REAL ANALYSIS

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Upper bound:

Let $S \subseteq \mathbb{R}$ if $\exists$ a real number m such that $m \geq s$ $\forall s \in S$, then m is an upper bound of $S \rightarrow$ Bounded above.

Lower bound:

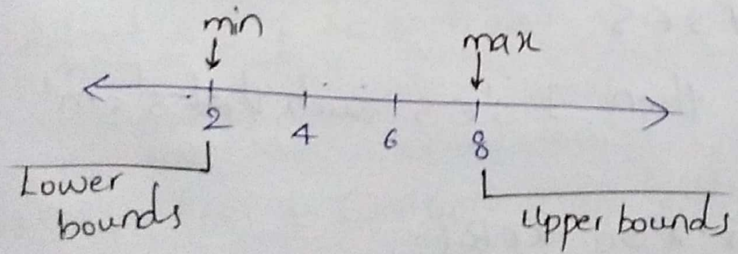
If $m \leq s$, $\forall s \in S$ then m is a lower bound of S & S is bounded below.

→ S is bounded if it is bounded above & bounded below.

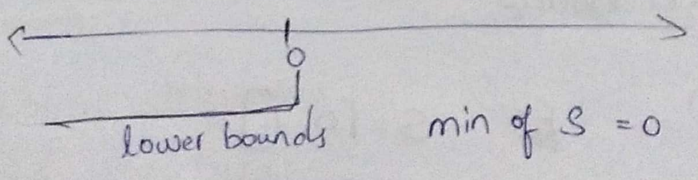
→ If an upper bound 'm' of S is a member of S, m is called maximum/greatest element/largest element of S

→ If a lower bound 'm' of S is a member of S, m is called minimum/least element/smallest element of S.

Ex: $S = \{2, 4, 6, 8\}$



Ex: $S = [0, \infty)$ is not bounded above



min of $S = 0$

Ex: $S = (0, 4]$ max of $S = 4$

There is no min of S

Ex: ϕ

$m \geq s \ \forall s \in \phi \equiv$ if $s \in \phi$ then $m \geq s$

ϕ is bounded below & above by any real number

Supremum:

Let S be an non-empty subset of R if S is bounded above then the least upper bound (lub) of S is called its supremum and is denoted by $m = \sup S$

iff (a) $m \geq s \ \forall s \in S$ [m is upper bound of S]

(b) if $m' < m$ then $\exists s' \in S$ such that $s' > m'$

Nothing smaller than m is an upper bound of S

Infimum:

If S is bounded below then greatest lower bound (g.l.b) of S is called its infimum

$m = \inf S$ iff

(a) $m \leq s, \ \forall s \in S$

(b) if $m' > m$, then $\exists s' \in S$ such that $s' < m'$

Ex: $S = (0, \infty) = \{x : x > 0, x \in R\}$

$\inf S = 0$

a) 0 is a lower bound

Ex: $S = (0, 1)$

$\sup S = 1$

$\inf S = 0$

max → doesn't exist

min → doesn't exist

Ex: $S = [0, 1]$

$\sup S = 1$ max = 1

$\inf S = 0$ min = 0

Ex: $S = \{\frac{1}{n} : n \in N\}$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$\sup S = 1$

$\inf S = 0$

L.U.B Axiom (Completeness Axiom)

Every non-empty subsets of R that is bounded above has a l.u.b that is sup S exists and that is a real number.

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Real Sequences:

Sequence: A function whose ~~ele~~ domain is set N and range in a set of real numbers R is called as real sequence

$$S : N \longrightarrow R$$

Domain: N $\quad S_n : n \in N$

$$\{S_n\} = \{ \underset{\substack{\downarrow \\ 1^{st}\ term}}{S_1}, \underset{\substack{\downarrow \\ 2^{nd}\ term}}{S_2}, S_3 \ldots \underset{\substack{\downarrow \\ n^{th}\ term}}{S_n}, \ldots \underset{\substack{\downarrow \\ m^{th}\ term}}{S_m} \ldots \}$$

* n^{th} term and m^{th} term are treated distinct though $S_n = S_m$
* Sequence is an ordered set of Real numbers
* No. of terms in a sequence is always infinite.

Ex: $\{S_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$

Distinct elements $= \{-1, 1\}$

Range $= \{-1, 1\}$

Ex: $\{S_n\} = \{\frac{1}{n}\} \quad n \in N$

$$= \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$

Range: The range or range set consists of all the distinct elements of a sequence without repetition and without regard of position. Range set is finite/infinite

Range = not set ϕ

$S_n = \{\frac{1}{n}\} \ n \in N \quad = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Bounds of a sequence:

Bounded above sequence:

A sequence $\{S_n\}$ is said to be bounded above if $\exists$ real number k such that

$$S_n \leq k \quad \forall n \in N$$

Bounded below sequence.

A sequence $\{S_n\}$ is said to be bounded below if $\exists$ real number k such that

$$S_n \geq k \quad \forall n \in N$$

* Sequence is bounded if it is both bounded above and bounded below.

* Evidently if sequence is bounded iff range is bounded.

$$S_n = (-1)^n = \{-1, 1, -1, 1, \dots\} \qquad S_1 = -1,\ S_2 = 1,\ S_3 = -1,\ S_4 = 1$$

Range $\{-1, 1\}$

* Bounds of range are bounds of sequence

Convergence of sequences:

A sequence $\{S_n\}$ is said to converge to a real number 'l' (to have real number l as its limit) if for each $\varepsilon > 0$ $\exists$ positive integer m (depends on ε) such that

$$|S_n - l| \leq \varepsilon \quad \forall n \geq m$$

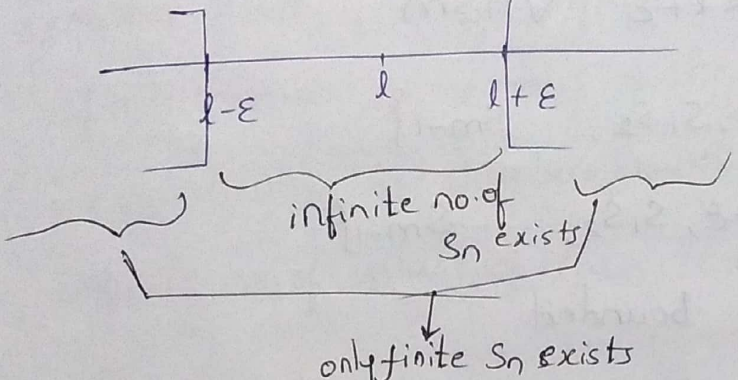
$$l - \varepsilon \leq S_n \leq l + \varepsilon$$

