Bisimulation on General Probability Spaces

Diego Lopez

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1 Background

1.1 Motivation

In the study of state transition systems, a notion of behavioral similarity called bisimilarity was first defined for finite state spaces. In practice, however, many spaces of interest are not finite. In order to extend this definition to infinite state spaces one can no longer consider all possible transitions and instead must restrict one's view to maps which are measurable. This was done for example in [3] where the authors defined a notion of bisimulation for measurable spaces, however this notion fails to be transitive for spaces which are not analytic, that is spaces which are not continuous images of Polish spaces, as explained in [1].

In [1], which we shall review, the authors define a notion of bisimilarity for general probability spaces. The resulting theory is built on top of an abstract theory of cones, and, as we shall see, once again restricts which state transitions are under consideration. In addition, this theory considers state transitions from the point of view of how they act on function spaces, rather than points or measurable sets.

1.2 Notation

We keep the same notation used in the paper. For instance, the set $\mathbb{R}^+$ denotes the set $[0, \infty) \subseteq \mathbb{R}$.

Given a measurable space (X, Σ), a measure μ on X, and a non-negative measurable map $f : X \to [0, \infty]$, we denote the measure

$$A \in \Sigma \mapsto \int_A f \, \mathrm{d}\mu \in [0, \infty]$$

by $f \cdot \mu$.

Given two measures μ and ν on the same measurable space (X, Σ), if μ is absolutely continuous with respect to ν, i.e., if $\mu(A) = 0$ whenever $\nu(A) = 0$ for every $A \in \Sigma$, we write $\mu \ll \nu$.

Furthermore, given a measurable map between measurable space $f : (X_1, \Sigma_1) \to (X_2, \Sigma_2)$ and a measure μ on X_1, we denote the pushforward measure

$$A \in \Sigma_2 \mapsto \mu(f^{-1}(A)) \in [0, \infty]$$

by $M_f(p)$.

Lastly, in a category $\mathbf{C}$ we will denote the hom-set between objects X and Y as $\mathbf{C}[X, Y]$. All categories considered here are locally small so these indeed are sets.

1.3 Markov Processes

Definition (Markov Kernel)**.** Let (X, Σ) and (Y, Λ) be measurable spaces. A *Markov kernel* from X to Y is a map

$$\tau : X \times \Lambda \to [0, 1]$$

such that

(i) for every $x \in X$, the function $\tau(x, \cdot) : \Lambda \to [0, 1]$ is a sub-probability measure on Y; and

(ii) for every $B \in \Lambda$, the function $\tau(\cdot, B) : X \to [0,1]$ is a measurable function.

This is different from the standard definition. Indeed, it is more general in that the authors only require the maps $\tau(x, \cdot)$ to be sub-probability measures, instead of requiring them to be probability measures.

Definition (Labelled Markov Process)**.** Let (X, Σ) be a probability space. A *labelled Markov process* on (X, Σ) is a collection of Markov kernels τ_a from X to X indexed by a finite or countable set $\mathcal{A}$. The elements of $\mathcal{A}$ are called *actions*.

We note that this is distinct from the original definition given by [3] in that we do not require the spaces (X, Σ) to be analytic. The goal of the authors of [1] is to define bisimulation on a broader class of spaces.

2 Cones

2.1 Elementary Theory

Definition (Cone)**.** A cone is a set V together with two operations $(+) : V \times V \to V$ and $(\cdot) : \mathbb{R}^+ \times V \to V$. In addition is assumed to be commutative, associative and it has an identity, denoted $0 \in V$. Multiplication is associative and distributes over addition. Finally, for all $u, v, w \in V$, the following cancellation property holds

$$v + u = w + u \implies v = w$$

as well as the following strictness property:

$$v + w \implies v = w = 0.$$

Lastly, if $u + v = w$, then may we write $w - v$ for u.

These axioms mimic a concept with the same name, where instead of definining a structure abstractly we define a cone as a subset of a real vector space satisfying certain properties (i.e., strictness). However, we can see that the authors of [1] chose to take the abstract route as there are some spaces for which a cone structure naturally arises but a vector space structure does not. This is the case for finite measures on a measurable space (X, Σ), for instance.

