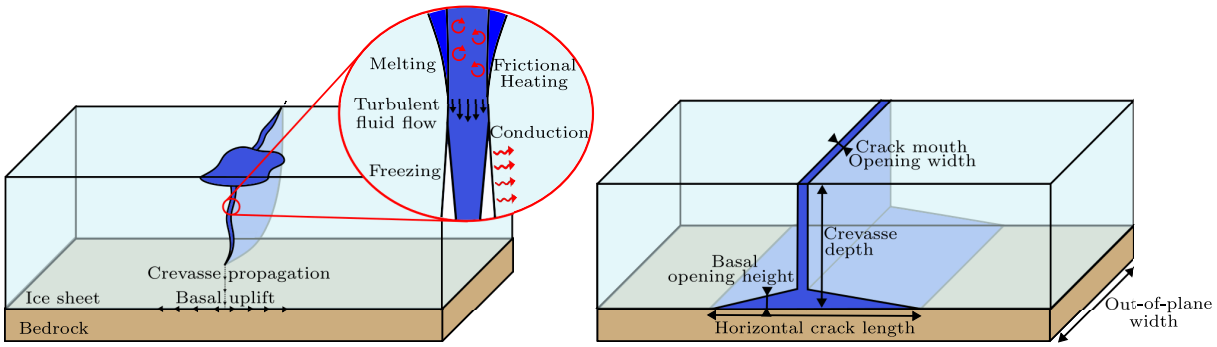

Figure 1: Overview of the system simulated using the presented MATLAB code

IceHydroFrac: A MATLAB code to simulate water-filled crevasse propagation and uplifting of ice sheets

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Abstract

Documentation that accompanies the *MATLAB* code IceHydroFrac, available from here. This documentation explains the usage of the implemented finite element framework, and highlight the main files.

If using this module, please cite: T Hageman, JZ Mejia,R Duddu, and E Martinez-Pañeda. *Ice Viscosity Governs Hydraulic Fracture Causing Rapid Drainage of Supraglacial Lakes.* The cryosphere [1].

Keywords: MATLAB, Hydraulic fracture, Greenland Ice Sheet, Crevasse, Numerical simulation, Viscous effects

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Contents

1. Introduction

Intro sentences

1.1. Basic usage

All parameters are set within and relevant functions are called from the matlab file "main.m", and running this file performs the full simulation. Parameters are also set within this file, for instance defining the visco-plastic model through:

main.m

```
64  %Interior of ice and rock: Momentum balance
65  physics_in{1}.type = "ViscoElastic";
66  physics_in{1}.Egroup = "Internal";
67  physics_in{1}.young = [9e9; 20e9];      %Youngs modulus of Ice and Rock [Pa]
68  physics_in{1}.poisson = [0.33; 0.25];   %Poisson ratio of ice and rock [-]
69  physics_in{1}.A0 = 5e-24;               %Glenns law creep coefficient [Pa^-3 s^-1]
70  physics_in{1}.Q = 150e3;                %Energy for scaling creep with temperature
        [J]
71  physics_in{1}.TRef = 273.15;            %Reference temperature at which A0 is
        determined [K]
72  physics_in{1}.n = 3;                    %Glenns law creep exponent [-]
73  physics_in{1}.T_Ice = T_Ice;            %Temperature profile for the ice
```

These parameters are automatically passed along to the relevant physics models once they are initialized. As such, no changes in other files are needed to adapt the simulation set-up for other parameters.

2. Summary of included files

The code is set up in a object-oriented manner, defining matlab classes for each sub-component and providing their accompanying methods. As a result, a clear distinction is made between different components, and each can be used and altered with limited/no impact on other components. Here, the different classes are described. The commenting style employed within the code is compatible with the matlab help function, as such information about all usable methods within a class can be accessed by including the relevant folders, and typing, for instance, "help Solver" to print all variables contained within and all function available from the solver.

2.1. main.m

This is the main file, from which all classes are constructed and the actual simulation is performed. Within it, all properties used within other classes are defined as inputs. These properties are then passed on to initialize the "physics" object, via

main.m

```
147 %initialize physics
148 physics = Physics(mesh, physics_in, dt0);
```

taking all relevant input parameters within the physics_in structure, and an initial time step increment dt0. In a similar manner, the mesh is also initialized from a structure of properties,

```
142 %load mesh from file
143 mesh = Mesh(mesh_in);
144 mesh.plot(true, true, false, false);
145 mesh.check();
```

which reads the mesh from a file, initializes the required elements and node groups, displays the mesh in a figure, and confirms the mesh is valid and prints statistics related to the area of each separate element group to the output. Finally, the main file performs the time-stepping, calling the function:

```
195         %solve current time increment
196         solver.Solve();
```

to solve the actual time increments.