$S_n \to l$ as $n \to \infty$

$\displaystyle \underset{n\to\infty}{Lt}\ S_n = l$

$n \geq m$ $\boxed{l - \varepsilon \leq S_n \leq l + \varepsilon}$ $\rightarrow$ converges to l
*

Infinite number of terms exists in this interval



Ex: $\{S_n\} = \dfrac{3n+1}{7n-4} \quad n \in N$

$S_1 = \dfrac{4}{3}$, $S_2 = \dfrac{7}{10}$

$S_3 = \dfrac{10}{17}$, $S_4 = \dfrac{13}{24}$

$S_5 = \dfrac{16}{31}$, $S_6 =$

$$\underset{n\to\infty}{Lt}\ S_n = \underset{n\to\infty}{Lt}\ \frac{3 + \frac{1}{n}}{7 - \frac{4}{n}} = \frac{3}{7} = l$$

$l = \dfrac{3}{7}$

$\exists\ \varepsilon > 0$ such that $\forall n \geq m$

$$|S_n - l| \leq \varepsilon \quad \forall n \geq m$$

$$\frac{3}{7} - \varepsilon \leq S_n \leq \frac{3}{7} + \varepsilon$$

$\varepsilon = 1 \quad \exists m \quad n > 0 \quad \left|\dfrac{3n+1}{7n-4} - \dfrac{3}{7}\right| < 1$

$\varepsilon = 0.1 \quad n > 4 \quad \left|\dfrac{3n+1}{7n-4} - \dfrac{3}{7}\right| < 0.1$

$\varepsilon = 0.01 \quad n > 39 \quad \left|\dfrac{3n+1}{7n-4} - \dfrac{3}{7}\right| < 0.01$

Theorem:

Every convergent sequence is bounded

Let $\{S_n\}$ converges to l

Let $\varepsilon > 0$ $\exists m$

$$|S_n - l| < \varepsilon \quad \forall n \geq m$$

$$l - \varepsilon < S_n < l + \varepsilon \quad \forall n \geq m$$

$$g = \min\{l-\varepsilon, S_1 S_2, \ldots S_{m-1}\}$$

$$G = \max\{l+\varepsilon, S_1 S_2, \ldots S_{m-1}\}$$

$g \leq S_n \leq G$ bounded

$\therefore$ Every convergent sequence is bounded

* The converse of the above theorem may not be true.
* A sequence cannot converge to more than one limit.

Ex: $S_n = \dfrac{n}{2^n}$ $n \in N$ $\quad l = \underset{n \to \infty}{Lt} \dfrac{n}{2^n} = 0$

$\frac{1}{2}, \frac{2}{2^2}, \frac{3}{2^3}, \frac{4}{2^4}, \ldots$

n	$\frac{n}{2^n}$
1	0.5
2	0.5
3	0.375
4	0.25
5	0.

$\epsilon = 0.1$ $\quad \underset{m}{n>5}$ $\quad \left|\frac{n}{2^n} - 0\right| < 0.1$

$\epsilon = 0.01$ $\quad \underset{m}{n>9}$ $\quad \left|\frac{n}{2^n} - 0\right| < 0.01$

$\epsilon = 0.001$ $\quad n>14$ $\quad \left|\frac{n}{2^n} - 0\right| < 0.001$

$\epsilon = 0.00001$ $\quad n>22$ $\quad \left|\frac{n}{2^n} - 0\right| < 0.00001$

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* Every convergent sequence is bounded and has a unique limit.

Limit Points of a Sequence:

A real number ξ is said to be a limit point of a sequence $\{S_n\}$, if every neighbourhood of ξ contains an infinite number of members of the sequence.

i.e., given any +ve number ϵ, however small for an infinite no. of values of n. $S_n \in (\xi - \epsilon, \xi + \epsilon)$

~~$|S_n - \xi|$, for~~

$|S_n - \xi| < \epsilon$, for infinitely many values of n.

* Intutively S_n is an arbitrarily close to ξ for an infinite number of values of n.
* As in a set, a limit point is also called cluster point or a point of condensation.
* A number ξ is not a limit point of the sequence $\{S_n\}$ if $\exists$ a number $\epsilon > 0$ such that $S_n \in (\xi - \epsilon, \xi + \epsilon)$ for at most a finite values of n.
* A limit point of the range set of a sequence is also a limit point of the sequence converse may not be true.

Ex. $S_n = 1 \quad \forall\, n \in N$

Limit point of the sequence = 1

Range = $\{1\}$

Limit point of the range = No limit point.

$S_n = \{1\} \quad \forall n \in N$

for every $\epsilon > 0$

$S_n \in (1-\epsilon, 1+\epsilon), \forall n \in N \quad \boxed{\xi = 1}$

So limit point of S_n is 1.

Ex2: $S_n = \frac{1}{n}, \forall n \in N$

$= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

Limit point of sequence = 0

Range set = $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

Limit point of Range set = 0

Ex3 $S_n = 1 + (-1)^n \quad n \in N$

$\{0, 2, 0, 2, \dots\}$

Limit point of sequence = $\{0, 2\}$

Range = $\{0, 2\}$

Limit point of Range = No limit points.

Ex4 $S_n = (-1)^n \left(1 + \frac{1}{n}\right) \quad n \in N$

$\{-2, (1+\frac{1}{2}), -(1+\frac{1}{3}), (1+\frac{1}{4}), -(1+\frac{1}{5}), \dots\}$

Limit points of sequence = $\{-1, 1\}$

Limit points of Range = $\{-1, 1\}$

Relation between limit point & limit:

Ex: $S_n = \{1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \dots\}$

Limit point = 1

But the sequence is not convergent (so no limit)

Unique limit point is a limit?

NO

Bolzano-Dieðstrass Theorem: Every bounded sequence has atleast a limit point.

* The converse of the theorem may not be true.
* Bounded sequence may have many limit points.
* The greatest and the smallest of the limit points of a bounded sequence are respectively called the upper and lower limits.

Ex $\{1, 2, 1, 2, 1, 2, \dots\}$ Upper limit = 2
Lower limit = 1

$\{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$ Upper limit = 3
Lower limit = 1

Convergent sequence: Every bounded sequence with a unique limit point is convergent. (Th)

Th A necessary & sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.

