The Koopman Representation

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Abstract

We present the Koopman Representation and some applications while making little assumptions on the knowledge of the reader. Only knowledge of introductory group theory, topology and measure theory is assumed. We outline the basics of topological group theory, Hilbert spaces, unitary representations, square-integrable function spaces, Pontryagin duality and Hausdorff's embedding theorem. We then outline some parts of ergodic group theory while avoiding the study of Borel equivalence relations for brevity.

Contents

1 Background **2**

1.1 Topological Groups 2

1.2 The Haar Measure 2

1.3 Hilbert Spaces 3

1.4 Unitary Representations 5

1.5 Square-integrable function spaces 6

1.6 Pontryagin Duality 6

1.7 Hausdorff's Embedding Theorem 8

2 Ergodic Group Theory **8**

2.1 Group Actions on Probability Spaces 8

2.2 The Koopman Representation 9

2.3 Actions by Generalized Rotations 11

3 Applications **11**

1 Background

1.1 Topological Groups

Mathematical structures can be combined in order to make more complex mathematical structures at the intersection of two fields. They often also provide a better framework for mathematical objects one may encounter. For instance, the real numbers form a group $(\mathbb{R}, +)$, but they also have a standard topology. A natural question is whether the additive group structure is compatible with the topology. It turns out that it is, and we make this precise.

Definition 1.1. A *topological group* is a group G together with a topology on G such that the maps

- $x \mapsto gx$
- $x \mapsto xg$
- $x \mapsto x^{-1}$

are continuous for all $g \in G$.

Example 1.2. The groups $(\mathbb{R}, +)$ and the circle group $(\mathbb{T}, \cdot)$ together with their standard topology are topological groups. Moreover, any group can be turned into a topological group by giving it the discrete topology (but this topology is usually not very interesting).

Just as groups and topological spaces have a notion of equivalence or isomorphism, topological group also have their own notion which combines the algebraic and topological structures.

Definition 1.3. Two topological groups G, H are said to be *isomorphic* if there exists a map $\Phi : G \to H$ which is a group isomorphism and a homeomorphism.

Example 1.4. The group $\mathbb{R}/\mathbb{Z}$ can be given the quotient topology, and in this case we have an isomorphism of topological groups between $\mathbb{R}/\mathbb{Z}$ and the circle group $\mathbb{T}$.

1.2 The Haar Measure

It turns out that some topological groups can have a measure defined on them which is invariant under left or right multiplication. In fact very familiar group have such measure, such as $(\mathbb{R}, +)$ and $(\mathbb{T}, \cdot)$. We present the Haar measure as well as existence and uniqueness theorems.

Definition 1.5. Let G be a topological group. A *left Haar measure* is a nontrivial measure μ on the Borel sigma algebra of G such that

$$\mu(gA) = \mu(A)$$

for all $g \in G$ and all Borel measureable subsets A of G. A *right Haar measure* is similarly defined, using right translates rather than left translates. If a measure is both a left and a right Haar measure, then it is said to be a *Haar measure*.

Example 1.6. The standard measures on $\mathbb{R}$ and $\mathbb{T}$ are Haar measures. Given a finite group with the discrete topology, the counting measure is a left Haar measure.

Theorem 1.7 (Existence and Uniquess of the Haar Measure)**.** *Let G be a locally compact group. Then, there exists a unique left Haar measure on G, up to a multiplicative constant.*

Corollary 1.8. *Let G be an locally compact abelian group (LCA). Then, there exists a unique Haar measure on G, up to a multiplicative constant.*

1.3 Hilbert Spaces

Whenever one has a measure space, one may define the space of real or complex-valued square integrable functions on that space. It turns out that looking at properties on this space can give insights on the original measure space. Moreover, this space turns out to be a Hilbert space, and so we outline the basic theory here.

Definition 1.9. A *complex Banach* space is a complete normed vector space over $\mathbb{C}$.