After each time increment, outputs are saved into a single structure for later plotting,

```
198         %save timedata
199         physics.models{6}.updateSurfaceElevation(physics);
200         TimeSeries.tvec(end+1) = physics.time;%times of outputs[s]
201         TimeSeries.Lfrac(end+1) = mesh.Area(9);%crevasse depth/length
202         TimeSeries.Qvec(end+1,:) = physics.models{5}.QMeltTot;%thermal fluxes ('Ice
                desorbtion', 'Flow produced', 'Melting process')
203         TimeSeries.Qinflow(end+1) = physics.models{6}.QTotal(end);%total volume of fluid
                that has entered the crevasse
204         TimeSeries.qCurrent(end+1) = physics.models{6}.qCurrent(end);%current inflow rate
205         TimeSeries.upLift(end+1,1) = physics.models{6}.dxCurrent(end);%surface uplift at
                the centre of the surface
206         TimeSeries.upLift(end,2) = physics.models{6}.dyCurrent(end);%surface uplift at the
                centre of the surface
207         TimeSeries.SurfaceDisp(end+1,:,1) = physics.models{6}.surface_dX;%full uplift
                profile at top surface    (horizontal displacement)
208         TimeSeries.SurfaceDisp(end,:,2) = physics.models{6}.surface_dY;%full uplift profile
                 at top surface (vertical displacement)
```

These outputs are:

200. tvec: This contains all time increments at which the other elements within this structure have outputted data. Notably, as the time increment varies between the initialization period ($tvec(i) < 0$) and actual simulations ($tvec(i) \geq 0$) differs, this vector is also required to translate the number of the time step (as used within the naming of full output files) to the actual time of the outputs.
201. Lfrac: This is the length all fractured interfaces. When the crevasse has yet to reach the base, this corresponds to the depth of the crevasse. After reaching the bottom, this is the depth of the crevasse (=the ice thickness) plus the total length of the horizontal cracks (both directions summed together).
202. Qvec: This reports the total thermal energies produced/consumed throughout the simulation. The first element of this vector corresponds to the thermal energy conducted into the ice, the second to the heat produced by the turbulent flow due to friction, and the final element corresponds to the thermal energy used to cause freezing/melting of the crevasse walls. These values are the integrated totals for the complete crack, and are also integrated over the complete time.
203. Qinflow: The total volume of fluid that has entered the crevasse from the inlet at the surface. As with all other outputs, this is given per metre of unit depth.
204. qCurrent: The current fluid inflow at the top inlet, $\partial \text{Qinflow}/\partial t$.
205. upLift: Surface displacement at the centre of the top surface, coinciding with the crevasse. This contains the horizontal displacement (half the crevasse opening height), and the vertical displacement.
207. SurfaceDisp: Vectors containing horizontal and vertical displacements for the complete top surface of the ice-sheet. The coordinates that correspond to the given data points are saved within TimeSeries.SurfaceCoords.

In addition to these time series, full outputs are saved after every 10 time steps,

```
215         %save outputs for post-processing and restarting
216         if mod(tstep, 10) == 0
217             filename = savefolder+string(tstep);
218             save(filename, "mesh","physics","solver","t_max","TimeSeries");
219         end
```

These output files are appended with the number of the time increment during which the output is saved, and they contain all information required to restart a previously interrupted simulation.

2.2. Models

The models implement parts of the relevant physics. At initialization time, models are constructed by passing their relevant input parameters as:

Models/@ViscoElastic/ViscoElastic.m

```
47        function obj = ViscoElastic(mesh, physics, inputs)
```

This also saves pointers to the physics and mesh objects, allowing interacting with these whenever needed. The physics implemented by the model are added within the GetKf function, called whenever an updated tangent stiffness matrix and/or force vector is needed:

```
256       function getKf(obj, physics)
```

When models have history-dependent parameters, these are updated by calling the commit function,

```
127       function Commit(obj, physics, commit_type)
```

where the commit type indicates either "TimeDep" for committing time-irreversible data (e.g. plastic strains, time-integrated quantities), "irreversibles" for path-irreversible quantities (e.g. cohesive zone models), and "StartIt" for anything that needs to be performed at the start of each time increment.

2.2.1. ViscoElastic.m

This model implements the static momentum balance:

$$-\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \tag{1}$$

with the weak form of this contribution given by:

$$\int_{\Omega} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \boldsymbol{u}^{t+\Delta t} - \boldsymbol{B}^T \boldsymbol{D} \varepsilon_{\mathrm{v}}^{t+\Delta t} \, \mathrm{d}\Omega \tag{2}$$

with the plastic strains determined once per time increment, based on the displacements at the end of the previous time increment:

$$\varepsilon_{\mathrm{v}}^{t+\Delta t} = \varepsilon_{\mathrm{v}}^{t} + \Delta t A \left(\left(\mathbf{u}^{t^T} \boldsymbol{B}^T - \varepsilon_{\mathrm{v}}^{t+\Delta t} \right) \boldsymbol{D}^T \boldsymbol{J_2} \boldsymbol{D} \left(\boldsymbol{B} \mathbf{u}^t - \varepsilon_{\mathrm{v}}^{t+\Delta t} \right) \right)^{\frac{n-1}{2}} \boldsymbol{J_2} \boldsymbol{D} \left(\boldsymbol{B} \mathbf{u}^t - \varepsilon_{\mathrm{v}}^{t+\Delta t} \right) \tag{3}$$

with $\boldsymbol{B}$ the displacement to strain mapping matrix, $\epsilon = \boldsymbol{B}\mathbf{u}$, and $\boldsymbol{J}_2$ the deviatoric stress operator:

$$\sigma_{dev} = \boldsymbol{J}\sigma \qquad \boldsymbol{J}_2 = \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The input parameters for this model are defined by:

