Duality is All You Need

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1 Motivation

Automata minimization has been a rich and widely explored area since the 1950s, with numerous algorithms developed for the deterministic finite case [1]. The applications of these techniques are numerous and broad, including text processing, image analysis, and network intrusion detection. In particular, many problems seen in modern AI research involve learning and planning transition systems under uncertainty. This is seen under the framework of Partially Observable Markov Decision Processes (POMDPs) prevalent in Reinforcement Learning: available information about the environment is incomplete and thus the agent maps actions to "belief states", which are probability distributions over underlying model states [8]. Previous work has been done to find methods to learn these POMDPs, and minimization methods offer potential for more effective planning and learning algorithms that can be performed on these different representations of the same system.

What is this 'minimal transition system'? In the finite setting, this equates to having the fewest states while the system is still behaviourly equivalent to the original. When dealing with a probabilistic, infinite setting however, we use the process of quotienting to define the minimal realization: in other words, we seek an equivalence relation $\sim$ such that if S' is a quotient of S, we can further quotient it to reach $\tilde{S} = S/\sim$ [4]. The duality between logics and transition systems has been connected to this objective of finding the "maximally quotiented object" through mathematical dualities between categories, including — but certainly not limited to — Stone duality and its variants, as well as Gelfand duality. To be more precise, we start with a transition system that falls within a certain category, derive the dual object in the dual category, and seek a certain subobject such that bringing this back to the original category gives the minimal realization of the original system. This is the core concept discussed by the paper written by Bezhanishvili et al., titled *Minimization Via Duality* [2] ("[BKP]"), which we now aim to address. This paper is divided as follows: we present the general background that motivates the applicability of categorical duality to the minimization problem, and then we delve further into two specific examples that [BKP] discuss — finite Stone duality and partially observable deterministic finite automata, and Gelfand duality and belief automata.

2 Background

All of [BKP]'s results are consequences of Theorem 1, a very general statement about minimization of transition systems using categorical language. We will first state the following definition.

Definition. Let $\mathrm{T} : \mathcal{C} \to \mathcal{C}$ be a functor and let $\mathcal{S} = (S, \gamma)$ be a T-coalgebra. An epimorphism $e : S \to S_0$ of $\mathcal{S}$ is called a **minimization** of $\mathcal{S}$ if for all other quotients $e' : S \to S'$ there exists a unique map $g : S' \to S_0$ such that $g \circ e' = e$.

This definition states that if we have a transition system (S, γ) where S encodes the states and γ encodes the transitions and other structure, then we have that $e : (S, \gamma) \to (S_0, \gamma_0)$ is a minimization if given another canditate $e' : (S, \gamma) \to (S', \gamma')$, there is a unique map $e^* : S' \to S_0$ such that the following diagram commutes.

$$\begin{tikzcd}
(S,\gamma) \arrow[r, "e"] \arrow[d, "e'"] & (S_0,\gamma_0) \\
(S',\gamma') \arrow[ur, dotted, "e^*"] &
\end{tikzcd}$$

If there is an epimorphism $f : (S_1, \gamma_1) \to (S_2, \gamma_2)$, it indicates that the two machines agree on all inputs, but the second one is "smaller" than the first, hence the definition. Indeed, such an epimorphism is a T-coalgebra homomorphism. That is, it is a map such that the following diagram commutes.

$$\begin{tikzcd}
S_1 \arrow[r, "f"] \arrow[d, "\gamma_1"] & S_2 \arrow[d, "\gamma_2"] \\
\mathrm{T}S_1 \arrow[r, "\mathrm{T}f"] & \mathrm{T}S_2
\end{tikzcd}$$

This is how [BKP] characterizes minimization of automata under a categorical umbrella. Now, we state [BKP]'s Theorem 1. We give an explanation of the theorem and the idea behind the proof.

Theorem 1. *Let $\mathcal{C}$ be a co-wellpowered category with (Epi,Mono)-factorization. Let $\mathcal{D}$ be a category dually equivalent to $\mathcal{C}$ and $\mathrm{T} : C \to C$ be a functor that preserves monos. Assume also that $\mathrm{L} : \mathcal{D} \to \mathcal{D}$ is a functor such that* $\mathbf{Coalg}(\mathrm{T})$ *and* $\mathbf{Alg}(\mathrm{L})$ *are dually equivalent. Let*

$$\hat{F} : \mathbf{Coalg}(\mathrm{T}) \to \mathbf{Alg}(\mathrm{L})^{\mathrm{op}} \text{ and } \hat{G} : \mathbf{Alg}(\mathrm{L}) \to \mathbf{Coalg}(\mathrm{T})^{\mathrm{op}}$$

be functors establishing this duality. Let $\mathcal{S}$ be any T-coalgebra with C_m being the minimal subobject of $\hat{F}(\mathcal{S})$. Then, $\hat{G}(C_m)$ is the minimization of $\mathcal{S}$.