Example 1.10. The vector space $\mathbb{C}^n$ is a complex Banach space.

Definition 1.11. A *complex Hilbert space* is a Banach space with an inner product. That is, it is an inner product space such that the norm

$$\|f\| := \sqrt{\langle f, f\rangle}$$

is complete.

Example 1.12. The vector space $\mathbb{C}^n$ together with the standard inner product is a Hilbert space.

From now on, all definition will assume that the underlying vector space is $\mathbb{C}$. However, all of the definition are also valid for real Hilbert spaces.

Definition 1.13. Let $\mathcal{H}$ be a Hilbert space. A subset $S \subseteq H$ is said to be *orthonormal* if for all $x, y \in S$ with $x \neq y$ we have that $\langle x, y\rangle = 0$ and $\|x\| = 1$ for all $x \in S$.

So far, this is not too disimilar from studying finite dimensional inner product spaces. However, many Hilbert spaces of interest happen to be infinite dimensional, and for our setting, using only finite sums in the definition of basis poses many challenges. A much more useful definition is to allow for countable sums rather than finite ones. We first give an equivalent definition, which allows us to define orthonormal bases. To make this precise, however, we need to define unordered sums and their convergence.

Definition 1.14. Let $\mathcal{B}$ be a Banach space. Let $x_\alpha \in \mathcal{B}$ for all α in some index set I. The *unordered sum* or *infinite linear combination*

$$\sum_{\alpha \in I} x_\alpha$$

is said to *converge* or *unconditionally converge* to $x \in \mathcal{B}$ if for every $\varepsilon > 0$ there is a finite subset $J \subseteq I$ such that

$$\left\| x - \sum_{\alpha \in J'} x_\alpha \right\| < \varepsilon$$

for all finite sets J' containing J.

Proposition 1.15. *Let $\mathcal{B}$ be a Banach space. If an unordered sum unconditionally converged to $x, y \in \mathcal{B}$, then $x = y$. That is, limits sums of unconditionally convergent sums in Banach spaces are unique. Thus, we write*

$$x = \sum_{\alpha \in I} x_\alpha$$

for the unconditional sum of elements x_α for some index set I.

Definition 1.16. Let $\mathcal{H}$ be a Hilbert space. A subset $B \subseteq \mathcal{H}$ is said to be an *orthonormal basis* of $\mathcal{H}$ if it is an orthonormal subset and moreover there is no orthonormal subset strictly larger than it.

Theorem 1.17. *Let H be a Hilbert space. Let $B \subseteq H$. Then, B is an orthonormal basis if and only if*

$$\xi = \sum_{b \in B} \langle \xi, b \rangle b$$

for all $\xi \in H$.

Here we used the definition of unconditional convergence. Note that in this case B can be finite, countably infinite or even uncountable since the definition of unconditional convergence does not consider the cardinality of the index set.

Theorem 1.18. *Let $\mathcal{H}$ be a Hilbert space. Let B be an orthonormal basis. Then, every $\xi \in \mathcal{H}$ be expressed uniquely as an infinite linear combination of elements of ξ.*

Definition 1.19. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear map $T : \mathcal{H} \to \mathcal{K}$ is *bounded* if

$$\|T\| := \sup_{\xi \in \mathcal{H},\ \|\xi\|=1} \|T\xi\| < \infty.$$

Definition 1.20. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear map $T : \mathcal{H} \to \mathcal{K}$ is said to be *isometric* if

$$\langle \xi, \eta \rangle = \langle T\xi, T\eta \rangle$$

for all $\xi, \eta \in \mathcal{H}$.

Definition 1.21. Let $\mathcal{H}$ be a Hilbert space. A bounded operator $u : \mathcal{H} \to \mathcal{H}$ is *unitary* if it is bijective and isometric. The set of unitary maps is denoted $\mathbb{U}(\mathcal{H})$.