main.m

```
64        %Interior of ice and rock: Momentum balance
65        physics_in{1}.type = "ViscoElastic";
66        physics_in{1}.Egroup = "Internal";
67        physics_in{1}.young = [9e9; 20e9];      %Youngs modulus of Ice and Rock [Pa]
68        physics_in{1}.poisson = [0.33; 0.25];   %Poisson ratio of ice and rock [-]
69        physics_in{1}.A0 = 5e-24;               %Glenns law creep coefficient [Pa^-3 s^-1]
70        physics_in{1}.Q = 150e3;                %Energy for scaling creep with temperature
              [J]
71        physics_in{1}.TRef = 273.15;            %Reference temperature at which A0 is
              determined [K]
72        physics_in{1}.n = 3;                    %Glenns law creep exponent [-]
73        physics_in{1}.T_Ice = T_Ice;            %Temperature profile for the ice
74        physics_in{1}.Hmatswitch = ["FLeft", "FRight"];         %Element groups indicating
              ice-rock boundary
```

where “Type” is the name of this model, “Egroup” indicates the domain this model is applied to, and all other are the physical parameters for this model. The temperature of the ice is defined as an interpolation function, which is previously loaded from a file or defined to return a constant by:

```
12      %Temperature profile used
13      if true
14          T_Ice = @(y) min(0.0,1.0e-4*y*(y-250));
15      else
16          load("TProfile.mat","T_Ice");
17      end
```

For computational efficiency, Eq. (2) is split into two contributions: The first term contributes to the stiffness matrix and force vector, with the stiffness matrix only needing to be updated when the mesh changes (due to crack propagation):

Models/@ViscoElastic/ViscoElastic.m

```
263             if (length(obj.myK) == length(physics.fint))
264                 recalc = false;
265             else
266                 recalc = true;
267             end
```

, with the stiffness matrix assembled in a standard manner as:

```
286                         if (obj.Hmatswitch(xy(1,ip),xy(2,ip)) == false)
287                             K_el = K_el + B'*obj.D_el2*B*w(ip);
288                         else
289                             K_el = K_el + B'*obj.D_el*B*w(ip);
290                         end
```

where the height of integration points is checked to determine whether elements are located within the ice or rock parts of the domain. This tangent matrix is then used together with a similar force vector resulting for the viscous part of Eq. (2) to add contributions to the global foce vector as:

```
300             physics.fint = physics.fint + obj.myK*physics.StateVec ;
301             if (obj.A0>0)
302                 physics.fint = physics.fint + obj.FVisc;
303             end
304             physics.K = physics.K + obj.myK;
```

As a result, the force vector for the visco-plastic component and the tangent matrix for the linear-elastic components only need to be updated once per time increment, independent of the actual amount of non-linear iterations performed within that increment.

2.2.2. Inertia.m

This model adds inertia terms to the momentum balance, appending the momentum balance from Eq. (1) to read:

$$\rho_\pi \ddot{\mathbf{u}} - \nabla \cdot \sigma = \mathbf{0} \tag{5}$$

where the first term is the newly added term in this model (note: this model is additive to the ViscoElastic.m model, it does not add the stress term by itself). The inputs required to initialize this model are given by:

main.m

```
76          %Interior of ice and rock: Contribution due to inertia
77          physics_in{2}.type = "Inertia";
78          physics_in{2}.Egroup = "Internal";
79          physics_in{2}.density = 950;          %Density [kg/m^3]
80          physics_in{2}.beta    = 0.4;          %Time discretisation constant (Newmark scheme)
81          physics_in{2}.gamma   = 0.75;         %Time discretisation constant (Newmark scheme)
```

where the time discretisation parameters beta and gamma are used to define the acceleration in terms of history parameters (the old velocity and acceleration) and the current displacement as:

$$\dot{\mathbf{u}}^{t+\Delta t} = \frac{\gamma}{\beta \Delta t}\left(\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t}\right) - \left(\frac{\gamma}{\beta} - 1\right)\dot{\mathbf{u}}^{t} - \left(\frac{\Delta t \gamma}{2\beta} - \Delta t\right)\ddot{\mathbf{u}}^{t} \tag{6}$$

$$\ddot{\mathbf{u}}^{t+\Delta t} = \frac{1}{\beta \Delta t^2}\left(\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t}\right) - \frac{1}{\beta \Delta t}\dot{\mathbf{u}}^{t} - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{u}}^{t} \tag{7}$$

Similar to the viscoelastic model, the tangent matrix for this model is calculated once, and then re-used to assemble the global tangent matrix every iteration.

2.2.3. SelfWeight.m

This model adds the gravity contribution to the momentum balance, appending the momentum balance to:

$$\rho_\pi \ddot{\mathbf{u}} - \nabla \cdot \sigma - \rho_\pi \mathbf{g} = \mathbf{0} \tag{8}$$

For simplicity, this gravity force is added to the global force vector, with no differentiation made between internal and external forces. Inputs for this model are the element group the model is acting on, and the densities of ice and rock:

main.m

```
83          %Interior of ice and rock: Contribution due to gravity
84          physics_in{3}.type = "SelfWeight";
85          physics_in{3}.Egroup = "Internal";
86          physics_in{3}.density = [910; 2500];    %Density [kg/m^3]
```

2.2.4. FractureCZM.m

This model adds the cohesive fracture model, defined by the interface tractions normal to the fracture surface:

$$\tau_{CZM} = -f_{\mathrm{t}}\mathbf{n}\exp\left(-[\![\mathbf{u}]\!]\cdot\mathbf{n}\frac{f_{\mathrm{t}}}{G_{\mathrm{c}}}\right) \tag{9}$$

It also manages the propagation criteria for fracture propagation, allowing the crack to propagate when $\sigma \cdot \mathbf{n} - f_t > 0$.