Roughly speaking, the two assumptions on the category $\mathcal{C}$ are assumptions that it nicely behaves in some sense. A category is co-wellpowered if collections of subobjects in $\mathcal{C}^{\mathrm{op}}$ are small. For instance, in **Set**, an object's collection of subobjects is small since they correspond to members of the object's powerset. A category has (Epi,Mono)-factorization if every morphism $f : A \to B$ can be factored into an epimorphism $e : A \to C$ and a monomorphism $m : C \to B$ and it is unique up to isomorphism. For instance, the category **Set** has the following (Epi,Mono)-factorization.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"] & B \\
\mathrm{im}(f) \arrow[ur, "m"] &
\end{tikzcd}$$

We present a rough idea of the proof. Let $\mathcal{A}$ denote $\hat{F}(\mathcal{S})$. Let A_0 denote the minimal subobject of $\mathcal{A}$ and $m_0 : \mathcal{A}_0 \to \mathcal{A}$ be the corresponding monomorphism. Let S_0 denote $\hat{G}(\mathcal{A}_0)$. Let $e_0 : \mathcal{S} \to \mathcal{S}_0$ denote $\hat{G}(m_0)$. Let $e' : \mathcal{S} \to \mathcal{S}'$ be an epimorphism. Let $\mathcal{A}'$ denote $\hat{F}(\mathcal{S}')$. Thus, we have a monomorphism $m' : \mathcal{A}' \to \mathcal{A}$. Since $\mathcal{A}_0$ is the minimal subobject of $\mathcal{A}$, there is a unique monomorphism $m^* : \mathcal{A}_0 \to \mathcal{A}'$ such that $m' \circ m^* = m_0$. Thus, there is an epimorphism $e^* : \mathcal{S}' \to \mathcal{S}_0$ such that $e^* \circ e' = e_0$. It must be the case that e^* is the unique morphism with $e^* \circ e' = e_0$, otherwise it would contradict the uniqueness of m^*. Thus, $\mathcal{S}_0$ is the minimization of $\mathcal{S}$. The following pair of diagrams presents the idea behind the proof.

$$
\begin{tikzcd}
\mathcal{S} \arrow[r, two heads, "e_0"] \arrow[d, two heads, "e'"'] & \mathcal{S}_0 \\
\mathcal{S}' \arrow[ur, two heads, dotted, "e^*"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A}_0 \arrow[r, hook, "m_0"] \arrow[dr, hook, dotted, "m^*"'] & \mathcal{A} \\
& \mathcal{A}' \arrow[u, hook, "m'"']
\end{tikzcd}
$$

3 Partially Observable Deterministic Finite Automata and Stone Duality

The minimization of partially observable deterministic finite automata, a kind of deterministic automata which generalizes ordinary deterministic finite automata, uses a simple and well known duality between finite sets and finite boolean algebras called finite Stone duality.

3.1 Background

3.1.1 Relationship Between Finite Boolean Algebras and Finite Powersets

We present a list of results with which finite Stone duality can be derived.

Proposition. *Every set gives rise to a boolean algebra given by the powerset.*

Definition. We define an ordering on boolean algebras given by

$$a \leq b \text{ if } a \vee b = b.$$

Definition. An element of a boolean algebra $b \in B$ is called an **atom** if $\forall x \in B : b \wedge x = b$ or $b \wedge x = 0$.

Definition. A boolean algebra is **atomic** if every element is the least upper bound of a set of atoms.

Proposition. *Every finite boolean algebra is atomic.*

Proposition. *Every finite boolean algebra is isomorphic to the powerset of its atoms.*

Proposition. *Let S, T be finite sets. Every boolean algebra homomorphism $h : P(T) \to P(S)$ has a corresponding unique function $f : S \to T$ such that h is a function which sends subsets of T to their preimage under f.*

Finite Stone duality follows easily from these observations.

3.1.2 Finite Stone Duality

We present a concrete construction of the functors responsible for the duality between finite sets and finite boolean algebras. In it, sets are mapped to their power set boolean algebras and any function between two sets becomes the pre-image, which is a function between power sets. Likewise, finite boolean algebras are mapped to their atoms and any homomorphism between boolean algebras is given by a function which takes an atom from one boolean algebra and maps it to the pre-image under that homomorphism.

