```
> restart
>
```

Gröbner bases elimination for **Proposition 1** & **Proposition 2** in the paper:

On Rational Recursion for Holonomic Sequences

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Throughout $s(n+l+i) \rightarrow s_{l+i}$, $i = 1, 2$

Degree 1 case

```
>
> d := 1
```

$$d := 1 \tag{1}$$

Case l+1:

```
> p0 := add(alpha[0, k]·n^k, k = 0 ..d)
```

$$p0 := \alpha_{0,1}\, n + \alpha_{0,0} \tag{2}$$

```
> p1 := collect(add(add(binomial(d - k + j, j)·(alpha[1, d - j] + beta[l, d - j]·s[l + 1]), j
      = 0 ..k)·n^(d - k), k = 0 ..d), n, distributed)
```

$$p1 := \left(\beta_{l,1}\, s_{l+1} + \alpha_{1,1}\right) n + \beta_{l,0}\, s_{l+1} + \beta_{l,1}\, s_{l+1} + \alpha_{1,0} + \alpha_{1,1} \tag{3}$$

```
>
> L := [p0, normal(p1)]
```

$$L := \left[\alpha_{0,1}\, n + \alpha_{0,0}, n\, \beta_{l,1}\, s_{l+1} + n\, \alpha_{1,1} + \beta_{l,0}\, s_{l+1} + \beta_{l,1}\, s_{l+1} + \alpha_{1,0} + \alpha_{1,1}\right] \tag{4}$$

```
> vars := remove(has, indets([p1, p0]), beta) minus {n}
```

$$vars := \left\{\alpha_{0,0}, \alpha_{0,1}, \alpha_{1,0}, \alpha_{1,1}, s_{l+1}\right\} \tag{5}$$

```
> J := PolynomialIdeals:-PolynomialIdeal(L, parameters = select(has, indets(p1), beta))
```

$$J := \left\langle \alpha_{0,1}\, n + \alpha_{0,0}, n\, \beta_{l,1}\, s_{l+1} + n\, \alpha_{1,1} + \beta_{l,0}\, s_{l+1} + \beta_{l,1}\, s_{l+1} + \alpha_{1,0} + \alpha_{1,1} \right\rangle \tag{6}$$

```
> J := PolynomialIdeals:-EliminationIdeal(J, vars) :
> J := select(type, convert(J, list), polynom) :
> numelems(J)
```

(7)

$$1 \quad (7)$$

> degree(J[1], s[l + 1])

$$1 \quad (8)$$

> collect(J[1], s[l + 1])

$$\left(-\alpha_{0,0}\,\beta_{l,1} + \alpha_{0,1}\,\beta_{l,0} + \alpha_{0,1}\,\beta_{l,1}\right) s_{l+1} - \alpha_{0,0}\,\alpha_{1,1} + \alpha_{0,1}\,\alpha_{1,0} + \alpha_{0,1}\,\alpha_{1,1} \quad (9)$$

>

Case l+2:

>

Degree $d \leq 3$ case

To see the computations for d=3, replace 2 in the right-hand side below by 3.

>

> $d := 2$

$$d := 2 \quad (10)$$

> $p0 := add\left(\text{alpha}[0, k] \cdot n^k, k = 0..d\right)$

$$p0 := \alpha_{0,2}\,n^2 + \alpha_{0,1}\,n + \alpha_{0,0} \quad (11)$$

> $p1 := collect\left(add\left(add(\text{binomial}(d - k + j, j) \cdot \text{alpha}[1, d - j], j = 0..k) \cdot n^{d-k}, k = 0..d\right), n, distributed\right)$

$$p1 := \alpha_{1,2}\,n^2 + \left(\alpha_{1,2} + 2\,\alpha_{1,1}\right) n + \alpha_{1,2} + \alpha_{1,1} + \alpha_{1,0} \quad (12)$$

> $p2 := collect\left(add\left(add\left(\text{binomial}(d - k + j, j) \cdot 2^j \cdot (\text{alpha}[2, d - j] + \text{beta}[l, d - j] \cdot s[l + 2]), j = 0..k\right) \cdot n^{d-k}, k = 0..d\right), n, distributed\right)$

$$p2 := \left(\beta_{l,2}\,s_{l+2} + \alpha_{2,2}\right) n^2 + \left(4\,\beta_{l,1}\,s_{l+2} + \beta_{l,2}\,s_{l+2} + 4\,\alpha_{2,1} + \alpha_{2,2}\right) n + 4\,\beta_{l,0}\,s_{l+2} + 2\,\beta_{l,1}\,s_{l+2} + \beta_{l,2}\,s_{l+2} + 4\,\alpha_{2,0} + 2\,\alpha_{2,1} + \alpha_{2,2} \quad (13)$$

