Investigation of the Non-uniform Fast Fourier Transformation

Author: Toetu F. Lafoai
Supervisor: Professor. Andrew Ensor
TE WĀNANGA ARONUI O TĀMAKI MAKAU RAU

Introduction

"Any function of a variable, whether continuous or discontinuous can be represented as a sum of sinusoidal waves." ~ Jean-Babtiste Joseph Fourier. This claim was the discovery of what is known as the Discrete Fourier Transform (DFT), and it also paved the wave for some of the most ubiquitous algorithms known to mankind.

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{j2\pi kn}{N}}$$

2-Dimensional Fourier Transform.

$$X_{kl} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x_{mn} e^{-j2\pi(k\frac{m}{M}+l\frac{n}{N})}$$

The DFT has a time complexity of $O(n^2)$ which was not ideal when it came to solving for large samples of data. In the year 1965, a breakthrough was made by two mathematicians, *James W. Cooley* and *John W. Tukey* where they discovered an alternate way to solve for the DFT and that was the Fast Fourier Transform (FFT) algorithm which uses a divide and conquer approach to solve for the DFT of a given signal. The discovery of this new approach improved the time complexity for solving the DFT down to $O(n \cdot log \cdot n)$.

Nowadays, the FFT is widely used in signal reconstruction, Numerical analysis etc. In the context of reconstructing signals, the FFT will hold given complete and uniform samples of the source signal or sampling above the Nyquist Rate. Unfortunately, we do not always get to choose the number of samples we get for a given source signal. Therefore, we will need to find other approaches that can help us achieve a high-fidelity reconstruction of a signal given incomplete samples. This is where the Non-Uniform Fast Fourier Transform (NUFFT) approach comes in to play.

Objective

- Investigate and generalize the main intuition behind NUFFT approaches.

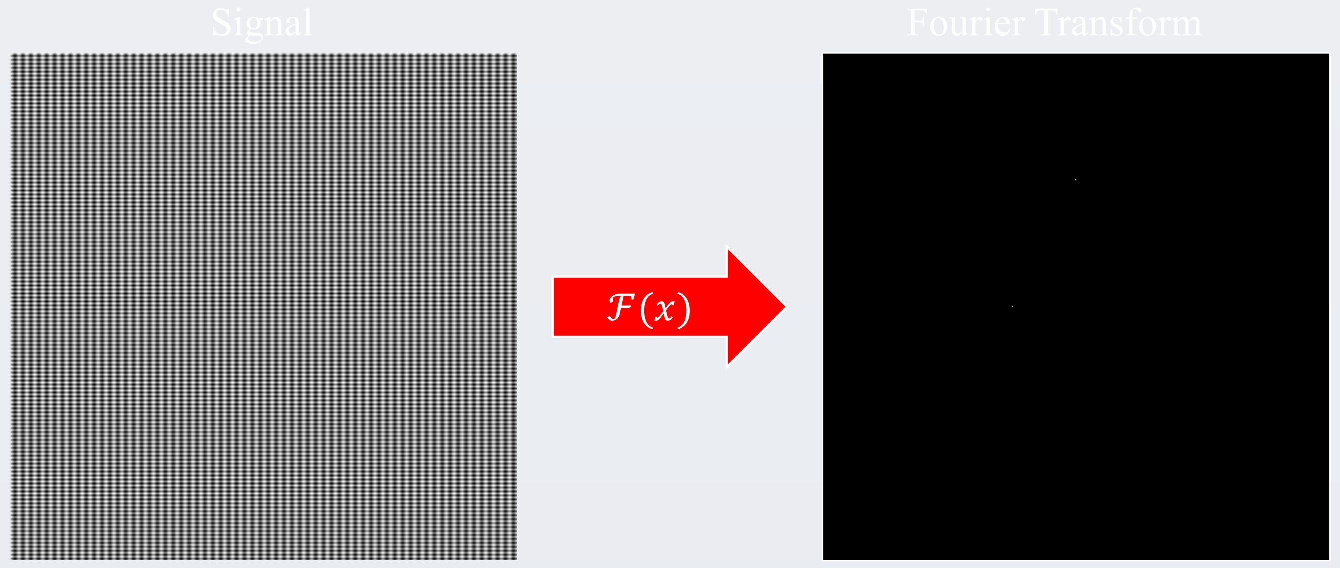


Figure 1: Complex sine wave. 64Hz (Horizontal) + 128Hz (Vertical). 262144 Channels

Figure 2: Fourier Transform of signal x

- Suppose we call **Figure 1**, x. Our goal is to reconstruct **Figure 2** which is a representation of the signal x in the Fourier Domain (frequency domain) using incomplete samples of x.

Methodology: Gridding Approach

- Suppose we are given incomplete samples of a continuous signal; one would first want to determine the connection amongst the sampled points. This is known mathematically as "Interpolation." There are many interpolating approaches out there, but in here, we will utilize an averaging sum approach known as the "Gaussian distribution function" and convolve it with our sampled points to get an approximation of our signal. This is known as **Kernel Convolution**.

