```
> restart
> with(HyperTypeSeq)
```

$$[AlgebraHolonomicSeq, HTS, HTSproduct, HolonomicRE, REtoHTS, mfoldInd] \quad (1)$$

mfoldInd

New (March 2024): command to evaluate an m-fold indicator term or write it symbolically

```
> mfoldInd(n,7,1)
```

$$\chi_{\{modp(n,7)=1\}} \quad (2)$$

```
> mfoldInd(15,7,1)
```

$$1 \quad (3)$$

```
> mfoldInd(23,7,1)
```

$$0 \quad (4)$$

HolonomiRE

```
> HolonomicRE(n!+1/n!,a(n))
```

$$(n+3)(n+1)^2 a(n) - (n^2+3n+1)(n^2+3n+3) a(n+1) + n(n^2+4n+4) a(n+2) = 0 \quad (5)$$

```
> HolonomicRE(sin(n*Pi/4)^2,a(n))
```

$$-a(n) + a(n+1) - a(n+2) + a(n+3) = 0 \quad (6)$$

```
> HolonomicRE(sin(cos(n*Pi)*Pi/6)*sin(n*Pi/4),a(n))
```

$$a(n) + \sqrt{2}\, a(n+1) + a(n+2) = 0 \quad (7)$$

```
> HolonomicRE(sin(cos(n*Pi)*Pi/6)*sin(n*Pi/4),a(n),reshift=2)
```

$$a(n) + a(n+4) = 0 \quad (8)$$

New (March 2024): HolonomicRE can now compute recurrence equations from normal forms. Moreover, one can use mfoldInd to write m-fold indicator terms (interlacements)

```
> s:=n!*chi[{modp*(n,3)=1}]
```

$$s := n!\, \chi_{\{modp(n,3)=1\}} \quad (9)$$

```
> HolonomicRE(s,a(n))
```

$$-(n+3)(n+2)(n+1) a(n) + a(n+3) = 0 \quad (10)$$

```
> s:=n^2*chi[{modp*(n,2)=1}]+1/n!*chi[{modp*(n,4)=3}]
```

$$s := n^2 \chi_{\{modp(n,2)=1\}} + \frac{\chi_{\{modp(n,4)=3\}}}{n!} \quad (11)$$

```
> HolonomicRE(s,a(n))
```

$$(n^6 + 30 n^5 + 371 n^4 + 2418 n^3 + 8747 n^2 + 16628 n + 12956) a(n) + (-n^6 - 18 n^5 \quad (12)$$

$$- 131 n^4 - 490 n^3 - 983 n^2 - 992 n - 384) a(n+2) - (n+4)(n+3)(n^6 + 30 n^5 + 371 n^4 + 2418 n^3 + 8747 n^2 + 16628 n + 12956)(n+1)(n+2) a(n+4) + (n+4)(n+3)(n+5)(n+6)(n^6 + 18 n^5 + 131 n^4 + 490 n^3 + 983 n^2 + 992 n + 384) a(n+6) = 0$$

Or equivalently using mfoldInd,

```
> s:=n!*mfoldInd(n,3,1)
```

$$s := n!\, \chi_{\{modp(n,3) = 1\}} \tag{13}$$

```
> HolonomicRE(s,a(n))
```

$$-(n+3)(n+2)(n+1) a(n) + a(n+3) = 0 \tag{14}$$

```
> HolonomicRE(n^2*mfoldInd(n,2,1)+1/n!*mfoldInd(n,4,3),a(n))
```

$$(n^6 + 30 n^5 + 371 n^4 + 2418 n^3 + 8747 n^2 + 16628 n + 12956) a(n) + (-n^6 - 18 n^5 - 131 n^4 - 490 n^3 - 983 n^2 - 992 n - 384) a(n+2) - (n+4)(n+3)(n^6 + 30 n^5 + 371 n^4 + 2418 n^3 + 8747 n^2 + 16628 n + 12956)(n+1)(n+2) a(n+4) + (n+4)(n+3)(n+5)(n+6)(n^6 + 18 n^5 + 131 n^4 + 490 n^3 + 983 n^2 + 992 n + 384) a(n+6) = 0 \tag{15}$$

REtoHTS

The sequence A212579

```
> RE:= a(n) = a(n-1)+2*a(n-2)-a(n-3)-2*a(n-4)-a(n-5)+2*a(n-6)+a
  (n-7)-a(n-8)
```

$$RE := a(n) = a(n-1) + 2\, a(n-2) - a(n-3) - 2\, a(n-4) - a(n-5) + 2\, a(n-6) + a(n-7) - a(n-8) \tag{16}$$

```
> REtoHTS(RE,a(n),[0, 1, 8, 31, 80, 171, 308, 509, 780, 1137, 1584,
  2143, 2812])
```

