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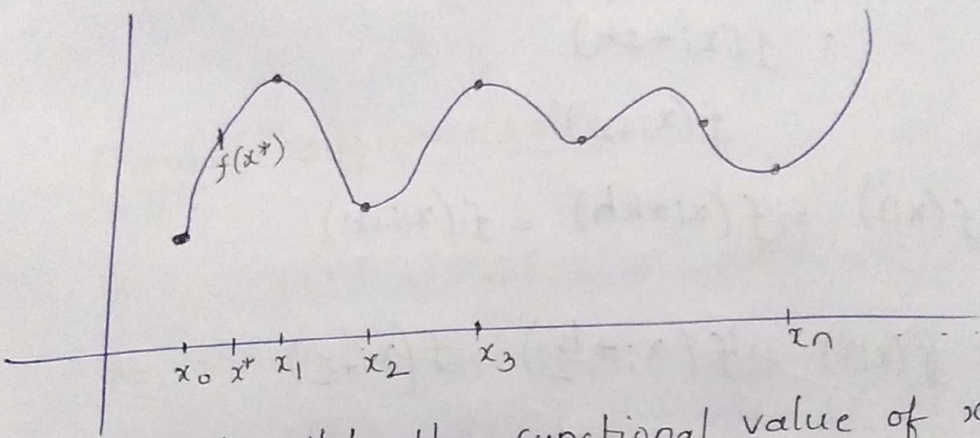
NUMERICAL ANALYSIS

Interpolation:

x	x_0	x_1	x_2		x_n
$y=f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	. . . .	$f(x_n)$

$\underbrace{x_i - x_{i-1} = h}_{\text{Equidistant points}}$, $\forall i = 1, 2, \cdots n$



What will be the functional value of x^* if it lies b/w x_0 & x_1?

This problem can be solved by interpolation.

If we estimate $y=f(x)$ using given points then $f(x^*)$ can be calculated.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Interpolation with evenly spaced points

Finite difference operators:

$(x_i, f(x_i))$, $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots n$

Shift operator E

$$E(f(x_i)) = f(x_i+h)$$
$$= f(x_{i+1})$$

$$E(f(x_0)) = f(x_1) = f(x_0+h)$$
$$E(f(x_1)) = f(x_2) = f(x_1+h)$$

$$E^2(f(x_i)) = E(E(f(x_i)))$$
$$= E[f(x_i+h)]$$
$$= f(x_i+2h)$$
$$= f(x_{i+2})$$

$$E^k(f(x_i)) = f(x_i+kh) = f(x_{(i+k)})$$

$k=\frac{1}{2}$

$$E^{1/2}(f(x_i)) = f\left(x_i+\frac{h}{2}\right) = f\left(x_{i+\frac{1}{2}}\right)$$

Forward difference operator:

$$\Delta f(x_i) = f(x_i+h) - f(x_i)$$
$$= f_{i+1} - f_i$$
$$= f(x_{i+1}) - f(x_i)$$

$$\Delta f(x_0) = f(x_1) - f(x_0) = f(x_0+h) - f(x_0)$$
$$\Delta f(x_1) = f(x_2) - f(x_1) = f(x_1+h) - f(x_1)$$

$$\Delta^2 f(x_i) = \Delta[\Delta f(x_i)]$$
$$= \Delta[f(x_i+h) - f(x_i)]$$
$$= \Delta f(x_i+h) - \Delta f(x_i)$$
$$= \cancel{\Delta} f(x_{i+2}) - f(x_{i+1}) - [f(x_{i+1}) - f x_i])$$
$$\Delta^2 f(x_i) = f(x_{i+2}) + f(x_i) - 2f(x_{i+1})$$

$$\Delta^3(f(x_i)) = f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)$$

Relation b/w Δ & E ?

$$\Delta f(x_i) = f(x_i+h) - f(x_i) = E(f(x_i)) - f(x_i)$$

$$\Delta f(x_i) = (E-1)\, f(x_i)$$

$$\Delta = E - 1 \quad \text{or} \quad E = \Delta + 1 \qquad \Delta(\text{constant}) = 0$$

$$\Delta^n f(x_i) = (E-1)^n f(x_i)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!} \cdot f_{i+n-k}$$

Forward difference table:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	$f(x_0)$	$\Delta f_0 = f_1 - f_0$	$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
x_1	$f(x_0)$	$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	
x_2	$f(x_2)$	$\Delta f_2 = f_3 - f_2$		
x_3	$f(x_3)$			

Backward difference operator: $\nabla \rightarrow$ nabla

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1}$$

$$\nabla f(x_1) = f(x_1) - f(x_0)$$

$$\nabla f(x_2) = f(x_2) - f(x_1)$$

$$\nabla^2 f(x_i) = \nabla[\nabla f(x_i)]$$

$$= f(x_i) - 2f(x_{i-1}) + f(x_{i-2})$$

$$\nabla^3 f(x_i) = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}$$

Relation b/w ∇ & E ?

