Monday, December 21, 2020
2:18 PM

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

exponentials yields

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

let $x = \sinh y = \frac{e^y - e^{-y}}{2}$ $\longrightarrow$ $x = \frac{e^y - e^{-y}}{2}$

$x = \sinh y$

$\boxed{\sinh^{-1} x = y}$

which can be rewritten as

$$e^y - 2x - e^{-y} = 0$$

Multiplying this equation through by e^y we obtain

$$e^{2y} - 2xe^y - 1 = 0$$

and applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $e^y > 0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$e^y = x + \sqrt{x^2+1}$$

Taking natural logarithms yields

$$\boxed{y = \ln(x + \sqrt{x^2+1}) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2+1})}$$

$2x = e^y - e^{-y}$

$\times$ b/s with e^y

$2xe^y = e^{2y} - e^{-y+y}$

$2xe^y = e^{2y} - e^0$

$2xe^y = e^{2y} - 1$

$\rightarrow e^{2y} - 2xe^y - 1 = 0$

$(e^y)^2 - 2x(e^y)' - 1 = 0$

$a = 1, \quad b = -2x, \quad c = -1$

$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$e^y = \frac{+2x \pm \sqrt{4x^2+4}}{2}$

$e^y = x \pm \sqrt{x^2+1}$

Taking ln b/s

$\ln e^y = \ln(x \pm \sqrt{x^2+1})$

$y = \ln(x \pm \sqrt{x^2+1})$

$\sinh^{-1} x = \ln(x \pm \sqrt{x^2+1})$

$\boxed{\sinh^{-1} x = \ln(x + \sqrt{x^2+1})}$

$x = e^y - e^{-y}$ 2

Arc Length $= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Example 4 A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution. From above, the wire forms a catenary curve with equation

$$\boxed{y = a\cosh\left(\frac{x}{a}\right) + c}$$

where the origin is on the ground midway between the poles. Using Formula (4) of Section 6.4 for the length of the catenary, we have

$$\begin{aligned}
100 &= \int_{-45}^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\int_0^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx && \text{By symmetry about the y-axis} \\
&= 2\int_0^{45} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\, dx \\
&= 2\int_0^{45} \cosh\left(\frac{x}{a}\right) dx && \text{By (1) and the fact that } \cosh x > 0 \\
&= 2a\sinh\left(\frac{x}{a}\right)\Big]_0^{45} = 2a\sinh\left(\frac{45}{a}\right)
\end{aligned}$$

Arc length

Arc length

$\int_{-25}^{25} f(x)$

$2\int_0^{25} f(x)$

Using a calculating utility's numeric solver to solve

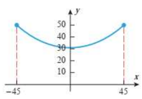
$$100 = 2a\sinh\left(\frac{45}{a}\right)$$

$\longrightarrow 100 - 2a\sinh\left(\frac{45}{a}\right) = 0$

for a gives $a \approx 56.01$. Then

$$50 = y(45) = 56.01\cosh\left(\frac{45}{56.01}\right) + c \approx 75.08 + c$$

so $c \approx -25.08$. Thus, the middle of the wire is $y(0) \approx 56.01 - 25.08 = 30.93$ ft above the ground (Figure 6.9.4).



$y = 56.01\cosh\left(\frac{x}{56.01}\right) - 25.08$

Figure 6.9.4