Theorem 2. *The categories* **FinSet** *and* **FinBA** *are dually equivalent. The functors* $F : \mathbf{FinSet} \to \mathbf{FinBA}^{\mathrm{op}}$ *and* $G : \mathbf{FinBA}^{\mathrm{op}} \to \mathbf{FinSet}$ *are given by*

$$F(A) = P(A), \quad F(f : A \to B) = f^{-1} : P(B) \to P(A)$$

and

$$G(B) = \mathrm{At}(B), \quad G(f : B \to C) = c \mapsto \bigwedge\{b \in B \,|\, b \leq f(c).\}$$

3.2 Partially Observable Deterministic Finite Automata

Definition. A **partially observable deterministic finite automata** (PODFA) is a quintuple $\mathcal{S} = (S, \mathcal{A}, \mathcal{O}, \delta : S \to S^{\mathcal{A}}, \gamma : S \to 2^{\mathcal{O}})$ where S is a finite set of states, $\mathcal{A}$ is a finite set of actions, $\mathcal{O}$ is a finite set of observations, δ is a transition function and γ is an observation function.

What differentiates these automata defined by [BKP] from finite deterministic automata is that we do not have an initial state and instead of assigning which states are final, we assign observations to each state. Adding an initial state and setting $\mathcal{O} = \{\text{accept}\}$, we get finite deterministic automata.

If we fix the set of actions and observations, then PODFAs are determinined by triples (S, δ, γ). This allows [BKP] to define PODFAs as T-coalgebras.

Definition. Let $\mathrm{T} : \mathbf{FinSet} \to \mathbf{FinSet}$ be defined as

$$\mathrm{T}(S) = S^{\mathcal{A}} \times 2^{\mathcal{O}}, \; T(f : S \to S') = \lambda\langle \alpha : \mathcal{A} \to S, O \subseteq \mathcal{O}\rangle.\langle f \circ \alpha, O\rangle.$$

Then, the category of T-coalgebras is called **PODFA**, the category of partially observable deterministic finite automata.

Indeed, a T-coalgebra is pair (S, σ) where S is a finite set and $\sigma : S \to \mathrm{T}S$ is a morphism. By the universal property of products, σ is uniquely determined by a morphism $\delta : S \to S^{\mathcal{A}}$ and a morphism $\gamma : S \to 2^{\mathcal{O}}$, as required.

Proposition. *The functor* $\mathrm{T} : \mathbf{FinSet} \to \mathbf{FinSet}$ *preserves monomorphisms.*

Proof. Let f be a monomorphism. We show that $\mathrm{T}f$ is also a monomorphism. We have

$$\begin{aligned} \mathrm{T}f(g_1, O_1) = \mathrm{T}f(g_2, O_2) &\implies (f \circ g_1, O_1) = (f \circ g_2, O_2) \\ &\implies f \circ g_1 = f \circ g_2 \text{ and } O_1 = O_2 \\ &\implies g_1 = g_2 \text{ and } O_1 = O_2 \\ &\implies (g_1, O_1) = (g_2, O_2), \end{aligned}$$

as required. □

Definition. The category **FBAO** of finite boolean algebras with operators has objects boolean algebras together with an endomorphism (a) for each action $a \in \mathcal{A}$ and a distinguished element $\underline{\omega}$ for each observation $\omega \in \mathcal{O}$. The morphisms are boolean algebra homomorphisms which preserve, in addition, the endomorphisms (a) for each $a \in \mathcal{A}$ and the observations $\underline{\omega}$ for each $\omega \in \mathcal{O}$. In other words, a homomorphism $h : (B_1, \mathbb{A}_1, \Omega_1) \to (B_2, \mathbb{A}_2, \Omega_2)$ must satisfy

$$\forall a \in \mathcal{A}\ \forall x \in B_1 : h((a)(x)) = (a)(h(x)) \quad \text{and} \quad \forall \omega \in \mathcal{O} : h(\underline{\omega}) = h(\omega).$$

Proposition. *The category* **FBAO** *is isomorphic to* **Alg**(L) *where* $\mathrm{L} : \mathbf{FinBA} \to \mathbf{FinBA}$ *is an endofunctor given by*

$$\mathrm{L}B = \coprod_{a \in \mathcal{A}} B + F_{\mathbf{FinBA}}(\mathcal{O}),$$

where $F_{\mathbf{FinBA}}(\mathcal{O})$ *is the free boolean algebra generated by* $\mathcal{O}$.