Proposition. *Let V be a cone. Then, V has a natural ordering given by $u \le v$ whenever there is some $w \in V$ such that $u + w = v$.*

A good example that illustrates this fact is considering the set $\mathbb{R}^+$ itself as a cone with the usual addition and multiplication operations. Then, the natural order coincides with the usual order.

Definition (Normed Cone)**.** A normed cone is a cone together with a function $\|\cdot\| : C \to \mathbb{R}^+$ such that

(1) $\|v\| = 0 \iff v = 0$;
(2) for every $r \in \mathbb{R}^+$ and $v \in C$, we have that $\|r \cdot v\| = r\|v\|$;
(3) $\|u + v\| \le \|u\| + \|v\|$;
(4) $u \le v \implies \|u\| \le \|v\|$;

Normed cones as defined here once again mimic a structure with the same name, were we explicitly look for a subset of a real normed vector space with a strictness property. We note again that the authors of [1] choose the abstract structure because, for instance, finite measures on a measurable space (X, Σ) naturally forms a normed cone structure (but it does not form a normed vector space). Indeed, the norm of a finite measure μ of (X, Σ) is taken to be $\mu(X)$.

Definition (ω-Complete Normed Cone)**.** An *ω-complete normed cone* is a normed cone such that

(1) if $\{u_i\}_{i \in I}$ is a chain with bounded norm, i.e. $\sup_{i \in I} \|u_i\| < \infty$, then the least upper bound

$$\bigvee_{i \in I} u_i$$

exists, and

(2) we have that

$$\bigvee_{i \in I} \|u_i\| = \Big\| \bigvee_{i \in I} u_i \Big\|.$$

Since cones do not have a well-defined substraction operation, we cannot talk about 'distances' between two points (as $\|x - y\|$ is not well-defined), so we must use ideas from order theory to define a kind of 'completeness' for normed cones. In order to work with ω-complete normed cones, the following lemma will prove useful.

Lemma 2.1. *Let $\{u_i\}_{i=1}^{\infty}$ be a countable chain with upper bound u such that $\lim_{i \to \infty} \|u - u_i\| = 0$. Then,*

$$u = \bigvee_{i=1}^{\infty} u_i.$$

Definition (Linear Map)**.** A map $f : C \to D$ is said to be *linear* if for every $u, v \in C$ and every $r, s \in \mathbb{R}^+$, we have

$$f(ru + sv) = rf(u) + sf(v).$$

Definition (ω-Continuous)**.** A linear map $f : C \to D$ is said to be *ω-continuous* if for every countable chain $\{u_i\}_{i=1}^{\infty}$ in C such that $\bigvee_{i=1}^{\infty} u_i$ exists, then we have that

$$\bigvee_{i=1}^{\infty} f(u_i) = f\Big(\bigvee_{i=1}^{\infty} u_i \Big).$$

Proposition. *Every ω-continuous linear map $f : C \to D$ is bounded in the sense that*

$$\sup_{\|u\|_C \leq 1} \|f(u)\|_D < \infty.$$

Definition (Dual Cone)**.** Let C be an ω-complete normed cone. The *dual cone*, denoted C^*, is the set of ω-continuous linear maps from C to $\mathbb{R}^+$ with pointwise addition, scalar multiplication, and norm

$$\|f\|_* = \sup_{\|u\| \leq 1} \|f(u)\|.$$

Definition (Dual Map)**.** Let $f : C \to D$ be an ω-continuous linear map. The dual map $f^* : D^* \to C^*$ is given by

$$f^*(L) = L \circ f.$$

It is illustrated by the following diagram.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "f^*(L)"'] & D \arrow[d, "L"] \\ & \mathbb{R}^+ \end{tikzcd}$$

Definition (The ω**CC** Category)**.** The category $\omega\mathbf{CC}$ has objects ω-complete normed cones and arrows ω-continuous linear maps.

