c = (-1, x_1, \ldots, x_n, 0, \ldots, 0)$, $b = (0, \ldots, 0, u_1, \ldots, u_n) \in \mathbb{R}^{m+n}$, with $m$ being the number of vertices, and

$$A = \begin{bmatrix} -\delta_1 & a_{11} & \cdots & a_{1n} & 0 \cdots 0 \\ -\delta_2 & a_{21} & \cdots & a_{2n} & 0 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \ddots \vdots \\ -\delta_m & a_{m1} & \cdots & a_{mn} & 0 \cdots 0 \\ 0 & 1 & \cdots & 0 & 1 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & \cdots & 1 & 0 \cdots 1 \end{bmatrix}$$

is an $(m+n) \times (1+2n)$-matrix with $a_{ie} = 1$ if $e \in E_i^+$, $a_{ie} = -1$ if $e \in E_i^-$ and 0 otherwise. By 5.41, the corresponding dual problem is maximizing $\langle b, (v, w) \rangle$ subject to $A^\top (v, w) \leq c$, where $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$. Explicitly, this amounts to

$$\underset{v, w}{\text{maximize}} \sum\nolimits_{e \in E} u_e w_e \;\; \text{subject to} \;\; v_i - v_j + w_e \leq x_e \;\; \forall (i, j) = e \in E$$
$$v_t - v_s \leq -1, \;\; w_e \leq 0 \;\; \forall e \in E.$$

Since the maximum value here is $-f(x)$ by 5.41, $f(x)$ coincides with the minimum value of $\sum_{e \in E} -u_e w_e$ under the same constraints. The overall problem can then be formulated as that of minimizing $\sum_{e \in E} -u_e w_e$ over $x, v, w$ subject to these constraints as well as the original constraints on $x$. We can also replace "$\leq -1$" by "$= -1$" because of the structure of the problem and this eliminates redundance. Putting all together, the maximum-flow network-interdiction problem can equivalently be stated as

$$\begin{aligned} \underset{x,v,w}{\text{minimize}} \sum\nolimits_{e\in E} -u_e w_e \ \text{ subject to } \ v_i - v_j + w_e &\le x_e \ \ \forall (i,j)=e\in E \\ v_t - v_s &= -1 \\ \sum\nolimits_{e\in E} r_e x_e &\le \alpha \\ w_e \le 0, \ \ x_e &\in \{0,1\} \ \ \forall e \in E. \end{aligned}$$

Thus, an objective function involving a supremum over an uncountable number of possibilities can be reformulated into a linear optimization problem with integer constraints. As mentioned in 4.3, there are well-developed algorithms for such problems that rely on branch-and-bound techniques; see also [114] for refinements and [20] for related defense applications.

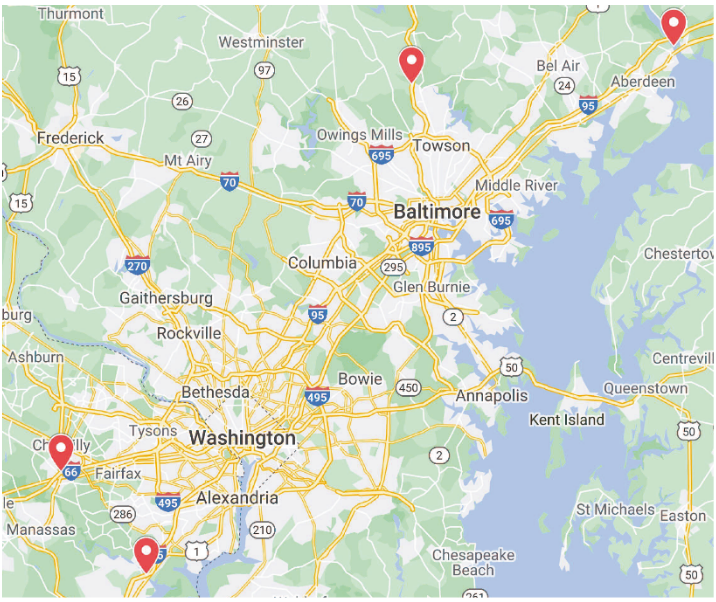


Fig. 5.12: Optimal points (marked) of interdiction in a maximum-flow network-interdiction problem involving 3672 vertices and 10031 edges.

As a concrete instance, let's consider the evacuation of Washington, D.C., after a terrorist attack and the amount of traffic that can "flow" out of Washington. A robust plan needs to consider the effect of destroyed roadways and bridges. The maximum-flow network-interdiction problem informs decision-makers about evacuation routes as well as points of vulnerability. A dataset involving all roads with speed limits of 30 miles per hour or higher in Maryland, Virginia and Washington, D.C., gives a network of vertices and edges with 152 vertices in the Baltimore-Washington region, to be aggregated into a source vertex $s$, representing points of departure. About the periphery 35 vertices representing evacuation centers are aggregated into a sink vertex $t$. The resulting network has 3672 vertices and 10031 edges. If $\alpha = 4$ so that only four edges can be interdicted, we obtain a minimizer of the maximum-flow network-interdiction problem visualized in Figure 5.12; four marked locations have been interdicted and the flow out of Washington to the evacuation centers is reduced from a nominal capacity of 194 to 146 units. The network is moderately resilient: the reduction from 194 to 146 is significant but not catastrophic. Moreover, the points of interdiction could be proposed as locations where hardening of the infrastructure might be especially beneficial; cf. [114] for further information. □

## 5.H L-Shaped Method

Stochastic optimization problems with linear recourse have large-scale formulations that may exceed our computational capacity; see §3.J and especially (3.22). However, duality theory can be brought in for the purpose of constructing tractable approximations that leverage the linear structure of the problems. Let's consider

$$\underset{x \in C_1}{\text{minimize}} \; f_0(x) + \mathbb{E}\big[f(\boldsymbol{\xi}, x)\big],$$

where $C_1 \subset \mathbb{R}^{n_1}$, $f_0 : \mathbb{R}^{n_1} \to \mathbb{R}$ is the first-stage cost and $f : \mathbb{R}^q \times \mathbb{R}^{n_1} \to \overline{\mathbb{R}}$ has

$$f(\xi, x) = \inf_y \big\{\langle a, y\rangle \,\big|\, Wy = d - Tx, \;\; y \geq 0\big\} \tag{5.16}$$

for $a \in \mathbb{R}^{n_2}$, $m_2 \times n_2$-matrix $W$, $m_2 \times n_1$-matrix $T$, $d \in \mathbb{R}^{m_2}$ and $\xi = (a, T_1, \dots, T_{m_2}, d)$, with $T_i$ being the $i$th row of $T$. The random vector $\boldsymbol{\xi}$ is then of dimension $q = n_2 + (1 + n_1)m_2$. We assume that $W$ is deterministic and this is critical in the following development. Since $a$ may or may not be random, our setting includes stochastic optimization problems with fixed recourse; see §3.J. Let's assume that the probability distribution of $\boldsymbol{\xi}$ is finite and $\{p_\xi > 0, \xi \in \Xi\}$ are the corresponding probabilities.

For fixed $\xi = (a, T_1, \dots, T_{m_2}, d) \in \mathbb{R}^q$, the function

$$(x, y) \mapsto \langle a, y\rangle + \iota_C(x, y), \;\; \text{with } C = \big\{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,\big|\, Wy - d + Tx = 0, y \geq 0\big\}$$

is epi-polyhedral because it's affine on its domain, which is a polyhedral set. Then,

$$x \mapsto f(\xi, x) = \inf_y \langle a, y\rangle + \iota_C(x, y)$$

is also epi-polyhedral by 2.66 and thus convex. As seen from 2.67, this means that the expectation function $Ef$ is epi-polyhedral and convex.