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = (1-E^{-1}) f x_i$$

$$\nabla = 1 - E^{-1} \quad \text{or} \quad E = (1-\nabla)^{-1}$$

$$\nabla^n f(x_i) = (1-E^{-1})^n f(x_i) = \sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{k!(n-k)!} \cdot f_{i-k}$$

Backward difference Table

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	$f(x_0)$			
x_1	$f(x_1)$	$\nabla f_1 = f_1 - f_0$		
x_2	$f(x_2)$	$\nabla f_2 = f_2 - f_1$	$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	
x_3	$f(x_3)$	$\nabla f_3 = f_3 - f_2$	$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$	$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$

$$\Delta f_0 = \nabla f_1$$

$$\Delta f_1 = \nabla f_2 \qquad \Delta^3 f_0 = \nabla^3 f_3$$

$$\Delta f_2 = \nabla f_3$$

Central difference operator:

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

$$= f_{i+1/2} - f_{i-1/2}$$

$$\delta\left(f\left(x_i + \frac{h}{2}\right)\right) = \delta f_{i+1/2} = f(x_i + h) - f(x_i)$$

$$= \cancel{f(i+} f_{i+1} - f_i$$

$$\delta f_{1/2} = f_1 - f_0$$

$$\delta f_{3/2} = f_2 - f_1$$

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$$\delta^2 f(x_i) = \delta[\delta(f(x_i))]$$
$$= f(x_{i+1}) - 2f(x_i) + f(x_{i-1})$$

$$\delta^3 f(x_i) = f_{i+3/2} - 3f_{i+1/2} + 3f_{i-1/2} - f_{i-3/2}$$

Relation between δ & E:

$$\delta f(x_i) = f_{i+1/2} - f_{i-1/2} = (E^{1/2} - E^{-1/2}) f_i$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\delta^n f(x_i) = (E^{1/2} - E^{-1/2})^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!\,k!} f_{i+(n/2)-k}$$

Central difference table

x	$f(x)$	δf	$\delta^2 f$	$\delta^3 f$
x_0	$f(x_0)$	$\delta f_{1/2} = f_1 - f_0$		$\delta^3 f_{3/2}$
x_1	$f(x_1)$	$f_{3/2} = f_2 - f_1$	$\delta^2 f_1 = \delta f_{3/2} - \delta f_{1/2}$	$= \delta^2 f_2 - \delta^2 f$
x_2	$f(x_2)$	$\delta f_{5/2} = f_3 - f_2$	$\delta^2 f_2 = \delta f_{5/2} - f_{3/2}$	
x_3	$f(x_3)$			

Relation b/w δ & Δ, ∇

$$\Delta^n f_i = \nabla^n f_{i+n} = \delta^n f_{i+(n/2)}$$

$$\nabla = 1 - E^{-1} = (E-1)E^{-1} = \Delta E^{-1}$$

$$\nabla^n f_{i+n} = \Delta^n E^{-n} f_{i+n} = \Delta^n f_i$$

$$\delta = E^{1/2} - E^{-1/2} = (E-1)E^{-1/2} = \Delta E^{-1/2}$$

$$\delta^n f_{i+n/2} = \Delta^n E^{-n/2} f_{i+n/2} = \Delta^n f_i$$

$$\Delta f(x) = f(x+h) - f(x)$$

Remark:

Let $P_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ $\quad$ n = degree of polynomial.

$$\Delta^k P_n(x) = \begin{cases} 0 & k > n \\ a_0 n! & k = n \end{cases}$$

$$\Delta P_n(x) = P_n(x+h) - P_n(x)$$

$$= \cancel{a_0(x+h)^n - a_0 x^n}$$

$$\nabla^k P_n(x) \begin{cases} 0 & k > n \\ a_0 n! & k = n \end{cases}$$

Mean Operator:

$$\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left(f_{i+1/2} + f_{i-1/2}\right)$$

Relation b/w μ & E

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

Ex: $\Delta^3\left[(1-2x)(1-3x)(1-4x)\right] = ?$

$$= \Delta^3\left[-24x^3 + 26x^2 - 9x + 1\right]$$

$$= -24 \times 3!$$

$$= -24 \times 6$$

$$= -144$$

Relations:

i) $\delta = \nabla (1-\nabla)^{-1/2}$

ii) $\mu = \left[1+\frac{\delta^2}{4}\right]^{1/2}$

iii) $\Delta (f_i^2) = (f_i + f_{i+1}) \Delta f_i$

Proof: $\Delta(f_i^2) = f_{i+1}^2 - f_i^2 = (f_{i+1}+f_i)(f_{i+1}-f_i)$

$= (f_{i+1}+f_i) \cdot \Delta f_i$

iv) $\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x) \cdot g(x+h)}$

Proof

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{g(x)f(x+h) - f(x)g(x+h)}{g(x) \cdot g(x+h)}$$