Proposition. *The dual operation is a contravariant functor $(-)^* : \omega\mathbf{CC} \to \omega\mathbf{CC}$.*

2.2 Cones of Measures and of Measurable Functions

Definition ($\mathcal{L}^+(X,\Sigma)$)**.** Let (X,Σ) be a measureable space. The normed cone $\mathcal{L}^+(X,\Sigma)$ is the set of measurable maps from X to $\mathbb{R}^+$.

Definition ($L_1^+(X,\Sigma,\mu)$ and $L_\infty^+(X,\Sigma,\mu)$)**.** Let (X,Σ,μ) be a measure space. Then, the normed cones $L_1^+(X,\Sigma,\mu)$ and $L_\infty^+(X,\Sigma,\mu)$ are the almost-everywhere non-negative functions in the normed vector spaces $L_1(X,\Sigma,\mu)$ and $L_\infty(X,\Sigma,\mu)$, respectively.

Proposition. *The normed cones $L_1^+(X,\Sigma,\mu)$ and $L_\infty^+(X,\Sigma,\mu)$ are ω-complete.*

Proposition. *The normed cones $L_1^+(X,\Sigma,\mu)$ and $L_\infty^{+,*}(X,\Sigma,\mu)$ are isomorphic in ω**CC**, as are the cones $L_\infty^+(X,\Sigma,\mu)$ and $L_1^{+,*}(X,\Sigma,\mu)$. Moreover, the isomorphism is even stronger in the sense that there is a isometric isomorphism between both pairs of normed cones.*

This isomorphism is very important, and much of the results rely on this fact. In functional analysis, the normed vector spaces $L_1^*(X,\Sigma,\mu)$ and $L_\infty(X,\Sigma,\mu)$ are isomorphic, but the spaces $L_1(X,\Sigma,\mu)$ and $L_\infty^*(X,\Sigma,\mu)$ are not. This is the reason why the present paper builds its theory on top of normed cone theory instead of standard functional analysis. We will see later how this duality allows us to define bisimulation on general probability spaces.

Definition (The $\mathbf{Rad}_\infty$ Category)**.** The $\mathbf{Rad}_\infty$ category has objects probability spaces and arrows measurable maps

$$\alpha : (X,\Sigma,p) \to (Y,\Lambda,p)$$

such that the pushforward measure $M_\alpha(p)$ satisfies $M_\alpha(p) \leq Kq$ for some $K > 0$. That is,

$$p(\alpha^{-1}(A)) \leq Kq(A)$$

for every $A \in \Lambda$.

Definition (The $\mathbf{Rad}_1$ Category)**.** The $\mathbf{Rad}_1$ category has objects probability spaces and arrows measurable maps

$$\alpha : (X,\Sigma,p) \to (Y,\Lambda,q)$$

such that the pushforward measure $M_\alpha(p)$ is absolutely continuous with respect to q.

The reason for defining these categories in the way that they are is due to the fact that we can map morphisms in $\mathbf{Rad}_1$ and $\mathbf{Rad}_\infty$ to elements of $L_1(X,\Sigma,p)$ or $L_\infty(X,\Sigma,p)$. It is clear that if $\alpha \in \mathbf{Rad}_1[(X,\Sigma,p),(Y,\Lambda,q)]$, then we have that

$$\frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p) \in L_1^+(Y,\Lambda,q).$$

However, for $\mathbf{Rad}_\infty$ we need the following lemma.