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

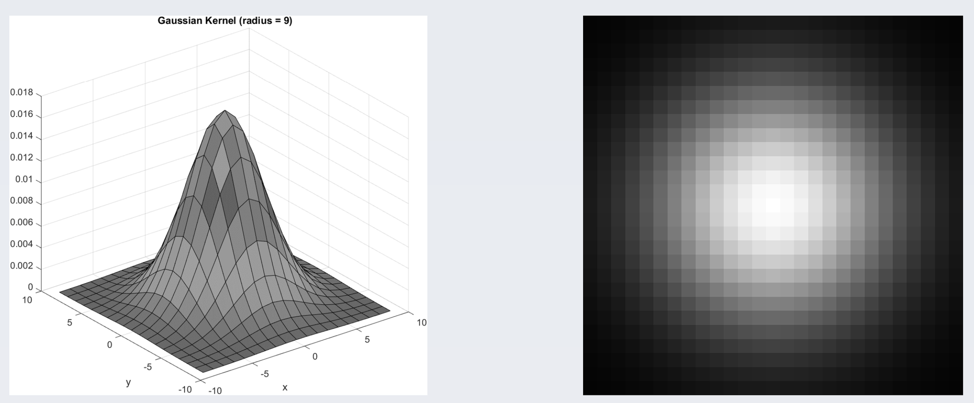


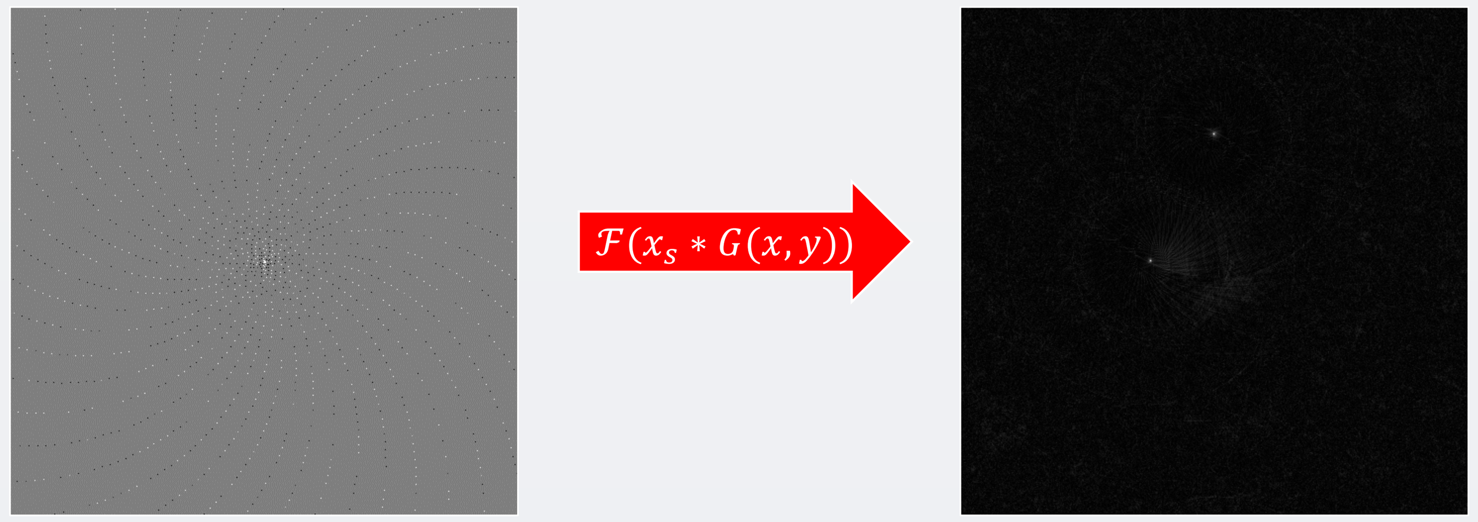
Figure 3: Gaussian Distribution Kernel

- After interpolating our sampled points, we use the 2-Dimensional FFT to represent our sampled data on a Fourier Plane and observe if we could possibly achieve a high-fidelity reconstruction of the original signal.

This overall approach is called "Gridding" and it is an approach that was initially used in radioastronomy and was later adapted by other fields like medical imaging.