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\left[f(x+h) - f(x)\right] - f(x)\left[g(x+h) - g(x)\right]}{g(x) \cdot g(x+h)}$$

Newton's Forward Difference Interpolation Formula:

$h = x_i - x_{i-1}$

x	$y = f(x)$
x_0	y_0
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

$$f(x) = f(x_0) + (x-x_0)\frac{\Delta f_0}{1!h} + (x-x_0)(x-x_1)\frac{\Delta^2 f_0}{2!h^2}$$

$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\frac{\Delta^n f_0}{n!h^n}$$

$x - x_i = x_0 + sh - (x_0 + ih)$

$= (s-i)h$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$\vdots$

$x_n = x_0 + nh$

$x = x_0 + sh$

$x - x_0 = sh$

$x - x_1 = (s-1)h$

$x - x_2 = (s-2)h$

$f(x) = f(x_0 + sh) = E^s f(x_0)$

$= (1+\Delta)^s f(x_0)$

$(1+\Delta)^s f(x_0) = ({}^sC_0 + {}^sC_1\Delta + {}^sC_2\Delta^2 + \cdots\cdots)f(x_0)$

${}^sC_2 = \frac{s(s-1)}{2!}$

$\equiv {}^sC_0 f(x_0) + {}^sC_1 \cdot \Delta f_0 + {}^sC_2 \cdot \Delta^2 f_0 + \cdots + {}^sC_n \cdot \Delta^n f_0$

$= f(x_0) + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \cdots$

$+ \frac{s(s-1)(s-2)\ldots(s-n+1)}{n!}\Delta^n f_0$

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Error of Interpolation:

$f(x) - P_n(x) = E_n(f, x)$ → Error function

$$E_n(f, x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} \cdot f^{n+1}(\xi)$$

$0 < \xi < n$

$f(\xi) = f(x_\xi)$

NFDI (s form):

$x = x_0 + sh$

$$f(x) = f(x_0) + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \dots +$$

$$\frac{s(s-1)(s-2)\dots(s-n+1)}{n!}\Delta^n f_0$$

Properties of NFDI:

1) Interpolating x's near the beginning of the table of the nodal points.
2) Applicable for evenly spaced nodal data points.

$x_i - x_{i-1} = h, \forall\ i = 1, 2, \dots n$

3) Has permanence property.

x	$f(x)$
x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	
x_n	$f(x_n)$

Add one more data point (x_{n+1}, f_{n+1})

All the earlier terms dont change (permanence property)

Just Add $\dfrac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!\,h^{n+1}}\Delta^{n+1} f_0$

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Newton's Backward Difference Interpolation Formula

Interpolating x's at the end of the table

x	$f(x)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
$\vdots$	
x_n	$f(x_n)$

$$f(x) = f(x_n) + \frac{(x-x_n)}{1!\,h} \nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2} \nabla^2 f(x_n) + \cdots + \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{n!\,h^n} \nabla^n f(x_n)$$

s form of NBDI:

$x = x_n + sh \qquad s<0$

$$f(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \cdots + \frac{s(s+1)(s+2)\cdots(s+n-1)}{n!}\nabla^n f(x_n)$$

$$f(x_n+sh) = E^s f(x_n) = (1-\nabla)^{-s} f(x_n)$$

$$E_n(f,x) = f(x) - P_n(x)$$

$$= \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_0)}{(n+1)!} f^{n+1}(\xi)$$

where $0<\xi<n$

If we add a point we need to change everything (does not obey permanence property)

Applicable for evenly spaced points

Ex 1

Construct the forward difference table for the data

x	$f(x)$	Δf (∇f)	$\Delta^2 f$ $(\nabla^2 f)$	$\Delta^3 f$ $(\nabla^3 f)$
-1	-8			
		11		
			-13	
0	3			26
		-2		
1	1		13	
		11		
2	12			

Same table for forward & backward difference table

Ex 2 Find $f(2.4)$

x	2	4	6	8	10
$f(x)$	9.68	10.96	12.32	13.76	15.28

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	9.68		
		1.28	0.08
4	10.96		
		1.36	0.08
6	12.32		
		1.44	0.08
8	13.76		
		1.52	
10	15.28		

$x = 2.4 \quad x_0 = 2 \quad , h = 2$

$$f(2.4) = 9.68 + \frac{(2.4-2)}{1!\,2}(1.28) + \frac{(2.4-2)(2.4-4)}{2!\,2^2}(0.08)$$