The structure offers an opportunity for approximation: A subgradient of $f(\xi, \cdot)$ at a point $x^k$ where the function is finite leads to a lower bound on $f(\xi, \cdot)$ through the subgradient inequality 2.17. Specifically, let $w_\xi^k \in \partial_x f(\xi, x^k)$ and then

$$f(\xi, x) \geq f(\xi, x^k) + \langle w_\xi^k, x - x^k\rangle \;\; \forall x \in \mathbb{R}^{n_1}.$$

After repeating this for each $\xi \in \Xi$ and summing both sides, we obtain the lower bound

$$Ef(x) \ge \sum\nolimits_{\xi\in\Xi} p_\xi f(\xi, x^k) + \Big\langle \sum\nolimits_{\xi\in\Xi} p_\xi w_\xi^k,\ x - x^k \Big\rangle \ \forall x \in \mathbb{R}^{n_1}.$$

If the process is carried out at the points $x^1, \dots, x^\nu$ and the lower bound is taken as the maximum value across the bounds, then we obtain an approximating problem

$$\begin{aligned} &\underset{x\in C_1, \alpha\in\mathbb{R}}{\text{minimize}}\ f_0(x) + \alpha \\ &\text{subject to } \sum\nolimits_{\xi\in\Xi} p_\xi f(\xi, x^k) + \Big\langle \sum\nolimits_{\xi\in\Xi} p_\xi w_\xi^k, x - x^k \Big\rangle \le \alpha, \quad k = 1, \dots, \nu. \end{aligned}$$

In contrast to alternative solution approaches such as those based on (3.22), the size of this problem doesn't depend on $\Xi$. The cardinality of $\Xi$ enters only in the preprocessing of the data.

An algorithm emerges from this development: For an initial guess $x^1$, compute subgradients

$$\big\{ w_\xi^1 \in \partial_x f(\xi, x^1),\ \xi \in \Xi \big\},$$

solve the approximating problem with $\nu = 1$ and hopefully obtain a minimizer $x^2$ of that problem. At $x^2$, compute subgradients

$$\big\{ w_\xi^2 \in \partial_x f(\xi, x^2),\ \xi \in \Xi \big\},$$

solve the approximating problem with $\nu = 2$, obtain $x^3$ and so forth. This is the idea behind the *L-shaped method*. When $C_1$ is polyhedral and $f_0$ is linear, the approach requires solving only a sequence of linear optimization problems. Beyond such cases, the approximating problem may remain tractable (especially compared to the actual problem) even though it grows in size by one constraint per iteration.

**Proposition 5.54** (properties of recourse function). *For $\xi = (a, T_1, \dots, T_{m_2}, d) \in \mathbb{R}^q$ and $m_2 \times n_2$-matrix $W$, let $f(\xi, \cdot)$ be given by (5.16). Then, $f(\xi, \cdot)$ is convex and*

$$f(\xi, x) \ge \sup\nolimits_{v\in V} \langle d - Tx, v\rangle \quad \forall x \in \mathbb{R}^{n_1}, \quad \text{where } V = \{v \in \mathbb{R}^{m_2} \mid W^\top v \le a\}.$$

*At a point $\bar{x}$ where $f(\xi, \cdot)$ is finite, one has*

$$\begin{aligned} &f(\xi, \bar{x}) = \sup\nolimits_{v\in V} \langle d - T\bar{x}, v\rangle \\ &-T^\top \bar{v} \in \partial_x f(\xi, \bar{x}) \quad \forall \bar{v} \in \operatorname{argmax}_{v\in V} \langle d - T\bar{x}, v\rangle, \end{aligned}$$

*with this set of maximizers being nonempty.*

**Proof.** The claim about convexity summarizes the discussion earlier in the section. The minimization problem defining $f(\xi, x)$ has maximizing $\langle d - Tx, v\rangle$ over $v \in V$ as a dual problem by 5.41, which also establishes the claimed inequality.

Since $f(\xi, \bar{x})$ is finite, one has

$$f(\xi, \bar{x}) = \sup\nolimits_{v\in V} \langle d - T\bar{x}, v\rangle \quad \text{and} \quad \operatorname{argmax}_{v\in V} \langle d - T\bar{x}, v\rangle \ne \emptyset$$

by 5.41. This means that $V$ is nonempty and the function $h : \mathbb{R}^{m_2} \to \overline{\mathbb{R}}$ given by $h(u) = \sup_{v \in V} \langle u, v \rangle$ is proper. It's convex by 1.18(a) and lsc because epi $h$ is an intersection of closed sets. Then, by 5.37 and 2.19,

$$\bar{v} \in \partial h(\bar{u}) \iff \bar{u} \in \partial h^*(\bar{v}) \iff \bar{v} \in \operatorname{argmin}\{h^* - \langle \bar{u}, \cdot \rangle\}.$$

From 5.29, we see that $h^*(v) = \iota_V(v)$. Thus, for $\bar{u} \in \operatorname{dom} h$,

$$\partial h(\bar{u}) = \operatorname{argmin}\{h^* - \langle \bar{u}, \cdot \rangle\} = \operatorname{argmax}_{v \in V} \langle \bar{u}, v \rangle.$$

Let $\bar{w} \in -T^\top \bar{v}$, with $\bar{v} \in \operatorname{argmax}_{v \in V} \langle d - T\bar{x}, v \rangle$. Then, $\bar{v} \in \partial h(d - T\bar{x})$. Consequently, by the subgradient inequality 2.17 as applied to $h$, one has

$$\begin{aligned} f(\xi, x) \geq h(d - Tx) &\geq h(d - T\bar{x}) + \big\langle \bar{v}, d - Tx - (d - T\bar{x}) \big\rangle \\ &= f(\xi, \bar{x}) + \langle -T^\top \bar{v}, x - \bar{x} \rangle \quad \forall x \in \mathbb{R}^{n_1}. \end{aligned}$$

This amounts to a subgradient inequality for $f(\xi, \cdot\,)$ and the claim holds by 2.17. □

While the proposition furnishes subgradients at points where $f(\xi, \cdot\,)$ is finite, the approximating problem may lead to other points too or even fail to produce a solution at all. An implementable algorithm needs to address these concerns. We make an assumption that helps us weed out some pathological cases:

$$f(\xi, x) > -\infty \quad \forall \xi \in \Xi,\ x \in C_1. \tag{5.17}$$

This implies that the second-stage problem isn't unbounded, which is reasonable because otherwise $\mathbb{E}[f(\boldsymbol{\xi}, \bar{x})] = -\infty$ for some $\bar{x} \in C_1$ and then the actual problem is unbounded. Still, we permit $f(\xi, \bar{x}) = \infty$, which corresponds to induced constraints; see §3.J. We may discover after the uncertainty is resolved that $\bar{x} \in C_1$ is infeasible due to the lack of an admissible second-stage decision $y$. Sometimes the second-stage model is formulated such that this obviously can't occur. However, in general, we need to check for such infeasibility and, in fact, the L-shaped method gradually learns the induced constraints and this is useful in many ways.

At a current point $x^\nu \in C_1$, the key quantity to examine is the feasible set

$$V(a) = \{v \in \mathbb{R}^{m_2} \mid W^\top v \leq a\}$$

in the dual problem

$$\underset{v \in V(a)}{\text{maximize}}\ \langle d - Tx^\nu, v \rangle \tag{5.18}$$

corresponding to the minimization problem behind $f(\xi, x^\nu)$, with $\xi = (a, T_1, \dots, T_{m_2}, d)$; see 5.54 and also 5.41.

**Infeasible dual problem.** If $V(a) = \emptyset$ for some $\xi = (a, T_1, \dots, T_{m_2}, d) \in \Xi$, then (5.18) is infeasible and $f(\xi, x) = \infty$ or $f(\xi, x) = -\infty$ by 5.41 for any $x \in \mathbb{R}^{n_1}$ because $V(a)$ is independent of $x$. Since the possibility $f(\xi, x) = -\infty$ for $x \in C_1$ violates (5.17), we must then have $f(\xi, x) = \infty$ for all $x \in C_1$ and the actual problem is infeasible due to induced constraints.