Th A necessary & sufficient condition for a sequence $\{S_n\}$ to converge to l is that to each $\epsilon > 0$, there corresponds to a +ve integer m such that $|S_n - l| < \epsilon$, $\forall n \geq m$.

Non-convergent sequences:

Bounded: A bounded sequence which does not converge and has atleast two limit points is said to oscillate

Ex: $\{1, 2, 1, 2, 1, 2, \ldots\}$

Unbounded: Limit point $-\infty, \infty$

~~Limit~~ $\lim S_n = \infty, -\infty$ (divergent)

Oscillates infinitely if it diverges neither to $-\infty$ or ∞

Show that $\lim_{n \to \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$

Let ϵ be any +ve number

$$\left| \frac{3 + 2\sqrt{n}}{\sqrt{n}} - 2 \right| < \epsilon$$

if $\left| \frac{3}{\sqrt{n}} \right| < \epsilon$

or $n > \frac{9}{\epsilon^2}$

Let m be a +ve integer gretater than $\frac{9}{\epsilon^2}$

Thus to $\epsilon > 0$, $\exists$ a +ve integer m,

such that $\left| \frac{3 + 2\sqrt{n}}{\sqrt{n}} - 2 \right| < \epsilon \quad \forall n \geq m$

$$\therefore \lim_{n \to \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2.$$

Th. If $\{a_n\}$ converges to L & $c \in R$ then the ~~converge~~ sequence $\{ca_n\}$ converges to $c.L$

$$\underset{n\to\infty}{Lt}\ c \cdot a_n = c \cdot \underset{n\to\infty}{Lt}\ a_n$$

Th $\underset{n\to\infty}{Lt}(a_n+b_n) = \underset{n\to\infty}{Lt}\ a_n + \underset{n\to\infty}{Lt}\ b_n$

Th $\underset{n\to\infty}{Lt}(a_n \cdot b_n) = \underset{n\to\infty}{Lt}\ a_n \cdot \underset{n\to\infty}{Lt}\ b_n$.

Th $\{a_n\}$ converges to L

& $\{b_n\}$ converges to M

$a_n \le b_n \ \forall\ n \ge m$

then $L \le M$.

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Sandwich Theorem (Squeeze theorem):

If $\{a_n\}, \{b_n\}, \{c_n\}$ are three sequences such that

if $a_n \le b_n \le c_n \quad \forall\ n \in N \rightarrow ①$

and $\underset{n\to\infty}{Lt}\ a_n = \underset{n\to\infty}{Lt}\ c_n = l$

then $\underset{n\to\infty}{Lt}\ b_n = l$.

$c_n = \{1+\frac{1}{n}\} \rightarrow 1$

$a_n = \{1-\frac{1}{n}\} \rightarrow 1$

$b_n = \{1\} \rightarrow 1$

Proof: Let $\epsilon > 0$ be given $\{a_n\}, \{c_n\}$ converge to l

$\exists$ +ve integers m_1, m_2 such that

$|a_n - l| < \epsilon \ \forall\ n \ge m_1$ & $|c_n - l| < \epsilon \ \forall\ n \ge m_2 \rightarrow ②$

Let $m = \max(m_1, m_2)$

then for all $n \geq m$ we have from ① & ②

$$l-\epsilon < a_n \leq b_n \leq c_n < l+\epsilon$$

$$l-\epsilon < b_n < l+\epsilon \quad \forall n \geq m$$

$$|b_n - l| < \epsilon \quad \forall n \geq m$$

Hence $\underset{n\to\infty}{Lt}\, b_n = l$.

Ex:

$$\{b_n\} = \left\{\frac{\sin(n)}{n}\right\} \quad n \in N$$

$$-1 \leq \sin(n) \leq 1 \quad \forall n \in N$$

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \quad \forall n \in N$$

We choose $\{a_n\} = \left\{-\frac{1}{n}\right\}$ and

$$\{c_n\} = \left\{\frac{1}{n}\right\} \quad \forall n \in N$$

$$a_n \leq b_n \leq c_n \quad \forall n \in N$$

Now $\underset{n\to\infty}{Lt}\, a_n = \underset{n\to\infty}{Lt}\, c_n = 0$

$\therefore \{b_n\}$ converges to 0.

Monotonic Sequence:

A sequence $\{S_n\}$ is said to be monotonic increasing if $S_{n+1} \geq S_n$ $\forall n \in N$ and monotonic decreasing if $S_{n+1} \leq S_n$ $\forall n \in N$. It is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

strictly increasing $S_{n+1} > S_n \quad \forall n \in N$

strictly decreasing $S_{n+1} < S_n \quad \forall n \in N$

* Monotonic sequences either converge or diverge, they can't oscillate

Ex: Monotonic increasing : $2^n, n^2, e^n$

Monotonic decreasing : $\frac{1}{n}, -n, e^{-n}$

Th: A necessary & sufficient condition for the converge of a monotonic sequence is that it is bounded.

Corollary 1

A monotonic increasing bounded above sequence converges to its least upper bound and a monotonic decreasing bounded below to the greatest lower bound.

Ex $\{\frac{1}{n}\} \to 0$ (glb) $\{-\frac{1}{n}\} \to 0$ (lub)

Corollary 2

Every monotonic increasing sequence which is not bounded above, diverges to $(+\infty)$ Ex: $\{n^2\} \to \infty$

Corollary 3

Every monotonic decreasing sequence which is not bounded below, diverges to $(-\infty)$

Ex: $\{-n^3\} \to -\infty$

Subsequences:

If $\{S_n\} = \{S_1, S_2, S_3 \dots\}$ be a sequence, then any infinite succession of its terms picked out in any way (but preserving the original order) is called a subsequence of $\{S_n\}$ or in otherwords if $\{n_k\}$ be a strictly monotonic increasing sequence of natural numbers

i.e., $n_1 < n_2 < n_3 < \ldots$ then $\{S_{n_k}\}$ is a subsequence of the sequence $\{S_n\}$

Ex:

$\{S_2, S_4, S_6, S_8, \ldots S_{2n}, \ldots\}$ is a subsequence of $\{S_n\}$

$\{S_1, S_4, S_9, \ldots S_{n^2}, \ldots\}$ is a subsequence of $\{S_n\}$

$\{S_7, S_8, S_9, \ldots\}$ obtained by removing a finite number of terms from the beginning is a subsequence of $\{S_n\}$

Th

A sequence $\{S_n\}$ converges to s iff its every subsequence converges to s. Similarly $\underset{n\to\infty}{Lt}\, S_n = \infty$ or $-\infty$ iff every subsequence of $\{S_n\}$ tends to ∞ or $-\infty$.

