Fast Evaluation of Nonviscously Damped System With A Kernel Independent Algorithm

Abstract

Keywords:

1. Nonviscous Damping With A Single Exponential Kernel

1.1. Nonviscous Damped System

Consider the equation of motion of a nonviscously damped inelastic multi-degree-of-freedom (MDOF) system,

$$\boldsymbol{Y}\left(\boldsymbol{u},\boldsymbol{v},\boldsymbol{a}\right)+\boldsymbol{F}\left(t\right)=\boldsymbol{P}\left(t\right),\tag{1}$$

where $\boldsymbol{u}=\boldsymbol{u}\left(t\right)$, $\boldsymbol{v}=\boldsymbol{v}\left(t\right)=\dot{\boldsymbol{u}}$ and $\boldsymbol{a}=\boldsymbol{a}\left(t\right)=\dot{\boldsymbol{v}}$ are the displacement, velocity and acceleration vectors, $\boldsymbol{Y}=\boldsymbol{Y}\left(\boldsymbol{u},\boldsymbol{v},\boldsymbol{a}\right)$ is the resistance vector of the system, $\boldsymbol{P}=\boldsymbol{P}\left(t\right)$ is the external load vector, and $\boldsymbol{F}$ is the nonviscous damping force which can be expressed in the form of the convolution of the kernel $f=f\left(t\right)$ and the vector $\boldsymbol{w}$, viz. $\boldsymbol{F}\left(t\right)=f*\boldsymbol{w}$.

Note here, $\boldsymbol{w}$ can be either the exact velocity vector $\boldsymbol{v}$, or the subset of $\boldsymbol{v}$ such that they share the same size but some velocity components in $\boldsymbol{v}$ are replaced by zeros in $\boldsymbol{w}$ on selected DoFs. This is beneficial when it comes to compositing flexible damping that will be discussed later in this work. Formally,

$$\boldsymbol{w}=\boldsymbol{T}\boldsymbol{v},\tag{2}$$

where $\boldsymbol{T}$ is a square diagonal matrix, the diagonal entries of which are either one or zero.

Since it is an inelastic system, the stiffness matrix $\boldsymbol{K}$, the viscous damping matrix $\boldsymbol{C}$ and the mass matrix $\boldsymbol{M}$ are

$$\dfrac{\partial\boldsymbol{Y}}{\partial\boldsymbol{u}}=\boldsymbol{K},\qquad\dfrac{\partial\boldsymbol{Y}}{\partial\boldsymbol{v}}=\boldsymbol{C},\qquad\dfrac{\partial\boldsymbol{Y}}{\partial\boldsymbol{a}}=\boldsymbol{M}.\tag{3}$$

The viscous damping matrix $\boldsymbol{C}$ may not be trivial as the system may consist of viscous damping components (e.g., viscous damper devices). Using $\boldsymbol{u}$ as the basic quantity, the effective stiffness matrix $\bar{\boldsymbol{K}}$

$$\bar{\boldsymbol{K}} = \frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}\boldsymbol{u}} = \boldsymbol{K} + \boldsymbol{C}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}} + \boldsymbol{M}\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}\boldsymbol{u}} \tag{4}$$

is the combination of the three, its specific form depends on the specific time integration method used.

1.2. A Single Exponential Kernel

For the moment, we focus on the scalar-valued exponential kernel function

$$f = f\left(t\right) = m\exp\left(-st\right), \tag{5}$$

where s is often denoted by the relaxation parameter μ, m is often denoted by $c\mu$ in which c is the damping constant. The convolution can be then expressed as

$$\boldsymbol{F}\left(t\right) = f * \boldsymbol{w} = \int_0^t f(t-\tau)\cdot\boldsymbol{w}\left(\tau\right)\ \mathrm{d}\tau = \int_0^t m\exp\left(-s\left(t-\tau\right)\right)\cdot\boldsymbol{w}\left(\tau\right)\ \mathrm{d}\tau. \tag{6}$$

Assuming trivial initial condition $\boldsymbol{v}\left(0\right) = \boldsymbol{0}$, Eq. (6) corresponds to the solution of the following ODE,

$$\boldsymbol{F}' = -s\boldsymbol{F} + m\boldsymbol{w}. \tag{7}$$

It can be validated by solving Eq. (7) with the assist of the integrating factor $\exp\left(st\right)$.

1.3. An Efficient Algorithm

Instead of directly integrating Eq. (6) using higher-order methods (such as the Runge–Kutta family), Eq. (7) can be combined with Eq. (1) to develop an efficient algorithm.