**Feasible dual problem.** If $V(a) \neq \emptyset$ for $\xi = (a, T_1, \ldots, T_{m_2}, d) \in \Xi$, then $f(\xi, x^\nu) = \infty$ when (5.18) is unbounded (cf. 5.41) and otherwise $f(\xi, x^\nu) \in \mathbb{R}$. The dual problem is unbounded for $\xi$ if there's a direction $\bar{z} \in \mathbb{R}^{m_2}$ and $\bar{v} \in V(a)$ such that

$$W^\top(\bar{v} + \lambda\bar{z}) \leq a \ \ \forall \lambda \in [0, \infty) \quad \text{and} \quad \langle d - Tx^\nu, \bar{z}\rangle > 0$$

because then the objective function value of the dual problem keeps on increasing as we move along the line from $\bar{v}$ in the direction of $\bar{z}$. Since the length of a direction doesn't matter and the feasibility condition reduces to $W^\top \bar{z} \leq 0$, we can check for unboundedness by solving for each $\xi = (a, T_1, \ldots, T_{m_2}, d) \in \Xi$:

$$\operatorname*{maximize}_{z \in \mathbb{R}^{m_2}} \ \langle d - Tx^\nu, z\rangle \ \text{ subject to } \ W^\top z \leq 0, \ \ z \in [-1, 1]^{m_2}. \tag{5.19}$$

If the maximum value here is 0 for all $\xi \in \Xi$, then the dual problems (5.18) across all $\xi \in \Xi$ aren't unbounded and $f(\xi, x^\nu) \in \mathbb{R}$ for all $\xi \in \Xi$. If there's $\xi \in \Xi$ producing a positive maximum value, then the corresponding (5.18) is unbounded, $f(\xi, x^\nu) = \infty$ and $Ef(x^\nu) = \infty$; the current point $x^\nu$ is infeasible due to an induced constraint. Let $z_\xi^\nu$ be a direction that attains such a positive maximum value. We can eliminate $x^\nu$ from further consideration as well as other $x$ that allows unbounded growth in (5.18) along $z_\xi^\nu$ by imposing the constraint

$$\langle d - Tx, z_\xi^\nu\rangle \leq 0 \quad \text{or, equivalently,} \quad \langle T^\top z_\xi^\nu, x\rangle \geq \langle d, z_\xi^\nu\rangle.$$

These refinements ensure a well-defined algorithm.

**L-Shaped Method.**

Data. Tolerance $\varepsilon \geq 0$.
Step 0. Set $\nu = \tau = \sigma = 0$.
Step 1. Replace $\nu$ by $\nu + 1$.
If $\sigma = 0$, then solve the *master problem*

$$\operatorname*{minimize}_{x \in C_1} \ f_0(x) \ \text{ subject to } \ \langle B_k, x\rangle \geq \beta_k, \quad k = 1, \ldots, \tau.$$

If this master problem doesn't have a minimizer, then stop.
Compute a minimizer $x^\nu$ of this master problem and set $\alpha^\nu = -\infty$.
Else, solve the *master problem*

$$\begin{aligned}\operatorname*{minimize}_{x \in C_1, \alpha \in \mathbb{R}} \ f_0(x) + \alpha \ \text{ subject to } \ & \langle B_k, x\rangle \geq \beta_k, \quad k = 1, \ldots, \tau \\ & \langle D_k, x\rangle + \alpha \geq \delta_k, \quad k = 1, \ldots, \sigma.\end{aligned}$$

If this master problem doesn't have a minimizer, then stop.
Compute a minimizer $(x^\nu, \alpha^\nu)$ of this master problem.

Step 2. Solve (5.19) for $\xi = (a, T_1, \ldots, T_{m_2}, d) \in \Xi$ and obtain maximizers $\{z_\xi^\nu, \xi \in \Xi\}$.
If the corresponding maximum values are all 0, then go to Step 3.
For some $\xi = (a, T_1, \ldots, T_{m_2}, d)$ with a positive maximum value, set

$$B_{\tau+1} = T^\top z_\xi^\nu, \qquad \beta_{\tau+1} = \langle d, z_\xi^\nu \rangle,$$

replace $\tau$ by $\tau + 1$ and go to Step 1.

Step 3. Solve (5.18) for $\xi = (a, T_1, \ldots, T_{m_2}, d) \in \Xi$.
If no maximizers exist, then stop.
Let $\{v_\xi^\nu, \xi \in \Xi\}$ be maximizers and set

$$D_{\sigma+1} = \sum\nolimits_{\xi \in \Xi} p_\xi T_\xi^\top v_\xi^\nu, \qquad \delta_{\sigma+1} = \sum\nolimits_{\xi \in \Xi} p_\xi \langle d_\xi, v_\xi^\nu \rangle,$$

where $T_\xi$ and $d_\xi$ are the $T$-matrix and $d$-vector corresponding to $\xi$.
If

$$\delta_{\sigma+1} - \langle D_{\sigma+1}, x^\nu \rangle - \alpha^\nu \leq \varepsilon,$$

then stop.
Replace $\sigma$ by $\sigma + 1$ and go to Step 1.

The master problems correspond to the approximating problem discussed above, with $\sigma$ constraints of the kind described there and also $\tau$ constraints deriving from induced constraints. If the algorithm stops in Step 1 because a master problem is infeasible, then the actual problem is infeasible because the master problems are relaxations of the actual problem. The algorithm may also stop in Step 1 because a master problem is unbounded or fails to have a minimizer for some other reason (cf. for example 4.9), but these situations are usually avoided by imposing assumptions on $C_1$ and $f_0$.

Step 2 checks for unboundedness in the dual problems as described around (5.19). If we discover unboundedness for some $\xi$, then we return to Step 1 with an additional constraint. Otherwise, we proceed to Step 3 with the confidence that (5.18) isn't unbounded for any $\xi \in \Xi$.

If we encounter $\xi \in \Xi$ without a maximizer in Step 3, then (5.18) is infeasible for this $\xi$ because unboundedness has been ruled out in the previous step. Since the feasible set in (5.18) is independent of $x^\nu$, this is then the case for any $x \in C_1$ and the actual problem is infeasible as discussed above. If every $\xi \in \Xi$ returns a maximizer, then we leverage 5.54 to compute subgradients of $f(\xi, \cdot)$ at $x^\nu$ and construct a constraint of the form in the approximating problem, which is then added to the master problems.

A master problem furnishes a lower bound $f_0(x^\nu) + \alpha^\nu$ on the minimum value of the actual problem because it relies on a lower approximation of $f(\xi, \cdot)$. Step 3 provides an upper bound on that minimum value because $x^\nu$ is feasible in the actual problem whenever we get past the initial stopping condition. This upper bound is

$$f_0(x^\nu) + \sum\nolimits_{\xi \in \Xi} p_\xi f(\xi, x^\nu) = f_0(x^\nu) + \delta_{\sigma+1} - \langle D_{\sigma+1}, x^\nu \rangle.$$

Thus, the optimality gap is bounded by $\delta_{\sigma+1} - \langle D_{\sigma+1}, x^\nu\rangle - \alpha^\nu$ and the algorithm terminates if this is sufficiently small.

Compared to the expanded formulation (3.22), the L-shaped method considers many smaller problems repeatedly: linear optimization problems in Steps 2 and 3 and then the slowly growing master problems. An important advantage is that the upper and lower bounds provide an estimate of the optimality gap that may allow us to terminate after just a few iterations. The need for obtaining (global) minimizers of the master problems is a bottleneck but can be relaxed to near-minimizers with only minor adjustments to the lower bound calculation. The linear optimization problems are usually quick to solve, but a large number of $\xi \in \Xi$ may require further refinements. For example, in Step 2, the only variation between the linear problems is in the coefficients of the objective function; the feasible sets remain the same and this can be utilized. We refer to [119] for such developments and for further theory in the case of linear $f_0$ and polyhedral $C_1$.

**Example 5.55** (L-shaped method for simple recourse). Let's consider the special case when $f(\xi, x)$ in (5.16) is defined by

$$f(\xi, x) = \inf_y \left\{\langle a^+, y^+\rangle + \langle a^-, y^-\rangle \,\middle|\, Iy^+ - Iy^- = d - Tx,\ y^+ \geq 0,\ y^- \geq 0\right\},$$

where $\xi = (T_1, \dots, T_{m_2}, d) \in \mathbb{R}^{m_2(n_1+1)}$ and $T_i$ is the $i$th row of $T$. The second-stage decision vector $y = (y^+, y^-) \in \mathbb{R}^{n_2}$ is broken into two equally long vectors. Thus, the identity matrix $I$ is of dimension $m_2 \times n_2/2$ so we necessarily have $m_2 = n_2/2$. In the mold of the earlier development, $a = (a^+, a^-)$ is deterministic and $W = (I, -I)$. This results in simple recourse by 3.39. However, let's reach the same conclusion using duality theory and also examine the implications for the L-shaped method.