Proof. Given a finite boolean algebra with operators $(B, \mathbb{A}, \Omega)$, we wish to find an L-algebra which corresponds to it. We observe that $\mathrm{L}B$ is well-defined, since coproducts and free boolean algebras exist and are finite. Thus, a homomorphism $h : \mathrm{L}B \to B$ is uniquely determined by morphisms $h_a : B \to B$ for each $a \in \mathcal{A}$ and a morphism $h_{\mathcal{O}} : F_{\mathbf{FinBA}}(\mathcal{O}) \to B$, by the universal property of coproducts. However, since $F_{\mathbf{FinBA}}(\mathcal{O})$ is a free boolean algebra generated by $\mathcal{O}$, by the universal property of free objects we have that a morphism $h_{\mathcal{O}} : F_{\mathbf{FinBA}}(\mathcal{O})$ is determined by a morphism in **Set** given by $f : \ \to \mathrm{U}B$ where U is the forgetful functor $\mathbf{FinBA} \to \mathbf{Set}$. In otherwords, a morphism $h : F_{\mathbf{FinBA}}(\mathcal{O}) \to B$ is determined by picking an element $\underline{\omega} \in B$ for each $\omega \in \mathcal{O}$. Likewise, given an L-algebra, we can find a boolean algebra with operators given by the respective morphisms in **FinBA** and **Set**. □

In the case of PODFAs, it happens that the category **FinBA** has free objects. This is not the case for every category. In the case where this does not happen, an object with a property similar to the free object is needed so that the conditions of Theorem 1 are satisfied. This is the case, for example, in the next section.

Proposition. *The categories* **PODFA** *and* **FBAO** *are dually equivalent. The duality is given by the contravariant functors* $\hat{F} : \mathbf{PODFA}^{\mathrm{op}} \to \mathbf{FBAO}$ *and* $\hat{G} : \mathbf{FBAO} \to \mathbf{PODFA}^{\mathrm{op}}$. *The functor* $\hat{F}$ *is defined by*

$$(S, \delta, \gamma) \mapsto (2^S, \mathbb{A}, \Omega),$$

where $\underline{\omega} = \{s \,|\, \omega \in \gamma(s)\}$ *and* $(a)b = \{s \,|\, \delta(s, a) \in b\}$ *and the arrow part of the functor is given by inverse images, that is, given* $f : (S_1, \delta_1, \gamma_1) \to (S_2, \delta_2, \gamma_2)$*, we have* $\hat{F}(f) : (2^{S_2}, \mathbb{A}_2, \Omega_2) \to (2^{S_1}, \mathbb{A}_1, \Omega_1)$

$$F(f)(O \subseteq S_1) = f^{-1}(O).$$

The functor $\hat{G}$ *is defined by*

$$(B, \mathbb{A}, \Omega) \mapsto (\mathrm{At}(B), \delta, \gamma),$$

where

$$\delta(b, a) = \bigwedge \{b' \in B \,|\, b \leq (a)b'\}$$

and

$$\gamma(b) = \{\omega \in \mathcal{O} \,|\, b \leq \underline{\omega}\}.$$

The arrow part of the functor is given by

$$f : B_1 \to B_2, \quad \hat{G}(f)(b \in \mathrm{At}(B_2)) = \bigwedge \{c \in B_1 \,|\, b \leq f(c)\}.$$

Definition. Consider the language $\mathcal{L}$:

$$t ::= \top \mid \hat{\omega} \mid \langle a \rangle t \mid t_1 \wedge t_2 \mid \neg t.$$

We define a satisfaction relation for a given automaton $\mathcal{S} = (S, \delta, \gamma)$ as follows

$$\begin{array}{ll} s \vDash \top & \text{always} \\ s \vDash t_1 \wedge t_2 & \text{if } s \vDash t_1 \text{ and } s \vDash t_2 \\ s \vDash \langle a \rangle t & \text{if } \delta(s,a) \vDash t \end{array} \qquad \begin{array}{ll} s \vDash \hat{\omega} & \text{if } \omega \in \gamma(s) \\ s \vDash \neg t & \text{if } s \nvDash t. \end{array}$$

We say that a subset U of S is **definable** by $\mathcal{L}$ if $U = [\![t]\!] := \{s \in S : s \vDash t\}$ for some $t \in \mathcal{L}$.