In the context of a discretised iterative solving schema, Eq. (7) can be rewritten as follows using the backward (implicit) Euler method,

$$\frac{\boldsymbol{F}_{n+1} - \boldsymbol{F}_n}{\Delta t} = -s\boldsymbol{F}_{n+1} + m\boldsymbol{w}_{n+1}, \tag{8}$$

in which subscripts $(\cdot)_{n+1}$ and $(\cdot)_n$ denote the corresponding quantity at t_n and $t_{n+1} = t_n + \Delta t$. Rearranging Eq. (8) yields

$$(1 + s\Delta t)\, \boldsymbol{F}_{n+1} - \boldsymbol{F}_n - m\Delta t \boldsymbol{w}_{n+1}. \tag{9}$$

Assuming Eq. (1) is satisfied at t_{n+1}[1], then, accounting for both Eq. (1) and Eq. (9), the residual $\boldsymbol{R}$ (with the subscript $(\cdot)_{n+1}$ dropped for brevity) is

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{W} \end{bmatrix} = \begin{cases} \boldsymbol{Y} + \boldsymbol{F} - \boldsymbol{P}, \\ (1 + s\Delta t)\, \boldsymbol{F} - \boldsymbol{F}_n - m\Delta t \boldsymbol{w}. \end{cases} \tag{10}$$

The unknown quantity is $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{F} \end{bmatrix}^{\mathrm{T}}$. Linearisation results in the following Jacobian.

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial \boldsymbol{Q}}{\partial \boldsymbol{u}} & \dfrac{\partial \boldsymbol{Q}}{\partial \boldsymbol{F}} \\ \dfrac{\partial \boldsymbol{W}}{\partial \boldsymbol{u}} & \dfrac{\partial \boldsymbol{W}}{\partial \boldsymbol{F}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{K}} & \boldsymbol{I} \\ -m\Delta t \boldsymbol{T} \dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}} & (1 + s\Delta t)\, \boldsymbol{I} \end{bmatrix}. \tag{11}$$

Typically, $\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}}$ reduces to a scalar constant (multiplied by an identity matrix), for example, in the Newmark method, it is $\dfrac{\gamma}{\beta \Delta t}$.

Noting that $\dfrac{\partial \boldsymbol{W}}{\partial \boldsymbol{F}}$ is a diagonal matrix that can be easily inverted, there is no need to explicitly formulate the Jacobian. Instead, one could perform static condensation such that, from the second expression,

$$-m\Delta t \boldsymbol{T} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}} \delta \boldsymbol{u} + (1 + s\Delta t)\, \delta \boldsymbol{F} = \boldsymbol{W}, \tag{12}$$

the increment $\delta \boldsymbol{F}$ is

$$\delta \boldsymbol{F} = \frac{1}{1 + s\Delta t} \left(\boldsymbol{W} + m\Delta t \boldsymbol{T} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}} \delta \boldsymbol{u} \right), \tag{13}$$

[1]This assumption is not always valid as some time integration methods establish the EOM elsewhere, see, for example, the generalised-α method, the GSSSS method, the Bathe two-step method, the OALTS method, etc.

substituting it into the first expression yields

$$\bar{\boldsymbol{K}}\delta\boldsymbol{u} + \frac{1}{1+s\Delta t}\left(\boldsymbol{W} + m\Delta t\boldsymbol{T}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}}\delta\boldsymbol{u}\right) = \boldsymbol{Q}. \tag{14}$$

Rearranging gives

$$\left(\bar{\boldsymbol{K}} + \frac{m\Delta t}{1+s\Delta t}\boldsymbol{T}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}}\right)\delta\boldsymbol{u} = \boldsymbol{Q} - \frac{1}{1+s\Delta t}\boldsymbol{W}. \tag{15}$$

By denoting

$$\hat{\boldsymbol{K}} = \bar{\boldsymbol{K}} + \frac{m\Delta t}{1+s\Delta t}\boldsymbol{T}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{u}}, \qquad \hat{\boldsymbol{Q}} = \boldsymbol{Q} - \frac{1}{1+s\Delta t}\boldsymbol{W}, \tag{16}$$

the system to be solved is simply

$$\hat{\boldsymbol{K}}\delta\boldsymbol{u} = \hat{\boldsymbol{Q}}. \tag{17}$$

The revised effective load vector $\hat{\boldsymbol{Q}}$ can be explicitly written as

$$\hat{\boldsymbol{Q}} = \boldsymbol{Y} - \boldsymbol{P} + \frac{1}{1+s\Delta t}\boldsymbol{F}_n + \frac{1}{1+s\Delta t}m\Delta t\boldsymbol{w}. \tag{18}$$

References