Lemma 2.2. *If $\alpha : (X,\Sigma,p) \to (Y,\Lambda,q)$ satisfies $M_\alpha(p) \leq K(q)$ for some real positive constant K, then*

$$\frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p) \in L_\infty^+(Y,\Lambda,q).$$

3 Conditional Expectation Functor

Definition (The Pairing)**.** Let (X,Σ,p) be a probability space. We define the following map $\langle -,- \rangle : L_\infty^+(X,\Sigma,p) \times L_1^+(X,\Sigma,p) \to \mathbb{R}^+$ by

$$\langle f,g \rangle_X = \int_X fg \, \mathrm{d}p.$$

We can now express the isomorphism between $L_\infty^+(X,\Sigma,p)$ and $L_1^{+,*}(X,\Sigma,p)$ with the pairing. The isomorphism $L_\infty^+(X,\Sigma,p) \to L_1^{+,*}(X,\Sigma,p)$ is given by

$$f \mapsto \langle f, - \rangle_X.$$

Definition (Precomposition Functors P_1 and P_∞)**.** We define the contravariant functor $P_1 : \mathbf{Rad}_\infty \to \omega\mathbf{CC}$. On objects it maps probability spaces (X,Σ,p) to $L_1^+(X,\Sigma,p)$ and maps a morphism

$$\alpha \in \mathbf{Rad}_\infty[(X,\Sigma,p),(Y,\Lambda q)]$$

to the morphism

$$P_1(\alpha) \in \omega\mathbf{CC}[L_1^+(Y,\Lambda,p), L_1^+(X,\Sigma,q)]$$

given by precomposition, i.e.,

$$P_1(\alpha)(f) = f \circ \alpha.$$

On the other hand, the contravariant functor $P_\infty : \mathbf{Rad}_1 \to \omega\mathbf{CC}$ maps probability spaces (X,Σ,p) to $L_\infty^+(X,\Sigma,p)$ and maps morphisms $\alpha \in \mathbf{Rad}_1[(X,p),(Y,q)]$ to the map

$$P_\infty(\alpha) \in \omega\mathbf{CC}[L_\infty^+(Y,\Lambda,p), L_\infty^+(X,\Sigma,q)]$$

also given by precomposition, i.e.,

$$P_\infty(\alpha)(f) = f \circ \alpha.$$

Proposition. *The precomposition functors P_1 and P_∞ are well defined.*

Proof. We wish to show that if $g \in L_1^+(Y,\Lambda,p)$ and $\alpha \in \mathbf{Rad}_\infty[(X,\Sigma,p),(Y,\Lambda,q)]$, then $P_1(\alpha)(g) \in L_1^+(X,\Sigma,q)$. We observe that $\frac{\mathrm{d}}{\mathrm{d}q}M_\alpha(p) \in L_\infty^+(X,\Sigma,q)$ by lemma 2.2. So, we have

$$\begin{aligned}
\int_X P_1(\alpha)(g)\ \mathrm{d}p &= \int_X g \circ \alpha\ \mathrm{d}p \\
&= \int_X g\ \mathrm{d}M_\alpha(p) \\
&= \int_Y g \frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p)\ \mathrm{d}q \\
&= \left\langle \frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p), g \right\rangle_Y.
\end{aligned}$$

Thus, $P_1(\alpha)(g) \in L_1^+(X,\Sigma,p)$. It remains to show that $P_1(\alpha)$ is ω-continuous as it is clearly linear. Consider a chain $\{g_i\}_{i=1}^\infty$ in $L_1^+(Y,p)$ such that

$$g = \bigvee_{i=1}^\infty g_i.$$

We wish to show that

$$\bigvee_{i=1}^\infty P_1(\alpha)(g_i) = P_1(\alpha)(g).$$

We have that $g_i \leq g$ so $g_i + h_i = f$ for some $h_i \in L_1^+$. Thus,

$$\begin{aligned}
P_1(\alpha)(g) &= P_1(\alpha(g_i + h_i)) \\
&= P_1(\alpha)(g_i) + P_1(\alpha)(h_i),
\end{aligned}$$

which implies that $P_1(\alpha)(g_i) \leq P_1(\alpha(g))$. Using the monotone convergence theorem, we have that

$$\begin{aligned}
\lim_{i\to\infty} \|P_1(\alpha)(g) - P_1(\alpha)(g_i)\|_X &= \lim_{i\to\infty} \int_X P_1(\alpha)(g) - P_1(\alpha)(g_i) \, \mathrm{d}p \\
&= \int_X \lim_{i\to\infty} P_1(\alpha)(g) - P_1(\alpha)(g_i) \, \mathrm{d}p \\
&= \int_X \lim_{i\to\infty} (g - g_i) \circ \alpha \, \mathrm{d}p \\
&= \int_X \lim_{i\to\infty} (g - g_i) \, \mathrm{d}M_\alpha(p) \\
&= \int_Y \lim_{i\to\infty} (g - g_i) \frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p) \, \mathrm{d}q.
\end{aligned}$$