Finally, we make precise the notion of equivalence or isomorphism of Hilbert spaces.

Definition 1.22. Two Hilbert spaces $\mathcal{H}, \mathcal{K}$ are said to be *isomorphic* if there exists a linear map $U : \mathcal{H} \to \mathcal{K}$ which is bijective and isometric.

1.4 Unitary Representations

It is often useful to consider groups acting on sets by automorphisms. In our case, we are ultimately interested in groups acting on Hilbert spaces which preserve the structure in a sense that we will make precise. First, we define representations which are a kind of group action on general vector spaces.

Definition 1.23. A *representation* of a group G is a homomorphism $\pi : G \to \mathrm{GL}(V, \mathbb{F})$. That is, a representation is a homomorphisim into the endomorphism group of a vector space V over a field $\mathbb{F}$. Equivalently, a representation is an action $G \circlearrowright V$ which is compatible with the vector space structure. That is,

- $e \cdot v = v$ for all $v \in V$;
- $g \cdot (h \cdot v) = (gh) \cdot v$ for all $g, h \in G$ and $v \in V$;
- the maps $v \mapsto g \cdot v$ are linear for all $g \in G$;

Example 1.24. Given a symmetric group S_n, we can map every element of this group to a permutation matrix in $\mathbb{F}^n$ permuting the corresponding coordinates. This map is a representation of S_n.

Similarly to group actions, these two definitions are related by currying or uncurying one map to get the other.

Definition 1.25. A *unitary representation* is a representation $\pi : G \to \mathrm{GL}(H, \mathbb{C})$ where the underlying vector space is a complex Hilbert space and the representation can be restricted to the set of unitary endomorphisms of H, that is $\pi : G \to \mathbb{U}(H)$.

Example 1.26. In the previous example, taking $\mathbb{F} = \mathbb{C}$ results in a unitary representation of S_n.

1.5 Square-integrable function spaces

The set of square integrable complex-valued functions on a measure space is a complex Hilbert space. We give a precise definition and a sufficient condition on the measure space.

Definition 1.27. Let (X, μ) be a measure space. Then, $L^p(X, \mu)$ is defined as the normed space of functions $f : X \to \mathbb{C}$, considered up to a.e. equality, whose pth power is integrable, that is

$$\int_X |f|^p \, \mathrm{d}\mu < \infty.$$

The norm is defined as

$$\|f\| = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Theorem 1.28. *Let (X, μ) be a measure space. Then, $L^p(X, \mu)$ is a Banach space.*

Example 1.29. Consider $\mathbb{N}$ with the counting measure. Then $L^2(\mathbb{N}, \mu)$ is the space of square-summable sequences.

Theorem 1.30. *Let (X, μ) be a measure space. Then, $L^2(X, \mu)$ is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle = \int_X f \bar{g} \, \mathrm{d}\mu.$$

1.6 Pontryagin Duality

It turns that locally compact abelian groups exhibit a duality which is very similar to finite dimensional vector spaces over arbitrary fields. Indeed, for finite dimensional vector spaces, we define the dual of a vector space to be the set of morphisms (linear maps) from that space into a particular object (the field). In our case, the type of morphism under consideration is continuous homomorphisms and the particular object is the circle group $\mathbb{T}$. This duality allows us to define the fourier transform on locally compact abelian groups. Moreover, this Fourier transform corresponds with the usual Fourier transform on $\mathbb{R}$ and $\mathbb{T}$, which themselves are locally compact abelian groups.

Recall. Let X and Y be two topological spaces. Let $C(X, Y)$ denote the set of continuous maps $f : X \to Y$. This set can be given a topology called the *compact-open* topology with subbase consisting of the sets

$$\{f \in C(X, Y) \mid f(K) \subseteq U\}$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

Definition 1.31. Let G be a locally compact abelian group (LCA). The dual of G is defined as

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{T}),$$

where $\mathbb{T}$ is the circle group and $\mathrm{Hom}(G, \mathbb{T})$ denotes the set of continuous homomorphisms $\chi : G \to \mathbb{T}$. The group operation is pointwise multiplication.