To initialize the model, the input parameters required are:

main.m

```
88          %Fracture interface: Cohesive zone model and propagation
89          physics_in{4}.type = "FractureCZM";
90          physics_in{4}.Egroup = "Fracture";
91          physics_in{4}.energy = 10;          %Fracture release energy [J/m^2]
92          physics_in{4}.dummy = 0*1e10;       %Dummy stiffness to prevent walls from
                penetrating
93          physics_in{4}.Hmatswitch = ["FLeft", "FRight"];     %Depth of ice-rock interface
94          physics_in{4}.T_ice = T_Ice;        %Temperature profile of ice
```

where the temperature profile that is given as input is used to obtain the tensile strength via:

Models/@FractureCZM/FractureCZM.m

```
45          obj.ft = @(T) 2.0-0.0068*(T+273.15);
```

For the surface tractions, the maximum displacement is saved as a history variable, saved during the assembly of the force and tangent stiffness matrix:

```
155                 if (h>=hstOld)
156                     hloc = h;
157                     tau(1) = f_t * exp(-f_t*h/obj.energy);
158                     dtaudh(1,1) = -f_t.^2/obj.energy * exp(-f_t*h/obj.energy);
```

```
                elseif (h>0)
                     hloc = hstOld;
                     tau(1) = f_t * exp(-f_t*hstOld/obj.energy)*h/hstOld;
                     dtaudh(1,1) = f_t * exp(-f_t*hstOld/obj.energy)/hstOld;
                else
                    hloc = hstOld;
                    %tractions via no-pen condition
                end
                newHist(n_el, ip) = hloc;
```

which also produces the surface traction normal to the surface, tau(1), and its derivative dtaudh(1,1). The tangent component of this traction is set to zero, and during assembly this vector is rotated from the local to the global coordinate system. To improve the stability of the no-penetration condition, used to enforce contact between the crevasse walls, a lumped integration scheme is used, with sets of nodes contributing on a set-by-set basis to the overall force vector as:

```
            for cp=1:length(C_Lumped)
                NL = zeros(size(N, 2), 1);
                NL(cp) = 1.0;
                Nd = obj.getNd(NL);

                n_est = nvec(1,:);
                if (n_est*Nd*XY < 0)
                    f_el = f_el + C_Lumped(cp)*obj.dummy*(Nd'*(n_est'*n_est)*Nd)*XY;
                    K_el = K_el + C_Lumped(cp)*obj.dummy*(Nd'*(n_est'*n_est)*Nd);
                end
            end
```

with this condition only activating when the crevasse opening height is negative.

The fracture criterion is evaluated in the function

```
function [fc, dofsXY] = get_fc(obj, physics)
```

This function queries the elements ahead of the crack tip to obtain its stresses

```
        [direction, elem, ips] = obj.mesh.getNextFracture();
        stresses = physics.Request_Info("stresses",elem,"Interior");
```

which are compared to the temperature-dependent tensile strength. If the propagation criterion is exceeded, the mesh is addapted to include a newly inserted interface element:

```
        [fc, ~] = obj.get_fc(physics);
        if (fc>=0 && physics.time>=0)
            obj.mesh.Propagate_Disc_New(physics);
            Irr = true;
        end
```

2.2.5. FractureFluid.m

FractureFluid.m implements all the fluid flow and thermal related equations for the water within the crevasse, and is defined by the input parameters:

main.m

```
%Fracture interface: Fluid flow and melting
physics_in{5}.type = "FractureFluid";
physics_in{5}.Egroup = "Fracture";
physics_in{5}.visc = 1.0e-3;        %water viscosity [Pa s]
physics_in{5}.Kf   = 1.0e9;         %Water compressibility [Pa]
physics_in{5}.FlowModel = "FrictionFactor";   %"CubicLaw";"FrictionFactor", Model
    used for fluid flow within crevasse
physics_in{5}.melt = true;          %Include wall melting
physics_in{5}.freeze = true;        %Include wall freezing
physics_in{5}.rho_ice = 910;        %Density of ice [kg/m^3]
physics_in{5}.rho_water = 1000;     %Density of water [kg/m^3]
```

```
106         physics_in{5}.cp_ice = 2115;          %heat capacity of ice [J/kg]
107         physics_in{5}.melt_heat = 335000;     %latent heat of ice-water tansition [J/kg]
108         physics_in{5}.T_ice = T_Ice;          %Temperature profile
109         physics_in{5}.k_ice = 2;              %Thermal conductivity of ice [J/K/m^2]
```

Notably, the input "FlowModel" allows selecting either turbulent flow, based on a friction factor approach via:

$$q = -2\rho_{\mathrm{w}}^{-\frac{1}{2}} k_{\mathrm{wall}}^{-\frac{1}{6}} f_0^{-\frac{1}{2}} h^{\frac{5}{3}} \left| \frac{\partial p}{\partial \xi} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right|^{-\frac{1}{2}} \left(\frac{\partial p}{\partial \xi} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right) \tag{10}$$

or laminar flow, using the analytic solution for uni-directional pressure-driven fluid flow between flat plates:

$$q = \frac{h^3}{12\mu} \left(\frac{\partial p}{\partial \xi} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right) \tag{11}$$

with this choice of model being set as either "FrictionFactor" for turbulent flow, or "CubicLaw" for laminar flow. This choice is applied to the complete crevasse, and no checking is performed whether the fluid velocity warrants laminar or turbulent flow (based on simulations performed for our paper, the Reynolds number indicates turbulent for our use cases).