**Detail.** For any $\xi = (T_1, \dots, T_{m_2}, d)$ and $x$, it's always possible to find a feasible $y$ so $f(\xi, x) < \infty$, i.e., there are no induces constraints. Moreover, (5.18) specializes to

$$\operatorname*{maximize}_{v} \langle d - Tx^\nu, v\rangle \ \text{ subject to } \ -a^- \leq v \leq a^+.$$

Via 5.41, $f(\xi, x^\nu) < \infty$ implies that the maximization problem can't be unbounded and Step 2 in the L-shaped method can be skipped altogether. The maximization problem could still be infeasible, but this is avoided if $-a^- \leq a^+$, which is easily checked and corresponds to the condition $\gamma_i \leq \delta_i$ imposed in 3.39. Under this assumption, $f(\xi, x)$ is always finite. In Step 3, a maximizer of the dual problem is analytically available. □

**Exercise 5.56** (implementation of L-shaped method). A grocer sells four types of fruits—nectarines, bananas, oranges and apples—that are obtained from two different suppliers. The shipments from Supplier 1 come in lots of 4 kg of nectarines, 9 kg of bananas, 3 kg of oranges and 10 kg of apples, while those from Supplier 2 consist of 1 kg of nectarines, 1 kg of bananas, 1.85 kg of oranges and 2.7 kg of apples. Supplier 1 can deliver up to 40 lots at \$5 per lot if the order is placed early enough and, if additional supplies are required, Supplier 1 can guarantee same-day delivery of another 40 lots at \$8 per lot. Supplier 2 can deliver up to 800 lots at \$7 per lot if the order is placed early enough, and can also guarantee same-day delivery of another 200 lots at \$15 per lot. The objective of the grocer is to satisfy the demand for these four types of fruits at a minimum expected cost. The uncertain (daily) demand is given by Table 5.1, with each of the $3^4 = 81$ possible values of the demand $\xi \in \mathbb{R}^4$ having equal probability of occurring. Formulate and solve the grocer's problem using the L-shaped method.

Table 5.1: Demand levels in the grocer's problem.

| | demand (kg) | | |
|---|---|---|---|
| nectarines | 1050 | 1150 | 1200 |
| bananas | 1000 | 1250 | 1500 |
| oranges | 1250 | 1750 | 2000 |
| apples | 2500 | 3000 | 3250 |

**Guide.** The first-stage variables are $x_1$ and $x_2$, with $x_i$ representing the number of lots ordered "early" from Supplier $i$. The second-stage variables $y_1$ and $y_2$ specify the number of same-day lots ordered from the two suppliers. Let's ignore any need for ordering only integer number of lots and allow these variables to be a nonnegative number. (Integer restrictions on the first-stage variables are easily incorporated into $C_1$ and only make the master problems a bit harder to solve.) □

## 5.I Monitoring Functions

A problem needs to be sufficiently structured to be tractable theoretically and computationally. A promising kind in this regard is

$$\underset{x \in X}{\text{minimize}}\ f_0(x) + h\big(F(x)\big) \tag{5.20}$$

for some closed $X \subset \mathbb{R}^n$, proper, lsc and convex $h : \mathbb{R}^m \to \overline{\mathbb{R}}$, smooth $f_0 : \mathbb{R}^n \to \mathbb{R}$ and smooth $F : \mathbb{R}^n \to \mathbb{R}^m$. However, a bottleneck might be aspects related to the function $h$. For example, the Rockafellian in 5.28 produces a Lagrangian with

$$l(x, y) = \iota_X(x) + f_0(x) + \big\langle F(x), y\big\rangle - h^*(y)$$

and we then need an expression for the conjugate of $h$. Let's examine choices of $h$ that result in explicit expressions for the corresponding conjugate functions and dual problems.

For a nonempty polyhedral set $Y \subset \mathbb{R}^m$ and a symmetric positive semidefinite $m \times m$-matrix $B$, we say that $h_{Y,B} : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a *monitoring function* when

$$h_{Y,B}(u) = \sup_{y \in Y} \big\{\langle u, y\rangle - \tfrac{1}{2}\langle y, By\rangle\big\}.$$

Monitoring functions track quantities of interest and appropriately assign penalties to undesirable values. Setting $h = h_{Y,B}$ in (5.20) is often a viable modeling choice and turns out to be computationally tractable. Common situations are covered by specific choices of $Y$ and $B$:

$$h_{Y,B}(u) = \max_{i=1,\dots,m} u_i \quad \text{for } Y = \left\{y \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^m y_i = 1,\ y \geq 0\right\} \text{ and } B = 0$$

$$h_{Y,B}(u) = \iota_D(u) \quad \text{for } D = \operatorname{pol} Y, \text{ polyhedral cone } Y \text{ and } B = 0.$$

The first choice can be used to penalize various values of $F(x)$, while the second one enforces the constraint $F(x) \in D$.

A monitoring function is convex by 1.18(a), lsc by virtue of having an epigraph that's the intersection of closed sets and also proper because $Y$ is nonempty and then $h_{Y,B}(u) > -\infty$ for all $u$ and

$$h_{Y,B}(0) = \sup_{y \in Y} -\tfrac{1}{2}\langle y, By\rangle \leq 0$$

by the positive semidefiniteness of $B$. Moreover,

$$h^*_{Y,B}(y) = \tfrac{1}{2}\langle y, By\rangle + \iota_Y(y)$$

as can be seen from the definition of a conjugate function and the Fenchel-Moreau theorem 5.23. Thus, under the choice of $h = h_{Y,B}$ in (5.20), the Lagrangian from 5.28 has

$$l(x, y) = \iota_X(x) + f_0(x) + \left\langle F(x), y\right\rangle - \tfrac{1}{2}\langle y, By\rangle - \iota_Y(y),$$

which is rather explicit since $Y$ is presumably a known polyhedral set and $X$ is usually also simple. Via the inverse rule 5.37 and the optimality condition 2.19, we obtain that

$$\partial h_{Y,B}(u) = \operatorname{argmin}_{y \in Y} \tfrac{1}{2}\langle y, By\rangle - \langle u, y\rangle \quad \forall u \in \operatorname{dom} h_{Y,B}.$$

The horizon subgradients, important in the chain rule 4.64 and elsewhere, are given by (see [105, Example 11.18])

$$\partial^\infty h_{Y,B}(u) = \left\{y \in Y^\infty \cap \operatorname{null} B \;\middle|\; \langle y, u\rangle = 0\right\} \quad \forall u \in \operatorname{dom} h_{Y,B},$$

where for an arbitrarily selected $\bar{y} \in Y$,

$$Y^\infty = \left\{y \in \mathbb{R}^m \;\middle|\; \bar{y} + \lambda y \in Y \quad \forall \lambda \in [0, \infty)\right\}.$$

For example, if $Y$ is bounded, then $Y^\infty = \{0\}$ and $\partial^\infty h_{Y,B}(u) = \{0\}$ for all $u \in \operatorname{dom} h_{Y,B}$. Alternatively, if $B$ has full rank, then $\operatorname{null} B = \{0\}$ and we reach the same conclusion.

**Example 5.57** (duality for quadratic problem with monitoring). For a nonempty polyhedral set $X \subset \mathbb{R}^n$, a symmetric positive semidefinite $n \times n$-matrix $Q$, an $m \times n$ matrix $A$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, consider the problem

$$\underset{x \in X}{\text{minimize}} \ \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + h_{Y,B}(b - Ax),$$

where $h_{Y,B}$ is a monitoring function, which necessarily means that $Y$ is nonempty and polyhedral and $B$ is symmetric and positive semidefinite. For a Rockafellian given by

$$f(u,x) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + h_{Y,B}(b - Ax + u),$$

which recovers the actual problem as minimizing $f(0,\cdot)$, the dual problem is

$$\underset{y\in Y}{\text{maximize}}\ \langle b, y\rangle - \tfrac{1}{2}\langle y, By\rangle - h_{X,Q}(A^\top y - c).$$

If either the actual problem or the dual problem has a finite optimal value, then the other one has the same value and strong duality holds. Under such circumstances, there's a minimizer for the actual problem and a maximizer for the dual problem.

