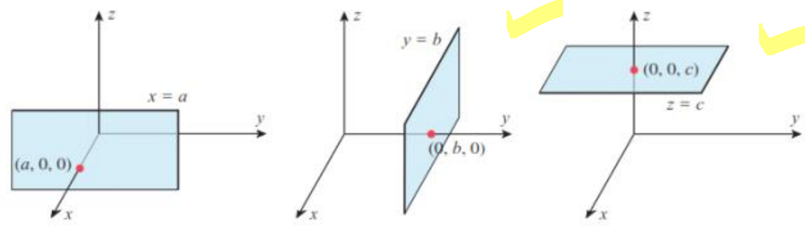
11.6 PLANES IN 3-SPACE

PLANES PARALLEL TO THE COORDINATE PLANES



▲ Figure 11.6.1

Equation of Plane

two normal vector or normal & point

$\vec{n} = \langle a, b, c \rangle$

normal vector $\leftarrow$ $\overrightarrow{PP_0} = \langle x - x_0, y - y_0, z - z_0 \rangle$

line
parallel
vector

$\vec{n} \cdot \overrightarrow{PP_0} = 0$

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$

$ax + by + cz - ax_0 - by_0 - cz_0 = 0$

$ax + by + cz + \underbrace{(-ax_0 - by_0 - cz_0)}_{d} = 0$

equation $\leftarrow$ $\boxed{ax + by + cz + d = 0}$

equation of plane in 3-space

$$\boxed{ax + by + cz + d = 0}$$

▸ **Example 1** Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5\rangle$.

$\vec{n} = \langle 4, 2, -5 \rangle$ ($\langle a, b, c\rangle$)

$P = (3, -1, 7)$ ((x_0, y_0, z_0))

$ax + by + cz - ax_0 - by_0 - cz_0 = 0$

$4x + 2y - 5z - 12 + 2 + 35 = 0$

$$\boxed{4x + 2y - 5z + 25 = 0}$$

$-x + 7y + 6z - 1 + 7 - 12 =$

$-x + 7y + 6z - 13 + 7 = 0$

$-x + 7y + 6z - 6 = 0$

3–6 Find an equation of the plane that passes through the point P and has the vector $\mathbf{n}$ as a normal.

3. $P(2, 6, 1)$; $\mathbf{n} = \langle 1, 4, 2\rangle$ ((x_0, y_0, z_0), $\langle a, b, c\rangle$)
4. $P(-1, -1, 2)$; $\mathbf{n} = \langle -1, 7, 6\rangle$
5. $P(1, 0, 0)$; $\mathbf{n} = \langle 0, 0, 1\rangle$
6. $P(0, 0, 0)$; $\mathbf{n} = \langle 2, -3, -4\rangle$

Q.6

$2x - 3y - 4z = 0$

Q.3

$x + 4y + 2z - 2 - 24 - 2 = 0$

$x + 4y + 2z - 28 = 0$

▸ **Example 3** Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

$\vec{P_1P_2}$

$\vec{P_1P_3}$

Solution. Since the points P_1, P_2, and P_3 lie in the plane, the vectors $\overrightarrow{P_1P_2} = \langle 1, 1, 2\rangle$ and $\overrightarrow{P_1P_3} = \langle 2, -3, 3\rangle$ are parallel to the plane. Therefore,

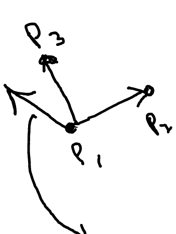
$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

is normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. By using this normal and the point $P_1(1, 2, -1)$ in the plane, we obtain the point-normal form

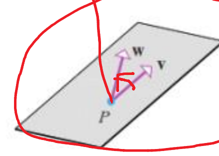
$$9(x - 1) + (y - 2) - 5(z + 1) = 0$$

which can be rewritten as

$$9x + y - 5z - 16 = 0 \;◂$$

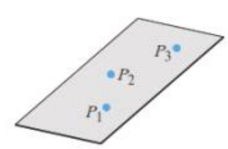


$\vec{P_1P_3} \times \vec{P_1P_2} =$



There is a unique plane through P that is parallel to both $\mathbf{v}$ and $\mathbf{w}$.