Th

If ξ is a limit point of a sequence $\{S_n\}$ then there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ which converge to ξ i.e., $\underset{k\to\infty}{Lt}\, S_{n_k} = \xi$.

Ex:

$$S_n = \left\{ \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2^9}, \frac{1}{2^{16}}, \ldots \frac{1}{2^{n^2}}, \ldots \right\}$$

Monotonically decreasing and bounded.

$\therefore \{S_n\}$ is convergent. $\underset{n\to\infty}{Lt}\, S_n = 0$

$$\underset{n\to\infty}{Lt}\, \{x^n\} = 0 \text{ if } |x| < 1$$

Show that the sequence $\{b_n\}$, where

$$b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2} \right]$$ converges to zero

$$\frac{1}{(n+1)^2} \geq \frac{1}{(2n)^2}$$

$$\frac{1}{(n+2)^2} \geq \frac{1}{(2n)^2}$$

$$\vdots$$

$$\frac{1}{(2n)^2} \geq \frac{1}{(2n)^2}$$

sum $\downarrow$ sum

$$\frac{n}{(2n)^2} \leq b_n \leq \frac{n}{n^2}$$

$$\frac{1}{4n} \leq b_n \leq \frac{1}{n}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{1}{4n} = 0, \quad \operatorname*{Lt}_{n\to\infty} \frac{1}{n} = 0$$

$$\therefore \operatorname*{Lt}_{n\to\infty} b_n = 0$$

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1. $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$ show by definition.

$$|s_n - l| = \left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1-n-1}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1}$$

$\frac{2}{n+1} < \epsilon$ if $n+1 > \frac{2}{\epsilon}$

i.e., if $n > N$ where $N = \frac{2}{\epsilon} - 1$

$\forall \epsilon > 0 \quad |s_n - l| < \epsilon \quad \forall n \geq m$

Given $\epsilon > 0$ $\left|\frac{n-1}{n+1} - 1\right| = \frac{2}{n+1} < \epsilon$ if $n > \underbrace{\frac{2}{\epsilon} - 1}_{m}$

$\therefore \lim_{n\to\infty} \frac{n-1}{n+1} = 1$

2. $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$ show by definition

$$(\sqrt{n+1} - \sqrt{n}) = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

Given $\epsilon > 0$, $\frac{1}{2\sqrt{n}} < \epsilon$ if $\frac{1}{4n} < \epsilon^2$

i.e., $n > \frac{1}{4\epsilon^2}$

for any $\epsilon > 0$

$$|\sqrt{n+1} - \sqrt{n} - 0| < \epsilon \quad \forall n > \underbrace{\frac{1}{4\epsilon^2}}_{m}$$

$\therefore$ By definition $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

3. $S_n = \{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, \ldots\}$

Does this converge? Limit points?

One subsequence is monotonically increasing and not bounded above

$\therefore$ It is non converging

One subsequence is monotonically decreasing

Limit point $= 0$

4. $$\lim_{n\to\infty} \frac{2-\cos n}{n+3} = ?$$

$$-1 \leq \cos n \leq 1$$

$$-1 \leq -\cos n \leq 1$$

$$1 \leq 2-\cos n \leq 3$$

$$\frac{1}{n+3} \leq \frac{2-\cos n}{n+3} \leq \frac{3}{n+3}$$

$$\lim_{n\to\infty} \frac{1}{n+3} = 0 \qquad \lim_{n\to\infty} \frac{3}{n+3} = 0$$

$$\therefore \lim_{n\to\infty} \frac{2-\cos n}{n+3} = 0$$

5. $S_n = \{\sqrt{30}, \sqrt{30+\sqrt{30}}, \sqrt{30+\sqrt{30+\sqrt{30}}}, \ldots\}$

Does it converge?

$a_1 = \sqrt{30}$, $\quad a_{n+1} = \sqrt{30+a_n} \quad \forall n \geq 1$

$a_1 = \sqrt{30} < \sqrt{36} \qquad a_1 < 6$

$a_2 = \sqrt{30+a_1} < \sqrt{30+6}$, $\quad a_2 < 6$

$a_3 = \sqrt{30+a_2} < \sqrt{30+6} \qquad a_3 < 6$

$\therefore$ Monotonically increasing and bounded above $\therefore$ Converging.

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Infinite Series:

Def: If $u_1, u_2, u_3, \ldots u_n, \ldots$ be an infinite sequence of real numbers then $u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$ is called an infinite series denoted by $\sum u_n$ and the sum of first n terms is denoted by S_n. (Sequence of partial sums)

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$\vdots$$
$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$\{S_n\} \rightarrow n^{th}$ partial sum.

Convergence, divergence and oscillation of a series:

Consider the infinite series

$$\sum u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$$

and let the sum of the first 'n' terms be

$$S_n = u_1 + u_2 + \ldots + u_n.$$

clearly, S_n is a function of 'n' and as 'n' increases indefinitely there are 3 possibilities.

1) If S_n tends to a finite limit as $n \rightarrow \infty$, the series $\sum u_n$ is said to be convergent.

2) If S_n tends to $\pm \infty$ as $n \rightarrow \infty$ the series $\sum u_n$ is said to be divergent.

3) If S_n does not tend to a unique limit as $n \to \infty$ then the series Σu_n is said to be oscillatory or non-convergent

Ex: Examine the convergence of the series

$$1+2+3+\cdots+n+\cdots\infty$$

$$S_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\underset{n\to\infty}{Lt}\ S_n = \frac{1}{2}\cdot\underset{n\to\infty}{Lt}\ n(n+1) = \infty \text{ is divergent}$$

$\therefore \Sigma u_n$ is divergent

Ex: Examine the convergence of the series

$$5-4-1+5-4-1+5-4-1+\cdots\infty$$

$$S_n = 5-4-1+5-4-1+\cdots n \text{ terms}$$

$$S_n = 5 \quad \overset{n}{(3m+1)}$$
$$= 1 \quad (3m+2)$$
$$= 0 \quad (3m)$$

Σu_n is oscillatory.

