Explanation function

DeformationalModes_1dom.m

The idea behind this function is as follows, see

Given $\boldsymbol{A}_d^e$ (matrix of displacement snapshots, see Eq. 21) and $\widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^e$ (self-equilibrated modes, computed in expression 40), find $\widetilde{\boldsymbol{\Phi}}^e$ such that $span(\widetilde{\boldsymbol{\Phi}}^e) \subseteq span(\widetilde{\boldsymbol{\mathcal{P}}}^e \boldsymbol{A}_d^e)$) and $\boldsymbol{H}^e = \widetilde{\boldsymbol{\Psi}}^{e^T} \widetilde{\boldsymbol{\Phi}}^e$ is invertible.

1. Compute the deformational component of $\boldsymbol{A}_d^e$:

$$\widetilde{\boldsymbol{A}}_d^e = \widetilde{\boldsymbol{\mathcal{S}}}^e \boldsymbol{A}_d^e \tag{43}$$

2. Determine the SVD of the row block of $\widetilde{\boldsymbol{A}}_d^e$ corresponding to the interfaces (DOFs $\mathbf{f}^e$)

$$[\boldsymbol{U}_a, \boldsymbol{S}_a, \boldsymbol{V}_a] = \mathtt{SVD}(\widetilde{\boldsymbol{A}}_d^e(\mathbf{f}^e, :), 0)$$

3. Obtain the SVD of the coefficients of the projection of $\widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^e$ onto $span(\boldsymbol{U}_a)$:

$$[\boldsymbol{U}_u, \boldsymbol{S}_u, \boldsymbol{V}_u] = \mathtt{SVD}(\boldsymbol{U}_a^T \widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^e, 0) \tag{44}$$

4. If $\mathtt{ncol}(\boldsymbol{U}_u) = p^e = \mathtt{ncol}(\widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^e)$ (here $\mathtt{ncol}(\bullet)$ denotes number of columns), then go to step 5; otherwise, make $\widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^e \leftarrow \widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^e(:, 1 : p^e - 1)$ and return to step 3.
5. Compute

$$\boldsymbol{B} = \widetilde{\boldsymbol{A}}_d^e(\boldsymbol{S}_a \boldsymbol{V}_a) \boldsymbol{U}_u \tag{45}$$

6. Orthogonalize $\boldsymbol{B}$ with respect to $\boldsymbol{M}^e$ by applying the weighted SVD (see Algorithm 1):

$$[\widetilde{\boldsymbol{\Phi}}^e, \bullet, \bullet] = \mathtt{WSVD}(\boldsymbol{B}, \boldsymbol{M}^e, 0) \tag{46}$$

Box 5.1: Computation of deformational modes $\widetilde{\boldsymbol{\Phi}}^e$ (for a given subdomain Ω^e, $e = 1, 2 \ldots M$)

5.3.2. Detailed explanation

- The idea here is to write $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{A}\boldsymbol{c}$, and then propose a criterion for determining $\boldsymbol{c}$.
- One option would be to simply apply the truncated SVD:

$$\boldsymbol{A} \approx \widetilde{\boldsymbol{\Phi}} \boldsymbol{S} \boldsymbol{V}^T \tag{47}$$

In this case, notice that

$$\boldsymbol{c} = \boldsymbol{V} \boldsymbol{S}^{-1} \tag{48}$$

- However, this may include modes where $\widetilde{\boldsymbol{\Phi}}_{\mathsf{f}} \approx \boldsymbol{0}$.
- To avoid the above singular case, we can decompose, rather than the original matrix, the matrix defined only at the boundary DOFs

$$\boldsymbol{A}(f, :) \approx \boldsymbol{U}_a \boldsymbol{S}_a \boldsymbol{V}_a^T \tag{49}$$

In this case, the desired set of coefficients would adopt the form

$$\boldsymbol{c} = \boldsymbol{V}_a \boldsymbol{S}_a^{-1} \tag{50}$$

Notice that, in doing so:

$$\widetilde{\boldsymbol{\Phi}} = \boldsymbol{A}\boldsymbol{c} = \boldsymbol{A} \boldsymbol{V}_a \boldsymbol{S}_a^{-1} \tag{51}$$

Thus

$$\widetilde{\boldsymbol{\Phi}}_{\mathsf{f}} = \boldsymbol{A}(f, :)\boldsymbol{c} = \boldsymbol{A}(f, :) \boldsymbol{V}_a \boldsymbol{S}_a^{-1} = \boldsymbol{U}_a \boldsymbol{S}_a \boldsymbol{V}_a^T \boldsymbol{V}_a \boldsymbol{S}_a^{-1} = \boldsymbol{U}_a \tag{52}$$

Thus, the deformational basis matrix at the boundary coincides with the left singular matrix of $\boldsymbol{A}(f, :)$.

- Yet in proceeding this way, we do not ensure one of the conditions upon which the proposed approach is founded, namely, that $\boldsymbol{H} = \widetilde{\boldsymbol{\Psi}}_{\mathsf{f}}^T \widetilde{\boldsymbol{\Phi}}_{\mathsf{f}}$ must be square. To ensure this condition, we have to seek the subspace of the column space of $\boldsymbol{U}_a$ that "does work", which is simply given by

$$\widetilde{\mathbf{\Phi}}_{\mathbf{f}} = \boldsymbol{U}_a(\boldsymbol{U}_a^T \widetilde{\mathbf{\Psi}}_{\mathbf{f}}) \tag{53}$$

The situation in which $(\boldsymbol{U}_a^T \widetilde{\mathbf{\Psi}}_{\mathbf{f}})$ is not full rank deserves special attention; it implies that there are reaction modes in $\widetilde{\mathbf{\Psi}}_{\mathbf{f}}$ which does not work at all. To detect such cases, we calculate the SVD of $\boldsymbol{U}_a^T \widetilde{\mathbf{\Psi}}_{\mathbf{f}}$, and proceed as indicated in steps 3 and 4. This leads to

$$\widetilde{\mathbf{\Phi}}_{\mathbf{f}} = \boldsymbol{U}_a \boldsymbol{U}_u \tag{54}$$

and therefore

$$\boxed{\widetilde{\mathbf{\Phi}} = \boldsymbol{A}\boldsymbol{V}_a \boldsymbol{S}_a^{-1} \boldsymbol{U}_u} \tag{55}$$

Notice that

$$\widetilde{\mathbf{\Phi}}_{\mathbf{f}} = \boldsymbol{A}(f,:)\boldsymbol{V}_a \boldsymbol{S}_a^{-1} \boldsymbol{U}_u = \boldsymbol{U}_a \boldsymbol{S}_a \boldsymbol{V}_a^T \boldsymbol{V}_a \boldsymbol{S}_a^{-1} \boldsymbol{U}_u = \boldsymbol{U}_a \boldsymbol{U}_u \tag{56}$$

as stated in Eq.(54).

□ [**JAHO:** *NOTICE THAT THERE IS A MISTAKE IN THE ORIGINAL DOCUMENT (THE TERM* $\boldsymbol{S}_a \boldsymbol{V}_a$ *in EQUATION Eq.(45)) SHOULD BE REPLACED BY* $\boldsymbol{V}_a \boldsymbol{S}_a^{-1}$. *IT SHOULD BE NOTED, HOWEVER, THAT THIS DOES NOT AFFECT THE SUBSPACE SPANNED BY THE FINAL SET OF MODES.*] □