Example 1.32. The maps $t \mapsto t^n$ are members of the dual of $\mathbb{T}$ for all $n \in \mathbb{Z}$.

Proposition 1.33. *Let G be an LCA group. Then $\widehat{G}$ can be turned into a topological group by considering $\widehat{G} \subseteq C(G, \mathbb{T})$, that is considering $\widehat{G}$ as a subset of the set of continuous functions $G \to \mathbb{T}$, and then giving $\widehat{G}$ the subspace topology.*

Proposition 1.34. *The dual $\widehat{G}$ of an LCA group G is also LCA.*

Theorem 1.35. *Let G be an LCA group. We have that*

1. *if G is compact, then $\widehat{G}$ is discrete; and*
2. *if G is discrete, then $\widehat{G}$ is compact.*

The following theorem makes precise the equivalence between a locally compact abelian group and its double dual. We include it for completion but we do not use it directly.

Theorem 1.36 (Pontryagin Duality)**.** *Let G be an LCA group. There is a natural isomorphism between G and its double dual $\widehat{\widehat{G}}$ given by*

$$\mathrm{ev}_G(x)(\chi) = \chi(x) \in \mathbb{T},$$

where $\chi \in \mathrm{Hom}(G, \mathbb{T})$.

Definition 1.37. Let G be an LCA group. Let μ be a Haar measure for G. Let $f : G \to \mathbb{C}$ be a complex valued-function. The *Fourier transform* of f is the function $\hat{f} : \widehat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, d\mu(x).$$

Proposition 1.38. *Let G be an LCA group. Let $f \in L^1(G, \mu)$ for some Haar measure μ. Then, the Fourier transform $\hat{f}$ is well-defined.*

Theorem 1.39 (Plancherel)**.** *Let G be an LCA group. Let μ be a Haar measure for G. The Fourier transform can be extended to define a unitary isomorphism $\mathscr{F} : L^2(G, \mu) \to L^2(\widehat{G}, \hat{\mu})$, for some choice of Haar measure $\hat{\mu}$ for $\widehat{G}$.*

1.7 Hausdorff's Embedding Theorem

One of the applications of the Koopman representation which we present is related the the special orthogonal group, the two-sphere and the free group in two variables. We outline their relationship here.

Definition 1.40. The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is a topological group defined as

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in M_{n \times n}(\mathbb{R}) \mid \det A \neq 0\},$$

where $M_{n \times n}(\mathbb{R})$ denotes the set of $n \times n$ real matrices. The topology is given by the subspace topology of $M_{n \times n}(\mathbb{R})$ viewed as the euclidean space $\mathbb{R}^{n^2}$.

Definition 1.41. The *orthogonal group* $\mathrm{O}(n, \mathbb{R})$ is a topological group defined as

$$\mathrm{O}(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid AA^{\mathrm{T}} = I\}.$$

It is given the subspace topology as a subset of $\mathrm{GL}(n, \mathbb{R})$.

Definition 1.42. The *special orthogonal group* $\mathrm{SO}(n)$ is a topological group defined as

$$\mathrm{SO}(n) := \{A \in O(n) \mid \det A = 1\}.$$

It is given the subspace topology as a subset of $\mathrm{GL}(n, \mathbb{R})$.