**Detail.** By 5.28, the Lagrangian in this case has

$$l(x,y) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + \langle b - Ax, y\rangle - \tfrac{1}{2}\langle y, By\rangle - \iota_Y(y).$$

The minimization of the Lagrangian produces a dual objective function with

$$\begin{aligned}\psi(y) &= -\iota_Y(y) + \langle b, y\rangle - \tfrac{1}{2}\langle y, By\rangle - \sup_x \left\{\langle A^\top y - c, x\rangle - \iota_X(x) - \tfrac{1}{2}\langle x, Qx\rangle\right\}\\ &= -\iota_Y(y) + \langle b, y\rangle - \tfrac{1}{2}\langle y, By\rangle - h_{X,Q}(A^\top y - c).\end{aligned}$$

The claim about strong duality is supported by [105, Theorem 11.42, Example 11.43]. □

If $h$ in (5.20) is a monitoring function that penalizes certain values of $F$ and these penalties are assigned componentwise, then it suffices to consider a separable monitoring function, i.e.,

$$h_{Y,B}(u) = \sum\nolimits_{i=1}^m h_{Y_i,\beta_i}(u_i) = \sum\nolimits_{i=1}^m \sup_{y_i\in Y_i}\left\{u_i y_i - \tfrac{1}{2}\beta_i y_i^2\right\}$$

for $B$, a diagonal matrix with nonnegative elements $\beta_1,\dots,\beta_m$, and $Y = Y_1\times\cdots\times Y_m$, with $Y_i$ being a nonempty closed interval.

**Example 5.58** (one-dimensional monitoring function). If $\beta > 0$ and $Y = [\sigma,\tau]$ for $\sigma,\tau \in \overline{\mathbb{R}}$ with $\sigma \le \tau$, then we obtain a smooth monitoring function given by

$$h_{Y,\beta}(u) = \begin{cases} \sigma u - \frac{1}{2}\beta\sigma^2 & \text{if } u < \sigma\beta\\ \frac{1}{2\beta}u^2 & \text{if } \sigma\beta \le u \le \tau\beta\\ \tau u - \frac{1}{2}\beta\tau^2 & \text{if } u > \tau\beta.\end{cases}$$

Figure 5.13(left) illustrates the possibilities.

**Detail.** In particular, $h_{[0,\infty),\beta}(u) = 0$ if $u \le 0$ and $(2\beta)^{-1}u^2$ otherwise and thus penalizes positive values. For $\tau \in [0,\infty)$, a nonsmooth monitoring function is given by

$$h_{[0,\tau],0}(u) = \begin{cases} 0 & \text{if } u \le 0\\ \tau u & \text{if } u > 0;\end{cases}$$

see Figure 5.13(right). An actual constraint is produced by $h_{[0,\infty),0}(u) = \iota_{(-\infty,0]}(u)$. □

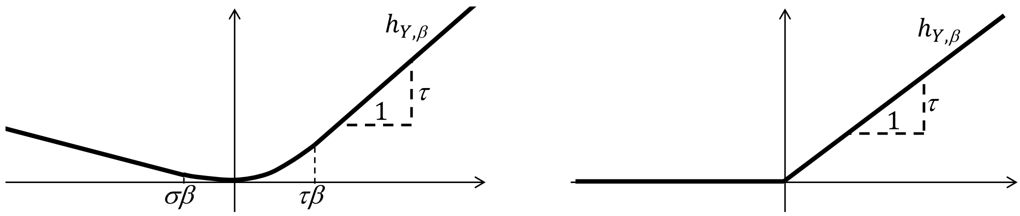


Fig. 5.13: Monitoring functions $h_{Y,\beta}$ in 5.58 with $\beta > 0$ and $Y = [\sigma, \tau] \subset \mathbb{R}$ (left); $\beta = 0$ and $Y = [0, \tau] \subset \mathbb{R}$ (right).

**Exercise 5.59** (duality with separable monitoring). For an $m_1 \times n$-matrix $A$, with rows $A_1, \dots, A_{m_1}$, an $m_2 \times n$-matrix $T$, with rows $T_1, \dots, T_{m_2}$, nonnegative scalars $q_1, \dots, q_n$, positive scalars $\beta_1, \dots, \beta_{m_2}$, $r_1, \dots, r_{m_2}$ as well as $s, c \in \mathbb{R}^n$, $b \in \mathbb{R}^{m_1}$ and $d \in \mathbb{R}^{m_2}$, consider the problem

$$\begin{aligned}&\operatorname*{minimize}_{x\in\mathbb{R}^n}\ \langle c, x\rangle + \tfrac{1}{2}\sum\nolimits_{j=1}^{n} q_j x_j^2 + \sum\nolimits_{i=1}^{m_2} h_{[0,r_i],\beta_i}\big(d_i - \langle T_i, x\rangle\big)\\ &\text{subject to } \ Ax \geq b, \quad 0 \leq x \leq s.\end{aligned}$$

Show that the dual problem corresponding to a Rockafellian of the kind supporting 5.57 can be written as

$$\begin{aligned}&\operatorname*{maximize}_{v\in\mathbb{R}^{m_1}, w\in\mathbb{R}^{m_2}}\ \langle b, v\rangle + \langle d, w\rangle - \tfrac{1}{2}\sum\nolimits_{i=1}^{m_2} \beta_i w_i^2 - \sum\nolimits_{j=1}^{n} h_{[0,s_j],q_j}\big(\langle A^j, v\rangle + \langle T^j, w\rangle - c_j\big)\\ &\text{subject to } \ v \geq 0, \quad 0 \leq w \leq r,\end{aligned}$$

where $A^1, \dots, A^n$ and $T^1, \dots, T^n$ are the columns of $A$ and $T$, respectively.

**Guide.** Following 5.57, set $X = \{x \in \mathbb{R}^n \mid 0 \leq x \leq s\}$ and $Q$ to be the diagonal matrix with elements $q_1, \dots, q_n$. The effect of the monitoring functions and the constraint $Ax \geq b$ can be incorporated by setting

$$h_{Y,B}\big(\tilde{b} - \tilde{A}x\big), \quad \text{with } \ Y = [0,\infty)^{m_1} \times [0, r_1] \times \cdots \times [0, r_{m_2}],$$

$B$ being the diagonal $(m_1+m_2)\times(m_1+m_2)$-matrix with elements $0, \dots, 0, \beta_1, \dots, \beta_{m_2}$, $\tilde{b} = (b, d) \in \mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$ and $\tilde{A}$ being the $(m_1+m_2)\times n$-matrix with rows $A_1, \dots, A_{m_1}, T_1, \dots, T_{m_2}$. By 5.57, the dual problem is

$$\operatorname*{maximize}_{y\in Y}\ \langle \tilde{b}, y\rangle - \tfrac{1}{2}\langle y, By\rangle - h_{X,Q}(\tilde{A}^\top y - c),$$

which can be simplified as indicated using $y = (v, w)$. □

There are many solution approaches for (5.20) when $h$ is a monitoring function, even if the dual problem may not be quite as explicit as in 5.57 and 5.59. One possibility that leverages existing algorithms for equality constraints is supported by the following fact.

**Proposition 5.60** (alternative expression for monitoring function). *If a monitoring function $h_{Y,B} : \mathbb{R}^m \to \overline{\mathbb{R}}$ has*

$$Y = \{y \in \mathbb{R}^m \mid A^\top y \leq b\} \quad \textit{and} \quad B = DJ^{-1}D^\top$$

*for some $m \times q$-matrix $A$, $b \in \mathbb{R}^q$, $m \times m$-matrix $D$ and symmetric positive definite $m \times m$-matrix $J$, then*

$$h_{Y,B}(u) = \inf_{v,w} \left\{ \langle b, v\rangle + \tfrac{1}{2}\langle w, Jw\rangle \;\middle|\; Av + Dw = u,\; v \geq 0 \right\} \;\; \forall u \in \mathbb{R}^m.$$

**Proof.** Fix $u \in \mathbb{R}^m$. Let's view the minimization problem in the asserted formula for $h_{Y,B}$ as an instance of the problem examined in 5.28 with $u - Av - Dw$ and $[0,\infty)^q \times \mathbb{R}^m$ playing the roles of $F(x)$ and $X$, respectively. Under the Rockafellian adopted there, we obtain a Lagrangian of the form

$$l(v, w, y) = \iota_{[0,\infty)^q \times \mathbb{R}^m}(v, w) + \langle b, v\rangle + \tfrac{1}{2}\langle w, Jw\rangle + \langle u - Av - Dw, y\rangle.$$