Ex: 3

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

Find $f(1.0)$

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
0.1	-1.699					
		0.626				
0.3	-1.073		0.072			
		0.698		0.048		
0.5	-0.375		0.120		-0.0[illegible]	
		0.818		-0.048		0.0[illegible]
0.7	+0.443		0.168		0.04	
		0.986		0.048		
0.9	+1.429		0.216			
		1.202				
1.1	2.631					

$x = 1$, $h = 0.2$, $x_n = 1.1$

$$f(1.0) = 2.631 + \frac{(1-1.1)}{1!(0.2)}(1.202) + \frac{(1-1.1)(1-0.9)}{2!(0.2)^2}(0.216)$$

$$+ \frac{(1-1.1)(1-0.9)(1-0.7)}{3!(0.2)^3}(0.048)$$

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Central Difference Interpolation (sterling)

x	$f(x)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
$\vdots$	
x_r	$f(x_r)$
$\vdots$	
x_n	$f(x_n)$

$$\delta f(x_r) = f(x_{r+\frac{1}{2}}) - f(x_{r-\frac{1}{2}})$$

$$\mu f(x_r) = \frac{1}{2}\left[f(x_{r+\frac{1}{2}}) + f(x_{r-\frac{1}{2}})\right]$$

$$\mu = \frac{E^{-1/2} + E^{1/2}}{2}$$

$$f(x) = f(x_{r_0} + kh)$$

$$= \left[1 + k\mu\delta + \frac{k^2}{2!}\delta^2 + \frac{k(k-1)(k+1)}{3!}\mu\delta^3 + \frac{k^2(k^2-4)}{4!}\delta^4 + \ldots\right] f(x_{r_0})$$

Ex

0

1

2

3

4

5

Interpolate at 2.2

$x_r = 2$ (previous nodal point)

$h = 1$ (diff b/w nodal points)

$K = 0.2$ (diff b/w interpolating value and x_r) $\frac{\quad}{h} = \frac{x - x_r}{h}$

$$x = x_r + kh$$

2nd term

$$\mu\delta f(x_{r_0}) = \frac{E^{-1/2} + E^{1/2}}{2}\delta f_r$$

$$= \frac{1}{2}\left[\delta f_{r-1/2} + \delta f_{r+1/2}\right]$$

4th term

$$\mu\delta^3 f(x_0) = \frac{1}{2}\left[\delta^3 f_{r_0-1/2} + \delta^3 f_{r_0+1/2}\right]$$

$x_0 \quad f(x_0)$

$x_1 \quad f(x_1) \qquad \delta f_{1/2}$

$\vdots \qquad \vdots$

$x_{r-2} \quad f(x_{r-2})$

$x_{r-1} \quad f(x_{r-1}) \qquad \delta f_{r-3/2}$

$\qquad\qquad\qquad \delta^2 f_{r-1}$

$x_r \quad f(x_r) \qquad \boxed{\begin{matrix}\delta f_{r-1/2} \\ \delta f_{r+1/2}\end{matrix}} \qquad \delta^2 f_r \qquad \boxed{\begin{matrix}\delta^3 f_{r-1/2} \\ \delta^3 f_{r+1/2}\end{matrix}} \qquad \delta^4 f_{r_0}$

$x_{r+1} \quad f(x_{r+1})$

$\qquad\qquad\qquad \delta f_{r+3/2} \qquad \delta^2 f_{r+1}$

$x_{r+2} \quad f(x_{r+2})$

Ex: $f(x) = e^x \qquad x = 0.1, 0.6, 1.1, 1.6, 2.1$

$f(1.3) = ?$

x	$f(x)$	δ	δ^2	δ^3	δ^4
0.1	1.1051				
		0.7169			
0.6	1.8221		0.4652		
		1.1821		0.3015	
1.1	3.0042		0.7667		0.1962
		1.9488		0.4977	
1.6	4.9530		1.2644		
		3.2132			
2.1	8.1662				

$x_{r_0} = 1.1$

$h = 0.5$

$k = 0.4$

$$f(1.3) = 3.0042 + 0.4 \times \left(\frac{1.1821 + 1.9488}{2}\right) + \frac{(0.4)^2}{2}\, 0.7667$$

$$+ \frac{(0.4)(0.4-1)(0.4+1)}{3!}\left(\frac{0.3015 + 0.4977}{2}\right)$$

$$+ \frac{(0.4)^2\left((0.4)^2 - 4\right)}{4!}\,(0.1962)$$

$f(1.3) \simeq 3.6682$

$e^{1.3} = 3.6693$

$f(1.3) \cong e^{1.3}$