However, we know that $\lim_{i\to\infty}(g - g_i)$ is zero q-almost everywhere and so we have that

$$\lim_{i\to\infty} \|P_1(\alpha)(g) - P_1(\alpha)(g_i)\| = 0$$

Thus, by 2.1 we have that $P_1(\alpha)$ is ω-continuous.

Now, we show that if $f \in L^+_\infty(Y, p)$ and $\alpha \in \mathbf{Rad}_1[(X, p), (Y, q)]$, then $P_\infty(\alpha)(f) \in L_1(X, \Sigma, p)$. We have

$$\begin{aligned}
\int_X P_\infty(\alpha)(g) \, \mathrm{d}p &= \int_Y f \circ \alpha \, \mathrm{d}p \\
&= \int_Y f \, \mathrm{d}M_\alpha(p) \\
&= \int_Y f \frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p) \, \mathrm{d}q \\
&= \Big\langle f, \frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(p) \Big\rangle.
\end{aligned}$$

Thus, we have that $P_\infty(\alpha)(f) \in L_1(X, \Sigma, p)$. The proof that $P_\infty(\alpha)$ is also ω continuous is similar to the one for $P_1(\alpha)$. □

Definition ($\mathbb{E}_\infty(\cdot)$ Functor)**.** We define the functor $\mathbb{E}_\infty(\cdot)$ from the category $\mathbf{Rad}_\infty$ to the category $\omega\mathbf{CC}$. On objects, the functor maps the probability space (X, Σ, p) to $L^+_\infty(X, \Sigma, p)$. On arrows, the functor maps the arrow $\alpha : (X, \Sigma, p) \to (Y, \Lambda, q)$ to the map $\mathbb{E}_\infty(\alpha)$ such that the following diagram commutes

$$\begin{tikzcd}
L_1^{+,*}(X, \Sigma, p) \arrow[d, "(P_1(\alpha))^*"'] & L^+_\infty(X, \Sigma, p) \arrow[l, dotted] \arrow[d, "\mathbb{E}_\infty(\alpha)"] \\
L_1^{+,*}(Y, \Lambda, q) \arrow[r, dotted] & L^+_\infty(Y, \Lambda, q)
\end{tikzcd}$$

where the dotted lines correspond to the isomorphisms discussed earlier.

This is an instance where we make explicit use of the fact that the normed cones $L_1^{+,*}(X, \Sigma, p)$ and $L^+_\infty(X, \Sigma, p)$ are isomorphic.

Proposition. *For every arrow $\alpha \in \mathbf{Rad}_\infty[(X, \Sigma, p), (Y, \Lambda, q)]$ and every $f \in L^+_\infty(X, \Sigma, p)$, we have that*

$$\mathbb{E}_\infty(\alpha)(f) = \frac{\mathrm{d}}{\mathrm{d}q} M_\alpha(f \cdot p).$$

Proof. The isomorphism from $L_\infty^+(X,\Sigma,p)$ to $L_1^{+,*}(X,\Sigma,p)$ sends f to the map $\langle f,-\rangle_X : L_1^+(X,\Sigma,p) \to \mathbb{R}^+$. Then, applying the morphism $(P_1(\alpha))^*$ we have