Figure 11.6.5



There is a unique plane through three noncollinear points.

Figure 11.6.6

plane through three noncollinear points.

Figure 11.6.6

$\vec{n} = \vec{P_1P_2} \times \vec{P_1P_3} =$

11–12 Find an equation of the plane that passes through the given points.

11. $(-2, 1, 1)$, $(0, 2, 3)$, and $(1, 0, -1)$

12. $(3, 2, 1)$, $(2, 1, -1)$, and $(-1, 3, 2)$

Q 11

$P_1(-2,1,1)$

$P_2(0,2,3)$

$P_3(1,0,-1)$

$\vec{P_1P_2} = \langle 2, 1, 2 \rangle$

$\vec{P_1P_3} = \langle 3, -1, -2 \rangle$

$$\vec{P_1P_2} \times \vec{P_1P_3} = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix}$$

$= i(-2+2) - j(-\cancel{4}-6) + k(-2-3)$

$= 0 + 10j - 5k$

$\vec{n} = \langle 0, 10, -5 \rangle$

$P_1(-2,1,1)$

$ax + by + cz - ax_0 - by_0 - cz_0 = 0$

$+10y - 5z - 10 + 5 = 0$

$\boxed{10y - 5z - 5 = 0}$

▶ Example 2 Determine whether the planes

$$3x - 4y + 5z = 0 \quad \text{and} \quad -6x + 8y - 10z - 4 = 0$$

are parallel.

Solution. It is clear geometrically that two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is

$$\mathbf{n}_1 = \langle 3, -4, 5 \rangle$$

and a normal to the second plane is

$$\mathbf{n}_2 = \langle -6, 8, -10 \rangle$$

Since $\mathbf{n}_2$ is a scalar multiple of $\mathbf{n}_1$, the normals are parallel, and hence so are the planes. ◀

$\rightarrow \vec{n}_1 = \langle 3, -4, 5 \rangle$

$\rightarrow n_2 = \langle -6, 8, -10 \rangle$

$\vec{n}_2 = -2\vec{n}_1$

normal parallel = Plane parallel

▶ Example 6 Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 6 \quad \text{and} \quad 6x + 2y - 3z = 4$$

Example 6 Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 6 \quad \text{and} \quad 6x + 2y - 3z = 4$$

Solution. The given equations yield the normals $\mathbf{n}_1 = \langle 2, -4, 4\rangle$ and $\mathbf{n}_2 = \langle 6, 2, -3\rangle$. Thus, Formula (9) yields

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21}$$

from which we obtain

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$$

11–12 Find an equation of the plane that passes through the given points.

11. $(-2, 1, 1)$, $(0, 2, 3)$, and $(1, 0, -1)$

12. $(3, 2, 1)$, $(2, 1, -1)$, and $(-1, 3, 2)$

Q.12

$P_1(3,2,1)$
$P_2(2,1,-1)$
$P_3(-1,3,2)$

$\overrightarrow{P_1P_2} = \langle -1, -1, -2 \rangle$

$\overrightarrow{P_1P_3} = \langle -4, 1, 1 \rangle$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ -1 & -1 & -2 \\ -4 & 1 & 1 \end{vmatrix}$$

$$= i(-1+2) - j(-1-8) + k(-1-4)$$

$\vec{n} = i + 9j - 5k$

$\vec{n} = \langle 1, 9, -5 \rangle$

$\langle a, b, c \rangle$

$P_1(3,2,1)$
(x_0, y_0, z_0)

$x + 9y - 5z - 3 - 18 + 5 = 0$

$$\boxed{x + 9y - 5z - 16 = 0}$$

DISTANCE PROBLEMS INVOLVING PLANES

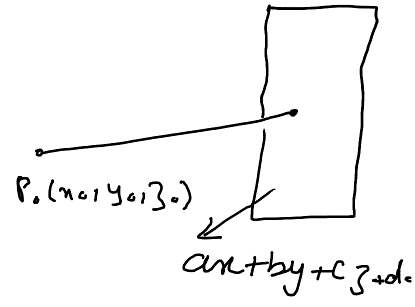
Next we will consider three basic distance problems in 3-space:

- Find the distance between a point and a plane. → CASE I
- Find the distance between two parallel planes. → CASE II
- Find the distance between two skew lines. → CASE III

11.6.2 THEOREM *The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is*

$d = -ax_0 - by_0 - cz_0$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{10}$$



Case I

Example 8 Find the distance D between the point $(1, -4, -3)$ and the plane

$$\boxed{2x - 3y + 6z = -1} \quad (x_0, y_0, z_0)$$

Solution. Formula (10) requires the plane be rewritten in the form $ax + by + cz + d = 0$. Thus, we rewrite the equation of the given plane as

$$\underbrace{2x - 3y + 6z + 1}_{} = 0 \qquad \overset{a}{2x}\ \overset{b}{-3y}\ \overset{c}{+6z}\ \overset{d}{+1}$$

from which we obtain $a = 2$, $b = -3$, $c = 6$, and $d = 1$. Substituting these values and the coordinates of the given point in (10), we obtain

$$D = \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

→ point to plane

Case II

Example 9 The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $\langle 1, 2, -2\rangle$ and $\langle 2, 4, -4\rangle$, are parallel vectors. Find the distance between these planes.

Solution. To find the distance D between the planes, we can select an *arbitrary* point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (10), the distance from P_0 to the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|(2)(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$

$x - 4 = 9t$

$y - 5 = 6t$

$z - 4 = 6t$

two parallel plane

$$d = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Plane 1 → $\underline{x + 2y - 2z = 3}$

$x + 2y - 2z - 3 = 0$
a b c d_1

Plane2 → $\underline{x + 2y - 2z = 7/2}$

$x + 2y - 2z - 7/2 = 0$
a b c d_2

$$d = \left|\frac{-3 + \frac{7}{2}}{\sqrt{(1)^2 + (2)^2 + (-2)^2}}\right|$$

$$\boxed{d = 1/6}$$

Case III distance b/w two Skew lines

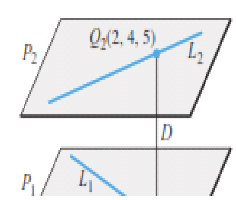
~~Example 10~~ It was shown in Example 3 of Section 11.5 that the lines

$$L_1: x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2: x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

are skew. Find the distance between them.

Solution. Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively (Figure 11.6.10). To find the distance D between L_1 and L_2, we will calculate the distance from a point in P_1 to the plane P_2. Since L_1 lies in plane P_1, we can find a point in P_1 by finding a point on the line L_1; we can do this by substituting any convenient value of t in the parametric equations of L_1. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.

The next step is to find an equation for the plane P_2. For this purpose, observe that the



from a point in P_1 to the plane P_2. Since L_1 lies in plane P_1, we can find a point in P_1 by finding a point on the line L_1; we can do this by substituting any convenient value of t in the parametric equations of L_1. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.

The next step is to find an equation for the plane P_2. For this purpose, observe that the vector $\mathbf{u}_1 = \langle 4, -4, 5\rangle$ is parallel to line L_1, and therefore also parallel to planes P_1 and P_2. Similarly, $\mathbf{u}_2 = \langle 8, -3, 1\rangle$ is parallel to L_2 and hence parallel to P_1 and P_2. Therefore, the cross product

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

is normal to both P_1 and P_2. Using this normal and the point $Q_2(2, 4, 5)$ found by setting $t = 0$ in the equations of L_2, we obtain an equation for P_2:

$$11(x-2) + 36(y-4) + 20(z-5) = 0$$

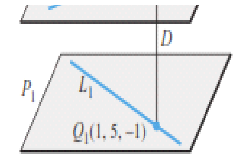
or

$$11x + 36y + 20z - 266 = 0$$

The distance between $Q_1(1, 5, -1)$ and this plane is

$$D = \frac{|(11)(1) + (36)(5) + (20)(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}$$