Geometric series:

i) Converge if $|r|<1$

ii) diverges if $r \geq 1$ and ~~osci~~

iii) oscillates if $r \leq -1$

Let $S_n = 1+r+r^2+\cdots+r^{n-1}$

* If $|r|<1$ $\quad \underset{n\to\infty}{Lt}\ r^n = 0$

$$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r}$$

∴ The series is convergent

* When $r>1$ $\quad \underset{n\to\infty}{Lt}\ r^n$ diverges to ∞

$$S_n = \frac{r^n-1}{r-1} = \frac{r^n}{r-1} - \frac{1}{r-1}$$

∴ $\underset{n\to\infty}{Lt}\ S_n$ diverges to ∞

∴ The series is divergent.

* When $r<-1$, Let $r=-\rho$, so that $\rho>1$ then $r^n = (-1)^n \rho^n$

$$S_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n \rho^n}{1+\rho}$$

As $\lim_{n\to\infty} \rho^n \to \infty$ $\quad \underset{n\to\infty}{Lt}\ S_n \to -\infty$ or ∞

according to n (n is even or odd)

∴ The series is oscillating.

Necessary condition for convergence of infinite series

A necessary condition for convergence of an infinite series $\sum u_n$ is that $\lim_{n\to\infty} u_n = 0$

proof: Let $S_n = u_1 + u_2 + \cdots + u_n$ $\{S_n\}$

As $\sum u_n$ converges, then the sequence $\{S_n\}$ also converges

Let $\lim_{n\to\infty} S_n = s$ (say)

Now $u_n = S_n - S_{n-1}, \quad n > 1$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0$$

Converse may not be true.

* $\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \infty$ (Harmonic series)

$u_n = \frac{1}{n}$

$\lim_{n\to\infty} u_n = 0$ but not convergent.

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2}$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2}$$

$$S_{2^n} > \frac{1}{2}(n+2)$$

$\sum u_n$ is divergent.

* $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \cdots$

$|r| = \frac{1}{2} = 0.5$

$|r| < 1$ so convergent.

$S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$

$= \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{\frac{1}{2}} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

$\lim_{n \to \infty} S_n = 1$

* $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

$1 - \frac{1}{2} + 1 - \frac{1}{3} + 1 - \frac{1}{4} + \cdots$

$(1 + 1 + 1 + \cdots) - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$

$u_n = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$

$\lim_{n \to \infty} u_n = 1 \neq 0$

Σu_n is not convergent.

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$1 - 1 + 1 - 1 + 1 - 1 + \cdots \infty$ is convergent or non convergent or divergent?
(Grandi's series)

$S_n = 1 - 1 + 1 - 1 + \cdots$ (n terms)

$S_n = 0$ n is even
$= 1$ n is odd

$S = 1 - (1 - 1 + 1 - \cdots + \infty)$

$S = 1 - S$

$2S = 1$

$S = \frac{1}{2}$

sum is neither 0, 1 nor $\frac{1}{2}$

Series is non convergent.

General properties of series:

1) The convergence or divergence of ~~of~~ an infinite series remains unaffected by the addition or removal of a finite number of its terms.

2) If a series in which all the terms are positive, is convergent, the series remains ~~constar~~ convergent even when some or all of its terms are negative.

Ex: $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \infty$

$-1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \cdots \infty$

3) The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.

Ex: $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots\infty$

$\sum \frac{1}{2n}$

$\sum \frac{1}{n}$ is divergent (Harmonic series)

Series $\sum \frac{1}{2n}$ is divergent

Series of positive terms:

An infinite series in which all the terms after some particular terms are +ve, is a positive term series

Ex: $-7-5-2+2+7+13+20+\cdots\infty$

Integral Test:

A positive term series $f(1)+f(2)+\cdots+f(n)+\cdots$ where $f(n)$ decreases as n increases, converges or diverges as the integral $\int_1^\infty f(x)dx$ is finite or infinite.

p-test:

Test for comparison

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

converges for $p>1$

diverges for $p\leq 1$

Ex: $p = 2$

$\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is convergent (p-test)

$p = \frac{1}{2}$ $\sum \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ is divergent (p-test)

Comparison Tests:

1) If two positive term series $\sum u_n$ and $\sum v_n$ be such that

i) $\sum v_n$ converges and ii) $u_n \le v_n \ \forall n$

then $\sum u_n$ also converges.

* If $\sum v_n$ converges & $u_n \ge v_n$ we cannot judge anything

Ex: $\sum u_n = \sum \frac{1}{2^n + n}$ $\qquad v_n = \sum \frac{1}{2^n}$ (geometric series $|r| < 1$) converges

$2^n + n > 2^n \quad \forall n$

$\frac{1}{2^n + n} < \frac{1}{2^n} \quad \forall n$

$u_n \le v_n \quad \forall n$

$\sum u_n$ converges by comparison test.

2) If two positive term series $\sum u_n$ and $\sum v_n$ be such that

i) $\sum v_n$ diverges and ii) $v_n \le u_n \ \forall n$

than $\sum u_n$ also diverges

$\sum u_n = \sum \frac{1}{n+1}$ $\qquad \sum v_n = \sum \frac{1}{2n}$ diverges

$n + 1 < 2n \ \forall n$ $\qquad u_n \ge v_n \ \forall n$

$\frac{1}{n+1} > \frac{1}{2n} \ \forall n$ $\qquad \sum u_n$ diverges

Limit form (Comparison test):

If two positive term series $\sum u_n$ & $\sum v_n$ be such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = k \; (\neq 0) \text{ finite number then}$$

$\sum u_n$ and $\sum v_n$ converge or diverge together.

Ex: $\sum u_n = \sum \sin\frac{1}{n}$ $\quad v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \neq 0$$

$\sum v_n$ diverges so $\sum u_n$ also diverges.

D' Alembert's Ratio test:

In a positive term series $\sum u_n$

if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, then the series converges

if $\lambda < 1$ & diverges if $\lambda > 1$

The test fails if $\lambda = 1$

Ex: $\sum \frac{n^2-1}{n^2+1} x^n \quad , x > 0$

Let $u_n = \frac{n^2-1}{n^2+1} \cdot x^n$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{\frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}}{\frac{n^2-1}{n^2+1} \cdot x^n}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = x$$

By D'Alemberts Ratio tes $\sum u_n$ converges

if $x < 1$ and diverges if $x > 1$

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* In a positive term series $\sum u_n$

If $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = k$

$\sum u_n$ converges for $k > 1$

diverges for $k < 1$

* Ratio test for $\lambda = 1$ [D'Alemberts Ratio Test]

$\sum u_n = \sum \frac{1}{n^p}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^p}$$

$$= 1 = \lambda$$

$\lambda = 1$ for any p.