$$\begin{aligned}
(P_1(\alpha))^*(\langle f,-\rangle_X) &= \langle f,-\rangle_X \circ (P_1(\alpha)) \\
&= g \in L_1^{+,*}(Y,q) \mapsto \langle f,-\rangle_X \circ (P_1(\alpha))(g) \\
&= g \mapsto \langle f,-\rangle_X(P_1(\alpha)(g)) \\
&= g \mapsto \langle f, P_1(\alpha)(g)\rangle_X \\
&= g \mapsto \int_X fP_1(\alpha)(g))\,\mathrm{d}p \\
&= g \mapsto \int_X P_1(\alpha)(g)\,\mathrm{d}f\cdot p \\
&= g \mapsto \int_X g\circ\alpha\,\mathrm{d}f\cdot p \\
&= g \mapsto \int_Y g\,\mathrm{d}M_\alpha(f\cdot p) \\
&= g \mapsto \int_Y g\frac{\mathrm{d}}{\mathrm{d}q}M_\alpha(f\cdot p)\,\mathrm{d}q \\
&= \left\langle \frac{\mathrm{d}}{\mathrm{d}q}M_\alpha(f\cdot p), - \right\rangle_Y.
\end{aligned}$$

Taking the isomorphism from $L_1^{+,*}(Y,q)$ to $L_\infty^+(Y,q)$ gives us

$$\mathbb{E}_\infty(\alpha)(f) = \frac{\mathrm{d}}{\mathrm{d}q}M_\alpha(f\cdot p),$$

as desired. □

4 Labelled Abstract Markov Processes

Definition (Abstract Markov Kernel)**.** An *abstract Markov kernel* from (X,Σ,p) to (Y,Λ,q) is an ω-continuous linear map

$$\tau : L_\infty^+(Y,\Lambda,q) \to L_\infty^+(X,\Sigma,p)$$

such that $\|\tau\| \le 1$.

This definition is central to the definition of bisimulation on probability spaces. It accomplishes two things: firstly, it changes the perspective from thinking about points and measurable sets to thinking about function spaces, and secondly it considers only Markov kernels which are compatible with the measure in some sense.

We start by discussing the former. Given a Markov kernel from (X,Σ) to (Y,Λ) $\hat{\tau} : X\times\Lambda \to [0,1]$ we can define an operator $\tau : \mathcal{L}^+(Y,\Lambda) \to \mathcal{L}^+(X,\Sigma)$ given by

$$\tau(f)(x) = \int_Y f\mathrm{d}\hat{\tau}(x,-).$$

This operator can be easily shown to have norm at most 1. We may also go backwards. That is, given an operator

$$\tau : \mathcal{L}^+(Y,\Lambda) \to \mathcal{L}^+(X,\Sigma)$$

with norm at most 1, we can define a Markov kernel informally as

$$\hat{\tau}(x,B) = \tau(\mathbf{1}_B)(x).$$

Here we note the usefulness of using subprobability measures rather than strictly using probability measures in the definition of a Markov kernel.

Now, we discuss the latter. It is tempting to simply consider τ as acting on equivalence classes of functions in $\mathcal{L}^+(Y,\Lambda)$ and $\mathcal{L}^+(X,\Sigma)$ with respect to almost everywhere equality in the corresponding measure. However, this process is not well-defined for all Markov kernels. The Markov kernels that we need are called *nonsingular Markov kernels.* The setup is that we now have a measure p on (X,Σ) and a measure q on (Y,Λ). A Markov kernel $\hat{\tau}$ from X to Y is said to be nonsingular if for every measurable set $B\in\Lambda$ such that $q(B)=0$, we have that $\hat{\tau}(x,B)=0$ for p-almost every $x\in X$. In fact, there is a bijection between nonsingular Markov kernels and abstract Markov kernels, and it can be described informally as

$$\hat{\tau}(x,B)=\tau(\mathbf{1}_B)(x).$$

However, this needs to be made precise as $\tau(\mathbf{1}_B)$ is strictly speaking an equivalence class of measureable functions in X and so evaluation is only defined p-almost everywhere on X.

Rigor aside, the important facts are that this definition captures exactly what we need since it only considers the Markov kernels which are compatible with the measure and views them in a perspective in which we can develop the theory.