$$\frac{4}{9} + \frac{31}{12} n - 3 n^2 + \frac{67}{36} n^3 - \frac{1}{4} n\, \chi_{\{modp(n,2) = 0\}} - \frac{4}{9} \chi_{\{modp(n,3) = 0\}} - \frac{8}{9} \chi_{\{modp(n,3) = 1\}} \tag{17}$$

```
> U:=proc(n) U(n):=subs([n=n-1,a=U],RE) end proc:
  U(0):=0: U(1):=1:U(2):=8:U(3):=31:U(4):=80:U(5):=171:U(6):=308:U
  (7):=509:U(8):=780:U(9):=1137:U(10):=1584:U(11):=2143:U(12):=
  2812:
> REtoHTS(RE,a(n),U)
```

$$\frac{4}{9} + \frac{31}{12} n - 3 n^2 + \frac{67}{36} n^3 - \frac{1}{4} n\, \chi_{\{modp(n,2) = 0\}} - \frac{4}{9} \chi_{\{modp(n,3) = 0\}} - \frac{8}{9} \chi_{\{modp(n,3) = 1\}} \tag{18}$$

The sequence [A033481]
Requirement: [FPS]

```
> f:=(21 + 64*z + 32*z^2 - 5*z^3 - 56*z^4 - 28*z^5 - 14*z^6 - 7*
  z^7) / ((1 - z)*(1 + z + z^2))
```

$$f := \frac{-7 z^7 - 14 z^6 - 28 z^5 - 56 z^4 - 5 z^3 + 32 z^2 + 64 z + 21}{(1 - z)\,(z^2 + z + 1)} \qquad (19)$$

```
> FPS(f,z,n)
```

$$7 z^4 + 14 z^3 + 28 z^2 + 63 z + 19 + \left(\sum_{n=0}^{\infty} 4 z^n\right) + \left(\sum_{n=0}^{\infty} -2 z^{3n}\right) + \left(\sum_{n=0}^{\infty} -3 z^{3n+1}\right) \qquad (20)$$

```
> RE:=FPS:-FindRE(f-(7*z^4 + 14*z^3 + 28*z^2 + 63*z + 19),z,u(n))
```

$$RE := (-n + 1)\, u(n) + (4 n - 12)\, u(n - 4) + (n - 1)\, u(n - 3) + (2 n + 2)\, u(n - 2) + (-4 n + 12)\, u(n - 1) + (-2 n - 2)\, u(n + 1) = 0 \qquad (21)$$

```
> U:=proc(n) U(n):=subs([n=n-1,u=U],solve(RE,u(n+1))) end proc:
  U(0):=2:U(1):=1:U(2):=4:U(3):=2:U(4):=1:U(5):=4:
> REtoHTS(RE,u(n),U)
```

$$4 - 2\,\chi_{\{modp(n,3)=0\}} - 3\,\chi_{\{modp(n,3)=1\}} \qquad (22)$$

```
> REtoHTS(RE,u(n),[2,1,4,2,1,4])
```

$$4 - 2\,\chi_{\{modp(n,3)=0\}} - 3\,\chi_{\{modp(n,3)=1\}} \qquad (23)$$

```
> REtoHTS(a(n+2)=a(n+1)+a(n),a(n),[0,1,1])
```

$$-\frac{\sqrt{5}\left(-\frac{\sqrt{5}}{2} + \frac{1}{2}\right)^n}{5} + \frac{\sqrt{5}\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)^n}{5} \qquad (24)$$

```
> U:=  proc(n) U(n):=2*(2*n-1)*U(n-1)/(n+1) end proc: U(0):=1: U(1)
  :=1:
> REtoHTS((n+2)*C(n+1)=2*(2*n+1)*C(n),C(n),U)
```

$$\frac{(2 n)!\,(n + 1)}{(n + 1)!^2} \qquad (25)$$

HTS

```
> HTS(sin(n*Pi/4)^2,n)
```

$$\frac{1}{2} - \frac{(-1)^{\frac{n}{2}}\,\chi_{\{modp(n,2)=0\}}}{2} \qquad (26)$$

```
> HTS(sin(Pi*cos(n*Pi)/6)*cos(n*Pi/4),n)
```

$$\frac{(-1)^{\frac{n}{4}}\,\chi_{\{modp(n,4)=0\}}}{2} - \frac{\sqrt{2}\,(-1)^{\frac{n}{4} - \frac{1}{4}}\,\chi_{\{modp(n,4)=1\}}}{4} + \frac{\sqrt{2}\,(-1)^{\frac{n}{4} - \frac{3}{4}}\,\chi_{\{modp(n,4)=3\}}}{4} \qquad (27)$$

```
> HTS(sin(cos(n*Pi/3)*Pi),n)
```

$$(-1)^{\frac{n}{3}-\frac{1}{3}}\chi_{\{modp(n,3)=1\}}-(-1)^{\frac{n}{3}-\frac{2}{3}}\chi_{\{modp(n,3)=2\}} \tag{28}$$

```
> HTS(tan(n*Pi/4),n)
```

$$\chi_{\{modp(n,4)=1\}}+\left(\lim_{n\to 2}\tan\left(\frac{n\pi}{4}\right)\right)\chi_{\{modp(n,4)=2\}}-\chi_{\{modp(n,4)=3\}} \tag{29}$$

```
> HTS(tan(n*Pi/3),n)
```