which is also the distance between L_1 and L_2. ◄



▲ Figure 11.6.10

$Q_1(1, 5, -1)$ | $Q_2(2, 4, 5)$

$\vec{U}_1 = \langle 4, -4, 5\rangle$ | $\vec{U}_2 = \langle 8, -3, 1\rangle$

Step I Cross product of U_1 & U_2

$$U_1 \times U_2 = \begin{vmatrix} i & j & k \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix}$$

$$= i(-4+15) - j(4-40) + k(-12+32)$$

$$\vec{n} = \vec{U}_1 \times \vec{U}_2 = +11i + 36j + 20k$$

Step II equation of plane

$$\vec{n} = \langle 11, 36, 20\rangle \qquad Q_1(1, 5, -1)$$

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$

$$11x + 36y + 20z - 11 - 180 + 20 = 0$$

$$\underset{a}{11}x + \underset{b}{36}y + \underset{c}{20}z \underset{d}{-171} = 0$$

Step III distance formula ($Q_2(2, 4, 5)$
(x_0, y_0, z_0))

$$d = \left|\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left| \frac{(11)(2)+(36)(4)+(20)(5)-171}{\sqrt{11^2+36^2+20^2}} \right|$$

$$d = \underline{2.22 \text{ unit}}$$

47–48 Find the distance between the given skew lines.

47. $x = 1 + 7t,\ y = 3 + t,\ z = 5 - 3t$
$x = 4 - t,\ y = 6,\ z = 7 + 2t$

48. $x = 3 - t,\ y = 4 + 4t,\ z = 1 + 2t$
$x = t,\ y = 3,\ z = 2t$

Q. $Q_1(1, 3, 5)$ $\quad$ $Q_2(4, 6, 7)$

$\vec{U}_1 = \langle 7, 1, -3 \rangle$ $\quad$ $\vec{U}_2 = \langle -1, 0, 2 \rangle$

<u>Step I</u>

$$\vec{U}_1 \times \vec{U}_2 = \begin{vmatrix} i & j & k \\ 7 & 1 & -3 \\ -1 & 0 & 2 \end{vmatrix}$$

$$= i(2-0) - j(14-3) + k(0+1)$$

$$\vec{n} = \vec{U}_1 \times \vec{U}_2 = 2i - 11j + k$$

<u>Step II</u> equation plane $Q_1(1, 3, 5)$

$\vec{n} = \langle 2, -11, 1 \rangle$

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$

$$2x - 11y + z - 2 + 33 - 5 = 0$$

$$\boxed{\begin{matrix} 2x & -11y & +z & +26 & = 0 \\ a & b & c & d & \end{matrix}}$$

<u>Step III</u> $Q_2(4, 6, 7)$

$$d = \left| \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}} \right| = \frac{(2)(4)+(-11)(6)+(7)(1)+26}{\sqrt{2^2+(-11)^2+(1)^2}}$$

$$= \underline{\qquad}$$

45–46 Find the distance between the given parallel planes.

45. $-2x + y + z = 0$
$6x - 3y - 3z - 5 = 0$

46. $x + y + z = 1$
$x + y + z = -1$

$a = 1$
$b = 1$
$c = 1$
$d_1 = -1$
$d_2 = 1$

$$-2x+y+z=0$$

$$-2x+y+3z-\frac{5}{-3} = 0$$

$$-2x+y+z+\frac{5}{3} = 0$$

$$= \left| \frac{0 - 5/3}{\sqrt{(-2)^2+(1)^2+(1)^2}} \right|$$

$$=$$

$$d = \left| \frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}} \right| \to = \left| \frac{-1-1}{\sqrt{1^2+1^2+1^2}} \right|$$

$$= \frac{2}{\sqrt{3}}$$

$$-2x + y + z = 0$$

$$-2x+y+z=0$$

$$-2x+y+z=0$$
$a \quad b \quad c \quad d_1 = 0$

$$6x-3y-3z-5=0$$

$$-3\left(-2x+y+z+\frac{5}{3}\right)=0$$

$$-2x+y+z+5/3=0$$
$a \quad b \quad c \quad d_2 = 5/3$