Definition (Labelled Abstract Markov Process)**.** Let (X,Σ,p) be a probability space. A labelled abstract Markov process on (X,Σ,p) with respect to a given set of labels or actions $\mathcal{A}$ is a family of abtract Markov kernels

$$\tau_a : L^+_\infty(X,\Sigma,p)\to L^+_\infty(X,\Sigma,p)$$

indexed by elements of $\mathcal{A}$.

Definition (Projections of Markov Processes)**.** Let (X,Σ,p) be a probability space. Given an AMP τ_a on X and a morphism $\alpha\in\mathbf{Rad}_\infty[(X,\Sigma,p),(Y,\Lambda,q)]$, we define the AMP $\alpha(\tau_a)$ to be given by the following family of commutative diagrams indexed by $\mathcal{A}$:

$$\begin{tikzcd}
L^+_\infty(X,\Sigma,p) \arrow[r, "\tau_a"] & L^+_\infty(X,\Sigma,p) \arrow[d, "\mathbb{E}_\infty(\alpha)"] \\
L^+_\infty(Y,\Lambda,q) \arrow[u, "P_\infty(\alpha)"] \arrow[r, dotted, "\alpha(\tau_a)"'] & L^+_\infty(Y,\Lambda,q)
\end{tikzcd}$$

This AMP is called the *projection of* τ_a *on* (Y,Λ,q).

Definition (The **AMP** Category)**.** The category **AMP** has objects probability spaces together with abstract Markov process τ_a on X. We note that the set of actions here is fixed. An arrow $\alpha:(X,\Sigma,p,\tau_a)\to(Y,\Lambda,q,\rho_a)$ is a surjective, measure-preserving map from X to Y such that

$$\alpha(\tau_a)=\rho_a,$$

i.e. the following diagram commutes for every $a\in\mathcal{A}$:

$$\begin{tikzcd}
L^+_\infty(X,\Sigma,p) \arrow[r, "\tau_a"] & L^+_\infty(X,\Sigma,p) \arrow[d, "\mathbb{E}_\infty(\alpha)"] \\
L^+_\infty(Y,\Lambda,q) \arrow[u, "P_\infty(\alpha)"] \arrow[r, "\rho_a"'] & L^+_\infty(Y,\Lambda,q)
\end{tikzcd}$$

5 Bisimulation

Definition (Event Bisimulation)**.** Given an object of **AMP** (X,Σ,p,τ_a), an *event-bisimulation* is a sub-σ-algebra Λ of Σ such that the following diagram commutes:

$$\begin{tikzcd} L^+_\infty(X,\Sigma,p) \arrow[r, "\tau_a"] & L^+_\infty(X,\Sigma,p) \\ L^+_\infty(X,\Lambda,q) \arrow[u, hook] \arrow[r, "\tau_a"'] & L^+_\infty(X,\Lambda,p) \arrow[u, hook] \end{tikzcd}$$

Definition (Zigzag)**.** A *zigzag* from an abstract Markov process (X, Σ, p, τ_a) to another abstract Markov process (Y, Λ, q, ρ_a) is a measurable, measure-preserving surjective function α from X to Y such that the following family of diagrams indexed by $\mathcal{A}$ commute:

$$\begin{tikzcd} L^+_\infty(X,\Sigma,p) \arrow[r, "\tau_a"] & L^+_\infty(X,\Sigma,p) \\ L^+_\infty(Y,\Lambda,q) \arrow[u, "P_\infty(\alpha)"] \arrow[r, "p_a"'] & L^+_\infty(Y,\Lambda,q) \arrow[u, "P_\infty(\alpha)"'] \end{tikzcd}$$

Proposition. *If $\alpha : (X, \Sigma, p, \tau_a) \to (Y, \Lambda, q, \rho_a)$ is a zigzag, then it is a morphism in* **AMP***.*

Definition (Bisimulation on **AMP**)**.** Two objects of **AMP**, (X, Σ, p, τ_a) and (Y, Λ, q, ρ_a) are said to be *bisimular* if there exists another object (Z, Γ, r, π_a) and a pair of zigzags

$$\alpha : (X, \Sigma, p, \tau_a) \to (Z, \Gamma, r, \pi_a)$$
$$\beta : (Y, \Lambda, q, \rho_a) \to (Z, \Gamma, r, \pi_a).$$