This fluid flux is used within the mass balance:

$$\frac{\partial q}{\partial \xi} + \dot{h} - \frac{\rho_{\mathrm{i}}}{\rho_{\mathrm{w}}} \dot{h}_{\mathrm{melt}} + \frac{h}{K_{\mathrm{w}}} \dot{p} = 0 \tag{12}$$

which also requires changes in local opening height, and melting rate. These are obtained by solving the coupled system of equations within each integration point:

$$q = -2\rho_{\mathrm{w}}^{-\frac{1}{2}} k_{\mathrm{wall}}^{-\frac{1}{6}} f_0^{-\frac{1}{2}} h^{\frac{5}{3}} \left| \frac{\partial p}{\partial \xi} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right|^{-\frac{1}{2}} \left(\frac{\partial p}{\partial \xi} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right) \tag{13}$$

$$0 = \rho_{\mathrm{i}} \mathcal{L} \dot{h}_{\mathrm{melt}} + \frac{k^{\frac{1}{2}} T_\infty \rho_{\mathrm{i}}^{\frac{1}{2}} c_{\mathrm{p}}^{\frac{1}{2}}}{\pi^{\frac{1}{2}} \left(t - t_0\right)^{\frac{3}{2}}} + q \left(\frac{\partial p}{\partial \xi} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right) \tag{14}$$

$$h = h_{\mathrm{melt}} + \mathbf{n} \cdot [\![\mathbf{u}]\!] \tag{15}$$

Defining the momentum balance of the fluid, thermal balance, and total opening height respectively. This system is solved in a coupled manner, using:

$$\boldsymbol{C}_{ip} \begin{bmatrix} \mathrm{d}q_{ip}^{t+\Delta t} \\ \mathrm{d}{h_{\mathrm{melt}}}_{ip}^{t+\Delta t} \\ \mathrm{d}h_{ip}^{t+\Delta t} \end{bmatrix}_{i+1} = - \begin{bmatrix} f_{1,ip} \\ f_{2,ip} \\ f_{3,ip} \end{bmatrix}_i , \tag{16}$$

where the integration-point level tangent matrix given by:

$$\boldsymbol{C}_{ip} = \begin{bmatrix} 1 & 0 & \frac{10}{3} \rho_{\mathrm{w}}^{-\frac{1}{2}} k_{\mathrm{wall}}^{-\frac{1}{6}} f_0^{-\frac{1}{2}} h_{ip}^{t+\Delta t\,\frac{2}{3}} \left| \boldsymbol{\nabla} \mathbf{N}_{\mathrm{f}} \mathbf{p}^{t+\Delta t} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right|^{-\frac{1}{2}} \left(\boldsymbol{\nabla} \mathbf{N}_{\mathrm{f}} \mathbf{p}^{t+\Delta t} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right) \\ \boldsymbol{\nabla} \mathbf{N}_f \mathbf{p}^{t+\Delta t} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} & \frac{\rho_{\mathrm{i}} \mathcal{L}}{\Delta t} & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{17}$$

and during the assembly of the tangent system matrix for the global system, consistent tangent matrices are obtained via:

$$\begin{bmatrix} \frac{\partial q}{\partial \mathbf{p}} & \frac{\partial q}{\partial \mathbf{u}} \\ \frac{\partial h_{\mathrm{melt}}}{\partial \mathbf{p}} & \frac{\partial h_{\mathrm{melt}}}{\partial \mathbf{u}} \\ \frac{\partial h}{\partial \mathbf{p}} & \frac{\partial h}{\partial \mathbf{u}} \end{bmatrix} = -\boldsymbol{C}_{ip}^{-1} \begin{bmatrix} \rho_{\mathrm{w}}^{-\frac{1}{2}} k_{\mathrm{wall}}^{-\frac{1}{6}} f_0^{-\frac{1}{2}} h_{ip}^{t+\Delta t\,\frac{5}{3}} \left| \boldsymbol{\nabla} \mathbf{N}_{\mathrm{f}} \mathbf{p}^{t+\Delta t} - \rho_{\mathrm{w}} \mathbf{g} \cdot \mathbf{s} \right|^{-\frac{1}{2}} \boldsymbol{\nabla} \mathbf{N}_{\mathrm{f}} & 0 \\ q_{ip}^{t+\Delta t} \boldsymbol{\nabla} \mathbf{N}_{\mathrm{f}} & 0 \\ 0 & -\boldsymbol{N}_{\mathrm{d}} \end{bmatrix} \tag{18}$$

