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> restart
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Gröbner bases elimination for **Proposition 1** and **Theorem 2** in the paper:

On Rational Recursion for Holonomic Sequences

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Throughout $s(n + l + i) \rightarrow s_{l+i}, i = 1, 2$

Degree 1 case

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```
> d := 1
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$$d := 1 \quad (1)$$

Case l+1:

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> p0 := add(alpha[0, k]·nk, k=0..d)
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$$p0 := \alpha_{0,1} n + \alpha_{0,0} \quad (2)$$

```
> p1 := collect(add(add(binomial(d - k + j, j) · (alpha[1, d - j] + beta[l, d - j]·s[l + 1]), j  
= 0..k)·nd-k, k=0..d), n, distributed)
```

$$p1 := (\beta_{l,1} s_{l+1} + \alpha_{1,1}) n + \beta_{l,0} s_{l+1} + \beta_{l,1} s_{l+1} + \alpha_{1,0} + \alpha_{1,1} \quad (3)$$

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```
> L := [p0, normal(p1)]
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$$L := [\alpha_{0,1} n + \alpha_{0,0}, n \beta_{l,1} s_{l+1} + n \alpha_{1,1} + \beta_{l,0} s_{l+1} + \beta_{l,1} s_{l+1} + \alpha_{1,0} + \alpha_{1,1}] \quad (4)$$

```
> vars := remove(has, indets([p1, p0]), beta) minus {n}
```

$$vars := \{\alpha_{0,0}, \alpha_{0,1}, \alpha_{1,0}, \alpha_{1,1}, s_{l+1}\} \quad (5)$$

```
> J := PolynomialIdeals:-PolynomialIdeal(L, parameters = select(has, indets(p1), beta))
```