In other words, two objects are bisimilar if there exists a cospan diagram

$$\begin{tikzcd} (X,\Sigma,p,\tau_a) \arrow[dr, "\alpha"'] & & (Y,\Lambda,q,\rho_a) \arrow[dl, "\beta"] \\ & (Z,\Gamma,r,\pi_a) & \end{tikzcd}$$

where α and β are zigzags.

Definition (Bisimulation-Minimal Realization)**.** A bisimulation-minimal raelization of an object of **AMP** (X, Σ, p, τ_a) is another object of **AMP** $(\tilde{X}, \Gamma, r, \pi_a)$ and a zigzag $\eta : X \to \tilde{X}$ such that for any third object (Y, Λ, q, ρ_a), whenever there is a zigzag β from X to Y, there is a unique zigzag γ from (Y, Λ, q, ρ_a) to $(\tilde{X}, \Gamma, r, \pi_a)$ such that $\eta = \gamma \circ \beta$.

Theorem 5.1 (Main Results)**.**

(1) Bisimulation is an equivalence relation on the objects of **AMP***.*

(2) Every object of **AMP** *(X, Σ, p, τ_a) has a corresponding object $(\tilde{X}, \Gamma, r, \pi_a)$ and a zigzag $\eta : X \to \tilde{X}$ giving a bisimulation-minimal realization of (X, Σ, p, τ_a).*

(3) Two objects of **AMP** *(X, Σ, p, τ_a) and (Y, Λ, q, ρ_a) are bisimilar if and only if their bisimulation-minimal realizations are isomorphic in* **AMP***.*

(4) Let $\mathcal{L}$ be a logic given by

$$\mathcal{L} \coloneqq \mathbf{T} \mid \phi \wedge \varphi \mid \langle a \rangle_q \varphi$$

where $q \in \mathbb{Q}$ and $a \in \mathcal{A}$. Then, given an object of **AMP** *(X, Σ, p, τ_a) we assign each formula φ to a measurable set $[\![\varphi]\!] \in \Sigma$ defined recursively by*

$$[\![\mathbf{T}]\!] = X$$
$$[\![\phi \wedge \varphi]\!] = [\![\phi]\!] \cap [\![\varphi]\!]$$
$$[\![\langle a \rangle_q \varphi]\!] = \{s \in X \mid \tau_a(\mathbf{1}_{[\![\varphi]\!]})(s) > q\}.$$

Let $[\![\mathcal{L}]\!]$ denote the set of all such measurable sets. Then, the σ-algebra $\sigma([\![\mathcal{L}]\!])$ generated by $[\![\mathcal{L}]\!]$ is the smallest event-bisimulation on (X, Σ, p, τ_a) in the sense that given any zigzag $\alpha : (X, \Sigma, p, \tau_a) \to (Y, \Lambda, q, \rho_a)$ we have that $\sigma([\![\mathcal{L}]\!]) \subseteq \alpha^{-1}(\Lambda)$.

All these theorems were proven in [1], except the last one which was proven in the case of labelled Markov processes originally in [2], where it is a consequence of the minimal-realizability of bisimulation on labelled Markov processes.

References

[1] Philippe Chaput, Vincent Danos, Prakash Panangaden, and Gordon Plotkin. Approximating markov processes by averaging. *J. ACM*, 61(1), jan 2014.

[2] Vincent Danos, Josée Desharnais, François Laviolette, and Prakash Panangaden. Bisimulation and cocongruence for probabilistic systems. *Information and Computation*, 204(4):503 – 523, 2006. Seventh Workshop on Coalgebraic Methods in Computer Science 2004.

[3] Josée Desharnais, Abbas Edalat, and Prakash Panangaden. Bisimulation for labelled markov processes. *Information and Computation*, 179(2):163 – 193, 2002.