Proposition 1.43. *The group* $\mathrm{SO}(\mathrm{n} + 1)$ *acts transitively on* $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ *by matrix-vector multiplication. Moreover, the stabilizer of the north pole is isomorphic to* $\mathrm{SO}(n)$ *and we have*

$$\mathbb{S}^n \cong \mathrm{SO}(n + 1)/\,\mathrm{SO}(n).$$

where $\mathrm{SO}(n + 1)/\,\mathrm{SO}(n)$ *is the set of cosets of* $\mathrm{SO}(n)$ *together with the quotient topology.*

Theorem 1.44 (Hausdorff)**.** *There exists an embedding (an injective homomorphism)* $F_2 \hookrightarrow SO(3)$ *where* F_2 *is the free group in two variables. Moreover, this embedding has dense image.*

2 Ergodic Group Theory

2.1 Group Actions on Probability Spaces

Now we outline some parts of Ergodic Group Theory. A complete treatment typically includes discussion of Borel equivalence relations. For brevity, however, we avoid this and instead give equivalent definitions. We begin by definining probability measure preserving group actions.

Definition 2.1. A group action $G \circlearrowright (X, \mathcal{B}, \mu)$ on a probability space is said to be *probability measure preserving* or *pmp* if for every $B \in \mathcal{B}$ and every $g \in G$ we have

$$\mu(gB) = \mu(B).$$

That is, the maps $x \mapsto g \cdot x$ preserve the probability measure for all $g \in G$.

Now we define the notion of an ergodic group action. Ergodic group theory in part studies these actions and their properties. We aim to provide examples of ergodic pmp group actions in the next section.

Definition 2.2. A group action $G \circlearrowright (X, \mathcal{B}, \mu)$ on a probability space is said to be *ergodic* if every Borel set which is invariant under the action of G has either null or full measure. That is, for every $B \in \mathcal{B}$, if $g \cdot B = B$ for all $g \in G$, then we have that $\mu(B) \in \{0, 1\}$.

2.2 The Koopman Representation

In order to study pmp actions, we define a unitary representation called the Koopman representation which captures some but not all of the internal structure of pmp actions. We also present a lemma which allows us to study ergodicity of pmp actions by studying their Koopman representation.

Definition 2.3. Let G be a countable group. Let $G \circlearrowright (X\mu)$ be a pmp action on a probability space. The Koopman representation of the action $G \circlearrowright (X, \mu)$ is the unitary representation $G \circlearrowright L^2(X, \mu)$ given by

$$\kappa_s \cdot f = (x \mapsto f(s^{-1} \cdot x)).$$

That is, we can view the *Koopman representation* as a map $\kappa : G \to \mathbb{U}(L^2(X, \mu))$ given by $s \mapsto \kappa_s$.

Lemma 2.4. *A pmp action $G \circlearrowright (X, \mu)$ is ergodic if and only if every function $f \in L^2(X, \mu)$ which is invariant under the Koopman representation is essentially constant. By invariance under the Koopman representation we mean that $\kappa_s \cdot f = f$ for all $s \in G$.*

Proof. Let $G \circlearrowright (X, \mu)$ be an ergodic pmp action. We show that every function which is invariant under the Koopman representation is essentially constant. Let $f \in L^2(X, \mu)$ such that $\kappa_s \cdot f = f$ for all $s \in G$. Let $A_\alpha = \{x \in X \mid \mathrm{Re}(f(x)) \geq \alpha\}$ and $B_\beta = \{x \in X \mid \mathrm{Im}(f(x)) \geq \beta\}$. We show that A_α is invariant under the action of G for all $\alpha \in \mathbb{R}$. Let $g \in G$. We have

$$\begin{aligned} g \cdot A_\alpha &= \{g \cdot x \in X \mid x \in A_\alpha\} \\ &= \{g \cdot x \in X \mid \operatorname{Re}(f(x)) \geq \alpha\} \\ &= \{x \in X \mid \operatorname{Re}(f(g^{-1} \cdot x)) \geq \alpha\} \\ &= \{x \in X \mid \operatorname{Re}((\kappa_g \cdot f)(x)) \geq \alpha\} \\ &= \{x \in X \mid \operatorname{Re}(f(x)) \geq \alpha\} \\ &= A_\alpha. \end{aligned}$$

We can similarly show that B_β is invariant under the action of G for all $\beta \in \mathbb{R}$. Thus, for all $\alpha, \beta \in \mathbb{R}$, we have that $\mu(A_\alpha) \in \{0, 1\}$ and $\mu(B_\beta) \in \{0, 1\}$. If f was not essentially constant, then there would be be a $t \in \mathbb{R}$ such that $0 < \mu(A_t) < 1$ or $0 < \mu(B_t) < 1$, which is a contradiction.