$$\sqrt{3}\,\chi_{\{modp(n,3)=1\}}-\sqrt{3}\,\chi_{\{modp(n,3)=2\}} \tag{30}$$

```
> HTS(cos(n*arccos(x)),n)
```

$$\frac{\left(x-\sqrt{x^2-1}\right)^n}{2}+\frac{\left(x+\sqrt{x^2-1}\right)^n}{2} \tag{31}$$

```
> HTS(sin(n*Pi/6)*cos(n*Pi/3)-sin(n*Pi/2),n)
```

$$-\frac{\mathrm{I}\left(\frac{\sqrt{3}}{2}-\frac{\mathrm{I}}{2}\right)^n}{4}+\frac{\mathrm{I}\left(\frac{\sqrt{3}}{2}+\frac{\mathrm{I}}{2}\right)^n}{4}-\frac{(-1)^{\frac{n}{2}-\frac{1}{2}}\chi_{\{modp(n,2)=1\}}}{2} \tag{32}$$

```
> HTS(sin(n*Pi/4)^2*cos(n*Pi/6)^2,n)
```

$$\frac{1}{4}-\frac{(-\mathrm{I})^n}{8}+\frac{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^n}{8}+\frac{\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^n}{8}-\frac{\mathrm{I}\,(-1)^{\frac{n}{2}-\frac{1}{2}}\chi_{\{modp(n,2)=1\}}}{8}$$
$$-\frac{3\,(-\mathrm{I})^{\frac{n}{3}}\chi_{\{modp(n,3)=0\}}}{8}-\frac{3\,\mathrm{I}\,(-1)^{\frac{n}{6}-\frac{1}{2}}\chi_{\{modp(n,6)=3\}}}{8} \tag{33}$$

```
> HTS(sin(n*Pi/4)^2*cos(n*Pi/6)^4,n,maxreorder=12)
```

$$\frac{9}{32}+\frac{\left(4\,\mathrm{I}\,(-1)^{\frac{n}{2}}\sqrt{3}+8\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{\frac{n}{2}}+3\,(-1)^{\frac{n}{2}}\right)\chi_{\{modp(n,2)=0\}}}{32}$$
$$-\frac{9\,\chi_{\{modp(n,3)=0\}}}{32}+\frac{\mathrm{I}\sqrt{3}\,(-\mathrm{I})^{\frac{n}{3}-\frac{2}{3}}\chi_{\{modp(n,3)=2\}}}{4}$$
$$+\frac{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{\frac{n}{4}}\chi_{\{modp(n,4)=0\}}}{2}-\frac{\left(4\,\mathrm{I}\sqrt{3}+27\right)(-1)^{\frac{n}{6}}\chi_{\{modp(n,6)=0\}}}{32}$$
$$-\frac{\sqrt{3}\,(-1)^{\frac{n}{6}-\frac{5}{6}}\chi_{\{modp(n,6)=5\}}}{4} \tag{34}$$

HTSproduct

New (March 2024): computing product of hypergeometric type terms

```
> s1:=HTS(sin(n*Pi/4)^2,n)
```

$$s1 := \frac{1}{2} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,2)=0\}}}{2} \quad \textbf{(35)}$$

```
> s2:=HTS(cos(n*Pi/3)^2,n)
```

$$s2 := \chi_{\{modp(n,3)=0\}} + \frac{\chi_{\{modp(n,3)=1\}}}{4} + \frac{\chi_{\{modp(n,3)=2\}}}{4} \quad \textbf{(36)}$$

```
> s:=HTSproduct(s1,s2,n)
```

$$s := \frac{\chi_{\{modp(n,3)=0\}}}{2} + \frac{\chi_{\{modp(n,3)=1\}}}{8} + \frac{\chi_{\{modp(n,3)=2\}}}{8} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,6)=0\}}}{2} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,6)=4\}}}{8} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,6)=2\}}}{8} \quad \textbf{(37)}$$

*And we can now have fun with finding equivalent representations by combining **HTS** and **HTSproduct***

```
> HTS((sin(n*Pi/4)*cos(n*Pi/3))^2,n)
```

$$\frac{1}{8} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,2)=0\}}}{8} + \frac{3\,\chi_{\{modp(n,3)=0\}}}{8} - \frac{3\,(-1)^{\frac{n}{6}} \chi_{\{modp(n,6)=0\}}}{8} \quad \textbf{(38)}$$

What recurrence equation for

$$s := \frac{\chi_{\{modp(n,3)=0\}}}{2} + \frac{\chi_{\{modp(n,3)=1\}}}{8} + \frac{\chi_{\{modp(n,3)=2\}}}{8} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,6)=0\}}}{2} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,6)=4\}}}{8} - \frac{(-1)^{\frac{n}{2}} \chi_{\{modp(n,6)=2\}}}{8}$$

```
> HolonomicRE(s,a(n))
```

$$-a(n) + a(n+3) - a(n+6) + a(n+9) = 0 \quad \textbf{(39)}$$

```
>
>
```