We note that the term appearing at the end of the Lagrangian in 5.28 vanishes in this case because it represents the conjugate of $\iota_{\{0\}^m}$; see 5.29. The corresponding dual objective function then has

$$\psi(y) = \inf_{v,w} l(v, w, y) = \langle u, y\rangle + \inf_{v \geq 0, w} \left\{ \langle v, b - A^\top y\rangle + \tfrac{1}{2}\langle w, Jw\rangle - \langle w, D^\top y\rangle \right\}.$$

For $y \in Y$, the minimization in $v$ brings the term $\langle v, b - A^\top y\rangle$ to 0, while minimization in $w$ is achieved at $J^{-1}D^\top y$ using the optimality condition 2.19. For $y \notin Y$, $\psi(y) = -\infty$. This results in

$$\psi(y) = -\iota_Y(y) + \langle u, y\rangle - \tfrac{1}{2}\langle D^\top y, J^{-1}D^\top y\rangle.$$

Thus, the optimal dual value

$$\sup_{y \in Y} \left\{ \langle u, y\rangle - \tfrac{1}{2}\langle y, DJ^{-1}D^\top y\rangle \right\} = h_{Y,B}(u).$$

Since the minimization problem in the asserted formula for $h_{Y,B}$ and the maximization of $\psi$ fit the setting of 5.57, which in fact we could have used to obtain the expression for the dual problem, strong duality holds when the dual optimal value is finite. Since it defines a monitoring function, $Y$ is nonempty. Thus, the dual problem is always feasible and its maximum value is either finite or the problem is unbounded. In the latter case, by weak duality 5.25, the minimization problem is infeasible and the asserted formula holds even in that case. $\square$

The required representation of $B$ isn't a significant limitation. If $B = 0$, we can select $D = 0$ and $J = I$, the identity matrix. If $B$ is positive definite, then spectral decomposition gives $D$ and $J$. Specifically, there are an $m \times m$-matrix $D$ (consisting of orthonormal eigenvectors) and an $m \times m$-matrix $\Lambda$, with the eigenvalues of $B$ along its diagonal and 0 elsewhere, such that $B = D\Lambda D^\top$; see [22, Theorem 4.2]. Since $B$ is symmetric and positive definite, its eigenvalues are positive and we can take $J$ to be $\Lambda^{-1}$.

**Example 5.61** (reformulation of problems with monitoring). For $f_0 : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^m$ and $X \subset \mathbb{R}^n$, consider the problem

$$\underset{x \in X}{\text{minimize}}\ f_0(x) + h_{Y,B}\big(F(x)\big),$$

with $Y = \{y \in \mathbb{R}^m \mid A^\top y \le b\}$ and $B = DJ^{-1}D^\top$ for some $m \times q$-matrix $A$, $b \in \mathbb{R}^q$, $m \times m$-matrix $D$ and symmetric positive definite $m \times m$-matrix $J$. Then, the problem is equivalently stated as

$$\underset{x \in X, v \in \mathbb{R}^q, w \in \mathbb{R}^m}{\text{minimize}}\ f_0(x) + \langle b, v\rangle + \tfrac{1}{2}\langle w, Jw\rangle \ \text{ subject to }\ Av + Dw = F(x),\ v \ge 0.$$

**Detail.** The reformulation follows from 5.60. If $f_0$ and $F$ are smooth and $X$ is described by smooth constraint functions, then we can apply SQP and interior-point methods (§4.K) to the reformulation. Although such algorithms generally only obtain solutions satisfying KKT conditions, they're available in many software packages. □

## 5.J Lagrangian Finite-Generation Method

Large-scale problems aren't easily solved due to their excessive computing times and memory requirements. For example, the reformulation in 5.61 may simply involve too many constraints for SQP and interior-point methods. We then need to examine the structural properties of the problem and attempt to identify tractable approximations. Separability is one such property that dramatically improves tractability. However, the property may not be immediately present: a Lagrangian can be separable with neither the primal nor the dual problem possessing this property. We'll now examine such a situation, which leads to an algorithm for a class of large-scale problems.

For a nonempty polyhedral set $X \subset \mathbb{R}^n$, a symmetric positive semidefinite $n \times n$-matrix $Q$, an $m \times n$ matrix $A$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, let's consider the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ \varphi(x) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + h_{Y,B}(b - Ax),$$

where $h_{Y,B}$ is a monitoring function, $B$ is a diagonal matrix with nonnegative diagonal elements $\beta_1, \dots, \beta_m$ and $Y = Y_1 \times \cdots \times Y_m$ is a box, with each $Y_i$ being a nonempty closed interval, potentially unbounded. Suppose that $m$ is large, making a direct solution impractical or even impossible. In the setting of 5.57, the actual problem can be viewed as a primal problem paired with the dual problem

$$\underset{y \in \mathbb{R}^m}{\text{maximize}}\ \psi(y) = -\iota_Y(y) + \langle b, y\rangle - \tfrac{1}{2}\sum\nolimits_{i=1}^m \beta_i y_i^2 - h_{X,Q}(A^\top y - c).$$

We might attempt to solve the dual problem in lieu of the primal one, but there's no clear advantage to this strategy. In particular, neither problem is separable because of $X$ and $Q$. As a way forward, let's consider an approximation.

Suppose that $Y$ is approximated by

$$Y^\nu = \mathrm{con}\{\bar{y}^1, \dots, \bar{y}^\nu\},$$

the convex hull of a finite collection of points $\bar{y}^1, \dots, \bar{y}^\nu \in Y$, where $\nu$ is typically much less than $m$. This leads to the approximating dual problem

$$\underset{y\in\mathbb{R}^m}{\text{maximize}}\ \psi^\nu(y) = -\iota_{Y^\nu}(y) + \langle b, y\rangle - \tfrac{1}{2}\sum_{i=1}^m \beta_i y_i^2 - h_{X,Q}(A^\top y - c). \tag{5.21}$$

As seen from 5.57, this maximization problem can be viewed, in turn, as a dual problem of the approximating primal problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \varphi^\nu(x) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + h_{Y^\nu,B}(b - Ax).$$

Thus, we've replaced the actual primal-dual pair with an approximating primal-dual pair. The idea of the *Lagrangian finite-generation method* is to obtain a saddle point for the underlying Lagrangian of the approximating pair and use this as an approximation of a saddle point for the Lagrangian of the actual pair, which in turn furnishes a minimizer of the actual problem; see the saddle point theorem 5.40.

By 5.57, the Lagrangians of the actual and approximating problems are

$$l(x, y) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + \langle b - Ax, y\rangle - \tfrac{1}{2}\langle y, By\rangle - \iota_Y(y)$$
$$l^\nu(x, y) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + \langle b - Ax, y\rangle - \tfrac{1}{2}\langle y, By\rangle - \iota_{Y^\nu}(y).$$

Suppose that $(x^\nu, y^\nu)$ is a saddle point of $l^\nu$. Is it one for $l$ as well? Since $Y^\nu \subset Y$, $l^\nu(x, y) = l(x, y)$ for all $x \in \mathbb{R}^n$ and $y \in Y^\nu$. Moreover,

$$x^\nu \in \operatorname{argmin} l^\nu(\cdot, y^\nu) = \operatorname{argmin} l(\cdot, y^\nu).$$

Thus, we've satisfied one "half" of the saddle point condition for $l$. The other "half," $y^\nu \in \operatorname{argmax} l(x^\nu, \cdot)$, is typically *not* satisfied, however.