On the other hand, assume that every function $f \in L^2(X, \mu)$ which is invariant under the Koopman representation is essentially constant. Let $A \subseteq X$ be a G-invariant set. That is $GA = A$. Then, consider the square integrable function χ_A. We have

$$\begin{aligned} \kappa_s \chi_A(x) &= \chi_A(s^{-1}x) \\ &= \begin{cases} 1 & \text{if } s^{-1}(x) \in A \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

almost surely for all $s \in G$. But, since A is G-invariant, $x \in A$ if and only if $s^{-1}(x) \in A$ and so we have that

$$\begin{aligned} \kappa_s \chi_A(x) &= \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_A(x) \end{aligned}$$

almost surely for all $s \in G$. Thus, χ_A is invariant under the Koopman representation and so it is essentially constant. So either $\chi_A(x) = 1$ almost surely or $\chi_A(x) = 0$ almost surely. Thus, we have that $\mu(A) = 0$ or $\mu(A) = 1$, and so the pmp action $G \circlearrowright (X, \mu)$ is ergodic. $\square$

2.3 Actions by Generalized Rotations

We present a lemma which provides examples of ergodic pmp group actions. More specifically, this lemma allows us to define ergodic pmp group actions on coset spaces given a second countable compact group. It is a consequence of the fact that the Koopman representation of a compact group (as acting on itself by left multiplication) is strongly continuous, which we shall not discuss.

Lemma 2.5. *Let K be a second countable compact group. Let G be a countable set. Let $f : G \to K$ be a homomorphism with dense image. Let $H \subseteq K$ be a closed subgroup. Consider the action $G \circlearrowright K/H$ given by*

$$(g, kH) \mapsto (f(g)k)H.$$

Then, if μ is a Haar measure on K and $\pi : K \to K/H$ is the canonical projection, then the action $G \circlearrowright K/H$ is ergodic with respect to $\pi_*\mu \coloneqq \mu \circ \pi^{-1}$.

3 Applications

We outline two examples of ergodic pmp actions. The first one is an action of $\mathbb{Z}$ on the circle group (or more generally any compact abelian group) and the second one is an action of the free group on two variables F_2 on the sphere $\mathbb{S}^2$.

Proposition 3.1. *Let K be a compact abelian group and let $s \in K$ such that $\{s^n\}_{n\in\mathbb{N}}$ is dense. Then, the rotation action*

$$\mathbb{Z} \times K \to K$$
$$(n, t) \mapsto s^n t$$

is ergodic with respect to the Haar measure.

Proof. By theorem 1.39, there is a unitary isomorphism

$$\mathscr{F} : L^2(K, \mu) \to L^2(\widehat{K}, \hat{\mu})$$

given by extending the Fourier transform for some choice of Haar measure $\hat{\mu}$ on $\widehat{K}$. We observe that since K is compact, we have that μ is a probability measure and so we have $L^2(K, \mu) \subseteq L^1(K, \mu)$ and so the Fourier transform is well-defined for all $f \in L^2(K, \mu)$ by proposition 1.38. Since K is compact, by theorem 1.35 we have that $\widehat{K}$ is discrete. Since $\widehat{K}$ is discrete, we have that $\hat{\mu}$ is a multiple of the counting measure.