Within the code, this procedure is implemented through

Models/@FractureFluid/FractureFluid.m

```
343         function [qx, hmelt, h, derivs, Qprod] = get_qx_hmelt_h(obj, dp_dx, u, tfr, tOld,
                dt, hMelt_hist, initGuess, Ice_temp)
```

which takes the pressure gradient and displacement jump (and saved history variables), and returns the fluid flux, melting height, and total opening height and their derivatives. This function is aided by

```
418         function [f, C, D, Qres] = getFracK(obj, sol, dp_dx, u, tfr, tOld, dt, hMelt_hist,
                Ice_temp)
```

which returns the C and D matrices defined above, and the force vector. Additionally, this uses the function

```
479         function [qxFlow, dqx_dh, dqx_dpdx] = getFlow(obj, dp_dx, h)
480             if (h<0)
481                 h=0;
482             end
483             if (obj.FlowModel == "CubicLaw")
484                 qxFlow = -h^3/(12*obj.visc)*dp_dx;
485                 dqx_dh = -3*(h)^2/(12*obj.visc)*dp_dx;
486                 dqx_dpdx = -(h)^3/(12*obj.visc);
487             end
488             if (obj.FlowModel == "FrictionFactor")
489                 k = 1e-2; f0 = 0.143;
490                 preFac = obj.rho_water^(-0.5)*k^(-1/6)*sqrt(4)*f0^(-0.5);
491                 qxFlow   = -preFac    *(max(1,abs(dp_dx)))^(-0.5)*dp_dx*h^(5/3);
492                 dqx_dh   = -preFac*5/3*(max(1,abs(dp_dx)))^(-0.5)*dp_dx*h^(2/3);
493                 dqx_dpdx = -preFac*0.5*(max(1,abs(dp_dx)))^(-0.5)*h^(5/3);
494             end
495         end
```

to determine the fluid flux, depending on the used model.

2.2.6. LakeBoundary.m

Adds the pressure boundary condition at the top surface, and performs processing for saving fracture inflows, and surface uplifts. Input parameters:

main.m

```
112         physics_in{6}.type = "LakeBoundary";
113         physics_in{6}.Egroup = "Fracture";
114         physics_in{6}.Surface = "Top";
115         physics_in{6}.p0 = 1.0e5;       %Reference pressure imposed at the top [Pa]
116         physics_in{6}.dummy = 1e-6;     %dummy constant used to enforce this pressure
```

2.2.7. Constrainer.m

Applies boundary conditions to the defined element group:

main.m

```
119         physics_in{7}.type = "Constrainer";
120         physics_in{7}.Egroup = "Bottom";
121         physics_in{7}.dofs = {"dy"};
122         physics_in{7}.conVal = [0];
```

providing the name of the degree of freedom being constrained to all nodes of the element group, and the value to which it is being constrained to. These constraints are integrated within the tangential matrix and force vector through allocation matrices $\boldsymbol{C}_{con}$ and $\boldsymbol{C}_{uncon}$, reordering the system into a constrained and unconstrained part. This allows the constrained system to be solved as:

$$\boldsymbol{C}_{uncon}^T \boldsymbol{K} \boldsymbol{C}_{uncon} \mathbf{y} = -\left(\boldsymbol{C}_{uncon}^T \boldsymbol{f} + \boldsymbol{C}_{uncon}^T \boldsymbol{K} \boldsymbol{C}_{con} \mathbf{c}\right) \tag{19}$$

with the values of the boundary constraints contained in the vector $\mathbf{c}$. After solving, the state vector is then incremented through:

$$\mathbf{x}^{new} = \mathbf{x}^{old} + \boldsymbol{C}_{uncon} \mathbf{y} + \boldsymbol{C}_{con} \mathbf{c} \tag{20}$$

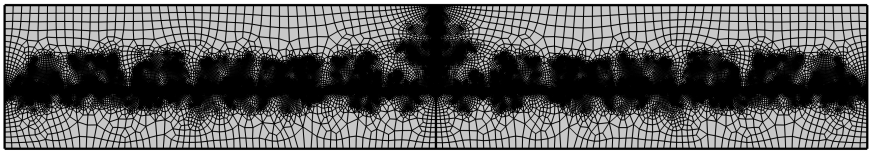


Figure 2: Example of a mesh used within simulations (and included in the file mesh_ 3000x300m.mphtxt), consisting of elements with size 2.5 m around the interface, and elements up to 30 m further from this interface.

2.3. Meshes

Meshes are generated using external meshing software (in the case of this studies, COMSOL, but this could be any other software), using quadratic Lagrangian quadrilateral elements. This mesh is then saved to a text file (containing lists of all nodal coordinates, and all interior and boundary elements). These meshes are created such that the path of the crevasse consists of element boundaries, enabling new interface elements to be easily inserted once the crevasse propagates. An example of such mesh is shown in Fig. 2.

Within the code, this mesh is loaded into mesh.m via the input parameters:

main.m

```
55          mesh_in.type = "File";
56          mesh_in.FileName = meshName;
57          mesh_in.nfrac = 30/dx_elem;     %number of initial interface elements
58          mesh_in.ipcount1D = 3;          %number of integration points per direction
59          mesh_in.zeroWeight = true;      %must be set to true, add zero-weight integration
                points to boundary of elements
```

where the field "type" is used to indicate the mesh is externally generated and loaded from a file, with the path to this file indicated by FileName. nfrac indicates the number of interface elements present at the start of the simulation. These are inserted during the initialization of the mesh, and not yet present in the mesh represented by the mesh contained within the input file. Finally, ipcount and zeroWeight indacte the number of integration points used to perform the numerical integration, and if additional integration points are inserted at element boundaries. These additional integration points do not have any integration weights, thus do not alter the obtained solutions, but they do allow for history parameters (such as the plastic strains) to be recorded at element boundaries, facilitating the post-processing and enabling stresses to be evaluated at these boundaries (required to enable the crack to evaluate its propagation criterion).