$$J := \langle \alpha_{0,1} n + \alpha_{0,0}, n \beta_{l,1} s_{l+1} + n \alpha_{1,1} + \beta_{l,0} s_{l+1} + \beta_{l,1} s_{l+1} + \alpha_{1,0} + \alpha_{1,1} \rangle \quad (6)$$

```
> J := PolynomialIdeals:-EliminationIdeal(J, vars) :
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```
> J := select(type, convert(J, list), polynom) :
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> numelems(J)
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(7)

$$1 \quad (7)$$

> degree(J[1], s[l + 1])

$$1 \quad (8)$$

> collect(J[1], s[l + 1])

$$(-\alpha_{0,0} \beta_{l,1} + \alpha_{0,1} \beta_{l,0} + \alpha_{0,1} \beta_{l,1}) s_{l+1} - \alpha_{0,0} \alpha_{1,1} + \alpha_{0,1} \alpha_{1,0} + \alpha_{0,1} \alpha_{1,1} \quad (9)$$

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Case l+2:

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Degree $d \leq 3$ case

To see the computations for d=3, replace 2 in the right-hand side below by 3.

>

> d := 2

$$d := 2 \quad (10)$$

> p0 := add(alpha[0, k] * n^k, k = 0 .. d)

$$p0 := \alpha_{0,2} n^2 + \alpha_{0,1} n + \alpha_{0,0} \quad (11)$$

> p1 := collect(add(add(binomial(d - k + j, j) * alpha[1, d - j], j = 0 .. k) * n^{d-k}, k = 0 .. d), n, distributed)

$$p1 := \alpha_{1,2} n^2 + (\alpha_{1,2} + 2 \alpha_{1,1}) n + \alpha_{1,2} + \alpha_{1,1} + \alpha_{1,0} \quad (12)$$

> p2 := collect(add(add(binomial(d - k + j, j) * 2^j * (alpha[2, d - j] + beta[l, d - j] * s[l + 2]), j = 0 .. k) * n^{d-k}, k = 0 .. d), n, distributed)

$$p2 := (\beta_{l,2} s_{l+2} + \alpha_{2,2}) n^2 + (4 \beta_{l,1} s_{l+2} + \beta_{l,2} s_{l+2} + 4 \alpha_{2,1} + \alpha_{2,2}) n + 4 \beta_{l,0} s_{l+2} + 2 \beta_{l,1} s_{l+2} + \beta_{l,2} s_{l+2} + 4 \alpha_{2,0} + 2 \alpha_{2,1} + \alpha_{2,2} \quad (13)$$

> L := [p0, p1, normal(p2)] :

> vars := remove(has, indets([p2, p1, p0]), beta) minus {n}

$$vars := \{\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,0}, \alpha_{2,1}, \alpha_{2,2}, s_{l+2}\} \quad (14)$$

> J := PolynomialIdeals:-PolynomialIdeal(L, parameters = select(has, indets(p2), beta))

$$J := \langle \alpha_{1,2} n^2 + (\alpha_{1,2} + 2 \alpha_{1,1}) n + \alpha_{1,2} + \alpha_{1,1} + \alpha_{1,0}, n^2 \alpha_{0,2} + n \alpha_{0,1} + \alpha_{0,0}, n^2 \beta_{l,2} s_{l+2} + n^2 \alpha_{2,2} + 4 n \beta_{l,1} s_{l+2} + n \beta_{l,2} s_{l+2} + 4 n \alpha_{2,1} + n \alpha_{2,2} + 4 \beta_{l,0} s_{l+2} + 2 \beta_{l,1} s_{l+2} + \beta_{l,2} s_{l+2} + 4 \alpha_{2,0} + 2 \alpha_{2,1} + \alpha_{2,2} \rangle \quad (15)$$

> J := PolynomialIdeals:-EliminationIdeal(J, vars) :

> J := select(type, convert(J, list), polynom) :

$$\textcolor{red}{>} \text{ numelems}(J) \qquad \qquad \qquad \textcolor{blue}{7} \qquad \qquad \qquad \textbf{(16)}$$

$$> \text{degree}(J[2], s[l+2]) \tag{17}$$

$$\begin{aligned} & \textcolor{blue}{> collect(J[2], s[l+2])} \\ & (-2 \alpha_{0,0} \alpha_{1,1} \beta_{l,2} + 4 \alpha_{0,0} \alpha_{1,2} \beta_{l,1} + \alpha_{0,1} \alpha_{1,0} \beta_{l,2} + \alpha_{0,1} \alpha_{1,1} \beta_{l,2} - 4 \alpha_{0,1} \alpha_{1,2} \beta_{l,0} \\ & - 2 \alpha_{0,1} \alpha_{1,2} \beta_{l,1} - 4 \alpha_{0,2} \alpha_{1,0} \beta_{l,1} - \alpha_{0,2} \alpha_{1,0} \beta_{l,2} + 8 \alpha_{0,2} \alpha_{1,1} \beta_{l,0} + \alpha_{0,2} \alpha_{1,1} \beta_{l,2} \\ & + 4 \alpha_{0,2} \alpha_{1,2} \beta_{l,0} - 2 \alpha_{0,2} \alpha_{1,2} \beta_{l,1}) s_{l+2} - 2 \alpha_{0,0} \alpha_{1,1} \alpha_{2,2} + 4 \alpha_{0,0} \alpha_{1,2} \alpha_{2,1} + \alpha_{0,1} \alpha_{1,0} \alpha_{2,2} \\ & + \alpha_{0,1} \alpha_{1,1} \alpha_{2,2} - 4 \alpha_{0,1} \alpha_{1,2} \alpha_{2,0} - 2 \alpha_{1,2} \alpha_{0,1} \alpha_{2,1} - 4 \alpha_{0,2} \alpha_{1,0} \alpha_{2,1} - \alpha_{0,2} \alpha_{2,2} \alpha_{1,0} \\ & + 8 \alpha_{0,2} \alpha_{1,1} \alpha_{2,0} + \alpha_{0,2} \alpha_{1,1} \alpha_{2,2} + 4 \alpha_{0,2} \alpha_{1,2} \alpha_{2,0} - 2 \alpha_{0,2} \alpha_{1,2} \alpha_{2,1} \end{aligned} \quad (18)$$