Let's quantify by how much $(x^\nu, y^\nu)$ falls short under the assumption that strong duality holds for the actual primal-dual pair, i.e., $\inf \varphi = \sup \psi$. Since $(x^\nu, y^\nu)$ is a saddle point of $l^\nu$, the saddle point theorem 5.40 ensures that

$$x^\nu \in \operatorname{argmin} \varphi^\nu, \qquad y^\nu \in \operatorname{argmax} \psi^\nu, \qquad \varphi^\nu(x^\nu) = \psi^\nu(y^\nu) = l^\nu(x^\nu, y^\nu).$$

The latter quantities equal $l(x^\nu, y^\nu)$ as well. Since $\psi(y) \geq \psi^\nu(y)$ for all $y$,

$$\inf \varphi = \sup \psi \geq \psi^\nu(y^\nu) = l(x^\nu, y^\nu).$$

An upper bound on the minimum value of the actual problem follows by

$$\inf \varphi \le \varphi(x^\nu) = \sup\, l(x^\nu, \cdot\,) = l(x^\nu, \bar{y}^{\nu+1}) \quad \text{for} \;\; \bar{y}^{\nu+1} \in \operatorname{argmax} l(x^\nu, \cdot\,),$$

where the first equality is a consequence of 5.24. Hence, the optimality gaps of $x^\nu$ and $y^\nu$ can be quantified as

$$\varphi(x^\nu) - \inf \varphi \le \varepsilon^\nu, \quad \sup \psi - \psi(y^\nu) \le \varepsilon^\nu, \quad \text{where} \;\; \varepsilon^\nu = l(x^\nu, \bar{y}^{\nu+1}) - l(x^\nu, y^\nu).$$

These calculations require maximization over the high-dimensional $Y$ to obtain $\bar{y}^{\nu+1}$, which seems to bring us back to the original concern about a large $m$. However, now we maximize the Lagrangian $l(x^\nu, \cdot\,)$ and not $\psi$. This makes a big difference: The former is separable, but the latter isn't. Specifically,

$$\begin{aligned}
&(\bar{y}_1^{\nu+1}, \dots, \bar{y}_m^{\nu+1}) \in \operatorname{argmax} l(x^\nu, \cdot\,) = \operatorname{argmax}_{y \in Y} \big\{\langle b - Ax^\nu, y\rangle - \tfrac{1}{2}\langle y, By\rangle\big\} \\
&\iff \bar{y}_i^{\nu+1} \in \operatorname{argmax}_{y \in Y_i} \Big\{\big(b_i - \langle A_i, x^\nu\rangle\big) y_i - \tfrac{1}{2}\beta_i y_i^2\Big\}, \;\; i = 1, \dots, m,
\end{aligned} \tag{5.22}$$

where $A_i$ is the $i$th row of $A$. Thus, the formidable optimization of $y$ is reduced to many trivial problems.

If $\varepsilon^\nu$ isn't sufficiently small, we've in the process identified a point $\bar{y}^{\nu+1} \notin Y^\nu$ that at least for $x^\nu$ is most sorely missed in the approximating $Y^\nu$. Naturally, we amend the approximation by setting $Y^{\nu+1} = \operatorname{con}\{\bar{y}^1, \dots, \bar{y}^\nu, \bar{y}^{\nu+1}\}$ and the process is repeated.

**Lagrangian Finite-Generation Method.**

Data. $\bar{y}^1 \in Y$, $\varepsilon \in [0, \infty)$.
Step 0. Set $Y^1 = \{\bar{y}^1\}$ and $\nu = 1$.
Step 1. Compute a saddle point $(x^\nu, y^\nu)$ of $l^\nu$.
Step 2. Compute $\bar{y}^{\nu+1} = (\bar{y}_1^{\nu+1}, \dots, \bar{y}_m^{\nu+1})$ using (5.22).
Step 3. If $l(x^\nu, \bar{y}^{\nu+1}) - l(x^\nu, y^\nu) \le \varepsilon$, then stop.
Step 4. Set

$$Y^{\nu+1} = \operatorname{con}\big\{\bar{y}^1, \dots, \bar{y}^{\nu+1}\big\}.$$

Step 5. Replace $\nu$ by $\nu + 1$ and go to Step 1.

Under the assumptions that there always exist a saddle point $(x^\nu, y^\nu)$ in Step 1 and a maximizer $\bar{y}^{\nu+1}$ in Step 2 and that strong duality holds for the actual primal-dual pair, the algorithm is well defined and terminates after a finite number of iterations with a solution satisfying any positive tolerance $\varepsilon$; see [103] for a proof and further details. It's clear what it takes to have a maximizer in Step 2: Each $i$ must have either $\beta_i > 0$ or $Y_i$ bounded. Thus, let's focus the discussion on Step 1.

Finding a saddle point of $l^\nu$ involves the high-dimensional vector $y$, which is problematic. However, a reformulation leveraging the structure of $Y^\nu$ overcomes this difficulty. Every $y \in Y^\nu$ can be expressed as

$$y = \sum\nolimits_{k=1}^{\nu} \lambda_k \bar{y}^k \;\; \text{for some} \;\; (\lambda_1, \dots, \lambda_\nu) \in \mathbb{R}^\nu \;\; \text{with} \;\; \sum\nolimits_{k=1}^{\nu} \lambda_k = 1, \;\; \lambda_k \ge 0 \;\; \forall k;$$

see the discussion above 2.62. Concisely, $y = \bar{Y}_\nu \lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_\nu)$ and $\bar{Y}_\nu$ is the $m \times \nu$-matrix with $\{\bar{y}^k, k = 1, \ldots, \nu\}$ as columns. Thus,

$$\langle b, y\rangle = \langle b^\nu, \lambda\rangle, \qquad \text{where } b^\nu = \bar{Y}_\nu^\top b$$
$$\langle y, By\rangle = \langle \lambda, B_\nu \lambda\rangle, \qquad \text{where } B_\nu = \bar{Y}_\nu^\top B \bar{Y}_\nu$$
$$A^\top y = A_\nu^\top \lambda, \qquad \text{where } A_\nu^\top = A^\top \bar{Y}_\nu.$$

Implementing this change of variables, we obtain a reformulation of (5.21):

$$\underset{\lambda \in \Lambda}{\text{maximize}} \ \langle b^\nu, \lambda\rangle - \tfrac{1}{2}\langle \lambda, B_\nu \lambda\rangle - h_{X,Q}(A_\nu^\top \lambda - c), \tag{5.23}$$

where

$$\Lambda = \Big\{\lambda \in \mathbb{R}^\nu \;\Big|\; \sum\nolimits_{k=1}^{\nu} \lambda_k = 1, \;\; \lambda \geq 0\Big\}.$$

By 5.57, the reformulation can be viewed as a dual problem of

$$\underset{x \in X}{\text{minimize}} \ \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + h_{\Lambda, B_\nu}(b^\nu - A_\nu x) \tag{5.24}$$

and this primal-dual pair corresponds to the Lagrangian

$$\hat{l}^\nu(x, \lambda) = \iota_X(x) + \langle c, x\rangle + \tfrac{1}{2}\langle x, Qx\rangle + \langle b^\nu - A_\nu x, \lambda\rangle - \tfrac{1}{2}\langle \lambda, B_\nu \lambda\rangle - \iota_\Lambda(\lambda).$$

Thus, we can carry out Step 1 by obtaining a saddle point $(x^\nu, \lambda^\nu)$ of $\hat{l}^\nu$ and then setting $y^\nu = \bar{Y}_\nu \lambda^\nu$. Since $\nu$ is much smaller than $m$, it's typically easier to find a saddle point of $\hat{l}^\nu$ than of $l^\nu$.

The problem (5.24) is feasible because $X$ is nonempty and $h_{\Lambda, B_\nu}$ is real-valued. If $X$ is bounded, then (5.24) isn't unbounded either, which implies via 5.57 that both (5.23) and (5.24) have solutions and the pair satisfies strong duality. Consequently, there's a saddle point of $\hat{l}^\nu$, and then also for $l^\nu$, furnished by the pair of solutions; cf. the saddle point theorem 5.40. Step 1 therefore amounts to solving (5.23) and (5.24), for example leveraging 5.61. Typically, the solution of one of the problems also furnishes a multiplier vector that's a solution of the other problem, but the ease by which the multipliers can be accessed depends on the algorithm (software) used. In any case, (5.23) and (5.24) share by 5.36 the optimality condition

$$-c - Qx + A_\nu^\top \lambda \in N_X(x) \qquad b^\nu - A_\nu x - B_\nu \lambda \in N_\Lambda(\lambda).$$

Thus, when $x^\nu$ solves (5.24), then these relations with $x = x^\nu$ furnish $\lambda^\nu$, which is computable by solving a quadratic optimization problem. Similarly, the relations recover $x^\nu$ from a solution $\lambda^\nu$ of (5.23).