Proposition. *The set of subsets definable by $\mathcal{L}$ of a PODFA $\mathcal{S}$ with the following restriction of the operators gives a minimal subalgebra C_m of $\hat{F}(\mathcal{S})$.*

Theorem 3. *The automaton $\hat{G}(C_m)$ is the minimal realization of $\mathcal{S}$.*

Proof. We verify that the conditions of Theorem 1 are satisfied. We have that **FinSet** and **FinBA** are dually equivalent by finite Stone duality. The collections of subobjects in **FinBA** are small, and so **FinSet** is co-wellpowered. Futhermore, **FinSet** has (Epi,Mono)-factorization given by $f : A \to B$ equal to $m \circ e$ where $e : A \to \operatorname{im} f$ is the restriction of f to its image and $m : \operatorname{im}(f) \to B$ is the inclusion map. Moreover, T preserves monomorphisms. We have that **PODFA** and **FBAO** are T-coalgebras and L-algebras, respectively. By the above theorems, we have that **Coalg**(T) and **Alg**(L) are dually equivalent. Thus, by Theorem 1, $\hat{G}(C_m)$ is the minimal realization of $\hat{F}(S)$. ☐

Definition. Let $\mathcal{L}_0$ be a sublanguage of $\mathcal{L}$ defined as follows

$$t ::= \hat{\omega} \mid \langle a \rangle t.$$

Lemma. *If a formula of $\mathcal{L}$ distinguishes two states then so does a formula of $\mathcal{L}_0$.*

Proposition. *Given a PODFA, the boolean algebra with operators generated by the subsets definable by $\mathcal{L}_0$ gives the zero-generated subobject of the dual* **FBAO** *object. Thus one can construct the minimal PODFA by using formulas of $\mathcal{L}_0$.*

The previous propositions allows [BKP] to provide an algorithm which first finds subsets definable by a smaller language and then find C_m by generating it. That is, subsets definable by $\mathcal{L}$ can be expressed in terms of subsets definable by $\mathcal{L}_0$. Finding the atoms of this subalgebra gives us the states of the minimization. Then, δ and γ can be computed via their definition in the above proposition.

4 Belief Automata and Gelfand Duality

The core idea outlined by [BKP] can be wielded as a versatile tool due to the multitude of different categorical dualities, which particularly lends a use case for probabilistic transition systems— commonly dealt with in reinforcement learning as partially observable Markov decision processes. [BKP] show how minimization of these belief automata fit into the framework of the category of compact Hausdorff spaces **KHaus** and dually the category of real commutative C*-algebras **C*Alg** using results that follow from Gelfand duality. We aim to rigorously prove the underlying theory of Gelfand duality that can then be extended without breaking any of the added structure defined on the compact Hausdorff automata and their dual real commutative C*-algebras with added operators.

4.1 Background

4.1.1 Relationship between Compact Hausdorff Spaces and the Ring of Continuous Real-Valued Functions

To offer some motivation, we turn our attention to general compact Hausdorff spaces X and the set of continuous, real-valued functions, denoted $C(X)$ in a non-categorical setting. $C(X)$ is a space of interest because it has a wealth of properties: it is a commutative ring under pointwise addition and multiplication, a Banach space equipped with the sup norm, and a unital $\mathbb{R}$-algebra. It is in fact a C*-algebra:

Definition (Real Commutative C*-Algebra)**.** A real commutative unital C*-Algebra A is a commutative ring with unit equipped with multiplication by scalars and a (trivial) involution operation $a \mapsto a^*$. In addition, A is given a norm for which it's a Banach algebra, satisfying $\forall\ a,\ b \in A,\ \lambda \in \mathbb{R}$

$$||\lambda a|| = |\lambda|||a|, \qquad ||a + b|| \leq ||a|| + ||b||,$$
$$||ab|| \leq ||a||\,||b||, \qquad ||a^* a|| = ||a||^2.$$

Definition (Maximal Spectrum)**.** Let A be a commutative ring with unity. The maximal spectrum of A, denoted $\mathrm{Spec}_M(A)$ is the set of maximal ideals of A.

Considering a point $x \in X$ a compact Hausdorff space, we identify the set $M(x) = \{f \in C(X) : f(x) = 0\}$. For each f we have a corresponding surjective $\mathbb{R}$-algebra 'evaluation' homomorphism $\phi_x : f \mapsto f(x)$ into $\mathbb{R}$, which is a field. By the First Isomorphism Theorem, it follows that $M(x)$ is a maximal ideal. Looking upstream, we wish to recover the points of X and its topology by looking at the space of maximal ideals, $\mathrm{Spec}_M C(X)$. In fact, we will show that the map $\Phi : X \to \mathrm{Spec}_M C(X)$ is a homeomorphism; in other words, there is a one-to-one correspondance between points in X and maximal ideals of $C(X)$!

We define the cozero sets of X that correspond to the preimages $f_x^{-1}(\mathbb{R} \setminus \{0\})$, which form a base for the topology on a completely regular space— in fact, it's a characterization! (see [5])

Definition. Let $f \in C(X)$. Then $Z(f) = f^{-1}(\{0\}) = \{x \in X : f(x) = 0\}$ is the zero set of f in X. The complement of a zero set in X is called a cozero set: $coz(f) = X \setminus Z(f) = \{x \in X : f(x) \neq 0\}$.