3. Post-processing and Sample results

Due to the long duration of full ice-sheet simulations, the code is set up to use a smaller geometry as example case. However, both mesh files are included, and this can be switched by referring to Meshes/mesh_ Das_ 25.mphtxt as input, using the 980 m thick geometry used for comparison with observations from Das et al. [2]. While the code is running, some results will be plotted, and outputs for post-processing will be saved every 10 time steps to the Results folder.

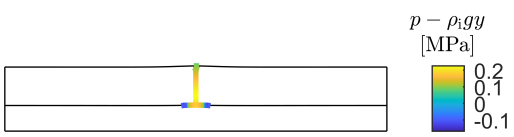
The matlab script Postprocessing.m can perform visualisation of these output files. this script requires the name of an output file:

PostProcessing.m

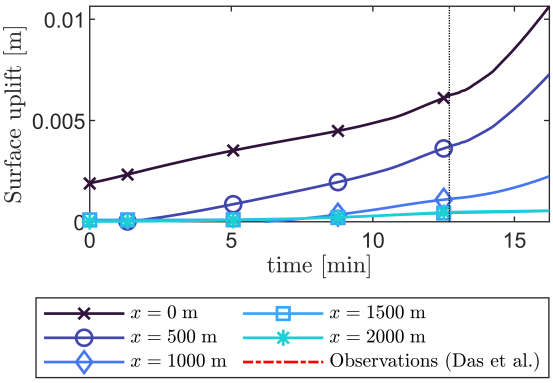
```
9   %Define file name for inputs, and file name for refernce displacements
10  fileName = "Results/640.mat";
11  fileZero = "Results/100.mat";
```

it also requires an output file from the end of the initialization time, which is used to define the relative displacements that have occurred during the actual crevasse propagation.

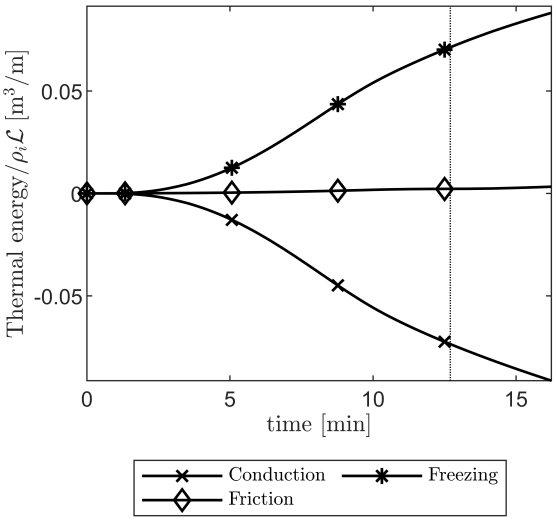
Within this file, the deformation of the ice-sheet and pressures within the crack, see Fig. 3a, are produced via the function:



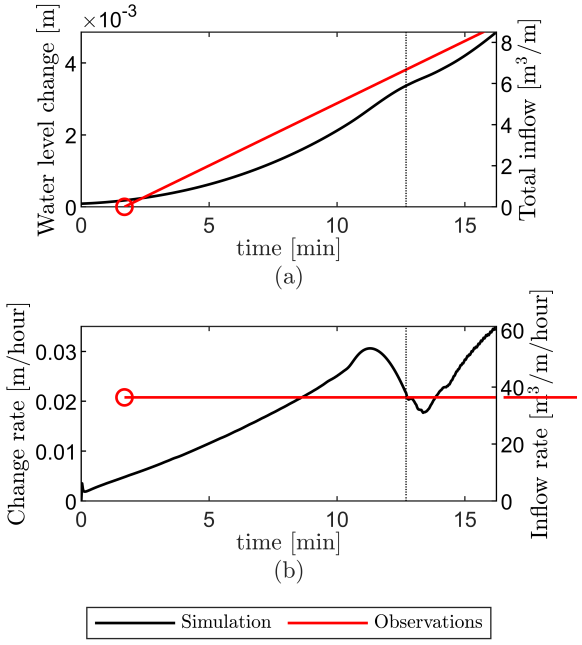
(a) Pressure and displacements



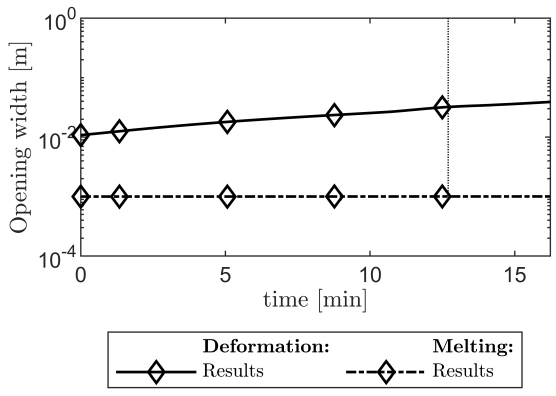
(b) Vertical displacement



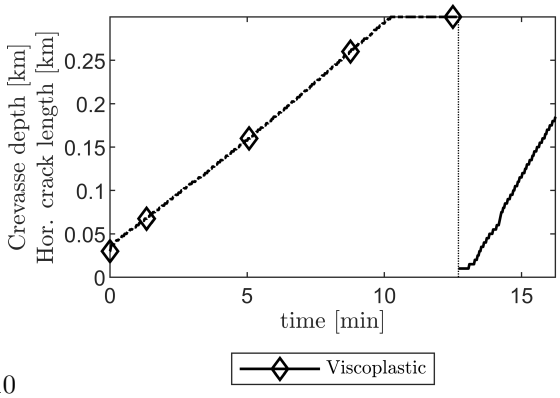
(c) Thermal energy produced by friction, consumed by conduction, and the resulting effect on freezing