But we know that from p-test

$\sum u_n$ converges for $p > 1$

and diverges for $p \leq 1$

Cauchy's Root Test: In a positive term series $\sum u_n$

if $\lim\limits_{n\to\infty} (u_n)^{1/n} = \lambda$, then the series converges for

$\lambda < 1$ and diverges for $\lambda > 1$.

This test fails when $\lambda = 1$.

Ex: $\sum (\log n)^{-2n}$ $\quad u_n = (\log n)^{-2n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} (\log n)^{-2} = 0\ (<1)$$

Hence by Cauchy's root test, the given series converges.

Alternating series: A series in which the terms are alternatively +ve or −ve is called an alternating series

Leibnitz's series: An alternating series $u_1 - u_2 + u_3 - u_4 + \cdots \infty$ converges if

i) each term is numerically less than its preceeding term

ii) $\lim_{n\to\infty} u_n = 0$

Ex: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \infty$ converges or not?

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Leibnitz's rule

The terms of the given series are alternatively +ve & −ve.

$$u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} < 0$$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Hence by Leibnitz's rule

Σu_n converges.

* **Series of positive & Negative Terms (general)**

The +ve term series & alternating series are special types of these series with arbitrary signs

Absolutely convergent: If the series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots \infty$ be such that the series $|u_1| + |u_2| + |u_3| + \cdots + |u_n| + \cdots \infty$ is convergent, then the original series $\sum u_n$ is said to be absolutely convergent.

Ex:

$$\sum u_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \cdots - (-1)^n \frac{1}{n^2} + \cdots \infty$$

is absolutely convergent

Conditionally convergent:

If $\sum |u_n|$ is divergent but $\sum u_n$ is convergent then $\sum u_n$ is said to be conditionally convergent.

Ex:

$\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \infty$ is convergent (By Leibnitz's rule)

$$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \infty$$

$= \sum \frac{1}{n}$ is divergent

So the original series $\sum u_n$ is conditionally convergent.

Note: An absolutely convergent series is necessarily convergent but not conversely.

Ex: Test whether the following series is absolutely convergent or not?

$$\sum u_n = \sum \frac{(-1)^{n-1}}{2n-1}$$

$$\sum u_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \frac{(-1)^{n-1}}{2n-1}$$

(Alternating series)

$$\underset{n\to\infty}{Lt}\ u_n = \underset{n\to\infty}{Lt}\ \frac{(-1)^{n-1}}{2n-1}$$

$$= \underset{n\to\infty}{Lt}\ \frac{(-1)^{n-1}}{n\left(2-\frac{1}{n}\right)}$$

$$\underset{n\to\infty}{Lt}\ u_n = 0$$

$$u_n - u_{n-1} = \frac{(-1)^{n-1}}{2n-1} - \frac{(-1)^{n-2}}{2(n-1)-1}$$

$$= \frac{(-1)^{n-1}}{2n-1} - \frac{(-1)^{n-2}}{2n-3}$$

$$= \frac{1}{2n-1} - \frac{1}{2n-3}$$

$$= \frac{-2}{(2n-1)(2n-3)}$$

$$u_n - u_{n-1} < 0$$

By leibnitz rule Σu_n is convergent

comparison test $\Sigma|u_n| = \Sigma\left|\frac{1}{2n-1}\right|$ $\quad \Sigma v_n = \frac{1}{n}$

$$\underset{n\to\infty}{Lim}\ \frac{u_n}{v_n} = \underset{n\to\infty}{Lim}\ \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \frac{1}{2} \ (\neq 0$$

Σv_n diverges so $\Sigma|u_n|$ also diverges.

Σu_n is convergent & $\Sigma|u_n|$ diverges

$\therefore$ Σu_n is conditionally convergent.

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Power Series: A series of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ where the a's are independent of x, is called a power series in x.

This series may converge or diverge for different values of x.

Interval of convergence:

In the power series, $u_n = a_n x^n$

The interval in which power series is convergent

How do you test for convergence?

Ratio test

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{a_{n+1} x^{n+1}}{a_n x^n}$$

$$= \underset{n\to\infty}{Lt} \left(\frac{a_{n+1}}{a_n}\right) x$$

$$\underset{n\to\infty}{Lt} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{Lt} \left|\frac{a_{n+1}}{a_n} x\right| = |x| \underset{n\to\infty}{Lt} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= l \cdot |x| \qquad \left[\underset{n\to\infty}{Lt} \left|\frac{a_{n+1}}{a_n}\right| = l\right]$$

Σu_n converges if $l \cdot |x| < 1$

$$|x| < \frac{1}{l}$$

(or) $$-\frac{1}{l} < x < \frac{1}{l}$$

$$\frac{1}{(1-x)} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \cdots \infty$$

$$u_n = \frac{1}{n(1-x)^n}$$

$$\underset{n\to\infty}{Lt} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{Lt} \left|\frac{\frac{1}{(n+1)(1-x)^{n+1}}}{\frac{1}{n(1-x)^n}}\right|$$

$$= \underset{n\to\infty}{Lt} \left| \frac{n(1-x)^n}{(n+1)(1-x)^{n+1}} \right| = \underset{n\to\infty}{Lt} \left| \frac{n}{n+1} \frac{1}{(1-x)} \right|$$

$$= \left| \frac{1}{(1-x)} \right| < 1 \implies -1 < \frac{1}{1-x} < 1$$

$$x-1 \ngtr 1 \qquad 1-x > 1$$

Interval of convergence $\qquad x \ngtr 2 \qquad x \nless 0$

$x < 0$ & $x \geq 2$

For $x=2$ $\sum u_n = \sum \frac{(-1)^n}{n}$ converges by leibnitz rule. For $x=0$ $\sum u_n = \sum \frac{1}{n}$ divergent

Convergence of Exponential series. (e^x)

The series is $1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \infty$

Find the interval of convergence

Let $u_n = \frac{x^n}{n!}$, $u_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{n+1}$$

$$= x \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

$\sum u_n$ coverges for any value of x.

Convergence of logarithmic series;

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots \infty$$

Find the interval of convergence.

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{n+1}}{(n+1)} \cdot \frac{n}{(-1)^{n+1} \cdot x^n} \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{n}{n+1} \right|$$

$$= |x| < 1 \implies -1 < x < 1$$

$x=1 \Rightarrow \quad 1-\frac{1}{2}+\frac{1}{3} \cdots \frac{(-1)^{n+1}}{n} \cdots$

converges by leibnitz rule

Interval of convergence $-1 \le x \le 1$

Convergence of Binomial Series:

$$1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{r!}\cdot x^r+\cdots$$

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(1) $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$ converges or diverges?