Proposition. *The map $\Phi : X \to \mathrm{Spec}_M C(X)$ defined as $\Phi(x) = M_x = \{f \in C(X) : f(x) = 0\}$ is a homeomorphism.*

Proof. We will first prove surjectivity; suppose I is an ideal of $C(X)$ that is not contained in any M_x (and is thus contained in another maximal ideal M). Thus $\forall\ x \in X\ \exists f_x \in I : f_x(x) \neq 0$. Since each f_x is continuous, the collection of preimages $\{{f_x}^{-1}(\mathbb{R} \setminus \{0\})\}_{x \in X}$ form an open cover of X. Thus we have a finite subcover of X, $\{{f_{x_i}}^{-1}(\mathbb{R} \setminus \{0\})\}_{1 \leq i \leq k}$. It follows that the function $f = \sum_{i=1}^{k} f_x^2$ is nowhere zero, i.e. a unit contained in M; thus $M = C(X)$. Hence all maximal ideals are of the form M_x for some $x \in X$.

Since X is compact Hausdorff, it is a normal space; thus by Urysohn's Lemma, disjoint closed sets in X can be separated by a continuous function $X \to \mathbb{R}$. Thus the maximal ideals M_x, M_y coincide exactly when $x = y$, proving injectivity.

Letting $\mathcal{O}_f = \{x \in X : f(x) \neq 0\}$ be a base for a topology on X and $U_f = \{M \in \operatorname{Spec}_M C(X) : f \notin M\}$ as a base for a topology on $\operatorname{Spec}_M C(X)$, we can see that $\Phi(\mathcal{O}_f) = U_f$, and thus we conclude Φ is a homeomorphism. ☐

The Gelfand-Kolmogorov Theorem follows from the above proposition, as isomorphic rings have homeomorphic maximum ideal spaces.

Theorem 4 (Gelfand-Kolmogorov)**.** *If X, Y are compact, completely regular spaces such that $C(X)$, $C(Y)$ are isomorphic, then X and Y are homeomorphic spaces.*

4.1.2 Gelfand Duality

We now extend this observation to establish the duality between the spaces **KHaus** and **C*Alg** by characterizing a real commutative C*-algebra A that occurs as $C(X)$ for some X. This is what the commutative **Gelfand-Naimark Theorem** states: the maximal spectrum of any such algebra is a compact Hausdorff space.

We will first establish a bijection between the maximal ideals of A and the set of structure-preserving maps from $A \to \mathbb{R}$, denoted as X. Considering this set of structure-preserving maps, there are two remaining claims to prove: there is a bijection between A and $C(X)$, and X is in fact a compact Hausdorff space when given a certain topology.

Definition (Characters)**.** The set of characters on a commutative ring A, denoted $X = \operatorname{Hom}(A, \mathbb{R})$, is the set of non-zero homomorphisms $\tau : A \to \mathbb{R}$.

Proposition (Gelfand)**.** *Let A be a commutative Banach algebra. Then there is a bijection between X and $\operatorname{Spec}_M(A)$, defined by the mapping $\tau \mapsto \ker(\tau)$.*

It remains to prove that X is in fact a compact Hausdorff space. In other words, we seek the coarsest topology that makes all functions $\hat{f} : X \to \mathbb{R}$ continuous — this importance we place on the evaluation maps bears similar notions as to when we identified other dualities, such as Stone duality or that between a vector space and its dual.

Proposition. *Define a topology on X with sets of the form*

$$coz(\hat{f}) = \{M \in X : f \notin M\} = \{M \in X : \hat{f}(M) \neq 0\}$$

as its base. Then X is a compact Hausdorff space under this topology.

Proof. First we will show that X is compact under this topology. Let $\mathcal{C} = \{\operatorname{coz}(\hat{f}) | f \in S, S \subseteq A\}$ be a cover of X; in other words, no maximal ideal contains all elements of S. By the Maximal Ideal Theorem, this means the ideal generated by S is the whole of A. Thus $\exists\, \hat{S} \subseteq S, |\hat{S}| < \infty$ such that the ideal generated by $\hat{S}$ is A. It follows that $\{\operatorname{coz}(\hat{f}) | f \in \hat{S}\}$ is a finite subcover of $\mathcal{C}$.

To prove X is a compact Hausdorff space, consider two maximal ideals $M, N \in X$. We want to find two disjoint neighbourhoods of M and N; taking $f \in N - M$ and $g \in M - N$ yields two continuous, real-valued functions $\hat{f}, \hat{g}$ such that the open sets $N \in \{P \in X : |\hat{f}| > |\hat{g}|\}$, $M \in \{P \in X : |\hat{g}| > |\hat{f}|\}$ are disjoint. ☐

The basic open sets in the previous proposition bear striking resemblance to the sets $U_f = \{M \in \operatorname{Spec}_M C(X) : f \notin M\}$ which served as a topology on $\operatorname{Spec}_M C(X)$ in 4.1.1.