Results

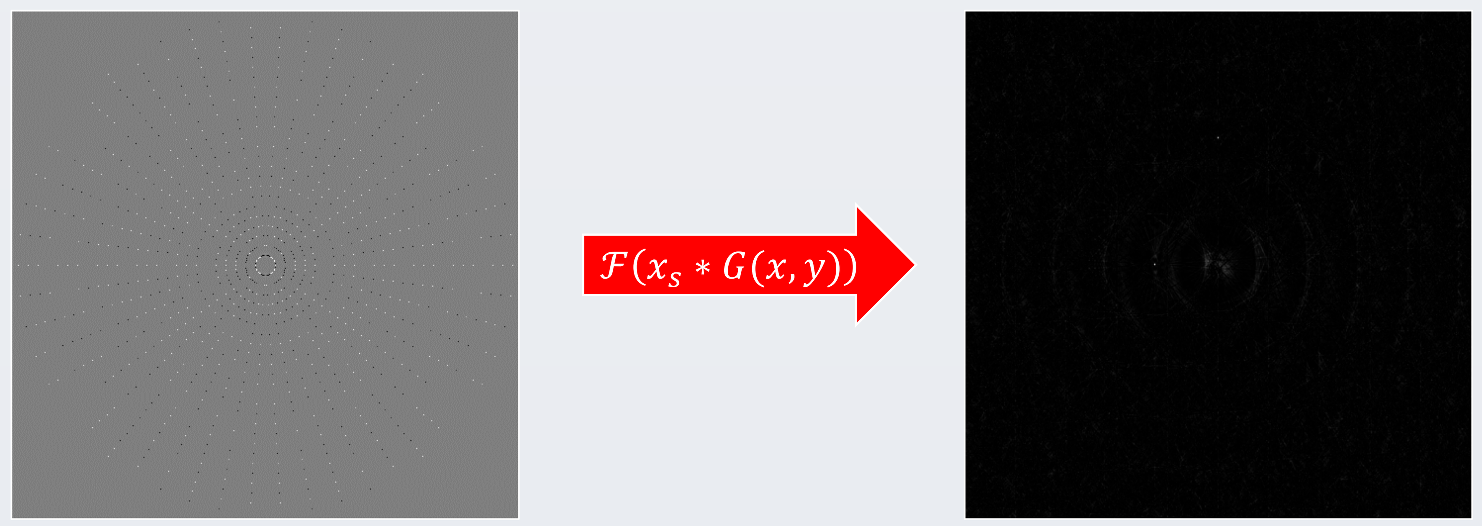
Spiral Samples



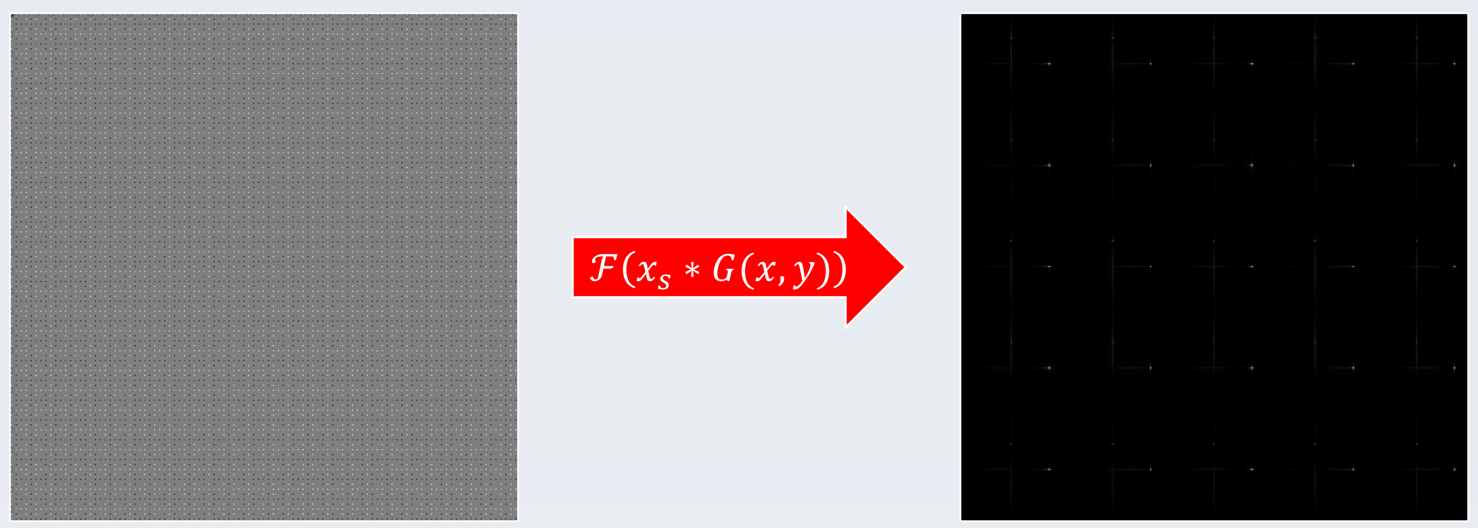
An Archimedean spiral was used to sample the figure above (Left). The overall number of samples is around 3,594. This is about 1.37% of our continuous signal given that we 262144 channels altogether. This is well below the Nyquist rate and therefore violates the "Shannon-Nyquist" criterion. In our case, the Nyquist rate is about 131,072Hz.

Shannon-Nyquist Criterion:
If a signal x(t) contains no frequency higher than ω (bandwidth), then therefore a sufficient sample rate is anything higher than 2ω samples per second.

Radial Samples



Rectilinear Samples



Conclusion