$u_n = \frac{n^3}{n^5+3}$ $\quad$ $v_n = \frac{n^3}{n^5}$ $\quad$ [+ve term series]

$v_n = \frac{1}{n^2}$

$\sum v_n = \sum \frac{1}{n^2}$ converges by p-test

$u_n < v_n \quad \forall n \in N$

By comparison test

$\sum u_n$ also converges.

(2). $\sum_{n=1}^{\infty} \frac{3^n}{4^n+4}$

$u_n = \frac{3^n}{4^n+4}$ $\quad$ $v_n = \frac{3^n}{4^n}$

$\sum v_n = \sum \left(\frac{3}{4}\right)^n$

$\sum v_n$ converges geometric series with $|r|<1$

$u_n < v_n \quad \forall n \in N$

By comparison test

$\sum u_n$ also converges.

(3). $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$ converges or diverges?

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!\,(n+2)!}{(3n+3)!}}{\frac{n!(n+1)!}{(3n)!}}$$

$\therefore \sum u_n$ converges

$$= \lim_{n\to\infty} \frac{(n+1)(n+2)}{(3n+1)(3n+2)(3n+3)} = 0$$

④ $\sum_{n=0}^{\infty} \frac{n^3 x^{3n}}{n^4+1}$ → Power series

Interval of convergence?

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3 x^{3(n+1)}}{(n+1)^4+1}}{\frac{n^3}{n^4+1} \cdot x^{3n}} \right| = |x|^3 \lim_{n\to\infty} \left| \frac{(n^4+1)(n+1)^3}{n^3((n+1)^4+1)} \right|$$

$$= |x|^3$$

$$|x|^3 < 1$$

$$-1 < x < 1$$

$x = -1$ $\quad \sum_{n=0}^{\infty} -\frac{n^3}{n^4+1}$

$x = 1$ $\quad \sum_{n=} \frac{n^3}{n^4+1}$ diverges by comparison test $\left(v_n = \frac{n^3}{n^4}\right)$

⑤ $\sum (-1)^n \frac{1}{\sqrt{n^2+1}}$ Does this series converge absolutely, conditionally or diverge?

$\sum u_n = \sum (-1)^n \frac{1}{\sqrt{n^2+1}}$ converges by Leibnitz's series

$$\sum |u_n| = \left| \frac{(-1)^n}{\sqrt{n^2+1}} \right|$$

$v_n = \frac{1}{n}$

$\sum v_n$ is divergent

limit form. $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = 1 \neq 0$

$\sum u_n$ converges $\quad \sum |u_n|$ diverges

$\therefore \sum u_n$ is conditionally convergent.

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Sequence & Series of Functions:

Sequence of real numbers : $f : N \to R$

Sequence of functions

$E \subseteq R \quad E = [a,b]$

For each $n \in N$ Let $f_n : E \to R$ be a function

then $\{f_n\}$ is a sequence of functions on E to R

$x \in E$ then $\{f_n(x)\}$ is a sequence of real numbers

for some $x \in E$, $\{f_n(x)\}$ may converge, for other $x \in E$ it may not.

N				
1	$\to$	$f_1 : E \to R$		$f_1(x)$
2	$\to$	$f_2 : E \to R$		$f_2(x)$
3	$\to$	$f_3 : E \to R$	$x \in E$	$f_3(x)$
$\vdots$				$\vdots$
n	$\to$	$f_n : E \to R$		$f_n(x)$
$\vdots$				$\vdots$

Ex: $f_n(x) = x + n \qquad E = [1,3]$

$x \in [1,3]$

$x = 2 \quad f_n(2) = 2 + n$

$\{f_n(2)\} = \{3, 4, 5, 6, \ldots\}$

Point wise convergence:

* If $\{f_n(x)\}$ converges $\forall x \in E$ then $f : E \to R$

$$f(x) = \lim_{n \to \infty} f_n(x), \quad x \in E$$

$\{f_n\}$ converges to f pointwise

$f_n \to f$ pointwise $\equiv f_n(x) \to f(x) \quad \forall x \in E$

Ex: $E = [0,1]$, $f_n(x) = x^n \quad 0 \le x \le 1$

$\forall x_1 \in E \quad f_n(x) \to f$ pointwise

$x_1 \in E \qquad \lim_{n\to\infty} f_n(x_1) = l_{x_1}$

$x_2 \in E \qquad \lim_{n\to\infty} f_n(x_2) = l_{x_2}$

$$f(x) = l_x = \lim_{n\to\infty} f_n(x)$$

then $f_n(x) \longrightarrow f$ pointwise.

Ex: $E = [0,1] \qquad f_n(x) = x^n \quad , 0 \le x \le 1$

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

$f_n(0) = 0$

$f_n\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^n$

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

$f_n \longrightarrow f$ pointwise

$E = [-1,1] \qquad f_n(x) = x^n$

~~$f(x) = 0 \quad -1 < x < 1$~~

$$f(x) = \begin{cases} 0 & -1 < x < 1 \\ 1 & x = 1 \end{cases}$$

$f_n(-1) = (-1)^n \to$ non convergent

f_n converges to f pointwise in $(-1, 1]$

Uniform convergence:

Let $f: E \to R$ be a sequence of functions & $f: E \to R$ be a function we say that $f_n \to f$ uniformly if $\forall \epsilon > 0$ $\exists n_0 \overset{\text{belongs to}}{\in} N$ such that $\forall n \geq n_0$

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in E$$

If $f_n \to f$ uniformly

$x = c$, $\forall \epsilon > 0$, $\exists n_0 \in N$, such that $n \geq n_0$

$$|f_n(c) - f(c)| < \epsilon$$

$$f_n(c) \to f(c)$$

$$\forall x \in E \quad f_n(x) \to f(x)$$

* If $f_n \underset{\text{converges}}{\longrightarrow} f$ uniformly then $f_n \to f$ pointwise

* If $f_n \to f$ pointwise $\equiv f_n(x) \to f(x) \quad \forall x \in E$

$\forall x \in E \quad \forall \epsilon > 0 \quad \exists n_0 \in N$

such that $\underset{\downarrow}{n \geq n_0} \Rightarrow |f_n(x) - f(x)| < \epsilon$

$n_0(x)$

Let $f_n \to f$ uniformly $\{|f_n(x) - f(x)| < \epsilon \; ; \; x \in E\}$

n_0 changes for every x in pointwise convergence and n_0 is fixed in uniform convergence. $\forall x$.