Now for all $f \in A$, consider the map $\hat{f} : X \to \mathbb{R}$ defined by $\hat{f}(x) = x(f) = \mathrm{eval}_x(f)$.

In the next proposition, we claim that the map $f \mapsto \hat{f}$ is an isomorphism from $A \to C(X)$. To prove injectivity, we will require a result that follows from the Maximal Ideal Theorem, and to prove surjectivity, we will use a variant of the Stone-Weierstrass Theorem (both of which are outlined below but will not be proven explicitly— proofs can be found in [6]).

Lemma (Maximal Ideal Theorem)**.** *Every non-unit of a nonzero ring is contained in some maximal ideal.*

Lemma (Stone-Weierstrass Theorem)**.** *Let X be a compact Hausdorff space and A be a subalgebra of C(X) that separates points (i.e. $\forall\, x, y \in X\ \exists\, f \in A : f(x) \neq f(y)$). Then $A = C(X)$.*

We require that the map $A \mapsto C(\operatorname{Spec}_M(A))$ to be an isometry, which doesn't hold for norm-decreasing morphisms between general Banach algebras; C*-algebras, however, satisfy this norm-preserving condition:

Lemma. *$A \mapsto C(\operatorname{Spec}_M(A))$ is a norm-preserving map.*

Proposition. *The map $\phi : A \to C(X)$ defined as $\phi(f) = \hat{f}$ is a structure-preserving, bijective map.*

Sketch of Proof. It follows from some algebraic manipulation that $\phi(f) = \hat{f}$ is a ring homomorphism from $A \mapsto C(\operatorname{Spec}_M(A))$. Since this mapping is norm-preserving and A is a complete normed space, the image of $\phi(A)$ is a closed subalgebra of $C(\operatorname{Spec}_M(A))$; by the Stone-Weierstrass Theorem, we can conclude that $A = C(\operatorname{Spec}_M(A))$. □

We can now state the major theorem establishing Gelfand duality with all of these results:

Theorem 5 (Gelfand-Naimark)**.** *Let A be a C*-algebra. Then the map $A \mapsto \operatorname{Spec}_M(A)$ is an isomorphism. It follows that the categories* $\mathbf{KHaus}$ *and* $\mathbf{C^*Alg}$ *are dually equivalent given the functors* $\mathbf{H} : \mathbf{C^*Alg} \to \mathbf{KHaus}^{\mathrm{op}}$ *where* $A \mapsto \max(A)$ *and* $\mathbf{A} : \mathbf{KHaus}^{\mathrm{op}} \to \mathbf{C^*Alg}$ *where* $X \mapsto C(X)$.

4.2 Minimizing Belief Automata

With Gelfand duality, we can now establish contravariant functors between the categories **CHA** and $\mathbf{CAO}^{\mathrm{op}}$ of compact Hausdorff automata, denoted (K, Δ, Γ), and real commutative C*-algebras with two operators, denoted as $(C, \{(a)\}, \{\underline{\omega}\})$. [BKP] show the isomorphisms $\mathbf{CHA} \cong \mathbf{Coalg}(\mathrm{T})$ where $T : \mathbf{KHaus} \to \mathbf{KHaus}$ and $\mathbf{CAO} \cong \mathbf{Alg}(\mathrm{L})$ where $L : \mathbf{C^*Alg} \to \mathbf{C^*Alg}$.

Recalling [BKP]'s Theorem 1, we have $\mathbf{Coalg}(\mathrm{T})$ and $\mathbf{Alg}(\mathrm{L})$ are dually equivalent with $\hat{F} : \mathbf{Coalg}(\mathrm{T}) \to \mathbf{Alg}(\mathrm{L})^{\mathrm{op}}$ and $\hat{G} : \mathbf{Alg}(\mathrm{L})^{\mathrm{op}} \to \mathbf{Coalg}(\mathrm{T})$ as the corresponding contravariant functors. These functors are defined by [BKP] as $T = (-)^{\mathcal{A}} \times \mathrm{Sub}(\mathcal{O})\ :\ \mathbf{KHaus} \to \mathbf{KHaus}$ and $L\ :\ \mathbf{C^*Alg} \to \mathbf{C^*Alg}$ where $LC = \coprod_{a \in \mathcal{A}} C + F_{\mathbf{C^*Alg}}(\mathcal{O})/J$. The functor $F_{\mathbf{C^*Alg}}$ was stated to be the left-adjoint of the functor from $\mathbf{C^*Alg}$ to **Set** mapping a C*-Algebra to its unit interval $I = \{c \in C : c \in [0, 1]\}$. Although not explicitly given in the minimization paper, Negrepontis [7] gives a definition which we include in the following proposition:

Proposition. *Let $D : \mathbf{C^*Alg} \to \mathbf{Set}$ be the functor defined by $DA = \{a \in A : 0 \le a \le 1\}$ and $Df = f_{|DA}$, for $A \in \mathbf{C^*Alg}$, $f : A \to A'$. Then D has a left adjoint, denoted $\mathbf{F_{C^*Alg}} : \mathbf{Set} \to \mathbf{C^*Alg}$. Let $X, Y \in \mathbf{Set}$, $\alpha : X \to Y$, $g \in C(I^X)$, and $\gamma \in I^Y$ where elements of I^X are functions from X to the unit interval $X \to [0,1]$. Then $\mathbf{F_{C^*Alg}}$ is defined as $\mathbf{F_{C^*Alg}}(X) = C(I^X)$ and $\mathbf{F_{C^*Alg}}\,\alpha(g)(\gamma) = g(\gamma \circ \alpha)$.*

Given $(K, \Delta, \Gamma) \in \mathbf{CHA}$, the dual object $\mathbf{A}(K, \Delta, \Gamma) = (C, \{(a)\}, \{\underline{\omega}\})$ follows by Gelfand duality; from the previous section, we know that C is the C*-algebra of continuous functions from $K \to \mathbb{R}$. In the reverse direction, given $(C, \{(a)\}, \{\underline{\omega}\})$, we can assume that $C \cong C(X)$ for some compact Hausdorff space. Thus, we can define the dual compact Hausdorff automaton $\mathbf{H}(C, \{(a)\}, \{\underline{\omega}\}) = (K, \Delta, \Gamma)$ where $K = \mathrm{Hom}(C, \mathbb{R})$. In the category of real commutative C*-algebras with operators, we retrieve C_M, the minimal subobject(subalgebra) of $\hat{F}(\mathcal{K})$; it follows that $\hat{G}(C_M)$ is the minimization of S. We explore how to construct this minimal subalgebra C_M. [BKP] identifies this as a C*-algebra generated by the set $[\![\mathcal{T}]\!] = \{[\![t]\!] : t \in \mathcal{T}\}$ by adding constant functions, closing under the ring operations, and under limits in the sup norm. This guarantees the resultant C_M is the smallest sub-algebra containing $[\![\mathcal{T}]\!]$, but it is also equivalent to

$$C_m = \bigcap_{[\![\mathcal{T}]\!] \subseteq B_{\mathcal{T}}} B_{\mathcal{T}},$$

where each $B_{\mathcal{T}}$ is a subalgebra of A given the operators $(a), \underline{\omega}$. It is an algebraic/topological exercise to prove that the intersection of two C*-algebras is another C*-algebra— we can check it satisfies all conditions given its definition in 4.1.1. We can also check that $(a), \underline{\omega}$ are well-defined because any $B_{\mathcal{T}}$ can interpret tests by $[\![\mathcal{T}]\!]$ based on its construction as a subalgebra containing the test functions. Retrieving the subalgebras containing $\mathcal{T}$ is a computationally much less practical task, but may allow for computations of 'minimized' automata without having reached the minimal sub-algebra.

4.3 Extensions

We have explored the derivation of real Gelfand duality and its application to minimizing belief automata. This minimization algorithm is a more unorthodox approach in the widely explored field of automata minimization. While this application of duality introduces a new class of algorithms— considering different automata and modelling them to fit under the framework of other types of dualities, for instance— the practicality of returning the minimized automaton is a topic to be studied further. The computational difficulty of the process is multifaceted considering the transformation of our automaton and identifying the minimal C*-algebra. The contravariant adjoint functors defined by Gelfand duality are concisely described as mapping a compact Hausdorff space to its space of real-valued continuous functions and dually mapping a C*-algebra to its maximal ideal space, but this is far from trivial to compute!

Compared to several existing DFA minimization algorithms, this bears resemblance to Brzozowski's algorithm [3] in motivation. By reversing the transitions twice, we can obtain a minimized deterministic automaton using a 'powerset construction' in the intermediate step. This algorithm is an example of how worst-case exponential complexity can produce surprising results comparable to more conventional algorithms like Hopcroft's, as explored in [9]. There is still room to explore with automata minimization via duality, equipping the exact algorithm with approximation metrics or the identification of incrementally smaller subalgebras satisfying the equivalence classes defined by the test equivalences.

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