**Exercise 5.62** (Lagrangian finite-generation method). Implement the algorithm and solve the instance with

$$X = \{x \in \mathbb{R}^6 \mid \langle a, x\rangle \le 24,\ 0 \le x \le s\}$$

$$a = (1, 1.5, 0.5, 2, 1, 1),\ s = (4, 20, 4, 10, 3, 2),\ c = (1, 2, 1, 4, 1, 3)$$

$$B = \mathrm{diag}(0.5/60, 0.4/75, 0.3/80, 0.005/120),\quad Q = \mathrm{diag}(1, 2, 1, 0.25, 1, 0.5)$$

$$Y = [0, 60] \times [0, 75] \times [0, 80] \times [0, 120]$$

$$A = \begin{bmatrix} 0.29 & 0.4 & 0 & 0.11 & 0 & 0 \\ 0.1 & 0.0975 & 0.315 & 0.51 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.4875 & 0.1925 \\ 0 & 0 & 0 & 0 & 0.3267 & 0.4833 \end{bmatrix}, \qquad b = \begin{bmatrix} 3.825 \\ 0.9667 \\ 3.1 \\ 1.5 \end{bmatrix}.$$

Here, $\mathrm{diag}(v)$ is the diagonal matrix with the vector $v$ as its diagonal elements.

**Guide.** The minimizer is $x^\star = (4, 4.25, 0, 4.31, 3, 2)$, with $y^\star = (58.87, 0, 80, 0)$. □

**Example 5.63** (control of water pollution). In the model described in §3.I for controlling water pollution, let's assume that the direct cost associated with decision $x$ is

$$f_0(x) = \langle c, x\rangle + \tfrac{1}{2}\sum\nolimits_{j=1}^{n} q_j x_j^2$$

and this leads to

$$\underset{x\in X}{\text{minimize}}\ \langle c, x\rangle + \tfrac{1}{2}\sum\nolimits_{j=1}^{n} q_j x_j^2 + \mathbb{E}\Big[\sum\nolimits_{i=1}^{m_2} h_{[0,\tau_i],\beta_i}\big(\boldsymbol{d}_i - \langle \boldsymbol{T}_{i,x}\rangle\big)\Big],$$

where $X = \{x \in \mathbb{R}^n \mid \langle a, x\rangle \le \alpha,\ 0 \le x \le s\}$ and the rest of the data is explained in §3.I. In contrast to that section, we here use the notation for monitoring functions (see 5.58 specifically) and also generalize slightly by having $m_2$ terms in the second sum; $m_2 = 4$ in §3.I. If the probability distribution of $(\boldsymbol{d}, \boldsymbol{T}_1, \dots, \boldsymbol{T}_{m_2})$ is finite, then the Lagrangian finite-generation method applies.

**Detail.** For each $i = 1, \dots, m_2$, suppose that $p_i^k > 0$ is the probability that $(\boldsymbol{d}_i, \boldsymbol{T}_i)$ takes the value $(d_i^k, T_{i^k})$, $k = 1, \dots, \nu_i$. Then, $\sum_{k=1}^{\nu_i} p_i^k = 1$ and

$$\begin{aligned} \mathbb{E}\Big[\sum\nolimits_{i=1}^{m_2} h_{[0,\tau_i],\beta_i}\big(\boldsymbol{d}_i - \langle \boldsymbol{T}_{i,x}\rangle\big)\Big] &= \sum\nolimits_{i=1}^{m_2}\sum\nolimits_{k=1}^{\nu_i} p_i^k h_{[0,\tau_i],\beta_i}\big(d_i^k - \langle T_i^k, x\rangle\big) \\ &= \sum\nolimits_{i=1}^{m_2}\sum\nolimits_{k=1}^{\nu_i} h_{[0,\tau_i^k],\beta_i^k}\big(d_i^k - \langle T_i^k, x\rangle\big), \end{aligned}$$

where $\tau_i^k = p_i^k \tau_i$ and $\beta_i^k = \beta_i / p_i^k$. Thus, the term is indeed of the form $h_{Y,B}$, where $Y$ is a box and $B$ is a diagonal matrix. The dimension of $Y$ is $\sum_{i=1}^{m_2} \nu_i$, with $\nu_i$ being the number of possible outcomes of $(\boldsymbol{d}_i, \boldsymbol{T}_i)$, and thus tends to be large. This is exactly a situation in which the Lagrangian finite-generation method can be efficient. □

**Exercise 5.64** (pollution control). Using the Lagrangian finite-generation method, solve the problem in 5.63 with the data $X$, $c$, $q_1, \dots, q_6$ (diagonal elements of $Q$) and $\beta_1, \dots, \beta_4$ (diagonal elements of $B$) as in 5.62. Moreover $(\tau_1, \dots, \tau_4) = (60, 75, 80, 120)$ and the probability distribution of $(\boldsymbol{d}_i, \boldsymbol{T}_i)$ is listed below.

| $d_1^k$ | $T_1^k$ | $p_1^k$ |
|---|---|---|
| 3.5 | (0.26 0.35 0 0.08 0 0) | 1/12 |
| 3.8 | (0.26 0.35 0 0.08 0 0) | 1/12 |
| 4 | (0.26 0.35 0 0.08 0 0) | 1/6 |
| 3.5 | (0.28 0.40 0 0.10 0 0) | 1/24 |
| 3.8 | (0.28 0.40 0 0.10 0 0) | 1/24 |
| 4 | (0.28 0.40 0 0.10 0 0) | 1/12 |
| 3.5 | (0.30 0.40 0 0.10 0 0) | 1/24 |
| 3.8 | (0.30 0.40 0 0.10 0 0) | 1/24 |
| 4 | (0.30 0.40 0 0.10 0 0) | 1/12 |
| 3.5 | (0.32 0.45 0 0.15 0 0) | 1/12 |
| 3.8 | (0.32 0.45 0 0.15 0 0) | 1/12 |
| 4 | (0.32 0.45 0 0.15 0 0) | 1/6 |

| $d_2^k$ | $T_2^k$ | $p_2^k$ |
|---|---|---|
| 0.8 | (0.10 0.05 0.27 0.48 0 0) | 1/8 |
| 1 | (0.10 0.05 0.27 0.48 0 0) | 1/12 |
| 1.4 | (0.10 0.05 0.27 0.48 0 0) | 1/24 |
| 0.8 | (0.10 0.10 0.30 0.52 0 0) | 1/8 |
| 1 | (0.10 0.10 0.30 0.52 0 0) | 1/12 |
| 1.4 | (0.10 0.10 0.30 0.52 0 0) | 1/24 |
| 0.8 | (0.10 0.12 0.33 0.52 0 0) | 1/8 |
| 1 | (0.10 0.12 0.33 0.52 0 0) | 1/12 |
| 1.4 | (0.10 0.12 0.33 0.52 0 0) | 1/24 |
| 0.8 | (0.10 0.12 0.36 0.52 0 0) | 1/8 |
| 1 | (0.10 0.12 0.36 0.52 0 0) | 1/12 |
| 1.4 | (0.10 0.12 0.36 0.52 0 0) | 1/24 |

| $d_3^k$ | $T_3^k$ | $p_3^k$ |
|---|---|---|
| 3 | (0 0 0 0.10 0.40 0.15) | 1/8 |
| 3.2 | (0 0 0 0.10 0.40 0.15) | 1/8 |
| 3 | (0 0 0 0.20 0.50 0.20) | 1/4 |
| 3.2 | (0 0 0 0.20 0.50 0.20) | 1/4 |
| 3 | (0 0 0 0.30 0.55 0.22) | 1/8 |
| 3.2 | (0 0 0 0.30 0.55 0.22) | 1/8 |

| $d_4^k$ | $T_4^k$ | $p_4^k$ |
|---|---|---|
| 0.5 | (0 0 0 0 0.28 0.40) | 1/12 |
| 1.5 | (0 0 0 0 0.28 0.40) | 1/6 |
| 2.5 | (0 0 0 0 0.28 0.40) | 1/12 |
| 0.5 | (0 0 0 0 0.30 0.50) | 1/12 |
| 1.5 | (0 0 0 0 0.30 0.50) | 1/6 |
| 2.5 | (0 0 0 0 0.30 0.50) | 1/12 |
| 0.5 | (0 0 0 0 0.40 0.55) | 1/12 |
| 1.5 | (0 0 0 0 0.40 0.55) | 1/6 |
| 2.5 | (0 0 0 0 0.40 0.55) | 1/12 |

**Guide.** The minimizer is $x^\star = (4, 4.10, 0, 4.43, 3, 2)$. $\square$