> $L := [p0, p1, normal(p2)]:$

> $vars := remove(has, indets([p2, p1, p0]), \text{beta})$ **minus** $\{n\}$

$$vars := \left\{\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,0}, \alpha_{2,1}, \alpha_{2,2}, s_{l+2}\right\} \quad (14)$$

> $J := PolynomialIdeals{:-}PolynomialIdeal(L, parameters = select(has, indets(p2), \text{beta}))$

$$J := \left\langle \alpha_{1,2}\,n^2 + \left(\alpha_{1,2} + 2\,\alpha_{1,1}\right) n + \alpha_{1,2} + \alpha_{1,1} + \alpha_{1,0}, n^2\,\alpha_{0,2} + n\,\alpha_{0,1} + \alpha_{0,0}, n^2\,\beta_{l,2}\,s_{l+2} + n^2\,\alpha_{2,2} + 4\,n\,\beta_{l,1}\,s_{l+2} + n\,\beta_{l,2}\,s_{l+2} + 4\,n\,\alpha_{2,1} + n\,\alpha_{2,2} + 4\,\beta_{l,0}\,s_{l+2} + 2\,\beta_{l,1}\,s_{l+2} + \beta_{l,2}\,s_{l+2} + 4\,\alpha_{2,0} + 2\,\alpha_{2,1} + \alpha_{2,2}\right\rangle \quad (15)$$

> $J := PolynomialIdeals{:-}EliminationIdeal(J, vars):$

> $J := select(type, convert(J, list), polynom):$

> $numelems(J)$

$$7 \tag{16}$$

> $degree(J[2], s[l+2])$

$$1 \tag{17}$$

> $collect(J[2], s[l+2])$

$$\begin{aligned}(&-2\,\alpha_{0,0}\,\alpha_{1,1}\,\beta_{l,2} + 4\,\alpha_{0,0}\,\alpha_{1,2}\,\beta_{l,1} + \alpha_{0,1}\,\alpha_{1,0}\,\beta_{l,2} + \alpha_{0,1}\,\alpha_{1,1}\,\beta_{l,2} - 4\,\alpha_{0,1}\,\alpha_{1,2}\,\beta_{l,0} \\ &- 2\,\alpha_{0,1}\,\alpha_{1,2}\,\beta_{l,1} - 4\,\alpha_{0,2}\,\alpha_{1,0}\,\beta_{l,1} - \alpha_{0,2}\,\alpha_{1,0}\,\beta_{l,2} + 8\,\alpha_{0,2}\,\alpha_{1,1}\,\beta_{l,0} + \alpha_{0,2}\,\alpha_{1,1}\,\beta_{l,2} \\ &+ 4\,\alpha_{0,2}\,\alpha_{1,2}\,\beta_{l,0} - 2\,\alpha_{0,2}\,\alpha_{1,2}\,\beta_{l,1})\,s_{l+2} - 2\,\alpha_{0,0}\,\alpha_{1,1}\,\alpha_{2,2} + 4\,\alpha_{0,0}\,\alpha_{1,2}\,\alpha_{2,1} + \alpha_{0,1}\,\alpha_{1,0}\,\alpha_{2,2} \\ &+ \alpha_{0,1}\,\alpha_{1,1}\,\alpha_{2,2} - 4\,\alpha_{0,1}\,\alpha_{1,2}\,\alpha_{2,0} - 2\,\alpha_{1,2}\,\alpha_{0,1}\,\alpha_{2,1} - 4\,\alpha_{0,2}\,\alpha_{1,0}\,\alpha_{2,1} - \alpha_{0,2}\,\alpha_{2,2}\,\alpha_{1,0} \\ &+ 8\,\alpha_{0,2}\,\alpha_{1,1}\,\alpha_{2,0} + \alpha_{0,2}\,\alpha_{1,1}\,\alpha_{2,2} + 4\,\alpha_{0,2}\,\alpha_{1,2}\,\alpha_{2,0} - 2\,\alpha_{0,2}\,\alpha_{1,2}\,\alpha_{2,1}\end{aligned} \tag{18}$$

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