Let $\{\chi_n\}_{n\in\mathbb{N}}$ be an enumeration of $\widehat{K}$ such that $\chi_1 \equiv 1$ is the trivial character. An orthonomal basis for the space $L^2(\widehat{K}, \hat{\mu})$ is the set $\{g_m\}_{m\in\mathbb{N}}$ given by

$$g_m(\chi_n) \coloneqq \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

We define the basis $\{e_m\}_{m\in\mathbb{N}}$ given by

$$e_m \coloneqq \mathscr{F}^{-1}(g_m).$$

We observe that $\widehat{G} \subseteq L^2(G, \mu)$. Moreover, we show that, $e_m = \chi_m$ for all $m \in \mathbb{N}$. We have

$$g_m(\chi_n) = \hat{e}_n(\chi_n) = \int_K e_m(x)\overline{\chi_n(x)}\,\mathrm{d}\mu(x) = \langle e_m, \chi_n\rangle$$

that is, we have that $\langle e_m, \chi_n\rangle = 1$ if $m = n$. But, since e_i, χ_n are unit vectors, this implies that $e_i = \chi_n$. Thus, $e_i = \chi_i$ for all $i \in \mathbb{N}$. Let $f \in L^2(G, \mu)$ be a function which is invariant under the Koopman representation.

$$\begin{aligned} \langle f, \chi_n\rangle &= \langle \kappa_1 f, \chi_n\rangle \\ &= \int_K f(s^{-1}x)\overline{\chi_n(x)}\,\mathrm{d}\mu(x) \\ &= \int_K f(x)\overline{\chi_n(sx)}\,\mathrm{d}\mu(x) \\ &= \overline{\chi_n(s)}\int_K f(x)\overline{\chi_n(x)}\,\mathrm{d}\mu(x) \\ &= \overline{\chi_n(s)}\langle f, \chi_n\rangle \end{aligned}$$

since the Haar measure in invariant under translation. Thus, for all $n \in \mathbb{N}$ we have that $\overline{\chi_n(s)} = 1$ or $\langle f, \chi_n\rangle = 0$. If $\overline{\chi_n(s)} = 1$, then this implies that $\chi_n(s) = 1$ and so $\chi_n(s^m) = 1$ for all $m \in \mathbb{Z}$. But, the set $\{s^m\}_{m\in\mathbb{Z}}$ is dense in K and so we have that $\chi_n \equiv 1$. In otherwords, the only $n \in N$ for which $\overline{\chi_n(s)} = 1$ is $n = 1$. For all other $n \in N$, we must have $\langle f\chi_n\rangle = 0$. Then, by theorem 1.17 we have

$$f = \sum_{n\in N}\langle f, \chi_n\rangle\chi_n.$$

Thus, f is essentially constant. Hence, by lemma 2.4, we have that the action by rotations is ergodic. □

Corollary 3.2. *Let $\mathbb{Z}$ act on $\mathbb{T}$ by irrational rotations. That is, let $s \in \mathbb{T}$ such that*

$$Z \times \mathbb{T} \to \mathbb{T}$$

$$(n, t) \mapsto s^n t.$$

is an action by rotations of an irrational multple of π. Then, the action $\mathbb{Z} \circlearrowright \mathbb{T}$ is ergodic.

Proof. Since $\mathbb{T}$ is compact and $\{s_n\}_{n\in\mathbb{Z}}$ is dense if and only if $s \in \mathbb{T}$ is an irrational multiple of π, the corollary follows from proposition 3.1 $\square$

Proposition 3.3. *There exists an ergodic action $F_2 \circlearrowright \mathbb{S}^2$.*

Proof. Consider the second countable compact group SO(3). By theorem 1.44 there is a homomorphism $f : F_2 \to \mathrm{SO}(3)$ with dense image. Let H be the stabilizer of the north pole. We observe that H is closed. Then, by lemma 2.5 there is an action $F_2 \circlearrowright \mathrm{SO}(3)/H$. However by proposition 1.43 we have that $\mathrm{SO}(3)/H \cong \mathbb{S}^2$. Thus, we have an ergodic action $F_2 \circlearrowright \mathbb{S}^2$. $\square$