(d) Water inflow into the crevasse (a), and rate of water inflow (b)



(e) Mechanical and melting induced crevasse opening heights



(f) Crevasse depth/vertical crack length

Figure 3: Generated figures during postprocessing

```
309 function plotDeformationsPressure(physics, physicsOffset, sc, fillICE)
```

using the state of the system, saved within the physics objects, a deformation scale sc with with the shown displacements are enhanced, and allowing for the ice to be either filled in (white for ice, grey for rock), or for this to be left transparent (much faster).

Within the output file, surface displacements are saved for convenient plotting, and can be plotted at several locations as:

```
56      for l=1:length(lcs)
57          hp = plot(TimeSeries.tvec(iStart:end)/60, TimeSeries.SurfaceDisp(iStart:end,lcs(l)
                ,2)-TimeSeries.SurfaceDisp(iStart,1,2),'Color',cc(l,:),'LineWidth',1,'
                DisplayName',"$x="+string(xLoc(l))+"\;\mathrm{m}$");
58          hold on
59          plotsparsemarkers(hp, [], m(l), 6, false);
60      end
```

Producing figure Fig. 3b. In a similar manner, the total thermal energy lost into the ice due to conduction, produced due to friction, and used to freeze the crevasse walls are plotted (See Fig. 3c):

```
78      QScale = 1/(910*335000);
79      hp = plot(TimeSeries.tvec(iStart:end)/60, QScale*TimeSeries.Qvec(iStart:end,1),'k','
            LineWidth',1,'DisplayName',"Conduction");
80      hold on
81      plotsparsemarkers(hp, [], "x", 6, false);
82
83      hp = plot(TimeSeries.tvec(iStart:end)/60, QScale*TimeSeries.Qvec(iStart:end,2),'k','
            LineWidth',1,'DisplayName',"Friction");
84      hold on
85      plotsparsemarkers(hp, [], "d", 6, false);
86
87      hp = plot(TimeSeries.tvec(iStart:end)/60, QScale*TimeSeries.Qvec(iStart:end,3),'k','
            LineWidth',1,'DisplayName',"Freezing");
88      hold on
89      plotsparsemarkers(hp, [], "*", 6, false);
```

which also uses the script plotsparsemarkers [3] to plot evenly spaced markers that do not correspond to all data points (which are saved every 2 seconds). Flow rates are plotted similarly, Fig. 3d:

```
113     plot(TimeSeries.tvec(iStart:end)/60, TimeSeries.Qinflow(iStart:end)*m3_to_mWaterHeigt
            ,"-k",'LineWidth',1,'DisplayName',"Simulation");


141     plot(TimeSeries.tvec(iStart:end)/60, TimeSeries.qCurrent(iStart:end)*m3_msec_to_m_hour
            ,"k-",'LineWidth',1,'DisplayName',"Simulation");
```

using data extracted from [2] as reference via the function:

```
235 function [tVec,yVec] = GetRef(whichToPlot,T0)
```

Finally, crevasse opening height, Fig. 3e, and crack lengths, Fig. 3f as:

```
176     hp = semilogy(TimeSeries.tvec(iStart:end)/60, -2*TimeSeries.upLift(iStart:end,1),'k-','
            LineWidth',1,'DisplayName',"Results");


195     LBase = TimeSeries.Lfrac-iceHeight; LBase(LBase<=0)=nan;
196     LCrev = TimeSeries.Lfrac-iceHeight; LCrev(LCrev>0)=nan; LCrev=LCrev+iceHeight;
197     hp = plot(TimeSeries.tvec(iStart:end)/60, LBase(iStart:end)*1e-3,'k-','LineWidth',1,'
            DisplayName',"Viscoplastic");
198     hold on
199     plotsparsemarkers(hp, [], "d", 6, false);
200     hp = plot(TimeSeries.tvec(iStart:end)/60, LCrev(iStart:end)*1e-3,'k-.','LineWidth',1,'
            DisplayName',"Viscoplastic",'HandleVisibility','off');
201     plotsparsemarkers(hp, [], "d", 6, false);
```

References

[1] T. Hageman, J. Z. Mejia, R. Duddu, E. Martinez-Pañeda, Ice viscosity governs hydraulic fracture causing rapid drainage of supraglacial lakes [Submitted], The Cryosphere.

[2] S. B. Das, I. Joughin, M. D. Behn, I. M. Howat, M. A. King, D. Lizarralde, M. P. Bhatia, Fracture propagation to the base of the Greenland ice sheet during supraglacial lake drainage, Science 320 (5877) (2008) 778–781.

[3] Jacob Jonsson, plotsparsemarkers, matlab central file exchange (2009).
URL `https://www.mathworks.com/matlabcentral/fileexchange/37165-plotsparsemarkers`