Ex: $\forall n \in N$ Let $f_n : R \to R$ $f_n(x) = \frac{x}{n}$, $x \in R$

$f_n \to f$ pointwise $f(x) = 0$ $\forall x \in R$

* $f_n \to f$ pointwise $\underbrace{\max\{n_0(x)\}}$

$\forall x \in E$, $f_n(x) \to f(x)$ $\forall n \geq n_0$

$\forall x \in E$, $\forall \epsilon > 0$, $\exists n_0 \in N$ such that $\forall n \geq n_0$

$|f_n(x) - f(x)| < \epsilon$

$x = c$ $\forall \epsilon > 0$ $\exists \underset{1000}{n_0} \in N$, such that $n \geq 1000$

$x = a$ $\forall \epsilon > 0$ $\exists \underset{10000}{n_0} \in N$, such that $|f_n(x) - f(x)| < \epsilon$

* $f_n \to f$ convergence

$\forall \epsilon > 0$, $\exists n_0 \in N$, $\forall n \geq n_0$

$|f_n(x) - f(x)| < \epsilon$, $\forall x \in E$

* $f_n \to f$ uniformly $\{\underbrace{|f_n(x) - f(x)|}_{\text{This is Bounded.}} : x \in E\}$

Let $M_n = \operatorname{Sup}\{|f_n(x) - f(x)| : x \in E\}$

$f_n \to f$ uniformly $\equiv M_n \to 0$ as $n \to \infty$

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* $\{f_n\} \to f$ pointwise on $E \subseteq R$

$\forall x \in E \quad f(x) = \lim_{n\to\infty} f_n(x)$

* $\{f_n\} \to f$ uniformly

$\forall \epsilon > 0$, $\exists n_0 \in N$ such that $n \geq n_0$

$|f_n(x) - f(x)| < \epsilon$, $\forall x \in E$

Ex: $f_n(x) = x^n \qquad 0 \leq x \leq 1$

$f(x) = 0 \qquad 0 \leq x < 1$

$\quad = 1 \qquad x = 1$

$M_n = \sup\{|x^n - f(x)| : x \in [0,1]\}$

$= \sup\{x^n : 0 \leq x < 1\}$

$x = 1 \qquad M_1 = \sup\{x : 0 \leq x < 1\} = 1$

$x = 2 \qquad M_2 = \sup\{x^2 : 0 \leq x < 1\} = 1$

$M_n \to 1$ as $n \to \infty$

But for uniform convergence $M_n \to 0$

$f_n(x) = x^n$ is pointwise convergente but not uniformly convergent.

Ex: Let $D = \{x \in R : x \geq 0\}$ and for each $n \in N$ let

$f_n : D \to R$ be defined by $f_n(x) = \dfrac{x}{1+nx}$, $x \geq 0$

Does it converge?

Sol $f(x) = 0$, $x \geq 0$

$f_n \to f$ pointwise

$\Rightarrow$ Is it uniform convergence?

$\forall x \geq 0$ $\quad |f_n(x) - f(x)| = \frac{x}{1+nx} < \frac{1}{n}$ $\qquad \frac{1}{n} < \epsilon$, $n > \frac{1}{\epsilon}$

Let $\epsilon > 0$ then $|f_n(x) - f(x)| < \epsilon$, $\forall n > \frac{1}{\epsilon}$

Let $k = \left[\frac{1}{\epsilon}\right] + 1$ then $k \in N$ & $\forall x \geq 0$

$$|f_n(x) - f(x)| < \epsilon \quad , \forall n \geq k$$

This proves that $\{f_n\} \to f$ uniformly on $[0, \infty)$

Ex: $f_n(x) = 1 + x + x^2 + \dots + x^{n-1}$, $x \in [0, 1)$ (Geometric series)

$\lim_{n \to \infty} f_n(x) = \frac{1}{1-x}$, $x \in [0, 1)$ $\qquad f(x) = \frac{1}{1-x}$ $\downarrow$ is not bounded.

$f_n \to f$ pointwise

$|f_n(x)| = |1 + x + x^2 + \dots + x^{n-1}| \leq 1 + |x| + |x^2| + \dots + |x^{n-1}|$

$\leq n \qquad \forall x \in [0, 1)$

Each f_n is bounded on $[0, 1)$ but f is unbounded on $[0, 1)$

Th Let $D \subseteq R$ and for each $n \in N$, $f_n : D \to R$ is bounded on D. If the sequence $\{f_n\}$ be uniformly convergent on D, then the limit function f is bounded on D.

Th$^{\underline{m}}$ Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ is continous on D; If the sequence $\{f_n\}$ be uniformly convergent on D to a function f, then f is continous on D.

* If $f_n \to f$ and f, f_n is continous that does not mean f_n is uniformly convergent.

Ex: $f_n(x) = \dfrac{x^n}{1+x^n}$ $\quad x \in [0, 2]$

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & 1 < x \leq 2 \end{cases}$$

$f_n \to f$ pointwise

Each f_n is continous on $[0, 2]$

f is not continous on $[0, 2]$

The convergence of the sequence is not uniform.

__Series of functions:__

Let $D \subseteq \mathbb{R}$, Let $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$. Then $f_1 + f_2 + f_3 + \cdots$ is said to be a series of functions on D. $(\sum f_n)$

$\{S_n\}$ is defined for $x \in D$ by

$S_1(x) = f_1(x)$

$S_2(x) = f_1(x) + f_2(x)$

$S_3(x) = f_1(x) + f_2(x) + f_3(x)$

$\vdots$

$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

The sequence $\{S_n\}$ is said to be sequence of partial sums of the infinite series Σf_n

If $\{S_n\}$ is pointwise convergence on D to a function then Σf_n is said to be pointwise convergence on D and S is said to be the sum function of the series Σf_n on D.

If $\{S_n\} \to S$ uniformly on D then

Σf_n is said to be uniformly convergent on D to the sum function S

<u>Prob</u>: Prove that the series of f^ns

$1 + x + x^2 + \cdots + x^{n-1}$ $0 \le x < 1$ is pointwise convergent on $0 \le x < 1$ but the convergence is not uniform on $[0, 1)$

<u>Sol</u> $S_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$

$$\lim_{n \to \infty} S_n(x) = \frac{1}{1-x}, \quad x \in [0, 1)$$

$\Sigma f_n \to S$ pointwise

$S_n(x) \to S$ pointwise

The convergence is not uniform since S is not bounded